Stability properties for quasilinear parabolic equations with measure data

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Abstract

Let Ω be a bounded domain of $\mathbb{R}^N$, and $Q = \Omega \times (0,T)$. We study problems of the model type

$$\begin{cases} u_t - \Delta_p u = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases}$$

where $p > 1$, $\mu \in \mathcal{M}_b(Q)$ and $u_0 \in L^1(\Omega)$. Our main result is a *stability theorem* extending the results of Dal Maso, Murat, Orsina, Prignet, for the elliptic case, valid for quasilinear operators $u \longmapsto \mathcal{A}(u) = \mathrm{div}(A(x,t,\nabla u))$.

1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$, and $Q = \Omega \times (0,T)$, $T > 0$. We denote by $\mathcal{M}_b(\Omega)$ and $\mathcal{M}_b(Q)$ the sets of bounded Radon measures on Ω and Q respectively. We are concerned with the problem

$$\begin{cases} u_t - \mathrm{div}(A(x,t,\nabla u)) = \mu & \text{in } Q, \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\mu \in \mathcal{M}_b(Q)$, $u_0 \in L^1(\Omega)$ and A is a Caratheodory function on $Q \times \mathbb{R}^N$, such that for *a.e.* $(x,t) \in Q$, and any $\xi, \zeta \in \mathbb{R}^N$,

$$A(x,t,\xi).\xi \geq \Lambda_1 |\xi|^p, \qquad |A(x,t,\xi)| \leq a(x,t) + \Lambda_2 |\xi|^{p-1}, \qquad \Lambda_1, \Lambda_2 > 0, a \in L^{p'}(Q), \tag{1.2}$$

$$(A(x,t,\xi) - A(x,t,\zeta)).(\xi - \zeta) > 0 \qquad \text{if } \xi \neq \zeta, \tag{1.3}$$

for $p > 1$.This includes the model problem where $\mathrm{div}(A(x,t,\nabla u)) = \Delta_p u$, where Δ_p is the p-Laplacian.

The corresponding elliptic problem:

$$-\Delta_p u = \mu \qquad \text{in } \Omega, \qquad u = 0 \qquad \text{on } \partial\Omega,$$

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with $\mu \in \mathcal{M}_b(\Omega)$, was studied in [9, 10] for $p > 2 - 1/N$, leading to the existence of solutions in the sense of distributions. For any $p > 1$, and $\mu \in L^1(\Omega)$, existence and uniqueness are proved in [4] in the class of *entropy solutions*. For any $\mu \in \mathcal{M}_b(\Omega)$ the main work is done in [14, Theorems 3.1, 3.2], where not only existence is proved in the class of *renormalized solutions*, but also a stability result, fundamental for applications.

Concerning problem (1.1), the first studies concern the case $\mu \in L^{p'}(Q)$ and $u_0 \in L^2(\Omega)$, where existence and uniqueness are obtained by variational methods, see [19]. In the general case $\mu \in \mathcal{M}_b(Q)$ and $u_0 \in \mathcal{M}_b(\Omega)$, the pionner results come from [9], proving the existence of solutions in the sense of distributions for

$$p > p_1 = 2 - \frac{1}{N+1}, \tag{1.4}$$

see also [11]. The approximated solutions of (1.1) lie in Marcinkiewicz spaces $u \in L^{p_c,\infty}(Q)$ and $|\nabla u| \in L^{m_c,\infty}(Q)$, where

$$p_c = p - 1 + \frac{p}{N}, \qquad m_c = p - \frac{N}{N+1}. \tag{1.5}$$

This condition (1.4) ensures that u and $|\nabla u|$ belong to $L^1(Q)$, since $m_c > 1$ means $p > p_1$ and $p_c > 1$ means $p > 2N/(N+1)$. Uniqueness follows in the case $p = 2$, $A(x,t,\nabla u) = \nabla u$, by duality methods, see [21].

For $\mu \in L^1(Q)$, uniqueness is obtained in new classes of *entropy solutions*, and *renormalized solutions*, see [5, 26, 27].

A larger set of measures is studied in [15]. They introduce a notion of parabolic capacity initiated and inspired by [24], used after in [22, 23], defined by

$$c_p^Q(E) = \inf\left(\inf_{E \subset U \text{ open} \subset Q}\{||u||_W : u \in W, u \geq \chi_U \quad a.e. \text{ in } Q\}\right),$$

for any Borel set $E \subset Q$, where setting $X = L^p((0,T); W_0^{1,p}(\Omega) \cap L^2(\Omega))$,

$$W = \left\{z : z \in X, \quad z_t \in X'\right\}, \text{ embedded with the norm } ||u||_W = ||u||_X + ||u_t||_{X'}.$$

Let $\mathcal{M}_0(Q)$ be the set of Radon measures μ on Q that do not charge the sets of zero c_p^Q-capacity:

$$\forall E \text{ Borel set } \subset Q, \quad c_p^Q(E) = 0 \Longrightarrow |\mu|(E) = 0.$$

Then existence and uniqueness of renormalized solutions of (1.1) hold for any measure $\mu \in \mathcal{M}_b(Q) \cap \mathcal{M}_0(Q)$, called *soft (or diffuse, or regular) measure*, and $u_0 \in L^1(\Omega)$, and $p > 1$. The equivalence with the notion of entropy solutions is shown in [16]. For such a soft measure, an extension to equations of type $(b(u))_t - \Delta_p u = \mu$ is given in [6]; another formulation is used in [23] for solving a perturbed problem from (1.1) by an absorption term.

Next consider an *arbitrary measure* $\mu \in \mathcal{M}_b(Q)$. Let $\mathcal{M}_s(Q)$ be the set of all bounded Radon measures on Q with support on a set of zero c_p^Q-capacity, also called *singular*. Let $\mathcal{M}_b^+(Q), \mathcal{M}_0^+(Q), \mathcal{M}_s^+(Q)$ be the positive cones of $\mathcal{M}_b(Q), \mathcal{M}_0(Q), \mathcal{M}_s(Q)$. From [15], μ can be written (in a unique way) under the form

$$\mu = \mu_0 + \mu_s, \qquad \mu_0 \in \mathcal{M}_0(Q), \quad \mu_s = \mu_s^+ - \mu_s^-, \qquad \mu_s^+, \mu_s^- \in \mathcal{M}_s^+(Q), \tag{1.6}$$

and $\mu_0 \in \mathcal{M}_0(Q)$ admits (at least) a decomposition under the form

$$\mu_0 = f - \operatorname{div} g + h_t, \qquad f \in L^1(Q), \quad g \in (L^{p'}(Q))^N, \quad h \in X, \tag{1.7}$$

and we write $\mu_0 = (f, g, h)$. Conversely, any measure of this form, *such that* $h \in L^\infty(Q)$, lies in $\mathcal{M}_0(Q)$, see [23, Proposition 3.1]. The solutions of (1.1) are searched in a renormalized sense linked to this decomposition, introduced in [15, 22]. In the range (1.4) the existence of a renormalized solution relative to the decomposition (1.7) is proved in [22], using suitable approximations of μ_0 and μ_s. Uniqueness is still open, as well as in the elliptic case.

In *all the sequel* we suppose that p satisfies (1.4). Then the embedding $W_0^{1,p}(\Omega) \subset L^2(\Omega)$ is valid, that means

$$X = L^p((0,T); W_0^{1,p}(\Omega)), \qquad X' = L^{p'}((0,T); W^{-1,p'}(\Omega)).$$

In Section 2 we recall the definition of renormalized solutions, given in [22], that we call R-solutions of (1.1), relative to the decomposition (1.7) of μ_0, and study some of their properties. Our main result is a *stability theorem* for problem (1.1), proved in Section 3, extending to the parabolic case the stability result of [14, Theorem 3.4]. In order to state it, we recall that a sequence of measures $\mu_n \in \mathcal{M}_b(Q)$ converges to a measure $\mu \in \mathcal{M}_b(Q)$ in the *narrow topology* of measures if

$$\lim_{n\to\infty} \int_Q \varphi d\mu_n = \int_Q \varphi d\mu \qquad \forall \varphi \in C(Q) \cap L^\infty(Q).$$

Theorem 1.1 *Let $A : Q \times \mathbb{R}^N \to \mathbb{R}^N$ satisfy (1.2),(1.3). Let $u_0 \in L^1(\Omega)$, and*

$$\mu = f - \operatorname{div} g + h_t + \mu_s^+ - \mu_s^- \in \mathcal{M}_b(Q),$$

with $f \in L^1(Q), g \in (L^{p'}(Q))^N$, $h \in X$ and $\mu_s^+, \mu_s^- \in \mathcal{M}_s^+(Q)$. Let $u_{0,n} \in L^1(\Omega)$,

$$\mu_n = f_n - \operatorname{div} g_n + (h_n)_t + \rho_n - \eta_n \in \mathcal{M}_b(Q),$$

with $f_n \in L^1(Q), g_n \in (L^{p'}(Q))^N, h_n \in X$, and $\rho_n, \eta_n \in \mathcal{M}_b^+(Q)$, such that

$$\rho_n = \rho_n^1 - \operatorname{div} \rho_n^2 + \rho_{n,s}, \qquad \eta_n = \eta_n^1 - \operatorname{div} \eta_n^2 + \eta_{n,s},$$

with $\rho_n^1, \eta_n^1 \in L^1(Q), \rho_n^2, \eta_n^2 \in (L^{p'}(Q))^N$ and $\rho_{n,s}, \eta_{n,s} \in \mathcal{M}_s^+(Q)$. Assume that

$$\sup_n |\mu_n|(Q) < \infty,$$

and $\{u_{0,n}\}$ converges to u_0 strongly in $L^1(\Omega)$, $\{f_n\}$ converges to f weakly in $L^1(Q)$, $\{g_n\}$ converges to g strongly in $(L^{p'}(Q))^N$, $\{h_n\}$ converges to h strongly in X, $\{\rho_n\}$ converges to μ_s^+ and $\{\eta_n\}$ converges to μ_s^- in the narrow topology; and $\{\rho_n^1\}, \{\eta_n^1\}$ are bounded in $L^1(Q)$, and $\{\rho_n^2\}, \{\eta_n^2\}$ bounded in $(L^{p'}(Q))^N$.

Let $\{u_n\}$ be a sequence of R-solutions of

$$\begin{cases} u_{n,t} - div(A(x,t,\nabla u_n)) = \mu_n & \text{in } Q, \\ u_n = 0 & \text{on } \partial\Omega \times (0,T), \\ u_n(0) = u_{0,n} & \text{in } \Omega. \end{cases} \tag{1.8}$$

relative to the decomposition $(f_n + \rho_n^1 - \eta_n^1, g_n + \rho_n^2 - \eta_n^2, h_n)$ of $\mu_{n,0}$. Let $U_n = u_n - h_n$.

Then up to a subsequence, $\{u_n\}$ converges a.e. in Q to a R-solution u of (1.1), and $\{U_n\}$ converges a.e. in Q to $U = u - h$. Moreover, $\{\nabla u_n\}, \{\nabla U_n\}$ converge respectively to $\nabla u, \nabla U$ a.e. in Q, and $\{T_k(U_n)\}$ converge to $T_k(U)$ strongly in X for any $k > 0$.

