Pseudohermitian invariants and classification of CR mappings in generalized ellipsoids*

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Abstract

Given a strictly pseudoconvex hypersurface $M \subset \mathbb{C}^{n+1}$, we discuss the problem of classifying all local CR diffeomorphisms between open subsets $N, N' \subset M$. Our method exploits the Tanaka–Webster pseudohermitian invariants of a contact form ϑ on M, their transformation formulae, and the Chern–Moser invariants. Our main application concerns a class of generalized ellipsoids where we classify all local CR mappings.

0 Introduction

In this paper, we give a contribution to the problem of classifying local CR mappings between real hypersurfaces in $\mathbb{C}^{n+1}$. Namely, given a surface $M := b\Omega$, where $\Omega \subset \mathbb{C}^{n+1}$ is a smooth open set, we consider the problem of classifying all CR mappings $f : N \to N'$, where N and N' are open subsets of M. The question is rather natural, because biholomorphic mappings of Ω that extend smoothly to the boundary define CR mappings on M.

Our approach is mainly based on CR differential geometry of strictly pseudoconvex manifolds. We fix a contact form ϑ on M, we calculate the Tanaka–Webster invariants (see [**?**, **?**]) and we exploit Lee's transformation formulae, see [**?**]. The idea is reminiscent of known techniques in the study of conformal mappings in Riemannian manifolds, see [**?**, **?**, **?**]. Our point of view is described in Section 1, in the setting of CR surfaces in $\mathbb{C}^{n+1}$, $n \geq 2$.

We also exploit the connection between pseudohermitian invariants and the classical Cartan–Chern–Moser CR invariants, see [**?**]. In particular, we introduce a new Chern-invariant cone bundle. Namely, starting from the Chern tensor, we define a subset $\mathscr{H} := \bigcup_{P \in M} \mathscr{H}_P$ of the holomorphic tangent bundle which is preserved by CR mappings. The definition of $\mathscr{H}$ is given in Section 1. We believe that the study of this cone bundle may by of some interest in similar or related situations.

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These ideas are applied to the model given by a *generalized ellipsoid*

$$M := bE, \quad E = \left\{ z \in \mathbb{C}^{n+1} : |z_1|^{2m_1} + \cdots + |z_{s-1}|^{2m_{s-1}} + |z_s|^2 = 1 \right\}, \tag{0.1}$$

where $z_1, z_2, \ldots, z_s$ are groups of variables and the numbers m_j satisfy suitable hypoteses. The automorphism group of E is studied in [**?**] and the model is considered also in [**?**]. In [**?**] the authors prove that all local CR mappings of bE extend to global biholomorphic mappings of E, under suitable hypotheses on the dimension of the groups of variables z_j.

In the present paper, we study local CR mappings on bE. We recover both the results in [?] and [?] on the model (0.1). Our arguments are completely different and new. The statement of our classification result for CR mappings on generalized ellipsoids is contained in Section 2, Theorem 2.2. Subsections 3.1 and 3.2 contain the computation of the pseudohermitian and Chern–Moser invariants in our model. Section 4 is devoted to the computation of the CR factor of a mapping and the to the classifications of CR mappings which are "Levi-isometric".

We mainly use differential geometric arguments, which require a certain computational effort. On the other hand, they provide a good understanding of the geometry of the manifold M. Several other strategies are available in the study of CR mappings. For a complete account, we refer the reader to the monograph [**?**], where *Segre varieties*, *infinitesimal CR mappings* and other tools are widely discussed.

1 Chern-invariant cones

Let $M \subset \mathbb{C}^{n+1}$ be a strictly pseudoconvex real hypersurface. Fix on M a contact form ϑ and let $L := -i d\vartheta$ denote the Levi form. For a fixed frame of holomorphic vector fields Z_α, $\alpha = 1, \ldots, n$, let $h_{\alpha\overline{\beta}} = L(Z_\alpha, Z_{\overline{\beta}})$.

A diffeomorphism $f : N \to N'$ between open subsets N and N' of M is by definition a *CR mapping* if $f^*\vartheta = \lambda\vartheta$ for some function $\lambda > 0$ on N and the tangent mapping f_* preserves the complex structure. We call λ the *CR factor* of f. Observe that CR mappings preserve orthogonality with respect to the Levi form:

$$L(f_* Z, f_* \overline{W}) = \lambda L(Z, \overline{W}) \qquad \text{for all } Z, W \in T^{1,0}N, \tag{1.1}$$

where $T^{1,0}N \subset \mathbb{C}TN$ denotes the holomorphic tangent bundle. Observe also that $L(f_* Z, f_* W) = L(Z, W) = 0$ for all $Z, W \in T^{1,0}N$. Therefore CR mappings are somewhat similar to conformal mappings in Riemannian manifolds and (1.1) is an overdetermined system analogous to the system satisfied by conformal mappings in the Riemannian setting.

Let us denote by $u = \lambda^{-1}$ the inverse of the CR factor. Given a contact form $\widetilde{\vartheta} = u^{-1}\vartheta$, the pseudohermitian Ricci curvature $R_{\alpha\overline{\beta}}$ and the preusohermitian torsion $A_{\alpha\beta}$ transform according to Lee's formulae [?]:

$$\widetilde{R}_{\alpha\overline{\beta}} = R_{\alpha\overline{\beta}} + \frac{n+2}{2u}\left\{ u_{,\alpha\overline{\beta}} + u_{,\overline{\beta}\alpha} - \frac{2}{u} u_{,\alpha} u_{,\overline{\beta}} \right\} + \frac{1}{2u}\left\{ \Delta u - \frac{2(n+2)}{u} |\nabla u|^2 \right\} h_{\alpha\overline{\beta}}, \tag{1.2}$$

and

$$\widetilde{A}_{\alpha\beta} = A_{\alpha\beta} - \frac{i}{u} u_{,\alpha\beta}. \tag{1.3}$$

We refer the reader to [**?**, ?, **?**] for definition and basic properties of these tensors. Here, $u_{,\alpha\overline{\beta}} = \nabla_{\overline{\beta}}\nabla_\alpha u$ denote second order covariant derivatives with respect to the Webster connection ∇, while $\Delta u = u_{,\gamma}{}^{\gamma} + u_{,\overline{\gamma}}{}^{\overline{\gamma}}$ and $|\nabla u|^2 = u_\gamma u^\gamma$. As usual, we raise an lower indices by $h^{\alpha\overline{\beta}}$, the matrix defined by $h^{\alpha\overline{\beta}} h_{\gamma\overline{\beta}} = \delta^\alpha_\gamma$, so that $u_,{}^{\gamma} = h^{\gamma\overline{\beta}} u_{,\overline{\beta}}$ and $u_{,\gamma}{}^{\gamma} = h^{\gamma\overline{\beta}} u_{,\gamma\overline{\beta}}$. Here and henceforth, we omit summation on repeated indices. For future reference, recall the contracted version of (1.2)

$$\frac{1}{u}\widetilde{R} = R + \frac{n+1}{u}\left\{\Delta u - \frac{n+2}{u}|\nabla u|^2\right\}, \tag{1.4}$$

where $R := h^{\alpha\overline{\beta}} R_{\alpha\overline{\beta}} = R_\alpha{}^\alpha$ is the pseudohermitian scalar curvature.

Formulae (1.2) and (1.3) are relevant in the study of the CR Yamabe problem, see [**?**]. The Riemannian version of (1.2) is also important in some regularity questions for conformal mappings, see [?].

Equations (1.2) and (1.3) form a system of nonlinear PDEs for the inverse of the CR factor u. This system also involves f, in a way that becomes clear in the coordinate-free notation:

$$\begin{aligned} \mathrm{Ric}(f_*Z, f_*\overline{W}) = \mathrm{Ric}(Z,\overline{W}) &+ \frac{n+2}{2u}\left\{\nabla^2 u(Z,\overline{W}) + \nabla^2 u(\overline{W}, Z) - \frac{2}{u} Z u \overline{W} u\right\} \\ &+ \frac{1}{2u}\left\{\Delta u - \frac{2(n+2)}{u}|\nabla u|^2\right\} L(Z,\overline{W}), \end{aligned} \tag{1.5}$$

and

$$A(f_*Z, f_*W) = A(Z,W) - \frac{i}{u}\nabla^2 u(Z,W), \tag{1.6}$$

for any $Z, W \in T^{1,0}N$. The function f appears through the geometric terms with Ric and A in the left-hand side of (1.5) and (1.6).

When M has dimension $2n+1 \geq 5$, i.e. $n \geq 2$, the Chern tensor $S_{\alpha\overline{\beta}\lambda\overline{\mu}}$ introduced in [?] is a nontrivial relative CR invariant which satisfies

$$\widetilde{S}_{\alpha\overline{\beta}\lambda\overline{\mu}} = \frac{1}{u} S_{\alpha\overline{\beta}\lambda\overline{\mu}}, \tag{1.7}$$

see [**?**]. The tensor $S_{\alpha\overline{\beta}\lambda\overline{\mu}}$ can be expressed in terms of the pseudohermitian curvatures by means of the *Webster's formula*, see [?, ?],

$$\begin{aligned} S_{\alpha\overline{\beta}\lambda\overline{\mu}} = R_{\alpha\overline{\beta}\lambda\overline{\mu}} &- \frac{1}{n+2}\{h_{\alpha\overline{\beta}} R_{\lambda\overline{\mu}} + h_{\lambda\overline{\beta}} R_{\alpha\overline{\mu}} + h_{\alpha\overline{\mu}} R_{\lambda\overline{\beta}} + h_{\lambda\overline{\mu}} R_{\alpha\overline{\beta}}\} \\ &+ \frac{R}{(n+1)(n+2)}\{h_{\alpha\overline{\beta}} h_{\lambda\overline{\mu}} + h_{\lambda\overline{\beta}} h_{\alpha\overline{\mu}}\}. \end{aligned} \tag{1.8}$$

Let us introduce a Chern-invariant cone bundle $\mathscr{H} \subset T^{1,0}M$ which is preserved by CR mappings. Namely, let $\mathscr{H} := \bigcup_{P\in M} \mathscr{H}_P$, where

$$\mathscr{H}_P := \Big\{U \in T^{1,0}_P M : R(U,\overline{V},Z,\overline{W}) = 0 \text{ for all } V,Z,W \in T^{1,0}_P M \text{ such that } L(U,\overline{V}) = L(U,\overline{W}) = L(Z,\overline{V}) = L(Z,\overline{W}) = 0\Big\}. \tag{1.9}$$

The set $\mathscr{H}_P$ is a cone in the vector space $T^{1,0}_P M$. If $U,V,Z,W \in T^{1,0}_P M$ satisfy the orthogonality relations in (1.9), then we have $R(U,\overline{V},Z,\overline{W}) = S(U,\overline{V},Z,\overline{W})$, by (1.8). In particular, $\mathscr{H}$ does not depend on ϑ and, moreover, any CR mapping $f : N \to N'$ satisfies

$$f_*(\mathscr{H}_P) = \mathscr{H}_{f(P)} \qquad \text{for any } P \in N. \tag{1.10}$$

In general, the set $\mathscr{H}_P$ is not closed under addition. If ϑ is any contact form on the standard sphere or on the Siegel domain, then $\mathscr{H} = T^{1,0}M$ and (1.10) carries no information. On the other hand, if the Chern tensor has a nontrivial structure, then $\mathscr{H}$ can provide some useful information on how a CR mapping f transforms the holomorphic tangent space.

