Concentration phenomena in the optimal design of thin rods

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Abstract

In this paper we analyze the concentration phenomena which occur in thin rods, solving the following optimization problem: a given fraction of elastic material must be distributed into a cylindrical design region with infinitesimal cross section in an optimal way, so that it maximizes the resistance to a given external load. For small volume fractions, the optimal configuration of material is described by a measure which concentrates on 2-rectifiable sets. For some choices of the external charging, the concentration phenomena turn out to be related to some new variants of the Cheeger problem of the cross section of the rod. The same study has already been carried out in the particular case of pure torsion regime in [6]. Here we extend those results by enlarging the class of admissible loads.

Keywords: thin rods, optimization, compliance, duality, dimension reduction, Cheeger problem, generalized Cheeger sets

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1 Introduction

In this paper we study the problem of characterizing the most robust configurations of an isotropic elastic material when the design region is a thin rod subject to a given external load. Moreover, at the same time, we let the ratio between the volume of the elastic material and the volume of the design region tend to zero.

We represent a rod as a thin straight cylinder of the form

$$Q_\delta := \delta\overline{D} \times I \ ,$$

where the cross section $D \subset \mathbb{R}^2$ is an open bounded domain, the axis $I \subset \mathbb{R}$ is a closed bounded interval and $\delta > 0$ is a vanishing parameter describing the small ratio between the diameter of the cross section and the length of the axis.

We let the design region Q_δ be subject to an external load $F^\delta \in H^{-1}(Q_\delta;\mathbb{R}^3)$, which we assume to be a suitable scaling of a fixed field $F \in H^{-1}(Q;\mathbb{R}^3)$, being $Q := \overline{D} \times I$.

The optimization problem under consideration takes the form of a double limit process of a 2-parameters family of variational problems: we study

$$\phi^\delta(k) := \inf_{\Omega\subset Q_\delta} \left\{ \mathcal{C}^\delta(\Omega) + k\,\frac{|\Omega|}{|Q_\delta|} \right\} \ , \tag{1.1}$$

as $\delta \to 0^+$ and $k \to +\infty$.

The first term of (1.1) is a shape functional, called *compliance*, which, in the framework of small displacements, describes the resistance to the external load of an isotropic elastic material that occupies a certain region $\Omega \subset Q_\delta$. More precisely, the smaller is the compliance, the higher is the resistance. Such shape functional is defined as

$$\mathcal{C}^\delta(\Omega) := \sup \left\{ \langle F^\delta, u \rangle_{\mathbb{R}^3} - \int_\Omega j(e(u))\, dx \ : \ u \in H^1(Q_\delta, \mathbb{R}^3) \right\} , \tag{1.2}$$

where, as usual in linear elasticity, $e(u)$ denotes the symmetric part of the gradient ∇u, and the strain potential $j : \mathbb{R}^{3\times 3}_{\mathrm{sym}} \to \mathbb{R}$, assumed to be isotropic, is strictly convex and has the form

$$j(z) := \frac{\lambda}{2} \operatorname{tr}^2(z) + \eta |z|^2 ,$$

$\lambda, \mu > 0$ being the Lamé coefficients of the material. The scaling F^δ is chosen so that in the limit process the infimum remains finite and, as it is customary in the literature, it depends on the assumptions made on the type of applied loads (see *e.g.* [13]). Clearly, in order that $\mathcal{C}^\delta(\Omega)$ remains finite, the load must have support contained into $\overline{\Omega}$, moreover it has to be *balanced*, *i.e.*

$$\langle F^\delta, u \rangle_{\mathbb{R}^3} = 0 , \quad \text{whenever } e(u) = 0 .$$

The second term of (1.1) is the product of the volume fraction $|\Omega|/|Q_\delta|$ occupied by the material in the design region and a Lagrange multiplier $k \in \mathbb{R}$.

In the first passage to the limit we let $\delta \to 0^+$, keeping k fixed: this corresponds to look for the most robust configuration in a rod-like set, when the ratio between the volume of material and the volume of the thin design region is prescribed.

The asymptotical study of $\phi^\delta(k)$ is based on the comparison with the "fictitious counterpart", namely their relaxed formulation in $L^\infty(Q_\delta; [0,1])$: it is well known that the infimum problems (1.1) are in general ill-posed, due to occurrence of homogenization phenomena which prevent the existence of an optimal domain (see [1]); thus we need to enlarge the class of admissible materials, passing from "real" materials, represented by characteristic functions, to "composite" materials, represented by densities with values in $[0,1]$.

The limit $\phi(k) := \lim_{\delta\to 0^+} \phi^\delta(k)$ is a variational problem set over the space of densities and turns out to be solvable section by section.

In the second limit process we study the asymptotic behavior of problem $\phi(k)$ as $k \to +\infty$, namely when the "filling ratio" is infinitesimal. We show that the sequence $\phi(k)$ is asymptotically equivalent to $\sqrt{2k}$, and we determine the limit $\overline{m} := \lim_{k\to+\infty} \frac{\phi(k)}{\sqrt{2k}}$. Such limit is again a variational problem solvable section by section, and it is set in the space $\mathcal{M}^+(Q)$ of positive measures on $\mathbb{R}^3$, compactly supported in Q. This new setting is well adapted to the limit problem, since we wish to detect concentration phenomena, which may occur in lower dimensional parts.

The problem we treat, and consequently the approach we adopt to solve it, draws its inspiration from the recent works by G. Bouchitté, I. Fragalà, I. Lucardesi and P. Seppecher: in [7] the authors studied the compliance optimization problem when the design region is a thin plate, described by a family of cylinders having infinitesimal thickness; while in [6] the authors studied the problem of thin rods just in torsion regime and found that, when the cross section is convex, the optimal material concentrates, section by section, on the boundary of the so called Cheeger set of the section (see [6, Theorem 5.4]). We recall that a Cheeger set of D is a minimizer for the quotient perimeter/area among all the measurable subsets of D (for more details see [3, 8, 9, 11]):

$$\inf_{E \subset \overline{D}\,,\, \mathbb{1}_E \in BV(\mathbb{R}^2)} \frac{\int_{\mathbb{R}^2} |\nabla \mathbb{1}_E|}{|E|}\ . \tag{1.3}$$

In this paper we enlarge the class of admissible loads, letting the bending, twisting and stretching energies interplay. Our generalization is not merely a technical variant of [6] and the asymptotics change significantly: the generalized limit problem reads

$$\overline{m} = \int_I m(r(x'), t, s_0, s_1, s_2)\ , \tag{1.4}$$

where

$$m(r,t,s_0,s_1,s_2) := \inf_{\sigma \in \mathcal{M}(D;\mathbb{R}^3)} \left\{ \int_D |\sigma| \ :\ \mathrm{div}_{x'}(\sigma_1,\sigma_2) = r(x')\,,\ \int_D (x_1\sigma_2 - x_2\sigma_1) = t\,, \right. \\ \left. \int_D \sigma_3 = s_0\,,\ \int_D x_\alpha \sigma_3 = s_\alpha \right\}\ , \tag{1.5}$$

with $r \in H^{-1}(D)$ and $t, s_i \in \mathbb{R}$ functions depending on the particular load F chosen. The expression (1.4) means that an optimal configuration is represented by a measure which solves, section by section, the variational problem m in (1.5). The description of concentration phenomena is in general a difficult task; here we provide some examples for which the solution of the variational problem $\overline{m}$ can be written explicitly, or characterized in terms of new variants of the Cheeger problem.

For vertical loads such that $r = t = 0$, the optimal configurations concentrate on a subset of the boundary of the cylinder Q (see §5.1). When $r = s_i = 0$, corresponding to pure torsion chargings (see §5.2), problem (1.5) can be written equivalently as

$$\inf \left\{ \int_D |Du| \ :\ u \in BV_0(D)\,,\ \int_D u = 1 \right\}\ , \tag{1.6}$$

which is the relaxed formulation of the Cheeger problem (1.3). As already pointed out, this means that, on each section, the support of an optimal measure coincides with the boundary of the Cheeger set of D. In general, the optimal measure cannot be written explicitly. However, under different assumptions on the load, (1.5) reduces to one of the following variants of the Cheeger problem (1.6): the first variant comes into play when $s_i = 0$ and it is a sort of perturbation with a translation term (see §5.3):

$$\inf \left\{ \int_D |Du + q| \ :\ u \in BV_0(D)\,,\ \int_D u = 1 \right\}\ , \tag{1.7}$$

being q a fixed vector field; while the second variant, appearing when $r = 0$, is a weighted version (see §5.4):

$$\inf \left\{ \int_D \alpha |Du| \ :\ u \in BV_0(D)\,,\ \int_D u = 1 \right\}\ , \tag{1.8}$$

being α a non negative function in D. We point out that problem (1.8) has been treated some years ago by Ionescu and Lachand-Robert: in [10] the authors, motivated by applications to landslides modeling, study the case in which both the integral to minimize and the integral in the constraint are weighted.

The paper is organized as follows.

In Section 2 we list the notation adopted throughout the paper and introduce the classes of displacement fields.
In Section 3 we introduce the admissible loads and provide some examples.
In Section 4 we carry out the asymptotic analysis of (1.1) in the double limit process, as $\delta \to 0^+$ and $k \to +\infty$, specifying the scaling we adopt: in Theorem 4.1 we write the limit $\phi(k) := \lim_{\delta\to 0^+} \phi^\delta(k)$, while in Theorem 4.8 we characterize the limit $\overline{m} := \lim_{k\to+\infty} \phi(k)/\sqrt{2k}$. Moreover we rewrite both $\phi(k)$ and $\overline{m}$ in a dual form (Propositions 4.4 and 4.9) and compare the solutions of the primal and dual problems deriving optimality conditions (Propositions 4.7 and 4.10).
The proof of these results is quite technical and follows the same approach performed in the pure torsion regime (treated in [6]). However, for the benefit of the reader, we recall the main steps in Section 6, the Appendix.
In Section 5 we analyze the limit problem $\overline{m}$ and try to characterize the optimal configurations, in terms of optimal measures. Under particular assumptions on the load chosen, we are able to characterize it explicitly or to provide qualitative properties which turn out to be related to some new variants of the Cheeger problem (see Theorem 5.7, Remark 5.8 and Proposition 5.10).

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2 Preliminaries

2.1 Notation

We let the Greek indices α and β run from 1 to 2 and the Latin indices i and j run from 1 to 3. As usual, we omit to indicate the sum over repeated indices.

Given $a \in \mathbb{R}^n$ we denote by δ_a the Dirac mass at $x = a$.
Given a set $A \subset \mathbb{R}^n$ we introduce two functions: we denote by $\mathbb{1}_A$ the characteristic function which equals 1 in A and 0 outside, and by χ_A the indicator function which equals 0 in A and $+\infty$ outside. We denote by $\mathrm{Int}\,(A)$ the interior of the set A, and by $\overline{A}$ its closure.

For every measurable set $A \subset \mathbb{R}^n$ we denote by $|A|$ its Lebesgue measure, namely $|A| := \int_A 1\,dx$.
In the integrals, unless otherwise indicated, integration is made with respect to the n-dimensional Lebesgue measure. Furthermore, in all the circumstances when no confusion may arise, we omit to indicate the integration variable.
We write any $x \in \mathbb{R}^3$ as $(x', x_3) \in \mathbb{R}^2 \times \mathbb{R}$.
Derivation of functions depending only on x_3 will be denoted by a prime. Given $\psi \in H^1(\mathbb{R}^2)$ we denote by $\nabla^R\psi$ its rotated gradient, namely

$$\nabla^R\psi(x') := (-\partial_2\psi(x'), \partial_1\psi(x')) \ .$$

We adopt the same convention for the weak derivatives and write D^R to denote the differential operator $(-D_2, D_1)$.

We recall that the design region under study is a suitable scaling of the fixed right cylinder $Q := \overline{D} \times I$, whose cross section $D \subset \mathbb{R}^2$ is an open, bounded, connected set with Lipschitz boundary, and whose axis is a bounded closed interval $I \subset \mathbb{R}$. Without loss of generality, we may assume that $I=[-1/2,1/2]$ and $|D|=1$, so that $|Q|=1$. Finally, we chose the coordinate axes so that $\int_D x_\alpha\,dx'=0$.