In Section 4 we check that any measure $\mu \in \mathcal{M}_b(Q)$ can be approximated in the sense of the stability Theorem, hence we find again the existence result of [22]:

Corollary 1.2 *Let $u_0 \in L^1(\Omega)$ and $\mu \in \mathcal{M}_b(Q)$. Then there exists a R-solution u to the problem (1.1) with data (μ, u_0).*

Moreover we give more precise properties of approximations of $\mu \in \mathcal{M}_b(Q)$, fundamental for applications, see Propositions 4.1 and 4.2. As in the elliptic case, Theorem 1.1 is a key point for obtaining existence results for more general problems, and we give some of them in [2, 3, 20], for measures μ satisfying suitable capacitary conditions. In [2] we study perturbed problems of order 0, of type

$$u_t - \Delta_p u + \mathcal{G}(u) = \mu \qquad \text{in } Q, \tag{1.9}$$

where $\mathcal{G}(u)$ is an absorption or a source term with a growth of power or exponential type, and μ is a good in time measure. In [3] we use potential estimates to give other existence results in case of absorption with $p > 2$. In [20], one considers equations of the form

$$u_t - \operatorname{div}(A(x,t,\nabla u)) + \mathcal{G}(u, \nabla u) = \mu$$

under (1.2),(1.3) with $p = 2$, and extend in particular the results of [1] to nonlinear operators.

2 Renormalized solutions of problem (1.1)

2.1 Notations and Definition

For any function $f \in L^1(Q)$, we write $\int_Q f$ instead of $\int_Q f dxdt$, and for any measurable set $E \subset Q$, $\int_E f$ instead of $\int_E f dxdt$. For any open set ϖ of $\mathbb{R}^m$ and $F \in (L^k(\varpi))^\nu$, $k \in [1, \infty]$, $m, \nu \in \mathbb{N}^*$, we set $\|F\|_{k,\varpi} = \|F\|_{(L^k(\varpi))^\nu}$

We set $T_k(r) = \max\{\min\{r, k\}, -k\}$, for any $k > 0$ and $r \in \mathbb{R}$. We recall that if u is a measurable function defined and finite *a.e.* in Q, such that $T_k(u) \in X$ for any $k > 0$, there exists a measurable function w from Q into $\mathbb{R}^N$ such that $\nabla T_k(u) = \chi_{|u| \le k} w$, *a.e.* in Q, and for any $k > 0$. We define the gradient ∇u of u by $w = \nabla u$.

Let $\mu = \mu_0 + \mu_s \in \mathcal{M}_b(Q)$, and (f, g, h) be a decomposition of μ_0 given by (1.7), and $\widehat{\mu_0} = \mu_0 - h_t = f - \operatorname{div} g$. In the general case $\widehat{\mu_0} \notin \mathcal{M}(Q)$, but we write, for convenience,

$$\int_Q w d\widehat{\mu_0} := \int_Q (fw + g.\nabla w), \qquad \forall w \in X \cap L^\infty(Q).$$

Definition 2.1 *Let $u_0 \in L^1(\Omega)$, $\mu = \mu_0 + \mu_s \in \mathcal{M}_b(Q)$. A measurable function u is a* ***renormalized solution,*** *called* ***R-solution*** *of (1.1) if there exists a decompostion (f, g, h) of μ_0 such that*

$$U = u - h \in L^\sigma((0,T); W_0^{1,\sigma}(\Omega)) \cap L^\infty((0,T); L^1(\Omega)), \quad \forall \sigma \in [1, m_c); \qquad T_k(U) \in X, \quad \forall k > 0, \tag{2.1}$$

and:

(i) for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support on $\mathbb{R}$, and $S(0) = 0$,

$$-\int_\Omega S(u_0)\varphi(0)dx - \int_Q \varphi_t S(U) + \int_Q S'(U)A(x,t,\nabla u).\nabla\varphi + \int_Q S''(U)\varphi A(x,t,\nabla u).\nabla U = \int_Q S'(U)\varphi d\widehat{\mu_0}, \tag{2.2}$$

for any $\varphi \in X \cap L^\infty(Q)$ such that $\varphi_t \in X' + L^1(Q)$ and $\varphi(., T) = 0$;

(ii) for any $\phi \in C(\overline{Q})$,

$$\lim_{m\to\infty} \frac{1}{m} \int\limits_{\{m \le U < 2m\}} \phi A(x,t,\nabla u).\nabla U = \int_Q \phi d\mu_s^+ \tag{2.3}$$

$$\lim_{m\to\infty} \frac{1}{m} \int\limits_{\{-m \ge U > -2m\}} \phi A(x,t,\nabla u).\nabla U = \int_Q \phi d\mu_s^-. \tag{2.4}$$

Remark 2.2 *As a consequence,* $S(U) \in C([0,T]; L^1(\Omega))$ *and* $S(U)(.,0) = S(u_0)$ *in* Ω*; and* u *satisfies the equation*

$$(S(U))_t - \operatorname{div}(S'(U)A(x,t,\nabla u)) + S''(U)A(x,t,\nabla u).\nabla U = fS'(U) - \operatorname{div}(gS'(U)) + S''(U)g.\nabla U, \tag{2.5}$$

in the sense of distributions in Q*, see [22, Remark 3]. Moreover assume that* $[-k,k] \supset suppS'$*. then from (1.2) and the Hölder inequality, we find easily that*

$$\begin{aligned}\|S(U)_t\|_{X'+L^1(Q)} \le C\, \|S\|_{W^{2,\infty}(\mathbb{R})} (\, \left\||\nabla u|^p \chi_{|U|\le k}\right\|_{1,Q}^{1/p'} + \left\||\nabla u|^p \chi_{|U|\le k}\right\|_{1,Q} + \||\nabla T_k(U)|\|_{p,Q}^p \\ + \|a\|_{p',Q} + \|a\|_{p',Q}^{p'} + \|f\|_{1,Q} + \|g\|_{p',Q} \left\||\nabla u|^p \chi_{|U|\le k}\right\|_{1,Q}^{1/p} + \|g\|_{p',Q}\,),\end{aligned} \tag{2.6}$$

where $C = C(p, \Lambda_2)$*. We also deduce that, for any* $\varphi \in X \cap L^\infty(Q)$*, such that* $\varphi_t \in X' + L^1(Q)$*,*

$$\begin{aligned}\int_\Omega S(U(T))\varphi(T)dx - \int_\Omega S(u_0)\varphi(0)dx - \int_Q \varphi_t S(U) + \int_Q S'(U)A(x,t,\nabla u).\nabla\varphi \\ + \int_Q S''(U)A(x,t,\nabla u).\nabla U\varphi = \int_Q S'(U)\varphi d\widehat{\mu_0}.\end{aligned} \tag{2.7}$$

Remark 2.3 *Let* u, U *satisfy (2.1). It is easy to see that the condition (2.3) (resp. (2.4)) is equivalent to*

$$\lim_{m\to\infty} \frac{1}{m} \int\limits_{\{m \le U < 2m\}} \phi A(x,t,\nabla u).\nabla u = \int_Q \phi d\mu_s^+ \tag{2.8}$$

resp.

$$\lim_{m\to\infty} \frac{1}{m} \int\limits_{\{m \ge U > -2m\}} \phi A(x,t,\nabla u).\nabla u = \int_Q \phi d\mu_s^-. \tag{2.9}$$

In particular, for any $\varphi \in L^{p'}(Q)$ *there holds*

$$\lim_{m\to\infty} \frac{1}{m} \int\limits_{m \le |U| < 2m} |\nabla u|\varphi = 0, \qquad \lim_{m\to\infty} \frac{1}{m} \int\limits_{m \le |U| < 2m} |\nabla U|\varphi = 0. \tag{2.10}$$

Remark 2.4 *(i) Any function* $U \in X$ *such that* $U_t \in X' + L^1(Q)$ *admits a unique* c_p^Q*-quasi continuous representative, defined* c_p^Q*-quasi a.e. in* Q*, still denoted* U*. Furthermore, if* $U \in L^\infty(Q)$*, then for any* $\mu_0 \in \mathcal{M}_0(Q)$*, there holds* $U \in L^\infty(Q, d\mu_0)$*, see [22, Theorem 3 and Corollary 1].*

(ii) Let u *be any R- solution of problem (1.1). Then,* $U = u - h$ *admits a* c_p^Q*-quasi continuous functions representative which is finite* c_p^Q*-quasi a.e. in* Q*, and* u *satisfies definition 2.1 for every decomposition* $(\tilde{f}, \tilde{g}, \tilde{h})$ *such that* $h - \tilde{h} \in L^\infty(Q)$*, see [22, Proposition 3 and Theorem 4].*

2.2 Steklov and Landes approximations

A main difficulty for proving Theorem 1.1 is the choice of admissible test functions (S,φ) *in (2.2), valid for any R-solution.* Because of a lack of regularity of these solutions, we use two ways of approximation adapted to parabolic equations:

Definition 2.5 *Let* $\varepsilon \in (0,T)$ *and* $z \in L^1_{loc}(Q)$*. For any* $l \in (0,\varepsilon)$ *we define the* ***Steklov time-averages*** $[z]_l, [z]_{-l}$ *of* z *by*

$$[z]_l(x,t) = \frac{1}{l}\int_t^{t+l} z(x,s)ds \qquad \text{for a.e. } (x,t) \in \Omega \times (0, T-\varepsilon),$$

$$[z]_{-l}(x,t) = \frac{1}{l}\int_{t-l}^{t} z(x,s)ds \qquad \text{for a.e. } (x,t) \in \Omega \times (\varepsilon, T).$$

The idea to use this approximation for R-solutions can be found in [7]. Recall some properties, given in [23]. Let $\varepsilon \in (0,T)$, and $\varphi_1 \in C_c^\infty(\overline{\Omega} \times [0,T))$, $\varphi_2 \in C_c^\infty(\overline{\Omega} \times (0,T])$ with $\text{Supp}\varphi_1 \subset \overline{\Omega} \times [0, T-\varepsilon]$, $\text{Supp}\varphi_2 \subset \overline{\Omega} \times [\varepsilon, T]$. There holds:

(i) If $z \in X$, then $\varphi_1[z]_l$ and $\varphi_2[z]_{-l} \in W$.

(ii) If $z \in X$ and $z_t \in X' + L^1(Q)$, then, as $l \to 0$, $(\varphi_1[z]_l)$ and $(\varphi_2[z]_{-l})$ converge respectively to $\varphi_1 z$ and $\varphi_2 z$ in X, and *a.e.* in Q; and $(\varphi_1[z]_l)_t, (\varphi_2[z]_{-l})_t$ converge to $(\varphi_1 z)_t, (\varphi_2 z)_t$ in $X' + L^1(Q)$.

(iii) If moreover $z \in L^\infty(Q)$, then from any sequence $\{l_n\} \to 0$, there exists a subsequence $\{l_\nu\}$ such that $\{[z]_{l_\nu}\}, \{[z]_{-l_\nu}\}$ converge to z, c_p^Q-quasi everywhere in Q.

Next we recall the approximation used in several articles [8, 12, 11], first introduced in [17].

Definition 2.6 *Let* $k > 0$*, and* $y \in L^\infty(\Omega)$ *and* $Y \in X$ *such that* $||y||_{L^\infty(\Omega)} \le k$ *and* $||Y||_{L^\infty(Q)} \le k$*. For any* $\nu \in \mathbb{N}$*, a* ***Landes-time approximation*** $\langle Y\rangle_\nu$ *of the function* Y *is defined as follows:*

$$\langle Y\rangle_\nu(x,t) = \nu\int_0^t Y(x,s)e^{\nu(s-t)}ds + e^{-\nu t}z_\nu(x), \qquad \forall (x,t) \in Q.$$

where $\{z_\nu\}$ *is a sequence of functions in* $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$*, such that* $||z_\nu||_{L^\infty(\Omega)} \le k$*,* $\{z_\nu\}$ *converges to* y *a.e. in* Ω*, and* $\nu^{-1}||z_\nu||^p_{W_0^{1,p}(\Omega)}$ *converges to* 0.

