A nonstandard free boundary problem arising in the shape optimization of thin torsion rods

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Abstract

We study a $2d$-variational problem, in which the cost functional is an integral depending on the gradient through a convex but *not strictly convex* integrand, and the admissible functions have zero gradient on the complement of a given domain D. We are interested in establishing whether solutions exist whose gradient "avoids" the region of non-strict convexity. Actually, the answer to this question is related to establishing whether homogenization phenomena occur in optimal thin torsion rods. We provide some existence results for different geometries of D, and we study the nonstandard free boundary problem with a gradient obstacle, which is obtained through the optimality conditions.

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1 Introduction

Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be the following convex but non-strictly convex integrand:

$$\varphi(y) := \begin{cases} \frac{|y|^2}{2} + \frac{1}{2} & \text{if } |y| \geq 1 \\ |y| & \text{if } |y| < 1 \,. \end{cases} \tag{1.1}$$

Let D be a bounded and connected domain in $\mathbb{R}^2$, let s be a real parameter, and consider the variational problem

$$m(s) := \inf \left\{ \int_{\mathbb{R}^2} \varphi(\nabla u) \ : \ u \in H^1_c(D) \,, \ \int_{\mathbb{R}^2} u = s \right\} \,, \tag{1.2}$$

where

$$H^1_c(D) := \left\{ u \in H^1(\mathbb{R}^2) \ : \ \nabla u = 0 \text{ on } \mathbb{R}^2 \setminus \overline{D} \right\} \,.$$

Notice that functions in $H^1_c(D)$ must vanish identically on the unique unbounded connected component of $\mathbb{R}^2 \setminus \overline{D}$; in particular, if D is simply connected, functions in $H^1_c(D)$ are elements of the usual Sobolev space $H^1_0(D)$, extended to zero out of D. More generally, if $D = D_0 \setminus \cup_{i=1}^k \overline{D_i}$, where

D_i $(i = 0, 1, \ldots, k)$ are Jordan domains with mutually disjoint boundaries, functions in $H^1_c(D)$ are extensions to zero of elements of $H^1_0(D_0)$ which are constant on each D_i for $i = 1, \ldots, k$.
We say that u is a *special solution* to $m(s)$ if it minimizes (1.2) and satisfies

$$|\nabla u| \in \{0\} \ \cup \ (1, +\infty\,) \text{ a.e. in } D\ .$$

This paper is focused on the following question:

$$\textit{Does problem } m(s) \textit{ admit a special solution?} \tag{1.3}$$

This question, which was raised in [9], appears as an important issue in the context of shape optimization of thin torsion rods. We postpone to Section 2 a brief description of the mechanical motivation and of the precise meaning of problem (1.2). In this framework, question (1.3) corresponds to ask whether an optimal design contains some homogenization region made by a composite material.
Below and throughout the paper, we adopt the following notation: if u is a special solution to problem $m(s)$, we call the *plateau* of u, and we denote it by $\Omega(u)$, the set $\{\nabla u = 0\}$ minus the unbounded connected component of $\mathbb{R}^2 \setminus \overline{D}$ (where $u \equiv 0$). The set $\Gamma(u) := \partial\Omega(u) \cap D$ will be called the *free boundary* of u (see Figure 1).

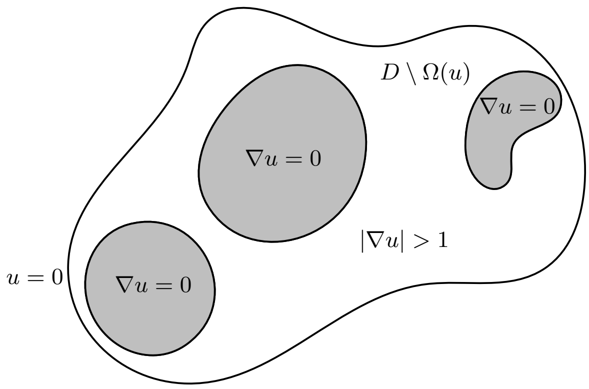

Figure 1: *Behaviour of special solutions.*

Our main results concern the study of question (1.3) in relation with the geometry of the domain D and also with the value of the parameter s. They are listed below.

- When D is a ball or a ring, through explicit computations and exploiting the optimality conditions, we show that $m(s)$ admits a special solution, and there is no other solution, see Proposition 5.1 and Proposition 5.3.

- Balls and rings are *not* the unique domains for which $m(s)$ admits a special solution. At this point it is worth to compare with the case considered by Murat and Tartar in [25] where two linearly elastic materials have to be mixed in fixed proportions in a given cross-section in order to maximize torsional rigidity. In their case the variational problem under study is quite similar to ours except that it involves a *differentiable* integrand and they proved that classical solutions (i.e. with no homogenization regions) cannot exist unless the cross-section D is a disk. In our case the integrand φ is non-differentiable at zero and the conclusion goes in a quite opposite direction : we prove that there exists a non circular domain D with analytic boundary such that, for some s, problem $m(s)$ admits a special solution. Moreover this solution has a convex plateau with analytic boundary, see Theorem 6.1. To achieve this result, we use as a key tool the relationship between $m(s)$ and the Cheeger problem (whose definition is recalled in Section 4). Let us remark that the crucial role played by the Cheeger set of D in the study of the asymptotic behaviour of $m(s)$ as $s \to 0^+$ already emerged in [9, 10].

- After providing some elementary properties on the sign and the support of generic solutions (see Propositions 7.1 and 7.2), we derive some information on qualitative properties of special solutions, when the latter exist. This amounts to study a nonstandard free boundary problem with an obstacle on the gradient, see (7.4) below. Assuming that D is simply connected, and that there exists a special solution u with a smooth free boundary,
  - we prove that each connected component of $D \setminus \Omega(u)$ must touch ∂D (see Proposition 7.3);
  - we provide some sufficient conditions for the convexity of the plateau $\Omega(u)$ (see Proposition 7.4);
  - we show that, when D is not Cheeger set of itself, the plateau $\Omega(u)$ cannot be compactly contained into D for arbitrarily small filling ratios (see Proposition 7.5).

We point out that characterizing domains D where a special solution to $m(s)$ exists seems to be a very challenging problem, which remains by now open: in this respect we believe that, at least when D is convex, the existence of special solutions is likely related to the regularity of ∂D, and also to whether or not D coincides with its Cheeger set. Let us notice that the latter criterium would exclude the existence of a special solution in the case when D is a square. This guess seems to be confirmed by the numerical results performed in [23] for a very similar problem, in which homogenization regions are observed.

Outline of the paper.

In Section 2 we provide a physical motivation for the study of problem $m(s)$, namely we describe its relationship with the optimal design of thin rods in torsion regime.

In Section 3 we find necessary and sufficient optimality conditions, we deduce some consequences on the behaviour of solutions to $m(s)$ (including a uniqueness criterion), and we study $m(s)$ as a function of s.

In Section 4 we give some preliminary results about Cheeger sets.

In Section 5 we prove the existence and uniqueness of special solutions to $m(s)$ when D is a ball or a ring.

In Section 6 we prove the existence of special solutions to $m(s)$ for some domain D, different from those considered in Section 5.

In Section 7 we obtain qualitative properties of solutions and of special solutions.

Notation. Throughout the paper, unless otherwise stated, D is assumed to be open, bounded, and connected; when further assumptions are needed, we specify them in each statement. We denote by $|x|$ the Euclidean modulus of a vector $x \in \mathbb{R}^2$, and also by $|A|$ the Lebesgue measure of a Borel set

$A \subset \mathbb{R}^2$. We simply write $\int_A f$ to denote the integral of the function f with respect to the Lebesgue measure on A.

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2 An application in shape optimization

Question (1.3) draws its origin from an optimal design problem for thin elastic torsion rods, studied in [9, 10]. Below we briefly describe such problem and we sketch how it leads to (1.2), referring to the above quoted papers for more details.
Let Q_δ be a cylinder with infinitesimal cross section, of the form $Q_\delta := \delta D \times I$, where D is an open, bounded and connected domain in $\mathbb{R}^2$, $I = [-1/2, 1/2]$, and $\delta > 0$ is a small parameter.
Assume that a given amount of elastic material has to be distributed into the design region Q_δ, in such a way to maximize the resistance to a horizontal twisting load $G \in H^{-1}(Q; \mathbb{R}^3)$. Then one has to find the domain Ω, of prescribed volume, which minimizes the following shape functional, usually called elastic compliance:

$$\mathcal{C}^\delta(\Omega) := \sup \left\{ \langle G^\delta, u \rangle_{\mathbb{R}^3} - \int_\Omega j(e(u)) \ : \ u \in H^1(Q_\delta; \mathbb{R}^3) \right\} .$$

Here G^δ is a suitable scaling of G, $e(u)$ denotes the symmetric part of ∇u, and j is the strain potential, that for simplicity we take as $j(z) = |z|^2/4$.
After including the volume constraint into the cost functional through a Lagrange multiplier $k \in \mathbb{R}$, the shape optimization problem reads

$$\phi^\delta(k) := \inf \left\{ \mathcal{C}^\delta(\Omega) + \frac{k}{\delta^2} |\Omega| \ : \ \Omega \subseteq Q_\delta \right\} .$$

It is well known that such problem is in general ill-posed: homogenization phenomena prevent the existence of an optimal domain (see [1]), so that relaxed solutions must be searched under the form of densities with values in $[0, 1]$.
Now, keeping k fixed, one can consider the variational problem $\phi(k)$ obtained from $\phi^\delta(k)$ in the small cross section limit $\delta \to 0^+$. In [9, Theorem 3.2] it is proved that, under suitable assumptions on the load G, $\phi(k)$ is the following variational problem for varying densities in $L^\infty(Q; [0, 1])$:

$$\phi(k) := \inf \left\{ \mathcal{C}^{lim}(\theta) + k \int_Q \theta \ : \ \theta \in L^\infty(Q; [0, 1]) \right\} . \tag{2.1}$$