We denote by $\mathcal{D}'(Q)$ the subset of distributions on $\mathbb{R}^3$ whose support is contained in the compact set Q. These distributions are in duality with $\mathcal{C}^\infty(Q)$, the space of restrictions to Q of functions in $\mathcal{C}^\infty(\mathbb{R}^3)$, and $\langle T,\varphi\rangle_{\mathbb{R}^3}$ represents the duality bracket.
For any $T\in\mathcal{D}'(Q)$, we denote by $[[T]]\in\mathcal{D}'(\mathbb{R})$ the $1d$ distribution obtained by "averaging" T with respect to the cross section variable, which is characterized by

$$\langle[[T]],\varphi\rangle_{\mathbb{R}}:=\langle T,\varphi\rangle_{\mathbb{R}^3}\qquad\forall\varphi=\varphi(x_3)\in\mathcal{C}_0^\infty(\mathbb{R})\ ,$$

where $C_0^\infty(\mathbb{R})$ denotes the function belonging to $C^\infty(\mathbb{R})$ having compact support.
Given a tensor field $\Sigma\in L^2(Q;\mathbb{R}^{n\times n})$, by $\operatorname{div}\Sigma$ we mean the its divergence (in the sense of distributions) with respect to lines, namely

$$(\operatorname{div}\Sigma)_i:=\sum_{j=1}^n\partial_j\Sigma_{ij}\ .$$

In the following $H^1(Q)$ denotes the space of restrictions to Q of elements of the Sobolev space $H^1(\mathbb{R}^3)$, equipped with the usual norm $\|u\|^2_{H^1(Q)}=\int_Q(|u|^2+|\nabla u|^2)$. Notice that, by the boundary regularity assumed on D, it coincides with the usual Sobolev space $H^1(D\times I)$. The dual space, denoted by $H^{-1}(Q)$, can be identified to the subspace of distributions $T\in\mathcal{D}'(Q)$ verifying the inequality $|\langle T,\varphi\rangle|\le C\|\varphi\|_{H^1(Q)}$ for every $\varphi\in H^1(Q)$, being C a suitable constant. Similar conventions will be adopted for functions or distributions on D or on I. It is easy to check that $[[T]]$ belongs to $H^{-1}(I)$ whenever $T\in H^{-1}(Q)$.
We recall that in dimension 1, a distribution $S\in\mathcal{D}'(I)$ satisfying $\langle S,1\rangle_{\mathbb{R}}=0$ has a unique primitive belonging to $\mathcal{D}'(I)$, that we denote by $\mathcal{P}_0(S)$. In the general case, given $S\in\mathcal{D}'(I)$, we denote by $\mathcal{P}(S)$ the primitive $\mathcal{P}(S):=\mathcal{P}_0(S-\langle S,1\rangle_{\mathbb{R}})$.
We denote by $BV_0(Q)$ (respectively $BV_0(D)$ and $BV_0(I)$) the space of bounded variation functions which vanish identically outside Q (resp. $\overline{D}$ and I). We denote by $\mathcal{M}(Q)$ the space of measures on $\mathbb{R}^3$ compactly supported in Q, and by $\mathcal{M}^+(Q)$ the subspace of positive measures.
When we add a subscript $_m$ to a functional space, we are considering the subspace of its elements which have zero integral mean.
In the particular case of $H^2_m(I)$ we require that also the distributional derivative has zero integral mean, *i.e.*

$$H^2_m(I):=\left\{\zeta\in H^2(I)\ :\ \int_I\zeta=\int_I\zeta'=0\right\}\ .$$

When needed, further notations will be introduced throughout the paper.

2.2 Displacement fields

Let us introduce the classes of displacement fields we consider, which are subspaces of $H^1(Q;\mathbb{R}^3)$. As usual, by *rigid motion* we mean the space

$$R(Q):=\left\{r\in H^1(Q;\mathbb{R}^3)\ :\ e(r)=0\right\}$$

namely vector fields of the form $r(x) = a + b \wedge x$, with a, $b \in \mathbb{R}^3$.
We define the space of *Bernoulli-Navier fields*

$$BN(Q) := \Big\{ u \in H^1(Q;\mathbb{R}^3) \;:\; e_{ij}(u) = 0 \quad \forall (i,j) \neq (3,3) \Big\}$$

and the space

$$TW(Q) := \Big\{ v = (v_\alpha, v_3) \in H^1(Q;\mathbb{R}^2) \times L^2(I; H^1_m(D)) \;:\; e_{\alpha\beta}(v) = 0 \quad \forall \alpha, \beta \in \{1,2\} \Big\} \;.$$

It is easy to check that, up to subtracting a rigid motion, any $u \in BN(Q)$ admits the following representation:

$$u(x) = (\zeta_1(x_3), \zeta_2(x_3), \zeta_3(x_3) - x_\alpha \zeta'_\alpha(x_3)) \quad \text{for some } (\zeta_\alpha, \zeta_3) \in (H^2_m(I))^2 \times H^1_m(I) \;. \tag{2.1}$$

Similarly, up to subtracting a Bernoulli-Navier field, any $v \in TW(Q)$ can be written as a *twist field*, namely a displacement of the form

$$v(x) = (-x_2 c(x_3), x_1 c(x_3), w(x)) \;\; \text{for some } c \in H^1_m(I) \;, \quad w \in L^2(I; H^1_m(D)) \;. \tag{2.2}$$

We remark that, up to rigid motions, any $v \in TW(Q)$ can be written as $v = \overline{u} + \overline{v}$, with $\overline{u}$ as in (2.1) and $\overline{v}$ as in (2.2), and the decomposition is unique.
We notice that the third component w of a field belonging to $TW(Q)$ is not necessarily in $H^1(Q)$; nevertheless, using the representation (2.2), we see that

$$(e_{13}(v), e_{23}(v)) = \frac{1}{2} \left(c'(x_3)(-x_2, x_1) + \nabla_{x'} w \right) \in L^2(Q;\mathbb{R}^2) \;. \tag{2.3}$$

Finally, exploiting Korn inequality and Poincaré-Wirtinger inequality, it is easy to show that the quotients $BN(Q)/R(Q)$ and $TW(Q)/BN(Q)$, endowed with the norms $\|e_{33}(\cdot)\|_{L^2(\Omega)}$ and $(\|e_{1,3}(\cdot)\|^2_{L^2(Q)} + \|e_{2,3}(\cdot)\|^2_{L^2(Q)})^{1/2}$ respectively, are Banach spaces.

3 Admissible loads

3.1 Definition and decomposition

In the mechanics of beams it is customary to distinguish between stretching, bending and torsion loads. The general case is difficult to handle, due to the interplay between these contributions of the charging. Here we focus our attention on the loads for which the contribution of bending and torsion may be decoupled in a suitable way, and we choose two different scalings for these two components (see §4.1).

Let $F \in H^{-1}(Q;\mathbb{R}^3)$ be an external load. In the asymptotic procedure, it turns out that the load F enters in the limit problem merely with its *resultant* and *momentum* averaged on each section. In particular, the normal component of the load to the section gives the average *axial load* $[[F_3]]$, while the component lying on the section gives the average *shear force* $[[F_1]]e_1 + [[F_2]]e_2$. Similarly, the normal and planar components of the average of the momentum $[[x \wedge F]]$ give the *torsion*

$$m_F := [[x_1 F_2 - x_2 F_1]] \in H^{-1}(I;\mathbb{R}) \tag{3.1}$$

and the average *bending moment*

$$\underline{m}_F^{(b)} := ([[x_2F_3 - x_3F_2]]; [[-x_1F_3 + x_3F_1]]) \in H^{-1}(I;\mathbb{R}^2) \ , \tag{3.2}$$

respectively.

We now fix the type of exterior loads F we consider.
With any $\Sigma \in L^2(Q;\mathbb{R}^{3\times3}_{sym})$, that we extend to zero over $\mathbb{R}^3 \setminus Q$, we associate the distribution $\operatorname{div}\Sigma$. As an element of $H^{-1}(Q;\mathbb{R}^3)$, it is characterized by

$$\langle \operatorname{div}\Sigma, u\rangle_{\mathbb{R}^3} = -\int_Q \Sigma\cdot\nabla u = -\int_Q \Sigma\cdot e(u) \quad \forall\, u \in H^1(Q;\mathbb{R}^3) \ . \tag{3.3}$$

Definition 3.1. *We say that $F \in H^{-1}(Q;\mathbb{R}^3)$ is an admissible load if it satisfies the following conditions:*

(h1) there exists $\Sigma \in L^2(Q;\mathbb{R}^{3\times3}_{sym})$ such that $F = \operatorname{div}\Sigma$ in $\mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3)$;

(h2) either $F_3 = \partial_1\Sigma_{13} + \partial_2\Sigma_{23}$ *or* $\begin{cases} F_3 = \partial_3\Sigma_{33} \\ [[F_\alpha]] = 0 \end{cases}$ *;*

(h3) the set $\{x \in Q \ : \ \operatorname{dist}(x, \operatorname{spt}(F)) < \delta\}$ has vanishing Lebesgue measure as $\delta \to 0$.

Remark 3.2. Assumption (h1) is equivalent to require that the load is balanced, namely it satisfies

$$\langle F, u\rangle_{\mathbb{R}^3} = 0 \quad \text{whenever } e(u) = 0 \ .$$

Indeed, as already observed in the Introduction, this condition is necessary in order that the compliance remains finite.
Assumption (h3) is needed to ensure that the load can be supported by a small amount of material. From a technical point of view, (h3) enables us to apply Proposition 2.8 in [7]. This condition on the topological support of F is satisfied for instance when $\operatorname{spt}(F)$ is a 2-rectifiable set, and in particular in the standard case when F is applied at the boundary of Q.
In order to better understand the condition (h2), let us compute the resultant of the forces on the sections, the torque and the bending momentum for a load F that admits the divergence representation (h1).
Let $\Sigma \in L^2(Q;\mathbb{R}^{3\times3}_{sym})$ be associated to F. Hence, according to the definitions (3.1) and (3.2), with a direct computation we obtain that the components of the average resultant read

$$[[F_i]] = [[\partial_3\Sigma_{i3}]] \ , \tag{3.4}$$

the normal component of the average momentum $[[x\wedge F]]$ equals

$$m_F = [[x_1F_2 - x_2F_1]] = [[x_1\partial_3\Sigma_{23} - x_2\partial_3\Sigma_{13}]] \ , \tag{3.5}$$

and the planar component of $[[x\wedge F]]$ is given by

$$\begin{aligned} \underline{m}_F^b &= ([[x_2F_3 - x_3F_2]]; [[-x_1F_3 + x_3F_1]]) \\ &= (-x_3[[F_2]]; x_3[[F_1]]) + ([[x_2(\partial_1\Sigma_{13} + \partial_2\Sigma_{23})]]; [[-x_1(\partial_1\Sigma_{13} + \partial_2\Sigma_{23})]]) + \\ &\quad + ([[x_2\partial_3\Sigma_{33}]]; -[[x_1\partial_3\Sigma_{33}]]) \ , \end{aligned} \tag{3.6}$$

where we have used the symmetry of the tensor Σ and the fact that, for every distribution $T \in \mathcal{D}'(Q)$, the average $[[T]]$, being an element of $\mathcal{D}'(I)$, satisfies

$$\langle [[\partial_\alpha T]], \varphi(x_3) \rangle_{\mathbb{R}} = -\langle T, \partial_\alpha \varphi(x_3) \rangle_{\mathbb{R}^3} = 0 \quad \forall\, \varphi \in \mathcal{D}(I) \ .$$

In view of (3.4)-(3.6), we see that all the quantities above do not depend on $\Sigma_{\alpha\beta}$; moreover we notice that imposing (h2) we require that either the bending moment *does not* depend on Σ_{33}, or it depends *only* on Σ_{33}.

We decompose an admissible load F as the sum $F = G + H$ of two loads belonging to $H^{-1}(Q;\mathbb{R}^3)$ defined as follows:

$$G := \operatorname{div} \Sigma_G \quad \text{with} \quad (\Sigma_G)_{ij} = \begin{cases} \Sigma_{ij} & \text{if } (i,j) \neq (3,3) \\ 0 & \text{if } (i,j) = (3,3) \end{cases} \tag{3.7}$$

$$H := \operatorname{div} \Sigma_H \quad \text{with} \quad (\Sigma_H)_{ij} = \begin{cases} 0 & \text{if } (i,j) \neq (3,3) \\ \Sigma_{33} & \text{if } (i,j) = (3,3) \end{cases} \ , \tag{3.8}$$

where $\Sigma \in L^2(Q;\mathbb{R}^{3\times3}_{sym})$ is associated to F as in (h1).

In view of condition (h2), the admissible loads satisfy

$$\text{either} \quad \begin{cases} F & = G \\ H & = 0 \\ m_F & = m_G \\ \underline{m}_F^{(b)} & = \underline{m}_G^{(b)} \end{cases} \qquad \text{or} \quad \begin{cases} F_\alpha & = G_\alpha\,, \ [[G_\alpha]] = 0 \\ F_3 & = H_3 \\ m_F & = m_G \\ \underline{m}_F^{(b)} & = \underline{m}_H^{(b)} \end{cases}$$

In Section 4 we will consider two different scalings for the two components G and H. We point out that the first case, corresponding to $H = 0$, has been presented in the paper [6].

Remark 3.3. The essential feature of our decomposition (3.7)-(3.8) is that the component G does not act on Bernoulli-Navier displacements (see (ii) in Proposition 3.4 below): this will ensure that the scaled functional $\mathcal{C}^\delta$ introduced in (4.4) is finite.
Assumption (h2) guarantees that the action of the components over the displacements (in particular of G on $TW(Q)$) do not depend on the representative Σ chosen among the tensors associated to F. In the general case, when the bending moment depends on both Σ_{33} and $\{\Sigma_{13}, \Sigma_{23}\}$, it is necessary to find a different decomposition, which is invariant under sum with a divergence free tensor. We believe that the difficulty in treating the problem is just in this first step and we expect that, possibly introducing a third component, our techniques still apply, leading to similar results characterized by the same structure.

The properties of the action of an admissible load over the displacements are summarized in the next proposition.

