Existence of Calibrations for the Mumford-Shah Functional and the Reinitialization of the Distance Function in the Framework of Chan and Vese Algorithms

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Contents

Introduction

This thesis contains the results I have obtained during my PhD at the Max-Planck Institute in Leipzig. It could be ideally divided in two parts (the first one embracing Chapters 1 to 3 and the second one Chapter 4) and interestingly enough the order of the topics of this thesis follows very closely the chronological order of my research in these years.

The first part is devoted to the notion of calibration for the Mumford-Shah functional and it is structured in the first three chapters. Chapter 1 is a list of well-known results and definitions regarding the Mumford-Shah functional and the theory of calibrations related to it that will be used in the rest of the thesis as a reference. Chapter 2 collects the issues regarding the calibration of the crack-tip and Chapter 3 is devoted instead to the topic of the existence of calibration and the lifting of the Mumford-Shah functional to the space of graphs of SBV functions. More precisely Chapter 3 contains a review of Chambolle's approach to this topic and my contribution to that, available in [14] as well.
On the other hand the second part of the thesis encompasses the last part of my PhD and it is built around the Chan and Vese algorithm for approximating the Mumford-Shah functional. In particular in the first section the reader will find a review of the main features of this algorithm and a formal derivation of the gradient flow associated to it. The main contribution that can be found in this chapter regards the Hamilton-Jacobi equation that is commonly used in the framework of Chan and Vese-type algorithms in order to reinitialize the initial data to be the distance function from its zero level set ([13]). For this reason, Section 3.4 is dedicated to a review of the main definitions and results related to the theory of viscosity solutions for Hamilton-Jacobi equation and Section 3.5 proposes the result about the long time behaviour for the viscosity solution of the reinitialization of the distance function.
In the next part of this introduction I am going to outline the most important aspects of each chapter as well as a clarification of my contributions, the ideas behind that and some references.

The Mumford-Shah functional is one of the most important variational model for image segmentation and edge detection. The main goal of image segmentation is the following: suppose that we are given an image in a bounded domain $\Omega \subset \mathbb{R}^2$ represented by its level of gray as a function $g \in L^\infty(\Omega)$; then we would like to create a copy of g that is simpler and therefore easier to process. A possible solution of this vague problem is to build a functional and to find the candidate as a minimizer of it. Models like the one just described are called variational models for image processing ([45], [25], [39]) and the Mumford-Shah is one of the mathematically most interesting and well-known example in this framework.

It was introduced in the late 80's by Mumford and Shah ([41],[40]) as

$$J(u, K) = \int_{\Omega\setminus K} |\nabla u|^2 \, dx + \beta \mathcal{H}^{n-1}(K) + \alpha \int_{\Omega\setminus K} |u - g|^2 \, dx, \tag{1}$$

where $\Omega \in \mathbb{R}^n$ is open, $K \subset \Omega$ is closed, $g \in L^\infty(\Omega)$, $u \in W^{1,2}(\Omega \setminus K)$ and β and α are tuning parameters.
In this particular variational model the fidelity term is the L^2 distance from g and the regularizing terms are the Dirichlet energy and the lenght of the contour K. The minimum will be close to the original image g thanks to the fidelity term and it will be simpler due to the presence of the regularizing terms that penalize the oscillation and the complexity of the contour K. The Mumford-Shah functional is also a model for edge detection as the minimum K represents the boundary of the image detected by the model.
The existence of minimizers for (1.1) was proved in [2] introducing a weak formulation of the Mumford-Shah functional obtained considering $u \in SBV(\Omega)$ and replacing the set K with S_u, i.e. the singular set of u:

$$\mathrm{MS}(u) = \int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^{n-1}(S_u) + \int_\Omega |u - g|^2 \, dx. \tag{2}$$

On the other hand the regularity issues are not yet fully understood, as the well-known conjecture proposed in [41] is still open in its full generality.

Conjecture (Mumford, Shah)**.** *Let (u, K) a pair minimizing* (1)*. Then K is locally union of finitely many $C^{1,1}$ embedded arcs.*

The conjecture and in general the regularity properties of the minimizers of the Mumford-Shah functional are intimately connected, by a proper blow-up argument due to Bonnet ([9]), to the local minimizers of the leading part of MS:

$$F(u) = \int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^{n-1}(S_u).$$

Unfortunately the characterization of local minimizers of F is far from being clear and when one looks at $\mathbb{R}^2$ it coincides with the conjecture stated by De Giorgi in [28]:

Conjecture (De Giorgi)**.** *The only non constant global minimizer in $\mathbb{R}^2$ is the crack-tip (here written in polar coordinates):*

$$u(\rho, \theta) = \sqrt{\frac{2\rho}{\pi}} \sin\left(\frac{\theta}{2}\right) \qquad \rho \geq 0, \quad -\pi < \theta < \pi. \tag{3}$$

This conjecture was partially established in [10] by Bonnet and David. In this fairly long and complicated article they proved that the crack-tip is actually a

global minimizer, but the uniqueness issue, that is the key point to solve the Mumford-Shah conjecture, remains open in its generality.
This is one of the main reason why in [1] Alberti, Bouchitté and Dal Maso introduced a notion of calibration for the Mumford-Shah functional (see also [38] and [37] for further development). Apart from the theoretical interest of having a concept of calibration for free discontinuity problems resembling the classical one of minimal surfaces by Harvey and Lawson ([31]), one of the main goals was to prove the minimality of functions as the crack-tip that are universally known to be minimizers. In particular in [1] they succeed in proving that the triple junction is a local minimizer, but the problem of finding a calibration for the crack-tip remained unsolved. The problem has a long history of attempts (failed), so that some mathematicians believe that it is not actually possible to find such a calibration with the current definition.

> *It would be nice if one could prove (the minimality of the crack-tip) with a simple calibration argument, but the author doubts that this will be possible* ([21]).

This is the open question that guided me in the first part of my PhD, where we tried to exhibit an explicit calibration for the crack-tip. Unfortunately we achieved only partial results that I decided not to include in this revised version. In any case the interested reader is more than welcomed to get in contact with me for further material.

As a consequence of the difficulties in calibrating the crack-tip and the skepticism around it, the natural problem to consider is the existence of calibration for a given minimizer. As the case of the calibrations for minimimal surfaces suggests (see [24]), this is a complex problem strongly related to the possibility of extending the Mumford-Shah functional to a larger vector space, in a convex way and preserving the minimizers (this procedure is called lifting). Chapter 3 is devoted to this specific topic.
A consequence of the theory of calibration for the Mumford-Shah functional already noted in [1] is that MS can be expressed as a supremum over a convex set of functionals corresponding to a set K of vector fields in the following way:

$$\mathrm{MS}(u) = \max_{\phi \in K} \int_{\Omega \times \mathbb{R}} \phi \cdot D\mathbf{1}_{\{u(x)>t\}}, \tag{4}$$

where $u \in SBV(\Omega)$ and $\mathbf{1}_{\{u(x)>t\}}$ is the characteristic function of the subgraph of u. Therefore replacing $\mathbf{1}_{\{u(x)>t\}}$ with $v(x,t)$, of bounded variation and non-decreasing in t one can construct a convex functional that extends MS; this is Chambolle's approach to the relaxation of the Mumford-Shah functional ([15]). More precisely in [15] he is able to prove that in dimension one the minimizers of the Mumford-Shah are also minimizers of the lifted Mumford-Shah and as a

consequence concluding the existence of calibration in a weak and asymptotical sense; section 2.2.1 of Chapter 3 is devoted to review Chambolle's approach. It is interesting to notice that the same result of Chambolle and the analogous for higher dimension, could be obtained proving of a coarea formula for the convexification of MS. Unfortunately it is easy to see that such a coarea formula cannot hold as it is shown in Section 2.2.2. However the counterexample proposed in Section 2.2.2 is the starting point to understand how a coarea formula could be modified in order to be suitable for the Mumford-Shah functional. This is indeed the idea behind the generalized coarea formula contained in [14] and presented in the last part of this chapter.
In Section 2.2.3 we propose our approach to the lifting of the Mumford-Shah functional. In particular we extend MS to the space of rectifible currents $T = (\mathcal{M}, \nu, \theta)$ as

$$G(T) = \sup_{\phi \in K} \int_{\mathcal{M}} \theta \langle \phi, \nu \rangle d\mathcal{H}^{n-1}. \tag{5}$$

Then in Section 2.3 we consider the functional (5) in one dimension taking values in the cone composed of all the linear combination of SBV graphs with multiplicity:

$$C := \left\{ T = \sum_{i=1}^{k} \lambda_i \Gamma_{u_i} : k \in \mathbb{N}, \lambda_i \in \mathbb{R}_+, u_i \in SBV(I) \right\}.$$

Our goal is to prove the equivalence of the minimum problems for local minimizers of MS and G. As anticipated, in order to achieve such a result, we employ a generalized coarea for G in one dimension. As we cannot count on the classical coarea formula (see Section 2.2.2), the idea is to find a proper graphs decomposition of a current $T \in C$ such that a coarea formula holds. In particular we prove the following theorem (see [14]):

Theorem (Coarea-type formula)**.** *Let I be open interval. Given $T = \sum_{i=1}^{k} \lambda_i \Gamma_{u_i}$ with $u_i \in SBV(I)$ and $\lambda_i > 0$ such that $|\bigcup S_{u_i}| < +\infty$ there exists $k' \in \mathbb{N}$, $\{\mu_i\}_{i=1 \dots k'} > 0$ and $\{w_i\}_{i=1 \dots k'} \subset SBV(I)$ such that $T = \sum_{i=1}^{k'} \mu_i \Gamma_{w_i}$ and*

$$G(T) = \sum_{i=1}^{k'} \mu_i G(\Gamma_{w_i}).$$

Furthermore we use it to prove the equivalence of the local minimum problems (see [14]).

Theorem. *Given $u \in SBV(I)$ a minimizer of the Mumford-Shah functional, Γ_u (i.e. the graph associated to u) is a minimizer of G among all the linear combinations of graphs of the form $T = \sum_{i=1}^{k} \lambda_i \Gamma_{u_i}$ with $\partial \Gamma_u = \partial T$.*

Then we come back to the orgininal problem of proving the existence of a calibration for minimizers of the Mumford-Shah functional. Thanks to the previous theorem, the functional G is a convex functional having the same minima as MS. Therefore, readapting Federer's argument for existence of calibrations for mass minimizing currents ([24]) we find, as an application of Hahn-Banach theorem, a linear functional defined on C that stays below the functional G and touches it only at the minimum point. This functional can be seen as a weak notion of calibration for our problem according to Definition 2.4.1.

In the second part of the thesis that is included in Chapter 4 we change our point of view on the Mumford-Shah functional. In particular we will examine some classical algorithms for computing an approximation of the minimizers of simple variants of the Mumford-Shah functional called: *piecewise constant* and *piecewise smooth* Mumford-Shah functional. To be more precise the strong simplification of these models lies in the fact that the discontinuity set of an SBV function is concentrated on the boundary of some set of finite perimeter E and the function is constant or smooth in the interior and in the exterior of E.
These algorithms are named Chan and Vese algorithms and they were introduced in ([16]) and ([17]). The main idea is that, to compute a minimizer (or at least a stationary point) of the piecewise smooth Mumford-Shah functional is possible by decoupling the minimization of the different terms of the functional. More precisely one can initialize the algorithm with a pair (u_0, K_0) and then using the level set method ([43]) to compute one step of the evolution by mean curvature (with an external force given by u_0) of the set K_0, obtaining the set K_1 and then updating the function u_0 by the classical Euler-Lagrange equation of the Mumford-Shah functional. This procedure can be iterate indefinitely and heuristically it should converge to a stationary point of the Mumford-Shah functional. Clearly, if we construct the same algorithm to the piecewise constant version of the Mumford-Shah functional, the operations simplify as the Euler-Lagrange equation reduces to the computation of the optimal constant on the interior and exterior of the discontinuity K.
The Chan and Vese algorithm employs the level set method to compute the mean curvature flow of the discontinuity in order to get rid of the lack of smoothness of K as well as on the change of topology of K during the evolution. Indeed embedding the interface K into the level set of a function $\phi : \mathbb{R}^2 \to \mathbb{R}$ it is possible to define a well-posed evolution of ϕ in the framework of viscosity solution for Hamilton-Jacobi equation. The drawback is that the equation becomes more and more degenerate (and numerically unstable) when $|\nabla \phi|$ approaches zero. This is the reason way it is customary in the Chan and Vese-type algorithm to reinitialize the function ϕ to be the signed distance function from the interface K.
The most classical way to obtain such a result is an Hamilton-Jacobi equation introduced in [47] that has the following form:

$$\begin{cases} \phi_t + f_\delta(\phi_0(x))(|\nabla \phi| - 1) = 0 & \text{in } \mathbb{R}^2 \times [0, +\infty) \\ \phi(x, 0) = \phi_0(x) & \text{in } \mathbb{R}^2 \end{cases} \tag{6}$$

where the initial data $\phi_0(x)$ is a function that is zero on some Lipschitz interface Γ and it is greater than zero in the external part of Γ and less than zero in the internal part; moreover f_δ is a smooth approximation of the sign function. As $t \to +\infty$ the viscosity solution of (6) should converge heuristically to the solution of $f_\delta(\phi_0(x))(|\nabla\phi| - 1) = 0$ that is the signed distance function from Γ.

In Section 3.4 of Chapter 4 we introduce the technical tools needed to study equation (6) in the framework of viscosity solution. In particular we will require also the theory of discontinuous viscosity solution and the stability results that are collected in Section 3.4.3 for the reader's convenience.

In Section 3.5 we prove rigorously the uniform convergence of the viscosity solution to the signed distance function from Γ (denoted by $d_s(x, \Gamma)$) obtaining the following result ([13]) :

Theorem. *Let $\phi(x, t)$ be a viscosity solution of (6) then $\phi(x, t)$ converges uniformly to $d_s(x, \Gamma)$ as $t \to \infty$ on every compact set of $\mathbb{R}^2$.*

The key point to prove such a convergence is to show the preservation of the zero level set of the viscosity solution $\phi(x, t)$. This can be done by adapting an argument of Namah and Roquejouffre in [42] to our specific Hamiltonian deducing the following proposition:

Proposition. *Let $\phi(x, t)$ a viscosity solution of (3.51) then letting*

$$\Gamma_t := \{x \in \mathbb{R}^2 : \phi(x, t) = 0\}$$

one has $\Gamma_t = \Gamma$ for every $t > 0$.

The difficult part is to infer the same preservation in the limit as $t \to 0$ without relying on uniform estimates of the gradient $\nabla\phi$ due to the lack of coercivity of the Hamiltonian close to Γ. We overcome this obstacle constructing barriers (see also [30]) for the viscosity solution that close to Γ do not depend on t in such a way that even without a control on the gradient of the solution we can still infer the uniform convergence as $t \to +\infty$. The proof of the uniform convergence of the viscosity solution to the signed distance function is done by a classical homogeneization procedure used in [42] and inspired by [35]; in particular one can consider the rescaled viscosity solutions $\phi^\varepsilon(x, t) = \phi(x, t/\varepsilon)$ that solve the following Hamilton-Jacobi equation:

$$\begin{cases} \epsilon\phi^\varepsilon_t + f(x)(|\nabla\phi^\varepsilon| - 1) = 0 & \text{in } \mathbb{R}^2 \times (0, +\infty), \\ \phi^\varepsilon(x, 0) = \phi_0(x) & \text{in } \mathbb{R}^2. \end{cases}$$

Then passing to the limit as $\varepsilon \to 0$ and using stability properties for discontinuous viscosity solutions and the strong comparison principle one can infer the uniform convergence of $\phi(x, t)$ to $d_s(s, \Gamma)$ as $t \to +\infty$.

Chapter 1

Preliminaries

1.1 The Mumford-Shah functional

1.1.1 The model

The Mumford-Shah functional is one of the most important variational model for image segmentation and edge detection. It was introduced in the late 80's by Mumford and Shah ([41],[40]) as a free discontinuity problem and it can be considered the evolution of simpler well-known models for image processing ([45], [25], [39]).
Given $\Omega \subset \mathbb{R}^n$ a bounded, regular and open set, $g \in L^\infty(\Omega, [0, 1])$ and α and β two positive tuning parameters the Mumford-Shah functional is defined as

$$J(u, K) \coloneqq \int_{\Omega\setminus K} |\nabla u|^2 \, dx + \beta\mathcal{H}^{n-1}(K) + \alpha \int_{\Omega\setminus K} |u - g|^2 \, dx, \tag{1.1}$$

where

$$(u, K) \in \mathcal{A}(\Omega) \coloneqq \left\{(u, K) : u \in W^{1,2}(\Omega \setminus K) \text{ and } K \subset \Omega \text{ is closed}\right\}. \tag{1.2}$$

In particular the function g represents the level of gray of an image and it associates to each pixel of Ω a value between zero and one. The functional is composed by a fidelity term, the L^2 distance between g and u, that penalizes how *far* is a competitor from the real image. Moreover it has two regularizing terms: the Dirichlet energy and the length of the contour K. By minimizing (1.1) one obtains a closed set K_{min} and a function $u_{min} \in W^{1,2}(\Omega \setminus K)$ with the following properties: K_{min} is the edge of the image detected by the model and u_{min} is a smoother copy of g.

Figure 1.1: Segmentation of a cagou ([34])

After the introduction of the functional there have been a huge effort in understanding the most basic questions regarding existence and regularity of the minimizers. The first observation in order to tackle these questions is that the value of the functional is not changing if we add to K a closed set of $\mathcal{H}^{n-1}$ negligible

Hausdorff measure. Therefore one needs a definition of minimizers of (1.1) that is *minimal* in this sense:

Definition 1.1.1 (Reduced minimizers)**.** *A* reduced minimizer *for J is a pair $(\tilde{u}, \tilde{K}) \in \mathcal{A}(\Omega)$ such that*

$$J(\tilde{u}, \tilde{K}) = \inf_{(u,K)\in\mathcal{A}(\Omega)} J(u, K)$$

and for all $K \subset \tilde{K}$ relatively closed in Ω such that if $K \neq \tilde{K}$ one has that $\tilde{u} \notin W^{1,2}_{loc}(\Omega \setminus K)$.

1.1.2 Weak formulation and existence of minimizers

The existence of minimizers for (1.1) was proved in [2] introducing a weak formulation of the Mumford-Shah functional obtained considering $u \in SBV(\Omega)$. We refer to Section 4.1 in Appendix for the basic definitions regarding BV and SBV functions and to [5] for more specific notions.
One can consider the functional defined in $SBV(\Omega)$ obtained replacing the set K with S_u in the definition of J:

$$\mathrm{MS}(u) = \int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^{n-1}(S_u) + \int_\Omega |u - g|^2 \, dx. \tag{1.3}$$

Remark 1.1.1. *The functional would make sense also if $u \in BV(\Omega)$, but the restriction to $SBV(\Omega)$ is natural, because the infimum in BV is always zero for every $g \in L^\infty(\Omega, [0, 1])$.*

Using the SBV compactness theorem due to Ambrosio ([5]) the functional MS has a minimizer in $SBV(\Omega)$. Then, one would like to infer from this, the existence of minimizers for (1.1) or in other words to build a minimizer of J in $\mathcal{A}(\Omega)$ knowing a minimizer of MS.
It is easy to prove that

$$\inf_{(u,K)\in\mathcal{A}(\Omega)} J(u, K) \geq \inf_{v\in SBV(\Omega)} \mathrm{MS}(v),$$

where the infimum on the right side of the inequality is actually a minimum. On the other hand given $u \in SBV(\Omega)$ a minimum for MS, a natural candidate for being the minimum of J would be $(u, \overline{S}_u) \in \mathcal{A}(\Omega)$. Therefore it is enough to prove that

$$\mathrm{MS}(u) = J(u, \overline{S}_u),$$

or equivalently

$$\mathcal{H}^{n-1}(\overline{S}_u \setminus S_u) = 0, \tag{1.4}$$

for every $u \in SBV(\Omega)$ minimizer of MS.
Equation (1.4) is a consequence of a celebrated regularity result for the Mumford-Shah functional due to De Giorgi, Carriero and Leaci ([29]) that gives a lower bound for the $\mathcal{H}^{n-1}$ density of the singular set of a minimizer.

Theorem 1.1.2 (De Giorgi, Carriero, Leaci)**.** *Let $u \in SBV(\Omega)$ be a minimizer for MS then there exists $\mu > 0$ a dimensional constant, such that for every $x \in \overline{S}_u$*

$$\mathcal{H}(S_u \cap B_r(x)) \geq \mu r^{n-1} \tag{1.5}$$

for every $r < \min\{dist(x, \partial\Omega), 1\}$.

Thanks to Theorem 1.1.2 and standard properties of rectifiable sets (1.4) follows.

1.1.3 Local minimizers and regularity

As for the regularity properties for the minimizers of (1.1) and (1.3) the main goal is to prove the following conjecture proposed by Mumford and Shah in their seminal paper:

Conjecture 1.1.3 (Mumford, Shah)**.** *Let (u, K) be a reduced minimizer of* (1.1)*. Then K is locally union of finitely many $C^{1,1}$ embedded arcs.*

Even if this conjecture remains open in its full generality, there have been a lot of partial regularity results in this direction (see for example [3],[4],[9],[20]).
An usual way to get regularity information for minimizers of functionals (for example in the context of minimal surfaces and obstacle problems) is to perform a blow-up analysis in a specific point of the domain and then to deduce the regularity properties of the minimizers from the structure of the blow-up limits. There exists an analogous procedure for the Mumford-Shah functional due to Bonnet ([9]) that introduced a proper rescaling from which it is possible to get relevant information in the blow-up limits. We present here the heuristic of the rescaling.
First of all it is natural to localize the Mumford-Shah functional: given $A \subset \Omega$ a Borel set we define

$$\mathrm{MS}(u, A) := \int_A |\nabla u|^2 \, dx + \mathcal{H}^{n-1}(S_u \cap A) + \int_A |u - g|^2 \, dx.$$

Then, given $r > 0$, $x \in \Omega$ and $v \in SBV(B_r(x))$ we consider the following rescaling. Defining $v_r \in SBV(B_1(0))$ as

$$v_r(y) := \frac{1}{\sqrt{r}} v(x + ry), \tag{1.6}$$

one obtains

$$\int_{B_1(0)} |\nabla v_r|^2 \, dx + \mathcal{H}^{n-1}(S_{v_r}) = \frac{1}{r} \left(\int_{B_r(x)} |\nabla v|^2 \, dx + \mathcal{H}^{n-1}(S_v) \right)$$

and rescaling in an analogous way $g \in L^\infty(\Omega)$

$$\int_{B_1(0)} |v_r - g_r|^2 \, dx = \frac{1}{r^3} \int_{B_r(x)} |v - g|^2 \, dx.$$

Thus

$$\int_{B_1(0)} |\nabla v_r|^2\,dx + \mathcal{H}^{n-1}(S_{v_r}) + r^2 \int_{B_1(0)} |v_r - g_r|^2\,dx =$$
$$= \frac{1}{r}\left(\int_{B_r(x)} |\nabla v|^2\,dx + \mathcal{H}^{n-1}(S_v) + \int_{B_r(x)} |v-g|^2\,dx\right). \tag{1.7}$$

So if one defines the so called *homogeneous* Mumford-Shah functional as

$$F(v,A) := \int_A |\nabla v|^2\,dx + \mathcal{H}^{n-1}(S_v \cap A)$$

for $v \in SBV(\mathbb{R}^n)$ and $A \subset \mathbb{R}^n$, the equation (1.7) reduces to

$$F(v_r, B_1(0)) + r^2 \int_{B_1(0)} |v_r - g_r|^2\,dx = \frac{1}{r}\left(F(v, B_r(x)) + \int_{B_r(x)} |v-g|^2\,dx\right).$$

Therefore, it turns out that under this rescaling the homogeneous Mumford-Shah F is the leading part of MS. Following this heuristic computation, Bonnet in [9] has shown that a rescaling of a sequence of minimizers for MS converges to a proper notion of minimizer of F usually called *topological global minimizer*. We will not enter into the details of his definition, as in the remaining part of this thesis we will use the classical notion of *local and global minimizer*, that is slightly stronger than the one of Bonnet, due to the fact that we are not imposing any topological constraint on the set competitors. We refer to [21] for a complete list of possible notions of minimizers for the Mumford-Shah functional.

Definition 1.1.2 (Local and global minimizers)**.** *Given $u \in SBV(\Omega)$ such that $F(u) < \infty$ we say that u is a local minimizer of the Mumford-Shah functional if*

$$F(u) \le F(v) \quad \forall v \in SBV(\Omega) \text{ s.t. } \{u \neq v\} \subset\subset \Omega$$

and if $\Omega = \mathbb{R}^n$ we say that u is a global minimizer.