In the case of the generalized ellipsoids (0.1), $\mathscr{H}$ can be decomposed as $\mathscr{H} = \mathscr{G} \oplus \mathscr{G}^\perp$, where $\mathscr{G}$ and $\mathscr{G}^\perp$ are orthogonal with respect to the Levi form. This is discussed in Section 3.2. A cone similar to $\mathscr{H}$ was used in the Riemannian setting in [**?**]. In that case, the cone bundle is related to umbilical surfaces in the underlying structure. It could be of interest to understand whether also in the CR setting the cone $\mathscr{H}$ is related to significant geometric objects.

2 The generalized ellipsoid model: main result and skeleton of the proof

In this section, we state the classification theorem for CR mappings on generalized ellipsoids and we indicate the scheme of the proof. Let

$$p(z,\overline{z}) := |z_1|^{2m_1} + \cdots + |z_{s-1}|^{2m_{s-1}} + |z_s|^2,$$

where $z = (z_1,\dots,z_s) \in \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_s} = \mathbb{C}^n$. We assume that the integers $m_j, n_j,\ j = 1,\dots,s$, satisfy

$$\begin{cases} m_j > 1 \text{ and } n_j \geq 2, & \text{if } 1 \leq j \leq s-1, \\ n_s \geq 0. \end{cases} \tag{2.1}$$

We denote by z^α the αth variable in $\mathbb{C}^n$ and we partition the indexes $\{1,\dots,n\}$ into the following sets

$$I_1 = \{1,\dots,n_1\}, \quad I_2 = \{n_1+1,\dots,n_1+n_2\}, \quad \dots \quad I_s = \{n_1+\cdots+n_{s-1}+1,\dots,n\},$$

so that $|z_j|^2 = \sum_{\alpha\in I_j} |z^\alpha|^2$. It may be $I_s = \emptyset$, if $n_s = 0$. Two indexes $\alpha, \beta \in \{1,...,n\}$ are said to be equivalent, and we write $\alpha \sim \beta$, if they belong to the same set I_j. Two indexes α, β are said to be orthogonal, and we write $\alpha \perp \beta$, if $\alpha \in I_j$ and $\beta \in I_k$ with $k \neq j$.

Let

$$\begin{aligned} &\Omega := \left\{(z, z^{n+1}) \in \mathbb{C}^{n+1} : \operatorname{Im}(z^{n+1}) > p(z, \overline{z})\right\},\\ &M_0 := b\Omega := \left\{(z, z^{n+1}) \in \mathbb{C}^{n+1} : \operatorname{Im}(z^{n+1}) = p(z, \overline{z})\right\}, \qquad \text{and}\\ &M := \left\{(z, z^{n+1}) \in M_0 : \prod_{j=1}^{s-1} |z_j| \neq 0\right\}. \end{aligned} \tag{2.2}$$

We will see that M is the strictly pseudoconvex part of the surface M_0. The unbounded open set Ω is biholomorphically equivalent to the bounded generalized ellipsoid

$$E := \left\{(w, w^{n+1}) \in \mathbb{C}^{n+1} : \sum_{j=1}^{s} |w_j|^{2m_j} + |w^{n+1}|^2 < 1\right\}$$

via the map

$$\Omega \ni (z, z^{n+1}) \longmapsto \left(\frac{2^{1/m_1} z_1}{(i + z^{n+1})^{1/m_1}}, , \dots, \frac{2^{1/m_{s-1}} z_{s-1}}{(i + z^{n+1})^{1/m_{s-1}}}, \frac{2z_s}{i + z^{n+1}}, \frac{i - z^{n+1}}{i + z^{n+1}}\right) \in E.$$

In the rest of the paper, we will work on the unbounded model, where the computations are easier.

Remark 2.1. In (2.1) we require $m_j \geq 2$ and $n_j \geq 2$ for $j = 1, \dots, s-1$. The case $n_j = 1$ for all j is discussed in the recent paper [**?**]. Assumption (2.1) ensures that all local CR mappings extend to global biholomorphic mappings, see [?]. If (2.1) is violated, this is in general not true.

Let us consider the following biholomorphic mappings. The mapping $I : \Omega \to \Omega$

$$I(z_1, \dots, z_{s-1}, z_s, z^{n+1}) = \left(\frac{z_1}{(z^{n+1})^{1/m_1}}, \dots, \frac{z_{s-1}}{(z^{n+1})^{1/m_{s-1}}}, \frac{z_s}{z^{n+1}}, -\frac{1}{z^{n+1}}\right) \tag{2.3}$$

is the inversion. For any $r > 0$, the mappings $\delta_r : \Omega \to \Omega$,

$$\delta_r(z_1, \dots, z_{s-1}, z_s, z^{n+1}) = \left(r^{1/m_1} z_1, \dots, r^{1/m_{s-1}} z_{s-1}, r z_s, r^2 z^{n+1}\right) \tag{2.4}$$

form a one-parameter group of dilatations. Finally, consider the mappings ψ of the form

$$\psi(z, z^{n+1}) = \left(B_1 z_{\sigma(1)}, \dots, B_{s-1} z_{\sigma(s-1)}, B_s z_s + b_s, b^{n+1} + z^{n+1} + 2i(B_s z_s \cdot \overline{b}_s)\right), \tag{2.5}$$

where σ is a permutation of $\{1, \dots, s-1\}$ such that $m_{\sigma(j)} = m_j$ and $n_{\sigma(j)} = n_j$ for any $j = 1, \dots, s-1$, $B_j \in U(n_j)$ are unitary matrices, $b_s \in \mathbb{C}^{n_s}$, and $b^{n+1} = t_0 + i|b_s|^2 \in \mathbb{C}$ for some $t_0 \in \mathbb{R}$.

For $a = (a_s, a^{n+1}) \in C^{n_s} \times \mathbb{C}$ with $a^{n+1} = t_0 + i|a_s|^2$ for some $t_0 \in \mathbb{R}$, let ϕ_a be the mapping

$$\phi_a(z_1, \dots, z_{s-1}, z_s, z^{n+1}) = (z_1, \dots, z_{s-1}, z_s + a_s, z^{n+1} + a^{n+1} + 2i z_s \cdot \overline{a}_s). \tag{2.6}$$

The mapping ϕ_a is a particular case of (2.5).

A composition of the mappings (2.3)–(2.5) extends to a CR mapping of M_0, possibly off one point. Our main theorem states that any local *CR* mapping of M is such a composition.

Theorem 2.2. *Let $f : N \to N'$ be a CR mapping between connected open subsets of M. Then, for a suitable choice of ψ as in* (2.5)*, $r > 0$, and $a = (a_s, t_0 + i|a_s|^2) \in \mathbb{C}^{n_s} \times \mathbb{C}$ we have*

$$f = \psi \circ \delta_r \circ J \circ \phi_a, \tag{2.7}$$

where either $J = I$ as in in (2.3) *or J is the identity map.*

Scheme of the proof of Theorem 2.2.

Step 1. For a suitable contact form ϑ on M, we show that the CR factor λ_f of a CR function f between open, connected subsets of M either is a constant or it has the form

$$\lambda_f = k^{-2}|z^{n+1} + a^{n+1} + 2iz_s \cdot \overline{a}_s|^{-2}, \tag{2.8}$$

for some $k > 0$, $a_s \in \mathbb{C}^{n_s}$ and $a^{n+1} = t_0 + i|a_s|^2 \in \mathbb{C}$, where $a_s = 0$ if $n_s = 0$. This is proved in Theorem 4.2. The proof requires the study of the overdetermined system in (1.5) and (1.6). To solve the system, we exploit the structure of the Chern-invariant cone bundle $\mathscr{H}$. This is carried out in Subsection 3.2.

Step 2. Once the form of the CR factor λ_f is known, we consider the mappings ϕ_a in (2.6) and δ_r in (2.4) with $r = 1/k$. Elementary computations on the CR factors give:

$$\lambda_{\phi_a}(z) = 1, \qquad \lambda_I(z) = |z^{n+1}|^{-2}, \quad \text{and} \quad \lambda_{\delta_r} = r^2. \tag{2.9}$$

Let $G := \delta_{1/k} \circ I \circ \phi_a$ and define the mapping ψ via the identity $f = \psi \circ G$. By (2.9), ψ satisfies $\psi^*\vartheta = \vartheta$. Indeed, the CR factor λ_G of G is

$$\lambda_G(z) = \lambda_{\delta_{1/k}}(I(\phi_a(z))\lambda_I(\phi_a(z))\lambda_{\phi_a}(z) = k^{-2}|z^{n+1} + a^{n+1} + 2iz_s \cdot \overline{a}_s|^{-2},$$

and therefore

$$\lambda_f(z) = \lambda_\psi(G(z))\lambda_G(z) = \lambda_\psi(G(z))k^{-2}|z^{n+1} + a^{n+1} + 2iz_s \cdot \overline{a}_s|^{-2}.$$

Thus, by the form of λ_f in (2.8) we deduce that the CR factor of ψ is $\lambda_\psi = 1$. In Subsection 4.2 we show that all such mappings, that we call *Levi-isometric*, have the form (2.5).

This concludes the proof of the classification result. □

3 Pseudohermitian and Chern–Moser invariants in generalized ellipsoids

3.1 Computation of the pseudohermitian invariants

Fix on M_0 the pseudohermitian structure $\vartheta = i(\partial F - \overline{\partial} F)$, where $F(z, \overline{z}, z^{n+1}, \overline{z}^{n+1}) = \mathrm{Im}(z^{n+1}) - p(z, \overline{z})$ is a defining function for M:

$$\vartheta := \frac{dz^{n+1} + d\overline{z}^{n+1}}{2} - i(p_\alpha dz^\alpha - p_{\overline{\alpha}} d\overline{z}^\alpha) = dt - i(p_\alpha dz^\alpha - p_{\overline{\alpha}} d\overline{z}^\alpha). \tag{3.1}$$

We use the notation $p_\alpha = \frac{\partial p}{\partial z^\alpha}$ and we let $t = \mathrm{Re}(z^{n+1})$. On M_0 we fix the coordinates $(z, t) \in \mathbb{C}^n \times \mathbb{R}$, i.e., we identify $(z, t) \in \mathbb{C} \times \mathbb{R}$ with $\big(z, t + ip(z, \overline{z})\big) \in M_0$. Fix the holomorphic frame

$$Z_\alpha = \partial_\alpha + i p_\alpha \partial_t \qquad \text{for } \alpha = 1, \dots, n, \tag{3.2}$$

and let $Z_{\overline{\alpha}} = \overline{Z}_\alpha$. We clearly have $\vartheta(Z_\alpha) = \vartheta(Z_{\overline{\alpha}}) = 0$ for any $\alpha = 1, \dots, n$.