Proposition 3.4. *Let F be a load satisfying (h1), and let $F = G + H$ with G and H defined in (3.7) and (3.8) respectively. Then the following facts hold:*

(i) the loads F, G and H are balanced, namely they do not act on rigid motions;

(ii) G does not act on Bernoulli-Navier displacements, whereas it acts on $TW(Q)$ being an element of $H^{-1}(Q;\mathbb{R}^2) \times L^2(I;H^{-1}(D))$. More precisely, for any $v \in TW(Q)$, there holds

$$\langle G, v\rangle_{\mathbb{R}^3} = \langle m_G, c\rangle_{\mathbb{R}} + \langle G_3, w\rangle_{\mathbb{R}^3} \ , \tag{3.9}$$

where c and w are associated to v according to (2.2);

(iii) the action of H on any Bernoulli-Navier displacement u is

$$\langle H, u\rangle_{\mathbb{R}^3} = -\langle \overline{H_\alpha}, \zeta'_\alpha\rangle_{\mathbb{R}} + \langle \overline{H_3}, \zeta_3\rangle_{\mathbb{R}} \ , \tag{3.10}$$

where ζ_i are associated to u according to (2.1), and $\overline{H_i} \in H^{-1}(I)$ are defined by

$$\overline{H_\alpha} := [[x_\alpha H_3]] \, , \qquad \overline{H_3} := [[H_3]] \, . \tag{3.11}$$

Proof. (i) By definition (h1), (3.7) and (3.8), F, G and H are defined as the divergence of suitable L^2 tensors in the sense of distributions. In view of (3.3) it is then clear that they vanish on rigid motions.
(ii) Let $\Sigma_G \in L^2(Q;\mathbb{R}^{3\times 3}_{sym})$ be associated to G according to (3.7). By (3.3), since $(\Sigma_G)_{33} = 0$, we infer that G vanishes on Bernoulli-Navier displacements. On the other hand, the action of G on $TW(Q)$ is well-defined through the equality

$$\langle G, v\rangle_{\mathbb{R}^3} = -2\,\langle \Sigma_{\alpha 3}, e_{\alpha 3}(v)\rangle_{\mathbb{R}^3} \tag{3.12}$$

for every $v \in TW(Q)$.
The right hand side of (3.12) makes sense as a scalar product in $L^2(Q;\mathbb{R}^2)$ thanks to (2.3). In particular, by taking $v = (0, 0, v_3)$, one can see that $G_3 \in L^2(I;H^{-1}(D))$. Finally, writing v using the representation (2.2), equality (3.12) can be rewritten under the form (3.9).
(iii) Let u be a Bernoulli-Navier displacement. Since by (i) H vanishes on rigid motions, and by definition $H_\alpha = 0$, there holds

$$\langle H, u\rangle_{\mathbb{R}^3} = \langle H_3, \zeta_3 - x_\alpha \zeta'_\alpha\rangle_{\mathbb{R}^3} \ ,$$

where ζ_i are associated to u according to (2.1). By construction, the functions ζ_i depend only on x_3, then we infer

$$\langle H, u\rangle_{\mathbb{R}^3} = -\langle [[x_\alpha H_3]], \zeta'_\alpha\rangle_{\mathbb{R}} + \langle [[H_3]], \zeta_3\rangle_{\mathbb{R}} \ ,$$

that gives (3.10), thanks to definition (3.11) of $\overline{H_i}$. □

Remark 3.5. Notice that, from the definition (3.1) of m_G and the assumption (3.7) on G, it follows that $\langle m_G, 1\rangle_{\mathbb{R}} = 0$. Indeed,

$$\langle m_G, 1\rangle_{\mathbb{R}} = \langle [[x_1 G_2 - x_2 G_1]], 1\rangle_{\mathbb{R}} = \langle x_1 G_2 - x_2 G_1, 1\rangle_{\mathbb{R}^3} = \langle \partial_1 \Sigma_{21}, x_1\rangle_{\mathbb{R}^3} - \langle \partial_2 \Sigma_{12}, x_2\rangle_{\mathbb{R}^3} = 0 \ ,$$

where the last equality holds since Σ is symmetric.
Similarly, since $H_3 = \partial_3 \Sigma_{33}$, there holds $\langle \overline{H}_3, 1\rangle_{\mathbb{R}} = 0$.

3.2 Examples of admissible loads

Let us introduce some examples of admissible loads: in the first we present a family of possible H and in the following ones some choices for G (already introduced in [6, Examples 2.5-2.7])
Other admissible loads can be obtained by combining G and H introduced above, provided that the resulting load $G+H$ satisfies assumption (h2).
In Section 5 we will analyze the behavior of optimal configurations when the design region is subject to these particular loads.

Example 3.6. *Component H concentrated on the top and bottom faces $D \times \{\pm 1/2\}$:*

$$H_\alpha = 0 \ , \quad H_3 = f(x')(\delta_{-1/2} - \delta_{1/2})(x_3) \ ,$$

with $f \in L^2(\mathbb{R}^2;\mathbb{R}^2)$.
It is easy to see that condition (3.8) is satisfied by considering

$$\Sigma_{ij} = 0 \ \text{ if } (i,j) \neq (3,3) \quad \text{and} \quad \Sigma_{33} = f(x')\, \mathbb{1}_Q \ .$$

In particular, if we take

$$f(x') = \frac{a_\alpha}{\int_D x_\alpha^2}\, x_\alpha + \frac{b}{|D|} \ , \tag{3.13}$$

with a_α and b arbitrary real constants, we obtain a load F such that

$$\begin{cases} \overline{H_\alpha} = [[x_\alpha H_3]] = \ a_\alpha\,(\delta_{-1/2} - \delta_{1/2})(x_3) \ , \\ \overline{H_3} = [[H_3]] = b(\delta_{-1/2} - \delta_{1/2})(x_3) \ . \end{cases} \tag{3.14}$$

Example 3.7. *Component G horizontal and concentrated on the "top and bottom faces" $D \times \{\pm 1/2\}$:*

$$(G_1, G_2) = (\delta_{-1/2} - \delta_{1/2})(x_3)\, \nabla^R \psi(x') \ , \quad G_3 = 0 \ ,$$

with $\psi \in H_0^1(D)$ (we recall that ∇^R denotes the rotated gradient operator $(-\partial_2, \partial_1)$). In general, by varying the choice of ψ, for every $c \in \mathbb{R}$ we can construct a load G such that

$$\begin{cases} m_G = c\,(\delta_{-1/2} - \delta_{1/2})(x_3) \ , \\ G_3 = 0 \ . \end{cases} \tag{3.15}$$

Example 3.8. *Component G horizontal and concentrated on the "lateral surface" $\partial D \times I$:*

$$(G_1, G_2) = \eta(x_3)\tau_{\partial D}(x')\mathcal{H}^1 \llcorner \partial D \ , \quad G_3 = 0 \ ,$$

with $\tau_{\partial D}$ the unit tangent vector at ∂D and $\eta \in L^2_m(I)$.
We notice that in this example the average momentum is absolutely continuous with respect to the Lebesgue measure, more precisely

$$m_G = -2|D|\eta(x_3) \ .$$

Example 3.9. *Component G concentrated on the whole boundary of Q:*

$$(G_1, G_2) = (\delta_{-1/2} - \delta_{1/2})(x_3)\,\nabla_{x'}\psi(x')\ , \quad G_3 = -h\,\mathcal{H}^1 \llcorner \partial D\ ,$$

with $h \in L^2_m(\partial D)$ and $\psi \in H^1(D)$ the solution of the two-dimensional Neumann problem

$$\begin{cases} \Delta\psi = 0 & \text{in } D \\ \partial_\nu \psi = h & \text{on } \partial D \end{cases} .$$

We remark that the average momentum of the load G reads

$$m_G = \left(\int_D \nabla_{x'}\psi \cdot (-x_2, x_1)\,dx'\right)(\delta_{-1/2} - \delta_{1/2})(x_3)\ .$$

4 The limit rod-model

In this Section we carry out the asymptotics of the family of variational problems introduced in (1.1)

$$\phi^\delta(k) := \inf_{\Omega \subset Q_\delta} \left\{ \mathcal{C}^\delta(\Omega) + k\,\frac{|\Omega|}{|Q_\delta|} \right\}\ , \tag{4.1}$$

as $\delta \to 0^+$ and $k \to +\infty$. As already explained in the Introduction, keeping k fixed and letting $\delta \to 0$ corresponds to study the compliance optimization problem in a rod-like set, when the relative amount of elastic material is prescribed (see Theorem 4.1). Letting then $k \to +\infty$ corresponds to look for the best configurations which minimize the "filling ratio" (see Theorem 4.8).
The proofs of the results stated in §4.2 and §4.3 follow the same scheme of the ones treated in [6], in which the authors analyzed the case $F = G$. For the benefit of the reader, we postpone to the Appendix the main steps of the proofs, enlightening the main differences due to the presence of the extra-term H.

4.1 Reformulation of $\phi^\delta(k)$ over the fixed domain Q

The first step in the study of the asymptotics of $\phi^\delta(k)$ is their reformulation on subsets of the the fixed domain $Q = \overline{D} \times I$ instead of the thin cylinders $Q_\delta = \delta\overline{D} \times I$. This operation corresponds to chose a suitable change of variables for the displacements and a suitable scaling for the loads.

Let us first rewrite the compliance term $\mathcal{C}^\delta(\Omega)$, with Ω varying among the subsets of Q_δ.
Given an admissible load F and its decomposition $F = G + H$ defined according to (3.7) and (3.8), we consider the following scaling $F^\delta := G^\delta + H^\delta$: for every $x \in Q_\delta$ we set

$$\begin{aligned} G^\delta(x) &:= (\delta^{-1}G_1, \delta^{-1}G_2, \delta^{-2}G_3)(\delta^{-1}x', x_3)\,, \\ H^\delta(x) &:= (0, 0, \delta^{-1}H_3)(\delta^{-1}x', x_3)\ . \end{aligned}$$

In what follows we deal with both the components G and H, implicitly meaning that one of the conditions of (h2) holds true. We remark that all the following results, stated in Sections 4 and 5, are valid also without such assumption.
For every test function $\tilde{u} \in H^1(Q_\delta; \mathbb{R}^3)$ it holds

$$\widetilde{u}(x) = (\delta^{-2}u_1, \delta^{-2}u_2, \delta^{-1}u_3)(\delta^{-1}x', x_3) \qquad \text{in } Q_\delta$$

for some $u \in H^1(Q;\mathbb{R}^3)$.
This change of variables induces a 1-1 correspondence between the subsets of Q_δ and the subsets of Q: every $\Omega \subset Q^\delta$ is associated to a unique $\omega \subset Q$ such that

$$\Omega = \{(\delta x', x_3) \ : \ x' \in \omega\} \ . \tag{4.2}$$

Further, let us introduce the operator $e^\delta : H^1(Q;\mathbb{R}^3) \to L^2(Q;\mathbb{R}^{3\times3}_{\rm sym})$ defined by

$$e^\delta_{\alpha\beta}(u) := \delta^{-2} e_{\alpha\beta}(u) \ , \quad e^\delta_{\alpha 3}(u) := \delta^{-1} e_{\alpha 3}(u) \ , \quad e^\delta_{33}(u) := e_{33}(u) \ , \tag{4.3}$$

as it is usual in the literature on $3d-1d$ dimension reduction. By combining these definitions and scalings, with a direct computation we infer that

$$\mathcal{C}^\delta(\Omega) = \sup \left\{ \frac{1}{\delta}\langle G, u\rangle_{\mathbb{R}^3} + \langle H, u\rangle_{\mathbb{R}^3} - \int_\omega j(e^\delta(u))\, dx \ : \ u \in H^1(Q;\mathbb{R}^3) \right\} , \tag{4.4}$$

with $\omega \subset Q$ satisfying (4.2). Since ω is uniquely determined, we denote the expression above as $\mathcal{C}^\delta(\omega)$, namely a shape functional posed on a subset of the fixed domain Q.
Finally we observe that $|\Omega|/|Q_\delta| = |\omega|/|Q|$. Hence, recalling that Q is assumed to have volume 1, we conclude that problem $\phi^\delta(k)$ defined in (4.1) can be rewritten as

$$\phi^\delta(k) = \inf \left\{ \mathcal{C}^\delta(\omega) + k|\omega| \ : \ \omega \subset Q \right\} \ . \tag{4.5}$$

4.2 The small cross section limit

In order to write the limit problem as $\delta \to 0^+$, we need to introduce another energy density $\overline{j} : \mathbb{R}^3 \to \mathbb{R}$, the so called *reduced potential*:

$$\overline{j}(y) := \inf_{A \in \mathbb{R}^{2\times2}} j \begin{pmatrix} A_{11} & A_{12} & y_1 \\ A_{21} & A_{22} & y_1 \\ y_1 & y_2 & y_3 \end{pmatrix} .$$

With a direct computation it is easy to show that

$$\overline{j}(y) = 2\eta \sum_\alpha |y_\alpha|^2 + (Y/2)|z_3|^2 \ , \tag{4.6}$$

where $Y := \eta\, \frac{3\lambda+2\eta}{\lambda+\eta}$ is called the *Young modulus*.

The behavior of the optimal design problem (4.1) in the dimension reduction process is described by the following

Theorem 4.1 (Limit as $\delta \to 0^+$). *For every $k \in \mathbb{R}$,*

(i) as $\delta \to 0^+$ the sequence $\phi^\delta(k)$ in (4.5) converges to the following limit:

$$\phi(k) := \inf \left\{ \mathcal{C}^{lim}(\theta) + k \int_Q \theta\, dx \ : \ \theta \in L^\infty(Q;[0,1]) \right\} \ , \tag{4.7}$$

where

$$\mathcal{C}^{lim}(\theta) := \sup\left\{ \langle G, v\rangle_{\mathbb{R}^3} + \langle H, u\rangle_{\mathbb{R}^3} - \int_Q \overline{j}(e_{13}(v), e_{23}(v), e_{33}(u))\,\theta\,dx \ :\ v \in TW,\ u \in BN \right\}\ ; \tag{4.8}$$

(ii) moreover, if ω^δ *is a sequence of minima for* $\phi^\delta(k)$ *then, up to subsequences,* $\mathbb{1}_{\omega^\delta} \overset{*}{\rightharpoonup} \overline{\theta}$ *and* $\overline{\theta}$ *is optimal for* $\phi(k)$.

Proof. See Appendix. □

Remark 4.2. We observe that the limit $\phi(k)$ is again a variational problem with the same structure of $\phi^\delta(k)$ (compliance term + volume term), but it is a convex problem, well-posed over the space of densities $L^\infty(Q;[0,1])$.
The new setting (densities instead of sets) is natural: it is well known that the variational problem (4.4), and hence (4.5), is in general ill-posed, due to homogenization phenomena that prevent the existence of minimizers, so that we need to enlarge the class of admissible materials, passing from "real" materials, represented by characteristic functions, to "composite" materials, represented by densities with values in $[0,1]$.
The proof of Theorem 4.1 is based on the idea of considering the "fictitious counterpart" of problem (4.5), namely

$$\widetilde{\phi}^\delta(k) := \inf\left\{ \widetilde{\mathcal{C}}^\delta(\theta) + k\int_Q \theta\,dx \ :\ \theta \in L^\infty(Q;[0,1]) \right\}\ , \tag{4.9}$$

where $\widetilde{\mathcal{C}}^\delta(\theta)$ denotes the natural extension of the compliance $\mathcal{C}^\delta(\omega)$ to $L^\infty(Q;[0,1])$:

$$\widetilde{\mathcal{C}}^\delta(\theta) := \sup\left\{ \frac{1}{\delta}\langle G, u\rangle_{\mathbb{R}^3} + \langle H, u\rangle_{\mathbb{R}^3} - \int_Q j(e^\delta(u))\,\theta\,dx \ :\ u \in H^1(Q;\mathbb{R}^3) \right\}\ . \tag{4.10}$$

We show that, for every $k \in \mathbb{R}$, the two sequences $\phi^\delta(k)$ and $\widetilde{\phi}^\delta(k)$ have the same asymptotics and converge towards $\phi(k)$.