In a similar way one can define local minimizer of the functional MS. We will denote it by the denomination *Dirichlet minimizer*:

Definition 1.1.3 (Dirichlet minimizer)**.** *Given $u \in SBV(\Omega)$ such that $MS(u) < \infty$ we say that u is a Dirichlet minimizer of the Mumford-Shah functional if*

$$MS(u) \le MS(v) \quad \forall v \in SBV(\Omega) \text{ s.t. } \{u \neq v\} \subset\subset \Omega$$

For the previous discussion it is a natural question to characterize local and global minimizers and the basic known results about that can be summarized in the following proposition:

Proposition 1.1.4. *Let $\Omega \subset \mathbb{R}^n$ be open. Consider $u \in SBV(\Omega)$, then*

i) *If u is harmonic then it is a local minimizer in every $A \subset \Omega$ provided that*

$$\left(\sup_A u - \inf_A u \right) \|\nabla u\|_{L^\infty} \leq 1.$$

ii) *If $\Omega = \mathbb{R}^n$ and $u = a$ in the half space $\{x_n > 0\}$ and $u = b$ in $\{x_n < 0\}$, then u is a local minimizer in every set $Q \times I$ where $Q \subset \mathbb{R}^{n-1}$ is a rectangle, I is in interval and $\mathscr{L}^1(I) \leq (b-a)^2$.*

iii) *If $\Omega = \mathbb{R}^2$ and u is a piecewise constant function equal to the real values a, b, c on each side of a triod centered in the origin, then u is a local minimizer in every ball $B_r(0)$ provided that*

$$r \leq \frac{1}{2} \min \left\{ |a-b|^2, |b-c|^2, |a-c|^2 \right\}.$$

The first two results can be proved easily by using standard arguments of calculus of variations. On the contrary (***iii***) was first proved in [1] by Alberti, Bouchitté and Dal Maso using a calibration method adapted to the Mumford-Shah functional. We are going to be more precise in the next section that is completely devoted to this technique.
As for global minimizers in the plane there is a long standing conjecture by De Giorgi stated in [28]

Conjecture 1.1.5 (De Giorgi)**.** *The only non constant global minimizer in $\mathbb{R}^2$ is the crack-tip (here written in polar coordinates):*

$$u(\rho, \theta) = \sqrt{\frac{2\rho}{\pi}} \sin\left(\frac{\theta}{2}\right) \qquad \rho \geq 0, \quad -\pi < \theta < \pi. \tag{1.8}$$

The uniqueness part of Conjecture 1.1.5 is related by Bonnet blow-up argument to the Mumford-Shah conjecture and it is still an open problem. However the fact that the crack-tip is a global minimizer in the plane was proved in [10] by Bonnet and David with a very deep result that uses most of the results known for the Mumford-Shah. Another open problem that we are going to discuss in Chapter **??** is proving the same result of Bonnet and David with a calibration argument.

1.1.4 Euler-Lagrange equation for the Mumford-Shah functional

The characterization of local and global minimizers of the Mumford-Shah functional is suggested by the Euler-Lagrange equations associated to it. For a general dimension n, two classical Euler-Lagrange equations are known: the first one describes the behavior of the minimizers far from the discontinuity and the second one, instead, relates the difference of the traces with the curvature of the discontinuity (see [5], Chapter 7). In dimension two there exists an additional one due to Léger ([33]), that allows to reconstruct a local minimizer from its discontinuity set. In this subsection we describes the two classical ones and in Chapter **??** we will review Léger's formula. We state the results for minimizers of MS, but we remark that being the Euler-Lagrage equations local, similar propositions hold for local minimizers and Dirichlet minimizers as well.

Proposition 1.1.6 (Euler-Lagrange formula 1)**.** *Let $u \in SBV(\Omega)$ be a minimizer of the Mumford-Shah functional. Suppose that there exists $A \subset\subset \Omega$ open such that*

$$A \cap \overline{S}_u = \{(x, f(x)) : x \in D\}$$

for some $D \subset \mathbb{R}^{n-1}$ and $f : D \to \mathbb{R}$. Then defining

$$A^+ := \{(x,t) \in A : f(x) > t\} \quad \text{and} \quad A^- := \{(x,t) \in A : f(x) < t\}$$

we have that u is a distributional solution of the following PDE:

$$\begin{cases} \Delta u = (u - g) & \text{in } A^\pm \\ \dfrac{\partial u}{\partial n} = 0 & \text{in } \partial A^\pm \cap \overline{S}_u. \end{cases} \tag{1.9}$$

The proof is a standard variational argument and it can be obtained comparing $\mathrm{MS}(u)$ with $\mathrm{MS}(v)$ where $v := u + \varepsilon\phi$, $\varepsilon > 0$ and ϕ is a test function vanishing in a neighbourhood of $\partial A^\pm \setminus \overline{S}_u$.

Proposition 1.1.7 (Euler-Lagrange formula 2)**.** *Let $u \in SBV(\Omega)$ be a minimizer of the Mumford-Shah functional and suppose $g \in C^1$. Assume that there exists $A \subset\subset \Omega$ open such that*

$$A \cap \overline{S}_u = \{(x, f(x)) : x \in D\}$$

for some $D \subset \mathbb{R}^{n-1}$ and $f : D \to \mathbb{R}$ of class C^2. Define

$$A^+ := \{(x,t) \in A : f(x) > t\} \quad \text{and} \quad A^- := \{(x,t) \in A : f(x) < t\}$$

and suppose in addition that $u \in W^{2,2}(A^+) \cap W^{2,2}(A^-)$. Then

$$-\operatorname{div}\left(\frac{\nabla f}{\sqrt{1 + |\nabla f|^2}}\right) = \left[|\nabla u|^2 + (u-g)^2\right]^+ - \left[|\nabla u|^2 + (u-g)^2\right]^- \tag{1.10}$$

on $S_u \cap A$.

Also in this case the proof is a quite standard computation that requires to compute the variation of the $\mathcal{H}^{n-1}$ measure of the singular set of u under a perturbation. We remark that the first term in (1.10) is the curvature of S_u and therefore this formula relates the curvature of S_u to the difference of the traces of u in a neighbourhood where S_u is at least of class C^2.

1.1.5 Calibration for the Mumford-Shah functional

Calibration for minimal surfaces

We start this section briefly recalling the theory of calibration for minimal surfaces in order to point out the similarities and the differences to the case of the Mumford-Shah functional.
The notion of calibration for manifold was firstly introduced in a seminal form by Federer in [24] and then developed by Harvey and Lawson in their article named *Calibrated geometries* ([31]).

Definition 1.1.4 (Calibration for manifolds)**.** *Let $M \subset \mathbb{R}^n$ an oriented manifold of dimension k. Then a* calibration *for M is k-dimensional form in $\mathbb{R}^n$ denoted by ω such that*

i) *ω is closed, i.e. $d\omega = 0$,*

ii) *$|\omega| \leq 1$ in $\mathbb{R}^n$,*

iii) *$\langle \omega, \xi \rangle = 1$ for every ξ unitary simple k-vector orienting M.*

The interest of this object lies in the fact that the existence of a calibration implies the minimality of M with respect the Area functional as it is stated by the following proposition

Proposition 1.1.8. *Let M be as in the previous definition. Suppose that there exists a calibration ω for M, then M is area minimizing among all the oriented manifold of dimension k having the same boundary of M.*

Proof. Let $N \subset \mathbb{R}^n$ a manifold of dimension k such that $\partial M = \partial N$. Then

$$\mathrm{Vol}(M) = \int_M \omega = \int_N \omega \leq \mathrm{Vol}(N)$$

where the first equality follows by property (***iii***) and the inequality from (***ii***). The second equality is a consequence of the Stokes theorem as M and N have the same boundary and ω is a closed form.

□

Calibration for the Mumford-Shah functional

One can give a very similar notion of calibration for functionals in the following way: given $H : L^1(\Omega) \to \mathbb{R}$ and $u \in L^1(\Omega)$, a calibration for u with respect to H is a functional $G : L^1(\Omega) \to \mathbb{R}$ such that

$$(\star) \quad G(u) = G(v), \qquad (\star\star) \quad H(v) \geq G(v), \qquad (\star\star\star) \quad H(u) = G(u) \qquad (1.11)$$

for all $v \in L^1(\Omega)$ such that $\{v \neq u\} \subset\subset \Omega$.
It is clear that if we can find a calibration for u, then u is automatically a local minimizer in Ω for H, in fact we have

$$H(u) \overset{(\star\star\star)}{=} G(u) \overset{(\star)}{=} G(v) \overset{(\star\star)}{\leq} H(v)$$

for all $v \in L^1(\Omega)$ such that $\{v \neq u\} \subset\subset \Omega$.
However it is easy to see that this definition is too generic to be meaningful. In fact it is clear that given $u \in L^1(\Omega)$ a local minimizer for H, then the constant functional defined as $G(v) = H(u)$ for all $v \in L^1(\Omega)$ is a calibration for u. So it is necessary to restrict the class of calibration we are searching for to find a notion of calibration the resembles the closed form satisfying (***i***), (***ii***) and (***iii***) for minimal surfaces.

In [1] Alberti, Bouchitté and Dal Maso introduced a suitable notion of calibration for the Mumford-Shah functional, building a functional defined on the graph of an SBV function considered as a n-rectifiable set. Before going into the description of the method we give some notations that will be useful in the sequel.
In what follows we assume $\Omega \subset \mathbb{R}^n$ to be open and bounded and we consider the Mumford-Shah functional in the SBV formulation with parameters $\alpha > 0$ and $\beta \geq 0$:

$$\mathscr{F}(v) = \int_\Omega |\nabla v|^2\, dx + \alpha \mathcal{H}^{n-1}(S_v) + \beta \int_\Omega |v - g|^2\, dx,$$

for $v \in SBV(\Omega)$ and $g \in L^\infty(\Omega)$. In this way we deal at the same time with the homogeneous Mumford-Shah functional F (when $\beta = 0$ and $\alpha = 1$) and non homogeneous version MS (when $\alpha = \beta = 1$).

Given $v \in SBV(\Omega)$, we denote by $v^-(x)$ and $v^+(x)$ the lower and the upper traces of v in $x \in \Omega$. Moreover let Γ_v be the extended graph of v defined as

$$\Gamma_v := \{(x,t) \in \Omega \times \mathbb{R} : v^-(x) \leq t \leq v^+(x)\}$$

and let $\mathscr{G}_v$ be the regular part of the graph of v defined as $\mathscr{G}_v := \Gamma_v \setminus (S_v \times \mathbb{R})$.
For standard theory of BV functions (see [27]) Γ_v is n-rectifiable and then it admits an approximate normal that we are going to denote by ν_{Γ_v}.
In [1] the authors searched for a calibration of the following form:

$$G(v) = \int_{\Gamma_v} \phi \cdot \nu_{\Gamma_v}\, d\mathcal{H}^n, \tag{1.12}$$

where $\phi : \Omega \times \mathbb{R} \to \mathbb{R}^{n+1}$ is an L^∞ vector field with additional regularity that will be made precise later. The key point is that, comparing this functional with $\mathscr{F}$, it is possible to find sufficient conditions on ϕ such that G satisfies properties

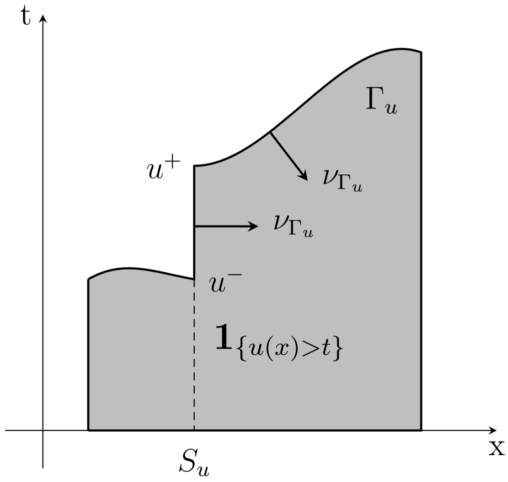

Figure 1.2: The complete graph of an SBV function

($\star$), ($\star\star$) and ($\star\star\star$) with respect to $\mathscr{F}$ for a given $u \in SBV(\Omega)$. Then they call calibration for u the vector field satisfying these properties.
We can split (1.12) in the integration over $\mathscr{G}_v$, the regular part of the graph of v, and the integration on the jump as

$$G(v) = \int_{\mathscr{G}_v} \phi \cdot \frac{(\nabla v, -1)}{\sqrt{|\nabla v|^2 + 1}} \, d\mathcal{H}^n + \int_{S_v} \left(\int_{v^-}^{v^+} \phi^x(x, s) \, ds \right) \cdot \nu_{S_v} \, d\mathcal{H}^{n-1},$$

where ∇v is the approximate gradient of v and ν_{S_v} is the approximate normal to S_v.
We apply the area formula for approximate differentiable functions ([27], Section 3.1.5) to the first term to get

$$G(v) = \int_{\Omega} (\phi^x(x, v(x)) \cdot \nabla v - \phi^t(x, v(x)) \, dx + \int_{S_v} \left(\int_{v^-}^{v^+} \phi^x(x, s) \, ds \right) \cdot \nu_{S_v} \, d\mathcal{H}^{n-1}, \tag{1.13}$$

where $\phi = (\phi^x, \phi^t) \in \mathbb{R}^n \times \mathbb{R}$.
At this point one can compare (1.13) with $\mathscr{F}(v)$ and deduce the following statement (using the inequality $ab \leq a^2/4 + b^2$ on the first term of the sum):

Proposition 1.1.9. *Let $\phi = (\phi^x, \phi^t) \in L^\infty(\Omega \times \mathbb{R}, \mathbb{R}^{n+1})$ be defined everywhere and such that*

a) $\phi^t(x, t) \geq \dfrac{|\phi^x(x, t)|^2}{4} - \beta(t - g(x))^2$ *for $\mathscr{L}^n$-a.e. $x \in \Omega$ and every $t \in \mathbb{R}$,*

b) $\left| \displaystyle\int_{t_1}^{t_2} \phi^x(x, s) \, ds \right| \leq \alpha$ *for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega$ and every $t_1, t_2 \in \mathbb{R}$.*

Then for every $v \in SBV(\Omega)$

$$\mathscr{F}(v) \geq \int_{\Gamma_v} \phi \cdot \nu_{\Gamma_v} \, d\mathcal{H}^{n-1}. \tag{1.14}$$

Moreover given ϕ *satisfying (**a**) and (**b**),* $\mathscr{F}(u) = G(u)$ *for a given* $u \in SBV(\Omega)$ *if and only if*

a') $\phi^x(x, u(x)) = 2\nabla u(x)$ *and* $\phi^t(x, u(x)) = |\nabla u(x)|^2 - \beta(u(x) - g(x))^2$ *for* $\mathscr{L}^n$*-a.e.* $x \in \Omega$,

b') $\displaystyle\int_{u^-}^{u+} \phi^x(x, s) \, ds = \alpha \nu_{S_u}(x)$ *for* $\mathcal{H}^{n-1}$*-a.e.* $x \in S_u$.

Remark 1.1.10. *It is worth to notice at this point that the conditions (**a**') and (**b**'), in absence of (**a**) and (**b**), are sufficient but not necessary to ensure equality in* (1.14)*. For example if* u *is a minimizer of* $\mathscr{F}$ *and* $\phi = (0, -\mathscr{F}(u)/\mathscr{L}^n(\Omega))$ *then* (1.14) *is satisfied for every* $v \in SBV(\Omega)$ *and the equality holds for* u*. However the vector field* ϕ *is not meaningful as a calibration because the functional just defined is the constant one as already remarked in the beginning of this subsection.*

Conditions (***a***), (***b***), (***a'***), (***b'***) are the analogous of the conditions (***i***) and (***iii***) for minimal surfaces. Therefore one needs just to find a Stoke's formula for the Mumford-Shah functional and so it is necessary to consider a slightly more regular class of vector fields than just L^∞. The notion of *approximately vector field* we are going to use was introduced in [1] and it allows the vector field to have jumps along some rectifiable set. This is useful when one wants to exhibit an explicit calibration, as sometimes it is natural to search for discontinuous vector field.

Definition 1.1.5 (Approximately regular vector field)**.** *Given* $A \subset \mathbb{R}^{n+1}$ *a vector field* $\phi : A \to \mathbb{R}^{n+1}$ *is said to be approximately regular it is bounded and for every Lipschitz hypersurface* M *in* $\mathbb{R}^{n+1}$ *there holds*

$$\lim_{r \to 0} \fint_{B_r(x_0) \cap A} |(\phi(x) - \phi(x_0)) \cdot \nu_M(x_0)| \, dx = 0 \tag{1.15}$$

for $\mathcal{H}^n$*-a.e.* $x_0 \in M \cap A$.

We state a divergence theorem for *approximately regular* vector fields. We refer to [1] for a detailed proof of this fact.

Proposition 1.1.11. *Let* U *be an open set in* $\mathbb{R}^{n+1}$ *with Lipschitz boundary. Consider* $u \in BV(\Omega)$ *and let* $\phi : \Omega \times \mathbb{R} \to \mathbb{R}^{n+1}$ *be an approximately regular vector field on* $\overline{U}$ *such that* $\operatorname{div} \phi \in L^\infty(U)$ *and* $u\phi \in L^1(\partial U, \mathcal{H}^n)$*. Then*

$$\int_U \phi \cdot Du = -\int_U u \operatorname{div} \phi \, dx - \int_{\partial U} u\phi \cdot \nu_{\partial U} \, d\mathcal{H}^n,$$

where $\nu_{\partial\Omega}$ *is the inner unit normal to* $\partial\Omega$.

Definition 1.1.6 (Calibration for the Mumford-Shah Functional)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and regular and $u \in SBV(\Omega)$. Given $\phi = (\phi^x, \phi^t) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}^{n+1}$ an approximately regular vector field in $\overline{\Omega} \times \mathbb{R}$, we say that it is a calibration for u if it is divergence free in $\Omega \times \mathbb{R}$ and conditions (**a**), (**b**), (**a'**) and (**b'**) hold.*

Because of Proposition 1.1.9 and Proposition 1.1.11 the following theorem holds (see [1]):

Theorem 1.1.12. *Let $u \in SBV(\Omega)$ and $\phi : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}^{n+1}$ be a calibration for u. Then u is a local minimizer in Ω of the Mumford-Shah functional.*

Remark 1.1.13. *If $\alpha = 1$ and $\beta = 0$ we reduce to the homogeneous Mumford-Shah functional and the conditions (**a**), (**b**), (**a'**) and (**b'**) read as follows:*

a) $\phi^t(x,t) \geq \dfrac{|\phi^x(x,t)|^2}{4}$ *for $\mathscr{L}^n$-a.e. $x \in \Omega$ and every $t \in \mathbb{R}$,*

b) $\left| \displaystyle\int_{t_1}^{t_2} \phi^x(x,s)\, ds \right| \leq 1$ *for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega$ and every $t_1, t_2 \in \mathbb{R}$,*

a') $\phi^x(x, u(x)) = 2\nabla u(x)$ *and* $\phi^t(x, u(x)) = |\nabla u(x)|^2$ *for $\mathscr{L}^n$-a.e. $x \in \Omega$,*

b') $\displaystyle\int_{u^-}^{u^+} \phi^x(x,s)\, ds = \nu_{S_u}(x)$ *for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$.*

Examples of calibrations

In what follows we give some explicit examples of calibrations of relevant minimizers for the homogeneous Mumford-Shah functional (denoted by F) that can be found in [1].
A direct consequence of the properties of the Mumford-Shah is the following remark:

Remark 1.1.14. *It is immediate to prove that given a function $u \in SBV(\Omega)$ and its truncation $u_{M,m}(x) := (u(x) \wedge M) \vee m$ for some $m < M$, we have $F(u_{M,m}) \leq F(u)$. Therefore when we want to prove the minimality of some candidates we can assume without loss of generality that they are bounded between two constants. In addition to that, it is clear that the calibration can be build just in $\overline{\Omega} \times [m, M]$ instead of in the whole domain $\overline{\Omega} \times \mathbb{R}$.*

Before going into some explicit examples we explain a general procedure to construct calibrations, that was employed in [1] and it is connected with the theory of extremal fields of scalar functionals (see [26], Section 6.3). The idea is that we want to embed the graph of the function we want to calibrate in a one-parameter family of graphs of solutions of the Euler-Lagrange equation associated to the

functional F (its absolutely continuous part). Then the calibration will be defined in a suitable way starting from this foliation.
To be precise let U be a subset of $\Omega\times\mathbb{R}$ and suppose that, given $I\subset\mathbb{R}$ an interval, there exists a family $\{u_\mu\}_{\mu\in I}$ of harmonic functions such that:

- the graphs $\mathscr{G}_{u_\mu}$ are pairwise disjoint,
- $\bigcup_{\mu\in I}\mathscr{G}_{u_\mu}=U$.

Define

$$\phi(x,t)\coloneqq(2\nabla u_{\mu(x,t)}(x),|\nabla u_{\mu(x,t)}(x)|^2),\tag{1.16}$$

where $\mu(x,t)$ is defined implicitly as the only $\mu\in I$ such that

$$t=u_\mu(x).\tag{1.17}$$

Clearly ϕ satisfies assumptions (***a***) and (***a'***) for every u_μ. Moreover a simple computation shows that the vector field ϕ constructed by this procedure is divergence free. We reproduce it here.

Lemma 1.1.15. *Suppose in addition to the previous assumptions that $u(\mu,x)\coloneqq u_\mu(x)$ and $\nabla u(\mu,x)$ are C^1 in μ. Moreover assume that $\partial_\mu u(\mu,x)\neq 0$ for every μ and x. Then* $\operatorname{div}\phi=0$.

Proof. Notice that by the implicit function theorem $\mu(x,t)$ is C^1 in both variables. Thus we can compute the divergence of ϕ classically and using the fact that the foliations is made by harmonic functions we obtain:

$$\begin{aligned}\operatorname{div}\phi &= 2\operatorname{div}_x\nabla u_{\mu(x,t)}(x)+2\nabla u_{\mu(x,t)}(x)\cdot\partial_t\nabla u_{\mu(x,t)}(x)\\ &= 2\Delta u_{\mu(x,t)}(x)+2(\partial_\mu\nabla u_{\mu(x,t)}(x))\cdot\nabla\mu(x,t)+2\nabla u_{\mu(x,t)}(x)\cdot\partial_\mu\nabla u_{\mu(x,t)}(x)\partial_t\mu\\ &= 2(\partial_\mu\nabla u_{\mu(x,t)}(x))\cdot(\nabla\mu(x,t)+\nabla u_{\mu(x,t)}(x)\partial_t\mu).\end{aligned}\tag{1.18}$$

By the definition of μ we can differentiate the identity $t=u_{\mu(x,t)}(x)$ in x and in t to get

$$\nabla u_{\mu(x,t)}(x)+\partial_\mu u_{\mu(x,t)}(x)\nabla\mu(x,t)=0\quad\text{and}\quad\partial_\mu u_{\mu(x,t)}(x)\partial_t\mu(x,t)=1.$$

This implies that (1.18) vanishes and so $\operatorname{div}\phi=0$.

□

It turns out that this construction will be useful in order to find calibrations for harmonic functions. However it is not possible to deal with conditions (***b***) and (***b'***) using such a foliation; this will be made clear when we will propose an attempt to calibrate the crack-tip.

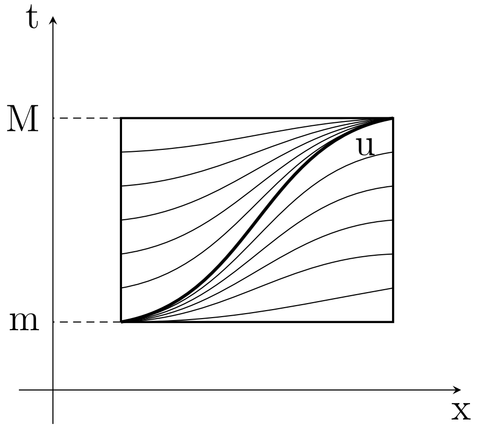

Figure 1.3: The foliation for an harmonic function

Calibration for harmonic functions in dimension n

We are going to prove the statement (***i***) of Proposition 1.1.4 by means of calibrations proposing an argument that can be found in [1]. Let $u : \Omega \to \mathbb{R}$ be an harmonic function in Ω and let $m, M \in \mathbb{R}$ be the minimum and the maximum (respectively) of u in $\overline{\Omega}$. Assuming that

$$\left(\sup_{\Omega} u - \inf_{\Omega} u\right) \|\nabla u\|_{L^\infty} \leq 1,$$

we will build a calibration for u in $\overline{\Omega} \times [m, M]$.
More precisely we will construct a foliation of $\overline{\Omega} \times [m, M]$ by graphs of harmonic functions. Define $u_\mu^m(x) := m + \mu(u(x) - m)$ and $u_\mu^M(x) = M + \mu(u(x) - M)$ defined for $\mu \in [0, 1]$ and consider the foliation induced by the graphs of the family of functions $\{u_\mu^m(x)\}_{\mu\in[0,1]} \cup \{u_\mu^M(x)\}_{\mu\in[0,1]}$. Moreover, as in the construction before, let us define $\mu(x, t)$ in such a way that it satisfies the equation $t = u_{\mu(x,t)}(x)$.
An easy computation shows that the vector field defined as in (1.16) is the following:

$$\phi(x,t) = \begin{cases} \left(2\frac{t-m}{u(x)-m}\nabla u(x), \left(\frac{t-m}{u(x)-m}\right)^2 |\nabla u(x)|^2\right) & \text{if } m \leq t \leq u(x) \\ \left(2\frac{M-t}{M-u(x)}\nabla u(x), \left(\frac{M-t}{M-u(x)}\right)^2 |\nabla u(x)|^2\right) & \text{if } u(x) \leq t \leq M. \end{cases}$$

Thanks to a modification of Lemma 1.1.15 (or by a direct computation) one can prove that $\operatorname{div} \phi = 0$ and assumptions (***a***), (***b***) and (***b'***) are trivially satisfied. In order to verify (***a'***) one needs to check that

$$\left| \int_{t_1}^{t_2} \phi^x(x, s)\, ds \right| \leq 1 \quad \forall t_1, t_2 \in [m, M].$$

It is easy to see that without loss of generality we can verify the previous one for $t_1 = m$ and $t_2 = M$. For this choice of t_1 and t_2 we have

$$\begin{aligned} \int_m^M \phi^x(x,s)\,ds &= \frac{2\nabla u(x)}{u(x)-m}\int_m^{u(x)}(s-m)\,ds + \frac{2\nabla u(x)}{M-u(x)}\int_{u(x)}^M (M-s)\,ds \\ &= \nabla u(x)(u(x)-m) + \nabla u(x)(M-u(x)) = \nabla u(x)(M-m). \end{aligned}$$

Therefore (***a'***) holds provided that $\|\nabla u\|_{L^\infty(\Omega)}\,(\sup_\Omega u - \inf_\Omega u) \leq 1$ as desired.