The limit compliance $\mathcal{C}^{lim}(\theta)$ appearing in the r.h.s. of (2.1) is defined by

$$\mathcal{C}^{lim}(\theta) := \sup \Big\{ \langle G, v \rangle_{\mathbb{R}^3} - \frac{1}{2} \int_Q \big| (e_{13}(v), e_{23}(v)) \big|^2 \theta \ : \ (v_1, v_2) = c(x_3)(-x_2, x_1) \ , \ c(x_3) \in H^1_m(I) \\ v_3 \in L^2(I; H^1_m(D)) \Big\} ,$$

being $H^1_m(D)$ (resp $H_1(I)$) the subspace of $H^1(D)$ (resp $H^1(I)$) of functions with zero integral mean. Clearly, as a variational problem on $L^\infty(Q;[0,1])$, $\phi(k)$ is well-posed (in contrast with problem $\phi^\delta(k)$ which in general, as mentioned above, has no solution). Then the following question arises in a natural way (*cf.* [9, Remark 4.6]):

$$\textit{Does problem } \phi(k) \textit{ admit a solution } \overline{\theta} \textit{ taking values into } \{0,1\}\textit{?} \tag{2.2}$$

An affirmative answer would mean that the optimal design problem $\phi(k)$ admits a classical solution, which may be identified with a set rather than a composite.
We claim that question (2.2) is equivalent to our initial question (1.3). In fact, let us explain the relationship betwen problems $\phi(k)$ and $m(s)$.
Decomposing the spatial variable $x \in \mathbb{R}^3$ as $x = (x', x_3) \in D \times I$ and writing $\phi(k)$ in dual form, one gets (see [9, Theorem 4.2])

$$\frac{\phi(k)}{2k} = \inf_{q \in L^2(\mathbb{R}^3;\mathbb{R}^2)} \left\{ \int_{\mathbb{R}^3} \varphi(q) \ : \ \mathrm{spt}(q) \subset \overline{Q}\,,\ \mathrm{div}_{x'}\, q = 0\,,\ \int_{\mathbb{R}^3} (x_1 q_2 - x_2 q_1) = -\frac{2M_G(x_3)}{\sqrt{2k}} \right\}, \tag{2.3}$$

where $M_G(x_3)$ can be written explicitly in terms of G. Here the divergence free constraint is due to the vanishing of the vertical component of G. It holds in the distributional sense on all $\mathbb{R}^3$, being implicitly assumed that q is zero on $\mathbb{R}^3 \setminus \overline{Q}$.
Now we observe that the constraints imposed on the admissible fields in (2.3) involve only the horizontal variable x'. Therefore, q solves (2.3) if and only if, for a.e. $x_3 \in I$, $q(\cdot, x_3)$ solves the following section problem, for $s = \frac{M_G}{\sqrt{2k}}$:

$$\alpha(s) := \inf_{q \in L^2(\mathbb{R}^2;\mathbb{R}^2)} \left\{ \int_{\mathbb{R}^2} \varphi(q) \ : \ \mathrm{spt}(q) \subset \overline{D}\,,\ \mathrm{div}\, q = 0\,,\ \int_{\mathbb{R}^2} (x_1 q_2 - x_2 q_1) = -2s \right\}. \tag{2.4}$$

Now we observe that any divergence free field $q \in L^2(\mathbb{R}^2;\mathbb{R}^2)$ compactly supported in $\overline{D}$ can be written in a unique way as the rotated gradient $(-\partial_2 u, \partial_1 u)$ of a suitable function $u \in H^1(\mathbb{R}^2)$. As q vanishes a.e in $\mathbb{R}^2 \setminus \overline{D}$, we have that $u \in H^1_c(D)$, as u is constant on each connected component of $\mathbb{R}^2 \setminus \overline{D}$. It follows that

$$\alpha(s) = m(s)\,. \tag{2.5}$$

The equalities (2.3), (2.4) and (2.5) show the link between the optimal design problem $\phi(k)$ and our initial problem $m(s)$.
To go farther, one can see that questions (1.3) and (2.2) are equivalent to each other, by exploiting the optimality conditions for problem $\phi(k)$ derived in [9, Theorem 4.5]. Actually their analysis reveals that, if problem $m(s)$ admits a special solution $\overline{u}$, then a solution $\overline{\theta}$ to problem $\phi(k)$ takes only the values 0 and 1, which means that no homogenization phenomenon occurs.
In the light of the above discussion, the results presented in the next sections can be applied to study the influence of the section's shape and of the filling ratio on the presence of homogenization regions in optimal thin torsion rods.
Let us emphasize that no precedent exists in this direction within the study of optimal thin plates. Indeed in that case the limit model obtained starting from three-dimensional elasticity through a $3d$-$2d$ dimension reduction process always admits classical "set" solutions, under the form of sandwich-like structures ([11], see also [8]).

3 Existence, uniqueness, optimality conditions, and dependence on s.

The contents of this section are organized as follows: in §3.1 we study the minimization problem $m(s)$ in its primal formulation (1.2): we prove the existence of solutions, and a necessary and sufficient condition for optimality; in §3.2 we give the dual form of problem $m(s)$, we derive the corresponding optimality conditions and some of their consequences; in §3.3 we show some properties of $m(s)$ seen as a function of the parameter s.

3.1 Primal problem

We begin by establishing the existence of minimizers for $m(s)$, and their characterization as solutions to a variational inequality.

Proposition 3.1. *For every $s \in \mathbb{R}$, the infimum $m(s)$ is achieved in $H^1_c(D)$. A function $u \in H^1_c(D)$ is optimal if and only if*

$$\int_{\{\nabla u=0\}} |\nabla v| + \int_{\{\nabla u\neq 0\}} \langle \nabla\varphi(\nabla u), \nabla v\rangle \geq 0 \qquad \forall v \in H^1_c(D) \ : \ \int_{\mathbb{R}^2} v = 0\,.$$

Proof. We observe that, since functions in $H^1_c(D)$ vanish in the unbounded connected component of $\mathbb{R}^2 \setminus \overline{D}$, by Poincaré inequality there exists a positive constant C such that

$$\|u\|_{H^1(\mathbb{R}^2)} \leq C\,\|\nabla u\|_{L^2(\mathbb{R}^2)} \qquad \forall\, u \in H^1_c(D)\ .$$

Combined with the coercivity of φ (in fact $\varphi(y) \geq \frac{|y|^2}{2}$), this ensures that every minimizing sequence for problem $m(s)$ is weakly relatively compact in $H^1(\mathbb{R}^2)$. Clearly any cluster point belongs to $H^1_c(D)$. On the other hand, by the convexity of φ, the integral functional $J_\varphi(u) := \int_{\mathbb{R}^2} \varphi(\nabla u)$ is weakly lower semicontinuous on $H^1(\mathbb{R}^2)$. Therefore the existence of at least one solution follows from the direct method of Calculus of Variations. Considering all the variations compatible with the integral constraint, it is straightforward to check that a minimizer u is characterized by the variational inequality $\delta J_\varphi(u,v) \geq 0$ for all $v \in H^1_c(D)$ such that $\int_{\mathbb{R}^2} v = 0$. Here the directional derivative $\delta J_\varphi(u,v)$ is given by

$$\delta J_\varphi(u,v) = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon}\left[J_\varphi(u+\varepsilon v) - J_\varphi(u)\right] = \int_{\{\nabla u=0\}} |\nabla v| + \int_{\{\nabla u\neq 0\}} \langle \nabla\varphi(\nabla u), \nabla v\rangle\,.$$

□

3.2 Dual problem

We are going to explicit the dual formulation of problem $m(s)$. Let us remark that the Fenchel conjugate of φ is given by

$$\varphi^*(\xi) = \begin{cases} \frac{1}{2}|\xi|^2 - \frac{1}{2} & \text{if } |\xi| > 1 \\ 0 & \text{if } |\xi| \leq 1\,. \end{cases} \tag{3.1}$$

Moreover let us introduce, for every $\lambda \in \mathbb{R}$, the class of vector fields

$$\mathcal{S}_\lambda(D) := \left\{\sigma \in L^2(\mathbb{R}^2;\mathbb{R}^2) \ : \ \mathrm{spt}(\sigma) \subseteq \overline{D}\,,\ \int_{\mathbb{R}^2} \sigma\cdot\nabla u = \lambda \int_{\mathbb{R}^2} u \quad \forall u \in H^1_c(D)\right\}. \tag{3.2}$$

By taking as test functions u in (3.2) elements of $H_0^1(D)$ extended to zero out of D, one can see that every $\sigma \in \mathcal{S}_\lambda(D)$ satisfies the condition $-\operatorname{div}\sigma = \lambda$ in D. In the special case when D is simply connected, all functions $u \in H_c^1(D)$ are of this type, so that $\mathcal{S}_\lambda(D)$ can be characterized as

$$\mathcal{S}_\lambda(D) = \Big\{\sigma \in L^2(\mathbb{R}^2;\mathbb{R}^2) \ : \ \operatorname{spt}(\sigma) \subseteq \overline{D}\,, \ -\operatorname{div}\sigma = \lambda \text{ in } D\Big\}\,. \tag{3.3}$$

More in general, if $D = D_0 \setminus \cup_{i=1}^k \overline{D_i}$, where D_i $(i = 0, 1, \dots, k)$ are Jordan domains with mutually disjoint boundaries, one has

$$\mathcal{S}_\lambda(D) = \Big\{\sigma \in L^2(\mathbb{R}^2;\mathbb{R}^2) \ : \ \operatorname{spt}(\sigma) \subseteq \overline{D}\,, \ -\operatorname{div}\sigma = \lambda \text{ in } D\,, \ \int_{\partial D_i} \sigma\cdot\nu_i = -\lambda|D_i| \ \ \forall i = 1,\dots,k\Big\}\,,$$

being ν_i the unit outer normal to ∂D_i.