The Levi form on M is the 2-form $L = -id\vartheta$. From the identities

$$h_{\alpha\overline{\beta}} := L(Z_\alpha, Z_{\overline{\beta}}) = -id\vartheta(Z_\alpha, Z_{\overline{\beta}}) = i\vartheta([Z_\alpha, Z_{\overline{\beta}}]) \quad \text{and} \tag{3.3}$$

$$[Z_\alpha, Z_{\overline{\beta}}] = -2ip_{\alpha\overline{\beta}}\partial_t \qquad \text{for } \alpha, \beta = 1, \dots, n, \tag{3.4}$$

we obtain

$$h_{\alpha\overline{\beta}} = \begin{cases} 2m_j|z_j|^{2(m_j-1)}\Big(\delta_{\alpha\beta} + (m_j - 1)\dfrac{\overline{z}^\alpha z^\beta}{|z_j|^2}\Big) & \text{if } \alpha \sim \beta \in I_j, \\ 0 & \text{if } \alpha \perp \beta. \end{cases} \tag{3.5}$$

The inverse matrix $h^{\lambda\overline{\beta}}$ has the form

$$h^{\lambda\overline{\beta}} = \begin{cases} \dfrac{1}{2m_j|z_j|^{2(m_j-1)}}\Big(\delta_{\lambda\beta} - \dfrac{m_j - 1}{m_j}\dfrac{z^\lambda\overline{z}^\beta}{|z_j|^2}\Big) & \text{if } \lambda \sim \beta \in I_j, \\ 0 & \text{if } \lambda \perp \beta. \end{cases} \tag{3.6}$$

The surface M defined in (2.2) is strictly pseudoconvex, because $\det h_{\alpha\overline{\beta}} > 0$ on M. The characteristic vector field of ϑ is $T = \partial_{n+1} + \overline{\partial}_{n+1} = \frac{\partial}{\partial t}$.

For $j = 1, \dots, s$, let us introduce the holomorphic vector fields

$$E_j = \frac{1}{m_j}\sum_{\alpha \in I_j} z^\alpha Z_\alpha. \tag{3.7}$$

We say that E_j is a vector field "of radial type". A short computation based on (3.5) shows that, for any $\alpha \in \{1, \dots, n\}$, $j, k \in \{1, \dots, s\}$,

$$\begin{aligned} L(Z_\alpha, \overline{E}_j) &= 2m_j|z_j|^{2(m_j-1)}\overline{z}^\alpha \qquad \text{if } \alpha \in I_j, \quad \text{and} \\ L(E_j, \overline{E}_k) &= 2|z_j|^{2m_j}\delta_{jk} \qquad \text{for all } j, k = 1, \dots, s. \end{aligned} \tag{3.8}$$

Here, δ_{jk} is the Kronecker's symbol. The vector fields $E_1, \dots, E_s$ span an s-dimensional subbundle $\mathscr{E} \subset T^{1,0}M$. If $n_s = 0$, we have no vector field E_s and $\mathscr{E}$ has dimension $s - 1$. We denote by $\mathscr{E}^\perp$ the orthogonal complement of $\mathscr{E}$ in $T^{1,0}M$ with respect to the Levi form.

Let $Q : T^{1,0}M \to \mathscr{E}^\perp$ be the projection

$$Q(Z) = Z - \sum_{j=1}^{s} \frac{L(Z, \overline{E}_j)}{L(E_j, \overline{E}_j)} E_j. \tag{3.9}$$

If $n_s = 0$, in the sum the index j ranges from 0 to $s-1$. In particular, for any $j \in \{1,\dots,s\}$ and $\alpha \in I_j$, let W_α be the holomorphic vector field

$$W_\alpha := Q(Z_\alpha) = \sum_{\beta\in I_j} Q_\alpha^\beta Z_\beta.$$

By (3.8), (3.9), and (3.7), we deduce that the coefficients Q_α^β are

$$Q_\alpha^\beta = \delta_\alpha^\beta - \frac{\overline{z}^\alpha z^\beta}{|z_j|^2}. \tag{3.10}$$

Let us introduce the hermitian form $Q_\flat$ on $T^{1,0}M$ associated with Q:

$$Q_\flat(U,\overline{V}) := L(Q(U),\overline{V}) = L(Q(U),\overline{Q(V)}) \qquad \text{for any } U,V \in T^{1,0}M. \tag{3.11}$$

Letting $Q_{\alpha\overline{\beta}} := Q_\flat(Z_\alpha, Z_{\overline{\beta}})$, we have for $\alpha \sim \beta \in I_j$,

$$Q_{\alpha\overline{\beta}} = Q_\alpha^\gamma h_{\gamma\overline{\beta}} = 2m_j|z_j|^{2(m_j-1)}\left(\delta_{\alpha\beta} - \frac{\overline{z}^\alpha z^\beta}{|z_j|^2}\right) = 2m_j|z_j|^{2(m_j-1)} Q_\alpha^\beta, \tag{3.12}$$

whereas $Q_{\alpha\overline{\beta}} = 0$ if $\alpha \perp \beta$.

Finally observe that (3.2) and (3.10) give

$$W_\alpha = \partial_\alpha - \sum_{\beta\in I_j} \frac{\overline{z}^\alpha z^\beta}{|z_j|^2}\partial_\beta \qquad \text{if } \alpha \in I_j. \tag{3.13}$$

Thus $W_\alpha p = 0$ for $\alpha = 1,\dots,n$, i.e., W_α is a holomorphic vector field in $\mathbb{C}^n$ which is tangent to the hypersurfaces of $\mathbb{C}^n$ given by $p(z,\overline{z}) =$ constant.

The bundle $\mathscr{E}^\perp$ is $n-s$ dimensional (in fact, $(n-s+1)$–dimensional if $n_s = 0$) and it is generated by the vector fields W_α with $\alpha = 1,\dots,n$. Let $\mathscr{W}_j \subset T^{1,0}M$ be the subbundle spanned by the vector fields W_α with $\alpha \in I_j$, where $\mathscr{W}_s = (0)$, if $n_s = 0$ or $n_s = 1$. Then we have $\mathscr{E}^\perp = \mathscr{W}_1 \oplus \cdots \oplus \mathscr{W}_{s-1} \oplus \mathscr{W}_s$. Finally, we have $\mathscr{W}_j \perp \mathscr{W}_k$ if $j,k \in \{1,\dots,s\}$ are different. Indeed, for any $\alpha \in I_j$ and $\beta \in I_k$, we have $L(W_\alpha, W_{\overline{\beta}}) = Q_\alpha^\gamma Q_{\overline{\beta}}^{\overline{\sigma}} h_{\gamma\overline{\sigma}} = 0$, because $h_{\gamma\overline{\sigma}} = 0$ if $\gamma \perp \sigma$. Therefore the decomposition

$$T^{1,0}M = \mathscr{W}_1 \oplus \cdots \oplus \mathscr{W}_{s-1} \oplus \mathscr{W}_s \oplus \mathscr{E} \tag{3.14}$$

is orthogonal. Observe also that $\mathscr{W}_s \oplus \mathscr{E} = \operatorname{span}\{E_1,\dots,E_{s-1}, Z_\alpha : \alpha \in I_s\}$.

Proposition 3.1. *The hermitian form $Q_\flat$ introduced in* (3.11) *satisfies:*

$$Q_\flat(E,\overline{Z}) = 0 \qquad \text{for all } E \in \mathscr{E} \text{ and } Z \in T^{1,0}M; \tag{3.15a}$$

$$Q_\flat(V,\overline{W}) = L(V,\overline{W}) \qquad \text{for all } j \in \{1,\dots,s\} \text{ and } V,W \in \mathscr{W}_j; \tag{3.15b}$$

$$Q_\flat(Z,\overline{Z}) \geq 0 \quad \text{for all } Z \in T^{1,0}M \quad \text{and} \quad \mathscr{E} = \left\{Z \in T^{1,0}M : Q_\flat(Z,\overline{Z}) = 0\right\}. \tag{3.15c}$$

Proof. Let $E \in \mathscr{E}$ and $Z \in T^{1,0}M$. Then $Q_\flat(E,\overline{Z}) = L(Q(E),\overline{Z}) = 0$, because $Q(E) = 0$. This proves the first line. To prove the second line, just observe that $Q_\flat(V,\overline{W}) = L(Q(V),\overline{W}) = L(V,\overline{W})$, because $Q(V) = V$. The third line (3.15c) follows from (3.11), letting $U = V$ and from strict pseudoconvexity. □

Let ∇ be the Tanaka-Webster connection of (M,ϑ). We refer to [?,?] and [?] for the relevant facts concerning ∇. The curvature operator of ∇ is $R(Z_\lambda, Z_{\overline{\mu}})Z_\alpha = \nabla_{Z_\lambda}\nabla_{Z_{\overline{\mu}}}Z_\alpha - \nabla_{Z_{\overline{\mu}}}\nabla_{Z_\lambda}Z_\alpha - \nabla_{[Z_\lambda,Z_{\overline{\mu}}]}Z_\alpha$. The curvature tensor have components $R_{\alpha\overline{\beta}\lambda\overline{\mu}} = L(R(Z_\lambda, Z_{\overline{\mu}})Z_\alpha, Z_{\overline{\beta}})$. It enjoys the symmetries

$$R_{\alpha\overline{\beta}\gamma\overline{\mu}} = R_{\gamma\overline{\beta}\alpha\overline{\mu}} \quad \text{and} \quad R_{\alpha\overline{\beta}\gamma\overline{\mu}} = \overline{R}_{\beta\overline{\alpha}\mu\overline{\gamma}}, \tag{3.16}$$

see [?, Section 1.4]. The pseudohermitian Ricci tensor is defined by $R_{\alpha\overline{\mu}} = R_{\alpha\ \lambda\overline{\mu}}^{\ \lambda}$, and the scalar curvature is $R = h^{\alpha\overline{\mu}}R_{\alpha\overline{\mu}}$. Finally, the pseudohermitian torsion of ∇ is defined by $\tau(Z_\beta) = \nabla_T Z_\beta - \nabla_{Z_\beta}T - [T, Z_\beta] =: A_\beta^{\overline{\alpha}} Z_{\overline{\alpha}}$, where $A_{\alpha\beta} := L(\tau(Z_\alpha), Z_\beta)$ satisfies $A_{\alpha\beta} = A_{\beta\alpha}$, as proved in [?].