Remark 4.3. At this point it is natural to ask whether $\phi(k)$ admits a solution associated to a "real" material or not, namely if there exists an optimal $\overline{\theta}$ taking values just in the set $\{0,1\}$. Such problem, in pure torsion regime, has been investigated in [2], in which the authors studied the the presence of homogenization regions in relation to the geometry of the section D and the value of k.

A useful tool in the study of homogenization phenomena is the reformulation of the variational problem $\phi(k)$ in a dual form (see Proposition 4.4 below). Furthermore, the solutions of the primal and dual problem are related by a set of optimality conditions, which are gathered in Proposition 4.7.

Proposition 4.4 (**Dual formulation**). *For every* $\theta \in L^\infty(Q;[0,1])$ *and every* $k \in \mathbb{R}$, *problems (4.7) and (4.8) admit respectively the dual formulations*

$$\phi(k) = \inf_{\sigma \in L^2(Q;\mathbb{R}^3)} \left\{ \int_Q [\overline{j}(\cdot) - k]^*_+(\sigma)\,dx \ :\ \partial_1\sigma_1 + \partial_2\sigma_2 = -2G_3\,,\ [[x_1\sigma_2 - x_2\sigma_1]] = -2\mathcal{P}_0(m_G)\,,\right.$$
$$\left. [[\sigma_3]] = -\mathcal{P}_0(\overline{H_3})\,,\ [[x_\alpha\sigma_3]] = -\mathcal{P}(\overline{H_\alpha}) \right\} \tag{4.11}$$

and

$$\mathcal{C}^{lim}(\theta) = \inf_{\sigma\in L^2(Q;\mathbb{R}^3)} \Big\{ \int_Q \theta^{-1}\overline{j}^*(\sigma)\,dx \ : \partial_1\sigma_1 + \partial_2\sigma_2 = -2G_3\,,\ [[x_1\sigma_2 - x_2\sigma_1]] = -2\mathcal{P}_0(m_G)\,, \\ [[\sigma_3]] = -\mathcal{P}_0(\overline{H_3})\,,\ [[x_\alpha\sigma_3]] = -\mathcal{P}(\overline{H_\alpha}) \Big\}\,. \tag{4.12}$$

Remark 4.5. By applying the definition of Fenchel conjugate and exploiting the convexity of $\overline{j}$, it is easy to prove (cf. [7, Lemma 4.4]) that $[\overline{j}(\cdot) - k]^*_+$ coincides with the function

$$f_k(\xi) := \begin{cases} 2\sqrt{k\,\overline{j}^*(\xi)} & \text{if } \overline{j}^*(\xi) \le k \\ \overline{j}^*(\xi) + k & \text{if } \overline{j}^*(\xi) \ge k \end{cases}$$

or, equivalently, with the convex envelope of

$$g_k(\xi) := \begin{cases} 0 & \text{if } \xi = 0 \\ \overline{j}^*(\xi) + k & \text{otherwise} \end{cases}\,.$$

Furthermore, a direct computation shows that

$$\overline{j}^*(\xi) = \frac{1}{8\eta}|\xi'|^2 + \frac{1}{2Y}\xi_3^2\,.$$

Thus, roughly speaking, the function $[\overline{j}(\cdot) - k]^*_+(\xi)$ depends, up to a linear transformation, on the modulus of the argument and grows linearly in a certain compact set containing the origin and quadratically outside it.

Remark 4.6. The dual formulation (4.11) reveals that problem $\phi(k)$ can be solved section by section, indeed the constraints involve only the first two variables.

Let us now compare the primal and dual formulation of $\phi(k)$.
We say that $(\overline{\theta}, \overline{v}, \overline{u}, \overline{\sigma}) \in L^\infty(Q;[0,1]) \times TW(Q) \times BN(Q) \times L^2(Q;\mathbb{R}^3)$ is optimal for $\phi(k)$ if

- $(\cdot)$ $\overline{\theta}$ is optimal for $\phi(k)$ in its primal formulation (4.7);
- $(\cdot)$ the couple $(\overline{v}, \overline{u})$ is optimal for $\mathcal{C}^{lim}(\overline{\theta})$ in its primal formulations given by (4.8);
- $(\cdot)$ $\overline{\sigma}$ is optimal for $\phi(k)$ and $\mathcal{C}^{lim}(\overline{\theta})$ in their dual formulations, given by (4.11) and (4.12) respectively.

Proposition 4.7 (**Optimality conditions**). *A vector* $(\overline{\theta}, \overline{v}, \overline{u}, \overline{\sigma}) \in L^\infty(Q;[0,1]) \times TW(Q) \times BN(Q) \times L^2(Q;\mathbb{R}^3)$ *is optimal for* $\phi(k)$ *if and only if it satisfies the following system:*

$$\begin{aligned} &\partial_1\overline{\sigma}_1 + \partial_2\overline{\sigma}_2 = -2G_3\,,\ [[x_1\overline{\sigma}_2 - x_2\overline{\sigma}_1]] = -2\mathcal{P}_0(m_G) \\ &[[\overline{\sigma}_3]] = -\mathcal{P}_0(\overline{H_3})\,,\ [[x_\alpha\overline{\sigma}_3]] = -\mathcal{P}(\overline{H_\alpha}) \\ &\overline{\sigma} = \overline{\theta}\ \overline{j}\,'(e_{13}(\overline{v}), e_{23}(\overline{v}), e_{33}(\overline{u})) \\ &\overline{\sigma} \in \partial[\overline{j} - k]_+(e_{13}(\overline{v}), e_{23}(\overline{v}), e_{33}(\overline{u})) \\ &\overline{\theta}\,[\overline{j}(e_{13}(\overline{v}), e_{23}(\overline{v}), e_{33}(\overline{u})) - k] = [\overline{j}(e_{13}(\overline{v}), e_{23}(\overline{v}), e_{33}(\overline{u})) - k]_+ \end{aligned}$$

Proof. The proof of this result follows by a direct computation, comparing the primal and dual formulations. □

4.3 The small volume fraction limit

We now perform the second passage to the limit, as $k \to +\infty$: namely we look for the most robust configurations as the filling ratio $\int_Q \theta$ (we recall that $|Q| = 1$) is infinitesimal. As already noticed, it is natural to set the problem over the space $\mathcal{M}^+(Q)$ of positive measures μ on $\mathbb{R}^3$ compactly supported in Q. First of all we extend the limit compliance $\mathcal{C}^{lim}(\theta)$ in (4.8) to the class $\mathcal{M}^+(Q)$ by setting

$$\begin{aligned}\mathcal{C}^{lim}(\mu) := \sup \Big\{ \langle G, v\rangle_{\mathbb{R}^3} + \langle H, u\rangle_{\mathbb{R}^3} - \int_Q \overline{j}\big(e_{13}(v), e_{23}(v), e_{33}(u)\big)\, d\mu \ : \\ v \in TW(Q) \cap \mathcal{C}^\infty(Q;\mathbb{R}^3)\,,\ u \in BN(Q) \cap \mathcal{C}^\infty(Q;\mathbb{R}^3) \Big\}\ .\end{aligned} \tag{4.13}$$

As already done in Proposition 4.4, we can rewrite $\mathcal{C}^{lim}(\mu)$ in a dual form as

$$\begin{aligned}\mathcal{C}^{lim}(\mu) = \inf_{\xi \in L^2_\mu(Q;\mathbb{R}^3)} \Big\{ \int_Q \overline{j}^*(\xi) \quad d\mu \ : \ \mathrm{div}_{x'}(\xi'\mu) = -2\, G_3 \\ [[x_1(\xi_2\mu) - x_2(\xi_1\mu)]] = -2\mathcal{P}_0(m_G), \\ [[\xi_3\mu]] = -\mathcal{P}_0(\overline{H}_3)\,,\ [[x_\alpha(\xi_3\mu)]] = -\mathcal{P}(\overline{H}_\alpha)\ \Big\}\ .\end{aligned} \tag{4.14}$$

Using definition (4.13), the limit problem $\phi(k)$ in (4.7) can be rewritten as

$$\begin{aligned}\phi(k) &= \inf \Big\{ \mathcal{C}^{lim}(\mu) + k \int d\mu \ : \ \mu = \theta\, dx\ ,\ \theta \in L^\infty(Q;[0,1]) \Big\} \\ &= \sqrt{2k}\, \inf \Big\{ \mathcal{C}^{lim}(\mu) + \frac{1}{2}\int d\mu \ : \ \mu = \theta\, dx\ ,\ \theta \in L^\infty(Q;[0,\sqrt{2k}]) \Big\}\ ,\end{aligned} \tag{4.15}$$

where the second equality is obtained multiplying μ by $\sqrt{2k}$ (for $k > 0$).
Thus we may guess that, as $k \to +\infty$, the sequence $\phi(k)/\sqrt{2k}$ converges towards the minimum problem

$$\overline{m} := \inf \Big\{ \mathcal{C}^{lim}(\mu) + \frac{1}{2}\int d\mu \ : \ \mu \in \mathcal{M}^+(Q) \Big\}\ . \tag{4.16}$$

This natural conjecture is proved in the following

Theorem 4.8 (Limit as $k \to +\infty$).

(i) For $k > 0$, the map $k \mapsto \frac{\phi(k)}{\sqrt{2k}}$ is non increasing and, as $k \to +\infty$, it converges decreasingly to $\overline{m}$;

*(ii) moreover, if θ_k is a solution to the density formulation (4.7) of $\phi(k)$, up to subsequences $\theta_k/\sqrt{2k}$ converges weakly * in $L^\infty(Q;[0,1])$ to a solution $\overline{\mu}$ of problem (4.16).*

Following the same strategy adopted for problem $\phi(k)$ we rewrite $\overline{m}$ in an equivalent dual formulation.

Proposition 4.9 (Dual formulation). *The limit problem $\overline{m}$ in (4.16) coincides with*

$$\begin{aligned}\min_{\sigma \in \mathcal{M}(Q;\mathbb{R}^3)} \Big\{ \int |\sigma| \ : \ \partial_1\sigma_1 + \partial_2\sigma_2 = -\frac{G_3}{\sqrt{\eta}}\ ,\ [[x_1\sigma_2 - x_2\sigma_1]] = -\frac{\mathcal{P}_0(m_G)}{\sqrt{\eta}}\ , \\ [[\sigma_3]] = -\frac{\mathcal{P}_0(\overline{H_3})}{\sqrt{Y}}\ ,\ [[x_\alpha\sigma_3]] = -\frac{\mathcal{P}(\overline{H_\alpha})}{\sqrt{Y}} \Big\}\ .\end{aligned} \tag{4.17}$$

By the convergence statement (ii) in Theorem 4.8, in order to understand which kind of concentration phenomenon occurs for small amounts of material, one needs to write explicitly, or at least characterize, the solutions $\overline{\mu}$. In this direction, let us first show that optimal measures $\overline{\mu}$ are strictly related to solutions $\overline{\sigma}$ to the dual problem (4.17). More precisely, we have:

Proposition 4.10 (Optimality conditions). *If $\overline{\sigma}$ is optimal for problem (4.17), then $\overline{\mu} := |\overline{\sigma}|$ is optimal for problem (4.16). Conversely, if $\overline{\mu}$ is optimal for problem (4.16), and $\overline{\xi}$ is optimal for the dual form (4.14) of $\mathcal{C}^{lim}(\overline{\mu})$, then $|\overline{\xi}| = 2\sqrt{\eta}\ \overline{\mu}$-a.e., and $\overline{\sigma} := \frac{\overline{\xi}}{2\sqrt{\eta}}\overline{\mu}$ is optimal for problem (4.17).*

Therefore, in view of Proposition 4.10, if we determine a solution $\overline{\sigma}$ of the dual problem (4.17), we can solve the primal problem $\overline{m}$, simply by taking $\overline{\mu} := |\overline{\sigma}|$ (and viceversa). The advantage is that the dual formulation sometimes happens to be more tractable than the primal one.

Moreover, we notice that the constraints imposed on the admissible measures σ in the minimization problem (4.17) only involve the behavior of $\sigma(\cdot, x_3)$ for each fixed $x_3 \in I$. This reveals that the problem can be solved section by section. More precisely $\overline{m}$ reads

$$\overline{m} = \int_I m\left(-\frac{G_3}{\sqrt{\eta}}, -\frac{\mathcal{P}_0(m_G)}{\sqrt{\eta}}, -\frac{\mathcal{P}_0(\overline{H_3})}{\sqrt{Y}}, -\frac{\mathcal{P}(\overline{H_1})}{\sqrt{Y}}, -\frac{\mathcal{P}(\overline{H_2})}{\sqrt{Y}}\right) dx_3 \ , \tag{4.18}$$

with m defined as

$$\begin{aligned} m(r(x'), t, s_0, s_1, s_2) := \inf_{\sigma\in\mathcal{M}(D;\mathbb{R}^3)} \Big\{ \int_D |\sigma| \ : \ \mathrm{div}_{x'}(\sigma_1, \sigma_2) = r(x')\,,\ \int_D (x_1\sigma_2 - x_2\sigma_1) = t\,, \\ \int_D \sigma_3 = s_0\,,\ \int_D x_\alpha \sigma_3 = s_\alpha \Big\} \end{aligned} \tag{4.19}$$

for any $r(x') \in H^{-1}(D)$ and t, s_i arbitrary real constants.