Lemma 3.2. *The map $s \mapsto m(s)$ is a convex even function on $\mathbb{R}$, whose Fenchel conjugate is given by*

$$m^*(\lambda) = \min\Big\{\int_{\mathbb{R}^2} \varphi^*(\sigma) \ : \ \sigma \in \mathcal{S}_\lambda(D)\Big\}\,. \tag{3.4}$$

Proof. Recalling definition (1.2), since the integrand φ is convex and even, we obtain immediately that the map $s \mapsto m(s)$ is a convex even function on $\mathbb{R}$. Its Fenchel conjugate is given by

$$m^*(\lambda) = \sup_{s\in\mathbb{R}} \Big\{\lambda s - m(s)\Big\} = \sup_{u \in H_c^1(D)} \Big\{\lambda \int_{\mathbb{R}^2} u - \int_{\mathbb{R}^2} \varphi(\nabla u)\Big\} \qquad \forall \lambda \in \mathbb{R}\,. \tag{3.5}$$

By seeing the constant λ as an element of the dual space of $H_c^1(D)$, we may rewrite (3.5) as the Fenchel conjugate of a composition:

$$m^*(\lambda) = \big(I_\varphi \circ A\big)^*(\lambda)\,,$$

where $I_\varphi : L^2(\mathbb{R}^2;\mathbb{R}^2) \to \mathbb{R}$ is the integral functional $I_\varphi(y) = \int_{\mathbb{R}^2} \varphi(y)$, and $A : H_c^1(D) \to L^2(\mathbb{R}^2;\mathbb{R}^2)$ is the gradient mapping $Au = \nabla u$. Then, since I_φ is convex continuous whereas A is a bounded linear operator, we have (see *e.g.*[7, Proposition 13])

$$m^*(\lambda) = \min\Big\{(I_\varphi)^*(\sigma) \ : \ \sigma \in L^2(\mathbb{R}^2;\mathbb{R}^2)\,, \ \operatorname{spt}(\sigma) \subseteq \overline{D}\,, \ A^*\sigma = \lambda\Big\}\,.$$

The above equality entails (3.4), by taking into account that $(I_\varphi)^* = I_{\varphi^*}$ (see [7, Example 4]), and by observing that $A^*\sigma = \lambda$ holds if and only if σ belongs to the subset $\mathcal{S}_\lambda(D)$ given in (3.2). $\square$

We can now give the optimality conditions which characterize solutions to problems $m(s)$ and $m^*(\lambda)$.

Proposition 3.3. *Let $s, \lambda \in \mathbb{R}$, $u \in H_c^1(D)$, and $\sigma \in L^2(\mathbb{R}^2;\mathbb{R}^2)$. There holds the following equivalence*

$$(i) \begin{cases} u \text{ solution to } m(s) \\ \sigma \text{ solution to } \ m^*(\lambda) \\ \lambda \in \partial m(s)\,. \end{cases} \iff (ii) \begin{cases} \int_{\mathbb{R}^2} u = s \\ \sigma \in \mathcal{S}_\lambda(D) \\ \sigma \in \partial\varphi(\nabla u) \ \text{a.e.} \end{cases}$$

Proof. [(i) $\Rightarrow$ (ii)] Let s, λ, u, σ satisfy (i). In particular, since by definition u and σ are admissible in problems (1.2) and (3.4) respectively, they satisfy $\int_{\mathbb{R}^2} u = s$ and $\sigma \in \mathcal{S}_\lambda(D)$. Thus we only have to show that $\sigma \in \partial\varphi(\nabla u)$ a.e. Since $\lambda \in \partial m(s)$, the Fenchel equality is satisfied

$$m(s) + m^*(\lambda) = s\lambda\,,$$

that is, thanks to the optimality of u and σ in (1.2) and (3.4),

$$\int_{\mathbb{R}^2} \varphi(\nabla u) + \int_{\mathbb{R}^2} \varphi^*(\sigma) = s\lambda = \lambda \int_{\mathbb{R}^2} u = \int_{\mathbb{R}^2} \nabla u \cdot \sigma\,, \tag{3.6}$$

which implies $\sigma \in \partial\varphi(\nabla u)$ a.e.
[(ii) $\Rightarrow$ (i)] Let s, λ, u, σ satisfy (ii). By the first two conditions in (ii), u and σ are admissible in problems (1.2) and (3.4) respectively. Moreover, the third condition $\sigma \in \partial\varphi(\nabla u)$ a.e. implies that (3.6) holds. Hence

$$\int_{\mathbb{R}^2} \varphi^*(\sigma) = \lambda s - \int_{\mathbb{R}^2} \varphi(\nabla u) \leq \lambda s - m(s) \leq m^*(\lambda)\,. \tag{3.7}$$

Therefore, σ is a solution to $m^*(\lambda)$, and all the inequalities in (3.7) hold with equality sign. This implies that u is a solution to $m(s)$ and that $\lambda \in \partial m(s)$. □

Let us examine more in detail the condition $\sigma \in \partial\varphi(\nabla u)$ a.e., appearing in Proposition 3.3. The convex integrand φ is differentiable at every $y \neq 0$, whereas its subdifferential at 0 is given by $\partial\varphi(0) = \{|y| \leq 1\}$. Therefore, the inclusion $\sigma \in \partial\varphi(\nabla u)$ always holds true on $\mathbb{R}^2 \setminus \overline{D}$, where $\sigma = 0$ and $\nabla u = 0$. On the other hand, the same inclusion can be rewritten more explicitly on the different regions of D as

$$\begin{cases} \sigma = \nabla u & \text{on } \big\{x \in D \,:\, |\nabla u(x)| > 1\big\} \\ \sigma = \frac{\nabla u}{|\nabla u|} & \text{on } \big\{x \in D \,:\, 0 < |\nabla u(x)| \leq 1\big\} \\ |\sigma| \leq 1 & \text{on } \big\{x \in D \,:\, |\nabla u(x)| = 0\big\}\,. \end{cases} \tag{3.8}$$

These equalities have several implications, which are listed in the next corollaries. First of all, the region where solutions u to problem $m(s)$ satisfy the condition $|\nabla u| > 1$ turns out to be uniquely determined by s, together with the value of ∇u on it. More precisely we have:

Corollary 3.4. *There exist a measurable subset Q_s of D and a function $\psi_s \in L^2(Q_s; \mathbb{R}^2)$ such that, for any solution u to problem $m(s)$ and any solution σ to problem $m^*(\lambda)$, with $\lambda \in \partial m(s)$, it holds*

$$\{|\nabla u| > 1\} = \{|\sigma| > 1\} = Q_s \qquad \text{and} \qquad \nabla u = \sigma = \psi_s \quad \text{a.e. on } Q_s\,, \tag{3.9}$$

where the first equality is intended up to Lebesgue negligible sets.
Moreover, $Q_s = Q_t$ and $\psi_s = \psi_t$ whenever $\partial m(s) \cap \partial m(t) \neq \emptyset$.

Proof. It is enough to observe that the equalities in (3.8) hold true, choosing $\lambda \in \partial m(s)$, an arbitrary solution u to problem $m(s)$, and an arbitrary solution σ to problem $m^*(\lambda)$: it follows that the sets where $\{|\nabla u| > 1\}$ and $\{|\sigma| > 1\}$, and the values of ∇u and σ on them, only depend on s. Moreover, such sets and values agree as soon as there exists some $\lambda \in \partial m(s) \cap \partial m(t)$. □

From Corollary 3.4 we derive the following uniqueness criterion:

Corollary 3.5. *If there exists a special solution to problem $m(s)$, then there is no other solution.*

Proof. Let u be a special solution to $m(s)$, and let $\tilde u$ be another solution. From (3.9) we infer

$$\begin{aligned} m(s) &= \int_{\{0<|\nabla\tilde u|\le 1\}} |\nabla\tilde u| + \int_{\{|\nabla\tilde u|>1\}} \varphi(\nabla\tilde u) = \int_{\{0<|\nabla\tilde u|\le 1\}} |\nabla\tilde u| + \int_{Q_s} \varphi(\psi_s) \\ &= \int_{\{0<|\nabla\tilde u|\le 1\}} |\nabla\tilde u| + \int_{\mathbb{R}^2} \varphi(\nabla u) = \int_{\{0<|\nabla\tilde u|\le 1\}} |\nabla\tilde u| + m(s)\ , \end{aligned}$$

hence the set $\{0 < |\nabla\tilde u| \le 1\}$ is Lebesgue negligible. Then $\nabla\tilde u = \nabla u$ a.e., *i.e.* the two solutions u and $\tilde u$ coincide a.e. up to an additive constant. As elements of $H^1_c(D)$, they are both compactly supported, hence the additive constant is zero. □

As a further consequence of the equalities in (3.8), we get some information on the gradient of special solutions on their free boundary:

Corollary 3.6. *If u is a special solution for $m(s)$ with a smooth free boundary $\Gamma(u)$, it holds*

$$|\nabla u| = 1 \quad \text{on } \Gamma(u)\ . \tag{3.10}$$

Proof. If σ is a solution to problem $m^*(\lambda)$, with $\lambda \in \partial m(s)$, by Proposition 3.3 we know that $\sigma \in \mathcal{S}_\lambda(D)$ and $\sigma \in \partial\varphi(\nabla u)$ a.e. The former condition implies $-\operatorname{div}\sigma = \lambda$ in D, the latter implies that $|\sigma| > 1$ or $|\sigma| \le 1$ according to whether $|\nabla u| > 1$ or $\nabla u = 0$ (see (3.8) above). We deduce that

$$\left|\sigma\cdot\nu_{\Gamma(u)}\right| = 1 \qquad \text{on } \Gamma(u)\,,$$

where $\nu_{\Gamma(u)}$ denotes the unit normal to $\Gamma(u)$, pointing outside $\Omega(u)$. This implies (3.10) since

$$|\sigma\cdot\nu_{\Gamma(u)}| = |\nabla u\cdot\nu_{\Gamma(u)}| = |\nabla u| \qquad \text{on } \Gamma(u)\,.$$

□

3.3 Properties of the map $s \mapsto m(s)$

Below we derive several properties of $m(s)$, seen as a function of the real parameter s. Firstly we give some bounds on it, and we determine its asymptotic behaviour as $s \to 0^+$ and $s \to +\infty$. To that aim we introduce two constants, τ_D and k_D, through the following variational problems set on the space $H^1_c(D)$:

$$\tau_D := \inf\Big\{\int_{\mathbb{R}^2} |\nabla u|^2 \ :\ u \in H^1_c(D)\ ,\ \int_{\mathbb{R}^2} u = 1\Big\} \tag{3.11}$$

$$k_D := \inf\Big\{\int_{\mathbb{R}^2} |\nabla u| \ :\ u \in H^1_c(D)\ ,\ \int_{\mathbb{R}^2} u = 1\Big\} \tag{3.12}$$

When D is simply connected, the constants τ_D and k_D are related to classical variational problems. More precisely, solving problem (3.11) allows to determine the torsional rigidity of a cylinder with cross section D; indeed the Saint-Venant torsional stiffness of D, namely the Dirichlet energy of the unique solution $u \in H^1_0(D)$ to the equation $-\Delta u = 2$, is given precisely by $\frac{4}{\tau_D}$.
On the other hand, the relaxation of problem (3.12) in the space of BV functions, leads to the theory of Cheeger sets; the relationship between the constant k_D and the Cheeger constant of D will be discussed more in detail in Section 4.