Therefore, we can verify that $(\langle Y\rangle_\nu)_t \in X$, $\langle Y\rangle_\nu \in X \cap L^\infty(Q)$, $||\langle Y\rangle_\nu||_{\infty,Q} \le k$ and $\{\langle Y\rangle_\nu\}$ converges to Y strongly in X and *a.e.* in Q. Moreover, $\langle Y\rangle_\nu$ satisfies the equation $(\langle Y\rangle_\nu)_t = \nu\left(Y - \langle Y\rangle_\nu\right)$ in the sense of distributions in Q, and $\langle Y\rangle_\nu(0) = z_\nu$ in Ω. In this paper, we only use the **Landes-time approximation** of the function $Y = T_k(U)$, where $y = T_k(u_0)$.

2.3 First properties

In the sequel we use the following notations: for any function $J \in W^{1,\infty}(\mathbb{R})$, nondecreasing with $J(0) = 0$, we set

$$\overline{J}(r) = \int_0^r J(\tau)d\tau, \qquad \mathcal{J}(r) = \int_0^r J'(\tau)\tau d\tau. \tag{2.11}$$

It is easy to verify that $\mathcal{J}(r) \ge 0$,

$$\mathcal{J}(r) + \overline{J}(r) = J(r)r, \quad \text{and} \quad \mathcal{J}(r) - \mathcal{J}(s) \ge s\left(J(r) - J(s)\right) \qquad \forall r, s \in \mathbb{R}. \tag{2.12}$$

In particular we define, for any $k>0$, and any $r\in\mathbb{R}$,

$$\overline{T_k}(r)=\int_0^r T_k(\tau)d\tau,\qquad \mathcal{T}_k(r)=\int_0^r T_k'(\tau)\tau d\tau, \tag{2.13}$$

and we use several times a truncature used in [14]:

$$H_m(r)=\chi_{[-m,m]}(r)+\frac{2m-|s|}{m}\chi_{m<|s|\le 2m}(r),\qquad \overline{H_m}(r)=\int_0^r H_m(\tau)d\tau. \tag{2.14}$$

The next Lemma allows to extend the range of the test functions in (2.2).

Lemma 2.7 *Let u be a R-solution of problem (1.1). Let $J\in W^{1,\infty}(\mathbb{R})$ be nondecreasing with $J(0)=0$, and $\overline{J}$ defined by (2.11). Then,*

$$\begin{aligned}&\int_Q S'(U)A(x,t,\nabla u).\nabla\left(\xi J(S(U))\right)+\int_Q S''(U)A(x,t,\nabla u).\nabla U\xi J(S(U))\\ &-\int_\Omega \xi(0)J(S(u_0))S(u_0)dx-\int_Q \xi_t\overline{J}(S(U))\le\int_Q S'(U)\xi J(S(U))d\widehat{\mu_0},\end{aligned} \tag{2.15}$$

for any $S\in W^{2,\infty}(\mathbb{R})$ such that S' has compact support on $\mathbb{R}$ and $S(0)=0$, and for any $\xi\in C^1(Q)\cap W^{1,\infty}(Q),\xi\ge 0$.

Proof. Let $\mathcal{J}$ be defined by (2.11). Let $\zeta\in C_c^1([0,T))$ with values in $[0,1]$, such that $\zeta_t\le 0$, and $\varphi=\zeta\xi[J(S(U))]_l$. Clearly, $\varphi\in X\cap L^\infty(Q)$; we choose the pair of functions (φ,S) as test function in (2.2). From the convergence properties of Steklov time-averages, we easily will obtain (2.15) if we prove that

$$\underline{\lim}_{l\to 0,\zeta\to 1}(-\int_Q (\zeta\xi[J(S(U))]_l)_t S(U))\ge -\int_Q \xi_t\overline{J}(S(U)).$$

We can write $-\int_Q (\zeta\xi[J(S(U))]_l)_t S(U)=F+G$, with

$$F=-\int_Q (\zeta\xi)_t[J(S(U))]_l S(U),\qquad G=-\int_Q \zeta\xi S(U)\frac{1}{l}\left(J(S(U))(x,t+l)-J(S(U))(x,t)\right).$$

Using (2.12) and integrating by parts we have

$$\begin{aligned}G&\ge -\int_Q \zeta\xi\frac{1}{l}\left(\mathcal{J}(S(U))(x,t+l)-\mathcal{J}(S(U))(x,t)\right)=-\int_Q \zeta\xi\frac{\partial}{\partial t}\left([\mathcal{J}(S(U))]_l\right)\\ &=\int_Q (\zeta\xi)_t[\mathcal{J}(S(U))]_l+\int_\Omega \zeta(0)\xi(0)[\mathcal{J}(S(U))]_l(0)dx\ge\int_Q (\zeta\xi)_t[\mathcal{J}(S(U))]_l,\end{aligned}$$

since $\mathcal{J}(S(U))\ge 0$. Hence,

$$-\int_Q (\zeta\xi[J(S(U))]_l)_t S(U)\ge\int_Q (\zeta\xi)_t[\mathcal{J}(S(U))]_l+F=\int_Q (\zeta\xi)_t\left([\mathcal{J}(S(U))]_l-[J(S(U))]_l S(U)\right).$$

Otherwise, $\mathcal{J}(S(U))$ and $J(S(U))\in C([0,T];L^1(\Omega))$, thus $\{(\zeta\xi)_t\left([\mathcal{J}(S(u))]_l-[J(S(u))]_l S(u)\right)\}$ converges to $-(\zeta\xi)_t\overline{J}(S(u))$ in $L^1(Q)$ as $l\to 0$. Therefore,

$$\underline{\lim}_{l\to 0,\zeta\to 1}(-\int_Q (\zeta\xi[J(S(U))]_l)_t S(U))\ge\underline{\lim}_{\zeta\to 1}\left(-\int_Q (\zeta\xi)_t\overline{J}(S(U))\right)\ge -\int_Q \xi_t\overline{J}(S(U)),$$

which achieves the proof. ■

Next we give estimates of the function and its gradient, following the first ones of [11], inspired by the estimates of the elliptic case of [4]. In particular we extend and make more precise the a priori estimates of [22, Proposition 4] given for solutions with smooth data; see also [15, 18].

Proposition 2.8 *If u is a R-solution of problem (1.1), then there exists $C_1 = C_1(p, \Lambda_1, \Lambda_2)$ such that, for any $k \geq 1$ and $\ell \geq 0$,*

$$\int\limits_{\ell \leq |U| \leq \ell+k} |\nabla u|^p + \int\limits_{\ell \leq |U| \leq \ell+k} |\nabla U|^p \leq C_1 k M, \tag{2.16}$$

$$\|U\|_{L^\infty(((0,T));L^1(\Omega))} \leq C_1(M + |\Omega|), \tag{2.17}$$

where $M = \|u_0\|_{1,\Omega} + |\mu_s|(Q) + \|f\|_{1,Q} + \|g\|_{p',Q}^{p'} + \|h\|_X^p + ||a||_{p',Q}^{p'}$.
As a consequence, for any $k \geq 1$,

$$\text{meas}\,\{|U| > k\} \leq C_2 M_1 k^{-p_c}, \qquad \text{meas}\,\{|\nabla U| > k\} \leq C_2 M_2 k^{-m_c}, \tag{2.18}$$

$$\text{meas}\,\{|u| > k\} \leq C_2 M_2 k^{-p_c}, \qquad \text{meas}\,\{|\nabla u| > k\} \leq C_2 M_2 k^{-m_c}, \tag{2.19}$$

where $C_2 = C_2(N, p, \Lambda_1, \Lambda_2)$, and $M_1 = (M+|\Omega|)^{\frac{p}{N}} M$ and $M_2 = M_1 + M$.

Proof. Set for any $r \in \mathbb{R}$, and $m, k, \ell > 0$,

$$T_{k,\ell}(r) = \max\{\min\{r - \ell, k\}, 0\} + \min\{\max\{r + \ell, -k\}, 0\}.$$

For $m > k + \ell$, we can choose $(J, S, \xi) = (T_{k,\ell}, \overline{H_m}, \xi)$ as test functions in (2.15), where $\overline{H_m}$ is defined at (2.14) and $\xi \in C^1([0,T])$ with values in $[0,1]$, independent on x. Since $T_{k,\ell}(\overline{H_m}(r)) = T_{k,\ell}(r)$ for all $r \in \mathbb{R}$, we obtain

$$-\int_\Omega \xi(0) T_{k,\ell}(u_0) \overline{H_m}(u_0) dx - \int_Q \xi_t \overline{T_{k,\ell}}(\overline{H_m}(U))$$
$$+ \int\limits_{\{\ell \leq |U| < \ell+k\}} \xi A(x,t,\nabla u).\nabla U - \frac{k}{m} \int\limits_{\{m \leq |U| < 2m\}} \xi A(x,t,\nabla u).\nabla U \leq \int_Q H_m(U) \xi T_{k,\ell}(U) d\widehat{\mu_0}.$$

And

$$\int_Q H_m(U) \xi T_{k,\ell}(U) d\widehat{\mu_0} = \int_Q H_m(U) \xi T_{k,\ell}(U) f + \int\limits_{\{\ell \leq |U| < \ell+k\}} \xi \nabla U.g - \frac{k}{m} \int\limits_{\{m \leq |U| < 2m\}} \xi \nabla U.g.$$

Let $m \to \infty$; then, for any $k \geq 1$, since $U \in L^1(Q)$ and from (2.3), (2.4), and (2.10), we find

$$-\int_Q \xi_t \overline{T_{k,\ell}}(U) + \int\limits_{\{\ell \leq |U| < \ell+k\}} \xi A(x,t,\nabla u).\nabla U \leq \int\limits_{\{\ell \leq |U| < \ell+k\}} \xi \nabla U.g + k(\|u_0\|_{1,\Omega} + |\mu_s|(Q) + \|f\|_{1,Q}). \tag{2.20}$$

Next, we take $\xi \equiv 1$. We verify that

$$A(x,t,\nabla u).\nabla U - \nabla U.g \geq \frac{\Lambda_1}{4}(|\nabla u|^p + |\nabla U|^p) - c_1(|g|^{p'} + |\nabla h|^p + |a|^{p'})$$

for some $c_1 = c_1(p, \Lambda_1, \Lambda_2) > 0$. Hence (2.16) follows. Thus, from (2.20) and the Hölder inequality, we get, for any $\xi \in C^1([0,T])$ with values in $[0,1]$,

$$-\int_Q \xi_t \overline{T_{k,\ell}}(U) \leq c_2 k M$$

for some $c_2 = c_2(p, \Lambda_1, \Lambda_2) > 0$.Thus $\int_\Omega \overline{T_{k,\ell}}(U)(t)dx \leq c_2 kM$, for *a.e.* $t \in (0, T)$. We deduce (2.17) by taking $k = 1, \ell = 0$, since $\overline{T_{1,0}}(r) = \overline{T_1}(r) \geq |r| - 1$, for any $r \in \mathbb{R}$.