Calibration for the triple junction

We also sketch the construction of a calibration for the triple junction proving the statement (***iii***) in Proposition 1.1.4 as it was done in [1].
Consider the triple junction defined as follows: let (A_1, A_2, A_3) be the partition of $B(0,R)$ in three sectors of 120 degree (Figure 1.4) and define the SBV function in $B(0,R)$ that is equal to a_i in A_i, where $a_1 < a_2 < a_3$ are distinct constants. We will denote by S_{ij} the discontinuity of u, i.e. three line segments meeting at the origin. Define $e_\pm := (\pm\sqrt{3}/2, -1/2)$ fix $\lambda > 0$. We will define the calibration for

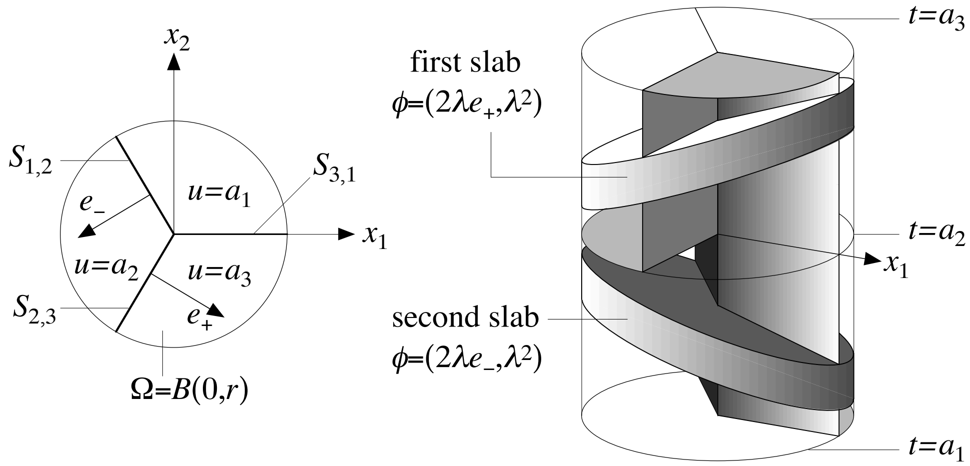

Figure 1.4: The calibration for the triple junction. Courtesy of [1]

the triple junction as follows:

$$\phi(x,t) := \begin{cases} (2\lambda e_-, \lambda^2) & (x,t) : \left| t - \frac{1}{2}(a_1+a_2) - \frac{\lambda}{2}x\cdot e_- \right| < \frac{1}{4\lambda} \\ (2\lambda e_+, \lambda^2) & (x,t) : \left| t - \frac{1}{2}(a_2+a_3) - \frac{\lambda}{2}x\cdot e_+ \right| < \frac{1}{4\lambda} \\ (0,0) & \text{otherwise.} \end{cases}$$

In other words $\phi(x,t)$ is a vector field defined in such a way that conditions (***a'***) and (***b'***) are satisfied in the two slabs in Figure 1.4. Moreover is easy to verify that $\phi(x,t)$ fulfills also conditions (***a***) and (***b***) and it is, due to the tangential jump, an approximately regular vector field provided that one is able to construct the two slabs such that the first one is contained in $B(0,R) \times [a_1, a_2]$ and the second one is contained in $B(0,R) \times [a_2, a_3]$. This condition is equivalent to require that there exists λ such that

$$a_{i+1} - a_i \geq \lambda R + \frac{1}{2\lambda}$$

that can be found iff

$$a_{i+1} - a_i \geq \sqrt{2R}.$$

This is exactly condition (***iii***) in Proposition 1.1.4 as we wanted to prove.

Chapter 2

Existence of calibrations

2.1 Overview

In this chapter we are addressing the issue of the existence of calibration for a minimizer of the Mumford-Shah functional. More precisely given u a Dirichlet minimizer (see Definition 1.1.3) for the Mumford-Shah functional (1.3) in Ω we ask if there exists a calibration according to Definition 1.1.6 for u in Ω.
The strategy to prove this fact is to find a convex functional extending the Mumford-Shah functional in a larger space that has the same minima. Then the conclusion follows by an application of Hahn-Banach Theorem based on the convexity of the extension. We are going to be more precise in what follows, but it is important to remark that the difficulty of this approach lies in proving the equivalence of the minimum problems and this can be overcome by a coarea-type formula for the convex extension of the Mumford-Shah functional. The expert reader can already see at this point the similarities of this approach with the classical theory of Federer about calibrations and flat chains ([24]).
It is worth to remark as well that even if we have faced the problem of relaxing the Mumford-Shah functional to a bigger space with the goal of proving the existence of calibration, the result has a strong practical relevance in the field of image processing. This is because, being able to relax the Mumford-Shah to a convex functional keeping the same minima would allow to run an efficient gradient descent method converging to a minimum ([44]).
The convex extension of the Mumford-Shah relies on a representation formula for the Mumford-Shah functional that is a consequence of the theory of calibration by Alberti, Bouchitté and Dal Maso.

2.2 Relaxation for the Mumford-Shah functional by lifting

Let $\Omega \subset \mathbb{R}^n$ be open, regular and bounded. Given $u : \Omega \to \mathbb{R}$ we denote by $\mathbf{1}_{\{u(x)>t\}}$ the characteristic function of the subgraph of u. Consider the Mumford-Shah functional defined for $u \in SBV(\Omega)$ as

$$\mathscr{F}(u) = \int_\Omega |\nabla u|^2 \, dx + \alpha \mathcal{H}^{n-1}(S_u) + \beta \int_\Omega |u-g|^2 \, dx,$$

with $\alpha > 0$ and $\beta \geq 0$. The following proposition is a representation formula for the Mumford-Shah functional proved in [1] as a consequence of the theory of calibration.

Proposition 2.2.1. *Given $u \in SBV(\Omega)$*

$$\mathscr{F}(u) = \max_{\phi \in K} \int_{\Gamma_u} \langle \phi, \nu_{\Gamma_u} \rangle \, d\mathcal{H}^n = \max_{\phi \in K} \int_{\Omega\times\mathbb{R}} \langle \phi, D\mathbf{1}_{\{u(x)>t\}} \rangle, \tag{2.1}$$

where

$$K = \{\phi : \Omega \times \mathbb{R} \to \mathbb{R}^{n+1}, Borel,\ such\ that\ (\boldsymbol{a})\ and\ (\boldsymbol{a}')\ hold\ pointwise\}.$$

and (***a***)*,* (***a***'*) are the properties of a calibration as defined in Proposition 1.1.9.*

The proof of this fact is immediate from the definition of calibration and one can notice that the maximum in (2.1) can be replaced by a supremum computed on $K \cap C_0(\Omega \times \mathbb{R}, \mathbb{R}^{n+1})$ where $C_0(\Omega \times \mathbb{R}, \mathbb{R}^{n+1})$ the set of continuous vector fields vanishing at $\partial\Omega \times \mathbb{R}$.

2.2.1 Chambolle setting for functional lifting

The representation formula (2.1) is the starting point for the proof of existence of calibration in dimension one, due to Chambolle ([15]). In this subsection we are going to review this results in order to underline the differences with our approach and to give a clear explanation of what is known in the field.
Let $\Omega \subset \mathbb{R}^n$ be a bounded, open and regular set. Define

$$X = \{v(x,t) = \mathbf{1}_{\{u(x)>t\}} : u \in L^1(\Omega, [0,1])\}$$

with the convention that $\mathbf{1}_{\{u(x)>0\}} = 1$ if $u(x) = 0$.
Let $\overline{X}$ be the closure of convex envelope of X in the $L^1(\Omega \times [0,1])$ topology, that is to say the set of Borel functions $v(x,t) : \Omega \times [0,1] \to [0,1]$, nonincreasing in t such that $v(\cdot, 0) = 1$ and $v(\cdot, 1) = 0$. Moreover let us define X_b and $\overline{X}_b$ to be $X \cap BV(\Omega, [0,1])$ and $\overline{X} \cap BV(\Omega, [0,1])$ respectively.
We will denote by $C_0(\Omega \times [0,1], \mathbb{R}^{n+1})$ the set of continuous vector fields vanishing at $\partial\Omega \times [0,1]$ and its dual $M(\Omega \times [0,1], \mathbb{R}^{n+1}) = [C_0(\Omega \times [0,1], \mathbb{R}^{n+1})]'$ the set of vector valued Radon measures.

Remark 2.2.2. *The next results hold for general functionals $F : L^1(\Omega) \to \mathbb{R}$ and can be applied for the particular case of the Mumford-Shah functional (extended to $+\infty$ outside SBV). We will present this theory in the general setting and in the end we will see the application to the Mumford-Shah functional case.*

Let $F : L^1(\Omega) \to \mathbb{R}$ be a functional. Let us define the lift of F in $\overline{X}$ the following way:

$$\mathcal{F}(v) = \begin{cases} F(u) & \text{if } v = \mathbf{1}_{\{u>t\}}\ a.e. \\ +\infty & \text{if } v \in \overline{X} \setminus X. \end{cases}$$

Define in addition $\overline{\mathcal{F}}$ to be the convex lower semicontinuous envelope of $\mathcal{F}$ in the L^1 topology. The content of the following proposition that we propose without a proof (we refer to [15]) is the description of the trace of $\overline{\mathcal{F}}$ on X.

Proposition 2.2.3. *Define*

$$\overline{F}(u) = \overline{\mathcal{F}}(\boldsymbol{1}_{\{u>t\}}) \tag{2.2}$$

for every $u \in L^1(\Omega, [0,1])$*; then* $\overline{F}$ *is the l.s.c. envelope of* F *in the* $L^1(\Omega, [0,1])$ *topology.*

The immediate consequence of the previous definitions and Proposition 2.2.3 is that

$$\inf_{u \in BV(\Omega,[0,1])} F(u) = \min_{v \in \overline{X}_b} \overline{\mathcal{F}}(v) \tag{2.3}$$

and every minimizer of $\overline{\mathcal{F}}$ is convex combination of $\mathbf{1}_{\{u>t\}}$ with u minimizing $\overline{F}$.

After lifting F to the space $\overline{X}$, Chambolle proved a representation formula for $\overline{\mathcal{F}}$ using convex duality arguments. We just state the result and we refer to [15] for the proof.

Theorem 2.2.4 (Chambolle)**.** *Suppose that there exists* $c_1, c_2 \geq 0$ *such that*

$$|Du|(\Omega) \leq c_1 + c_2 F(u) \quad \forall u \in L^1(\Omega, [0,1]), \tag{2.4}$$

then for every $v \in \overline{X}_b$

$$\overline{\mathcal{F}}(v) = \sup_{\phi \in H} \int_{\Omega \times [0,1]} \langle \phi, Dv \rangle \tag{2.5}$$

for some convex $H \subset C_0(\Omega \times \mathbb{R}, \mathbb{R}^{n+1})$*.*

Moreover the set H in Theorem 2.2.4 can be characterized by duality in the following way

$$H = \left\{ \phi \in C_0(\Omega \times \mathbb{R}, \mathbb{R}^{n+1}) : \int_{\Omega \times [0,1]} \langle \phi, Dv \rangle \leq \overline{\mathcal{F}}(v) \ \forall v \in \overline{X}_b \right\} \tag{2.6}$$

and it is not difficult to check using the definition of $\mathcal{F}$ that

$$H = \left\{ \phi \in C_0(\Omega \times \mathbb{R}, \mathbb{R}^{n+1}) : \int_{\Omega \times [0,1]} \langle \phi, D\mathbf{1}_{\{u>t\}} \rangle \leq F(u) \ \forall u \in BV(\Omega, [0,1]) \right\}. \tag{2.7}$$

One can clearly apply the previous relaxation to Mumford-Shah type functionals of the form

$$F(u) = \begin{cases} \displaystyle\int_\Omega f(x, u(x), \nabla u)\, dx + \int_{S_u} \psi(x, u^-, u^+)\, d\mathcal{H}^{n-1} & \text{if } u \in SBV(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{2.8}$$

where ψ and f satisfy the classical assumptions that make the previous functional lower semicontinuous and coercive according to condition (2.4) ([15]).

Moreover if we assume the following additional continuity hypothesis of f and ψ:

(R1) there exists a modulus of continuity $\omega(\xi, \tau)$ such that

$$|f(x,t,p) - f(x',t',p)| \le \omega(x'-x, t'-t)(1 + f(x,t,p)),$$

(R2) ψ continuous on $\Omega \times [0,1]^2$ and $\psi \ge C > 0$,

then there exists an analogous of the formula (2.1) for general free discontinuity functionals:

$$F(u) = \sup_{\phi \in \mathcal{K}} \int_{\Omega \times [0,1]} \langle \phi, D\mathbf{1}_{\{u>t\}} \rangle \tag{2.9}$$

with $\mathcal{K}$ defined as follows:

$$\mathcal{K} = \Big\{ \phi \in C_0(\Omega \times [0,1], \mathbb{R}^{n+1}) : \phi^t(x,t) \ge f^*(x,t,\phi^x(x,t)) \ , \\ \int_{t_1}^{t_2} \phi^x(x,s)\, ds \le \psi(x,t_1,t_2), \text{ for every } t_1, t_2 \in [0,1] \Big\},$$

where $f^*(x,t,\cdot)$ is the Legendre-Fenchel transform of $f(x,t,\cdot)$.
Therefore it is possible to define an extension of the Mumford-Shah functional simply substituing $\mathbf{1}_{\{u>t\}}$ with $v \in \overline{X}_b$ to obtain the convex funtional $\mathscr{G}$ defined as

$$\mathscr{G}(v) = \sup_{\phi \in \mathcal{K}} \int_{\Omega \times [0,1]} \langle \phi, Dv \rangle \tag{2.10}$$

for every $v \in \overline{X}_b$
It is easy to see that by the definition of K and the computations carried out in Section 1.1.5 we have $\mathcal{K} \subset H$, so that in general $\mathscr{G}(v) \le \overline{\mathcal{F}}(v)$ in $\overline{X}_b$. One can ask if the functionals are indeed the same and this is the question addressed by Chambolle in [15] in dimension one with the following theorem:

Theorem 2.2.5. *If $\Omega = I \subset \mathbb{R}$ is an interval, then $\mathscr{G}(v) = \overline{\mathcal{F}}(v)$ for every $v \in \overline{X}_b$.*

The first consequence is that one can recover standard relaxation results for coercive functional defined in BV (see [11]) and thanks to (2.3) an other important result is the following theorem that links minimizers of F with minimizers of $\mathscr{G}$.

Theorem 2.2.6. *Let $u \in SBV(I)$ and minimizer for the Mumford-Shah functional, then $\mathbf{1}_{\{u>t\}}$ is a minimizer for $\mathscr{G}$ in $\overline{X}_b$*

This implies the existence of calibrations in the following weak asymptotic sense.

Theorem 2.2.7. *Given $u \in SBV(I)$ a minimizer for F, there exists a sequence $\phi_n \in \mathcal{K}$ such that* $\operatorname{div} \phi_n \to 0$ *in the distributional sense and*

$$\int_{\Omega\times[0,1]} \langle \phi_n, D\mathbf{1}_{\{u>t\}}\rangle \to F(u).$$

Thanks to Theorem 2.2.7 one can infer that if the sequence ϕ_n has a limit in $\mathcal{K}$, then it is a calibration for F according to Definition 1.1.6.

2.2.2 Non-validity of a Coarea formula for the lifting of the Mumford-Shah functional

It is interesting to notice that one can prove the same result of Theorem 2.2.6 (even in higher dimension) if $\mathscr{G}$ satisfies a generalized coarea formula of the form

$$\mathscr{G}(v) = \int_0^1 \mathscr{G}(\mathbf{1}_{\{v(x,t)>s\}})\, ds \tag{2.11}$$

for $v \in \overline{X}_b$.
Unfortunately this is false even in dimension one. Indeed it is enough to consider

$$u_1(x) = \begin{cases} 0 & \text{if } x \le 1/2 \\ x & \text{if } x > 1/2, \end{cases} \qquad u_2(x) = \begin{cases} x & \text{if } x \le 1/2 \\ 1 & \text{if } x > 1/2 \end{cases}$$

and $v(x,t) = (1/2)\mathbf{1}_{\{u_1(x)>t\}} + (1/2)\mathbf{1}_{\{u_2(x)>t\}}$ to see that formula (2.11) does not hold.

In the next section we will present our setting for the relaxation of the Mumford-Shah functional by lifting to the space of rectifiable currents and we will show how to obtain a generalized coarea formula in order to prove the equivalent of Theorem 2.2.6.

2.2.3 A relaxation of the Mumford-Shah functional in the space of rectifiable currents

An alternative way to perform the extension presented in the previous section is to consider the functional taking values in $\mathcal{R}_n(\Omega \times \mathbb{R})$, the n-dimensional rectifiable currents with real multiplicity. We refer Section 4.2 in the Appendix for a short introduction on this topic and to [27] and [23] for a comprehensive treatise.
Given $T = (\mathcal{M}, \theta, \xi) \in \mathcal{R}_n(\Omega \times \mathbb{R})$ we define

$$G(T) := \sup_{\phi\in K} \int_{\mathcal{M}} \langle \phi, \star(-\xi)\rangle d\|T\| = \sup_{\phi\in K} \int_{\mathcal{M}} \theta\langle \phi, \nu_T\rangle d\mathcal{H}^{n-1} \tag{2.12}$$

where $\nu_T := -(\star\xi)$ and $\star$ is the Hodge star.

Proposition 2.2.8. *The functional G satisfies the following properties:*

(*i*) *It is convex on $\mathcal{R}_n(\Omega \times \mathbb{R})$.*

(*ii*) *It is lower semicontinous with respect to the mass bounded convergence.*

(*iii*) *Given $u \in SBV(\Omega)$, $G(\Gamma_u) = \mathscr{F}(u)$.*

Proof. Property (i) follows from the definition of G and (iii) is a consequence of the representation formula (2.1). Finally (ii) follows from an easy modification of a classical result in Calculus of Variation (see [27] Section 3.3.1).

□

In the next section we are going to prove a discrete coarea-type formula for the functional G in one dimension (we will work in the setting of currents, but without any modification the same formula holds for characteristic functions of sub level sets of graphs as in Chambolle approach). We will call it *discrete*, meaning that it holds for a finite linear combination of graphs. Then we apply this formula to prove that given u a Dirichlet minimizer for $\mathscr{F}$ then Γ_u is a Dirichlet minimizer of G in a suitably chosen vector space of rectifiable currents. In addition we will show that this result implies the existence of a calibration as a functional defined on currents in a weak sense (Definition 2.4.1) by an application of Hahn-Banach theorem.

2.3 A discrete Coarea-type formula for the Mumford-Shah functional in dimension one

We restrict our analysis to the case $n = 1$. We can also assume $\Omega = I$ an open interval and consider the Mumford-Shah functional in its general form in one dimension:

$$\mathscr{F}(u) := \int_I |u'(x)|^2 \, dx + \beta \int_I |u - g|^2 \, dx + \alpha \mathcal{H}^0(S_u) \tag{2.13}$$

where $\alpha > 0$, $\beta \geq 0$, $g \in L^\infty(I)$ and $u \in SBV(I)$.

Remark 2.3.1. *Even if we restrict our attention to* (2.13) *it is important to remark that as in [15] the results of this section hold for a more general class of functionals with minor modification of the proofs. Functionals of the form*

$$F(u) = \int_I f(u'(x), u(x), x) \, dx + \sum_{x \in S_u} \psi(x, u^+(x), u^-(x))$$

with suitable hypothesis on f and ψ necessary to ensure the lower semicontinuity of F and the existence of minimizers can be treated by this theory. We refer to [2] *for the precise assumptions and we stress the fact that in our setting f need not to be assumed more regular as in [15]. In particular we do not need assumptions (**R1**) and (**R2**) of the previous subsection. Moreover for the case of the Mumford-Shah functional g can be taken in L^∞ without affecting the proof, while in [15] the function g needs to have a l.s.c. and a u.s.c. representatives in L^∞.*

If we consider the functional $\mathscr{F}$ as defined in (2.13), its convex lift defined in (2.12) on $\mathcal{R}_1(I \times \mathbb{R})$ reads

$$G(T) = \sup_{\phi \in K} \int_{\mathcal{M}} \theta \langle \phi, \nu_T \rangle d\mathcal{H}^1 \tag{2.14}$$

for every $T = (\mathcal{M}, \theta, \xi)$.
In particular K is the set of $\phi : I \times \mathbb{R} \to \mathbb{R}^2$, Borel, such that

I) $\phi^t(x,t) \geq \dfrac{|\phi^x(x,t)|^2}{4} - \beta(t-g)^2$ for all $x \in I$ and for all $t \in \mathbb{R}$,

II) $\left| \displaystyle\int_{t_1}^{t_2} \phi^x(x,t)\, dt \right| \leq \alpha$ for all $x \in I$ and for all $t_1, t_2 \in \mathbb{R}$.

We are going to consider as the domain of G the cone $C \subset \mathcal{R}_1(I \times \mathbb{R})$ made by finite linear combination of SBV graphs:

$$C := \left\{ T = \sum_{i=1}^{k} \lambda_i \Gamma_{u_i} : k \in \mathbb{N}, \lambda_i \in \mathbb{R}_+, u_i \in SBV(I) \right\}. \tag{2.15}$$

In order to avoid any confusion we stress that u^- is the trace of u from the left and u^+ is the trace of u from the right.
Moreover for every $T \in C$ we will assume implicitly that, being a rectifiable current, it is defined by the triple $T = (\mathcal{M}, \theta, \xi)$.

2.3.1 Simplifying the cone C

From the definition of the cone C in (2.15) one easily notices that for every current $T \in C$ there exists different combinations of SBV graphs $\{u_i\}$ that represent it. In particular there are some configurations we would like to avoid and this subsection is devoted to make this simplifications for C.

Definition 2.3.1. *Given $\{u_i\}_{i=1\ldots k} \subset SBV(I)$. We say that the family $\{u_i\}_{i=1\ldots k}$ has* cancellation on the jumps *if there exists l_1, l_2 and $x_0 \in S_{u_{l_1}} \cap S_{u_{l_2}}$ such that*

$$u^-_{l_1}(x_0) < u^+_{l_1}(x_0), \quad u^-_{l_2}(x_0) > u^+_{l_2}(x_0), \quad u^+_{l_1}(x_0) > u^+_{l_2}(x_0).$$

We need a lemma that ensures that we can rearrange the graphs in order not to have this cancellation.

Lemma 2.3.2. *Given $T = \sum_{i=1}^{k} \lambda_i \Gamma_{u_i} \in C$ there exists $l \in \mathbb{N}$, $w_i \in SBV(I)$ and $\mu_i \in \mathbb{R}^+$ for $i = 1 \ldots l$ such that $T = \sum_{i=1}^{l} \mu_i \Gamma_{w_i}$ and there is no cancellation on the jumps.*

Proof. Given $T = \sum_{i=1}^{k} \lambda_i \Gamma_{u_i}$ let us suppose that we have cancellation between Γ_{u_1} and Γ_{u_2} in $A \subset S_{u_1} \cap S_{u_2}$ and $\lambda_1 \geq \lambda_2$ (without loss of generality). As A is countable we will denote it by the sequence $\{x_1, x_2, \ldots\}$ possibly infinite. Given $I = (a, b)$ consider the new sequence $\{a = x_0, x_1, x_2, \ldots\}$ and define two SBV functions in the following way:

$$w_1(x) = \begin{cases} u_1(x) & \text{for } x_{i-1} < x \leq x_i,\ i \geq 1 \text{ and odd} \\ u_2(x) & \text{for } x_{i-1} < x \leq x_i,\ i \geq 1 \text{ and even} \end{cases}$$

and

$$w_2(x) = \begin{cases} u_2(x) & \text{for } x_{i-1} < x \leq x_i,\ i \geq 1 \text{ and odd} \\ u_1(x) & \text{for } x_{i-1} < x \leq x_i,\ i \geq 1 \text{ and even.} \end{cases}$$

Then we have that $\lambda_2 \Gamma_{w_1} + \lambda_2 \Gamma_{w_2} + (\lambda_1 - \lambda_2) \Gamma_{u_1} = \lambda_1 \Gamma_{u_1} + \lambda_2 \Gamma_{u_2}$. Hence we produce a decomposition of $\lambda_1 \Gamma_{u_1} + \lambda_2 \Gamma_{u_2}$ that has no cancellation on the jumps. It is easy to check that one can repeat this operation for any pair of graphs that has cancellation on jumps and that this procedure ends in a finite number of steps.