Proposition 3.7. *The function $m(s)$ satisfies the following bounds:*

$$\max\left\{k_{_D}\,|s|, \tau_{_D}\frac{s^2}{2}\right\} \;\le\; m(s) \;\le \frac{1}{2}\left(\tau_{_D}\, s^2 + |D|\right) \,.$$

Furthermore, it holds

$$\lim_{s\to 0^+} \frac{m(s)}{s} \;=\; k_{_D}\,, \qquad \lim_{s\to +\infty} \frac{m(s)}{s^2} \;=\; \frac{\tau_{_D}}{2}\,. \tag{3.13}$$

Proof. The function φ defined by (1.1) satisfies the inequalities $\frac{1}{2}|y|^2 \le \varphi(y) \le \frac{1}{2}(|y|^2+1)$. Therefore, by homogeneity, we are led to:

$$\frac{1}{2}\,\tau_{_D}\, s^2 \;\le\; m(s) \;\le\; \frac{1}{2}\left(\tau_{_D}\, s^2 + |D|\right)\,,$$

which implies the second equality in (3.13).
On the other hand, since $\varphi(y) \ge |y|$, it holds $m(s) \ge k_{_D}|s|$, thus $\liminf_{s\to 0^+} \frac{m(s)}{s} \ge k_{_D}$.
Let $u \in H^1_c(D)$ such that $\int_{\mathbb{R}^2} u = 1$ and $s > 0$. Since su is admissible for $m(s)$ and $\varphi(s\nabla u) \le s^2\,|\nabla u|^2$ on the set $\{|\nabla u| > \frac{1}{s}\}$, we have

$$\frac{m(s)}{s} \;\le\; \frac{1}{s}\int_{\mathbb{R}^2} \varphi(s\nabla u) \;\le\; \int_{\{|\nabla u|\le\frac{1}{s}\}} |\nabla u| + s\int_{\{|\nabla u|>\frac{1}{s}\}} |\nabla u|^2.$$

Thus $\limsup_{s\to 0^+} \frac{m(s)}{s} \;\le\; \int_{\mathbb{R}^2} |\nabla u|$ and the first equality in (3.13) follows by taking the infimum with respect to u over $H^1_c(D)$.

□

We now turn attention to the differentiability properties of $m(s)$. Proposition 3.8 below shows in particular that, for any $s > 0$, the condition $\lambda \in \partial m(s)$ appearing in Proposition 3.3 turns out to determine λ uniquely, whereas this is not the case when $s = 0$.

Proposition 3.8. *(i) At every $s > 0$, $m(s)$ is differentiable, and*

$$m'(s) = \frac{1}{s}\Big[m(s) + \int_{Q_s} \big(\frac{1}{2}|\psi_s|^2 - \frac{1}{2}\big)\Big]\,, \tag{3.14}$$

where Q_s and ψ_s are defined according to Corollary 3.4.
(ii) The subdifferential of m at the origin is given by

$$\partial m(0) = [-k_{_D}, k_{_D}]\,,$$

where $k_{_D}$ is the constant defined in (3.12).

Remark 3.9. As a consequence of statement (ii) and of the convexity of m, we have that $m'(s) \ge k_{_D}$ for all positive s and the map $s \mapsto m(s)$ is strictly increasing on $(0, +\infty)$.

Proof. (i) Let $s > 0$ be fixed, and let $\lambda \in \partial m(s)$. If σ is a solution to $m^*(\lambda)$, by using the expressions of $m^*(\lambda)$ and φ^* given respectively by Lemma 3.2 and by (3.1), the Fenchel equality reads

$$\lambda s = m(s) + m^*(\lambda) = m(s) + \int_{\mathbb{R}^2} \varphi^*(\sigma) = m(s) + \int_{\{|\sigma|>1\}} \big(\frac{1}{2}|\sigma|^2 - \frac{1}{2}\big)\,.$$

In view of Corollary 3.4, we conclude that λ is uniquely determined by the equality

$$\lambda s = m(s) + \int_{Q_s} \big(\frac{1}{2}|\psi_s|^2 - \frac{1}{2}\big) \ .$$

Then $\partial m(s) = \{\lambda\}$, that is $m'(s) = \lambda$.

(ii) Since m is a convex even function, $\partial m(0)$ is a bounded closed interval of the form $[-c, c]$, for some positive constant c. Moreover, c agrees with the right derivative

$$m'_+(0) := \lim_{s \to 0^+} \frac{m(s) - m(0)}{s} \ .$$

Since $m(0) = 0$, by using the first equality in (3.13), we conclude that

$$m'_+(0) = \lim_{s \to 0^+} \frac{m(s)}{s} = k_{_D} \ .$$

□

Thanks to Proposition 3.8, we deduce that no special solutions can exist for s ranging in some open interval unless the map $s \mapsto m(s)$ is strictly convex on it.

Proposition 3.10. *Assume that the map $s \mapsto m(s)$ is affine on some interval $[a, b] \subset \{s \geq 0\}$. Then, for any $s \in (a, b]$, problem $m(s)$ does not admit a special solution. Moreover, if $a = 0$, for any $s \in [0, b]$ any solution u to $m(s)$ satisfies $|\nabla u| \leq 1$ a.e., and it holds $m(s) = k_{_D}\, s$.*

Proof. We recall that the sets Q_s and ψ_s are defined as in Corollary 3.4.
Let us assume that for some $s \in [a, b]$ problem $m(s)$ admits a special solution, so that $m(s) = \int_{Q_s} \varphi(\psi_s)$, and let us show that necessarily $s = a$. By the assumption that m is affine on $[a, b]$, it follows that $m'(s) = m'(t)$ for any other $t \in [a, b]$. Therefore, in view of the last assertion of Corollary 3.4, for any $t \in [a, b]$ it holds $Q_t = Q_s$ and $\psi_t = \psi_s$. Thus, denoting by u_t a solution to $m(t)$, we have

$$\begin{aligned} m(t) &= \int_{\{|\nabla u_t| \leq 1\}} |\nabla u_t| + \int_{Q_t} \varphi(\psi_t) = \int_{\{|\nabla u_t| \leq 1\}} |\nabla u_t| + \int_{Q_s} \varphi(\psi_s) \\ &= \int_{\{|\nabla u_t| \leq 1\}} |\nabla u_t| + m(s) \ . \end{aligned}$$

In particular this implies $m(t) \geq m(s)$ and in turn, since m is strictly increasing, that $t \geq s$. By the arbitrariness of $t \in [a, b]$, we conclude that $s = a$.
In the special case when $a = 0$, we get $Q_s = Q_0$, for any $s \in [0, b]$. Clearly the equality $m(0) = 0$ implies $|Q_0| = 0$. Therefore it holds $|Q_s| = 0$ for any $s \in [0, b]$, which means that any solution u to problem $m(s)$ satisfies $|\nabla u| \leq 1$ and $\varphi(\nabla u) = |\nabla u|$ a.e., hence the conclusion.

□

4 Link with the Cheeger problem

Recall that the *Cheeger constant* of a bounded and connected domain D is defined by

$$h_{_D} := \inf_{A \subset \overline{D}} \frac{|\partial A|}{|A|}\,, \tag{4.1}$$

where the infimum is taken over all the subsets A of $\overline{D}$ with finite perimeter.
In the last years, such minimization problem has captured the attention of many authors (see for instance [2, 3, 12, 14, 15, 16, 17, 18, 26]). In this section we present some related properties which shed some light on the link between the Cheeger constant $h_{_D}$ and the minimization problem $m(s)$.
The first result in this direction is the relationship between $h_{_D}$ and the constant $k_{_D}$ defined in (3.12):

Proposition 4.1. *The constants $h_{_D}$ and $k_{_D}$ defined respectively in* (4.1) *and* (3.12) *satisfy the inequality $h_{_D} \geq k_{_D}$, with equality in case D is simply connected.*

Proof. The Cheeger constant introduced in (4.1) can also be recast as

$$h_{_D} = \inf \left\{ \int_D |\nabla v| \ : \ v \in H^1_0(D)\ ,\ \int_{\mathbb{R}^2} v = 1 \right\}\ . \tag{4.2}$$

Then the statement follows by comparing (3.12) and (4.2). Indeed the space of extensions to zero of functions in $H^1_0(D)$ is included into $H^1_c(D)$, and coincides with it if D is simply connected. $\square$

Remark 4.2. The above statement can be generalized to the case when $D = D_0 \setminus \cup_{i=1}^k \overline{D_i}$, being D_i $(i = 0, 1, \ldots, k)$ Jordan domains with mutually disjoint boundaries. Indeed, thanks to the inclusions $H^1_0(D) \subset H^1_c(D) \subset H^1_0(D_0)$, there holds $h_{_D} \geq k_{_D} \geq h_{_{D_0}}$. Moreover, the equality $k_{_D} = h_{_{D_0}}$ holds as soon as there exists a Cheeger set C for D_0 such that

$$\partial C \cap \Big(\bigcup_{i=1}^k D_i \Big) = \emptyset\,, \tag{4.3}$$

and also the equality $h_{_D} = k_{_D}$ holds if in addition

$$\Big(\bigcup_{i=1}^k D_i \Big) \subset (D_0 \setminus C)\ . \tag{4.4}$$

Indeed, conditions (4.3) and (4.4) ensure respectively that the function $1\!\!1_C/|C|$ belongs not only to $H^1_0(D_0)$ but also to $H^1_c(D)$ and to $H^1_0(D)$. For instance, in Figure 2 below, the set D_0 is taken as a square, the grey region represents its Cheeger set, and conditions (4.3) and (4.4) are satisfied if the holes D_i are chosen respectively as in the left and in the right pictures.