Now we study the curvature tensors on the hypersurface M. Associated with the decomposition $\mathbb{C}TM = T^{1,0}M \oplus T^{0,1}M \oplus \mathbb{C}T$, we have the projections $\pi_+ : \mathbb{C}TM \to T^{1,0}M$ and $\pi_- : \mathbb{C}TM \to T^{0,1}M$. By definition, for U, V holomorphic vector fields, we have $\nabla_U\overline{V} := \pi_-([U,\overline{V}])$. In our case, from (3.4) we find

$$\nabla_{Z_\alpha} Z_{\overline{\beta}} = 0 \qquad \text{for all } \alpha,\beta = 1,...,n. \tag{3.17}$$

If U, V are holomorphic vector fields, then $\nabla_U V$ is defined by $L(\nabla_U V,\overline{W}) = UL(V,\overline{W}) - L(V,\nabla_U\overline{W})$ for all $W \in T^{1,0}M$. Thus, we obtain

$$\nabla_{Z_\lambda} Z_\alpha = (h^{\sigma\overline{\beta}}\partial_\lambda h_{\alpha\overline{\beta}})Z_\sigma \qquad \text{for all } \alpha,\lambda = 1,...,n. \tag{3.18}$$

Since $h_{\alpha\overline{\beta}} = 2p_{\alpha\overline{\beta}}$, we deduce from (3.18) that

$$\nabla_{Z_\lambda} Z_\alpha = 0 \qquad \text{for all } \alpha,\lambda \in 1,\dots,n \text{ with } \alpha \perp \lambda. \tag{3.19}$$

The Tanaka–Webster connection satisfies $\nabla T = 0$. Moreover, in our case we have $\nabla_T Z_\alpha = \pi_+([T, Z_\alpha]) = 0$. By (3.4), we deduce that

$$\nabla_{[Z_\lambda,Z_{\overline{\mu}}]} Z_\alpha = 0 \qquad \text{for all } \alpha,\lambda,\mu \in \{1,\dots,n\}. \tag{3.20}$$

By (3.17) and (3.20), the curvature operator reduces to $R(Z_\lambda, Z_{\overline{\mu}})Z_\alpha = -\nabla_{Z_{\overline{\mu}}}\nabla_{Z_\lambda}Z_\alpha$, and taking into account (3.18), we get the Riemann and Ricci tensors

$$R_{\alpha\overline{\beta}\lambda\overline{\mu}} = -\partial_{\overline{\mu}}(h^{\sigma\overline{\gamma}}\partial_\lambda h_{\alpha\overline{\gamma}})h_{\sigma\overline{\beta}}, \tag{3.21}$$

$$R_{\alpha\overline{\mu}} = R_{\alpha\ \lambda\overline{\mu}}^{\ \lambda} = -\partial_{\overline{\mu}}(h^{\lambda\overline{\gamma}}\partial_\lambda h_{\alpha\overline{\gamma}}). \tag{3.22}$$

Finally, since $\nabla_{Z_\alpha}T = \nabla_T Z_\alpha = [Z_\alpha, T] = 0$, the torsion vanishes identically,

$$A_{\alpha\beta} = 0. \tag{3.23}$$

Proposition 3.2. *The curvature tensor of* ∇ *of* (M,ϑ) *has the form*

$$R_{\alpha\overline{\beta}\lambda\overline{\mu}} = -\frac{m_j-1}{2m_j}|z_j|^{-2m_j}\{Q_{\lambda\overline{\mu}}Q_{\alpha\overline{\beta}} + Q_{\alpha\overline{\mu}}Q_{\lambda\overline{\beta}}\} \qquad \text{if } \alpha,\beta,\lambda,\mu \in I_j, \tag{3.24}$$

for some $j \in \{1,\dots,s\}$, *and* $R_{\alpha\overline{\beta}\lambda\overline{\mu}} = 0$ *if two of the indeces* α,β,λ,μ *are orthogonal. Moreover,* $R(U,\overline{V},Z,\overline{W}) = 0$ *as soon as one of the vector fields* $U,V,Z,W \in T^{1,0}M$ *belongs to* $\mathscr{E}\oplus\mathscr{W}_s$. *The pseudohermitian Ricci tensor has the form*

$$R_{\lambda\overline{\mu}} = -\frac{n_j(m_j-1)}{2m_j}|z_j|^{-2m_j}Q_{\lambda\overline{\mu}} \qquad \text{if } \lambda,\mu \in I_j, \tag{3.25}$$

and $R_{\lambda\overline{\mu}} = 0$ *if* $\lambda \perp \mu$. *Moreover,* $\mathrm{Ric}(U,\overline{V}) = 0$ *as soon as one of the vector fields* $U,V \in T^{1,0}M$ *belongs to* $\mathscr{E}\oplus\mathscr{W}_s$. *Finally, the scalar curvature is*

$$R = -\sum_{j=1}^{s-1}\frac{n_j(n_j-1)(m_j-1)}{2m_j}|z_j|^{-2m_j}. \tag{3.26}$$

Proof. We start from the formula (3.21). The components of the Levi form are given in (3.5) (and (3.6)). Note that

$$\begin{aligned}
\alpha\perp\lambda &\Rightarrow \partial_\lambda h_{\alpha\overline{\gamma}} = 0,\\
\alpha\perp\mu &\Rightarrow \partial_{\overline{\mu}}(h^{\sigma\overline{\gamma}}\partial_\lambda h_{\alpha\overline{\gamma}}) = 0,\\
\alpha\perp\beta &\Rightarrow \partial_{\overline{\mu}}(h^{\sigma\overline{\gamma}}\partial_\lambda h_{\alpha\overline{\gamma}})h_{\sigma\overline{\beta}} = 0.
\end{aligned}$$

Using the symmetries (3.16), we conclude that $R_{\alpha\overline{\beta}\lambda\overline{\mu}} = 0$ as soon as there are two orthogonal indexes.

Assume that α,β,λ,μ are in I_j. From (3.5) we get

$$\partial_\lambda h_{\alpha\overline{\gamma}} = 2m_j(m_j-1)|z_j|^{2(m_j-2)}\Big\{\overline{z}^\lambda\delta_{\alpha\gamma} + \overline{z}^\alpha\delta_{\lambda\gamma} + (m_j-2)\frac{\overline{z}^\alpha z^\gamma\overline{z}^\lambda}{|z_j|^2}\Big\},$$

and thus

$$h^{\sigma\overline{\gamma}}\partial_\lambda h_{\alpha\overline{\gamma}} = \frac{m_j-1}{|z_j|^2}\Big\{\overline{z}^\lambda\delta_{\alpha\sigma} + \overline{z}^\alpha\delta_{\lambda\sigma} - \frac{\overline{z}^\alpha z^\sigma\overline{z}^\lambda}{|z_j|^2}\Big\}.$$

After a short computation based on (3.10) and (3.12), we find

$$\begin{aligned}
-R_\alpha{}^\sigma{}_{\lambda\overline{\mu}} = \partial_{\overline{\mu}}\big(h^{\sigma\overline{\gamma}}\partial_\lambda h_{\alpha\overline{\gamma}}\big) &= \frac{m_j-1}{|z_j|^2}\{Q^\sigma_\alpha Q^\mu_\lambda + Q^\sigma_\lambda Q^\mu_\alpha\}\\
&= \frac{m_j-1}{2m_j}|z_j|^{-2m_j}\{Q^\sigma_\alpha Q_{\lambda\overline{\mu}} + Q^\sigma_\lambda Q_{\alpha\overline{\mu}}\},
\end{aligned} \tag{3.27}$$

and contracting with $h_{\sigma\overline{\beta}}$, we get (3.24).

Next we show that $R(Z,\overline{W},U,\overline{V}) = 0$ if $Z \in \mathscr{W}_s\oplus\mathscr{E}$. If $Z\in\mathscr{E}$, this follows trivially from the first line of (3.15a). If $Z\in\mathscr{W}_s$, then $R(Z,Z_{\overline{\beta}},Z_\lambda,Z_{\overline{\mu}}) = 0$ if at least one of the indexes β,λ,μ does not belong to I_s. If all $\beta,\lambda,\mu\in I_s$, then, by (3.24), $R(Z,Z_{\overline{\beta}},Z_\lambda,Z_{\overline{\mu}})=0$, because $m_j-1=0$ if $j=s$.

Identities (3.25) and (3.26) follow upon contracting indexes in (3.24), $R_{\lambda\overline{\mu}}=R_\alpha{}^\alpha{}_{\lambda\overline{\mu}}$. Recall that by (3.10) and (3.12) we have $\sum_{\alpha\in I_j}Q^\alpha_\alpha=n_j-1$ and $\sum_{\sigma\in I_j}Q^\sigma_\alpha Q_{\sigma\overline{\mu}}=Q_{\alpha\overline{\mu}}$, if $\alpha,\mu\in I_j$. □

Remark 3.3. Let $V\in\mathscr{E}^\perp$ be a vector such that $V=V_1+\ldots+V_{s-1}$ with $V_j\in\mathscr{W}_j$. The pseudohermitian sectional curvature of (M,ϑ) along $V\neq 0$ is

$$k(V):=\frac{R(V,\overline{V},V,\overline{V})}{|V|^4}=-\frac{1}{|V|^4}\sum_{j=1}^{s-1}\frac{m_j-1}{m_j}|z_j|^{-2m_j}|V_j|^4, \tag{3.28}$$

where $|V|:=L(V,\overline{V})^{1/2}$ denotes the Levi-lenght of V. This formula follows from (3.24) and (3.15b). Notice, in particular, that, since $m_j>1$ for all $j\le s-1$, then $k(V)\neq 0$ for any $V\in\mathscr{W}_1\oplus\cdots\oplus\mathscr{W}_{s-1}$ with $V\neq 0$.

3.2 Chern invariant cones in generalized ellipsoids

We describe the structure of the cones $\mathscr{H}_P$ introduced in (1.9). Here and hereafter, let $|U|:=L(U,\overline{U})^{1/2}$ denote the Levi-length of $U\in T^{1,0}M$.

Proposition 3.4. *Let $M\subset\mathbb{C}^{n+1}$ be the surface defined in* (2.2). *Then*

$$\mathscr{H}=\mathscr{W}_1\cup\ldots\cup\mathscr{W}_{s-1}\cup\big(\mathscr{W}_s\oplus\mathscr{E}\big). \tag{3.29}$$

Proof. We prove that $\mathscr{E}\oplus\mathscr{W}_s\subset\mathscr{H}$. In fact, if $U\in\mathscr{E}\oplus\mathscr{W}_s$, then $R(U,\overline{V},Z,\overline{W})=0$ for all $V,Z,W\in T^{1,0}M$, by Proposition 3.2.