In the next Section we will deal with problem (4.19) considering the loads introduced in §3.2, and determine the concentration phenomena that occur for small amounts of material, characterizing the behavior on the sections of the design region.

5 Concentration phenomena

In this Section we study the behavior of optimal configurations of the compliance optimization problem for small filling ratios, considering the loads introduced in §3.2. For simplicity, in the following we will consider D simply connected and assume that the Lamé coefficient η equals 1.
Thanks to Proposition 4.10, in order to determine optimal measures $\overline{\mu}$ for problem $\overline{m}$ in (4.16), one is reduced to study the solutions $\overline{\sigma}$ to the dual problem (4.17), more precisely there holds $\overline{\mu} = |\overline{\sigma}|$.
Moreover, in view of (4.18), $\overline{\sigma}$ solves, section by section, an infimum problem of the form

$$\begin{aligned} m(r(x'), t, s_0, s_1, s_2) := \inf_{\sigma\in\mathcal{M}(D;\mathbb{R}^3)} \Big\{ \int_D |\sigma| \ : \ \mathrm{div}_{x'}(\sigma_1, \sigma_2) = r(x')\,,\ \int_D (x_1\sigma_2 - x_2\sigma_1) = t\,, \\ \int_D \sigma_3 = s_0\,,\ \int_D x_\alpha \sigma_3 = s_\alpha \Big\} \end{aligned} \tag{5.1}$$

with $r(x') \in H^{-1}(D)$ and $t, s_i \in \mathbb{R}$ depending on the components G and H of the load.

More precisely, for every fixed $x_3 \in I$ there hold

$$r = -G_3(\cdot, x_3) \ , \quad t = -\mathcal{P}_0(m_G)(x_3) \ , \quad s_0 = -\frac{1}{\sqrt{Y}}\,\mathcal{P}_0(\overline{H_3})(x_3) \ , \quad s_\alpha = -\frac{1}{\sqrt{Y}}\,\mathcal{P}(\overline{H_\alpha})(x_3) \ .$$

We underline that, in view of assumption (3.7), the function r is the divergence, with respect to the first two variables, of the vector field $-(\Sigma_{13}, \Sigma_{23})$, with Σ associated to G according to (3.7).
For some choices of the loads, problem (5.1) is pretty tractable since some of the parameters $\{r, t, s_i\}$ vanish. We recall that, in view of (h2), in our case there holds either $r = 0$ or $s_i = 0$. In the proofs, when there is no ambiguity, we omit the parameters that vanish in the argument of m.

First, in §5.1 we deal with pure vertical loads H, namely when $r = t = 0$ in (5.1): it turns out that the material distribution tends to concentrate, section by section, near some portions of the lateral surface of the design region.
In §5.2, we present the case of pure torsion loads with null vertical component, namely the case $r = s_i = 0$: it turns out that, when the cross section D of the rod is a convex set, the material distribution tends to concentrate, section by section, near the boundary of its Cheeger set. Let us recall that, under the assumption that D is convex, the *Cheeger set* is unique (for more details see the Introduction and the references therein).
Finally, in §5.3 and §5.4, we study the cases $s_i = 0$ and $r = 0$ respectively: here the optimal measures can't be characterized explicitly, nevertheless the study brings into play two interesting variants of the Cheeger problem for D, introduced in (1.7) and (1.8).

5.1 The case $r = t = 0$

Let us now consider the case in which both G_3 and $\mathcal{P}_0(m_G)$ vanish, as it happens if $G \equiv 0$. As an example, let us consider a design region having as section D the square $[-1, 1]^2$. A similar procedure can be performed also for general geometries.
Before stating the result, let us introduce a family of subsets of the section D, depending on a triple of parameters $s = (s_0, s_1, s_2)$. For every $s \in \mathbb{R}^3 \setminus \{0\}$, we define the sets $M^+(s)$ and $M^-(s)$ according to the scheme (5.2).

$\max\lvert\mathbf{s_i}\rvert$	$\mathbf{M^+}$	$\mathbf{M^-}$
s_0	D	$\emptyset$
s_1	$\{+1\}\times[-1,1]$	$\{-1\}\times[-1,1]$
s_2	$[-1,1]\times\{+1\}$	$[-1,1]\times\{-1\}$
$s_0\,,\ s_1$	$\{+1\}\times[-1,1]$	$\emptyset$
$s_0\,,\ -s_1$	$\{-1\}\times[-1,1]$	$\emptyset$
$s_0\,,\ s_2$	$[-1,1]\times\{+1\}$	$\emptyset$
$s_0\,,\ -s_2$	$[-1,1]\times\{-1\}$	$\emptyset$

$\max\lvert\mathbf{s_i}\rvert$	$\mathbf{M^+}$	$\mathbf{M^-}$
$s_1\,,\ s_2$	$\{(1,1)\}$	$\{(-1,-1)\}$
$s_1\,,\ -s_2$	$\{(-1,1)\}$	$\{(1,-1)\}$
$s_0\,,\ s_1\,,\ s_2$	$\{(1,1)\}$	$\emptyset$
$s_0\,,\ -s_1\,,\ s_2$	$\{(-1,1)\}$	$\emptyset$
$s_0\,,\ -s_1\,,\ -s_2$	$\{(-1,-1)\}$	$\emptyset$
$s_0\,,\ s_1\,,\ -s_2$	$\{(1,-1)\}$	$\emptyset$

(5.2)

The cases not included in (5.2), corresponding to the opposite signature of the triples (s_0, s_1, s_2), can be deduced by interchanging the role of M^+ and M^-.

As we can notice from (5.2), depending on the sign of s_j and whether they agree with $\max|s_i|$ or not, the corresponding sets $M^\pm$ can be the empty set, the entire square D, one of the segments of ∂D, or one of the corners of the square $\{(i,j)\}_{i,j\in\{\pm1\}}$.

Theorem 5.1. *Assume that $G \equiv 0$ and D is the square $[-1,1]^2$. Hence a solution $\overline{\mu}$ to problem (4.16) is of the form*

$$\overline{\mu} = |\overline{\rho}|(x', x_3)\ ,$$

where $\overline{\rho} \in \mathcal{M}(Q)$ satisfies, for a.e. $x_3 \in I$, the following system:

$$\begin{cases} \int_D |\overline{\rho}| = \max_{i=0\ldots2} |s_i(x_3)|\ , \\ \int_D \overline{\rho} = s_0(x_3)\ , \\ \int_D x_\alpha \overline{\rho} = s_\alpha(x_3)\ , \end{cases} \tag{5.3}$$

with

$$s(x_3) := -\frac{1}{\sqrt{Y}}\left(\mathcal{P}_0(\overline{H_3}), \mathcal{P}(\overline{H_1}), \mathcal{P}(\overline{H_2})\right)\ . \tag{5.4}$$

Moreover, letting $\overline{\rho}^+$ and $\overline{\rho}^-$ denote the positive and negative parts of ρ, their supports satisfy

$$\mathrm{spt}\overline{\rho}^\pm \subseteq M^\pm(s(x_3))\ ,$$

$M^\pm(s)$ being the subsets of the section D introduced in (5.2).

Proof. Since $G \equiv 0$, it is easy to see that an optimal field $\overline{\sigma}$ for the dual problem (4.17) is vertical, namely of the form $\overline{\sigma} = (0, 0, \overline{\rho})$, for some $\overline{\rho} \in \mathcal{M}(Q)$. In particular, in view of Proposition 4.10, we infer $\overline{\mu} = |\overline{\rho}|$. Let us characterize the optimal density $\overline{\rho} \in \mathcal{M}(Q)$. By construction, for a.e. $x_3 \in I$, $\overline{\rho}(\cdot, x_3)$ solves the following infimum problem

$$m(s) := \inf_{\rho\in\mathcal{M}(D)} \left\{ \int_D |\rho|\ :\ \int_D \rho = s_0\,,\ \int_D x_\alpha \rho = s_\alpha \right\}\ , \tag{5.5}$$

with $s = s(x_3)$ defined as in (5.4).
In what follows we omit to denote the variable x_3, which we consider fixed in I.
The function m in (5.5) is convex, lower semicontinuous, positively 1-homogeneous and its Fenchel conjugate reads

$$\begin{aligned} m^*(s_0^*, s_1^*, s_2^*) &= \sup_{s\in\mathbb{R}^3} \{s\cdot s^* - m(s)\} \\ &= -\inf_{s\in\mathbb{R}^3} \inf \left\{ \int |\rho| - s\cdot s^* \ : \ \rho\in\mathcal{M}(D)\,,\ \int \rho = s_0\,,\ \int x_\alpha \rho = s_\alpha \right\} \\ &= -\inf_{\rho\in\mathcal{M}(D)} \left\{ \int_D (|\rho| - (s_0^* + s_\alpha^* x_\alpha)\rho) \right\} \\ &= \chi_{K(D)}\ , \end{aligned}$$

where $\chi_{K(D)}$ is the indicator function which equals 0 on the convex set

$$K(D) := \{s^* \in \mathbb{R}^3 \ : \ |s_0^* + x_\alpha s_\alpha^*| \le 1 \ \forall (x_1, x_2) \in D\}$$

and $+\infty$ on its complement. It is easy to prove that, in the case of the square $D = [-1, 1]^2$, the set $K(D)$ is given by

$$\left\{ s^* \in \mathbb{R}^3 \ : \ \sum |s_i^*| \le 1 \right\}\ .$$

As a consequence, we infer that m is the support function of $K(D)$:

$$m(s) = \sup_{s^*\in\mathbb{R}^3} \{s\cdot s^* - m^*(s^*)\} = \sup_{s^*\in K} \{s\cdot s^*\} = \max\{|s_i|\}\ , \tag{5.6}$$

in particular an optimal measure $\overline{\rho}$ is characterized by (5.3).
In order to deduce information about the support of the positive and negative part of $\overline{\rho}$, denoted respectively by $\overline{\rho}^{\,+}$ and $\overline{\rho}^{\,-}$, we compare m and m^*: by combining the Fenchel equality, formula (5.5) and formula (5.6), we infer that for every $s^* \in \partial m(s)$

$$\int_D |\overline{\rho}| = s\cdot s^* = \int_D (s_0^* + x_\alpha s_\alpha^*)\overline{\rho}\ ,$$

that is

$$\int_D |\overline{\rho}| - (s_0^* + x_\alpha s_\alpha^*)\overline{\rho} = 0\ .$$

Thus we have a precise information on the support of $\overline{\rho}^{\,+}$ and $\overline{\rho}^{\,-}$:

$$\begin{cases} \operatorname{spt}\overline{\rho}^{\,+} \subseteq \{x' \in D \ : \ s_0^* + x_\alpha s_\alpha^* = 1\}\ , \\ \operatorname{spt}\overline{\rho}^{\,-} \subseteq \{x' \in D \ : \ s_0^* + x_\alpha s_\alpha^* = -1\}\ . \end{cases}$$

By the arbitrariness of $s^* \in \partial m(s)$ we obtain

$$\operatorname{spt}\overline{\rho}^{\,\pm} \subseteq M^\pm(s) := \cap_{s^*\in\partial m(s)} \{x' \in D \ : \ s_0^* + x_\alpha s_\alpha^* = \pm 1\}\ . \tag{5.7}$$

We conclude by characterizing the sets $M^\pm$. By definition, the subdifferential $\partial m(s)$ reads

$$\partial m(s) = \{s^* \in \mathbb{R}^3 \ : \ s\cdot s^* = \max\{|s_i|\}\,,\ \sum |s_i^*| \le 1\}\ ,$$

and it can be characterized explicitly. Since $\partial m(s)$ is invariant under multiplication by positive constant, namely

$$\partial m(\alpha s) = \partial m(s) \quad \forall \alpha > 0 \ ,$$

we give its expression for s such that $m(s) = 1$. Let ξ and ζ denote two arbitrary constants with modulus less than 1, then

$$\begin{aligned} \partial m(\pm 1, \xi, \zeta) &= \{(\pm 1, 0, 0)\} \ , \\ \partial m(\pm 1, \pm 1, \xi) &= \{(\pm\alpha, \pm\beta, 0)\}_{\alpha+\beta=1\ ,\ \alpha,\beta\geq 0} \ , \\ \partial m(\pm 1, \pm 1, \pm 1) &= \{(\pm\alpha, \pm\beta, \pm\gamma)\}_{\alpha+\beta+\gamma=1\ ,\ \alpha,\beta,\gamma\geq 0} \ , \end{aligned}$$

and analogous expressions hold true exchanging the roles of s_i.
By combining these computations with (5.7) we obtain the representation (5.2) of $M^{\pm}(s)$. □

Remark 5.2. In general the solution $\overline{\mu}$ of problem (4.16) is not uniquely determined, unless its support is localized, section by section, in a single point of ∂D (see table (5.2)).
Moreover, we remark that in view of (5.2) it is clear that there is no superposition of solutions: a solution of problem $m(s)$ might not be the superposition of solutions of problems $m(s_0, 0, 0)$, $m(0, s_1, 0)$ and $m(0, 0, s_2)$, since its support must satisfy stricter constraints.

Example 5.3. Let us consider the admissible load introduced in Example 3.6, with f given by (3.13). Clearly it fulfills the assumptions of Theorem 5.1. In view of (3.14) we obtain that

$$\mathcal{P}_0(\overline{H_3}) = b 1\!\!1_I(x_3) \quad , \qquad \mathcal{P}(\overline{H_\alpha}) = a_\alpha 1\!\!1_I(x_3) \ .$$

Hence any optimal measure $\overline{\mu}$ is of the form

$$\overline{\mu} = 1\!\!1_I(x_3) \otimes |\overline{\rho}|(x')$$

with $\overline{\rho}$ solution for $m(s)$ defined in (5.5) with

$$s_0 = -\frac{b}{\sqrt{Y}} \ , \quad s_\alpha = -\frac{a_\alpha}{\sqrt{Y}} \ .$$

The subset of the boundary in which $\overline{\mu}$ concentrates depends on the values of the parameters a_α and b, according to the scheme in table (5.2). Some particular choices are represented in Figure 1 and 2.