□

From now on we will assume that given $T = \sum_i \lambda_i \Gamma_{u_i} \in C$, the graphs composing T have no cancellation on jumps.
In what follows we will need for technical reasons to have the graphs ordered. Clearly this is possible when we have superposition of graphs with the same multiplicity. In particular we need the following decomposition theorem ([3]) that we state for the reader convenience.

Theorem 2.3.3 (Ambrosio, Crippa, Le Floch)**.** *Let $T \in I^1(\mathbb{R}^2)$ be an integer rectifiable current satisfying the zero boundary condition $\partial T = 0$, the positivity condition $T \llcorner dx \geq 0$ and the cylindrical mass condition $\|T\|(B(0,R) \times \mathbb{R}) < \infty$ for every R. Then there exists a unique family of functions $w_i \in BV_{loc}(\mathbb{R})$ satisfying $w_1 \leq w_2 \leq \ldots \leq w_l$. Such that*

$$T = \sum_{i=1}^{l} \Gamma_{w_i} \quad \textit{and} \quad \|T\| = \sum_{i=1}^{l} \|\Gamma_{w_i}\|.$$

Proposition 2.3.4. *Given $T = \sum_{i=1}^{k} \Gamma_{u_i} \in C$ there exists $w_1 \leq \ldots \leq w_l \in SBV(I)$ such that $T = \sum_{i=1}^{l} \Gamma_{w_i}$.*

Proof. Notice that the current T is integer rectifiable and $T \llcorner dx \geq 0$. As we have assumed that T does not have cancellation on jumps thanks to Lemma 2.3.2 we have

$$\|T\| = \sum_{i=1}^{k} \|\Gamma_{u_i}\|. \tag{2.16}$$

Moreover by extending each function u_i as a constant outside I we can apply Theorem 2.3.3 to T to get the following representation:

$$T = \sum_{i=1}^{l} \Gamma_{w_i}$$

where $w_i \in BV(I)$ and they are ordered in an increasing way.
It remains to show that $w_i \in SBV(I)$, $\forall i$. By Theorem 2.3.3 and (2.16) one has that for every measurable set $C \subset I$ with $\mathscr{L}^1(C) = 0$

$$\sum_{i=1}^{k} \|\Gamma_{u_i}\|(C \times \mathbb{R}) = \|T\|(C \times \mathbb{R}) = \sum_{i=1}^{l} \|\Gamma_{w_i}\|(C \times \mathbb{R}). \tag{2.17}$$

By standard results on the graph of BV functions (see [27]) one has

$$\|\Gamma_{w_i}\|(C \times \mathbb{R}) = |\mu(Dw_i)|(C) \tag{2.18}$$

where $\mu(Dw_i) = (Dw_i, -\mathscr{L}^1)$. So from (2.17) and (2.18) and the fact the C is negligible it follows that

$$\sum_{i=1}^{l} |Dw_i|(C) = \sum_{i=1}^{k} |Du_i|(C)$$

and thus

$$\sum_{i=1}^{l} (|D^j w_i|(C) + |D^c w_i|(C)) = \sum_{i=1}^{k} (|D^j u_i|(C) + |D^c u_i|(C)).$$

Choose $C = \bigcup_{i=1}^{k} S_{u_i} = \bigcup_{i=1}^{l} S_{w_i}$ a countable measurable set; as the Cantor part of the derivative of a BV function is a diffuse measure we have

$$\sum_{i=1}^{l} |D^j w_i| = \sum_{i=1}^{k} |D^j u_i|.$$

Hence

$$\sum_{i=1}^{l} |D^c w_i| = \sum_{i=1}^{k} |D^c u_i|,$$

that implies that $w_i \in SBV(I)$ for every $i = 1, \ldots, k$.

□

2.3.2 Properties of the regular part of $G(T)$

Definition 2.3.2 (Regular part and singular part of T)**.** *We define the singular part of $T \in C$ as*

$$S_T := \bigcup S_{u_i} \tag{2.19}$$

and the regular part as $R_T := I \setminus S_T$.

Remark 2.3.5. *One can easily notice that if we assume that the graphs do not have cancellation according to Lemma 2.3.2, S_T is well defined, so it does not depend on the representation of T.*

Given a measurable set $A \subset I$ we define the localized version of G as

$$G(T, A) := \sup_{\phi \in K} \int_{\mathcal{M} \cap (A \times \mathbb{R})} \langle \phi, \nu_T \rangle d\|T\|.$$

Remark 2.3.6. *It is clear that given A_1, A_2 disjoint measurable sets we have*

$$G(T, A_1 \cup A_2) = G(T, A_1) + G(T, A_2)$$

so in particular

$$G(T) = G(T, S_T) + G(T, S_R). \tag{2.20}$$

Moreover when one computes the localized functional, it is possible to restrict the set K accordingly:

$$G(T, A) = \sup_{\phi \in K_A} \int_{\mathcal{M} \cap (A \times \mathbb{R})} \langle \phi, \nu_T \rangle d\|T\|.$$

where K_A is the set of $\phi : I \times \mathbb{R} \to \mathbb{R}$, Borel, such that

- $\phi^t(x,t) \geq \dfrac{|\phi^x(x,t)|^2}{4} - \beta(t-g)^2 \quad \forall x \in A$ *and* $\forall t \in \mathbb{R}$,
- $\left| \displaystyle\int_{t_1}^{t_2} \phi^x(x,t)\, dt \right| \leq \alpha$ *for every $x \in A$ and for all $t_1, t_2 \in \mathbb{R}$.*

We are presenting a proposition that allows us to split $G(T, R_T)$ as the sum of $\lambda_i G(\Gamma_{u_i}, R_T)$.

Proposition 2.3.7. *Given $T = \sum_{i=1}^k \lambda_i \Gamma_{u_i} \in C$, then*

$$G(T, R_T) = \sum_{i=1}^k \lambda_i G(\Gamma_{u_i}, R_T) = \sum_{i=1}^k \lambda_i \left(\alpha \int_I (u_i')^2\, dx + \beta \int_I |u_i - g|^2\, dx \right). \tag{2.21}$$

In order to give a proof of this fact we need some preliminary lemmas that simplifies the situation.

Lemma 2.3.8. *Given $T = \sum_{i=1}^{k} \lambda_i \Gamma_{u_i} \in C$ let $A \subset I$ be a measurable set such that $A \cap S_T = \emptyset$ and $\mathcal{H}^1(\Gamma_{u_i} \cap \Gamma_{u_j} \cap (A \times \mathbb{R})) = 0$ for every $i \neq j$. Then*

$$G(T, A) = \sum_i \lambda_i G(\Gamma_{u_i}, A).$$

Proof. By induction it is enough to show that given, $T_1 = \sum_{i=1}^{k-1} \lambda_i \Gamma_{u_i}$ and $T_2 = \lambda_k \Gamma_{u_k}$ one has

$$G(T_1 + T_2, A) = G(T_1, A) + G(T_2, A).$$

Fix $\varepsilon > 0$. For $i = 1, 2$ there exist $\phi_i \in K_A$ such that

$$\int_{\mathcal{M}_i \cap (A \times \mathbb{R})} \langle \phi_i, \nu_{T_i} \rangle \, d\|T_i\| \geq G(T_i, A) - \varepsilon.$$

Define the following vector field

$$\tilde{\phi} = \begin{cases} \phi_1 & (x,t) \in \mathcal{M}_1 \setminus \mathcal{M}_2 \\ \phi_2 & (x,t) \in \mathcal{M}_2 \setminus \mathcal{M}_1 \\ 0 & \text{otherwise.} \end{cases}$$

Let prove that $\tilde{\phi} \in K_A$.
For every $x \in A$ we have that $x \notin S_T$ by hypothesis, so that (II) is satisfied and (I) is trivial by definition. Moreover, as $\mathcal{H}^1(\mathcal{M}_1 \cap \mathcal{M}_2 \cap (A \times \mathbb{R})) = 0$, one has

$$\int_{(\mathcal{M}_1 \cup \mathcal{M}_2) \cap (A \times \mathbb{R})} \langle \tilde{\phi}, \nu_T \rangle \, d\mathcal{H}^1 = \int_{\mathcal{M}_1 \cap (A \times \mathbb{R})} \langle \phi_1, \nu_{T_1} \rangle \, d\mathcal{H}^1 + \int_{\mathcal{M}_2 \cap (A \times \mathbb{R})} \langle \phi_2, \nu_{T_2} \rangle \, d\mathcal{H}^1.$$

So

$$\begin{aligned} G(T_1, A) + G(T_2, A) &\leq \int_{\mathcal{M}_1 \cap (A \times \mathbb{R})} \langle \phi_1, \nu_{T_1} \rangle \, d\mathcal{H}^1 + \int_{\mathcal{M}_2 \cap (A \times \mathbb{R})} \langle \phi_2, \nu_{T_2} \rangle \, d\mathcal{H}^1 + 2\varepsilon \\ &\leq G(T_1 + T_2, A) + 2\varepsilon. \end{aligned}$$

Sending ε to zero we obtain the first inequality. The opposite one comes directly from the convexity of G.

□

Lemma 2.3.9. *Given $T = \sum_{i=1}^{k} \lambda_i \Gamma_{u_i} \in C$ let $A \subset I$ be a measurable set such that $A \cap S_T = \emptyset$. Then*

$$G(T, A) = \sum_i \lambda_i G(\Gamma_{u_i}, A).$$

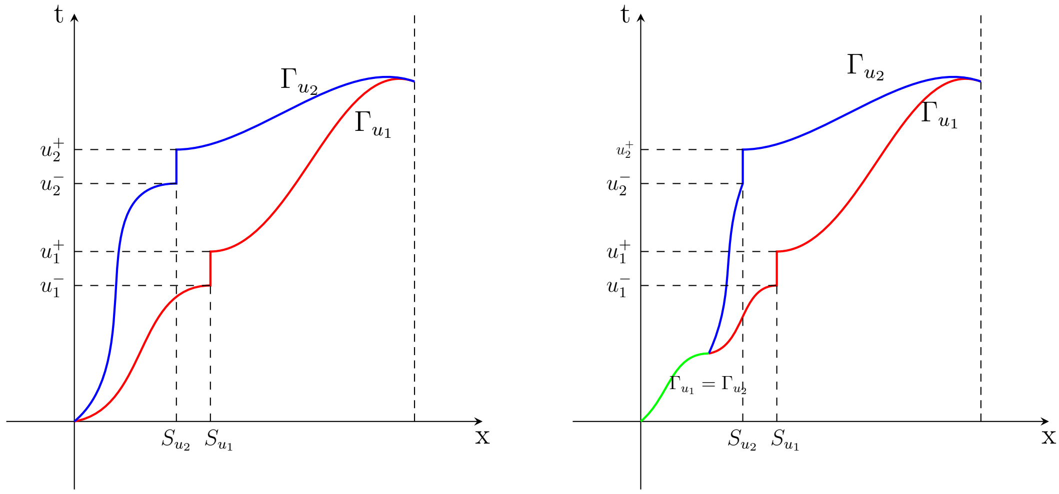

Figure 2.1: Configuration in Lemma 2.3.8 and 2.3.9

Proof. Given $T \in C$, let J be a set of indexes. Denote by $\Gamma = \bigcap_{i \in J} \Gamma_{u_i}$ an intersection of graphs and let $\theta = \sum_{i \in J} \lambda_i$ be the multiplicity on Γ. So

$$\begin{aligned}
\sup_{\phi \in K} \int_{\Gamma \cap (A \times \mathbb{R})} \langle \phi, \nu_T \rangle \, d\|T\| &= \sup_{\phi \in K} \int_{\Gamma \cap (A \times \mathbb{R})} \theta \langle \phi, \nu_T \rangle \, d\mathcal{H}^1 \\
&= \sup_{\phi \in K} \int_{\Gamma \cap (A \times \mathbb{R})} \sum_{i \in J} \lambda_i \langle \phi, \nu_T \rangle \, d\mathcal{H}^1 \\
&= \sum_{i \in J} \lambda_i \sup_{\phi \in K} \int_{\Gamma \cap (A \times \mathbb{R})} \langle \phi, \nu_T \rangle \, d\mathcal{H}^1 .
\end{aligned}$$

Clearly this can be repeated for every intersection of an arbitrary number of graphs. Combining this result with Lemma 2.3.8 we have the thesis.

□

Proof of Proposition 2.3.7

Proposition 2.3.7 is a direct consequence of Lemma 2.3.9 choosing $A = S_R$ and the second equality in (2.21) follows from Proposition 2.2.8.

□

2.3.3 Properties of the singular part of $G(T)$

In this section we are going to study the properties of $\mathcal{G}(T) := G(T, S_T)$.
Given $T = \sum_{i=1}^{k} \lambda_i \Gamma_{u_i} \in C$ and calling $\nu_T = ((\nu_T)^x, (\nu_T)^t)$, by (2.14) we have

$$\mathcal{G}(T) = \sup_{\phi \in K} \int_{\mathcal{M} \cap (S_T \times \mathbb{R})} \theta \phi^x (\nu_T)^x \, d\mathcal{H}^1$$

and it is easy to see that

$$(\nu_T)^x(x,t) = \begin{cases} +1 & (x,t) \in S_{u_i} \times (u_i^-, u_i^+) \\ -1 & (x,t) \in S_{u_i} \times (u_i^+, u_i^-). \end{cases}$$

Hence

$$\mathcal{G}(T) = \sup_{\phi \in K} \sum_{i=1}^{k} \int_{S_{u_i} \times (u_i^-, u_i^+)} \theta \phi^x \, d\mathcal{H}^1.$$

From now on we will work with linear combinations of graphs with the same multiplicity. We will see later the reason why we can reduce to this situation. We want to prove that, given $T = \sum_i \Gamma_{u_i}$, $\mathcal{G}(T)$ can be written as the sum of $\mathcal{G}(\Gamma_{u_i})$ in all the configurations in which there is non-adjacency of the jumps of the graphs.

Theorem 2.3.10. *Consider $T \in C$ such that $T = \sum_{i=1}^k \Gamma_{u_i}$ and u_i are ordered in an increasing way. Suppose that for every $i = 1 \dots k$*

$$\{x \in S_{u_i} \cap S_{u_{i+1}} : u_i^+(x) = u_{i+1}^-(x) \text{ or } u_i^-(x) = u_{i+1}^+(x)\} = \emptyset.$$

Then

$$\mathcal{G}\left(\sum_{i=1}^k \Gamma_{u_i}\right) = \sum_{i=1}^k \mathcal{G}\left(\Gamma_{u_i}\right).$$

Remark 2.3.11. *The assumption of the ordering of the graphs is not essential as given $T \in C$ with graphs of the same multiplicity, by Proposition 2.3.4 is always possible to find an alternative representation by ordered graphs.*

Remark 2.3.12. *Notice that without loss of generality we can prove the previous statement restricting the functional $\mathcal{G}$ to every $x \in S_T$. So the lemmas needed to prove Theorem 2.3.10 will be stated for a fixed point $x \in S_T$.*

For sake of clarity we propose two lemmas (Lemma 2.3.13 and 2.3.14) that deals with a simple situation that is enough to explain the general strategy (See Figure 2.2). Then, in Proposition 2.3.15 and 2.3.16, we generalize this procedure and finally we prove the Theorem.

Lemma 2.3.13. *Consider $T = \sum_{i=1}^k \Gamma_{u_i} \in C$ such that u_i are ordered in an increasing way. Fix $x \in S_T$ and suppose that we have $u_i^-(x) \le u_i^+(x)$ for every $i = 1 \dots k$. Suppose in addition that*

$$u_i^+(x) < u_j^-(x) \quad \text{for every} \quad i < j.$$

Then

$$\mathcal{G}(T, \{x\}) = \sum_{i=1}^k \mathcal{G}(\Gamma_{u_i}, \{x\}) = \alpha |\{i : x \in S_{u_i}\}|.$$

In addition the the maximum is achieved and letting ϕ_T be the vector field realizing the maximum for T

$$\phi_T^x(x,t) = \alpha/(u_i^+ - u_i^-) \quad \textit{for every} \quad t \in (u_i^-, u_i^+)$$

for every $i = 1 \dots k$ such that $x \in S_{u_i}$.

Proof. By induction it is enough to prove that for $T = T_1 + T_2$ where $T_1 = \sum_{i=1}^{k-1} \Gamma_{u_i}$ and $T_2 = \Gamma_{u_k}$ one has

$$\mathcal{G}(T_1 + T_2, \{x\}) = \mathcal{G}(T_1, \{x\}) + \mathcal{G}(T_2, \{x\})$$

and

$$\phi_T^x(x,t) = \alpha/(u_k^+ - u_k^-) \quad \text{for every} \quad t \in (u_k^-, u_k^+).$$

(We suppose $x \in S_{u_k}$ because if not, there is nothing to prove).
For the inductive hypothesis we have that for all $i = 1 \dots k-1$

$$\phi_{T_1}^x(x,t) = \alpha/(u_i^+ - u_i^-) \quad \text{for every} \quad t \in (u_i^-, u_i^+).$$

For the general theory of calibration we have that, calling ϕ_{T_2} the vector field realizing the maximum in $\mathcal{G}(T_2, \{x\})$,

$$\phi_{T_2}^x(x,t) = \alpha/(u_k^+ - u_k^-) \quad \text{for every} \quad t \in (u_k^-, u_k^+),$$

because

$$\int_{u_k^-}^{u_k^+} \phi_{T_2}^x(x) = \alpha \quad \text{for every } x \in S_{u_k}.$$

Define the following vector field on $\{x\} \times \mathbb{R}$:

$$\tilde{\phi} = \begin{cases} \phi_{T_1} & (x,t) \in \{x\} \times (u_1^-, u_{k-1}^+), \\ \phi_{T_2} & (x,t) \in \{x\} \times (u_k^-, u_k^+), \\ \{-\alpha/(u_k^- - u_{k-1}^+), \frac{(\tilde{\phi}^x)^2}{4} - \beta(t-g)^2\} & (x,t) \in \{x\} \times (u_{k-1}^+, u_k^-), \\ 0 & \text{otherwise}. \end{cases}$$

Let prove that $\tilde{\phi} \in K_{\{x\}}$.

$$\begin{aligned} \left| \int_{t_1}^{t_2} \tilde{\phi}(x,t)\, dt \right| &= \left| \int_{t_1}^{u_{k-1}^-} \phi_{T_1}^x(x,t)\, dt - \alpha + \int_{u_k^+}^{t_2} \phi_{T_2}^x(x,t)\, dt \right| \\ &= \left| \alpha \frac{(u_{k-1}^- - t_1)}{(u_{k-1}^- - u_1^-)} - \alpha + \alpha \frac{(t_2 - u_k^+)}{(u_k^+ - u_k^-)} \right| \le \alpha \end{aligned}$$

for every $t_1 \le u_1^-$, $t_2 \ge u_k^+$. As in all the other cases the computation is similar, then $\tilde{\phi} \in K_{\{x\}}$. Therefore

$$\mathcal{G}(T_1, \{x\}) + \mathcal{G}(T_2, \{x\}) = \int_{\mathcal{M} \cap (\{x\} \times \mathbb{R})} \langle \tilde{\phi}, \nu_T \rangle \theta \, d\mathcal{H}^1 \le \mathcal{G}(T, \{x\}).$$

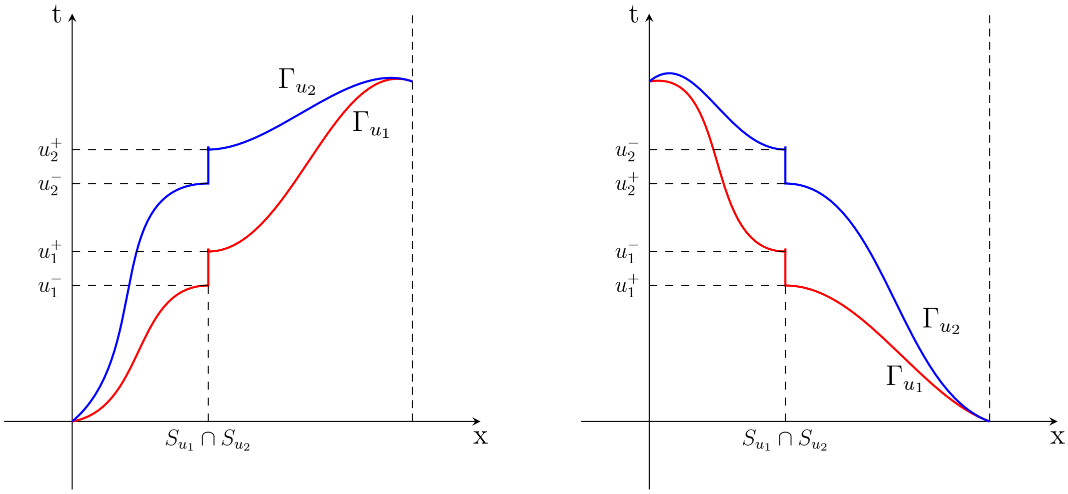

Figure 2.2: Configuration in Lemma 2.3.13 and in Lemma 2.3.14

On the other hand by convexity

$$\mathcal{G}(T, \{x\}) \leq \mathcal{G}(T_1, \{x\}) + \mathcal{G}(T_2, \{x\}) = \int_{\mathcal{M}\cap(\{x\}\times\mathbb{R})} \langle \tilde{\phi}, \nu_T \rangle \theta \, d\mathcal{H}^1.$$

So the thesis follows.

□

We can prove the analogue:

Lemma 2.3.14. *Given* $T = \sum_{i=1}^k \Gamma_{u_i} \in C$ *such that* u_i *are ordered in an increasing way. Fix* $x \in S_T$ *and suppose that we have* $u_i^+(x) \leq u_i^-(x)$ *for every* $i = 1 \dots k$*. Suppose in addition that*

$$u_i^+(x) > u_j^-(x) \quad \textit{for every} \quad i > j.$$

Then

$$\mathcal{G}(T, \{x\}) = \sum_{i=1}^k \mathcal{G}(\Gamma_{u_i}, \{x\}) = \alpha |\{i : x \in S_{u_i}\}|.$$

In addition the the maximum is achieved and letting ϕ_T *be the vector field realizing the maximum for* T

$$\phi_T^x(x,t) = \alpha/(u_i^+ - u_i^-) \quad \textit{for every} \quad t \in (u_i^-, u_i^+)$$

for every $i = 1 \dots k$ *such that* $x \in S_{u_i}$*.*

Proof. See Lemma 2.3.13.

□

We are now in position to prove two general statements that are generalizations of Lemmas 2.3.13 and 2.3.14.

Proposition 2.3.15. *Consider $T \in C$ such that $T = \sum_{i=1}^{k} \Gamma_{u_i}$ and u_i are ordered in an increasing way. Fix $x \in S_T$ and suppose that we have $u_i^-(x) \le u_i^+(x)$ for every $i = 1 \dots k$. Moreover assume that $u_i^+(x) \neq u_{i+1}^-(x)$ for every i such that $x \in S_{u_i}$.*
Then

$$\mathcal{G}(T, \{x\}) = \sum_{i=1}^{k} \mathcal{G}(\Gamma_{u_i}, \{x\}).$$

Proof. We can assume without loss of generality that $x \in S_{u_i}$ for every $i = 1 \dots k$. It is easy to see that $T \llcorner (\{x\} \times \mathbb{R}) = \sum_{i=1}^{k'} \lambda_i [\{x\} \times (a_i, a_{i+1})]$ for some $\lambda_i \in \mathbb{N}$ and $a_i \in \mathbb{R}$. Let denote by $\{\lambda_{M_j}\}$ the local maxima of the sequence $\{\lambda_i\}$ and let λ_{m_j} be the minimum multiplicity in $\{\lambda_{M_j}, \lambda_{M_j+1}, \dots, \lambda_{M_{j+1}-1}, \lambda_{M_{j+1}}\}$ for every j.
By the fact that the graphs are ordered, Lemma 2.3.2 and the current hypothesis we have

$$|\lambda_{i+1} - \lambda_i| = 1 \tag{2.22}$$

and

$$k = \sum_j \lambda_{M_j} - \sum_j \lambda_{m_j}. \tag{2.23}$$

Then the proof proceeds similarly to the proof of Lemma 2.3.13. One can build a vector field $\tilde{\phi}$ such that

$$\tilde{\phi}^x = \alpha/(a_{M_j+1} - a_{M_j}) \quad \text{in } \{x\} \times (a_{M_j+1}, a_{M_j}) \quad \forall j,$$

$$\tilde{\phi}^x = -\alpha/(a_{m_j+1} - a_{m_j}) \quad \text{in } \{x\} \times (a_{m_j+1}, a_{m_j}) \quad \forall j$$

and zero otherwise to get the thesis.