In order to show that $\mathscr{W}_j\subset\mathscr{H}$ for all $j=1,\ldots,s-1$, let $U\in\mathscr{W}_j$ and take $V,Z,W\in T^{1,0}M$ such that $L(U,\overline{V})=L(U,\overline{W})=L(Z,\overline{V})=L(Z,\overline{W})=0$. Observe that, writing $V=V_j+V_j^\perp$, where $V_j\in\mathscr{W}_j$ is the projection of V onto $\mathscr{W}_j$ and $V_j^\perp=V-V_j=E+\sum_{k\neq j}V_k$, for some $E\in\mathscr{E}$, by (3.14), we have

$$L(U,\overline{V})=L(U,\overline{V}_j)\quad\text{and}\quad L(U,\overline{W})=L(U,\overline{W}_j). \tag{3.30}$$

Here, we made for W the same decomposition as for V. Observe also that $Q_\flat(U,\overline{V})=Q_\flat(U,\overline{V}_j)$, and $Q_\flat(U,\overline{W})=Q_\flat(U,\overline{W}_j)$, by (3.11). Then we have by (3.24)

$$\begin{aligned}R(U,\overline{V},Z,\overline{W})&=-\frac{m_j-1}{2m_j}|z_j|^{-2m_j}\big\{Q_\flat(U,\overline{V}_j)Q_\flat(Z,\overline{W})+Q_\flat(U,\overline{W}_j)Q_\flat(Z,\overline{V})\big\}\\&=-\frac{m_j-1}{2m_j}|z_j|^{-2m_j}\big\{L(U,\overline{V}_j)Q_\flat(Z,\overline{W})+L(U,\overline{W}_j)Q_\flat(Z,\overline{V})\big\}=0,\end{aligned}$$

by (3.30) and by the definition of $\mathscr{H}$.

Now we show that any vector field $U=X+Y$ with $X\in\mathscr{E}\oplus\mathscr{W}_s$ and $Y\in\mathscr{W}_1\oplus\cdots\oplus\mathscr{W}_{s-1}$ such that $|X|\neq 0$ and $|Y|\neq 0$ does not belong to $\mathscr{H}$. We assume without loss of generality that $|Y| = 1$. We choose $Z = X + Y$ and $V = W = X - kY$ with $k = |X|^2 \neq 0$, in such a way that $L(U, \overline{V}) = L(X + Y, \overline{X} - k\overline{Y}) = 0$. By Proposition 3.2, $R(U, \overline{V}, Z, \overline{W}) = k^2 R(Y, \overline{Y}, Y, \overline{Y}) = k^2 |Y|^4 k(Y) \neq 0$, thanks to (3.28).

Finally, we prove that if $V \in (\mathscr{E} \oplus \mathscr{W}_s)^\perp \setminus (\mathscr{W}_1 \cup \cdots \cup \mathscr{W}_{s-1})$ then $V \notin \mathscr{H}$. Let $V = V_1 + \cdots + V_s$ with $V_j \in \mathscr{W}_j$ and assume without loss of generality that $|V_1| \neq 0$ and $|V_2| = 1$. Let $W = V_1 - \kappa V_2$ where $\kappa = |V_1|^2$ in such a way that $L(V, \overline{W}) = 0$. By Proposition 3.2 and Remark 3.3, we have $R(V, \overline{W}, V, \overline{W}) = R(V_1, \overline{V}_1, V_1, \overline{V}_1) + \kappa^2 R(V_2, \overline{V}_2, V_2, \overline{V}_2) \neq 0$, as claimed. □

Let $N, N' \subset M$ be connected open sets and let $f : N \to N'$ be a Cauchy–Riemann diffeomorphism. By (1.10) it must be $f_*(\mathscr{H}_P) = \mathscr{H}_{f(P)}$. Then there are two cases:

(A) $f_*(\mathscr{E} \oplus \mathscr{W}_s) = \mathscr{E} \oplus \mathscr{W}_s$;

(B) there exists $j = 1, \ldots, s-1$ such that $f_*(\mathscr{E} \oplus \mathscr{W}_s) = \mathscr{W}_j$.

Here and hereafter, with slight abuse of notation let $f_*(\mathscr{E} \oplus \mathscr{W}_s) = \mathscr{E} \oplus \mathscr{W}_s$ stand for $f_*(\mathscr{E} \oplus \mathscr{W}_s)_P = (\mathscr{E} \oplus \mathscr{W}_s)_{f(P)}$, for all $P \in N$.

Case (B) may occur only if $\dim(\mathscr{E} \oplus \mathscr{W}_s) = \dim(\mathscr{W}_j)$. Actually, the Case B cannot occur at all, as the following theorem states.

Theorem 3.5. *Let $N, N' \subset M$ be open sets. A CR diffeomorphism $f : N \to N' \subset M$ satisfies $f_*(\mathscr{E} \oplus \mathscr{W}_s) = \mathscr{E} \oplus \mathscr{W}_s$. In particular, there exists a permutation σ of $\{1, \ldots, s-1\}$ such that $f_*(\mathscr{W}_j) \subset \mathscr{W}_{\sigma(j)}$ for all $j = 1, \ldots, s-1$.*

We prove Theorem 3.5 in Section 5. In the following proposition, we prove that, in case (A), CR diffeomorphisms preserve the Ricci Tanaka–Webster curvature of ϑ. Diffeomorphisms that preserve the Ricci curvature of a Riemmanian metric are rather studied in Riemannian geometry, see [?, **?**]. It could be interesting to see whether such mappings enjoy any significant geometric property in the CR setting.

Proposition 3.6. *Let $N, N' \subset M$ be open sets. A CR diffeomorphism $f : N \to N'$ such that $f_*(\mathscr{E} \oplus \mathscr{W}_s) = \mathscr{E} \oplus \mathscr{W}_s$ preserves the Ricci curvature of ϑ, i.e.* $\mathrm{Ric}(f_* Z, f_* \overline{W}) = \mathrm{Ric}(Z, \overline{W})$ *for all $Z, W \in T^{1,0}N$.*

Proof. We divide the proof into two steps.

Step 1. We claim that $R = \lambda R \circ f$, where R is the scalar curvature of ϑ and $\lambda > 0$ is the CR factor of f, i.e., $f^* \vartheta = \lambda \vartheta$.

Let $Z \in \mathscr{E} \oplus \mathscr{W}_s$ with $|Z| = 1$. Proposition 3.2 and Webster formula (1.8) yield

$$S(Z, \overline{Z}, Z, \overline{Z}) = \frac{2R}{(n+1)(n+2)}. \tag{3.31}$$

Since $f_* Z \in \mathscr{E} \oplus \mathscr{W}_s$, we analogously have

$$S(f_* Z, f_* \overline{Z}, f_* Z, f_* \overline{Z}) = \frac{2R \circ f}{(n+1)(n+2)} |f_* Z|^4 = \frac{2\lambda^2 R \circ f}{(n+1)(n+2)}, \tag{3.32}$$

where we also used $|f_* Z|^2 = \lambda |Z|^2 = \lambda$. Thus, by the relative CR invariance (1.7) we have

$$S_\vartheta(f_* Z, f_* \overline{Z}, f_* Z, f_* \overline{Z}) = S_{f^*\vartheta}(Z, \overline{Z}, Z, \overline{Z}) = S_{\lambda\vartheta}(Z, \overline{Z}, Z, \overline{Z}) = \lambda S_\vartheta(Z, \overline{Z}, Z, \overline{Z}),$$

Here, $S_\vartheta, S_{f^*\vartheta}$ and $S_{\lambda\vartheta}$ denote the Chern tensors relative to ϑ, $f^*\vartheta$, and $\lambda\vartheta$, respectively. Comparing (3.31) and (3.32) we conclude *Step 1.*

Step 2. We claim that $\mathrm{Ric}(f_* V, f_* \overline{W}) = \mathrm{Ric}(V, \overline{W})$ for all $V, W \in (\mathscr{E} \oplus W_s)^\perp$.

Take vector fields $V, W \in (\mathscr{E} \oplus W_s)^\perp$. Let also $Z \in \mathscr{E} \oplus W_s$ be such that $|Z| = 1$. All terms in the Chern tensor containing curvature tensors along Z or terms of the form $L(V, \overline{Z})$ and $L(Z, \overline{W})$ vanish. Thus

$$S(V, \overline{W}, Z, \overline{Z}) = -\frac{1}{n+2}\left\{\mathrm{Ric}(V, \overline{W}) - \frac{R}{(n+1)} L(V, \overline{W})\right\}. \tag{3.33}$$

Since $f_* Z \in \mathscr{E} \oplus W_s$ and $f_* W, f_* V \in (\mathscr{E} \oplus W_s)^\perp$, we analogously have

$$S(f_* V, f_* \overline{W}, f_* Z, f_* \overline{Z}) = -\frac{\lambda}{n+2}\left\{\mathrm{Ric}(f_* V, f_* \overline{W}) - \frac{\lambda R \circ f}{n+1} L(V, \overline{W})\right\}. \tag{3.34}$$

Recall that $S(f_* V, f_* \overline{W}, f_* Z, f_* \overline{Z}) = S_{\lambda\vartheta}(V, \overline{W}, Z, \overline{Z}) = \lambda S(V, \overline{W}, Z, \overline{Z})$, by the CR invariance (1.7). Thus the *Step 2* can be accomplished on comparing (3.33), (3.34) and using the *Step 1.*

The proof is finished, because $\mathrm{Ric}(Z, \overline{W}) = 0$ for all $Z \in \mathscr{E} \oplus \mathscr{W}_s$ and $W \in T^{1,0}M$. □

Proposition 3.7. *Let $N, N' \subset M$ be open sets and let $f : N \to N'$ be a CR diffeomorphism such that $f_*(\mathscr{E} \oplus \mathscr{W}_s) = \mathscr{E} \oplus \mathscr{W}_s$. Then the CR factor λ of f satisfies $W_\alpha \lambda = 0$ for any $\alpha \in I_1 \cup \cdots \cup I_{s-1}$.*

Proof. Let $\alpha \in I_1 \cup \cdots \cup I_{s-1}$ and fix $W = W_\alpha$. We first observe that

$$d\vartheta(f_* T, f_* \overline{W}) = f^*(d\vartheta)(T, \overline{W}) = d(f^*\vartheta)(T, \overline{W}) = \big((d\lambda) \wedge \vartheta + \lambda d\vartheta\big)(T, \overline{W}) = -\overline{W}\lambda.$$

Therefore, it suffices to show that $d\vartheta(f_* T, f_* \overline{W}) = 0$.