Next, from the Gagliardo-Nirenberg embedding Theorem, see [13, Proposition 3.1], we have

$$\int_Q |T_k(U)|^{\frac{p(N+1)}{N}} \leq c_3 \|U\|^{\frac{p}{N}}_{L^\infty(((0,T));L^1(\Omega))} \int_Q |\nabla T_k(U)|^p,$$

where $c_3 = c_3(N, p)$. Then, from (2.16) and (2.17), we get, for any $k \geq 1$,

$$\text{meas}\{|U| > k\} \leq k^{-\frac{p(N+1)}{N}} \int_Q |T_k(U)|^{\frac{p(N+1)}{N}} \leq c_3 \|U\|^{\frac{p}{N}}_{L^\infty((0,T);L^1(\Omega))} k^{-\frac{p(N+1)}{N}} \int_Q |\nabla T_k(U)|^p \leq c_4 M_1 k^{-p_c},$$

with $c_4 = c_4(N, p, \Lambda_1, \Lambda_2)$. We obtain

$$\begin{aligned} \text{meas}\{|\nabla U| > k\} &\leq \frac{1}{k^p} \int_0^{k^p} \text{meas}(\{|\nabla U|^p > s\})\, ds \\ &\leq \text{meas}\left\{|U| > k^{\frac{N}{N+1}}\right\} + \frac{1}{k^p} \int_0^{k^p} \text{meas}\left(\left\{|\nabla U|^p > s, |U| \leq k^{\frac{N}{N+1}}\right\}\right) ds \\ &\leq c_4 M_1 k^{-m_c} + \frac{1}{k^p} \int\limits_{|U| \leq k^{\frac{N}{N+1}}} |\nabla U|^p \leq c_5 M_2 k^{-m_c}, \end{aligned}$$

with $c_5 = c_5(N, p, \Lambda_1, \Lambda_2)$. Furthermore, for any $k \geq 1$,

$$\text{meas}\{|h| > k\} + \text{meas}\{|\nabla h| > k\} \leq c_6 k^{-p} \|h\|^p_X,$$

where $c_6 = c_6(N, p)$. Therefore, we easily get (2.19). ∎

Remark 2.9 *If $\mu \in L^1(Q)$ and $a \equiv 0$ in (1.2), then (2.16) holds for all $k > 0$ and the term $|\Omega|$ in inequality (2.17) can be removed, where $M = ||u_0||_{1,\Omega} + |\mu|(Q)$. Furthermore, (2.19) is stated as follows:*

$$\text{meas}\{|u| > k\} \leq C_2 M^{\frac{p+N}{N}} k^{-p_c}, \qquad \text{meas}\{|\nabla u| > k\} \leq C_2 M^{\frac{N+2}{N+1}} k^{-m_c}, \forall k > 0. \tag{2.21}$$

with $C_2 = C_2(N, p, \Lambda_1, \Lambda_2)$.To see last inequality, we do in the following way:

$$\begin{aligned} \text{meas}\{|\nabla U| > k\} &\leq \text{meas}\left\{|U| > M^{\frac{1}{N+1}} k^{\frac{N}{N+1}}\right\} + \frac{1}{k^p} \int_0^{k^p} \text{meas}\left\{|\nabla U|^p > s, |U| \leq M^{\frac{1}{N+1}} k^{\frac{N}{N+1}}\right\} ds \\ &\leq C_2 M^{\frac{N+2}{N+1}} k^{-m_c}. \end{aligned}$$

Proposition 2.10 *Let $\{\mu_n\} \subset \mathcal{M}_b(Q)$, and $\{u_{0,n}\} \subset L^1(\Omega)$, such that*

$$\sup_n |\mu_n|(Q) < \infty, \text{ and } \sup_n ||u_{0,n}||_{1,\Omega} < \infty.$$

Let u_n be a R-solution of (1.1) with data $\mu_n = \mu_{n,0} + \mu_{n,s}$ and $u_{0,n}$, relative to a decomposition (f_n, g_n, h_n) of $\mu_{n,0}$, and $U_n = u_n - h_n$. Assume that $\{f_n\}$ is bounded in $L^1(Q)$, $\{g_n\}$ bounded in $(L^{p'}(Q))^N$ and $\{h_n\}$ bounded in X.

Then, up to a subsequence, $\{U_n\}$ converges a.e. to a function $U \in L^\infty((0,T); L^1(\Omega))$, such that $T_k(U) \in X$ for any $k > 0$and $U \in L^\sigma((0,T); W_0^{1,\sigma}(\Omega))$ for any $\sigma \in [1, m_c)$. And *(i)* $\{U_n\}$ *converges to* U *strongly in* $L^\sigma(Q)$ *for any* $\sigma \in [1, m_c)$, *and* $\sup \|U_n\|_{L^\infty((0,T);L^1(\Omega))} < \infty$,

(ii) $\sup_{k>0} \sup_n \frac{1}{k+1} \int_Q |\nabla T_k(U_n)|^p < \infty$,

(iii) $\{T_k(U_n)\}$ *converges to* $T_k(U)$ *weakly in* X, *for any* $k > 0$,

(iv) $\{A(x, t, \nabla (T_k(U_n) + h_n))\}$ *converges to some* F_k *weakly in* $(L^{p'}(Q))^N$.

Proof. Take $S \in W^{2,\infty}(\mathbb{R})$ such that S' has compact support on $\mathbb{R}$ and $S(0) = 0$. We combine (2.6) with (2.16), and deduce that $\{S(U_n)_t\}$ is bounded in $X' + L^1(Q)$ and $\{S(U_n)\}$ bounded in X. Hence, $\{S(U_n)\}$ is relatively compact in $L^1(Q)$. On the other hand, we choose $S = S_k$ such that $S_k(z) = z$, if $|z| < k$ and $S(z) = 2k\ \mathrm{sign} z$, if $|z| > 2k$. From (2.17), we obtain

$$\begin{aligned} \text{meas}\{|U_n - U_m| > \sigma\} &\leq \text{meas}\{|U_n| > k\} + \text{meas}\{|U_m| > k\} + \text{meas}\{|S_k(U_n) - S_k(U_m)| > \sigma\} \\ &\leq \frac{c}{k} + \text{meas}\{|S_k(U_n) - S_k(U_m)| > \sigma\}, \end{aligned}$$

where c does not depend of n, m. Thus, up to a subsequence $\{u_n\}$ is a Cauchy sequence in measure, and converges *a.e.* in Q to a function u. Thus, $\{T_k(U_n)\}$ converges to $T_k(U)$ weakly in X, since $\sup_n \|T_k(U_n)\|_X < \infty$ for any $k > 0$. And $\left\{|\nabla (T_k(U_n) + h_n)|^{p-2} \nabla (T_k(U_n) + h_n)\right\}$ converges to some F_k weakly in $(L^{p'}(Q))^N$. Furthermore, from (2.18), $\{U_n\}$ strongly converges to U in $L^\sigma(Q)$, for any $\sigma < p_c$. ■

3 The convergence theorem

We first recall some properties of the measures, see [22, Lemma 5], [14].

Proposition 3.1 *Let* $\mu_s = \mu_s^+ - \mu_s^- \in \mathcal{M}_b(Q)$, *where* μ_s^+ *and* μ_s^- *are concentrated, respectively, on two disjoint sets* E^+ *and* E^- *of zero* c_p^Q*-capacity. Then, for any* $\delta > 0$, *there exist two compact sets* $K_\delta^+ \subseteq E^+$ *and* $K_\delta^- \subseteq E^-$ *such that*

$$\mu_s^+(E^+ \backslash K_\delta^+) \leq \delta, \qquad \mu_s^-(E^- \backslash K_\delta^-) \leq \delta,$$

and there exist $\psi_\delta^+, \psi_\delta^- \in C_c^1(Q)$ *with values in* $[0, 1]$, *such that* $\psi_\delta^+, \psi_\delta^- = 1$ *respectively on* K_δ^+, K_δ^-, *and* $supp(\psi_\delta^+) \cap supp(\psi_\delta^-) = \emptyset$, *and*

$$||\psi_\delta^+||_X + ||(\psi_\delta^+)_t||_{X'+L^1(Q)} \leq \delta, \qquad ||\psi_\delta^-||_X + ||(\psi_\delta^-)_t||_{X'+L^1(Q)} \leq \delta.$$

There exist decompositions $(\psi_\delta^+)_t = \left(\psi_\delta^+\right)_t^1 + \left(\psi_\delta^+\right)_t^2$ *and* $(\psi_\delta^-)_t = \left(\psi_\delta^-\right)_t^1 + \left(\psi_\delta^-\right)_t^2$ *in* $X' + L^1(Q)$, *such that*

$$\left\|\left(\psi_\delta^+\right)_t^1\right\|_{X'} \leq \frac{\delta}{3}, \qquad \left\|\left(\psi_\delta^+\right)_t^2\right\|_{1,Q} \leq \frac{\delta}{3}, \qquad \left\|\left(\psi_\delta^-\right)_t^1\right\|_{X'} \leq \frac{\delta}{3}, \qquad \left\|\left(\psi_\delta^-\right)_t^2\right\|_{1,Q} \leq \frac{\delta}{3}. \tag{3.1}$$

Both $\{\psi_\delta^+\}$ *and* $\{\psi_\delta^-\}$ *converge to* 0, *weak-** *in* $L^\infty(Q)$, *and strongly in* $L^1(Q)$ *and up to subsequences, a.e. in* Q, *as* δ *tends to* 0.
Moreover if ρ_n *and* η_n *are as in Theorem 1.1, we have, for any* $\delta, \delta_1, \delta_2 > 0$,

$$\int_Q \psi_\delta^- d\rho_n + \int_Q \psi_\delta^+ d\eta_n = \omega(n, \delta), \qquad \int_Q \psi_\delta^- d\mu_s^+ \leq \delta, \qquad \int_Q \psi_\delta^+ d\mu_s^- \leq \delta, \tag{3.2}$$

$$\int_Q (1 - \psi_{\delta_1}^+ \psi_{\delta_2}^+) d\rho_n = \omega(n, \delta_1, \delta_2), \qquad \int_Q (1 - \psi_{\delta_1}^+ \psi_{\delta_2}^+) d\mu_s^+ \leq \delta_1 + \delta_2, \tag{3.3}$$

$$\int_Q (1 - \psi_{\delta_1}^- \psi_{\delta_2}^-) d\eta_n = \omega(n, \delta_1, \delta_2), \qquad \int_Q (1 - \psi_{\delta_1}^- \psi_{\delta_2}^-) d\mu_s^- \leq \delta_1 + \delta_2. \tag{3.4}$$

Hereafter, if $n,\varepsilon,...,\nu$ are real numbers, and a function ϕ depends on $n,\varepsilon,...,\nu$ and eventual other parameters $\alpha,\beta,..,\gamma$, and $n\to n_0,\varepsilon\to\varepsilon_0,..,\ \nu\to\nu_0$, we write $\phi=\omega(n,\varepsilon,..,\nu)$, then this means that, for fixed $\alpha,\beta,..,\gamma$, there holds $\overline{\lim}_{\nu\to\nu_0}..\overline{\lim}_{\varepsilon\to\varepsilon_0}\overline{\lim}_{n\to n_0}|\phi|=0$. In the same way, $\phi\leq\omega(n,\varepsilon,\delta,...,\nu)$ means $\overline{\lim}_{\nu\to\nu_0}..\overline{\lim}_{\varepsilon\to\varepsilon_0}\overline{\lim}_{n\to n_0}\phi\leq 0$, and $\phi\geq\omega(n,\varepsilon,..,\nu)$ means $-\phi\leq\omega(n,\varepsilon,..,\nu)$.

Remark 3.2 *In the sequel we recall a convergence property still used in [14]: If $\{b_{1,n}\}$ is a sequence in $L^1(Q)$ converging to b_1 weakly in $L^1(Q)$ and $\{b_{2,n}\}$ a bounded sequence in $L^\infty(Q)$ converging to b_2, a.e. in Q, then $\lim_{n\to\infty}\int_Q b_{1,n}b_{2,n}=\int_Q b_1b_2$.*

Next we prove Thorem 1.1.