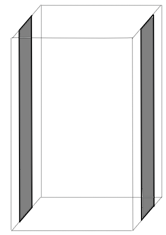

Figure 1: *Choosing F as a the vertical load $F = (0, 0, -\sqrt{Y}/12\, x_1)$, namely as in Example 3.6 with $a_2 = b = 0$ and $a_1 = -\sqrt{Y}$, it turns out that a particular solution of problem (4.16) is* $\overline{\mu} = \mathcal{H}^2 \lfloor_{\{\pm 1\}\times[-1/4,1/4]\times[-1/2,1/2]}$.

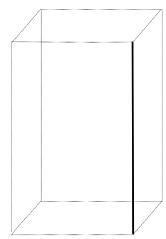

Figure 2: *Choosing F as a the vertical load $F = (0, 0, \sqrt{Y}/12(x_1 - x_2) + \sqrt{Y})$, namely as in Example 3.6 with $a_1 = b = \sqrt{Y}$ and $a_2 = -\sqrt{Y}$, it turns out that a particular solution of problem (4.16) is* $\overline{\mu} = \mathcal{H}^1 \lfloor_{\{1\}\times\{-1\}\times[-1/2,1/2]}$.

5.2 The case $r = s_i = 0$

This case has already been presented in [6, Theorem 5.4], however, for the benefit of the reader, we recall the result and its proof.

Lemma 5.4. *Let D be simply connected and let $r = s_i = 0$ in (5.1). Then a solution $\widehat{\sigma} \in \mathcal{M}(D; \mathbb{R}^3)$ for problem $m(0, t, 0, 0, 0)$ is of the form*

$$(\widehat{\sigma}_1, \widehat{\sigma}_2) = -\frac{t}{2}\, D^R \overline{u} \quad \textit{and} \quad \widehat{\sigma}_3 = 0 \ ,$$

with $\overline{u} \in BV_0(D)$ optimal for

$$\inf \left\{ \int_D |Du| \ : \ u \in BV_0(D) \, , \ \int_D u = 1 \right\} \ . \tag{5.8}$$

Proof. Since by assumption the parameters s_i vanish, it is easy to see that an optimal measure $\hat{\sigma}$ for problem m in (5.1) is horizontal, namely of the form

$$\widehat{\sigma} = (\overline{\nu}_1, \overline{\nu}_2, 0) \ ,$$

with $\overline{\nu} \in \mathcal{M}(D; \mathbb{R}^2)$ solution for problem

$$m(t) := \inf_{\nu \in \mathcal{M}(D;\mathbb{R}^2)} \left\{ \int_D |\nu| \ : \ \operatorname{div} \nu = 0 \, , \ \int_D (x_1 \nu_2 - x_2 \nu_1) = t \right\} \ . \tag{5.9}$$

We recall that D is simply connected, thus the constraint of zero divergence (namely $r = 0$) in (5.9) implies that every admissible field ν can be written as the rotated gradient $D^R u$ of a suitable function $u \in BV_0(D)$. Moreover, we can rewrite the second constraint of (5.9) in terms of u: integrating by parts we infer that

$$\int_D u = -\frac{t}{2} \, .$$

Therefore $m(t)$ can be rewritten as

$$m(t) = \inf \left\{ \int_D |Du| \ : \ u \in BV_0(D) \ , \quad \int_D u = -\frac{t}{2} \right\} \, .$$

After normalizing the class of admissible functions u si that $\int_D u = 1$, we conclude that $\overline{\nu}$ reads

$$\overline{\nu} = -\frac{t}{2} \, D^R \overline{u} \, ,$$

with $\overline{u} \in BV_0(D)$ optimal for the normalized problem (5.8). □

Theorem 5.5. *Assume that $G_3 = H = 0$ and that D is convex. Denote by C the Cheeger set of D. Then the unique solution to problem (4.17) is*

$$\overline{\sigma} := \frac{1}{2} \, \mathcal{P}_0(m_G)(x_3) \otimes \frac{1}{|C|} \tau_{\partial C}(x') \mathcal{H}^1 \llcorner \partial C \ ,$$

and hence the unique solution $\overline{\mu}$ to problem (4.16) is

$$\overline{\mu} = \frac{1}{2} \, |\mathcal{P}_0(m_G)(x_3)| \otimes \frac{1}{|C|} \mathcal{H}^1 \lfloor_{\partial C} \ .$$

Proof. By assumption $G_3 = H = 0$, then problem (4.17) reads

$$\min_{\sigma \in \mathcal{M}(Q;\mathbb{R}^3)} \left\{ \int_Q |\sigma| \ : \ \mathrm{div}_{x'}(\sigma_1, \sigma_2) = 0 \, , \ [[x_1 \sigma_2 - x_2 \sigma_1]] = -\mathcal{P}_0(m_G) \right\} \, .$$

Since the constraints depend only on x_3, the solutions are of the form

$$\overline{\sigma} = \mathcal{P}_0(m_G)(x_3) \otimes \widehat{\sigma}(x') \tag{5.10}$$

with $\widehat{\sigma}(x') \in \mathcal{M}(D;\mathbb{R}^3)$ optimal for problem $m(0, -1, 0, 0, 0)$. In view of Lemma 5.4 we infer that

$$\widehat{\sigma} = \frac{1}{2} \, (-D_2 \overline{u}, D_1 \overline{u}, 0) \ , \tag{5.11}$$

with $\overline{u}$ optimal for problem (5.8). We recall that (5.8) is the relaxed version of the Cheeger problem introduced in (1.3) which, under the assumption of D convex, admits a unique solution $\overline{u} := |C|^{-1} \mathbb{1}_C$, being C the Cheeger set of D. The rotated gradient of such characteristic function gives

$$D^R \overline{u} = \frac{1}{|C|} \, D^R \mathbb{1}_C = \frac{1}{|C|} \, \tau_{\partial C}(x') \mathcal{H}^1 \llcorner \partial C \ . \tag{5.12}$$

By combining (5.10), (5.11) and (5.12) the proof is achieved. □

Example 5.6. Let us consider G as in (3.15) of Example 3.7, with $c = 2|C|$. In view of Theorem 5.5 we obtain that the optimal measure $\overline{\mu}$ for problem (4.16) equals

$$\overline{\mu} = \mathbb{1}_I(x_3) \otimes \mathcal{H}^1 \lfloor_{\partial C}(x') \ ,$$

and its support is represented in Figure 3.

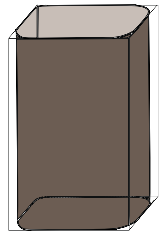

Figure 3: *Choosing F as the vertical load $F = (G_1, G_2, 0)$, being G_α as in Example 3.7 such that $m_G = (\delta_{-1/2} - \delta_{1/2})$, the optimal measure $\overline{\mu}$ of problem (4.16) turns out to be concentrated, section by section, on the boundary of the Cheeger set of the section D (here, a square).*

5.3 The case $s_i = 0$

In this paragraph we see that when the component H of the load vanishes, the limit problem (4.17) amounts to solve a "modified" Cheeger problem, in which the admissible functions are the same appearing in the classical version (1.6) of Cheeger problem, while the functional to minimize is perturbed by a fixed vector field. An example of such a load is given by taking G as in Example 3.9.

Theorem 5.7. *Assume that $H \equiv 0$ and that D is simply connected. Hence a solution $\overline{\mu} \in \mathcal{M}^+(Q)$ for problem (4.17) is of the form*

$$\overline{\mu} = |D\overline{u} + q| \ ,$$

where, for a.e. $x_3 \in I$, $\overline{u}(\cdot, x_3)$ is optimal for

$$\inf \left\{ \int_D |Du + q| \ : \ u \in BV_0(D) \, , \ \int_D u = \beta \right\} \tag{5.13}$$

being $q \in L^2(Q; \mathbb{R}^2)$ and $\beta \in \mathbb{R}$ depending on G.

Proof. Let $\overline{\sigma} \in \mathcal{M}(Q; \mathbb{R}^3)$ be optimal for the dual problem (4.17). Since by assumption $H \equiv 0$, in view of (5.1), we infer that, for a.e. $x_3 \in I$, $\overline{\sigma}(\cdot, x_3)$ solves

$$\inf_{\sigma \in \mathcal{M}(D;\mathbb{R}^3)} \left\{ \int_D |\sigma| \ : \ \operatorname{div}_{x'}(\sigma_1, \sigma_2) = r(x') \, , \ \int_D (x_1\sigma_2 - x_2\sigma_1) = t \right\} \ , \tag{5.14}$$

with $r = -G_3(\cdot, x_3)$, $t = -\mathcal{P}_0(m_G)(x_3)$. Clearly $\overline{\sigma}$ must be searched among the measures of the form $\overline{\sigma}(\cdot, x_3) = (\nu_1, \nu_2, 0)$, with $\nu := (\nu_1, \nu_2) \in \mathcal{M}(D; \mathbb{R}^2)$. Let us rewrite the constraints appearing in (5.14) in terms of ν. As already noticed at the beginning of the paragraph, we recall that $r = -\operatorname{div}_{x'}(\Sigma_{13}, \Sigma_{23})$, with $\Sigma \in L^2(Q; \mathbb{R}^{3\times 3}_{sym})$ associated to the load G according to (3.7). Therefore, since D is assumed to be simply connected, we infer that

$$\nu + (\Sigma_1, \Sigma_2) = D^R u \ ,$$

for some $u \in BV_0(D)$. Moreover

$$\int_D u = \beta := -\frac{t}{2} - \frac{1}{2} \int_D (x_1\Sigma_{23} - x_2\Sigma_{13}) \ .$$

By taking $q := (-\Sigma_{23}, \Sigma_{13})$ we know that $(\overline{\sigma}_1, \overline{\sigma}_2) = (-D_2\overline{u}, D_1\overline{u}) + (-q_2, q_1)$, with $\overline{u}$ optimal for

$$\inf\left\{\int_D |(-D_2u, D_1u) + (-q_2, q_1)| \ : \ u \in BV_0(D)\,,\ \int_D u = 0\right\}\ . \tag{5.15}$$

We remark that $|(-D_2u, D_1u) + (-q_2, q_1)| = |Du + q|$, thus (5.15) coincides with (5.13). Moreover, since by Proposition 4.10 an optimal measure $\overline{\mu}$ for problem (4.16) equals $|\overline{\sigma}|$ the proof is achieved. It only remains to show that β, which a priori is a function of x_3, is constant: rewriting β in terms of the load G, we infer

$$\begin{aligned}\beta(x_3) &= \frac{1}{2}\mathcal{P}_0(m_G)(x_3) - \frac{1}{2}[[x_1\Sigma_{23} - x_2\Sigma_{13}]](x_3)\\ &= \frac{1}{2}\mathcal{P}_0(\partial_3[[x_1\Sigma_{23} - x_2\Sigma_{13}]])(x_3) - \frac{1}{2}[[x_1\Sigma_{23} - x_2\Sigma_{13}]](x_3)\ ,\end{aligned}$$

therefore $\beta'(x_3) = 0$. □

Remark 5.8. It is easy to see that problem (5.13) can be written in the form (1.7). Indeed if $\beta \neq 0$ it is enough to "normalize" the admissible functions. If instead $\beta = 0$ it is enough to consider a function $v \in BV_0(D)$ such that $\int_D v = 1$ and replace the admissible functions u by $u + v$. Of course in both cases the vector field appearing in the reformulated infimum problem changes with respect to q by a dilation or a translation respectively.

5.4 The case $r = 0$

In this last paragraph we focus our attention on the case in which $G_3 = 0$, namely when the parameter r appearing in (4.17) vanishes. An example of such a load can be obtained by taking the component G as in Example 3.7 and the component H as in Example 3.6.
In view of the properties found in §5.1 and §5.2, in which we studied $m(0, 0, s_i)$ and $m(0, t, 0)$ respectively, we expect that a solution $\overline{\mu}$ for problem (4.16), section by section, is linked with some variant of the Cheeger problem.
Proceeding as in the previous cases, if D is simply connected, it is easy to see that an optimal measure $\overline{\mu} \in \mathcal{M}^+(Q)$, section by section, is of the form

$$\overline{\mu}(\cdot, x_3) = |(D\overline{u}, \overline{\rho})|(x')\ ,$$

with $(\overline{u}, \overline{\rho}) \in BV_0(D) \times \mathcal{M}(D)$ optimal for problem $m(t, s) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ defined by

$$m(t, s) := \inf\left\{\int_D |(Du, \rho)| \ : \ u \in BV_0(D)\,,\ \int_D u = t\,,\ \rho \in \mathcal{M}(D)\,,\ \int_D \rho = s_0\,,\ \int_D x_\alpha\rho = s_\alpha\right\}\ , \tag{5.16}$$

where $t \in L^2(I)$ and $s \in L^2(I; \mathbb{R}^3)$ are the following functions:

$$t := \frac{1}{2}\,\mathcal{P}_0(m_G)(x_3)\ ,\quad s_0 := -\frac{1}{\sqrt{Y}}\,\mathcal{P}_0(\overline{H_3})(x_3)\ ,\quad s_\alpha := -\frac{1}{\sqrt{Y}}\,\mathcal{P}(\overline{H_\alpha})(x_3)\ .$$

In Theorem 5.9 we will characterize an optimal couple $(D\overline{u}, \overline{\rho})$.
Before stating the result, it is useful to introduce the following convex subset of $\mathbb{R}^4$:

$$K(D) := \{(\lambda, s^*) \in \mathbb{R} \times \mathbb{R}^3 \ : \ \exists\sigma \in L^2(D; \mathbb{R}^2) \text{ st } -\operatorname{div}\sigma = \lambda\ ,\ |\sigma|^2 + |s_0^* + x_\alpha s_\alpha^*|^2 \le 1 \text{ in } D\}\ .$$