Using (3.4), we find for any $j, k = 1, ..., s-1$

$$[E_j, \overline{E}_k] = -\frac{1}{m_j^2} \sum_{\alpha,\beta \in I_j} 2i p_{\alpha\overline{\beta}} z^\alpha \overline{z}^\beta \delta_{jk} T = -2i|z_j|^{2m_j} \delta_{jk} T, \tag{3.35}$$

because $\sum_{\alpha \in I_j} p_\alpha z^\alpha = m_j |z_j|^{2m_j}$. Thus we have

$$f_* T = \frac{i}{2|z_j|^{2m_j}} f_*[E_j, \overline{E}_j] = \frac{i}{2|z_j|^{2m_j}} [f_* E_j, f_* \overline{E}_j] \qquad \text{for all } j = 1, \ldots, s-1.$$

Notice the commutation relations

$$\begin{aligned} &[E_j, Z_{\overline{\alpha}}] = 0 \qquad \text{if } \alpha \in I_s \text{ and } j \le s-1; \quad \text{and} \\ &[Z_\alpha, Z_{\overline{\beta}}] = -2i\delta_{\alpha\beta} T \qquad \text{if } \alpha, \beta \in I_s. \end{aligned} \tag{3.36}$$

In view of (3.36) and (3.35), we claim that for any vector field $Z \in \mathscr{E} \oplus \mathscr{W}_s$ there exist a real function σ and a vector field $U \in \mathscr{E} \oplus \mathscr{W}_s$ such that

$$[Z, \overline{Z}] = i\sigma T + U - \overline{U}. \tag{3.37}$$

Formula (3.37) can be checked by a routine computation. Here and hereafter, with slight abuse of notation we denote sections of a bundle with the same notation of the bundle. The claim applies to $Z = f_*(E_j) \in \mathscr{E} \oplus \mathscr{W}_s$, for $j = 1, \ldots, s-1$. The vector fields $f_* W \in \mathscr{W}_1 \oplus \ldots \oplus \mathscr{W}_{s-1}$ and $f_* T$ are then orthogonal with respect to the Levi form and the proof is concluded. □

4 CR mappings in generalized ellipsoids

4.1 Computation of the CR factor

Lemma 4.1. *Let $N \subset M$ be an open set and let v be a CR function in N such that $v_{,\alpha\beta} = 0$ for all $\alpha, \beta = 1, \ldots, n$. Then for any $\alpha \in I_j$, $j = 1, \ldots, s$, we have*

$$Z_\alpha T v = \frac{n_j(m_j - 1)}{2im_j(n+1)|z_j|^{2m_j}} W_\alpha v, \tag{4.1}$$

where $W_\alpha = Q(Z_\alpha)$ is the vector field (3.13).

Proof. We use the third order commutation formulae in [?, eq. (2.1)]. Because $v_{,\gamma\alpha} = 0$, we have

$$v_{,\gamma\overline{\beta}\alpha} = v_{,\gamma\alpha\overline{\beta}} - ih_{\alpha\overline{\beta}} v_{,\gamma 0} - R_\gamma{}^\varrho{}_{\alpha\overline{\beta}} v_{,\varrho} = -ih_{\alpha\overline{\beta}} v_{,\gamma 0} - R_\gamma{}^\varrho{}_{\alpha\overline{\beta}} v_{,\varrho}.$$

On the other hand, by the commutation formula $v_{,\gamma\overline{\beta}} - v_{,\overline{\beta}\gamma} = ih_{\gamma\overline{\beta}} v_{,0}$ and since $v_{,\overline{\beta}} = 0$, we get $ih_{\gamma\overline{\beta}} v_{,0\alpha} = -ih_{\alpha\overline{\beta}} v_{,\gamma 0} - R_\gamma{}^\varrho{}_{\alpha\overline{\beta}} v_{,\varrho}$. Contracting with $h^{\gamma\overline{\beta}}$ yields

$$i(n+1) v_{,0\alpha} = -R_\gamma{}^\varrho{}_\alpha{}^\gamma v_{,\varrho} = -R_\gamma{}^\gamma{}_\alpha{}^\varrho v_{,\varrho} = -R_\alpha{}^\varrho v_{,\varrho} = \frac{n_j(m_j-1)}{2m_j|z_j|^{2m_j}} Q_\alpha^\varrho v_{,\varrho}$$

and the proof is concluded. □

Theorem 4.2. *Let $N \subset M$ be a connected open set and let $f : N \to f(N) \subset M$ be a CR diffeomorphism with CR factor $\lambda = u^{-1}$. Then, either u is a constant or there exist $k \in \mathbb{R} \setminus \{0\}$ and $(a_s, a^{n+1}) = (a_s, t_0 + i|a_s|^2) \in \mathbb{C}^{n_s+1}$ such that $u = k^2|z^{n+1} + a^{n+1} + 2iz_s \cdot \overline{a}_s|^2$.*

Proof. The argument here is similar to [?]. The torsion A_ϑ of ϑ vanishes, as noted in (3.23), $A_{\alpha\beta} = 0$. On the other hand, denoting by $\widetilde{A} = A_{f^*\vartheta}$ the torsion of $\widetilde{\vartheta} = f^*\vartheta$, we have $A_{f^*\vartheta}(Z, W) = A_\vartheta(f_* Z, f_* W) = 0$, for all $Z, W \in T^{1,0}N$. Thus we also have $\widetilde{A}_{\alpha\beta} = 0$. From (1.3), we deduce that u satisfies the system of equations $u_{,\alpha\beta} = 0$.

By Theorem 3.5, the assumptions of Proposition 3.6 hold and therefore f preserves the Ricci curvature of ϑ. By (1.2), we have the system of equations

$$(n+2)\Big\{u_{,\alpha\overline{\beta}}+u_{,\overline{\beta}\alpha}-\frac{2}{u}u_{,\alpha}u_{,\overline{\beta}}\Big\}+\Big\{\Delta u-\frac{2(n+2)}{u}|\nabla u|^2\Big\}h_{\alpha\overline{\beta}}=0. \tag{4.2}$$

Then, the function $w=\log u$ satisfies the system of equations $w_{,\alpha\overline{\beta}}+w_{,\overline{\beta}\alpha}=\frac{\Delta w}{n}h_{\alpha\overline{\beta}}$. By [?, Proposition 3.3], w is locally the real part of a CR function F, i.e., we have locally

$$u=e^{(F+\overline{F})/2}=v\overline{v}=|v|^2,$$

where $v:=e^{F/2}$ is a CR function. Since $u_{,\alpha\beta}=0$, we also have $v_{,\alpha\beta}=0$.

The function $g:=Tv$ satisfies $g_{,\overline{\alpha}}=Z_{\overline{\alpha}}Tv=TZ_{\overline{\alpha}}v=0$. (Recall that T and $Z_{\overline{\alpha}}$ commute.) By Proposition 3.7, we have $W_\alpha u=0$ for all $\alpha\in I_1\cup\ldots\cup I_{s-1}$. This implies $W_\alpha v=0$ for the same indexes. By Lemma 4.1, we deduce that for any $\alpha=1,\ldots,n$ we also have $g_{,\alpha}=Z_\alpha Tv=0$. The equations $g_{,\alpha}=g_{,\overline{\alpha}}=0$ imply that g is locally constant. Therefore there exist a constant $k\in\mathbb{C}$ and a function $\psi=\psi(z,\overline{z})$ such that $v(z,\overline{z},t)=kt+\psi(z,\overline{z})$. Since v is CR, it must be $\partial_{\overline{\alpha}}\psi-ik\partial_{\overline{\alpha}}p=0$, which means

$$v=k\big(t+ip(z,\overline{z})\big)+\phi(z),$$

for some holomorphic function ϕ. Possibly multiplying v by a unitary complex number, we can assume that k is real.

Moreover, for any $\alpha\in I_1\cup\cdots\cup I_{s-1}$ we have $W_\alpha\phi=0$, because $W_\alpha v=0$. This fact implies that v depends locally only on $|z_1|,\ldots,|z_{s-1}|$ and, if $I_s\neq\varnothing$, on z_s. From $v_{,\alpha\beta}=0$ and since $\nabla_\alpha Z_\beta=0$ for all $\alpha,\beta\in I_s$, see (3.18), we deduce that for all $\alpha,\beta\in I_s$ we have $\frac{\partial^2\phi}{\partial z^\alpha\partial z^\beta}=0$, which finally yields

$$v(z,\overline{z},t)=k(t+ip(z,\overline{z}))+z_s\cdot\overline{d}+c \tag{4.3}$$

for some $d\in\mathbb{C}^{n_s}$ and $c\in\mathbb{C}$. In order to find the imaginary part of c, we observe that if u solves the system (4.2), it also solves the contracted equation $\Delta u-\frac{n+2}{u}|\nabla u|^2=0$. After a computation, this implies that v solves the equation

$$i\big(\overline{v}v_{,0}-v\overline{v}_{,0}\big)=h^{\gamma\overline{\mu}}v_{,\gamma}\overline{v_{,\mu}}. \tag{4.4}$$

Plugging (4.3) into (4.4), we get $k\mathrm{Im}(c)=|d|^2/4$. If $k=0$ then $d=0$, $\phi(z)=c$ and v is constant. If $k\neq0$ then, letting $t_0=\mathrm{Re}(c)/k$, we obtain $v=k\Big\{t+t_0+z_s\cdot\overline{d}/k+i\big(p+|d|^2/(4k^2)\big)\Big\}$. Letting $\overline{d}/k=2i\overline{a}_s$, we get

$$v=k\Big\{t+t_0+ip+2iz_s\cdot\overline{a}_s+i|a_s|^2\Big\}=k\big\{z^{n+1}+a^{n+1}+2iz_s\cdot\overline{a}_s\big\}, \tag{4.5}$$

where $z^{n+1}=t+ip(z,\overline{z})$ and $a^{n+1}=t_0+i|a_s|^2$. This concludes the proof. □

4.2 Levi-isometric mappings

Let M be the surface (2.2) endowed with the pseudohermitian structure ϑ introduced in (3.1). We say that a CR diffeomorphism $\psi : N \to N'$ is *Levi-isometric with respect to* ϑ if $\psi^*\vartheta = \vartheta$. A Levi isometric mapping ψ satisfies $L(\psi_* Z, \psi_* \overline{W}) = L(Z, \overline{W})$ for all $Z, W \in T^{1,0}N$. Moreover, since we trivially have $\psi_*(T^{\psi^*\vartheta}) = (T^{\vartheta})_{\psi}$, it turns out that a Levi isometric mapping satisfies

$$\psi_* T = T_{\psi}. \tag{4.6}$$

Theorem 4.3. *Let $N, N' \subset M$ be connected open sets and let $\psi : N \to N'$ be a Levi-isometric mapping with respect to ϑ. Then, there exists a permutation σ of $\{1,\dots,s-1\}$ such that $m_{\sigma(j)} = m_j$ and $n_{\sigma(j)} = n_j$ for any $j = 1,\dots,s-1$, there are unitary matrices $B_j \in U(n_j)$, and, if $n_s \geq 1$, there are $B_s \in U(n_s)$ and a vector $(b_s, b^{n+1}) = (b_s, t_0 + i|b_s|^2) \in \mathbb{C}^{n_s} \times \mathbb{C}$ such that for all $(z,t) \in N$ we have*

$$\psi(z, z^{n+1}) = \Big(B_1 z_{\sigma(1)}, \dots, B_{s-1} z_{\sigma(s-1)}, B_s z_s + b_s, b^{n+1} + z^{n+1} + 2i(B_s z_s \cdot \overline{b}_s)\Big). \tag{4.7}$$

We start with an easy lemma.