□

Proposition 2.3.16. *Consider $T \in C$ such that $T = \sum_{i=1}^{k} \Gamma_{u_i}$ and u_i are ordered in an increasing way. Fix $x \in S_T$ and suppose that we have $u_i^+(x) \le u_i^-(x)$ for every $i = 1 \dots k$. Moreover assume that $u_i^-(x) \neq u_{i+1}^+(x)$ for every i such that $x \in S_{u_i}$.*
Then

$$\mathcal{G}(T, \{x\}) = \sum_{i=1}^{k} \mathcal{G}(\Gamma_{u_i}, \{x\}).$$

Proof. See Proposition 2.3.15.

□

Now Theorem 2.3.10 is an immediate consequence of the previous propositions.

Proof of Theorem 2.3.10

Fix $x \in S_T$ and define

$$\mathcal{I} = \{i = 1 \dots k : u_i^-(x) \le u_i^+(x)\} \qquad \mathcal{J} = \{i = 1 \dots k : u_i^-(x) > u_i^+(x)\}$$

and call $T_{\mathcal{I}} = \sum_{i\in\mathcal{I}} \Gamma_{u_i}$ and $T_{\mathcal{J}} = \sum_{i\in\mathcal{J}} \Gamma_{u_i}$. Moreover let $\phi_{\mathcal{I}}$ ($\phi_{\mathcal{J}}$) be the vector field realizing the maximum in $\mathcal{G}(T_{\mathcal{I}}, \{x\})$ ($\mathcal{G}(T_{\mathcal{J}}), \{x\}$). From Proposition 2.3.15 and 2.3.16 it is easy to see that $\phi_{\mathcal{I}}^x \le 0$ outside the support of $T_{\mathcal{I}}$ restricted to $\{x\} \times \mathbb{R}$ and $\phi_{\mathcal{J}}^x \ge 0$ outside the support of $T_{\mathcal{I}}$ restricted to $\{x\} \times \mathbb{R}$. Therefore defining $\tilde{\phi} = \phi_{\mathcal{I}} + \phi_{\mathcal{J}}$, as we assumed that there is no cancellation on the jumps by Lemma 2.3.2, we have that $\tilde{\phi} \in K_{\{x\}}$ and

$$\mathcal{G}(T_{\mathcal{I}}, \{x\}) + \mathcal{G}(T_{\mathcal{J}}, \{x\}) = \int_{\{x\}\times\mathbb{R}} \langle \phi_{\mathcal{I}}^x + \phi_{\mathcal{J}}^x, \nu_T\rangle \, d\|T\| \le \mathcal{G}(T, \{x\}).$$

So by convexity

$$\mathcal{G}(T_{\mathcal{I}}, \{x\}) + \mathcal{G}(T_{\mathcal{J}}, \{x\}) = \mathcal{G}(T, \{x\}).$$

Finally we apply Proposition 2.3.15 and 2.3.16 to $T_{\mathcal{I}}$ and $T_{\mathcal{J}}$ to get the thesis.

□

We conclude this section with a lemma that shows that we can reduce any combination of graphs belonging to C to a combination of graphs, all with the same multiplicity. We are going to use this property in the proof of the coarea formula in the next section.

Lemma 2.3.17. *Consider $T_1, T_2 \in C$ and $x \in S_{T_1} \cap S_{T_2}$. Suppose that $T_1 \llcorner (\{x\} \times \mathbb{R}) = \sum_{i=1}^k \lambda_i [\{x\} \times (a_i, a_{i+1})]$ with $a_i \le a_{i+1}$ and let $\{M_j\}_{j\in J}$ be the indexes of the maximums of the multiplicities. Assume in addition that $T_2 \llcorner (\{x\} \times \mathbb{R}) = \nu \sum_{j\in J} [\{x\} \times (a_{M_j}, a_{M_j+1})]$ for some $\nu > 0$. Then we have*

$$\mathcal{G}(T_1 + T_2, \{x\}) = \mathcal{G}(T_1, \{x\}) + \mathcal{G}(T_2, \{x\}). \tag{2.24}$$

Proof. Given $\phi \in K$ define

$$\Lambda_\phi(s) := \int_{a_1}^s \phi^x(x,t)\, dt - \frac{1}{2},$$

so that

$$\mathcal{G}(T) = \sup_{\phi\in K} \sum_{i=1}^k \lambda_i \int_{a_i}^{a_{i+1}} \phi^x \, dt = \sup_{\phi\in K} \sum_{i=1}^k \lambda_i (\Lambda_\phi(a_{i+1}) - \Lambda_\phi(a_i)) =: \sup_{\phi \in K} \tilde{\mathcal{G}}(\Lambda_\phi).$$

Observe that for every $\phi \in K$, $|\Lambda_\phi(a_i) - \Lambda_\phi(a_j)| \le 1$. Define then the following set:

$$H = \{\Lambda_\phi : \phi \in K, \text{ such that } |\Lambda_\phi(a_i)| \le 1/2 \ \ \forall i = 1 \dots k\} .$$

As the value of the functional $\tilde{\mathcal{G}}$ depends only on the difference between $\Lambda_\phi(a_i)$ and $\Lambda_\phi(a_{i-1})$ we have that

$$\sup_{\phi \in K} \tilde{\mathcal{G}}(\Lambda_\phi) = \sup_{\Lambda_\phi \in H} \tilde{\mathcal{G}}(\Lambda_\phi). \tag{2.25}$$

Notice now that it is possible to rewrite the functional in the following form

$$\tilde{\mathcal{G}}(\Lambda_\phi) = -\lambda_1 \Lambda_\phi(a_1) + \sum_{i=2}^{k} (\lambda_{i-1} - \lambda_i)\Lambda_\phi(a_i) + \lambda_k \Lambda_\phi(a_{k+1}).$$

Hence the supremum in H is a maximum and thanks to (2.25) the maximum points in H are characterized by

$$\Lambda_\phi(a_1) = -1/2, \quad \Lambda_\phi(a_k) = 1/2, \quad \Lambda_\phi(a_i) = \frac{1}{2}\mathrm{sgn}(\lambda_{i-1} - \lambda_i). \tag{2.26}$$

Let us suppose without loss of generality that the maximums of the multiplicity $\{\lambda_{M_j}\}_{j \in J}$ correspond to intervals that are not adjacent (by changing a_i) and let Λ_ϕ be one of the maximum point in H of $\tilde{\mathcal{G}}$, then by (2.26) we get

$$1 = \Lambda_\phi(a_{M_j+1}) - \Lambda_\phi(a_{M_j}) = \int_{a_{M_j}}^{a_{M_j+1}} \phi^x(x,t)\, dt \quad \forall j \in J.$$

As the maximal multiplicities are located in the same interval both in T_1 and in $T_1 + T_2$, then the vector field realizing the maximum is the same and thus the thesis (2.24) follows.

□

Corollary 2.3.18. *Fix $1 \le k' < k$ and define $T_1, T_2 \in C$ such that $T_1 = \sum_{i=1}^{k} \lambda_i \Gamma_{u_i}$ with λ_i ordered in an increasing way and $T_2 = \sum_{i=k'+1}^{k} \nu \Gamma_{u_i}$ with $\nu > 0$. Then*

$$G(T_1 + T_2) = G(T_1) + G(T_2).$$

Proof. Notice that by Lemma 2.3.9 it is enough to prove the thesis for every $x \in S_{T_2} \cap S_{T_1}$. Thanks to Lemma 2.3.17 one has

$$G(T_1 + T_2, \{x\}) = G(T_1, \{x\}) + G(T_2, \{x\}).$$

□

2.3.4 Coarea-type decomposition formula

As anticipated in the introduction, this section is devoted to the proof of a decomposition formula for the Mumford-Shah functional in one dimension. This formula resembles closely a generalized coarea formula for functionals and it is performed for a finite combination of graphs with multiplicity. It is interesting to notice that the counterexample in Subsection 2.2.2 is "solved" by this decomposition, but it is difficult to generalize it to the continuous case. However it gives a strong indication on how this decomposition should be performed at least in dimension one. The higher dimensional case is a completely different issue, as the coarea-type formula we are going to present strongly relies on the one dimensional structure of the problem and cannot be extended in an easy way.

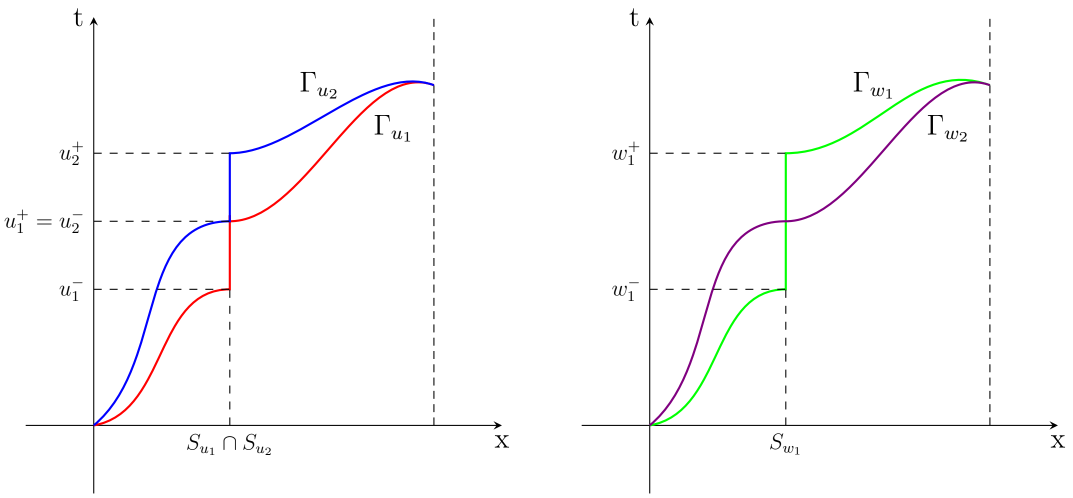

Figure 2.3: Coarea formula decomposition of two SBV graphs

Proposition 2.3.19. *Given* $T = \sum_{i=1}^{k} \Gamma_{u_i} \in C$ *such that* $|S_T| < +\infty$ *there exists* $\{w_i\}_{i=1\ldots k} \subset SBV(I)$ *such that* $T = \sum_{i=1}^{k} \Gamma_{w_i}$ *and*

$$G(T) = \sum_{i=1}^{k} G(\Gamma_{w_i}).$$

Proof. As a consequence of Proposition 2.3.4 we can suppose the graphs Γ_{u_i} ordered in an increasing way. Fix $x_0 \in S_T$ such that $x_0 \in \bigcap_{i=1}^{l} S_{u_i}$ with $l \leq k$. Thanks to Theorem 2.3.10 we can suppost wthout loss of generality that $u_i^+(x_0) = u_{i+1}^-(x_0)$ for every $i = 1 \ldots l$ (the case $u_i^-(x_0) = u_{i+1}^+(x_0)$ is analogous). Define the following functions (See Figure 2.3):

$$w_1 = \begin{cases} u_1 & \text{for } x \leq x_0 \\ u_l & \text{for } x \geq x_0 \end{cases}$$

and

$$w_i = \begin{cases} u_i & \text{for } x \le x_0 \\ u_{i-1} & \text{for } x \ge x_0 \end{cases} \qquad \forall i = 2 \dots l.$$

Clearly $\sum_{i=1}^{l} \Gamma_{w_i} = \sum_{i=1}^{l} \Gamma_{u_i}$ and $w_i^+(x_0) \ne w_{i+1}^-(x_0)$ for every $i = 1 \dots l$.
Hence using Theorem 2.3.10 and repeating this procedure for every $x_0 \in S_T$ one obtains the thesis.

□

Theorem 2.3.20 (Coarea-type formula)**.** *Given $T = \sum_{i=1}^{k} \lambda_i \Gamma_{u_i}$ such that $|S_T| < +\infty$ there exists $k' \in \mathbb{N}$, $\{\mu_i\}_{i=1\dots k'} \ge 0$ and $\{w_i\}_{i=1\dots k'} \subset SBV(I)$ such that $T = \sum_{i=1}^{k'} \mu_i \Gamma_{w_i}$ and*

$$G(T) = \sum_{i=1}^{k'} \mu_i G(\Gamma_{w_i}). \tag{2.27}$$

Proof. Consider $T = \sum_{i=1}^{k} \lambda_i \Gamma_{u_i} \in C$ with u_i ordered in an increasing way and suppose without loss of generality that also λ_i are ordered and λ_k is the maximum. Then T can be rewritten as

$$T = (\lambda_k - \lambda_{k-1})\Gamma_{u_k} + \lambda_{k-1}\Gamma_{u_k} + \sum_{i=1}^{k-1} \lambda_i \Gamma_{u_i}.$$

Hence by Corollary 2.3.18

$$G(T) = G((\lambda_k - \lambda_{k-1})\Gamma_{u_k}) + G\left(\lambda_{k-1}\Gamma_{u_k} + \sum_{i=1}^{k-1} \lambda_i \Gamma_{u_i}\right).$$

Then one can rewrite

$$\lambda_{k-1}\Gamma_{u_k} + \sum_{i=1}^{k-1} \lambda_i \Gamma_{u_i} = \lambda_{k-2}(\Gamma_{u_k} + \Gamma_{u_{k-1}}) + (\lambda_{k-1} - \lambda_{k-2})(\Gamma_{u_k} + \Gamma_{u_{k-1}}) + \sum_{i=1}^{k-2} \lambda_i \Gamma_{u_i}$$

and applying again Corollary 2.3.18

$$\begin{aligned} G\left(\lambda_{k-1}\Gamma_{u_k} + \sum_{i=1}^{k-1} \lambda_i \Gamma_{u_i}\right) &= G((\lambda_{k-1} - \lambda_{k-2})(\Gamma_{u_k} + \Gamma_{u_{k-1}})) \\ &+ G\left(\lambda_{k-2}(\Gamma_{u_k} + \Gamma_{u_{k-1}}) + \sum_{i=1}^{k-2} \lambda_i \Gamma_{u_i}\right). \end{aligned} \tag{2.28}$$

By Proposition 2.3.19 there exists u_k^2 and u_{k-1}^2 SBV functions such that $\Gamma_{u_k^2} + \Gamma_{u_{k-1}^2} = \Gamma_{u_k} + \Gamma_{u_{k-1}}$ and

$$\begin{aligned} (2.28) = G((\lambda_{k-1} - \lambda_{k-2})\Gamma_{u_k^2}) + G((\lambda_{k-1} - \lambda_{k-2})\Gamma_{u_{k-1}^2}) \\ + G\left(\lambda_{k-2}(\Gamma_{u_k} + \Gamma_{u_{k-1}}) + \sum_{i=1}^{k-2} \lambda_i \Gamma_{u_i}\right) \end{aligned}$$

and so on. Repeating this procedure k times one gets to

$$G(T) = \sum_{i=2}^{k} \sum_{j=i}^{k} (\lambda_i - \lambda_{i-1}) G(\Gamma_{u_j^{k-i+1}}) + G\left(\sum_{i=1}^{k} \lambda_1 \Gamma_{u_i}\right).$$

Hence, applying again Proposition 2.3.19 to the last term we obtain the desired decomposition (2.27).

□

2.4 Existence of calibration as a functional defined on currents

We now want to show an application of the coarea-type formula to the existence of calibration for the Mumford-Shah type functionals. Firstly we set the minimization problem associated to the previous functional G. Consider $S \in C$ and define

$$\psi_G(S) = \inf\{G(T) : T \in C,\ \partial T = \partial S\}.$$

Proposition 2.4.1. *The functional ψ_G is convex in C.*

Proof. As G is convex and the constraint is linear the proof is straightforward.

□

It is easy to see that by the coarea-type formula in Theorem 2.3.20 we have the following theorem:

Theorem 2.4.2. *If $u \in SBV(I)$ is a Dirichlet minimizer of $\mathscr{F}$, then $\psi_G(\Gamma_u) = G(\Gamma_u) = \mathscr{F}(u)$.*

Proof. Consider $T = \sum_{i=1}^{k} \lambda_i \Gamma_{u_i} \in C$ such that $\partial T = \partial \Gamma_u$. Without loss of generality we can suppose that $|S_T| < +\infty$. Then letting $I = (a, b)$ and $\pi : \mathbb{R}^2 \to \mathbb{R}$ the projection on the first component we have

$$\partial I = \pi^{\#}(\partial \Gamma_u) = \pi^{\#}(\partial T) = (\partial I) \sum_{i=1}^{k} \lambda_i. \tag{2.29}$$

Hence $\sum_{i=1}^{k} \lambda_i = 1$. By Theorem 2.3.20 there exist k' and $\{\mu_i\}_{i=1,\dots,k'} > 0$ such that

$$G(T) = G\left(\sum_{i=1}^{k} \lambda_i \Gamma_{u_i}\right) = \sum_{i=1}^{k'} \mu_i G(\Gamma_{w_i}) = \sum_{i=1}^{k'} \mu_i \mathscr{F}(w_i).$$

and $\sum_{i=1}^{k} \lambda_i \Gamma_{u_i} = \sum_{i=1}^{k'} \mu_i \Gamma_{w_i}$. Moreover applying the push forward as in equation (2.29) we have also $\sum_{i=1}^{k'} \mu_i = 1$.
Thus, it remains to prove that $w_{i,\partial I} = u_{\partial I}$ for every $i = 1, \dots, k'$, where $w_{i,\partial I}$ denotes the trace of w_i on ∂I. This is an easy adaptation of the theory of cartesian currents; we refer to Section 3.2.5 in [27] for a proof in a more general setting.

□

This will imply the existence of a calibration in the following sense: let

$$\hat{C} = \left\{ T = \sum_{i=1}^{k} \lambda_i \Gamma_{u_i} : k \in \mathbb{N}, \lambda_i \in \mathbb{R}, u_i \in SBV(I) \right\}$$

be the double cone and define the following:

Definition 2.4.1 (Calibration for minimal graphs)**.** *Given $u \in SBV(I)$ and Γ_u its associated graph, we say that $\xi \in Hom(\hat{C})$ is a calibration for Γ_u with respect to G if*

i) $\xi(\Gamma_u) = G(\Gamma_u) = \mathscr{F}(u)$,

ii) $\xi(T) = 0$ *for every* $T \in \hat{C}$ *such that* $\partial T = 0$,

iii) $\xi(T) \leq G(T)$ *for every* $T \in \hat{C}$.

Theorem 2.4.3. *Given $u \in SBV(I)$ a Dirichlet minimizer of $\mathscr{F}$ there exists a calibration for Γ_u with respect to G according to Definition 2.4.1.*

Proof. From Theorem 2.4.2 follows that

$$G(\Gamma_u) = \psi_G(\Gamma_u).$$

Consider the functional ψ_G defined on C and extend it to $+\infty$ for all the elements in $\hat{C} \setminus C$ (without renaming the extension). Clearly the extension is convex and $\psi_G(\Gamma_u) = G(\Gamma_u) > 0$.
Consider the vector subspace $L = \{a\Gamma_u : a \in \mathbb{R}\}$ and define $\psi : L \to \mathbb{R}$ as $\psi(a\Gamma_u) = a\psi_G(\Gamma_u)$ clearly linear. As we have that $\psi \leq \psi_G$ on L by Hahn-Banach theorem there exists $\xi \in Hom(\hat{C}, \mathbb{R})$ such that

$$\xi(\Gamma_u) = \psi(\Gamma_u) = \psi_G(\Gamma_u) \quad \text{and} \quad \xi(T) \leq \psi_G(T) \quad \forall T \in \hat{C}. \tag{2.30}$$

We want to prove that ξ is a calibration according to Definition 2.4.1. Let $T_0 \in \hat{C}$ be such that $\partial T_0 = 0$, then

$$\psi_G(T_0) = \inf\{G(S) : \partial S = \partial T_0 = 0\} \leq G(0) = 0.$$

In combination with (2.30) this implies $\xi(T) \leq 0$ for every $T_0 \in \hat{C}$ such that $\partial T_0 = 0$.
So, as ξ is an homeomorphism, one has also that $\xi(T_0) = 0$, so that (ii) holds. Moreover from (2.30), $\xi(\Gamma_u) = \psi_G(\Gamma_u) = \mathscr{F}(u)$ that is (i).
Let us show that also (iii) is satisfied: if $T \in \hat{C} \setminus C$ then $G(T) = +\infty$ and so there is nothing to prove. On the other hand given $T = \sum_{i=1}^{k} \lambda_i \Gamma_{u_i} \in C$ with $\lambda_i \in \mathbb{R}_+$ by (2.30) and using the definition of ψ_G

$$\xi(T) \leq \psi_G(T) \leq G(T).$$

Hence ξ is a calibration according to Definition 2.4.1. $\square$

Chapter 3

The Chan and Vese Algorithm and the reinitialization of the distance function

3.1 Overview

In the first part of this chapter we review the main features of a well known algorithm for edge detection and image segmentation. It was introduced in its seminal form by Chan and Vese in [16] as an algorithm for the edge detection of an image using the level set formulation for the mean curvature flow introduced by Osher and Sethian in their fundational paper [43]. Then, in the following years, this algorithm has been generalized to be an approximation of the piecewise smooth Mumford-Shah functional (see [17]).
The idea behind these approaches is that an approximation of the Mumford-Shah functional (piecewise constant or piecewise smooth) can be achieved by a two steps minimization procedure. Suppose that the we initialize the algorithm with a pair (u_0, K_0) and we want to find a procedure that produces a sequence of pairs (u_n, K_n) converging to a stationary point of the Mumford-Shah functional; the idea of Chan and Vese is to fix the function u_0 and then to perform a step of the mean curvature flow for the set K_0 (with external force given by u_0) using the level set method to obtain K_1. In the second step of the algorithm the function u_0 is updated to be the optimal one (with respect to K_1) via the classical Euler-Lagrange equation for the Mumford-Shah functional (see Section 1.1.4). This algorithm can be iterated indefinitely and heuristically it converges to a stationary point of the Mumford-Shah functional.
The structure of this chapter is the following: in Section 3.2 we present the level set formulation for evolving interfaces and in Section 3.3 we describe the Chan and Vese algorithm. Then, in the second part of this chapter, we focus on the so called *reinitialization of the distance function*; it is an Hamilton-Jacobi equation that is used in the framework of the Chan and Vese algorithm to ensured the numerical convergence of the algorithm. In particular we recall the standard theory of the Hamilton-Jacobi equation we are going to use and finally in Section 3.5 we study the long time behaviour of the *reinitialization of the distance function*.

3.2 Level set formulation

In this section we present the level set formulation for evolving interfaces as it was introduced by Osher and Sethian in [43]. When it is applied to the Chan and Vese algorithm the relevant dimension is 2, as we are working with images; however, as the level set theory is unchanged, we are going to describe it in a general dimension n.
We are given $\Gamma \subset \mathbb{R}^n$ a closed $n-1$ dimensional hypersurface and we want to produce an Eulerian formulation for the evolution of the surface $\Gamma(t)$ subjected to a normal velocity F, where F could be for example a function of the curvature or of the normal. Notice that if F is the curvature of Γ, then the evolution is just the mean curvature flow.

The fundamental idea of the level set formulation is to consider the hypersurface Γ as the zero level set of some function $\phi : \mathbb{R}^n \to \mathbb{R}$, so that Γ is embedded in a family of $n-1$ dimensional hypersurfaces that are level sets of ϕ. The goal is to write an equation for the evolution of the function ϕ and then to recover the evolution of $\Gamma(t)$ tracing the zero level set of ϕ.
We are going to derive the level set formulation for the particular case of the mean curvature flow. The evolution equation for Γ reads

$$\frac{\partial x}{\partial t} = H(x(t))\nu(x(t)), \tag{3.1}$$

where $x(t)$ is the evolution of each point of the hypersurface, $H(x(t))$ is the mean curvature and $\nu(x(t))$ is the normal to the surface both computed in the point $x(t)$.
We will initialize the procedure considering a function $\phi(x,0) := d_s(\Gamma, x)$ where $d_s(\Gamma, x)$ denotes the signed distance from Γ. In this way we have $\Gamma = \Gamma(0) = \{x \in \mathbb{R}^n : \phi(x,0) = 0\}$.
In order to derive the level set formulation we assume that the zero level set of the function ϕ has to be the evolving hypersurface we are tracing:

$$\phi(x(t), t) = 0 \quad \forall t > 0. \tag{3.2}$$

We can then differentiate (3.2) in t to get

$$\nabla\phi(x(t), t) \cdot \frac{\partial x}{\partial t} + \frac{\partial \phi}{\partial t} = 0. \tag{3.3}$$

Hence using equation (3.1) one obtains

$$\frac{\partial \phi}{\partial t} + \nabla\phi(x(t), t) \cdot H(x(t))\nu(x(t)) = 0. \tag{3.4}$$

Moreover one can verify that, as $\Gamma(t) = \{x : \phi(x,t) = 0\}$,

$$H(x(t)) = \operatorname{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)(x(t))$$

and $\nu(x(t)) = \frac{\nabla\phi}{|\nabla\phi|}(x(t))$. Hence equation (3.3) reduces to

$$\frac{\partial \phi}{\partial t} + |\nabla\phi| \operatorname{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right) = 0, \tag{3.5}$$

where the dependence on $x(t)$ is implicit.
Finally one can solve the mean curvature flow finding the solution of the Cauchy problem

$$\begin{cases} \frac{\partial \phi}{\partial t} + |\nabla\phi| \operatorname{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\ \phi(x,0) = d_s(\Gamma, x) \end{cases} \tag{3.6}$$

and then computing the zero level set of $\phi(\cdot, t)$ for every $t > 0$.