By combining Proposition 4.1 with Proposition 3.8 (ii) we obtain that, when D is simply connected, there holds

$$\partial m(0) = [-h_{_D}, h_{_D}]\,. \tag{4.5}$$

This identity allows to obtain Proposition 4.3 below, that will be exploited in Section 6. Though it is already known in the literature (see in particular [6, 19, 22]), we prefer to be self-contained and give a new proof of it, based on (4.5).

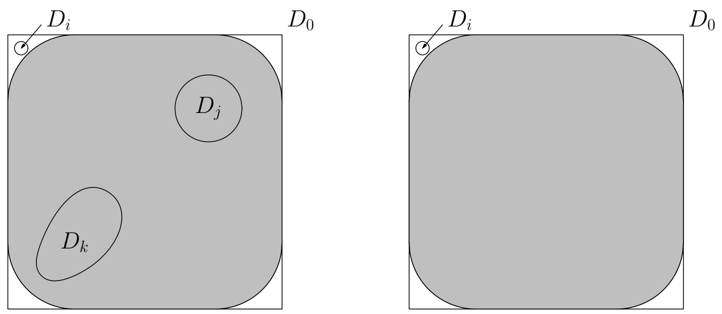

Figure 2: *About conditions* (4.3) *and* (4.4).

We need to introduce some definitions. Let $\Omega \subset \mathbb{R}^2$ be a bounded and connected set with finite perimeter. We say that Ω is a *Cheeger set of itself* if

$$h_\Omega = \frac{|\partial\Omega|}{|\Omega|} .$$

Some examples of Cheeger sets of themselves are the ball, the ellipse and the annulus.
We say that Ω is *calibrable* if there exists a *calibration*, namely a field $\sigma \in L^2(\Omega;\mathbb{R}^2)$ such that

$$-\operatorname{div}\sigma = h_\Omega \quad \text{in } \Omega\ , \qquad \|\sigma\|_{L^\infty(\Omega)} \le 1\ , \qquad [\sigma\cdot\nu_\Omega] = -1 \quad \mathcal{H}^1\text{-a.e. on } \partial\Omega\ .$$

Here $[\sigma\cdot\nu_\Omega]$ is meant as the weak notion of the trace of the normal component of σ on $\partial\Omega$, defined according to [5, Theorem 3.5] (see also [4, Theorem 1.2] for the same definition in case $\partial\Omega$ is Lipschitz).

Proposition 4.3. *Let Ω be a bounded and simply connected set with finite perimeter. Then*

$$\Omega \text{ is Cheeger set of itself} \iff \Omega \text{ is calibrable}\ . \tag{4.6}$$

Remark 4.4. Under the additional assumption that Ω is convex, it is known that each of the two equivalent conditions in (4.6) holds true if and only if the mean curvature of $\partial\Omega$ satisfies the uniform estimate $\|H_{\partial\Omega}\|_{L^\infty(\partial\Omega)} \le \frac{|\partial\Omega|}{|\Omega|}$, see [22].

Proof. Assume that Ω is calibrable, and let σ be a calibration. Integrating over Ω the equality $-\operatorname{div}\sigma = h_\Omega$, by the generalized divergence theorem proved in [5, Theorem 3.5], since $[\sigma\cdot\nu_\Omega] = -1$ $\mathcal{H}^1$-a.e. on $\partial\Omega$, we get $h_\Omega = |\partial\Omega|/|\Omega|$.
Conversely, assume that $h_\Omega = |\partial\Omega|/|\Omega|$. For every $s \in \mathbb{R}$, let $m(s)$ be the variational problem defined as in (1.2), settled on the domain $D = \Omega$. Using the equality $m(0) = 0$ and Lemma 3.2, we obtain

$$\partial m(0) = \Big\{\lambda\ :\ m^*(\lambda) = 0\Big\} = \Big\{\lambda\ :\ \exists\sigma \in \mathcal{S}_\lambda(\Omega)\,,\ \int_{\mathbb{R}^2} \varphi^*(\sigma) = 0\Big\}\,.$$

By recalling the expression of Fenchel conjugate of φ in (3.1), and the characterization of $\mathcal{S}_\lambda(\Omega)$ holding when Ω simply connected (*cf.* (3.3)), it follows

$$\partial m(0) = \Big\{\lambda\ :\ \exists\sigma \in L^2(\mathbb{R}^2;\mathbb{R}^2)\,,\ \operatorname{spt}(\sigma) \subseteq \overline{\Omega}\,,\ -\operatorname{div}\sigma = \lambda \text{ in } \Omega\,,\ \|\sigma\|_{L^\infty(\Omega)} \le 1\Big\}\,. \tag{4.7}$$

On the other hand, by (4.5), we know that $\partial m(0) = [-h_\Omega, h_\Omega]$, that is

$$h_\Omega = \max\{\lambda \in \mathbb{R} \ : \ \lambda \in \partial m(0)\}\,. \tag{4.8}$$

By combining (4.7) and (4.8), we infer that there exists $\sigma \in L^2(\mathbb{R}^2;\mathbb{R}^2)$ such that

$$\operatorname{spt}(\sigma) \subseteq \overline{\Omega}\,, \qquad -\operatorname{div}\sigma = h_\Omega \text{ in } \Omega\,, \qquad \|\sigma\|_{L^\infty(\Omega)} \le 1\,.$$

We claim that the restriction of such a field σ to Ω is a calibration for Ω (so that Ω is calibrable). We only have to show that $[\sigma \cdot \nu_\Omega] = -1$ $\mathcal{H}^1$-a.e. on $\partial\Omega$. By integrating again over Ω the equality $-\operatorname{div}\sigma = h_\Omega$, we obtain

$$\int_{\partial\Omega} [\sigma \cdot \nu_\Omega]\, d\mathcal{H}^1 = \int_\Omega \operatorname{div}\sigma = -h_\Omega|\Omega| = -|\partial\Omega|.$$

Since $\|\sigma\|_{L^\infty(\Omega)} \le 1$, the above equality implies $[\sigma \cdot \nu_\Omega] = -1$ $\mathcal{H}^1$-a.e. on $\partial\Omega$ as required.

$\square$

5 Existence and uniqueness of special solutions on a ball or a ring

In this section we show that, when D is a ball or a ring, problem $m(s)$ has a unique solution, which is a special one and has a circular plateau.

Proposition 5.1. *Let $R > 0$ and let $D = B_R(0)$ be the ball of radius R centered at the origin. Then, for every $s \in \mathbb{R}$, problem $m(s)$ admits a unique solution $\overline{u}$, which is a special solution. More precisely: if $s = 0$ then $\overline{u} \equiv 0$; if $s > 0$, there exists $r \in (0, R)$, uniquely determined by the values of s and R, such that*

$$\overline{u}(x) = \begin{cases} \dfrac{R^2 - \big(|x|^2 \vee r^2\big)}{2r} & \text{if } |x| < R \\ 0 & \text{otherwise}\ . \end{cases} \tag{5.1}$$

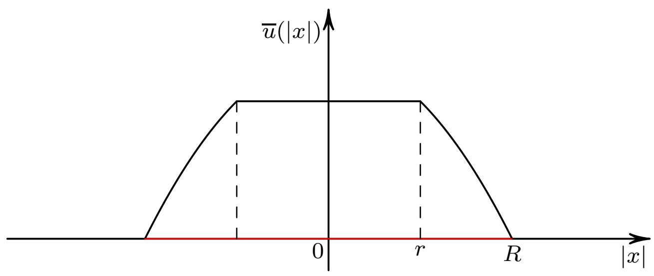

Figure 3: *The special solution $\overline{u}$ given by Proposition 5.1.*

Proof. If $s = 0$ the function $\overline{u} \equiv 0$ is clearly the unique solution to $m(0)$, and it is a special one. Assume $s > 0$. We begin by defining r as the unique number in the interval $(0, R)$ such that $f(r) = s$, where f is the map

$$f(t) := \frac{\pi}{4}\left(\frac{R^4}{t} - t^3\right)_+ \qquad \forall t \in (0, R)\,.$$

Notice that r is well-defined because f is strictly decreasing from $(0, R)$ onto $(0, +\infty)$. Using (5.1), the relation $f(r) = s$ and an integration by parts, it is straightforward to check that $\int_{\mathbb{R}^2} \overline{u} = s$. Moreover, $\overline{u}$ belongs to $H^1_c(D)$ since its gradient over $\mathbb{R}^2$ is given by

$$\nabla\overline{u}(x) = \begin{cases} -\dfrac{x}{r} & \text{if } |x| \in [\,r, R\,] \\ 0 & \text{otherwise}. \end{cases}$$

Hence $\overline{u}$ is admissible for problem $m(s)$. For every $v \in H^1_c(D)$ with $\int_{\mathbb{R}^2} v = 0$, it holds

$$\begin{aligned} \int_{\{\nabla\overline{u}=0\}} |\nabla v| + \int_{\{\nabla\overline{u}\neq 0\}} \langle \nabla\varphi(\nabla\overline{u}), \nabla v\rangle &\geq \int_{\{|x|<r\}} |\nabla v| - \int_{\{r<|x|<R\}} \langle \frac{x}{r}, \nabla v\rangle \\ &= \int_{\{|x|<r\}} \Big(|\nabla v| + \langle \frac{x}{r}, \nabla v\rangle \Big) \geq 0\,. \end{aligned}$$

Hence Proposition 3.1 implies that $\overline{u}$ is a solution to problem $m(s)$. It is a special solution as $|\nabla\overline{u}| = |\frac{x}{r}| > 1$ on the subset $\{r < |x| < R\}$. Finally, uniqueness follows from Corollary 3.5. ☐