Scheme of the proof. Let $\{\mu_n\},\{u_{0,n}\}$ and $\{u_n\}$ satisfy the assumptions of Theorem 1.1. Then we can apply Proposition 2.10. Setting $U_n=u_n-h_n$, up to subsequences, $\{u_n\}$ converges *a.e.* in Q to some function u, and $\{U_n\}$ converges *a.e.* to $U=u-h$, such that $T_k(U)\in X$ for any $k>0$, and $U\in L^\sigma((0,T);W_0^{1,\sigma}(\Omega))\cap L^\infty((0,T);L^1(\Omega))$ for every $\sigma\in[1,m_c)$. And $\{U_n\}$ satisfies the conclusions (i) to (iv) of Proposition 2.10. We have

$$\begin{aligned}\mu_n&=(f_n-\operatorname{div}g_n+(h_n)_t)+(\rho_n^1-\operatorname{div}\rho_n^2)-(\eta_n^1-\operatorname{div}\eta_n^2)+\rho_{n,s}-\eta_{n,s}\\&=\mu_{n,0}+(\rho_{n,s}-\eta_{n,s})^+-(\rho_{n,s}-\eta_{n,s})^-,\end{aligned}$$

where

$$\mu_{n,0}=\lambda_{n,0}+\rho_{n,0}-\eta_{n,0},\quad\text{with }\lambda_{n,0}=f_n-\operatorname{div}g_n+(h_n)_t,\quad\rho_{n,0}=\rho_n^1-\operatorname{div}\rho_n^2,\quad\eta_{n,0}=\eta_n^1-\operatorname{div}\eta_n^2.\tag{3.5}$$

Hence

$$\rho_{n,0},\eta_{n,0}\in\mathcal{M}_b^+(Q)\cap\mathcal{M}_0(Q),\quad\text{and}\quad\rho_n\geq\rho_{n,0},\quad\eta_n\geq\eta_{n,0}.\tag{3.6}$$

Let E^+,E^- be the sets where, respectively, μ_s^+ and μ_s^- are concentrated. For any $\delta_1,\delta_2>0$, let $\psi_{\delta_1}^+,\psi_{\delta_2}^+$ and $\psi_{\delta_1}^-,\psi_{\delta_2}^-$ as in Proposition 3.1 and set

$$\Phi_{\delta_1,\delta_2}=\psi_{\delta_1}^+\psi_{\delta_2}^++\psi_{\delta_1}^-\psi_{\delta_2}^-.$$

Suppose that we can prove the two estimates, near E

$$I_1:=\int_{\{|U_n|\leq k\}}\Phi_{\delta_1,\delta_2}A(x,t,\nabla u_n).\nabla\left(U_n-\langle T_k(U)\rangle_\nu\right)\leq\omega(n,\nu,\delta_1,\delta_2),\tag{3.7}$$

and far from E,

$$I_2:=\int_{\{|U_n|\leq k\}}(1-\Phi_{\delta_1,\delta_2})A(x,t,\nabla u_n).\nabla(U_n-\langle T_k(U)\rangle_\nu)\leq\omega(n,\nu,\delta_1,\delta_2).\tag{3.8}$$

Then it follows that

$$\overline{\lim}_{n,\nu}\int_{\{|U_n|\leq k\}}A(x,t,\nabla u_n).\nabla\left(U_n-\langle T_k(U)\rangle_\nu\right)\leq 0,\tag{3.9}$$

which implies

$$\overline{\lim}_{n\to\infty}\int_{\{|U_n|\leq k\}}A(x,t,\nabla u_n).\nabla\left(U_n-T_k(U)\right)\leq 0,\tag{3.10}$$

since $\{\langle T_k(U)\rangle_\nu\}$ converges to $T_k(U)$ in X. On the other hand, from the weak convergence of $\{T_k(U_n)\}$ to $T_k(U)$ in X, we verify that

$$\int_{\{|U_n|\leq k\}} A(x,t,\nabla(T_k(U)+h_n)).\nabla\left(T_k(U_n)-T_k(U)\right)=\omega(n).$$

Thus we get

$$\int_{\{|U_n|\leq k\}} \left(A(x,t,\nabla u_n)-A(x,t,\nabla(T_k(U)+h_n))\right).\nabla\left(u_n-(T_k(U)+h_n)\right)=\omega(n).$$

Then, it is easy to show that, up to a subsequence,

$$\{\nabla u_n\} \text{ converges to } \nabla u, \qquad a.e. \text{ in } Q. \tag{3.11}$$

Therefore, $\{A(x,t,\nabla u_n)\}$ converges to $A(x,t,\nabla u)$ weakly in $(L^{p'}(Q))^N$; and from (3.10) we find

$$\overline{\lim}_{n\to\infty}\int_Q A(x,t,\nabla u_n).\nabla T_k(U_n)\leq\int_Q A(x,t,\nabla u)\nabla T_k(U).$$

Otherwise, $\{A(x,t,\nabla\left(T_k(U_n)+h_n\right))\}$ converges weakly in $(L^{p'}(Q))^N$ to some F_k, from Proposition 2.10, and we obtain that $F_k=A(x,t,\nabla\left(T_k(U)+h\right))$. Hence

$$\begin{aligned}
&\overline{\lim}_{n\to\infty}\int_Q A(x,t,\nabla(T_k(U_n)+h_n)).\nabla(T_k(U_n)+h_n)\\
&\leq\overline{\lim}_{n\to\infty}\int_Q A(x,t,\nabla u_n).\nabla T_k(U_n)+\overline{\lim}_{n\to\infty}\int_Q A(x,t,\nabla(T_k(U_n)+h_n)).\nabla h_n\\
&\leq\int_Q A(x,t,\nabla(T_k(U)+h)).\nabla(T_k(U)+h).
\end{aligned}$$

As a consequence

$$\{T_k(U_n)\} \text{ converges to } T_k(U), \text{ strongly in } X, \qquad \forall k>0. \tag{3.12}$$

Then *to finish the proof we have to check that* u *is a solution of* (1.1). ■

In order to prove (3.7) we need a first Lemma, inspired of [14, Lemma 6.1]. It extends the results of [22, Lemma 6 and Lemma 7] relative to sequences of solutions with smooth data:

Lemma 3.3 *Let* $\psi_{1,\delta},\psi_{2,\delta}\in C^1(Q)$ *be uniformly bounded in* $W^{1,\infty}(Q)$ *with values in* $[0,1]$, *and such that* $\int_Q\psi_{1,\delta}d\mu_s^-\leq\delta$ *and* $\int_Q\psi_{2,\delta}d\mu_s^+\leq\delta$. *Let* $\{u_n\}$ *satisfying the assumptions of Theorem 1.1, and* $U_n=u_n-h_n$. *Then*

$$\frac{1}{m}\int_{\{m\leq U_n<2m\}}|\nabla u_n|^p\psi_{2,\delta}=\omega(n,m,\delta), \qquad \frac{1}{m}\int_{\{m\leq U_n<2m\}}|\nabla U_n|^p\psi_{2,\delta}=\omega(n,m,\delta), \tag{3.13}$$

$$\frac{1}{m}\int_{-2m<U_n\leq -m}|\nabla u_n|^p\psi_{1,\delta}=\omega(n,m,\delta), \qquad \frac{1}{m}\int_{-2m<U_n\leq -m}|\nabla U_n|^p\psi_{1,\delta}=\omega(n,m,\delta), \tag{3.14}$$

and for any $k>0$,

$$\int_{\{m\leq U_n<m+k\}}|\nabla u_n|^p\psi_{2,\delta}=\omega(n,m,\delta), \qquad \int_{\{m\leq U_n<m+k\}}|\nabla U_n|^p\psi_{2,\delta}=\omega(n,m,\delta), \tag{3.15}$$

$$\int\limits_{\{-m-k<U_n\leq -m\}}|\nabla u_n|^p\psi_{1,\delta}=\omega(n,m,\delta),\qquad \int\limits_{\{-m-k<U_n\leq -m\}}|\nabla U_n|^p\psi_{1,\delta}=\omega(n,m,\delta). \tag{3.16}$$

Proof. (i) Proof of (3.13), (3.14). Set for any $r\in\mathbb{R}$ and any $m,\ell\geq 1$

$$S_{m,\ell}(r)=\int_0^r\left(\frac{-m+\tau}{m}\chi_{[m,2m]}(\tau)+\chi_{(2m,2m+\ell]}(\tau)+\frac{4m+2h-\tau}{2m+\ell}\chi_{(2m+\ell,4m+2h]}(\tau)\right)d\tau,$$

$$S_m(r)=\int_0^r\left(\frac{-m+\tau}{m}\chi_{[m,2m]}(\tau)+\chi_{(2m,\infty)}(\tau)\right)d\tau.$$

Note that $S''_{m,\ell}=\chi_{[m,2m]}/m-\chi_{[2m+\ell,2(2m+\ell)]}/(2m+\ell)$. We choose $(\xi,J,S)=(\psi_{2,\delta},T_1,S_{m,\ell})$ as test functions in (2.15) for u_n, and observe that, from (3.5),

$$\widehat{\mu_{n,0}}=\mu_{n,0}-(h_n)_t=\widehat{\lambda_{n,0}}+\rho_{n,0}-\eta_{n,0}=f_n-\operatorname{div}g_n+\rho_{n,0}-\eta_{n,0}. \tag{3.17}$$

Thus we can write $\sum_{i=1}^{6}A_i\leq\sum_{i=7}^{12}A_i$, where

$$A_1=-\int_\Omega\psi_{2,\delta}(0)T_1(S_{m,\ell}(u_{0,n}))S_{m,\ell}(u_{0,n})dx,\quad A_2=-\int_Q(\psi_{2,\delta})_t\overline{T_1}(S_{m,\ell}(U_n)),$$

$$A_3=\int_Q S'_{m,\ell}(U_n)T_1(S_{m,\ell}(U_n))A(x,t,\nabla u_n)\nabla\psi_{2,\delta},\quad A_4=\int_Q(S'_{m,\ell}(U_n))^2\psi_{2,\delta}T'_1(S_{m,\ell}(U_n))A(x,t,\nabla u_n)\nabla U_n,$$

$$A_5=\frac{1}{m}\int\limits_{\{m\leq U_n\leq 2m\}}\psi_{2,\delta}T_1(S_{m,\ell}(U_n))A(x,t,\nabla u_n)\nabla U_n,$$

$$A_6=-\frac{1}{2m+\ell}\int\limits_{\{2m+\ell\leq U_n<2(2m+\ell)\}}\psi_{2,\delta}A(x,t,\nabla u_n)\nabla U_n,$$

$$A_7=\int_Q S'_{m,\ell}(U_n)T_1(S_{m,\ell}(U_n))\psi_{2,\delta}f_n,\qquad A_8=\int_Q S'_{m,\ell}(U_n)T_1(S_{m,\ell}(U_n))g_n.\nabla\psi_{2,\delta},$$

$$A_9=\int_Q\left(S'_{m,\ell}(U_n)\right)^2T'_1(S_{m,\ell}(U_n))\psi_{2,\delta}g_n.\nabla U_n,\qquad A_{10}=\frac{1}{m}\int\limits_{m\leq U_n\leq 2m}T_1(S_{m,\ell}(U_n))\psi_{2,\delta}g_n.\nabla U_n,$$

$$A_{11}=-\frac{1}{2m+\ell}\int\limits_{\{2m+\ell\leq U_n<2(2m+\ell)\}}\psi_{2,\delta}g_n.\nabla U_n,\quad A_{12}=\int_Q S'_{m,\ell}(U_n)T_1(S_{m,\ell}(U_n))\psi_{2,\delta}d\left(\rho_{n,0}-\eta_{n,0}\right).$$

Since $||S_{m,\ell}(u_{0,n})||_{1,\Omega}\leq\int\limits_{\{m\leq u_{0,n}\}}u_{0,n}dx$, we find $A_1=\omega(\ell,n,m)$. Otherwise

$$|A_2|\leq\|\psi_{2,\delta}\|_{W^{1,\infty}(Q)}\int\limits_{\{m\leq U_n\}}U_n,\qquad |A_3|\leq\|\psi_{2,\delta}\|_{W^{1,\infty}(Q)}\int\limits_{\{m\leq U_n\}}\left(|a|+\Lambda_2|\nabla u_n|^{p-1}\right),$$

which imply $A_2=\omega(\ell,n,m)$ and $A_3=\omega(\ell,n,m)$. Using (2.3) for u_n, we have

$$A_6=-\int_Q\psi_{2,\delta}d(\rho_{n,s}-\eta_{n,s})^++\omega(\ell)=\omega(\ell,n,m,\delta).$$

Hence $A_6 = \omega(\ell, n, m, \delta)$, since $(\rho_{n,s} - \eta_{n,s})^+$ converges to μ_s^+ as $n \to \infty$ in the narrow topology, and $\int_Q \psi_{2,\delta} d\mu_s^+ \leq \delta$. We also obtain $A_{11} = \omega(\ell)$ from (2.10).