The question that naturally arises from this discussion is why are we interested in solving an apparently more difficult problem obtained embedding the evolving surface in the level sets of a higher dimensional function. The advantages are various: first of all a surface moving accordingly to a mean curvature flow in $\mathbb{R}^n$ is subjected to changes of topology and development of singularities and for this reason equation (3.1) cannot describe this phenomena and it cannot be solved globally in time. On the other hand the function ϕ, solution of the level set formulation (3.5), remains a function throughout the evolution and equation (3.5) can be solved globally in time in the framework of viscosity solution ([22]); this permits to introduce discretization techniques to solve the PDE numerically in an efficient way called *first-order upwind schemes*([46, 43]).

3.3 The Chan and Vese Algorithm

3.3.1 Piecewise constant and piecewise smooth Mumford-Shah functional

Before going into the description of the algorithm let us introduce two variational problems that are "simplification" of the classical Mumford-Shah functional (1.1) that we recall here for reader convenience:

$$\mathscr{F}(u) := \int_\Omega |\nabla u|^2 \, dx + \beta \mathcal{H}^{n-1}(S_u) + \alpha \int_\Omega |u - g|^2 \, dx, \tag{3.7}$$

where we assume that $\Omega \subset \mathbb{R}^n$ is a bounded, regular and open set, $g \in L^\infty(\Omega, [0, 1])$, α and β two positive tuning parameters and $u \in SBV(\Omega)$.
In the piecewise constant and piecewise smooth Mumford-Shah functional we reduce the complexity of the set of competitors assuming that the minimizers have discontinuity on the reduced boundary of a set of finite perimeter E and they are smooth or constant in E and $\Omega \setminus E$. Therefore we are replacing the space of competitor $SBV(\Omega)$ with

$$\mathcal{A}_{const}(\Omega) := \{u \in SBV(\Omega) : u = c_1 \mathbf{1}_E + c_2 \mathbf{1}_{\Omega \setminus E},\ c_1, c_2 \in \mathbb{R} \text{ and } E \subset \Omega,\ P(E) < +\infty\} \tag{3.8}$$

for the piecewise constant Mumford-Shah and

$$\mathcal{A}_{smooth}(\Omega) := \{u \in SBV(\Omega) : u = u_1 \mathbf{1}_E + u_2 \mathbf{1}_{\Omega \setminus E},\ u_1 \in C^1(E), \\ u_2 \in C^1(\Omega \setminus E) \text{ and } , E \subset \Omega,\ P(E) < +\infty\}$$

for the piecewise smooth.
In this way the Mumford-Shah functional can be written a variational problem on

(E, c_1, c_2) for the piecewise constant version and on (E, u_1, u_2) for the piecewise smooth in the following way (we set $\alpha = \beta = 1$ for simplicity):

$$F_{const}(E, c_1, c_2) = \int_E |c_1 - g|^2\, dx + \int_{\Omega \setminus E} |c_2 - g|^2\, dx + P(E \cap \Omega) \tag{3.9}$$

for $c_1, c_2 \in \mathbb{R}$ and $E \subset \Omega$ of finite perimeter and

$$\begin{aligned} F_{smooth}(E, u_1, u_2) = \int_E |\nabla u_1|^2\, dx + & \int_{\Omega \setminus E} |\nabla u_2|^2\, dx + \int_E |u_1 - g|^2\, dx \\ & + \int_{\Omega \setminus E} |u_2 - g|^2\, dx + P(E \cap \Omega). \end{aligned} \tag{3.10}$$

for $u_1 \in C^1(E)$, $u_2 \in C^2(\Omega \setminus E)$ and $E \subset \Omega$ of finite perimeter.

Remark 3.3.1. *Notice that, on the contrary to the general Mumford-Shah functional, the existence of minimizers for the functionals F_{const} and F_{smooth} is an easy application of the direct method of calculus of variation as, in this setting, the perimeter is lower semicontinous with respect the L^1 convergence.*

3.3.2 The algorithm

The Chan and Vese algorithm was introduced in [16] for the case of piecewise constant Mumford-Shah functional and then generalized to the piecewise smooth case and to the multiphase setting in [17].
We are going to explain the approach related to the piecewise constant situation as it has the merit to be easier to visualize and then we will explain how to modify that to deal with more complex functionals. From now on we will assume that the

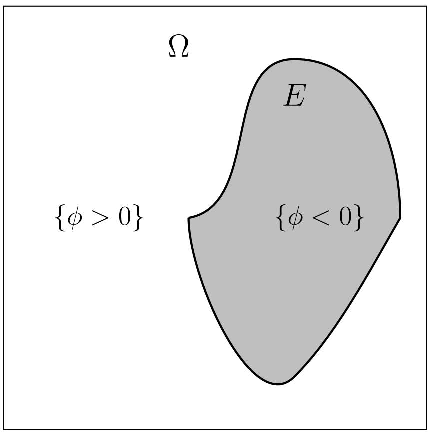

Figure 3.1: The function ϕ employed in the Chan and Vese algorithm

finite perimeter set E we want to find by the minimization procedure has boundary

∂E that is the zero level set of a Lipschitz function $\phi : \Omega \to \mathbb{R}$. Moreover we ask the function ϕ to be positive in $\Omega \setminus E$ and negative in E (see Figure 3.1). Thanks to the previous assumption F_{const} can be written as a functional dependent on ϕ and on c_1, c_2 as follows:

$$F_{const}(\phi, c_1, c_2) = \int_\Omega H(\phi)|c_1 - g|^2 \, dx + \int_\Omega (1 - H(\phi))|c_2 - g|^2 \, dx + |DH(\phi)|(\Omega), \tag{3.11}$$

where $H : \mathbb{R} \to \{0, 1\}$ is the Heavyside function and $|DH(\phi)|$ is the total variation of the BV function $H \circ \phi : \Omega \to \mathbb{R}$.
In order to explain the idea of the model, let us suppose that the image g is a black white image so that it splits Ω in two regions where $g = 0$ and $g = 1$ respectively, with interface Γ_g.
The goal of the Chan and Vese algorithm is to recover the boundary of the image, so in this case the interface between $\{g = 1\}$ and $\{g = 0\}$. It is easy to see that

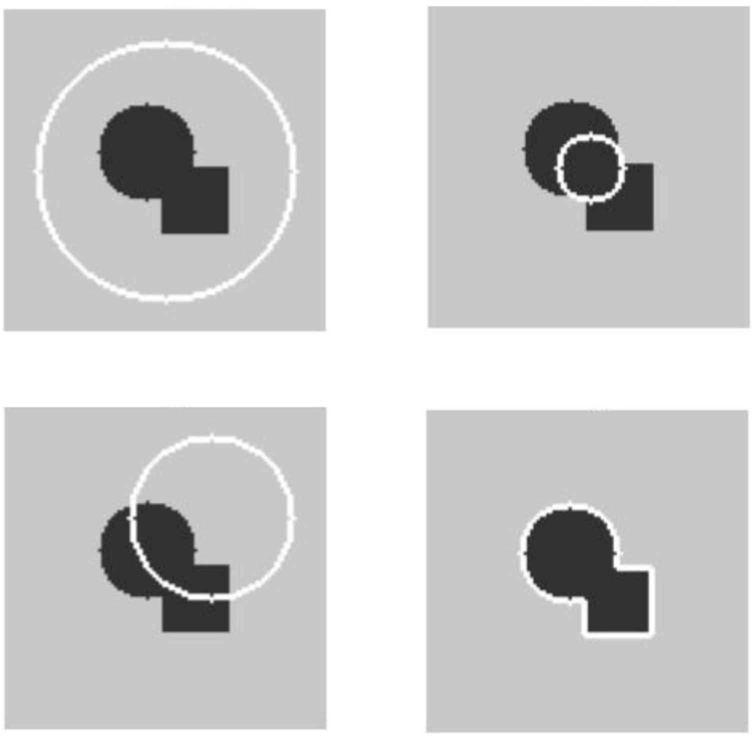

Figure 3.2: Example of the segmentation of an image using the Chan and Vese algorithm: starting from the picture up to the left and the proceeding in clockwise sense we have (i) $F_1 > 0$ and $F_2 = 0$, (ii) $F_1 = 0$ and $F_2 > 0$, (iii) $F_1 > 0$ and $F_2 > 0$, (iv) $F_1 = F_2 = 0$ ([16])

given $\phi_0 : \Omega \to \mathbb{R}$ Lipschitz with $\{\phi_0 = 0\} = \Gamma$, the energy

$$E(c_1, c_2) = \int_\Omega H(\phi_0)|c_1 - g|^2\,dx + \int_\Omega (1 - H(\phi_0))|c_2 - g|^2\,dx$$

is minimized when

$$c_1(\phi_0) = \fint_{\{\phi_0(x)>0\}} g(x)\,dx \qquad \text{and} \qquad c_2(\phi_0) = \fint_{\{\phi_0(x)<0\}} g(x)\,dx. \tag{3.12}$$

Moreover the functional

$$\begin{aligned} E(\phi) &= \int_\Omega H(\phi)|c_1(\phi) - g|^2\,dx + \int_\Omega (1 - H(\phi_0))|c_2(\phi) - g|^2\,dx \\ &= F_1(\phi) + F_2(\phi) \end{aligned}$$

is minimized when ϕ is such that $\{\phi(x) = 0\} = \Gamma_g$ and in all the other cases the situation is summed up in Figure 3.2. The length of the interface measured as $|DH(\phi)|(\Omega)$ can be seen as a regularizing term for the functional F_{const}.
The computational strategy is to recover Γ_g by a double step gradient descent. Suppose that we initialize the algorithm with a Lipschitz function $\phi_0 : \Omega \to \mathbb{R}$ such that $\Gamma_0 = \{\phi_0 = 0\}$. Then we compute one step of the gradient descent for the following functional with respect to ϕ with $c_1(\phi_0)$ and $c_2(\phi_0)$ fixed

$$\begin{aligned} F_{const}(\phi, c_1(\phi_0), c_2(\phi_0)) = &\int_\Omega H(\phi)|c_1(\phi_0) - g|^2\,dx \\ &+ \int_\Omega (1 - H(\phi))|c_2(\phi_0) - g|^2\,dx + |DH(\phi)|(\Omega) \end{aligned}$$

to get a new Lipschitz function $\phi_1 : \Omega \to \mathbb{R}$. Then we update the constants c_1 and c_2 with the new values $c_1(\phi_1)$ and $c_2(\phi_1)$ according to (3.12) and again we compute one step of the gradient descent for

$$\begin{aligned} F_{const}(\phi, c_1(\phi_1), c_2(\phi_1)) = &\int_\Omega H(\phi)|c_1(\phi_1) - g|^2\,dx \\ &+ \int_\Omega (1 - H(\phi))|c_2(\phi_1) - g|^2\,dx + |DH(\phi)|(\Omega) \end{aligned}$$

and so on. Heuristically this procedure should converge to the minimum or at least to a stationary point of the Mumford-Shah functional.
In order to perform a gradient descent method at each step, the Chan and Vese algorithm employs a regularized version of the Heavyside function H_δ defined as

$$H_\delta(t) = \begin{cases} 0 & t < -\delta \\ \frac{1}{2}\left(1 + \frac{t}{\delta} + \frac{1}{\pi}\sin(\pi t/\delta)\right) & -\delta < t < \delta \\ 1 & t > \delta, \end{cases} \tag{3.13}$$

in such a way that

$$H'_\delta(t) = \begin{cases} 0 & t < -\delta \\ \frac{1}{2\delta}\left(1+\cos(\pi t/\delta)\right) & -\delta < t < \delta \\ 0 & t > \delta. \end{cases}$$

The regularized functional will reduce to

$$F^\delta_{const}(\phi, c_1, c_2) = \int_\Omega H_\delta(\phi)|c_1 - g|^2\,dx + \int_\Omega (1 - H_\delta(\phi))|c_2 - g|^2\,dx + |DH_\delta(\phi)|(\Omega). \tag{3.14}$$

Remark 3.3.2. *One can perform a similar procedure to compute F^δ_{smooth} to obtain*

$$\begin{aligned} F^\delta_{smooth}(\phi, u_1, u_2) = &\int_\Omega H_\delta(\phi)|\nabla u_1|^2\,dx + \int_\Omega (1 - H_\delta(\phi))|\nabla u_2|^2\,dx \\ &+ \int_\Omega H_\delta(\phi)|u_1 - g|^2\,dx + \int_\Omega (1 - H_\delta(\phi))|u_2 - g|^2\,dx + |DH_\delta(\phi)|(\Omega). \end{aligned}$$

However in this case the function u_1 and u_2 are not constants and they cannot be optimized simply considering the average value. In this case the double step minimization is effective in the sense that we couple one step of the gradient descent with u_1 and u_2 fixed as in the previous explanation, with a step of the Euler-Lagrange equation of the Mumford-Shah functional for u_1 and u_2 keeping ϕ fixed. We will not enter too much into details, but we refer to [17].

The L^2 gradient flow for F^δ_{const}

We compute now in a formal way the L^2 gradient flow with respect to ϕ of F^δ_{const}. For the general theory of gradient flows in Banach and metric spaces we refer to [6].
We consider again

$$F^\delta_{const}(\phi) = \int_\Omega H_\delta(\phi)|c_1 - g|^2\,dx + \int_\Omega (1 - H_\delta(\phi))|c_2 - g|^2\,dx + |DH_\delta(\phi)|(\Omega) \tag{3.15}$$

with $c_1, c_2 \in \mathbb{R}$ and $\phi : \Omega \to \mathbb{R}$ Lipschitz. We suppose in addition that $H \in C^{2,1}(\mathbb{R})$ and that there exists $C > 0$ such that

$$\|\nabla\phi(x)\|_{L^\infty(\Omega)} \geq C. \tag{3.16}$$

In order to compute the L^2 gradient flow for F^δ_{const} we need to determine the Gateau derivative of it, defined as $DF_{\delta,\phi} : L^2(\Omega) \to \mathbb{R}$ as

$$DF_{\delta,\phi}(\psi) = \lim_{t\to 0} \frac{F^\delta_{const}(\phi) - F^\delta_{const}(\phi + t\psi)}{t} \tag{3.17}$$

for $\psi \in L^2(\Omega)$ when it exists.
Assume that $\psi \in C_c^\infty(\Omega)$. Then using Taylor expansion we obtain

$$H_\delta(\phi + t\psi) = H_\delta(\phi) + t\psi H'_\delta(\phi) + R_1(x,t), \tag{3.18}$$

where one can express the rest R_1 as

$$R_1(x,t) = \frac{1}{2} H''(\xi(x)) t^2 \psi^2(x)$$

for some $\xi(x) \in (0, t\psi(x))$. Therefore as $H \in C^{2,1}(\mathbb{R})$

$$\lim_{t\to 0} \frac{R_1(x,t)}{t} = 0 \quad \text{uniformly in } \Omega. \tag{3.19}$$

On the other hand for $\psi \in C_c^\infty(\Omega)$ we have

$$|\nabla H_\delta(\phi + t\psi)| = H'_\delta(\phi + t\psi)|\nabla\phi + t\nabla\psi|. \tag{3.20}$$

Using Taylor expansion on the first term of the product we get

$$H'_\delta(\phi + t\psi) = H'_\delta(\phi) + t\psi H''_\delta(\phi) + R_2(x,t), \tag{3.21}$$

where

$$\lim_{t\to 0} \frac{R_2(x,t)}{t} = 0 \quad \text{uniformly in } \Omega. \tag{3.22}$$

Moreover thanks to assumption (3.16)

$$\begin{aligned}
|\nabla\phi + t\nabla\psi| &= \sqrt{|\nabla\phi|^2 + t^2|\nabla\psi|^2 + 2t\langle\nabla\phi, \nabla\psi\rangle} = \\
&= |\nabla\phi| \sqrt{1 + t^2 \frac{|\nabla\psi|^2}{|\nabla\phi|^2} + 2t \frac{\langle\nabla\phi, \nabla\psi\rangle}{|\nabla\phi|^2}} \\
&= |\nabla\phi| \left(1 + \frac{t^2|\nabla\psi|^2}{|2\nabla\phi|^2} + t\frac{\langle\nabla\phi, \nabla\psi\rangle}{|\nabla\phi|^2}\right) + R_3(x,t),
\end{aligned}$$

where

$$\lim_{t\to 0} \frac{R_3(x,t)}{t} = 0 \quad \text{uniformly in } \Omega. \tag{3.23}$$

Hence

$$|\nabla\phi + t\nabla\psi| = |\nabla\phi| + t\frac{\langle\nabla\phi, \nabla\psi\rangle}{|\nabla\phi|} + R_4(x,t). \tag{3.24}$$

Therefore from (3.21), (3.24) and (3.20)

$$\begin{aligned}
\int_\Omega |\nabla H_\delta(\phi + t\psi)|\, dx - \int_\Omega |\nabla H_\delta(\phi)|\, dx &= t \int_\Omega H'_\delta(\phi) \frac{\langle\nabla\phi, \nabla\psi\rangle}{|\nabla\phi|}\, dx \\
&+ t \int_\Omega \psi(x) H''_\delta(\phi) |\nabla\phi|\, dx + \int_\Omega R(t,x)\, dx.
\end{aligned}$$

where $\frac{R(x,t)}{t} \to 0$ as $t \to 0$ uniformly in Ω.
So computing the limit as $t \to 0$ and integrating by part we obtain

$$\lim_{t\to 0}\frac{1}{t}\left(\int_\Omega |\nabla H_\delta(\phi+t\psi)|\,dx - \int_\Omega |\nabla H_\delta(\phi)|\,dx\right) = \int_\Omega H'_\delta(\phi)\frac{\langle\nabla\phi,\nabla\psi\rangle}{|\nabla\phi|}\,dx$$
$$+\int_\Omega \psi(x)H''_\delta(\phi)|\nabla\phi|\,dx = \int \psi(x)H''_\delta(\phi)|\nabla\phi|\,dx - \int_\Omega \psi(x)\,\mathrm{div}\left(H'_\delta(\phi)\frac{\nabla\phi}{|\nabla\phi|}\right).$$

Hence

$$DF_{\delta,\phi}(\psi) = \langle G_{\delta,\phi},\psi\rangle_{L^2},$$

where

$$G_{\delta,\phi}(x) := H'_\delta(\phi(x))\Big(|c_1-g|^2-|c_2-g|^2\Big) + H''_\delta(\phi)|\nabla\phi| - \mathrm{div}\left(H'_\delta(\phi)\frac{\nabla\phi}{|\nabla\phi|}\right).$$

Notice also that

$$\begin{aligned}\mathrm{div}\left(H'_\delta(\phi)\frac{\nabla\phi}{|\nabla\phi|}\right) &= \nabla H'_\delta(\phi)\cdot\left(\frac{\nabla\phi}{|\nabla\phi|}\right) + H'_\delta(\phi)\,\mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)\\ &= H''_\delta(\phi)|\nabla\phi| + H'_\delta(\phi)\,\mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right).\end{aligned}$$

So

$$G_{\delta,\phi}(x) = H'_\delta(\phi(x))\Big(-\mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right) + |c_1-g|^2 - |c_2-g|^2\Big). \tag{3.25}$$

By density arguments formula (3.25) is valid for every $\psi \in L^2(\Omega)$.
Therefore the gradient flow of the energy F^δ_{const} is

$$\langle\frac{\partial\phi}{\partial t},\psi\rangle_{L^2} = -\langle G_{\delta,\phi},\psi\rangle_{L^2}. \tag{3.26}$$

for every $\psi \in L^2(\Omega)$.

3.3.3 The reinitialization of the distance function

From the computation of the previous section and the gradient flow (3.26) it results clear the importance of the assumption (3.16). Indeed, also from a numerical point of view the Chan and Vese algorithm could not converge if the norm of the gradient of ϕ approaches zero during the two steps procedure.
This is the reason why the algorithm is usually coupled with an Hamilton-Jacobi equation called *reinitiliazion of the distance function*. It was introduced in [47] in the context of the level set methods for incompressible two phase-flow and it extends to the algorithms that involve the level set approach.
In particular when $|\nabla\phi|$ is small, it makes sense to update ϕ to be the distance

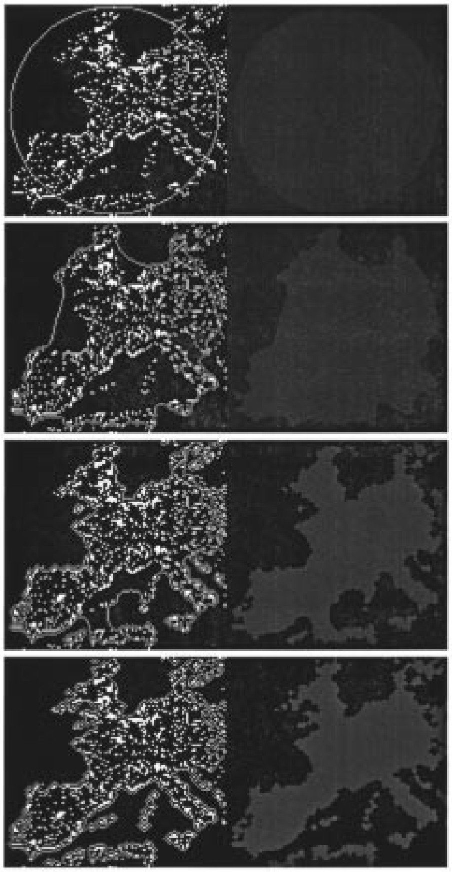

Figure 3.3: An example of the Chan and Vese algorithm for the segmentation of Europe ([16])

function from its zero level set. Indeed, in the level approach, we are interested only in the evolution of the surface $\Gamma = \{\phi = 0\}$. Therefore, as the behaviour of ϕ outside its zero level set is not relevant we can substitute ϕ with the distance function from Γ in such a way that the norm of gradient is equal to one almost everywhere.
The computation of the distance function from a set is too costly for the algorithm, therefore the strategy employed also in [16] and in [17] is to solve the following evolutive Hamilton-Jacobi equation, computing the viscosity solution as $\to +\infty$:

$$\begin{cases} \phi_t + f(x)(|\nabla\phi| - 1) = 0 & \text{in } \Omega \times [0, +\infty) \\ \phi(x, 0) = \phi_0(x) & \text{in } \Omega, \end{cases} \tag{3.27}$$

where ϕ_0 is the initial data, corresponding to the output of an intermediate step of the Chan and Vese algorithm and

$$f(x) = f_\delta(x) = \frac{\phi_0(x)}{\sqrt{\phi_0(x)^2 + \delta^2}}$$

for a fixed $\delta > 0$.
Notice that the function $f(x)$ is an approximation of the signed distance function that vasishes only on $\Gamma = \{x \in \Omega : \phi_0(x) = 0\}$. Therefore, at least heuristically, $\Gamma_t := \{x \in \Omega : \phi(x,t) = 0\}$ is preserved during the evolution and the solution of (3.51) should converge to the steady state of the equation, that is the signed distance function from $\{\phi_0(x) = 0\}$.
In Section 3.5 we will formalize these heuristical observations using the well estblished theory of viscosity solutions for Hamilton-Jacobi equations.

3.4 Hamilton-Jacobi equations

In this section we review some classical results on Hamilton Jacobi equation that we will need in Section 3.5. We refer to [7], [12] and [8] for comprehensive tractaction of the theory of viscosity solutions and optimal control.
Given $n \in \mathbb{N}$, let us consider $\Omega \subset \mathbb{R}^n$ an open, regular set. Throughout the rest of this thesis we will deal with PDE's of the following form

$$u_t + H(x, \nabla u) = 0 \quad \text{in } \Omega \times (0,T) \tag{3.28}$$

where $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is called Hamiltonian and $T \in (0, +\infty]$ is a fixed instant of time.
It is easy to see by the method of the characteristics that classical solutions of this equation develop discontinuity of the gradient in finite time. Moreover if we consider solutions that satisfy equation (3.28) almost everywhere, then the uniqueness of the solution of a Cauchy problem associated with (3.28) is irremediably lost as the following example shows.