Remark 5.2. With the same proof technique of Proposition 5.1, one can show that a similar result is valid also when $D = \bigcup_i B_i$ is the countable union of a family of pairwise disjoint balls B_i of radii R_i. Again, for every $s \in \mathbb{R}$ problem $m(s)$ admits a unique solution, which is a special one. More precisely: if $s = 0$ the solution is identically zero; if $s > 0$ there exists $r \in (0, \sup_i R_i)$, uniquely determined by the values of s and the radii R_i, such that on balls whose radius is smaller than r, the solution is identically zero, while on balls with a larger radius, it is of the form (5.1), with $R = R_i$. The critical radius r is the unique number in $(0, \sup_i R_i)$ such that $f(r) = s$, where

$$f(t) = \frac{\pi}{4} \sum_i \left(\frac{R_i^4}{t} - t^3 \right)_+ \qquad \forall t \in (0, \sup_i R_i)\ .$$

Proposition 5.3. *Let $R_2 > R_1 > 0$, and let $D := \{x \in \mathbb{R}^2 \ :\ R_1 < |x| < R_2\}$. Then, for every $s \in \mathbb{R}$, problem $m(s)$ admits a unique solution $\overline{u}$, which is a special solution. More precisely: if $s = 0$ then $\overline{u} \equiv 0$; if $s > 0$, there exists a unique $r \in (0, R_2)$, uniquely determined by the values of s and the radii R_1, R_2, such that*

$$\overline{u}(x) = \begin{cases} \dfrac{R_2^2 - \big(|x|^2 \vee (R_1 \vee r)^2\big)}{2r} & \text{if } |x| < R_2 \\ 0 & \text{otherwise}\ . \end{cases} \tag{5.2}$$

Proof. If $s = 0$ the function $\overline{u} \equiv 0$ is clearly the unique solution to $m(0)$, and it is a special one. For $s > 0$, we define r as the unique number in the interval $(0, R)$ such that $f(r) = s$, where f is the map

$$f(t) := \begin{cases} \dfrac{\pi}{4} \left(\dfrac{R_2^4 - R_1^4}{t} \right) & \text{if } t \in (0, R_1) \\ \dfrac{\pi}{4} \left(\dfrac{R_2^4 - t^4}{t} \right) & \text{if } t \in [R_1, R_2)\,. \end{cases}$$

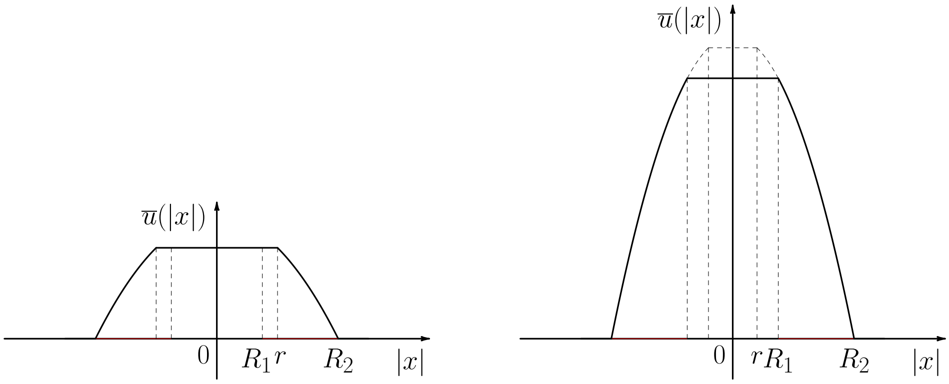

Figure 4: *The special solution $\overline{u}$ given by Proposition 5.3, respectively when $R_1 < r < R_2$ on the left, and when $0 < r < R_1$ on the right.*

Notice that r is well-defined since the map f is strictly decreasing from $(0, R_2)$ onto $(0, +\infty)$. Using the definition of r and an integration by parts, it is straightforward to obtain that $\int_{\mathbb{R}^2} \overline{u} = s$. Moreover, $\overline{u}$ belongs to $H^1_c(D)$ since it is constant on each connected component of $\mathbb{R}^2 \setminus \overline{D}$:

$$\overline{u} \equiv \frac{R_2^2 - (R_1 \vee r)^2}{2r} \quad \text{if } |x| \leq R_1 \qquad \text{and} \qquad \overline{u} \equiv 0 \quad \text{if } |x| \geq R_2 \; .$$

Hence $\overline{u}$ is admissible for problem $m(s)$. Let us show that it is optimal. We distinguish the two cases when $s < f(R_1)$ or $s \geq f(R_1)$, which correspond respectively to $r \in (R_1, R_2)$ or $r \in (0, R_1]$. If $r \in (R_1, R_2)$, $\overline{u}$ coincides with the function defined in (5.1), with $R = R_2$. The optimality of $\overline{u}$ for problem $m(s)$ set on the ball $B_{R_2}(0)$ implies the optimality also for problem $m(s)$ set on D, because of the inclusion $H^1_c(D) \subset H^1_c(B_{R_2}(0))$.
If $r \in (0, R_1]$, we apply Proposition 3.1: for every $v \in H^1_c(D)$ with $\int_{\mathbb{R}^2} v = 0$ it holds

$$\int_{\{\nabla \overline{u} = 0\}} |\nabla v| + \int_{\{\nabla \overline{u} \neq 0\}} \langle \nabla \varphi(\nabla \overline{u}), \nabla v \rangle = \int_{\mathbb{R}^2} \langle -\frac{x}{r}, \nabla v \rangle = \frac{2}{r} \int_{\mathbb{R}^2} v = 0 \; ,$$

where we have used the fact the gradient of $\overline{u}$ is given by

$$\nabla \overline{u}(x) = \begin{cases} -\dfrac{x}{r} & \text{if } |x| \in (R_1, R_2) \\ 0 & \text{otherwise} \, , \end{cases}$$

Thus $\overline{u}$ is a special solution, and uniqueness follows again from Corollary 3.5.

□

Remark 5.4. If in Proposition 5.3 we consider the case $s > f(R_1)$, when the solution $\bar{u}$ is given by (5.2) for a suitable $r \in (0, R_1)$ (see the above proof and Figure 4 at right), then the inequality $|\nabla \overline{u}(x)| > 1$ turns out to be strict up to $|x| = R_1$. This shows that, for a special solution, the equality (3.10) satisfied on the free boundary (lying in open set D) is in general false on $\partial \Omega(u) \cap \partial D$.

6 Existence of special solutions for some other domain D

By exploiting the results of Sections 3 and 4, we are going to prove that there exists some domain D, different from a ball, where problem $m(s)$ admits a special solution, see Theorem 6.1 below for a precise statement. Let us remark that the proof of Theorem 6.1, and in particular the construction of the vector field σ therein, has some similarity with results contained in [24, Sections 4-5].

Theorem 6.1. *There exists an open bounded simply connected set D, different from a ball, such that problem $m(s)$ admits a special solution u for some $s \in \mathbb{R} \setminus \{0\}$. Moreover, both D and the plateau of u have analytic boundary, and the latter is convex.*

Proof. Let us construct an open bounded simply connected set D with analytic boundary, and

– a function $u \in H_0^1(D)$ with

$$\begin{cases} \displaystyle\int_D u = s\,, \text{ for some } s \in \mathbb{R} \setminus \{0\} \\ \nabla u = 0 \quad \text{in a convex set } \ \Omega \subset D \\ |\nabla u| > 1 \quad \text{in } D \setminus \Omega\,, \end{cases} \tag{6.1}$$

– a field $\sigma \in L^2(D; \mathbb{R}^2)$ with

$$\begin{cases} -\operatorname{div} \sigma = \lambda \quad \text{in } D\,, \text{ for some } \lambda \in \mathbb{R} \\ |\sigma| \le 1 \quad \text{in } \Omega \\ \sigma = \nabla u \quad \text{in } D \setminus \Omega\,. \end{cases} \tag{6.2}$$

We recall that, since D is simply connected, functions in $H_c^1(D)$ are extensions to zero of elements in $H_0^1(D)$, and $\mathcal{S}_\lambda(D)$ is given by (3.3). Then u and σ (extended to zero out of D), satisfy conditions (ii) in Proposition 3.3. Since $|\nabla u| \in \{0\} \cup (1, +\infty)$, we conclude that u is a special solution to problem $m(s)$ (settled on D).

The construction is divided into 3 steps.

STEP 1. We choose a bounded convex set Ω, with analytic boundary, whose curvature satisfies the *strict* inequality $\|H\|_{L^\infty(\partial\Omega)} < \frac{|\partial\Omega|}{|\Omega|}$. In view of Remark 4.4, there exists a calibration for Ω, namely a vector field $\sigma_1 \in L^2(\Omega; \mathbb{R}^2)$ such that

$$-\operatorname{div} \sigma_1 = h_\Omega \quad \text{in } \Omega\,, \qquad \|\sigma_1\|_{L^\infty(\Omega)} \le 1 \quad \text{in } \Omega\,, \qquad [\sigma_1 \cdot \nu_\Omega] = -1 \quad \mathcal{H}^1\text{–a.e. on } \partial\Omega\,.$$

STEP 2. Since $\partial\Omega$ is analytic, Cauchy-Kowalevskaya Theorem ensures the existence of an analytic solution v in a neighbourhood $\mathcal{V}$ of $\partial\Omega$ to the initial value problem

$$\begin{cases} -\Delta v = h_\Omega \quad \text{in } \ \mathcal{V}\,, \\ v = 1\,, \ -v_\nu = 1 \quad \text{on } \ \partial\Omega\,, \end{cases}$$