It is easy to check that $K(D)$ can be represented as a union of intervals as follows

$$K(D) = \bigcup_{\{s^*: |s_0^* + x_\alpha s_\alpha^*| \le 1\}} [-\overline{\lambda}(s^*), \overline{\lambda}(s^*)] \times \{s^*\} \ ,$$

and $\overline{\lambda}(s^*)$ is defined as

$$\overline{\lambda}(s^*) := \sup\{\lambda \ : \ \exists \sigma \in L^2(D;\mathbb{R}^2) \text{ st } -\operatorname{div} \sigma = \lambda \ , \ |\sigma| \le \alpha_{s^*}(x') \ a.e. \text{ in } D\} \ , \tag{5.17}$$

with α_{s^*} the positive function

$$\alpha_{s^*}(x') := \sqrt{1 - |s_0^* + x_\alpha s_\alpha^*|^2} \ , \tag{5.18}$$

defined in D, for every $s^* \in \mathbb{R}^3$ such that $|s_0^* + x_\alpha s_\alpha^*| \le 1$ in D.
The function m defined in (5.16) is convex, lower semicontinuous and positively 1-homogeneous. It agrees with the support function of the set $K(D)$ and we have

$$m^*(\lambda, s^*) = \chi_{K(D)} \ .$$

Indeed, by definition of Fenchel transform, the dual problem of m reads

$$\begin{aligned} m^*(\lambda, s^*) &= \sup_{(t,s)}\{t\lambda + s \cdot s^* - m(t,s)\} \\ &= \sup_{(u,\rho)} \left\{ \int_D u\lambda + \int_D (s_0^* + s_\alpha x_\alpha^*)\rho - \int_D |(Du, \rho)| \right\} \ . \end{aligned}$$

Therefore $m^*(\lambda, s^*) = 0$ if the couple (λ, s^*) satisfies

$$\int_D |(Du, \rho)| - \int_D [\lambda u + (s_o^* + x_\alpha s_\alpha^*)\rho] \ge 0 \quad \forall (u, \rho) \in BV_0(D) \times \mathcal{M}(D) \tag{5.19}$$

and $+\infty$ otherwise. Exploiting the definition of total variation, it is easy to show that the set of (λ, s^*) satisfying (5.19) is given by $K(D)$.

Theorem 5.9. *Let $(\lambda, s^*) \in \partial m(t,s)$. Then an optimal couple $(D\overline{u}, \overline{\rho})$ for problem $m(t,s)$ defined in (5.16) satisfies*

(i) $-\operatorname{div}\left(\alpha_{s^*}(x')\frac{D\overline{u}}{|D\overline{u}|}\right) = \lambda$;

(ii) the singular part $\overline{\rho}_s$ of $\overline{\rho}$ with respect to $|D\overline{u}|$ concentrates on the straight lines $\{s_0^* + x_\alpha s_\alpha^* = \pm 1\}$;

(iii) the absolutely continuous part $\overline{\rho}_a$ of $\overline{\rho}$ with respect to $|D\overline{u}|$ satisfies

$$\overline{\rho}_a = \frac{s_0^* + x_\alpha s_\alpha^*}{\alpha_{s^*}(x')} |D\overline{u}| \ .$$

Proof. It is easy to prove that $(\overline{u}, \overline{\rho})$ is an optimal couple for problem $m(t,s)$ defined in (5.16) if and only if there exists $(\lambda, s^*) \in \partial m(t,s)$ such that

$$\int_D |(D\overline{u}, \overline{\rho})| = \int_D \sigma \cdot dD\overline{u} + \int_D (s_0^* + x_\alpha s_\alpha^*) \, d\overline{\rho} \ , \tag{5.20}$$

with σ associated to λ according to the definition of $K(D)$, namely such that

$$\begin{cases} -\operatorname{div}\sigma = \lambda \text{ in } D \\ |\sigma| \leq \alpha_{s^*}(x') \text{ in } D \end{cases} \tag{5.21}$$

Let us decompose the measures $|D\overline{u}|$ and $\overline{\rho}$ as follows:

$$\begin{aligned} &\overline{\rho} = \overline{\rho}_s + \overline{\rho}_a \ , \ \text{ with } \ \overline{\rho}_a << |D\overline{u}| \ , \ \overline{\rho}_s \perp |D\overline{u}| \ . \\ &\theta := \frac{d\overline{\rho}_a}{d|D\overline{u}|} \ , \\ &v := \frac{dD\overline{u}}{d|D\overline{u}|} \ . \end{aligned}$$

In view of (5.20), it is clear that that $\overline{\rho}_s$ concentrates on the straight lines $\{s_0^* + x_\alpha s_\alpha^* = \pm 1\}$. Let us consider the absolutely continuous part. Using the notation above, the integrand in the left hand side of (5.20) reads

$$|(D\overline{u}, \overline{\rho})| = \sqrt{1+\theta^2}\, d|D\overline{u}| \ . \tag{5.22}$$

Recalling that $|(\sigma, s_0^* + x_\alpha s_\alpha^*)| \leq 1$, the condition (5.20) implies that

$$(\sigma, s_0^* + x_\alpha s_\alpha^*) = \frac{(D\overline{u}, \overline{\rho})}{|(D\overline{u}, \overline{\rho})|} \ .$$

In view of (5.22) we obtain

$$\sigma = \frac{v}{\sqrt{1+\theta^2}} \ , \quad s_0^* + x_\alpha s_\alpha^* = \frac{\theta}{\sqrt{1+\theta^2}} \ ,$$

that is

$$\sigma = \alpha_{s^*}(x') \frac{dD\overline{u}}{|dD\overline{u}|} \ , \quad \theta = \frac{s_0^* + x_\alpha s_\alpha^*}{\alpha_{s^*}(x')} \ .$$

Hence, recalling that σ satisfies (5.21), we conclude that

$$-\operatorname{div}\left(\alpha_{s^*}(x') \frac{D\overline{u}}{|D\overline{u}|}\right) = \lambda \ .$$

$\square$

The role of the variant (1.8) of the Cheeger problem is enlightened in the next Proposition.

Proposition 5.10. *The variational problem (5.17) admits the following dual formulation:*

$$\inf \left\{ \int_D \alpha_{s^*} |Dw| \ : \ w \in BV_0(D), \ \int_D w = 1 \right\} \ , \tag{5.23}$$

where α_{s^*} is the non negative function defined in (5.18).

Proof. For every $p \in \mathbb{R}$, let us consider the infimum problem

$$f(p) := \inf \left\{ \int_D \varphi(Aw) \ : \ w \in BV_0(D), \ \int_D w = p \right\} .$$

with φ the convex function $\varphi(z) := \alpha_{s^*}|z|$ (we recall that α_{s^*} is assumed to be positive) and $Aw := Dw$. In particular problem (5.17) equals $f(1)$. An easy computation gives

$$f^*(p^*) = \inf \left\{ \int_D \varphi^*(\sigma) \ : \ -A^*\sigma = p^* \right\} .$$

Since $Aw = Dw$ we have $A^*\sigma = \operatorname{div}\sigma$, moreover $\varphi^*(z^*) = \chi_{|z^*| \le \alpha_{s^*}}$. Hence

$$f^*(p^*) = \begin{cases} 0 \ \text{ if } \exists \sigma \ : \ -\operatorname{div}\sigma = p^*, \ |\sigma| \le \alpha \\ \\ +\infty \text{ otherwise} \end{cases}$$

By definition of Fenchel transform we infer

$$f(p) = \sup_{p^*} \{ p\, p^* - f^*(p^*) \} = \sup \{ p\, p^* \ : \ \exists \sigma \ : \ -\operatorname{div}\sigma = p^*, \ |\sigma| \le \alpha \} . \tag{5.24}$$

Recalling that (5.17) equals $f(1)$, formula (5.24) with $p = 1$ gives (5.23). □

Problem (5.23) is a version of the relaxed formulation of the Cheeger problem with a weigh α that varies in D. For a deeper analysis of this problem, concerning the existence and qualitative properties of solutions, we refer the interested reader to [10].

6 Appendix

Before proving the statements of §4.2 and §4.3, let us introduce two key results. The first one, Lemma 6.1, concerns compactness and the proof can be found in [6, Proposition 3.4]. The second result, Lemma 6.2, is a standard convex duality lemma, whose proof can be found in [4, Proposition 14].

Lemma 6.1. *Let Ψ_D be the unique solution of the Dirichlet problem $-\Delta\Psi_D = 2$ in $H^1_0(D)$. Let $u^\delta \in C^\infty(Q;\mathbb{R}^3)$ be a sequence such that*

$$\int_Q u^\delta \, dx = \int_Q \Psi_D \operatorname{curl} u^\delta \, dx = 0 \qquad \forall\, \delta .$$

If $e^\delta(u^\delta)$ is bounded in $L^2(Q;\mathbb{R}^{3\times 3}_{\rm sym})$, then, up to subsequences,

(i) there exists $\overline{u} \in BN(Q)$ such that $\lim_{\delta\to 0} u^\delta = \overline{u}$ weakly in $L^2(Q;\mathbb{R}^3)$, moreover $\overline{u}$ is of the form (2.1);

(ii) setting

$$v^\delta_\alpha := \delta^{-1}(u^\delta - \overline{u})_\alpha - \delta^{-1}|D|^{-1}[[u^\delta - \overline{u}]]_\alpha$$

$$v^\delta_3 := \delta^{-1}(u^\delta - \overline{u})_3 - \delta^{-1}|D|^{-1}\Big([[u^\delta - \overline{u}]]_3 - x_\alpha[[u^\delta - \overline{u}]]'_\alpha\Big) ,$$

there exist $c \in H^1_m(I)$ and $w \in L^2(I; H^1_m(D))$ such that

$$\lim_{\delta \to 0} (v_1^\delta, v_2^\delta) = c(x_3)(-x_2, x_1) \text{ weakly in } L^2(Q; \mathbb{R}^2)$$

$$\lim_{\delta \to 0} v_3^\delta = w \text{ weakly in } H^{-1}(I; L^2(D)) \; ;$$

(iii) the weak limits in $L^2(Q)$ of $e^\delta_{i3}(u^\delta)$ are given by

$$\lim_{\delta \to 0} (e^\delta_{13}(u^\delta), e^\delta_{23}(u^\delta)) = \tfrac{1}{2} \left(c'(x_3)(-x_2, x_1) + \nabla_{x'} w \right)$$

$$\lim_{\delta \to 0} e^\delta_{33}(u^\delta) = e_{33}(\overline{u}) \; .$$

Lemma 6.2. *Let X, Y be Banach spaces. Let $A : X \to Y$ be a linear operator with dense domain $D(A)$. Let $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ be convex, and $\Psi : Y \to \mathbb{R} \cup \{+\infty\}$ be convex lower semicontinuous. Assume there exists $u_0 \in D(A)$ such that $\Phi(u_0) < +\infty$ and Ψ is continuous at $A(u_0)$. Let Y^* denote the dual space of Y, A^* the adjoint operator of A, and Φ^*, Ψ^* the Fenchel conjugates of Φ, Ψ. Then*

$$- \inf_{u \in X} \Big\{ \Psi(Au) + \Phi(u) \Big\} = \inf_{\sigma \in Y^*} \Big\{ \Psi^*(\sigma) + \Phi^*(-A^* \sigma) \Big\} \; , \tag{6.1}$$

where the infimum on the right hand side is achieved.

Proof of Theorem 4.1

STEP 1: *limit of $\widetilde{\phi}^\delta(k)$*

We claim that the sequence $\widetilde{\mathcal{C}}^\delta$ defined in (4.10) Γ-converges, with respect to the weak * topology of $L^\infty(Q; [0,1])$, to the limit compliance $\mathcal{C}^{lim}$ defined in (4.8). In particular, in view of the well-known properties of Γ-convergence, this implies that for every fixed $k \in \mathbb{R}$ the sequence $\widetilde{\phi}^\delta(k)$ defined in (4.9) tends to the limit problem $\phi(k)$ given by (4.7), as $\delta \to 0$.

The proof of the Γ-convergence is rather technical, therefore we omit it. We just underline that, for the Γ-limsup inequality, the key tool is the compactness Lemma 6.1.

STEP 2: *upper and lower bounds for $\phi^\delta(k)$*

We first remark that, for every k, there holds

$$\phi^\delta(k) = \inf \left\{ \overline{\mathcal{C}}^\delta(\theta) + k \int_Q \theta \; : \; \theta \in L^\infty(Q; [0,1]) \right\} \; ,$$

$\overline{\mathcal{C}}^\delta$ being the lower semicontinuous envelope, in the weak * topology of $L^\infty(Q; [0,1])$, of the functional which is defined as in (1.2) if θ is the characteristic function of a set ω, and $+\infty$ otherwise. Then, by the weak * lower semicontinuity of the fictitious compliance defined in (4.10), we immediately obtain the inequality

$$\widetilde{\phi}^\delta(k) \leq \phi^\delta(k) \; . \tag{6.2}$$

On the other hand, let us introduce another sequence of fictitious problems:

$$\widetilde{\phi}^\delta_0(k) := \inf \left\{ \widetilde{\mathcal{C}}^\delta_0(\theta) + k \int_Q \theta \, dx \; : \; \theta \in L^\infty(Q; [0,1]) \right\} \; . \tag{6.3}$$

associated to the fictitious compliance

$$\widetilde{\mathcal{C}}_0^\delta(\theta) := \sup\left\{\frac{1}{\delta}\langle G,u\rangle_{\mathbb{R}^3} + \langle F,u\rangle_{\mathbb{R}^3} - \int_Q j_0(e^\delta(u))\,\theta\,dx \ :\ u\in H^1(Q;\mathbb{R}^3)\right\}\,, \tag{6.4}$$

with $j_0:\mathbb{R}^{3\times3}_{\rm sym}\to\mathbb{R}$ the modified stored energy density defined by

$$j_0 := \sup\{z\cdot\xi - j^*(\xi)\ :\xi\in\mathbb{R}^{3\times3}_{\rm sym}\ ,\ \det(\xi)=0\}\ .$$

Under the assumption (h3) on the load, by applying [7, Proposition 2.8], it can be proved the upper bound

$$\phi^\delta(k)\le\widetilde{\phi}_0^\delta(k)\ . \tag{6.5}$$

Step 3: *limit of* $\widetilde{\phi}_0^\delta(k)$

The result proved in Step 1 is still valid replacing j by j_0, implying that also the functional $\widetilde{\mathcal{C}}_0^\delta(\theta)$ defined in (6.4) Γ-converges, in the weak * topology of $L^\infty(Q;[0,1])$, to the limit compliance $\mathcal{C}^{lim}(\theta)$: the estimate $j_0\le j$ gives the Γ-liminf inequality, and the coercivity, 2-homogeneity and the equality $\overline{j}=\overline{j_0}$ (cf. Lemma 3.9 in [6]) ensure the Γ-limsup inequality. As a consequence the fictitious problems $\widetilde{\phi}_0^\delta(k)$ defined in (6.3) converge to $\phi(k)$.