Lemma 4.4. *Let $D \subset \mathbb{C}^d$, $d \geq 2$, be an open connected set and let $\zeta : D \to \mathbb{C}^d$ be a nonconstant holomorphic mapping such that $|\zeta(z)|$ is constant if $|z|$ is constant, for $z \in D$. Then there exists $B \in GL(d, \mathbb{C})$ such that $\zeta(z) = Bz$ and $B^*B = \varrho^2 I$ for some $\varrho > 0$.*

Proof. Assume without loss of generality that there exists $z \in D$ such that $|\zeta(z)| = |z| = 1$. This can be achieved multiplying ζ by a positive constant. Then, by the Poincaré–Alexander theorem, see [**?**, ?, **?**], ζ is the restriction of an automorphism of the unit ball $B_1 := \{z \in \mathbb{C}^d : |z| < 1\}$. Thus, see [**?**, ?], there exist a unitary matrix $B \in U(d)$ and $a \in \mathbb{C}^d$ with $|a| < 1$ such that $\zeta(z) = B\phi_a(z)$ for all $z \in D$, where $\phi_a(z) = \frac{a - Pz - \sqrt{1-|a|^2}Qz}{1 - z\cdot\overline{a}}$, with $Pz = \frac{z\cdot\overline{a}}{|a|^2}a$ and $Qz = z - Pz$. When $a = 0$ we have $\phi_a(z) = -z$. If $a \neq 0$, ϕ_a takes bB_1 to bB_1 but it does not take any other (open piece of) sphere bB_r with $r \neq 1$ to a sphere centered at the origin. Then we have $a = 0$ and the Lemma follows. □

Proof of Theorem 4.3. All our claims along the proof are of a local nature. In the coordinates $(z,t) \in \mathbb{C}^n \times \mathbb{R}$ on M, we have $\psi = (\zeta^1, \dots, \zeta^n, \tau)$ with $\zeta^\beta : N \to \mathbb{C}$, $\beta = 1, \dots, n$, and $\tau : N \to \mathbb{R}$. We first notice that $Z_{\overline{\alpha}}\zeta^\beta = 0$ for all $\alpha, \beta = 1, \dots, n$, because ψ is a CR mapping. Moreover, we have $T\tau = 1$ and $T\zeta^\beta = 0$ because $\psi_* T = T$, by (4.6). Then, for $j = 1, \dots, s$ the functions $\zeta_j = \zeta_j(z)$ are holomorphic and $\tau = t + v(z, \overline{z})$ for some real function v.

By Theorem 3.5, there exists a permutation σ of $\{1,\dots,s-1\}$ such that $\psi_* \mathscr{W}_j = \mathscr{W}_{\sigma(j)}$. In the following we let $j' = \sigma(j)$. In particular, we have $n_{j'} = n_j$ for all $j = 1, \dots, s-1$.

Fix $j \in \{1, \dots, s-1\}$. Let $V \in \mathscr{W}_j$ with $|V| = 1$. Since ψ is Levi isometric, ψ preserves the sectional curvature of ϑ, $k(\psi_* V) = k(V)$. By (3.28), we deduce that

$$\frac{m_j - 1}{m_j}|z_j|^{-2m_j} = \frac{m_{j'} - 1}{m_{j'}}|\zeta_{j'}(z)|^{-2m_{j'}}. \tag{4.8}$$

With the notation $z_j^* = (z_1, ..., z_{j-1}, z_{j+1}, ..., z_s)$, consider for fixed z_j^* the mapping $z_j \mapsto \zeta_{j'}(z_j^*; z_j)$. This mapping is nonconstant and holomorphic from an open subset of $\mathbb{C}^{n_j}$ to $\mathbb{C}^{n_j}$. Moreover, by (4.8) it takes (pieces of) spheres of $\mathbb{C}^{n_j}$ centered at the origin into spheres centered at the origin. By Lemma 4.4, there exist $B_j \in GL(n_j, \mathbb{C})$ and $\varrho_j > 0$ such that $\zeta_{j'}(z) = B_j(z_j^*) z_j$ with $B_j^* B_j = \varrho_j^2 I$. Here $B_j = B(z_j^*)$ is holomorphic, while $\varrho_j = \varrho_j(z_j^*, \overline{z}_j^*)$. Therefore, (4.8) becomes

$$|z_j|^{2(m_j - m_{j'})} = \frac{m_j - 1}{m_j} \frac{m_{j'}}{m_{j'} - 1} \varrho_j(z_j^*, \overline{z}_j^*)^{2m_{j'}}.$$

Both the left-hand side and the right-hand side must be constant. Therefore $m_j = m_{j'}$ and $\varrho(z_j^*, \overline{z}_j^*) = 1$. Ultimately we have for any $j \le s-1$, $\zeta_{j'}(z) = B_j z_j$, for some constant matrix $B_j \in U(n_j)$.

Next we claim that, if $n_s \ge 1$ then ζ_s depends only on z_s. To prove the claim it suffices to show that for all $j \le s-1$ we have

$$W_\lambda \zeta^\gamma = 0 \qquad \text{for all } \lambda \in I_j, \gamma \in I_s; \quad \text{and} \tag{4.9a}$$

$$E_j \zeta^\gamma = 0 \qquad \text{for all } \gamma \in I_s. \tag{4.9b}$$

To prove (4.9a), fix $j \le s-1$ and $\lambda \in I_j$. Then

$$\begin{aligned} \psi_* W_\lambda &= \sum_{k=1}^{s-1} \sum_{\gamma \in I_k} (W_\lambda \zeta^\gamma)(\partial_\gamma)_\psi + \sum_{\gamma \in I_s} (W_\lambda \zeta^\gamma)(\partial_\gamma)_\psi + (W_\lambda \tau)(\partial_t)_\psi \qquad (\text{since } \psi \text{ is CR}) \\ &= \sum_{k=1}^{s-1} \sum_{\gamma \in I_k} (W_\lambda \zeta^\gamma)(Z_\gamma)_\psi + \sum_{\gamma \in I_s} (W_\lambda \zeta^\gamma)(Z_\gamma)_\psi. \end{aligned}$$

But $\psi_* W_\lambda \in \mathscr{W}_{j'}$. Then all the terms in the last sum vanish and (4.9a) is proved.

To prove (4.9b) start by computing $\psi_* E_j$ for $j \le s-1$:

$$\begin{aligned} \psi_* E_j &= \sum_{k'=1}^{s-1} \sum_{\substack{\gamma \in I_{k'} \\ \mu \in I_k}} B_{\gamma\mu} E_j z^\mu (Z_\gamma)_\psi + \sum_{\gamma \in I_s} E_j \zeta^\gamma (Z_\gamma)_\psi \\ &= \sum_{\substack{\gamma \in I_{j'} \\ \mu \in I_j}} B_{\gamma\mu} \frac{1}{m_j} z^\mu (Z_\gamma)_\psi + \sum_{\gamma \in I_s} E_j \zeta^\gamma (Z_\gamma)_\psi = (E_{j'})_\psi + \sum_{\gamma \in I_s} E_j \zeta^\gamma (Z_\gamma)_\psi. \end{aligned}$$

Taking the Levi-length, we find $|\psi_* E_j|^2 = |(E_{j'})_\psi|^2 + \sum_{\gamma, \varrho \in I_s} (E_j \zeta^\gamma)(\overline{E}_j \overline{\zeta}^\varrho)(h_{\gamma\overline{\varrho}})_\psi$. But we have $|\psi_* E_j|^2 = |E_j|^2 = 2|z_j|^{2m_j}$ and $|(E_{j'})_\psi|^2 = 2|\zeta_{j'}|^{2m_j} = 2|z_j|^{2m_j}$. Moreover it is $(h_{\gamma\overline{\varrho}})_\psi = 2\delta_{\gamma\varrho}$, because $\gamma, \varrho \in I_s$. Then we conclude that $E_j \zeta^\gamma = 0$, as required.

Next, we compute v and ζ_s. Let $\alpha \in I_j$, where $j \le s-1$. Recall that $\tau = t + v(z, \overline{z})$. Since

$\partial_\alpha \zeta_s = 0$, we have

$$\begin{aligned}\psi_* Z_\alpha &= \sum_{\beta\in I_{j'}} B_{\beta\alpha}(\partial_\beta)_\psi + \big(\partial_\alpha v + i m_j |z_j|^{2(m_j-1)}\overline{z}^\alpha\big)(\partial_t)_\psi \qquad \text{(since } \psi \text{ is CR)}\\ &= \sum_{\beta\in I_{j'}} B_{\beta\alpha}(Z_\beta)_\psi = \sum_{\beta\in I_{j'}} B_{\beta\alpha}\big\{(\partial_\beta)_\psi + i m_{j'}|\zeta_{j'}|^{2(m_{j'}-1)}\overline{\zeta}^\beta(\partial_t)_\psi\big\}\\ &= \sum_{\beta\in I_{j'}} B_{\beta\alpha}(\partial_\beta)_\psi + i m_j |z_j|^{2(m_j-1)}\overline{z}^\alpha(\partial_t)_\psi,\end{aligned}$$

where we used $|\zeta_{j'}| = |z_j|$, $m_{j'} = m_j$ and $\sum_{\beta\in I_{j'}} B_{\beta\alpha}\overline{B_{\beta\gamma}} = \delta_{\alpha\gamma}$, if $\alpha,\gamma \in I_j$. Comparing the first and third lines we get $v_\alpha = 0$. Thus v depends only on z_s, t.

Finally, we find ζ_s and v when $n_s \geq 1$. For $\alpha \in I_s$ we have

$$\psi_* Z_\alpha = \sum_{\beta\in I_s} (\partial_\alpha \zeta^\beta)(\partial_\beta)_\psi + \big\{i\overline{z}^\alpha + \partial_\alpha v\big\} T_\psi, \tag{4.10}$$

as well as

$$\psi_* Z_\alpha = \sum_{\beta\in I_s} (\partial_\alpha \zeta^\beta)(Z_\beta)_\psi = \sum_{\beta\in I_s} (\partial_\alpha \zeta^\beta)\big((\partial_\beta)_\psi + i\overline{\zeta}^\beta T_\psi\big). \tag{4.11}$$

Since ψ is Levi isometric, we have $2\delta_{\alpha\gamma} = L(\psi_* Z_\alpha, \psi_* Z_{\overline{\gamma}})$ for $\alpha,\gamma \in I_s$. Using formula (4.11), we obtain $\sum_{\beta\in I_s}(\partial_\alpha\zeta^\beta)(\partial_{\overline{\gamma}}\overline{\zeta}^\beta) = \delta_{\alpha\gamma}$, for all $\alpha,\gamma\in I_s$. Therefore it must be $\zeta_s(z) = b_s + Bz_s$, for some $B \in U(n_s)$ and $b_s \in \mathbb{C}^{n_s}$. Moreover, comparing the coefficients of T in (4.10) and (4.11), we obtain the equation $i\overline{z}^\alpha + \partial_\alpha v = i\sum_{\beta\in I_s}\overline{\zeta}^\beta \partial_\alpha \zeta^\beta$, which implies $\partial_\alpha v = i\sum_{\beta\in I_s} B_{\beta\alpha}\overline{b}^\beta$. Since v is a real function, we finally find $v = t_0 - 2\mathrm{Im}(Bz_s\cdot \overline{b}_s)$ for some $t_0 \in \mathbb{R}$.