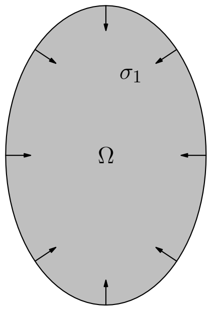

Figure 5: *Construction of the convex set Ω and the vector field σ_1 in Step 1.*

being ν the unit outer normal to $\partial\Omega$. We claim that, up to choosing a smaller neighbourhood $\mathcal{V}$, if we set $\mathcal{U} := \mathcal{V} \setminus \Omega$, it holds

$$v \leq 1 \quad \text{in } \mathcal{U} \tag{6.3}$$

and

$$|\nabla v| > 1 \quad \text{in } \mathcal{U}\,. \tag{6.4}$$

Indeed, (6.3) follows straightforward from the condition $v_\nu < 0$ on $\partial\Omega$. In order to prove (6.4), we exploit the equation $-\Delta v = h_{_\Omega}$, which may be rewritten pointwise on $\partial\Omega$ as

$$-\left(H_{\partial\Omega}\, v_\nu + v_{\nu\nu}\right) = h_{_\Omega} \qquad \text{on } \ \partial\Omega\,,$$

being $H_{\partial\Omega}$ the (signed) curvature of $\partial\Omega$. By construction, we have

$$|\nabla v| = 1\ , \quad v_\nu = -1, \quad |H_{\partial\Omega}| < h_\Omega \qquad \text{on } \partial\Omega\ .$$

Then (6.4) follows from the inequality

$$\partial_\nu(|\nabla v|^2) \ = \ 2v_\nu v_{\nu\nu} \ = \ -2v_{\nu\nu} \ = \ -2(H_{\partial\Omega} - h_\Omega) \ > \ 0 \qquad \text{on } \ \partial\Omega\ .$$

Next we choose $t_0 > 0$, independent of $y \in \partial\Omega$, such that the map

$$t \mapsto \phi_y(t) := v(y + t\nu(y))$$

is well-defined and satisfies the inequality $\phi_y'(t) < 0$ on $[0, t_0]$. Then, for some $\varepsilon_0 > 0$,

$$\max_{y\in\partial\Omega} \phi_y(t_0) = 1 - \varepsilon_0 < 1\ .$$

Therefore, if we fix $\varepsilon \in (0, \varepsilon_0)$, it holds:

$$\forall y \in \partial\Omega\,,\ \exists t_y \in [0, t_0]\ :\ \phi_y(t_y) = 1 - \varepsilon\ .$$

We set $\gamma := \{y + t_y\nu(y)\ :\ y \in \partial\Omega\}$, so that $\gamma = \partial D$, with

$$D := \Omega \cup \big\{1 - \varepsilon \leq v \leq 1\big\}\ .$$

Finally we define

$$\overline{v} := v - (1 - \epsilon) \quad \text{ and } \quad \sigma_2 = \nabla\overline{v} \qquad \text{on } D \setminus \Omega\ .$$

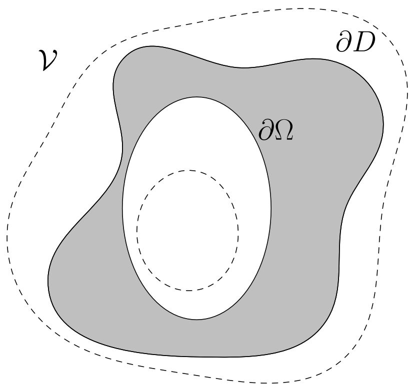

Figure 6: *Construction of the set D in Step 2.*

Notice in particular that σ_2 satisfies

$$-\operatorname{div}\sigma_2 = -\Delta v = h_{\Omega} \quad \text{in } D\setminus\Omega\,, \qquad [\sigma_2\cdot\nu_\Omega] = -1 \quad \mathcal{H}^1 - \text{a.e. on } \partial\Omega\,.$$

STEP 3. We set

$$u := \begin{cases} \varepsilon & \text{in } \Omega \\ \\ \overline{v} & \text{in } D\setminus\Omega \end{cases}\,, \qquad \sigma := \begin{cases} \sigma_1 & \text{in } \Omega \\ \\ \sigma_2 & \text{in } D\setminus\Omega \end{cases}\,,$$

where Ω and σ_1 have been defined in Step 1, while D, $\overline{v}$ and σ_2 have been defined in Step 2.
It is easy to check that, by construction, u and σ verify respectively (6.1) and (6.2).
So, as claimed at the beginning of the proof, u is a special solution to $m(s)$. Moreover, the plateau Ω was chosen to be convex with analytic boundary. And also ∂D is analytic by the implicit function Theorem for analytic functions (see *e.g.* [28]): indeed, γ is a level set of an analytic function whose gradient is nonzero along γ (because of (6.4) and since $\gamma\subset\mathcal{U}$).

□

7 Some qualitative properties of solutions and special solutions

We first state two results which concern arbitrary solutions to problem $m(s)$, and more precisely their sign (Proposition 7.1) and their support (Proposition 7.2).

Proposition 7.1. *For every $s\in\mathbb{R}^+$, any solution u to $m(s)$ satisfies the inequality $u\geq 0$ a.e.*

Proof. The unique solution to $m(0)$ is identically zero. Let $s>0$ and let u be a solution to $m(s)$. We set $u_+ := \max\{u,0\}$ and $\tilde{s} := \int_{\mathbb{R}^2} u_+$. Then

$$m(s) = \int_{\mathbb{R}^2}\varphi(\nabla u) \geq \int_{\mathbb{R}^2}\varphi(\nabla u_+) \geq m(\tilde{s})\,.$$

Since $\tilde{s}\geq s$ and m is strictly increasing (recall Remark 3.9), we infer that $s=\tilde{s}$, and hence that the set $\{u<0\}$ is Lebesgue negligible. □

Proposition 7.2. *Let s be positive and sufficiently small. Then any solution u to problem $m(s)$ satisfies*

$$\operatorname{spt}(u) \cap \partial D \neq \emptyset \; . \tag{7.1}$$

Proof. Assume that (7.1) is false for some $s > 0$. Then $\operatorname{spt}(u) \subset\subset D$ and, letting

$$u^\lambda(x) := \lambda \, u\Big(\frac{x}{\lambda}\Big) \qquad \forall x \in \mathbb{R}^2 \, , \; \forall \lambda > 0 \; ,$$

by continuity we have also $\operatorname{spt}(u^\lambda) \subset\subset D$ for λ close to 1. Accordingly, the function u^λ is admissible for problem $m(\lambda^3 s)$, whence we deduce

$$m(\lambda^3 s) \leq \int_{\mathbb{R}^2} \varphi(\nabla u^\lambda) = \lambda^2 \int_{\mathbb{R}^2} \varphi(\nabla u) = \lambda^2 m(s) \; . \tag{7.2}$$

Therefore the function $g(\lambda) = m(\lambda^3 s) - \lambda^2 m(s)$ achieves a local maximum at $\lambda = 1$, and $g'(1) = 0$. It follows that $3m'(s) = 2m(s)$. Thus, by applying Remark 3.9, we find $m(s) \geq \frac{3}{2} k_D$, which is not possible for s small. $\square$

We now turn our attention to investigate qualitative properties of *special* solutions, under the assumption that D is simply connected. The corresponding simplified formulation of $m(s)$, that we consider from now on, reads

$$m(s) = \inf \left\{ \int_D \varphi(\nabla u) \; : \; u \in H^1_0(D) \, , \; \int_D u = s \right\} \; . \tag{7.3}$$

The search for special solutions to problem (7.3) leads to study a nonstandard free boundary value problem. Indeed, by Proposition 3.3, (3.8), and Corollary 3.6, if u is a special solution to $m(s)$ with plateau $\Omega(u)$ and free boundary $\Gamma(u)$, there exist constants $\lambda (= m'(s))$ and $c_i \in \mathbb{R}$ such that

$$\begin{cases} -\Delta u = \lambda \; , \quad |\nabla u| > 1 & \text{in } D \setminus \Omega(u) \\ |\nabla u| = 1 & \text{on } \Gamma(u) \\ u = c_i & \text{on } \gamma_i \; , \end{cases} \tag{7.4}$$

where γ_i denote the different connected components of $\Gamma(u)$ (see Figure 7).
A full understanding of problem (7.4) seems to be a quite challenging task. To the best of our knowledge, it is not directly covered by the extensive literature on free boundary problems (see [13] and references therein). In particular, the available regularity results for free boundaries do not allow to obtain a priori the smoothness of $\Gamma(u)$. This is the reason why the results hereafter are stated under such smoothness assumption.

Proposition 7.3. *Assume that problem $m(s)$ admits a special solution u, with $\Gamma(u)$ smooth. Then each connected component of $D \setminus \Omega(u)$ meets the boundary ∂D.*

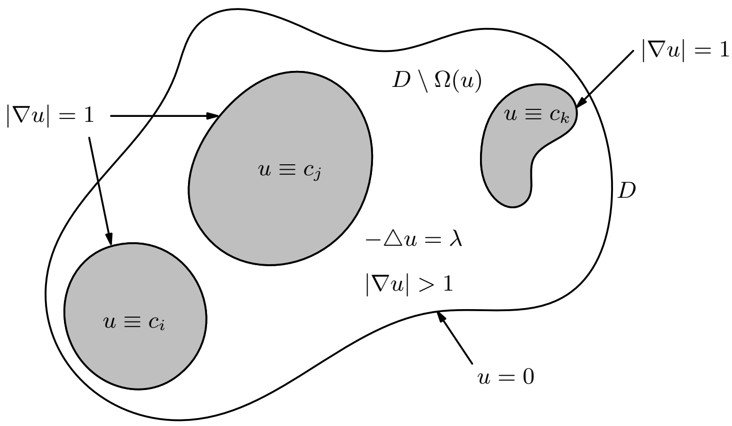

Figure 7: *The free boundary value problem* (7.4).