Now $\left\{S'_{m,\ell}(U_n)T_1(S_{m,\ell}(U_n))\right\}_\ell$ converges to $S'_m(U_n)T_1(S_m(U_n))$, $\{S'_m(U_n)T_1(S_m(U_n))\}_n$ converges to $S'_m(U)$ $T_1(S_m(U))$, $\{S'_m(U)T_1(S_m(U))\}_m$ converges to 0, weak-* in $L^\infty(Q)$ and $\{f_n\}$ converges to f weakly in $L^1(Q)$, $\{g_n\}$ converges to g strongly in $(L^{p'}(Q))^N$. From Remark 3.2, we obtain

$$A_7 = \int_Q S'_m(U_n)T_1(S_m(U_n))\psi_{2,\delta} f_n + \omega(\ell) = \int_Q S'_m(U)T_1(S_m(U))\psi_{2,\delta} f + \omega(\ell, n) = \omega(\ell, n, m),$$
$$A_8 = \int_Q S'_m(U_n)T_1(S_m(U_n))g_n.\nabla\psi_{2,\delta} + \omega(\ell) = \int_Q S'_m(U)T_1(S_m(U))g\nabla\psi_{2,\delta} + \omega(\ell, n) = \omega(\ell, n, m).$$

Otherwise, $A_{12} \leq \int_Q \psi_{2,\delta} d\rho_n$, and $\left\{\int_Q \psi_{2,\delta} d\rho_n\right\}$ converges to $\int_Q \psi_{2,\delta} d\mu_s^+$, thus $A_{12} \leq \omega(\ell, n, m, \delta)$.
Using Holder inequality and the condition (1.2), we have

$$g_n.\nabla U_n - A(x, t, \nabla u_n)\nabla U_n \leq c_1 \left(|g_n|^{p'} + |\nabla h_n|^p + |a|^{p'}\right)$$

with $c_1 = c_1(p, \Lambda_1, \Lambda_2)$, which implies

$$A_9 - A_4 \leq c_1 \int_Q \left(S'_{m,\ell}(U_n)\right)^2 T'_1(S_{m,\ell}(U_n))\psi_{2,\delta} \left(|g_n|^{p'} + |h_n|^p + |a|^{p'}\right) = \omega(\ell, n, m).$$

Similarly we also show that $A_{10} - A_5/2 \leq \omega(\ell, n, m)$. Combining the estimates, we get $A_5/2 \leq \omega(\ell, n, m, \delta)$.
Using Holder inequality we have

$$A(x, t, \nabla u_n)\nabla U_n \geq \frac{\Lambda_1}{2}|\nabla u_n|^p - c_2(|a|^{p'} + |\nabla h_n|^p).$$

with $c_2 = c_2(p, \Lambda_1, \Lambda_2)$, which implies

$$\frac{1}{m} \int_{\{m \leq U_n < 2m\}} |\nabla u_n|^p \psi_{2,\delta} T_1(S_{m,\ell}(U_n)) = \omega(\ell, n, m, \delta).$$

Note that for all $m > 4$, $S_{m,\ell}(r) \geq 1$ for any $r \in [\frac{3}{2}m, 2m]$; hence $T_1(S_{m,\ell}(r)) = 1$. So,

$$\frac{1}{m} \int_{\left\{\frac{3}{2}m \leq U_n < 2m\right\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(\ell, n, m, \delta).$$

Since $|\nabla U_n|^p \leq 2^{p-1}|\nabla u_n|^p + 2^{p-1}|\nabla h_n|^p$, there also holds

$$\frac{1}{m} \int_{\left\{\frac{3}{2}m \leq U_n < 2m\right\}} |\nabla U_n|^p \psi_{2,\delta} = \omega(\ell, n, m, \delta).$$

We deduce (3.13) by summing on each set $\left\{(\frac{4}{3})^i m \leq U_n \leq (\frac{4}{3})^{i+1} m\right\}$ for $i = 0, 1, 2$. Similarly, we can choose $(\xi, \psi, S) = (\psi_{1,\delta}, T_1, \tilde{S}_{m,\ell})$ as test functions in (2.15) for u_n, where $\tilde{S}_{m,\ell}(r) = S_{m,\ell}(-r)$, and we obtain (3.14).

(ii) Proof of (3.15), (3.16). We set, for any $k, m, \ell \geq 1$,

$$S_{k,m,\ell}(r) = \int_0^r \left(T_k(\tau - T_m(\tau))\chi_{[m,k+m+\ell]} + k\frac{2(k+\ell+m) - \tau}{k+m+\ell}\chi_{(k+m+\ell,2(k+m+\ell)]}\right) d\tau$$

$$S_{k,m}(r)=\int_0^r T_k(\tau-T_m(\tau))\chi_{[m,\infty)}d\tau.$$

We choose $(\xi,\psi,S)=(\psi_{2,\delta},T_1,S_{k,m,\ell})$ as test functions in (2.15) for u_n. In the same way we also obtain

$$\int\limits_{\{m\le U_n<m+k\}}|\nabla u_n|^p\psi_{2,\delta}T_1(S_{k,m,\ell}(U_n))=\omega(\ell,n,m,\delta).$$

Note that $T_1(S_{k,m,\ell}(r))=1$ for any $r\ge m+1$, thus $\int\limits_{\{m+1\le U_n<m+k\}}|\nabla u_n|^p\psi_{2,\delta}=\omega(n,m,\delta)$, which implies (3.15) by changing m into $m-1$. Similarly, we obtain (3.16). ∎

Next we look at the behaviour near E.

Lemma 3.4 *Estimate (3.7) holds.*

Proof. There holds

$$I_1=\int_Q\Phi_{\delta_1,\delta_2}A(x,t,\nabla u_n).\nabla T_k(U_n)-\int\limits_{\{|U_n|\le k\}}\Phi_{\delta_1,\delta_2}A(x,t,\nabla u_n).\nabla\langle T_k(U)\rangle_\nu.$$

From Proposition 2.10, (iv), $\{A(x,t,\nabla(T_k(U_n)+h_n)).\nabla\langle T_k(U)\rangle_\nu\}$ converges weakly in $L^1(Q)$ to $F_k\nabla\langle T_k(U)\rangle_\nu$. And $\left\{\chi_{\{|U_n|\le k\}}\right\}$ converges to $\chi_{|U|\le k}$, *a.e.* in Q , and Φ_{δ_1,δ_2} converges to 0 *a.e.* in Q as $\delta_1\to 0$, and Φ_{δ_1,δ_2} takes its values in $[0,1]$. From Remark 3.2, we have

$$\begin{aligned}\int\limits_{\{|U_n|\le k\}}\Phi_{\delta_1,\delta_2}A(x,t,\nabla u_n).\nabla\langle T_k(U)\rangle_\nu&=\int_Q\chi_{\{|U_n|\le k\}}\Phi_{\delta_1,\delta_2}A(x,t,\nabla(T_k(U_n)+h_n)).\nabla\langle T_k(U)\rangle_\nu\\&=\int_Q\chi_{|U|\le k}\Phi_{\delta_1,\delta_2}F_k.\nabla\langle T_k(U)\rangle_\nu+\omega(n)=\omega(n,\nu,\delta_1).\end{aligned}$$

Therefore, if we prove that

$$\int_Q\Phi_{\delta_1,\delta_2}A(x,t,\nabla u_n).\nabla T_k(U_n)\le\omega(n,\delta_1,\delta_2),\tag{3.18}$$

then we deduce (3.7). As noticed in [14, 22], it is precisely for this estimate that we need the double cut $\psi^+_{\delta_1}\psi^+_{\delta_2}$. To do this, we set, for any $m>k>0$, and any $r\in\mathbb{R}$,

$$\hat{S}_{k,m}(r)=\int_0^r(k-T_k(\tau))\,H_m(\tau)d\tau,$$

where H_m is defined at (2.14). Hence supp $\hat{S}_{k,m}\subset[-2m,k]$; and $\hat{S}''_{k,m}=-\chi_{[-k,k]}+\frac{2k}{m}\chi_{[-2m,-m]}$. We choose $(\varphi,S)=(\psi^+_{\delta_1}\psi^+_{\delta_2},\hat{S}_{k,m})$ as test functions in (2.2). From (3.17), we can write

$$A_1+A_2-A_3+A_4+A_5+A_6=0,$$

where

$$A_1=-\int_Q(\psi^+_{\delta_1}\psi^+_{\delta_2})_t\hat{S}_{k,m}(U_n),\quad A_2=\int_Q(k-T_k(U_n))H_m(U_n)A(x,t,\nabla u_n).\nabla(\psi^+_{\delta_1}\psi^+_{\delta_2}),$$

$$A_3=\int_Q\psi^+_{\delta_1}\psi^+_{\delta_2}A(x,t,\nabla u_n).\nabla T_k(U_n),\quad A_4=\frac{2k}{m}\int\limits_{\{-2m<U_n\le -m\}}\psi^+_{\delta_1}\psi^+_{\delta_2}A(x,t,\nabla u_n).\nabla U_n,$$

$$A_5=-\int_Q(k-T_k(U_n))H_m(U_n)\psi^+_{\delta_1}\psi^+_{\delta_2}d\widehat{\lambda_{n,0}},\quad A_6=\int_Q(k-T_k(U_n))H_m(U_n)\psi^+_{\delta_1}\psi^+_{\delta_2}d\left(\eta_{n,0}-\rho_{n,0}\right).$$

We first estimate A_3. As in [22, p.585], since $\left\{\hat{S}_{k,m}(U_n)\right\}$ converges to $\hat{S}_{k,m}(U)$ weakly in X, and $\hat{S}_{k,m}(U)\in L^{\infty}(Q)$, using (3.1), we find

$$A_1=-\int_Q(\psi_{\delta_1}^+)_t\psi_{\delta_2}^+\hat{S}_{k,m}(U)-\int_Q\psi_{\delta_1}^+(\psi_{\delta_2}^+)_t\hat{S}_{k,m}(U)+\omega(n)=\omega(n,\delta_1).$$

Next consider A_2. Notice that $U_n=T_{2m}(U_n)$ on supp $(H_m(U_n))$. From Proposition 2.10, (iv), the sequence $\left\{A(x,t,\nabla\left(T_{2m}(U_n)+h_n\right)).\nabla(\psi_{\delta_1}^+\psi_{\delta_2}^+)\right\}$ converges to $F_{2m}.\nabla(\psi_{\delta_1}^+\psi_{\delta_2}^+)$ weakly in $L^1(Q)$. From Remark 3.2 and the convergence of $\psi_{\delta_1}^+\psi_{\delta_2}^+$ in X to 0 as δ_1 tends to 0, we find

$$A_2=\int_Q(k-T_k(U))H_m(U)F_{2m}.\nabla(\psi_{\delta_1}^+\psi_{\delta_2}^+)+\omega(n)=\omega(n,\delta_1).$$

Then consider A_4. Then for some $c_1=c_1(p,\Lambda_2)$,

$$|A_4|\leq c_1\frac{2k}{m}\int\limits_{\{-2m<U_n\leq-m\}}\left(|\nabla u_n|^p+|\nabla U_n|^p+|a|^{p'}\right)\psi_{\delta_1}^+\psi_{\delta_2}^+.$$

Since $\psi_{\delta_1}^+$ takes its values in $[0,1]$, from Lemma 3.3, we get in particular $A_4=\omega(n,\delta_1,m,\delta_2)$.