Step 4: *limit of* $\phi^\delta(k)$

By combining the estimates (6.2) and (6.5) in Step 2, and the convergence results obtained in Step 1 and Step 3, we infer that also the sequence $\phi^\delta(k)$ converges to $\phi(k)$ as $\delta\to0$.

Let $\omega^\delta\subset Q$ be a sequence of domains such that $\phi^\delta(k)=\mathcal{C}^\delta(\omega^\delta)+k|\omega^\delta|+o(1)$. Since we know that the sequences $\widetilde{\phi}^\delta(k)$ and $\phi^\delta(k)$ have the same limit as $\delta\to0$, we deduce that $\widetilde{\phi}^\delta(k)=\widetilde{\mathcal{C}}^\delta(\mathbb{1}_{\omega^\delta})+k\int_Q\mathbb{1}_{\omega^\delta}\,dx+o(1)$. In view of Step 1, the sequence $\widetilde{\mathcal{C}}^\delta(\theta)+k\int_Q\theta\,dx$, being a continuous perturbation of $\widetilde{\mathcal{C}}^\delta(\cdot)$, Γ-converges to $\mathcal{C}^{lim}(\theta)+k\int_Q\theta\,dx$ in the the weak * topology $L^\infty(Q;[0,1])$; therefore any cluster point of $\mathbb{1}_{\omega^\delta}$ is a solution $\overline{\theta}$ to problem (4.7). □

Proof of Proposition 4.4

Let us consider problem $\mathcal{C}^{lim}(\theta)$. The formulation (4.12) follows by applying Lemma 6.2 to $X=TW(Q)\times BN(Q)$, $Y=L^2(Q;\mathbb{R}^3)$, $A(v,u)=(e_{13}(v),e_{23}(v),e_{33}(u))$, $\Phi(v,u)=-\langle G,v\rangle_{\mathbb{R}^3}-\langle F,u\rangle_{\mathbb{R}^3}$, and $\Psi(y)=\int_Q\overline{j}(y)\,\theta\,dx$.

We remark that the constraints appearing in (4.12) are due to the following equivalence: $\sigma\in L^2(Q;\mathbb{R}^3)$ satisfies

$$\int_Q\sigma\cdot(e_{13}(v),e_{23}(v),e_{33}(u))=\langle G,v\rangle_{\mathbb{R}^3}+\langle F,u\rangle_{\mathbb{R}^3}\quad\forall\,(v,u)\in TW(Q)\times BN(Q)$$

if and only if

$$\partial_1\sigma_1+\partial_2\sigma_2=-2G_3\,,\ [[x_1\sigma_2-x_2\sigma_1]]=-2\mathcal{P}_0(m_G)\,,\ [[\sigma_3]]=-\mathcal{P}_0(\overline{F_3})\,,\ [[x_\alpha\sigma_3]]=-\mathcal{P}(\overline{F_\alpha})\ ,$$

in the sense of distributions.

Let us now consider problem $\phi(k)$. Recalling the definitions (4.7) of $\phi(k)$ and (4.8) of $\mathcal{C}^{lim}(\theta)$, by a standarn inf-sup commutation argument (see *e.g.* [12, Proposition A.8]), we can rewrite $\phi(k)$ as

$$\phi(k)=\sup_{(v,u)\in TW\times BN}\left\{\langle G,v\rangle_{\mathbb{R}^3}+\langle F,u\rangle_{\mathbb{R}^3}-\int_Q\left[\overline{j}\big(e_{13}(v),e_{23}(v),e_{33}(u)\big)-k\right]_+dx\right\}\ . \tag{6.6}$$

Now, as already done in the first part of the proof, by applying Lemma 6.2 with X, Y, A and Φ as above, and $\Psi(y)=\int_Q[\overline{j}(y)-k]_+\,dx$, in view of (6.6) one obtains the dual form (4.11).

□

Proof of Theorem 4.8

Step 1: *equivalent formulation of* $\overline{m}$

We claim that $\overline{m}$ agrees with the following supremum:

$$m_0 := \sup_{(v,u)\in TW\times BN} \left\{ \langle G,v\rangle_{\mathbb{R}^3} + \langle F,u\rangle_{\mathbb{R}^3} \;:\; \|\overline{j}(e_{13}(v),e_{23}(v),e_{33}(u))\|_{L^\infty(Q)} \le \frac{1}{2} \right\}. \tag{6.7}$$

For every $t\in\mathbb{R}^+$, by the definition (4.13) of $\mathcal{C}^{\lim}(\mu)$ and a standard inf-sup commutation argument, we infer:

$$\inf_{\{\mu\,:\,\int d\mu\le t\}} \mathcal{C}^{\lim}(\mu) = \sup_{\substack{v\in TW\cap C^\infty\\ u\in BN\cap C^\infty}} \left\{ \langle G,v\rangle_{\mathbb{R}^3} + \langle F,u\rangle_{\mathbb{R}^3} - t\,\|\overline{j}(e_{13}(v),e_{23}(v),e_{33}(u))\|_{L^\infty(Q)} \right\} = \frac{m_0^2}{2t}.$$

The claim follows by a direct computation, passing to the infimum over $t\in\mathbb{R}^+$ and observing that

$$\overline{m} = \inf_{t\in\mathbb{R}^+} \left\{ \mathcal{C}^{\lim}(\mu) + \frac{t}{2} \;:\; \int d\mu \le t \right\}.$$

Step 2: *proof of (i)*

The second equality in (4.15) shows that the map $k\mapsto \frac{\phi(k)}{\sqrt{2k}}$ is nonincreasing and satisfies the inequality $\frac{\phi(k)}{\sqrt{2k}}\ge\overline{m}$. In order to show that it converges to $\overline{m}$ as $k\to+\infty$, we exploit the formulation of $\phi(k)$ given in (6.6) that, after a change of variables, reads

$$\frac{\phi(k)}{\sqrt{2k}} = \sup_{(v,u)\in TW\times BN} \left\{ \langle G,v\rangle_{\mathbb{R}^3} + \langle F,u\rangle_{\mathbb{R}^3} - \sqrt{2k}\int_Q \left[\overline{j}\big(e_{13}(v),e_{23}(v),e_{33}(u)\big) - \frac{1}{2}\right]_+ dx \right\}.$$

Given a maximizing sequence (v_k,u_k), by using the coercivity of $[\overline{j}(z)-k]_+$, the inequality $\phi(k)\ge 0$, and the assumption that F and G are admissible loads, we deduce that $\|e_{\alpha 3}(v_k)\|_{L^2(Q)}$ and $\|e_{33}(u_k)\|_{L^2(Q)}$ are bounded. With a direct computation, we can prove that such boundedness properties imply that, up to subsequences, there exists $(v,u)\in TW(Q)\times BN(Q)$ such that $\lim_k v_k = v$ weakly in $H^1(Q;\mathbb{R}^2)\times L^2(I;H^1_m(D))$, $\lim_k u_k = u$ weakly in $H^1(Q;\mathbb{R}^3)$, $\lim_k e_{\alpha 3}(v_k)=e_{\alpha 3}(v)$, $\lim_k e_{33}(u_k)=e_{33}(u)$ weakly in $L^2(Q)$; moreover the pair is admissible in the definition (6.7) of m_0. Thus we may conclude

$$\lim_{k\to+\infty}\frac{\phi(k)}{\sqrt{2k}} \le \lim_{k\to+\infty}\left(\langle G,v_k\rangle_{\mathbb{R}^3} + \langle F,u_k\rangle_{\mathbb{R}^3}\right) = \langle G,v\rangle_{\mathbb{R}^3} + \langle F,u\rangle_{\mathbb{R}^3} \le m_0 = \overline{m}.$$

Step 3: *proof of (ii)*

If θ_k is an optimal density for $\phi(k)$, setting $\mu_k := \sqrt{2k}\,\theta_k\,dx$ one has

$$\frac{\phi(k)}{\sqrt{2k}} = \mathcal{C}^{lim}(\mu_k) + \frac{1}{2}\int d\mu_k.$$

Since $\mathcal{C}^{lim}(\mu_k)\ge 0$ and since by monotonicity $\frac{\phi(k)}{\sqrt{2k}}\le\phi(1)$, the above equation implies that the integral $\int d\mu_k$ remains uniformly bounded. Then up to a subsequence there exists $\overline{\mu}$ such that

$\mu_k \overset{*}{\rightharpoonup} \overline{\mu}$. By using item (i) already proved, the weak * semicontinuity of the map $\mu \mapsto \mathcal{C}^{lim}(\mu)$, and the definition (4.16) of $\overline{m}$, we obtain

$$\overline{m} = \lim_{k\to+\infty} \frac{\phi(k)}{\sqrt{2k}} = \lim_{k\to+\infty} \left\{ \mathcal{C}^{lim}(\mu_k) + \frac{1}{2}\int d\mu_k \right\} \geq \mathcal{C}^{lim}(\overline{\mu}) + \frac{1}{2}\int d\overline{\mu} \geq \overline{m} \,.$$

Hence $\overline{\mu}$ is a solution to problem (4.16). □

Proof of Proposition 4.9

Let $X := (TW \times BN) \cap \mathcal{C}_0^1(Q;\mathbb{R}^6)$, $Y := \mathcal{C}_0(Q;\mathbb{R}^3)$, $A(v,u) := (e_{13}(v), e_{23}(v), e_{33}(u))$, $\Phi(v,u) := -\langle G, v\rangle_{\mathbb{R}^3} - \langle F, u\rangle_{\mathbb{R}^3}$, and $\Psi(y) = 0$ if $\|\overline{j}(y)\|_\infty \leq 1/2$, and $+\infty$ otherwise. By applying Lemma 6.2, we derive that the dual form of problem m_0 in (6.7) is given by (4.17). Since, in view of Step 1 in the proof of Theorem 4.8, $m_0 = \overline{m}$, we conclude that (4.17) agrees also with $\overline{m}$. □

Proof of Proposition 4.10

Let $\overline{\sigma}$ be optimal for the dual problem (4.17), and set $\overline{\mu} := |\overline{\sigma}|$. Then we have $\left|\frac{d\overline{\sigma}}{d\overline{\mu}}\right| = 1$ $\overline{\mu}$-a.e. and

$$\int \overline{\mu} = m_0 = \overline{m}\,. \tag{6.8}$$

Moreover, exploiting the optimality of $\overline{\sigma}$ and the constraints appearing in (4.17), we get

$$\begin{aligned} \mathcal{C}^{\lim}(\overline{\mu}) &= \sup\left\{ \langle \overline{\sigma}, x\rangle_{\mathbb{R}^3} - \frac{1}{2}\int_Q |x|^2\, d\overline{\mu} \ :\ x = (2\sqrt{\eta}e_{\alpha 3}(v), \sqrt{Y}e_{33}(u))\,,\ (v,u) \in TW \times BN \right\} \\ &\leq \frac{1}{2}\int_Q \left|\frac{d\overline{\sigma}}{d\overline{\mu}}\right|^2 d\overline{\mu} = \frac{1}{2}\int d\overline{\mu}\,. \end{aligned} \tag{6.9}$$

In view of (6.8) and (6.9) we conclude that

$$\mathcal{C}^{\lim}(\overline{\mu}) + \frac{1}{2}\int d\overline{\mu} \leq \int d\overline{\mu} \leq \overline{m}\,,$$

then $\overline{\mu}$ is optimal for the problem (4.16) .

Conversely, assume that $\overline{\mu}$ is optimal for the problem (4.16), and let $\overline{\xi}$ be optimal for the dual form (4.14) of $\mathcal{C}^{\lim}(\overline{\mu})$, that is

$$\int_Q \overline{j}^*(\overline{\xi})\, d\overline{\mu} = \mathcal{C}^{\lim}(\overline{\mu})\,. \tag{6.10}$$

Set $\overline{\sigma} := \left(\frac{\overline{\xi}'\,\overline{\mu}}{2\sqrt{\eta}}\,,\,\frac{\overline{\xi}_3\,\overline{\mu}}{\sqrt{Y}}\right)$ and notice that it is admissible for problem (4.17). Adopting the same strategy of [6, Proposition 5.3], we can prove that

$$\left|\left(\frac{\overline{\xi}'}{2\sqrt{\eta}}\,,\,\frac{\overline{\xi}_3}{\sqrt{Y}}\right)\right| \leq 1 \quad \overline{\mu}\text{−a.e.}\,. \tag{6.11}$$

Thus we conclude that $\overline{\sigma}$ is optimal for (4.17), since

$$\int |\overline{\sigma}| = \int \left|\frac{d\,\overline{\sigma}}{d\,\overline{\mu}}\right| d\overline{\mu} \leq \int d\overline{\mu} = \overline{m}\,.$$

□

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