Example 3.4.1. *Consider $\Omega = (-1, 1) \subset \mathbb{R}$ and the following Hamilton-Jacobi equation*

$$|u'(x)| = 1$$

with initial conditions $u(-1) = 0$ and $u(1) = 0$. Then it is easy to see that $u(x) = |x| - 1$ and $u(x) = -|x| + 1$ are both solutions as well as all the possible piecewise linear functions satisying the boundary conditions and such that $|u'(x)| = 1$.

Therefore it is natural to search for a criteria to select a unique solution. This was done in the 80's by the remarkable works of Crandall, Evans and Lions ([19], [18]). They introduced the notion of *viscosity solution* for PDE's and, as a consequence, they were able to prove existence, uniqueness and stability for equation (3.28). From now on we will consider uniformly continous functions defined on $\Omega \times (0,T)$ that we will denote by $UC(\Omega \times (0,T))$.

Definition 3.4.1 (Viscosity solution)**.** *Consider $u : \Omega \times (0,T) \to \mathbb{R}$, locally bounded and uniformly continuous.*

⋆) *u is a* viscosity subsolution *of* (3.28) *if for every* $(x_0, t_0) \in \Omega \times (0, T)$ *and for every* $v \in C^\infty(\Omega \times (0, T))$ *such that* $u - v$ *has a maximum in* (x_0, t_0) *we have*

$$v_t(x_0, t_0) + H(x_0, \nabla v(x_0)) \leq 0.$$

⋆) *u is a* viscosity supersolution *of* (3.28) *if for every* $(x_0, t_0) \in \Omega \times (0, T)$ *and for every* $v \in C^\infty(\Omega \times (0, T))$ *such that* $u - v$ *has a minimum in* (x_0, t_0) *we have*

$$v_t(x_0, t_0) + H(x_0, \nabla v(x_0)) \geq 0.$$

If u is both a subsolution and supersolution, we say that it is a viscosity solution of (3.28).

To be more precise this concept of solution enjoys the following desirable properties:

I) for every boundary data there exists a unique solution depending continuously on the boundary values and on the Hamiltonian,

II) the solution can be recovered by *vanishing viscosity* approximation. That is to say $u^\varepsilon \to u$ uniformly where u^ε are classical solutions of the following elliptic PDE
$$u_t + H(x, \nabla u^\varepsilon) = \varepsilon \Delta u^\varepsilon,$$

III) when the Hamilton Jacobi equation describes the value functional for some optimization problem, then the viscosity solution is exactly the value function.

We will not discuss (III) in this thesis but we will review (I) in Section 3.4.2 and (II) in Section 3.4.1.
An immediate consequence of the definition of viscosity solution and the application of Jensen inequality are the following propositions:

Proposition 3.4.1. *Let* $u(x, t)$ *be Lipschitz and let us suppose that* $H : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *is convex in the last variable and that*

$$u_t + H(x, p) \leq 0$$

for every $(x, t) \in \Omega \times (0, T)$. *Then* $u(x, t)$ *is a subsolution of* (3.28).

And analogously

Proposition 3.4.2. *Let* $u(x, t)$ *be Lipschitz and let us suppose that* $H : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *is convex in the last variable and that*

$$u_t + H(x, p) \geq 0$$

for every $(x, t) \in \Omega \times (0, T)$. *Then* $u(x, t)$ *is a supersolution of* (3.28).

3.4.1 Stability for viscosity solution of Hamilton-Jacobi equation

In this subsection we will discuss the property (II) of the viscosity solution and we will show in particular that under some hypothesis on the Hamiltonian H, the viscosity approximation indeed converges uniformly to the viscosity solution.
First of all we need a stability result that ensures that, assuming the uniform convergence of the viscosity approximation u^ε to u, then u is a viscosity solution

Proposition 3.4.3 (Stability). *Let u^ε be a family of a smooth solutions of*

$$u_t^\varepsilon + H(x, \nabla u^\epsilon) = \varepsilon \Delta u^\varepsilon \qquad \text{in } \Omega \times (0,T). \tag{3.29}$$

Assume in addition that $u^\varepsilon \to u$ uniformly in $\Omega \times (0,T)$ as $\varepsilon \to 0$. Then u is a viscosity solution of (3.28).

The proof of this fact is straightforward using the definition of viscosity solution. The difficult part normally relies on proving a priori bounds (independent on ε) on u^ε and ∇u^ε, in order to be able to show compacteness in the uniform topology and proving the existence of a limit u. Unfortunately the validity of these a priori bounds depends on the assumption on the Hamiltonian H. In particular we present the following result due to Lions in the context of semi-linear elliptic PDE ([36]) that shows a possible set of assumptions when $\Omega = \mathbb{R}^n$.
Consider the following initial value problem for (3.29):

$$\begin{cases} u_t^\varepsilon + H(x, \nabla u^\epsilon) = \varepsilon \Delta u^\varepsilon & \text{in } \mathbb{R}^n \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{3.30}$$

Then the following theorem holds:

Theorem 3.4.4. *Let the Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be Lipschitz and suppose that the following properties hold:*

a) $\lim_{|p|\to+\infty} H(x,p) = +\infty$ *uniformly for $x \in \mathbb{R}^n$,*

b) there exists $\alpha > 0$ such that

$$\langle H_p(x,p) - H_q(x,q), p-q\rangle \geq \alpha |p-q|^2 \quad \forall x, p, q, \tag{3.31}$$

c) there exists $\delta_0 > 0$ and $c_0 > 0$ such that

$$\liminf_{|p|\to+\infty} \left[\delta \nabla_x H(x,p)\cdot p + \delta^2 |p|^2 + \frac{H^2}{n} + c\delta \left(H - \nabla_p H \cdot p\right) \right] > 0, \tag{3.32}$$

uniformly for $x \in \mathbb{R}^n$ and for $0 \leq \delta \leq \delta_0$, $0 \leq c \leq c_0$.

Then, denoting by u^ε the solution of (3.30), *we have that for every $K \subset\subset \mathbb{R}^n$*

$$\|u^\varepsilon\|_{L^\infty(K)} < +\infty \tag{3.33}$$

and

$$\|Du^\varepsilon\|_{L^\infty(R^n)} < +\infty \tag{3.34}$$

independently on ε.

A consequence of Theorem 3.4.4 is that $u^\varepsilon \to u$ uniformly as $\varepsilon \to 0$ and therefore by Proposition 3.29 we have that u is a viscosity solution of 3.30.

3.4.2 Comparison principle for Hamilton-Jacobi equation

In this section we state and prove a classical comparison principle we will need in the next section. Notice that there are a lot of variants of this result that we are not going to threat here (see [7], [8]).

Theorem 3.4.5. *Consider the following Hamilton-Jacobi equation*

$$u_t + H(x, \nabla u) = 0 \quad (x,t) \in \mathbb{R}^n \times (0,T). \tag{3.35}$$

Assume $H \in UC(\mathbb{R}^n \times B_R(0))$ for every $R > 0$ and suppose that

(H1) *there exists a modulus of continuity $m : [0,+\infty) \to [0,+\infty)$ such that for every $x, y \in \mathbb{R}^n$ we have $|H(x,p) - H(y,p)| \leq m(|x-y|(1+|p|))$.*

Let $u_1, u_2 \in UC([0,T] \times \mathbb{R}^n)$ be viscosity sub- and supersolution of (3.35) *respectively; then we have*

$$\sup_{(x,t)\in\mathbb{R}^n\times(0,T)} (u_1(x,t) - u_2(x,t)) \leq \sup_{x\in\mathbb{R}^n} (u_1(x,0) - u_2(x,0)).$$

Proof. Set

$$A := \sup_{x\in\mathbb{R}^n} (u_1(x,0) - u_2(x,0)) < +\infty$$

and for $r \geq 0$ and $i = 1,2$ define

$$w_i(r) := \sup\{|u_i(x,t) - u_i(y,s)| : |t-s| + |x-y| \leq r\}. \tag{3.36}$$

As u_i are uniformly continuous we have

$$\sup_{r\geq 0} \frac{w_i(r)}{1+r} < +\infty. \tag{3.37}$$

Moreover

$$u_1(x,t) - u_2(y,s) = u_1(x,t) \pm u_1(x,0) - u_2(y,s) \pm u_2(x,0) \leq w_1(t) + A + w_2(s + |x-y|)$$

for all $(x,t),(y,s) \in \mathbb{R}^n \times (0,T)$.
Therefore, thanks to (3.37), there exists $C > 0$ such that we have

$$|u_1(x,t) - u_2(y,s)| \leq C(1 + |x-y|). \tag{3.38}$$

Set $Q := \mathbb{R}^n \times (0,T)$ and suppose by contradiction that

$$\sup_{(x,t)\in Q} (u_1(x,t) - u_2(x,t)) > A + \sigma_0$$

for some $\sigma_0 > 0$. For every $\varepsilon > 0$ consider the following function:

$$\Phi(x,t,y,s) := u_1(x,t) - u_2(y,s) - \frac{|x-y|^2}{\varepsilon}. \tag{3.39}$$

Thanks to (3.38) $\sup_{Q^2} \Phi < +\infty$; therefore for all $\delta > 0$ there exists $(x_0,t_0,y_0,s_0) \in Q^2$ such that

$$\Phi(x_0,t_0,y_0,s_0) + \delta > \sup_{Q^2} \Phi.$$

Consider now a cut off function $\xi \in C_0^\infty(\mathbb{R}^n)$ such that $\xi(x_0,y_0) = 1$, $0 \leq \xi \leq 1$ and $|\nabla\xi(x,y)| \leq 1$. Choosing $0 < \sigma < \sigma_0/2T$ and defining

$$\Psi(x,t,y,s) := \Phi(x,t,y,s) + \delta\xi(x,y) - \sigma t \tag{3.40}$$

it is clear that Ψ has a maximum in Q^2, so there exists $(\bar{x},\bar{t},\bar{y},\bar{s})$ such that $\Psi(\bar{x},\bar{t},\bar{y},\bar{s}) = \sup_{Q^2} \Psi$. Thus, in particular,

$$\Psi(\bar{x},\bar{t},\bar{x},\bar{t}) + \Psi(\bar{y},\bar{s},\bar{y},\bar{s}) \leq 2\Psi(\bar{x},\bar{t},\bar{y},\bar{s}). \tag{3.41}$$

Then using (3.41) and (3.38) we obtain easily

$$\frac{|\bar{x}-\bar{y}|^2}{\varepsilon} \leq C(1 + |\bar{x}-\bar{y}|) + 4\delta + \sigma \leq C(1 + |\bar{x}-\bar{y}|) + 2\sigma_0,$$

for $0 < \delta < \sigma_0/4$ and $0 < \sigma < \sigma_0/2T$. It is not difficult to see that there exists $D > 0$ independent on ε such that

$$|\bar{x}-\bar{y}| \leq D\sqrt{\varepsilon}, \tag{3.42}$$

and

$$\frac{|\bar{x}-\bar{y}|^2}{\varepsilon} \to 0, \qquad \text{as } \varepsilon \to 0. \tag{3.43}$$

The function $Z(x,t) := u_2(\bar{y},\bar{s}) + \frac{|x-\bar{y}|^2}{\varepsilon} - \delta\xi(x,\bar{y}) + \sigma t$ is such that $u_1 - Z$ has a maximum in $(\bar{x},\bar{t})$, so by definition of viscosity solution

$$\sigma + H(\bar{x}, \frac{1}{\varepsilon}(\bar{x}-\bar{y}) - \delta\nabla_x\xi(\bar{x},\bar{y})) \leq 0 \tag{3.44}$$

and analogously

$$H(\bar{y}, \frac{1}{\varepsilon}(\bar{x}-\bar{y}) + \delta\nabla_y \xi(\bar{x},\bar{y})) \leq 0. \tag{3.45}$$

Subtracting the previous inequalities we get

$$\begin{aligned}
\sigma &\leq H(\bar{y}, \frac{1}{\varepsilon}(\bar{x}-\bar{y}) + \delta\nabla_y \xi(\bar{x},\bar{y})) - H(\bar{x}, \frac{1}{\varepsilon}(\bar{x}-\bar{y}) - \delta\nabla_x \xi(\bar{x},\bar{y})) \\
&\leq \left| H(\bar{y}, \frac{1}{\varepsilon}(\bar{x}-\bar{y}) + \delta\nabla_y \xi(\bar{x},\bar{y})) - H(\bar{x}, \frac{1}{\varepsilon}(\bar{x}-\bar{y}) - \delta\nabla_y \xi(\bar{x},\bar{y}))\right| \\
&+ \left| H(\bar{x}, \frac{1}{\varepsilon}(\bar{x}-\bar{y}) - \delta\nabla_y \xi(\bar{x},\bar{y})) - H(\bar{x}, \frac{1}{\varepsilon}(\bar{x}-\bar{y}) - \delta\nabla_x \xi(\bar{x},\bar{y}))\right| \\
&\leq m\left(|\bar{x}-\bar{y}|\left(1+\frac{1}{\varepsilon}|\bar{x}-\bar{y}|+\delta\right)\right) + \omega\left(2\delta, \frac{1}{\varepsilon}|\bar{x}-\bar{y}|+\delta\right),
\end{aligned}$$

where

$$\omega(r,R) := \sup\{|H(x,q)-H(x,p)| : x \in \mathbb{R}^n,\ |q-p| \leq r,\ p,q \in B(0,R)\} \tag{3.46}$$

and in the last estimate we apply (**H1**).
Choosing δ and ε small enough and using (3.42) and (3.43) we get a contradiction. □

The comparison principle implies uniqueness of the viscosity solution of the Hamilton-Jacobi equation (3.35) as the following corollary shows.

Corollary 3.4.6. *Let $u_1, u_2 \in UC(\mathbb{R}^n \times (0,T))$ be viscosity solutions of the Cauchy problem*

$$\begin{cases} u_t + H(x,\nabla u) = 0 & \text{in } \mathbb{R}^n \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^n, \end{cases} \tag{3.47}$$

*with $H \in UC(\mathbb{R}^n \times B_R(0))$ for every $R > 0$ and $u_0 \in UC(\mathbb{R}^n)$. Suppose in addition that H satisfies hypothesis (**H1**); then $u_1 = u_2$.*

3.4.3 Discontinuous viscosity solutions

In this section we introduce the concept of discontinuous viscosity solution of (3.28). In particular we are going to focus on the stationary case, because it is the relevant one for the next section. However we remark that a similar theory can be carried out for the evolutive Hamilton-Jacobi equation and we refer to [8] for further details.
We consider Cauchy problems of the following form:

$$\begin{cases} H(x,\nabla u) = 0 & \text{in } \Omega \\ u(x) = u_0(x) & \text{on } \partial\Omega, \end{cases} \tag{3.48}$$

where Ω is an open, regular subset of $\mathbb{R}^n$ (possibly unbounded), $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is locally bounded and $u_0 : \partial\Omega \to \mathbb{R}$ is continous.
From now on given a function z locally bounded we will denote by z_* and z^* the lower and the upper semicontinous envelope of z respectively.
In order to provide a definition of viscosity solution for (3.48) we want to include the boundary condition into an Hamiltonian defined on $\overline{\Omega}$. Therefore (3.48) will be transformed into the equation $G(x, u, \nabla u) = 0$ where $G : \overline{\Omega} \to \mathbb{R}$ is defined as

$$G(x, u, \nabla u) = \begin{cases} H(x, \nabla u) & x \in \Omega \\ u - u_0 & x \in \partial\Omega. \end{cases} \tag{3.49}$$

Definition 3.4.2 (Discontinuous viscosity solutions)**.** *Consider $u : \overline{\Omega} \to \mathbb{R}$, locally bounded:*

- $\star$) *If u is upper semicontinous, we say that u is a subsolution of* (3.48) *if for every $x_0 \in \overline{\Omega}$ and for every $v \in C^\infty(\overline{\Omega})$ such that $u - v$ has a maximum in x_0 we have*
$$G_*(x_0, u(x_0), \nabla v(x_0)) \leq 0.$$
- $\star$) *If u is lower semicontinous, we say that u is a superolution of* (3.48) *if for every $x_0 \in \overline{\Omega}$ and for every $v \in C^\infty(\overline{\Omega})$ such that $u - v$ has a minimum in x_0 we have*
$$G^*(x_0, u(x_0), \nabla v(x_0)) \geq 0.$$

A viscosity solution (discontinous) of (3.48) *is a locally bounded function $u : \overline{\Omega} \to \mathbb{R}$ such that u^* is a subsolution of* (3.48) *and u_* is a supersolution of* (3.48).

Remark 3.4.7. *If $H : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is continous, then $G_* = G^* = H$ in $\Omega \times \mathbb{R}^n$ and for every $x \in \partial\Omega$*

$$G_*(x, u, \nabla u) = \min\{H(x, \nabla u), u - u_0\}, \quad G^*(x, u, \nabla u) = \max\{H(x, \nabla u), u - u_0\}.$$

Hence, in this case, Definition 3.4.2 can be reformulated in an explicit way, without the usage of the l.s.c. and u.s.c. envelopes of G.

Stability result for discontinous viscosity solutions and the method of the half-relaxed limits

Apart from the theorical interest of having a notion of viscosity solution extended to discontinous functions, one of the main goal of this theory is to be able to pass to the limit in a sequence of viscosity solutions depending on a parameter when it is not possible to obtain an uniform bound on the gradient.
We first define the concept of *half-relaxed limits* and we state a stability result for them, that is the counterpart of Proposition 3.4.3 for discontinous viscosity solutions.

Definition 3.4.3 (Half-relaxed limits)**.** *Suppose we have a sequence $(z^\varepsilon)_\varepsilon$ of uniformly locally bounded functions, then*

$$\bar{z}(y) = \limsup{}^* \mathrm{z}^\varepsilon(\mathrm{y}) = \limsup_{\tilde{\mathrm{y}}\to \mathrm{y},\varepsilon\to 0} \mathrm{z}^\varepsilon(\tilde{\mathrm{y}}) \quad \textit{and} \quad \underline{\mathrm{z}}(\mathrm{y}) = \liminf{}^* \mathrm{z}^\varepsilon(\mathrm{y}) = \liminf_{\tilde{\mathrm{y}}\to \mathrm{y},\varepsilon\to 0} \mathrm{z}^\varepsilon(\tilde{\mathrm{y}})$$

are respectively the upper and the lower relaxed limits of z.

Theorem 3.4.8 (Stability for relaxed limits)**.** *Suppose that for every $\varepsilon > 0$, u_ε is a subsolution (resp. supersolution) of the equation*

$$G_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) = 0 \quad \textit{in } \overline{\Omega}$$

where $G_\varepsilon : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a locally uniformly bounded sequence.
Suppose in addition that $(u_\varepsilon)_\varepsilon$ is locally uniformly bounded in $\overline{\Omega}$. Then $\overline{u}$ (resp. $\underline{u}$) is a subsolution (resp. supersolution) of

$$\underline{G}(x, u, \nabla u) = 0 \quad \textit{in } \overline{\Omega}.$$

$$(\textit{resp.} \quad \overline{G}(x, u, \nabla u) = 0 \quad \textit{in } \overline{\Omega})$$

A consequence of the previous theorem is the following lemma that links the equality of the half-relaxed limits with the uniform convergence of the sequence. For the proof we refer to [8].

Lemma 3.4.9. *If K is a compact set of $\mathbb{R}^n$ and $\underline{\phi} = \bar{\phi}$ on K, then $\phi^\varepsilon \to \underline{\phi} = \bar{\phi}$ uniformly on K.*

The strategy of the half-relaxed limits approach is the following: suppose that one wants to compute the uniform limit of the viscosity solution u^ε of

$$G_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) = 0 \quad \text{in } \overline{\Omega}.$$

without having a bound for ∇u_ε and supposing that G_ε is uniformly continous and it converges uniformly to G.
Then one can prove a L^∞ bound for u^ε and the define half-relaxed limits $\underline{u}$ and $\overline{u}$. By definition $\underline{u} \leq \overline{u}$ and by Theorem 3.4.8 $\overline{u}$ (resp. $\underline{u}$) is a viscosity subsolution (resp. supersolution) of

$$G(x, u, \nabla u) = 0 \quad \text{in } \overline{\Omega}.$$

If one is able to prove a comparison principle for discontinuous viscosity solutions (called *strong comparison principle*) then one can obtain the opposite inequality $\overline{u} \leq \underline{u}$ and therefore $u := \overline{u} = \underline{u}$. So, as $\overline{u}$ is u.s.c. and $\underline{u}$ is l.s.c., u turns out to be continous and moreover $u_\varepsilon \to u$ uniformly as $\varepsilon \to 0$ on every compact set.

Strong comparison principle for discontinous viscosity solutions

We state here a version of strong comparison principle for discontinous viscosity solution (see [8]).

Theorem 3.4.10 (Comparison principle for discontinuous viscosity solutions)**.** *Suppose that $\Omega \subset \mathbb{R}^n$ is an open, regular set of $\mathbb{R}^n$. Consider the following initial value problem for the Hamilton-Jacobi equation:*

$$\begin{cases} H(x, \nabla u) = 0 & \text{in } \Omega \\ u(x,0) = u_0(x) & \text{on } \partial\Omega. \end{cases} \tag{3.50}$$

Suppose that $H(x,p) \in UC(\mathbb{R}^n \times B_R(0))$ for every $R > 0$, it is convex in p for every $x \in \Omega$ and there exists a function ϕ of class C^1 on Ω and continuous on $\overline{\Omega}$ such that

$$H(x, \nabla\phi) \leq \alpha < 0.$$

Then given $u, v \in UC(\Omega)$ such that u is a subsolution of (3.50) *and v is a supersolution of* (3.50)*, we have*

$$u(x) \leq v(x) \quad \text{for every } x \in \Omega.$$

3.5 Long time behaviour for the reinitialization of the distance function

As anticipated in the previous section we are going to study the long time behaviour of the reinitialization of the distance function as introduced in Section 3.3.3 (see also [13]).

3.5.1 Setting of the problem

We are considering the following Cauchy problem for an Hamilton-Jacobi equation:

$$\begin{cases} \phi_t + f(x)(|\nabla\phi| - 1) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\ \phi(x,0) = \phi_0(x) & \text{in } \mathbb{R}^n \end{cases} \tag{3.51}$$

where $\phi_0 \in C^1(\mathbb{R}^n)$.
We will denote it by Γ the zero level set of ϕ_0:

$$\Gamma := \{x \in \mathbb{R}^n : \phi_0(x) = 0\}. \tag{3.52}$$

Moreover we will call D_+ and D_- the external and the internal part of Γ:

$$D_+ := \{x \in \mathbb{R}^2 : \phi_0(x) > 0\} \quad D_- := \{x \in \mathbb{R}^2 : \phi_0(x) < 0\}. \tag{3.53}$$

We assume the following hypothesis on $f(x)$ and on the initial data ϕ_0:

(**G1**) f is Lipschitz with Lipschitz constant $L > 0$,

(**G2**) $\|f\|_\infty \le C_1$,

(**G3**) $\Gamma = \{x \in \mathbb{R}^n : f(x) = 0\}$, $\qquad D_+ = \{x \in \mathbb{R}^n : f(x) > 0\}$, and $\qquad D_- = \{x \in \mathbb{R}^n : f(x) < 0\}$,

(**G4**) $\sup_{x\in\Gamma} \|\nabla\phi_0(x)\| > 0$,

(**G5**) $\|\nabla\phi_0\|_\infty \le C_2$.

From now on we will denote by $d_s(x,\Gamma)$ the signed distance from Γ.

Remark 3.5.1. *The Hamilton-Jacobi equation for the reinitialization of the distance function satisfies the previous hypothesis with*

$$f(x) = f_\delta(x) = \frac{\phi_0(x)}{\sqrt{\phi_0(x)^2 + \delta^2}}$$

for every $\delta > 0$*.*

3.5.2 Existence of solution and uniform Lipschitz property

The existence and uniqueness of the viscosity solution of (3.51) follow from the comparison principle (Theorem 3.4.5) and Perron's method developed by Ishii in [32]. In particular to apply Perron's method we have to exhibit a subsolution ϕ_* and a supersolution ϕ^* of (3.51) such that

$$\phi_*(x,0) \le \phi_0(x) \le \phi^*(x,0) \tag{3.54}$$

for every $x \in \mathbb{R}^n$.
In the current case one can readily verify that thanks to hypothesis (**G5**) there exists $C > 0$ such that, $\phi^*(x,t) = \phi_0(x) + Ct$ and $\phi_*(x,t) = \phi_0(x) - Ct$ are a supersolution and a subsolution of (3.51) respectively and they satisfy (3.54).