Now we estimate A_5. The sequence $\left\{(k-T_k(U_n))H_m(U_n)\psi_{\delta_1}^+\psi_{\delta_2}^+\right\}$ converges to $(k-T_k(U))H_m(U)\psi_{\delta_1}^+\psi_{\delta_2}^+$, weakly in X, and $\{(k-T_k(U_n))H_m(U_n)\}$ converges to $(k-T_k(U))H_m(U)$, weak-* in $L^{\infty}(Q)$ and *a.e.* in Q. Otherwise $\{f_n\}$ converges to f weakly in $L^1(Q)$ and $\{g_n\}$ converges to g strongly in $(L^{p'}(Q))^N$. From Remark 3.2 and the convergence of $\psi_{\delta_1}^+\psi_{\delta_2}^+$ to 0 in X and *a.e.* in Q as $\delta_1\to0$, we deduce that

$$A_5=-\int_Q(k-T_k(U_n))H_m(U)\psi_{\delta_1}^+\psi_{\delta_2}^+d\widehat{\nu_0}+\omega(n)=\omega(n,\delta_1),$$

where $\widehat{\nu_0}=f-\operatorname{div}g$.

Finally $A_6\leq 2k\int_Q\psi_{\delta_1}^+\psi_{\delta_2}^+d\eta_n$; using (3.2) we also find $A_6\leq\omega(n,\delta_1,m,\delta_2)$. By addition, since A_3 does not depend on m, we obtain

$$A_3=\int_Q\psi_{\delta_1}^+\psi_{\delta_2}^+A(x,t,\nabla u_n)\nabla T_k(U_n)\leq\omega(n,\delta_1,\delta_2).$$

Arguying as before with $(\psi_{\delta_1}^-\psi_{\delta_2}^-,\check{S}_{k,m})$ as test function in (2.2), where $\check{S}_{k,m}(r)=-\hat{S}_{k,m}(-r)$, we get in the same way

$$\int_Q\psi_{\delta_1}^-\psi_{\delta_2}^-A(x,t,\nabla u_n)\nabla T_k(U_n)\leq\omega(n,\delta_1,\delta_2).$$

Then, (3.18) holds. ∎

Next we look at the behaviour far from E.

Lemma 3.5 . *Estimate (3.8) holds.*

Proof. Here we estimate I_2; we can write

$$I_2=\int\limits_{\{|U_n|\leq k\}}(1-\Phi_{\delta_1,\delta_2})A(x,t,\nabla u_n)\nabla\left(T_k(U_n)-\langle T_k(U)\rangle_{\nu}\right).$$

Following the ideas of [25], used also in [22], we define, for any $r \in \mathbb{R}$ and $\ell > 2k > 0$,

$$R_{n,\nu,\ell} = T_{\ell+k}\left(U_n - \langle T_k(U)\rangle_\nu\right) - T_{\ell-k}\left(U_n - T_k\left(U_n\right)\right).$$

Recall that $\|\langle T_k(U)\rangle_\nu\|_{\infty,Q} \leq k$, and observe that

$$R_{n,\nu,\ell} = 2k\ \mathrm{sign}(U_n) \quad \text{in } \{|U_n| \geq \ell + 2k\}, \qquad |R_{n,\nu,\ell}| \leq 4k, \quad R_{n,\nu,\ell} = \omega(n,\nu,\ell)\ a.e. \text{ in } Q, \tag{3.19}$$

$$\lim_{n\to\infty} R_{n,\nu,\ell} = T_{\ell+k}\left(U - \langle T_k(U)\rangle_\nu\right) - T_{\ell-k}\left(U - T_k\left(U\right)\right), \qquad a.e. \text{ in } Q, \text{ and weakly in } X. \tag{3.20}$$

Next consider $\xi_{1,n_1} \in C_c^\infty([0,T)), \xi_{2,n_2} \in C_c^\infty((0,T])$ with values in $[0,1]$, such that $(\xi_{1,n_1})_t \leq 0$ and $(\xi_{2,n_2})_t \geq 0$; and $\{\xi_{1,n_1}(t)\}$ (resp. $\{\xi_{1,n_2}(t)\}$) converges to $1,$ for any $t \in [0,T)$ (resp. $t \in (0,T]$); and moreover, for any $a \in C([0,T];L^1(\Omega))$, $\left\{\int_Q a(\xi_{1,n_1})_t\right\}$ and $\int_Q a(\xi_{2,n_2})_t$ converge respectively to $-\int_\Omega a(.,T)dx$ and $\int_\Omega a(.,0)dx$. We set

$$\varphi = \varphi_{n,n_1,n_2,l_1,l_2,\ell} = \xi_{1,n_1}(1-\Phi_{\delta_1,\delta_2})[T_{\ell+k}\left(U_n - \langle T_k(U)\rangle_\nu\right)]_{l_1} - \xi_{2,n_2}(1-\Phi_{\delta_1,\delta_2})[T_{\ell-k}\left(U_n - T_k(U_n)\right)]_{-l_2}.$$

We observe that

$$\varphi - (1-\Phi_{\delta_1,\delta_2})R_{n,\nu,\ell} = \omega(l_1,l_2,n_1,n_2) \qquad \text{in norm in } X \text{ and } a.e. \text{ in } Q. \tag{3.21}$$

We can choose $(\varphi, S) = (\varphi_{n,n_1,n_2,l_1,l_2,\ell}, \overline{H_m})$ as test functions in (2.7) for u_n, where $\overline{H_m}$ is defined at (2.14), with $m > \ell + 2k$. We obtain

$$A_1 + A_2 + A_3 + A_4 + A_5 = A_6 + A_7,$$

with

$$A_1 = \int_\Omega \varphi(T)\overline{H_m}(U_n(T))dx, \qquad A_2 = -\int_\Omega \varphi(0)\overline{H_m}(u_{0,n})dx, \qquad A_3 = -\int_Q \varphi_t \overline{H_m}(U_n),$$
$$A_4 = \int_Q H_m(U_n)A(x,t,\nabla u_n).\nabla\varphi, \qquad A_5 = \int_Q \varphi H_m'(U_n)A(x,t,\nabla u_n).\nabla U_n,$$
$$A_6 = \int_Q H_m(U_n)\varphi d\widehat{\lambda_{n,0}}, \qquad A_7 = \int_Q H_m(U_n)\varphi d\left(\rho_{n,0} - \eta_{n,0}\right).$$

Estimate of A_4. This term allows to study I_2. Indeed, $\{H_m(U_n)\}$ converges to 1, $a.e.$ in Q; From (3.21), (3.19) (3.20), we have

$$\begin{aligned} A_4 &= \int_Q (1-\Phi_{\delta_1,\delta_2})A(x,t,\nabla u_n).\nabla R_{n,\nu,\ell} - \int_Q R_{n,\nu,\ell}A(x,t,\nabla u_n).\nabla\Phi_{\delta_1,\delta_2} + \omega(l_1,l_2,n_1,n_2,m) \\ &= \int_Q (1-\Phi_{\delta_1,\delta_2})A(x,t,\nabla u_n).\nabla R_{n,\nu,\ell} + \omega(l_1,l_2,n_1,n_2,m,n,\nu,\ell) \\ &= I_2 + \int_{\{|U_n|>k\}} (1-\Phi_{\delta_1,\delta_2})A(x,t,\nabla u_n).\nabla R_{n,\nu,\ell} + \omega(l_1,l_2,n_1,n_2,m,n,\nu,\ell) \\ &= I_2 + B_1 + B_2 + \omega(l_1,l_2,n_1,n_2,m,n,\nu,\ell), \end{aligned}$$

where

$$B_1 = \int_{\{|U_n|>k\}} (1-\Phi_{\delta,\eta})(\chi_{\left|U_n - \langle T_k(U)\rangle_\nu\right| \leq \ell+k} - \chi_{||U_n|-k| \leq \ell-k})A(x,t,\nabla u_n).\nabla U_n,$$
$$B_2 = -\int_{\{|U_n|>k\}} (1-\Phi_{\delta_1,\delta_2})\chi_{\left|U_n - \langle T_k(U)\rangle_\nu\right| \leq \ell+k}A(x,t,\nabla u_n).\nabla\langle T_k(U)\rangle_\nu.$$

Now $\{A(x,t,\nabla(T_{\ell+2k}(U_n)+h_n)).\nabla\langle T_k(U)\rangle_\nu\}$ converges to $F_{\ell+2k}\nabla\langle T_k(U)\rangle_\nu$, weakly in $L^1(Q)$. Otherwise $\left\{\chi_{|U_n|>k}\chi_{\left|U_n-\langle T_k(U)\rangle_\nu\right|\le\ell+k}\right\}$ converges to $\chi_{|U|>k}\chi_{\left|U-\langle T_k(U)\rangle_\nu\right|\le\ell+k}$, *a.e.* in Q. And $\{\langle T_k(U)\rangle_\nu\}$ converges to $T_k(U)$ strongly in X. From Remark 3.2 we get

$$\begin{aligned}B_2 &= -\int_Q (1-\Phi_{\delta_1,\delta_2})\,\chi_{|U|>k}\,\chi_{\left|U-\langle T_k(U)\rangle_\nu\right|\le\ell+k}F_{\ell+2k}.\nabla\langle T_k(U)\rangle_\nu+\omega(n)\\ &= -\int_Q (1-\Phi_{\delta_1,\delta_2})\,\chi_{|U|>k}\,\chi_{|U-T_k(U)|\le\ell+k}F_{\ell+2k}.\nabla T_k(U)+\omega(n,\nu)=\omega(n,\nu),\end{aligned}$$

since $\nabla T_k(U)\,\chi_{|U|>k}=0$. Besides, we see that, for some $c_1=c_1(p,\Lambda_2)$,

$$|B_1|\le c_1\int_{\{\ell-2k\le|U_n|<\ell+2k\}}(1-\Phi_{\delta_1,\delta_2})(|\nabla u_n|^p+|\nabla U_n|^p+|a|^{p'}).$$

Using (3.3) and (3.4) and applying (3.15) and (3.16) to $1-\Phi_{\delta_1,\delta_2}$, we obtain, for $k>0$,

$$\int_{\{m\le|U_n|<m+4k\}}(|\nabla u_n|^p+|\nabla U_n|^p)(1-\Phi_{\delta_1,\delta_2})=\omega(n,m,\delta_1,\delta_2). \tag{3.22}$$

Thus, $B_1=\omega(n,\nu,\ell,\delta_1,\delta_2)$, hence $B_1+B_2=\omega(n,\nu,\ell,\delta_1,\delta_2)$. Then

$$A_4=I_2+\omega(l_1,l_2,n_1,n_2,m,n,\nu,\ell,\delta_1,\delta_2). \tag{3.23}$$

Estimate of A_5. For $m>\ell+2k$, since $|\varphi|\le 2\ell$, and (3.21) holds, we get, from the dominated convergence Theorem,

$$\begin{aligned}A_5 &= \int_Q(1-\Phi_{\delta_1,\delta_2})R_{n,\nu,\ell}H'_m(U_n)A(x,t,\nabla u_n).\nabla U_n+\omega(l_1,l_2,n_1,n_2)\\ &= -\frac{2k}{m}\int_{\{m\le|U_n|<2m\}}(1-\Phi_{\delta_1,\delta_2})A(x,t,\nabla u_n).\nabla U_n+\omega(l_1,l_2,n_1,n_2);\end{aligned}$$

here, the final equality followed from the relation, since $m>\ell+2k$,

$$R_{n,\nu,\ell}H'_m(U_n)=-\frac{2k}{m}\chi_{m\le|U_n|\le 2m},\quad \textit{a.e. in } Q. \tag{3.24}$$

Next we go to the limit in m, by using (2.3), (2.4) for u_n, with $\phi=(1-\Phi_{\delta_1,\delta_2})$. There holds

$$A_5=-2k\int_Q(1-\Phi_{\delta_1,\delta_2})d\left((\rho_{n,s}-\eta_{n,s})^++(\rho_{n,s}-\eta_{n,s})^-\right)+\omega(l_1,l_2,n_1,n_2,m).$$

Then, from (3.3) and (3.4), we get $A_5=\omega(l_1,l_2,n_1,n_2,m,n,\nu,\ell,\delta_1,\delta_2)$.