Proof. Assume by contradiction that there exists a connected component A of $D \setminus \Omega(u)$ such that $A \subset\subset D$. Then $\partial A = \cup \gamma_i$, where γ_i are some of the connected components of $\Gamma(u)$. Then (*cf.* (7.4)), there exist constants $\lambda (= m'(s))$ and $c_i \in \mathbb{R}$ such that

$$\begin{cases} -\Delta u = \lambda \ , \quad |\nabla u| > 1 & \text{in } A \\ u = c_i \ , \quad |\nabla u| = 1 & \text{on } \gamma_i \ . \end{cases}$$

By standard regularity theory, u is smooth enough in order to apply [27, Lemma 5.1] (by taking therein $f(u) = \lambda$, $g(u) = 1$, and $h(u) = 0$). We deduce that the P-function

$$P(x) := |\nabla u|^2 \ , \qquad x \in A \ ,$$

is either constant in A or it attains its maximum on ∂A. In both cases, since we know that $|\nabla u| \geq 1$ in A, we infer that

$$|\nabla u| \equiv 1 \qquad \text{in } A \ . \tag{7.5}$$

We now consider another P-function,

$$\widetilde{P}(x) := |\nabla u|^2 + \lambda u \ , \qquad x \in A \ .$$

From (7.5) we obtain

$$\Delta \widetilde{P} = -\lambda^2 = -(m'(s))^2 < 0$$

(for the last inequality recall (3.14)). On the other hand, equality (5.17) in [27] (applied now with $f(u) = \lambda$, $g(u) = 1$, and $h(u) = \lambda u$) shows that $\Delta \widetilde{P} \geq 0$, a contradiction. $\square$

Proposition 7.4. *Let D be a convex set with a smooth boundary, and assume that problem $m(s)$ admits a special solution u, with $\Omega(u)$ connected, $\Omega(u) \subset\subset D$, and $\Gamma(u)$ smooth. Then Ω is convex.*

Proof. By applying Proposition 7.3, we obtain that $\Gamma(u)$ is connected (otherwise, some connected component of $D \setminus \Omega(u)$ would be compactly contained into D). Then (*cf.* (7.4)), there exist constants

$\lambda(= m'(s))$ and $c \in \mathbb{R}$ such that

$$\begin{cases} -\Delta u = \lambda \ , \quad |\nabla u| > 1 & \text{in } D \setminus \Omega(u) \\ \\ u = c \ , \quad |\nabla u| = 1 & \text{on } \Gamma(u) \ . \end{cases}$$

In order to prove that $\Omega(u)$ is convex, we follow the approach adopted in [21] (see also [20]): we consider the P-function

$$P(x) := |\nabla u|^2 + 2\lambda u \qquad \forall x \in D \setminus \Omega(u) \, .$$

By standard regularity theory, u is smooth enough in order to apply [27, Lemma 5.1]. Since by assumption u has no critical points in $D \setminus \Omega(u)$, we infer that one of the following facts occurs:

(a) P is constant;

(b) P attains its maximum on ∂D;

(c) P attains its maximum on $\Gamma(u)$.

Let us exclude the first two possibilities.
If P is constant, it holds

$$0 = P_\nu = 2(u_\nu u_{\nu\nu} + \lambda u_\nu) = -2(u_{\nu\nu} + \lambda) \qquad \text{on } \Gamma(u) \, . \tag{7.6}$$

On the other hand, since by assumption $\Gamma(u)$ is smooth, the equation $\Delta u + \lambda = 0$ can be rewritten pointwise on $\Gamma(u)$ as

$$H_\Gamma u_\nu + u_{\nu\nu} + \lambda = -H_\Gamma + u_{\nu\nu} + \lambda = 0 \qquad \text{on } \Gamma(u) \, , \tag{7.7}$$

where we have denoted by H_Γ the mean curvature of $\Gamma(u)$. Combining (7.6) and (7.7), we deduce that $H_\Gamma \equiv 0$ on $\Gamma(u)$, a contradiction.
If P attains its maximum at some point $x_0 \in \partial D$, since ∂D is smooth we may apply Hopf's boundary point lemma to infer that either P is constant or $P_\nu(x_0) > 0$ (here ν stands for the unit outer normal to ∂D). Since we have already excluded the first possibility, let us show that also the second one leads to a contradiction. We have

$$0 < P_\nu(x_0) = 2u_\nu(x_0) u_{\nu\nu}(x_0) + 2\lambda u_\nu(x_0) = -2(u_\nu(x_0))^2 H_{\partial D}(x_0) \, , \tag{7.8}$$

where the last equality follows by exploiting the pde $\Delta u + \lambda = 0$ on ∂D. In particular, (7.8) implies $H_{\partial D}(x_0) < 0$, against the convexity of D.
We conclude that (c) holds true, namely P assumes its maximum on $\Gamma(u)$. Since P is constant on $\Gamma(u)$, every point of the free boundary is a maximum point. Then, thanks to the smoothness of $\Gamma(u)$, Hopf's lemma applies and yields

$$0 > P_\nu = 2u_\nu u_{\nu\nu} + 2\lambda u_\nu = -2H_\Gamma \qquad \text{on } \Gamma(u) \, .$$

Hence $\Omega(u)$ is convex. □

Proposition 7.5. *Assume that D is* not *Cheeger set of itself, and let s_ε be an infinitesimal sequence of positive numbers. Then problem $m(s_\varepsilon)$ cannot admit for every ε a special solution u_ε with $\Omega(u_\varepsilon) \subset\subset D$ and $\Gamma(u_\varepsilon)$ smooth.*

Proof. Set for brevity $\Omega_\varepsilon := \Omega(u_\varepsilon)$ and $\Gamma_\varepsilon := \Gamma(u_\varepsilon)$. Assume by contradiction $\Omega_\varepsilon \subset\subset D$ and Γ_ε smooth. We set $\lambda_\varepsilon = m'(s_\varepsilon)$, and we take an optimal field $\sigma_\varepsilon \in \mathcal{S}_{\lambda_\varepsilon}(D)$ for the dual problem $m^*(\lambda_\varepsilon)$. By Proposition 3.3 and (3.8), σ_ε satisfies

$$\begin{cases} -\operatorname{div}\sigma_\varepsilon = \lambda_\varepsilon & \text{in } D \\ |\sigma_\varepsilon| \le 1 & \text{in } \Omega_\varepsilon \\ \sigma_\varepsilon = \nabla u_\varepsilon & \text{in } D \setminus \Omega_\varepsilon\,. \end{cases}$$

By Corollary 3.6 and the regularity assumed on Γ_ε, we infer that $|\sigma_\varepsilon \cdot \nu_{\Gamma_\varepsilon}|$ equals 1 and has constant sign on Γ_ε. Integrating on Ω_ε the equation $-\operatorname{div}\sigma_\varepsilon = \lambda_\varepsilon$, we obtain $\sigma_\varepsilon \cdot \nu_{\Gamma_\varepsilon} = -\operatorname{sgn}(\lambda_\varepsilon) = -1$ and

$$\lambda_\varepsilon = \frac{|\Gamma_\varepsilon|}{|\Omega_\varepsilon|}\,. \tag{7.9}$$

Since $\lambda_\varepsilon = m'(s_\varepsilon)$, by using (7.9), the continuity from the right of the right derivative $s \mapsto m'_+(s)$ as $s \to 0^+$, Proposition 3.8 and Proposition 4.1, we get

$$\lim_{\varepsilon\to 0} \lambda_\varepsilon = h_D\,. \tag{7.10}$$

Moreover, we have

$$|D \setminus \Omega_\varepsilon| \le \int_{D\setminus\Omega_\varepsilon} |\nabla u_\varepsilon|^2 = \int_{D\setminus\Omega_\varepsilon} \nabla u_\varepsilon \cdot \sigma_\varepsilon = \int_D \nabla u_\varepsilon \cdot \sigma_\varepsilon = s_\varepsilon \cdot \lambda_\varepsilon\,.$$

In view of (7.10), we infer that $\lim_{\varepsilon\to0} |D \setminus \Omega_\varepsilon| = 0$, which is equivalent to $\lim_{\varepsilon\to0} 1\!\!1_{\Omega_\varepsilon} = 1\!\!1_D$ in $L^1(D)$. By using the lower semicontinuity of the perimeter with respect to the L^1-convergence, and (7.9), we obtain

$$|\partial D| \le \liminf_{\varepsilon\to0} |\Gamma_\varepsilon| = \lim_{\varepsilon\to0} \lambda_\varepsilon |\Omega_\varepsilon| = h_D|D|\,,$$

hence D is Cheeger set of itself, against the assumption. □

Concluding comments. It is a quite challenging problem to identify the geometrical conditions on the design domain D under which a special solution with a smooth free boundary should exist or not. We point out that proving or disproving the existence of a special solution remains open even for simple geometries of D. For instance, when D is a square, in view of Propositions 7.3 and 7.4, it cannot happen that a special solution has the grey regions in Figure 8 as plateau. Actually, having in mind Proposition 7.5, at least for small s the set where a solution u is constant may be expected to be shaped as the grey region in Figure 9; but on its complement it is difficult to guess whether $|\nabla u| > 1$, or some homogenization phenomenon occurs.

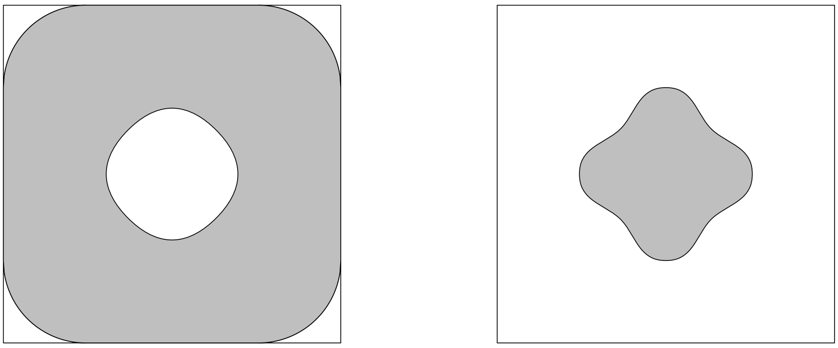

Figure 8: *Impossible plateaus for a special solution on the square.*

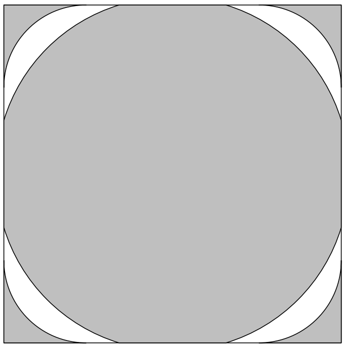

Figure 9: *A possible plateau for a special solution on the square.*

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