Proposition 3.5.2 (Lipschitz estimates)**.** *Given* ϕ *the viscosity solution of* (3.51) *there exists* $C > 0$ *such that*

$$\|\phi_t\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}_+)} \le C \tag{3.55}$$

and for almost every $x \in \mathbb{R}^n \setminus \Gamma$

$$\|\nabla\phi(x,\cdot)\|_{L^\infty(\mathbb{R}_+)} \le \frac{C}{|f(x)|} + 1. \tag{3.56}$$

Proof. Let $\phi(x,t)$ be a viscosity solution of (3.51). For $h>0$ we have that also $\phi(x,t+h)$ is viscosity solution of (3.51). Therefore by the comparison principle (Theorem 3.4.5) there holds

$$\|\phi(x,t+h)-\phi(x,t)\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}_+)} \leq \|\phi(x,h)-\phi(x,0)\|_{L^\infty(\mathbb{R}^n)}. \tag{3.57}$$

Moreover thanks to hypothesis (**G5**) there exists $C>0$ such that, $\phi^*(x,t)=\phi_0(x)+Ct$ and $\phi_*(x,t)=\phi_0(x)-Ct$ are a supersolution and a subsolution of (3.51) respectively and they satisfy (3.54). Hence by the comparison principle we obtain

$$\phi_*(x,t)\leq\phi(x,t)\leq\phi^*(x,t) \quad \text{in } \mathbb{R}^n\times[0,+\infty).$$

Thanks to (3.57) we have

$$\|\phi(x,t+h)-\phi(x,t)\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}_+)} \leq Ch \tag{3.58}$$

which implies that $\|\phi_t\|_{L^\infty(\mathbb{R}^n\times\mathbb{R}_+)}\leq C$.
The bound on $|\nabla\phi|$ follows from equation (3.51). □

Remark 3.5.3. *In contrast with [42] we cannot rely on uniform Lipschitz estimate up to* Γ*. This is a consequence of the lack of coercivity of the Hamiltonian close to* Γ*.*

3.5.3 Preservation of the zero level set

In the next result we want to show that the zero level set of the initial data Γ is preserved in the viscosity solution $\phi(x,t)$ for every $t>0$. This is an adaptation of the argument in [42].

Proposition 3.5.4. *Let* $\phi(x,t)$ *a viscosity solution of* (3.51) *then letting*

$$\Gamma_t := \{x\in\mathbb{R}^2 : \phi(x,t)=0\}$$

one has $\Gamma_t=\Gamma$ *for every* $t>0$.

Proof. Consider $x_0\in\Gamma$ and denote by $B(x_0,\varepsilon)$ a ball centered in x_0 and with radius ε. Moreover let us define $B^+(x_0,\varepsilon):=B(x_0,\varepsilon)\cap D_+$ and $B^-(x_0,\varepsilon):=B(x_0,\varepsilon)\cap D_-$.
As ϕ is a viscosity solution of (3.51), the equation holds almost everywhere. Therefore for every $t\geq 0$ and $h>0$ there holds

$$\begin{aligned}\frac{1}{\varepsilon^n}\int_{B^+(x_0,\varepsilon)}\phi(x,t+h)-\phi(x,t)\,dx &= \int_t^{t+h}\int_{B^+(x_0,\varepsilon)}\phi_t(x,s)\,ds\,dx\\ &= \frac{1}{\varepsilon^n}\int_t^{t+h}\int_{B^+(x_0,\varepsilon)} f(x)(1-|\nabla\phi|)\,dx\,ds\\ &\leq \frac{1}{\varepsilon^n}\int_t^{t+h}\int_{B^+(x_0,\varepsilon)} f(x)\,dx\,ds.\end{aligned}$$

Sending ε to zero we get that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^n}\int_{B^+(x_0,\varepsilon)}\phi(x,t+h)-\phi(x,t)\,dx=0.$$

On the other hand

$$\begin{aligned}\frac{1}{\varepsilon^n}\int_{B^-(x_0,\varepsilon)}\phi(t+h,x)-\phi(t,x)\,dx &= \frac{1}{\varepsilon^n}\int_t^{t+h}\int_{B^-(x_0,\varepsilon)}f(x)(1-|\nabla\phi|)\,dx\,ds\\ &\geq \frac{1}{\varepsilon^n}\int_t^{t+h}\int_{B^-(x_0,\varepsilon)}f(x)\,dx\,ds.\end{aligned}$$

Hence

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^n}\int_{B^-(x_0,\varepsilon)}\phi(x,t+h)-\phi(x,t)\,dx=0.$$

As ϕ is continuous this implies that $\phi(x,t+h)=\phi(x,t)$ for every $x\in\Gamma$ and in particular $\phi(x,t)=0$ for every $t\geq 0$.

□

Unfortunately this argument is not sufficient to obtain the preservation of the zero level set in the limit as $t\to+\infty$; this is due to the lack of coercivity of the Hamiltonian. Therefore, in order to overcome this difficulty we are going to build barriers for the viscosity solution in such a way that we can control the zero level set of the solution uniformly in time close to Γ (see [30] for an other example of application of this technique).

Proposition 3.5.5 (Construction of the barriers)**.** *There exists two locally Lipschitz functions $\phi^*,\phi_*:\mathbb{R}^n\times[0,+\infty)\to\mathbb{R}$ such that they are independent on t in a σ-neighborhood of Γ, $\phi_*(x,t)=\phi^*(x,t)=0$ for every $x\in\Gamma$ and $t\geq 0$ and*

$$\phi_*(x,t)\leq\phi(x,t)\leq\phi^*(x,t)$$

for every $(x,t)\in\mathbb{R}^n\times[0,+\infty)$.

Proof. Notice firstly that thanks to (**G4**) there exists $M>0$ such that

$$\sup_{x\in\Gamma}\|\nabla\phi_0(x)\|>2M.$$

Moreover as $\phi_0\in C^1(\mathbb{R}^n)$ there exists $\sigma>0$ such that $|\nabla\phi_0|\geq M$ in $\Gamma_{2\sigma}$.

Define $\Gamma_\sigma^+=\{x\in\mathbb{R}^2:0\leq\phi_0(x)\leq\sigma\}$ and let us consider the following function $\phi^*:\mathbb{R}^n\times[0,+\infty)\to\mathbb{R}$:

$$\phi^*(x,t)=\begin{cases}\frac{1}{M}\phi_0(x) & \text{in } (D_+\cap\Gamma_\sigma^+)\times[0,+\infty)\\ \frac{1}{M}\phi_0(x)e^{kt(\phi_0-\sigma)^2} & \text{in } (D_+\setminus\Gamma_\sigma^+)\times[0,+\infty)\\ \frac{1}{C_2}\phi_0(x) & \text{in } D_-\times[0,+\infty),\end{cases}\tag{3.59}$$

where $k > 0$ will be choosen later. It is easy to verify that ϕ^* is differentiable for every $x \notin \Gamma$. We want to prove that this defines a supersolution.
For $x \in D_+ \cap \Gamma^+_\sigma$ and $t \geq 0$ one has $\partial_t \phi^*(x,t) = 0$ and $|\nabla \phi^*| = \frac{1}{M}|\nabla \phi_0| \geq 1$, hence ϕ^* is a supersolution. Moreover for every $x \in D_+ \setminus \Gamma^+_\sigma$ and $t \geq 0$ we have

$$\partial_t \phi^* = \frac{2}{M} k \phi_0 (\phi_0 - \sigma)^2 e^{kt(\phi_0-\sigma)^2} \text{ and } |\nabla \phi^*| = \frac{1}{M} |\nabla \phi_0| e^{kt(\phi_0-\sigma)^2} |1 + 2\phi_0(\phi_0 - \sigma)kt|.$$

Therefore given a point $x \in D_+ \cap (\Gamma^+_{2\sigma} \setminus \Gamma^+_\sigma)$ we have that $\partial_t \phi^* \geq 0$ and

$$|\nabla \phi^*| \geq 1,$$

so that ϕ^* is a supersolution for $(x,t) \in D_+ \setminus \Gamma^+_\sigma \times [0, +\infty)$. On the other hand given a point $x \in D_+ \setminus \Gamma^+_{2\sigma}$

$$\partial_t \phi^* \geq \frac{2}{M} k \sigma^3.$$

So as a consequence of (**G2**) it is enough to choose $k \geq \frac{C_1 M}{2\sigma^3}$ to infer that ϕ^* is a supersolution in $D_+ \times [0, +\infty)$.
As for $x \in D_-$, using hypothesis (**G5**) we obtain

$$\partial_t \phi^* + f(x)(|\nabla \phi^*| - 1) = f(x) \left(\frac{|\nabla \phi_0|}{C_2} - 1 \right) \geq 0,$$

hence $\phi^\star$ is a subsolution in D_-.
Finally for $x \in \Gamma$ we have that $H(x,p) = 0$ for every $p \in \mathbb{R}^n$; therefore, as ϕ^* does not depend on t in a neighborhood of Γ, ϕ^* is a supersolution in $\mathbb{R}^n \times [0, +\infty)$.

Define $\Gamma^-_\sigma = \{x \in \mathbb{R}^2 : -\sigma \leq \phi_0(x) \leq 0\}$ and as before consider the following function $\phi_* : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}$:

$$\phi_*(x,t) = \begin{cases} \frac{1}{C_2}\phi_0(x) & \text{in } D_+ \times [0,+\infty) \\ \frac{1}{M}\phi_0(x) & \text{in } (D_- \cap \Gamma^-_\sigma) \times [0,+\infty) \\ \frac{1}{M}\phi_0(x) e^{kt(\phi_0+\sigma)^2} & \text{in } (D_- \setminus \Gamma^-_\sigma) \times [0,+\infty) \end{cases} \tag{3.60}$$

with $k \geq \frac{C_1 M}{2\sigma^3}$. With similar computation to the first part of the proof it is easy to prove that ϕ_* defines a subsolution.

Finally notice that $\phi_*(x,0) \leq \phi_0(x) \leq \phi^*(x,0)$. Therefore by comparison principle we obtain

$$\phi_*(x,t) \leq \phi(x,t) \leq \phi^*(x,t) \tag{3.61}$$

for every $(x,t) \in \mathbb{R}^n \times [0, +\infty)$.

□

Corollary 3.5.6. *Given $\phi(x,t)$ the viscosity of* (3.51) *we have that $\phi(x,t)$ is locally bounded in $\mathbb{R}^2$ uniformly in $t \in [0, +\infty)$.*

Proof. From Proposition 3.5.5 we have that

$$\phi_*(x,t) \le \phi(x,t) \le \phi^*(x,t) \tag{3.62}$$

for every $(x,t) \in \mathbb{R}^n \times [0,+\infty)$ where $\phi^\star$ and $\phi_\star$ are defined as in (3.59) and (3.60). Therefore $\phi(x,t)$ is bounded in Γ_σ for some $\sigma > 0$ and

$$|\phi(x,t)| \le C_M := \frac{1}{M} \max_{\partial\Gamma_\sigma} \phi_0(x) \qquad \text{in } \partial\Gamma_\sigma \times \mathbb{R}_+. \tag{3.63}$$

Moreover by Proposition 3.5.2 we have that

$$\|\nabla\phi\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}_+)} \le C \quad \text{ in } \mathbb{R}^2 \setminus \Gamma_\sigma. \tag{3.64}$$

Suppose by contraddiction that there exists x_0 in $D_- \setminus \Gamma_\sigma$ such that $|\phi(x_0,t)| \to +\infty$ as $t \to +\infty$. So we can fin $t_1 > 0$ such that $|\phi(x,t_1)| \ge Cd(x_0,\Gamma)+C_M+1$. By the estimate (3.64) we have also that there exists a segment l_{x_0} passing through x_0 such that $\nabla\phi(x,t_1)$ exists and $\|\nabla\phi(x,t_1)\| \le C$ for almost every $x \in l_{x_0} \cap (D_- \setminus \Gamma_\sigma)$. Hence, denoting by x_1 the first intersection of l_{x_0} with $\partial\Gamma_\sigma$ we have by (3.63)

$$\begin{aligned} Cd(x_0,\Gamma)+1 &\le \phi(x_0,t_1) - \phi(x_1,t_1) \\ &= \int_0^1 \langle \nabla\phi(x_0 + h(x_1-x_0),t_1), x - x_0 \rangle \, dh \\ &\le C|x-x_0|, \end{aligned}$$

that is a contradiction. In an analogous way one reaches a contradiction supposing that there exists $x_0 \in D_+ \setminus \Gamma_\sigma$ such that $|\phi(x_0,t)| \to +\infty$ as $t \to +\infty$.

□

3.5.4 Convergence to the signed distance function

In this subsection we are going to prove that for $t \to +\infty$ the viscosity solution of (3.51) converges to the signed distance function uniformly on the compact sets of $\mathbb{R}^n$.

We will homogenize (3.51) and then pass to the limit in the equation using relaxed limits. Given $\varepsilon > 0$ consider the rescaling $\phi^\varepsilon(x,t) = \phi(x,t/\varepsilon)$. Then it is easy to verify that if ϕ is a viscosity solution of the reinitialization problem, then ϕ^ε is a viscosity solution of

$$\begin{cases} \epsilon\phi^\varepsilon_t + f(x)(|\nabla\phi^\varepsilon| - 1) = 0 & \text{in } \mathbb{R}^n \times (0,+\infty), \\ \phi^\varepsilon(x,0) = \phi_0(x) & \text{in } \mathbb{R}^n \end{cases} \tag{3.65}$$

for every $\varepsilon > 0$.

In order to pass to the limit as $\varepsilon \to 0$ we employ the method of half-relaxed limits to ϕ^ε as described in Section 3.4.3. We will denote by $\bar{\phi}$ and $\underline{\phi}$ the upper and the lower relaxed limit of ϕ^ε respectively, that exists thanks to Corollary 3.5.6.

Theorem 3.5.7. *Let $\phi(x,t)$ a viscosity solution of* (3.51) *then $\phi(x,t)$ converges uniformly to $d_s(x,\Gamma)$ as $t\to\infty$ on every compact set of $\mathbb{R}^n$.*

Proof. From the preservation of the zero level set of ϕ follows that

$$\{x : \phi^\varepsilon(x,t) = 0\} = \Gamma \quad \forall t > 0.$$

Moreover thanks to Proposition 3.5.5

$$\phi_*\left(x, \frac{t}{\varepsilon}\right) \le \phi^\varepsilon(x,t) \le \phi^*\left(x, \frac{t}{\varepsilon}\right)$$

for every $(x,t) \in \mathbb{R}^n \times [0,+\infty)$ and $\varepsilon > 0$ and by construction ϕ^* and ϕ_* depend only on x in a neighborhood of Γ. Therefore taking the upper and the lower half relaxed limits on both sides we obtain that $\underline{\phi}(x_0,t) = \bar{\phi}(x_0,t) = 0$ for every $x_0 \in \Gamma$.
Moreover as $\phi^*(x,t)$ is strictly negative in $D_- \times [0,+\infty)$ and $\phi_*(x,t)$ is strictly positive $D_+ \times [0,+\infty)$ we infer that the preservation of the zero level set is inherited by the approximate limits, i.e.

$$\Gamma = \{x : \bar{\phi}(x,t) = 0\} = \{x : \underline{\phi}(x,t) = 0\}. \tag{3.66}$$

for every $t \in [0,+\infty)$.
Proposition 3.4.8 implies that $\bar{\phi}$ ($\underline{\phi}$) is a subsolution (supersolution) of

$$f(x)(|\nabla\phi(x,t))| - 1) = 0.$$

This yields that $\bar{\phi}$ ($\underline{\phi}$) is a subsolution (supersolution) of the following equation in D_+:

$$|\nabla\phi(x,t)| - 1 = 0.$$

Fixing $t_0 \in (0,+\infty)$ we have that $\bar{\phi}(x,t_0)$ is a subsolution and $\underline{\phi}(x,t_0)$ is a supersolution of the eikonal equation in D^+:

$$|\nabla u(x)| - 1 = 0. \tag{3.67}$$

Notice that $d(x,\Gamma)$ is a viscosity solution of (3.67) (here $d(x,\Gamma)$ is the usual distance). This implies by the strong comparison principle for discontinuous viscosity solutions (Theorem 3.4.10) and (3.66) that for every $t_0 > 0$

$$\bar{\phi}(x,t_0) \le d_s(\Gamma,x) \le \underline{\phi}(x,t_0) \quad \text{in } D_+.$$

So for every $(x,t) \in D_+ \times (0,+\infty)$ we have $\bar{\phi}(x,t) = \underline{\phi}(x,t) = d_s(x,\Gamma)$.

On the other hand $\bar{\phi}$ ($\underline{\phi}$) is a subsolution (supersolution) of the following equation in D_-

$$-|\nabla\phi(x,t)| + 1 = 0.$$

Fixing again $t_0 \in (0, +\infty)$, then $\bar{\phi}(x, t_0)$ is a subsolution and $\underline{\phi}(x, t_0)$ is a supersolution in D_- of

$$-|\nabla u(x)| + 1 = 0, \tag{3.68}$$

and consequently $-\bar{\phi}(x, t_0)$ is a supersolution and $-\underline{\phi}(x, t_0)$ is a subsolution in D_- of

$$|\nabla u(x)| - 1 = 0. \tag{3.69}$$

This implies by the strong comparison principle for discontinuity viscosity solutions (Theorem 3.4.10) and (3.66) that for every $t_0 > 0$

$$-\underline{\phi}(x, t_0) \leq d(\Gamma, x) \leq -\bar{\phi}(x, t_0) \quad \text{ in } D_-$$

and hence

$$\bar{\phi}(x, t_0) \leq -d(\Gamma, x) \leq \underline{\phi}(x, t_0) \quad \text{ in } D_-.$$

So for every $(x, t) \in D_- \times (0, \infty)$, we have $\bar{\phi}(x, t) = \underline{\phi}(x, t) = d_s(\Gamma, x)$.
Therefore

$$\underline{\phi}(x, t) = \bar{\phi}(x, t) = d_s(x, \Gamma) \quad \text{in } \mathbb{R}^n \times (0 + \infty). \tag{3.70}$$

By Lemma 3.4.9 we infer that

$$\phi^\varepsilon(x, t) \to d_s(x, \Gamma)$$

as $\varepsilon \to 0$, uniformly on every compact set of $\mathbb{R}^n$.
Finally recalling the definition of ϕ^ε one has the thesis.

□

Chapter 4

Appendix

4.1 Functions of bounded variation

Let $\Omega \subset \mathbb{R}^n$ be an open set.

Definition 4.1.1. *Given $u \in L^1(\Omega)$ we say that u is of bounded variation in Ω ($u \in BV(\Omega)$) if its distributional derivative is a finite Radon measure in Ω.*

From Definition 4.1.1 it follows immediately that if $u \in BV(\Omega)$, then

$$|Du|(\Omega) := \sup\left\{\int_\Omega u \,\mathrm{div}\,\phi\, dx : \phi \in C_c^1(\Omega), \|\phi\|_\infty \leq 1\right\} < +\infty \tag{4.1}$$

and $BV(\Omega)$ is a Banach space endowed with the norm

$$\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + |Du|(\Omega).$$

We define the singular set of a function of bounded variation as the complement of the set where an approximate limit exists.

Definition 4.1.2 (Approximate limit)**.** *Given $u \in BV(\Omega)$ we say that u has an approximate limit at $x \in \Omega$ if there exists $z \in \mathbb{R}$ such that*

$$\lim_{r\to 0} \fint_{B_r(x)} |u(y) - z|\, dx = 0.$$

Definition 4.1.3 (Singular set)**.** *We will call singular set of u, denoted by S_u, the set of points where u does not admit an approximate limit.*

In order to define the jump set, we introduce a convenient notation for the half ball of radius r and center x_0 with respect to a direction $\nu \in S_{n-1}$:

$B_r^+(x,\nu) = \{y \in B_r(x) : \langle y-x, \nu\rangle > 0\}$ and $B_r^-(x,\nu) = \{y \in B_r(x_0) : \langle y-x, \nu\rangle < 0\}$.

Definition 4.1.4 (Jump set)**.** *Given $u \in BV(\Omega)$ and $x \in \Omega$, we say that x belongs to the jump set of u, denoted by J_u, if there exists $u^+(x), u^-(x) \in \mathbb{R}$ and $\nu_u \in S_{n-1}$ such that*

$$\lim_{r\to 0} \fint_{B_r^+(x),\nu_u} |u(y) - u^+(x)|\, dx = 0 \quad \textit{and} \quad \lim_{r\to 0} \fint_{B_r^-(x),\nu_u} |u(y) - u^-(x)|\, dx = 0. \tag{4.2}$$

In particular it is possible to prove that in $u \in BV(\Omega)$, then S_u and J_u are rectifiable sets and ν_u defined as in Definition 4.1.4 is approximate normal to J_u, usually denoted by ν_{J_u}.
Moreover the values $u^+(x)$ and $u^-(x)$ are called upper and lower trace of u in a point $x \in \Omega$. The following theorem holds:

Theorem 4.1.1. *For any $u \in BV(\Omega)$ we have that $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$ and*

$$Du \llcorner J_u = (u^+ - u^-)\nu_{J_u}\mathcal{H}^{n-1} \llcorner J_u \tag{4.3}$$

and

$$|Du| \llcorner J_u = |u^+ - u^-|\mathcal{H}^{n-1} \llcorner J_u. \tag{4.4}$$

Decomposition of the derivative

Given a function $u \in BV(\Omega)$, its derivative can be decomposed by Radon Nikodym Theorem in the absolute continous part with respect to the Lebesgue measure $\mathscr{L}^n$ and the singular part:

$$Du = \nabla u \llcorner \mathscr{L}^n + D^s u,$$

where ∇u is the approximate gradient and it is defined as the density of Du with respect to $\mathscr{L}^n$ and $D^s u$ is the singular part of Du.

Definition 4.1.5 (Jump part and Cantor part). *Given $u \in BV(\Omega)$ we define the jump part and the Cantor part of Du as*

$$D^j u := D^s u \llcorner J_u \quad \text{and} \quad D^c u = D^s u \llcorner (\Omega \setminus S_u). \tag{4.5}$$

Therefore thanks to Theorem 4.1.1 it is possible to decompose the derivative of a function of bounded variation Du in the following way:

$$Du = \nabla u \llcorner \mathscr{L}^n + D^s u = \nabla u \llcorner \mathscr{L}^n + (u^+ - u^-)\nu_{J_u}\mathcal{H}^{n-1} \llcorner J_u + D^c u. \tag{4.6}$$

Special functions of bounded variation

We can then define the space of *special functions of bounded variation*, denoted by $SBV(\Omega)$.

Definition 4.1.6. *Given $u \in BV(\Omega)$, we say that $u \in SBV(\Omega)$ if $D^c u = 0$.*

4.2 Basic definitions and notations for currents

Let $U \subset \mathbb{R}^N$ be an open subset and let us denote by $\Lambda^k(U)$ the set of k-forms in U with coefficients in $C_c^\infty(U)$.

Definition 4.2.1. *A k-dimensional current on U is a linear continuous functional on $\Lambda^k(U)$. We will denote by $D_k(U)$ the space of all k-dimensional currents.*

We can define a notion of weak convergence in $D_k(U)$ as follows:

Definition 4.2.2. *We say that a sequence $(T_n)_{n\in\mathbb{N}} \subset D_k(U)$ weakly converges to $T \in D_k(U)$ if for every $\omega \in \Lambda^k(U)$ we have that $\lim_{n\to+\infty} T_n(\omega) = T(\omega)$.*

Moreover imposing the validity of the Stokes theorem for manifolds we obtain a notion for the boundary of a current.

Definition 4.2.3. *Given $T \in D_k(U)$, the boundary of T is the current $\partial T \in D_{k-1}(U)$ defined as*

$$\partial T(\omega) = T(d\omega)$$

for every $\omega \in \Lambda^{k-1}(U)$

In particular we are interested in a subset of $D_k(U)$. We define the space of k-dimensional rectifiable currents with real multiplicity (denote by $\mathcal{R}_k(U)$) as the triple $(\mathcal{M}, \theta, \xi)$ where $\mathcal{M} \subset U$ is a k-rectifiable set, $\theta : \mathcal{M} \to \mathbb{R}_+$ is called multiplicity and ξ is the k-vector giving an orientation of $\mathcal{M}$. We define the current $(\mathcal{M}, \theta, \xi)$ by its action on a k-diffential form $\omega \in \Lambda^k(U)$ in the following way:

$$(\mathcal{M}, \theta, \xi)(\omega) = \int_{\mathcal{M}} \langle \omega, \xi \rangle \theta \, d\mathcal{H}^k$$

where $\langle \cdot, \cdot \rangle$ denote the duality product between vectors and covectors. Moreover given $T = (\mathcal{M}, \theta, \xi)$ we can define the total variation measure associate to T as

$$\|T\|(A) = \int_{\mathcal{M} \cap A} \theta \, d\mathcal{H}^k$$

for every $A \subset U$ measurable. And we will call $\|T\|(U) = M(T)$ the mass of T.
We can define the restriction of a rectifiable current $T = (\mathcal{M}, \theta, \xi)$ on a measurable set as

$$T \llcorner A(\omega) = \int_{\mathcal{M} \cap A} \langle \omega, \xi \rangle \theta \, d\mathcal{H}^k$$

for every $A \subset U$ measurable. In addition given $\alpha \in \Lambda^h(U)$ with $h \leq k$, we define the restriction of $T \in R_k(U)$ to α as the $(k-h)$-dimensional current $T \llcorner \alpha$ defined as

$$T \llcorner \alpha(\omega) = T(\alpha \wedge \omega)$$

for every $\omega \in \Lambda^{k-h}(U)$.
Moreover let $I^k(U)$ be the subset of $\mathcal{R}^k(U)$ such that the multiplicity θ is integer valued. Each element of $I^k(U)$ is called k-dimensional integer rectifiable current.

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