The structure (4.7) of the isometry ψ is now determined locally. The proof is concluded because N is connected. □

5 Proof of Theorem 3.5

This section is devoted to the proof of Theorem 3.5. The proof is rather involved, but we were not able to find a more direct one. In many situations, the study of this case can be avoided for trivial dimensional reasons, see the discussion before the statement of Theorem 3.5.

Proposition 5.1. *Let $N \subset M$ be an open set and let $f : N \to f(N) \subset M$ be a CR diffeomorphism such that $f_*(\mathscr{E} \oplus \mathscr{W}_s) = \mathscr{W}_j$ for some $j = 1,\dots,s-1$. Then there exists $\nu \in \mathbb{N}\cup\{0\}$ such that*

$$M = \big\{(z_1, z_2, z^{n+1}) \in \mathbb{C}^{\nu+2}\times\mathbb{C}^\nu\times\mathbb{C} : \mathrm{Im} z^{n+1} = |z_1|^{2m_1} + |z_2|^2,\ \textit{and}\ z_1 \neq 0\big\}. \tag{5.1}$$

Moreover, we have $\lambda R\circ f = R$ where λ is the CR factor of f.

Proof. For some $k = 1,\dots,s-1$ we have $f_*(\mathscr{W}_k) = \mathscr{E}\oplus\mathscr{W}_s$. Since f is a CR diffeomorphism, it must be $\dim\mathscr{W}_j = \dim(\mathscr{E}\oplus\mathscr{W}_s) = \dim\mathscr{W}_k$. In other words,

$$n_j = n_s + s = n_k \tag{5.2}$$

For any $V \in \mathscr{W}_k$ with $|V| = 1$ we evaluate $S(V) := S(V, \overline{V}, V, \overline{V})$. By (1.8), (3.24), (3.25), and (3.15b) we get

$$S(V) = \frac{1}{n+2}\left\{(2n_k - n - 2)\frac{m_k - 1}{m_k}|z_k|^{-2m_k} + \frac{2R}{n+1}\right\}. \tag{5.3}$$

Since $f_* V \in \mathscr{E} \oplus \mathscr{W}_s$, all the terms involving curvature in $S(f_* V)$ vanish and we get

$$S(f_* V) = \frac{2R \circ f}{(n+1)(n+2)}|f_* V|^4 = \lambda^2 \frac{2R \circ f}{(n+1)(n+2)}. \tag{5.4}$$

By the CR invariance (1.7), we deduce from (5.3) and (5.4)

$$\frac{2\lambda R \circ f}{n+1} = (2n_k - n - 2)\frac{m_k - 1}{m_k}|z_k|^{-2m_k} + \frac{2R}{n+1}. \tag{5.5}$$

Let $Z \in \mathscr{E} \oplus \mathscr{W}_s$ with $|Z| = 1$. We have

$$S(Z) = \frac{2R}{(n+1)(n+2)}. \tag{5.6}$$

Since $f_* Z \in \mathscr{W}_j$ for some $j \leq s-1$, arguing as in (5.3) we find

$$S(f_* Z) = \frac{\lambda^2}{n+2}\left\{(2n_j - n - 2)\frac{m_j - 1}{m_j}|\zeta_j|^{-2m_j} + \frac{2R \circ f}{n+1}\right\}, \tag{5.7}$$

where we let $(\zeta, \zeta^{n+1}) = f(z, z^{n+1}) \in M$. By the CR invariance (1.7), we obtain

$$\lambda\left\{(2n_j - n - 2)\frac{m_j - 1}{m_j}|\zeta_j|^{-2m_j} + \frac{2R \circ f}{n+1}\right\} = \frac{2R}{n+1}. \tag{5.8}$$

Comparing (5.5) and (5.8) we get

$$(2n_k - n - 2)\frac{m_k - 1}{m_k}|z_k|^{-2m_k} + \lambda(2n_j - n - 2)\frac{m_j - 1}{m_j}|\zeta_j|^{-2m_j} = 0. \tag{5.9}$$

Recall that by (5.2), it must be $n_j = n_k$. Therefore, (5.9) becomes

$$(2n_j - n - 2)\left\{\frac{m_k - 1}{m_k}|z_k|^{-2m_k} + \lambda\frac{m_j - 1}{m_j}|\zeta_j|^{-2m_j}\right\} = 0.$$

But the curly bracket is positive. Then, we have $n_j = n_k = (n+2)/2$. Moreover, it must be $j = k$, because if $j \neq k$ the condition $n_j + n_k \leq n$ is not satisfied. Finally, using $n_i \geq 2$ for all $i \leq s-1$ and $n_s = n_j - s$, we get

$$n = n_1 + \ldots + n_s = n_j + n_s + \sum_{i \neq j} n_i \geq n_j + n_s + 2(s-2) = 2n_j + s - 4 = n + s - 2.$$

This gives $s \leq 2$. If $s = 1$, then M is the surface $\operatorname{Im}(z^3) = (|z^1|^2 + |z^2|^2)^{m_1}$. If $s = 2$, we have $n = n_1 + n_2 = n/2 + 1 + n_2$, which implies $n_1 = n_2 + 2$ and the domain has the form (5.1). □

Proposition 5.2. *Let M be as in* (5.1) *and let $f : N \to N'$ be a CR diffeomorpshism such that $f_*(\mathscr{E} \oplus \mathscr{W}_2) = \mathscr{W}_1$. Then the CR factor λ of f satisfies $E_1\lambda = -\lambda$.*

Proof. Fix $\sigma, \mu \in I_1$ and let $W = \overline{z}^\sigma Z_\mu - \overline{z}^\mu Z_\sigma = \overline{z}^\sigma \partial_\mu - \overline{z}^\mu \partial_\sigma$. Notice that $L(W, \overline{E}_1) = 0$ and thus $W \in \mathscr{W}_1$. We also have $[W, \overline{W}] = z^\sigma \partial_\sigma + z^\mu \partial_\mu - \overline{z}^\sigma \partial_{\overline{\sigma}} - \overline{z}^\mu \partial_{\overline{\mu}}$ and

$$[[W,\overline{W}],\overline{E}_1] = \Big[z^\sigma\partial_\sigma + z^\mu\partial_\mu - \overline{z}^\sigma\partial_{\overline{\sigma}} - \overline{z}^\mu\partial_{\overline{\mu}}, \frac{1}{m_1}\sum_{\beta\in I_1}\overline{z}^\beta\partial_{\overline{\beta}} - i|z_1|^{2m_1}\partial_t\Big] = 0. \tag{5.10}$$

Finally, we have $i\vartheta([W,\overline{W}]) = -id\vartheta(W,\overline{W}) = 2m_1|z_1|^{2(m_1-1)}\big(|z^\sigma|^2 + |z^\mu|^2\big)$.

Since $f_*W \in \mathscr{E}\oplus\mathscr{W}_2$, as in the proof of Proposition 3.7, see (3.37), we have

$$f_*[W,\overline{W}] = [f_*W, f_*\overline{W}] = F - \overline{F} + ikT, \tag{5.11}$$

for some $F \in \mathscr{E}\oplus\mathscr{W}_2$ and some real function k on $f(N)$. Since $f_*E_1 \in \mathscr{W}_1$, $f_*[W,\overline{W}]$ and $f_*\overline{E}_1$ are orthogonal by (5.11). Then, also using (5.10), we get

$$0 = d\vartheta(f_*[W,\overline{W}], f_*\overline{E}_1) = -(f_*\overline{E}_1)(\vartheta(f_*[W,\overline{W}])), \tag{5.12}$$

that is equivalent with $E_1\big(\lambda|z_1|^{2(m_1-1)}(|z^\sigma|^2+|z^\mu|^2)\big) = 0$. Since σ,μ are arbitrary, this implies $E_1\big(\lambda|z_1|^{2m_1}\big) = 0$ and eventually $E_1\lambda + \lambda = 0$, because $E_1|z_1|^{2m_1} = |z_1|^{2m_1}$. □

Proof of Theorem 3.5. Assume by contradiction that $f_*(\mathscr{E}\oplus\mathscr{W}_s) = \mathscr{W}_j$ for some $j = 1,\dots,s-1$. Then, by Proposition 5.1, M is of the form (5.1), and by Proposition 5.2 we have $E_1\lambda = -\lambda$, where $\lambda = u^{-1}$ is the CR factor of f. We have $f_*\mathscr{W}_1 = \mathscr{W}_2\oplus\mathscr{E}$ and $f_*(\mathscr{W}_2\oplus\mathscr{E}) = \mathscr{W}_1$.

In terms of u we have $E_1u = u$. Note that (3.17) implies that $\nabla_{\overline{E}_1}E_1 = 0$. Thus

$$\nabla^2u(E_1,\overline{E}_1) + \nabla^2u(\overline{E}_1,E_1) - \frac{2}{u}|E_1u|^2 = 0.$$

Since $\mathrm{Ric}(E_1,\overline{E}_1) = 0$, (1.5) becomes

$$\begin{aligned}\mathrm{Ric}(f_*E_1, f_*\overline{E}_1) &= \frac{1}{2u}\Big\{\Delta u - \frac{2(n+2)}{u}|\nabla u|^2\Big\}|E_1|^2\\ &= \frac{1}{2}\Big\{\Delta u - \frac{2(n+2)}{u}|\nabla u|^2\Big\}|f_*E_1|^2.\end{aligned} \tag{5.13}$$

On the other hand, since $f_*E_1 \in \mathscr{W}_1$, comparing (3.25) and (3.26), we get

$$\mathrm{Ric}(f_*E_1, f_*\overline{E}_1) = \frac{1}{n_1-1}(R\circ f)\left|f_*E_1\right|^2, \tag{5.14}$$

and therefore (5.13) becomes

$$2\frac{R\circ f}{n_1-1} = \Delta u - \frac{2(n+2)}{u}|\nabla u|^2. \tag{5.15}$$

By Proposition 5.1 we have $R \circ f = uR$, and from (1.4) we obtain

$$uR = R \circ f = \widetilde{R} = uR + (n+1)\Big\{\Delta u - \frac{n+2}{u}|\nabla u|^2\Big\},$$

that gives $\Delta u = \dfrac{n+2}{u}|\nabla u|^2$. Inserting this identity into (5.15) and using formula (3.26), we obtain

$$\frac{|\nabla u|^2}{u^2} = \frac{m_1 - 1}{m_1(n+2)|z_1|^{2m_1}} n_1 = \frac{m_1 - 1}{2m_1|z_1|^{2m_1}}, \tag{5.16}$$

because, $n_1 = \nu + 2$ and $n_2 = \nu$, so that $\frac{n_1}{n+2} = \frac{1}{2}$. On the other hand, by $E_1 u = u$ and $|E_1|^2 = 2|z_1|^{2m_1}$ we have $|\nabla u|^2 \geq \frac{|E_1 u|^2}{|E_1|^2} = \frac{u^2}{2|z_1|^{2m_1}}$, which contradicts (5.16). The proof is concluded. □

References

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