Estimate of A_6. Again, from (3.21),

$$\begin{aligned}A_6 &= \int_Q H_m(U_n)\varphi f_n+\int_Q g_n.\nabla(H_m(U_n)\varphi)\\ &= \int_Q H_m(U_n)(1-\Phi_{\delta_1,\delta_2})R_{n,\nu,\ell}f_n+\int_Q g_n.\nabla(H_m(U_n)(1-\Phi_{\delta_1,\delta_2})R_{n,\nu,\ell})+\omega(l_1,l_2,n_1,n_2).\end{aligned}$$

Thus we can write $A_6 = D_1 + D_2 + D_3 + D_4 + \omega(l_1, l_2, n_1, n_2)$, where

$$D_1 = \int_Q H_m(U_n)(1 - \Phi_{\delta_1,\delta_2})R_{n,\nu,\ell} f_n, \qquad D_2 = \int_Q (1 - \Phi_{\delta_1,\delta_2})R_{n,\nu,\ell} H'_m(U_n) g_n.\nabla U_n,$$
$$D_3 = \int_Q H_m(U_n)(1 - \Phi_{\delta_1,\delta_2}) g_n.\nabla R_{n,\nu,\ell}, \qquad D_4 = -\int_Q H_m(U_n)R_{n,\nu,\ell} g_n.\nabla \Phi_{\delta_1,\delta_2}.$$

Since $\{f_n\}$ converges to f weakly in $L^1(Q)$, and (3.19)-(3.20) hold, we get, from Remark 3.2,

$$D_1 = \int_Q (1 - \Phi_{\delta_1,\delta_2}) \left(T_{\ell+k}\left(U - \langle T_k(U)\rangle_\nu\right) - T_{\ell-k}\left(U - T_k\left(U\right)\right)\right) f + \omega(m, n) = \omega(m, n, \nu, \ell).$$

We deduce from (2.10) that $D_2 = \omega(m)$. Next consider D_3. Note that $H_m(U_n) = 1 + \omega(m)$, and (3.20) holds, and $\{g_n\}$ converges to g strongly in $(L^{p'}(Q))^N$, and $\langle T_k(U)\rangle_\nu$ converges to $T_k(U)$ strongly in X. Then we obtain successively that

$$\begin{aligned} D_3 &= \int_Q (1 - \Phi_{\delta_1,\delta_2}) g.\nabla \left(T_{\ell+k}\left(U - \langle T_k(U)\rangle_\nu\right) - T_{\ell-k}\left(U - T_k\left(U\right)\right)\right) + \omega(m, n) \\ &= \int_Q (1 - \Phi_{\delta_1,\delta_2}) g.\nabla \left(T_{\ell+k}\left(U - T_k(U)\right) - T_{\ell-k}\left(U - T_k\left(U\right)\right)\right) + \omega(m, n, \nu) \\ &= \omega(m, n, \nu, \ell). \end{aligned}$$

Similarly we also get $D_4 = \omega(m, n, \nu, \ell)$. Thus $A_6 = \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2)$.

Estimate of A_7. We have

$$\begin{aligned} |A_7| &= \left|\int_Q S'_m(U_n)\left(1 - \Phi_{\delta_1,\delta_2}\right) R_{n,\nu,\ell}\, d\left(\rho_{n,0} - \eta_{n,0}\right)\right| + \omega(l_1, l_2, n_1, n_2) \\ &\le 4k \int_Q \left(1 - \Phi_{\delta_1,\delta_2}\right) d\left(\rho_n + \eta_n\right) + \omega(l_1, l_2, n_1, n_2). \end{aligned}$$

From (3.3) and (3.4) we get $A_7 = \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2)$.

Estimate of $A_1 + A_2 + A_3$. We set

$$J(r) = T_{\ell-k}\left(r - T_k\left(r\right)\right), \qquad \forall r \in \mathbb{R},$$

and use the notations $\overline{J}$ and $\mathcal{J}$ of (2.11). From the definitions of ξ_{1,n_1}, ξ_{1,n_2}, we can see that

$$\begin{aligned} A_1 + A_2 &= -\int_\Omega J(U_n(T))\overline{H_m}(U_n(T))dx - \int_\Omega T_{\ell+k}(u_{0,n} - z_\nu)\overline{H_m}(u_{0,n})dx + \omega(l_1, l_2, n_1, n_2) \\ &= -\int_\Omega J(U_n(T))U_n(T)dx - \int_\Omega T_{\ell+k}(u_{0,n} - z_\nu)u_{0,n}dx + \omega(l_1, l_2, n_1, n_2, m), \end{aligned} \tag{3.25}$$

where $z_\nu = \langle T_k(U)\rangle_\nu(0)$. We can write $A_3 = F_1 + F_2$, where

$$F_1 = -\int_Q \left(\xi_{n_1}(1 - \Phi_{\delta_1,\delta_2})[T_{\ell+k}\left(U_n - \langle T_k(U)\rangle_\nu\right)]_{l_1}\right)_t \overline{H_m}(U_n),$$
$$F_2 = \int_Q \left(\xi_{n_2}(1 - \Phi_{\delta_1,\delta_2})[T_{\ell-k}\left(U_n - T_k\left(U_n\right)\right)]_{-l_2}\right)_t \overline{H_m}(U_n).$$

Estimate of F_2. We write $F_2 = G_1 + G_2 + G_3$, with

$$G_1 = -\int_Q (\Phi_{\delta_1,\delta_2})_t \xi_{n_2}[T_{\ell-k}(U_n - T_k(U_n))]_{-l_2}\overline{H_m}(U_n),$$

$$G_2 = \int_Q (1-\Phi_{\delta_1,\delta_2})(\xi_{n_2})_t[T_{\ell-k}(U_n - T_k(U_n))]_{-l_2}\overline{H_m}(U_n),$$

$$G_3 = \int_Q \xi_{n_2}(1-\Phi_{\delta_1,\delta_2})\big([T_{\ell-k}(U_n - T_k(U_n))]_{-l_2}\big)_t\overline{H_m}(U_n).$$

We find easily that

$$G_1 = -\int_Q (\Phi_{\delta_1,\delta_2})_t J(U_n)U_n + \omega(l_1,l_2,n_1,n_2,m),$$

$$G_2 = \int_Q (1-\Phi_{\delta_1,\delta_2})(\xi_{n_2})_t J(U_n)\overline{H_m}(U_n) + \omega(l_1,l_2) = \int_\Omega J(u_{0,n})u_{0,n}dx + \omega(l_1,l_2,n_1,n_2,m).$$

Next consider G_3. Setting $b = \overline{H_m}(U_n)$, there holds from (2.13) and (2.12),

$$(([J(b)]_{-l_2})_t b)(.,t) = \frac{b(.,t)}{l_2}(J(b)(.,t) - J(b)(.,t-l_2)).$$

Hence

$$\big([T_{\ell-k}(U_n - T_k(U_n))]_{-l_2}\big)_t\overline{H_m}(U_n) \geq \Big([\mathcal{J}(\overline{H_m}(U_n))]_{-l_2}\Big)_t = \big([\mathcal{J}(U_n)]_{-l_2}\big)_t,$$

since $\mathcal{J}$ is constant in $\{|r| \geq m+\ell+2k\}$. Integrating by parts in G_3, we find

$$\begin{aligned} G_3 &\geq \int_Q \xi_{2,n_2}(1-\Phi_{\delta_1,\delta_2})\big([\mathcal{J}(U_n)]_{-l_2}\big)_t = -\int_Q (\xi_{2,n_2}(1-\Phi_{\delta_1,\delta_2}))_t[\mathcal{J}(U_n)]_{-l_2} + \int_\Omega \xi_{2,n_2}(T)[\mathcal{J}(U_n)]_{-l_2}(T)dx \\ &= -\int_Q (\xi_{2,n_2})_t(1-\Phi_{\delta_1,\delta_2})\mathcal{J}(U_n) + \int_Q \xi_{2,n_2}(\Phi_{\delta_1,\delta_2})_t\mathcal{J}(U_n) + \int_\Omega \xi_{2,n_2}(T)\mathcal{J}(U_n(T))dx + \omega(l_1,l_2) \\ &= -\int_\Omega \mathcal{J}(u_{0,n})dx + \int_Q (\Phi_{\delta_1,\delta_2})_t\mathcal{J}(U_n) + \int_\Omega \mathcal{J}(U_n(T))dx + \omega(l_1,l_2,n_1,n_2). \end{aligned}$$

Therefore, since $\mathcal{J}(U_n) - J(U_n)U_n = -\overline{J}(U_n)$ and $\overline{J}(u_{0,n}) = J(u_{0,n})u_{0,n} - \mathcal{J}(u_{0,n})$, we obtain

$$F_2 \geq \int_\Omega \overline{J}(u_{0,n})dx \; - \int_Q (\Phi_{\delta_1,\delta_2})_t\overline{J}(U_n) + \int_\Omega \mathcal{J}(U_n(T))dx + \omega(l_1,l_2,n_1,n_2,m). \tag{3.26}$$

Estimate of F_1. Since $m > \ell + 2k$, there holds $T_{\ell+k}(U_n - \langle T_k(U)\rangle_\nu) = T_{\ell+k}\left(\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U))\rangle_\nu\right)$ on $\mathrm{supp}\overline{H_m}(U_n)$. Hence we can write $F_1 = L_1 + L_2$, with

$$L_1 = -\int_Q \Big(\xi_{1,n_1}(1-\Phi_{\delta_1,\delta_2})\big[T_{\ell+k}\left(\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U))\rangle_\nu\right)\big]_{l_1}\Big)_t\left(\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U)\rangle_\nu\right)$$

$$L_2 = -\int_Q \Big(\xi_{1,n_1}(1-\Phi_{\delta_1,\delta_2})\big[T_{\ell+k}\left(\overline{H_m}(U_n) - \langle T_k(\overline{H_m}(U))\rangle_\nu\right)\big]_{l_1}\Big)_t\langle T_k(\overline{H_m}(U))\rangle_\nu.$$

Integrating by parts we have, by definition of the Landes-time approximation,

$$L_2 = \int_Q \xi_{1,n_1}(1-\Phi_{\delta_1,\delta_2})\big[T_{\ell+k}\left(\overline{H_m}(U_n)-\langle T_k(\overline{H_m}(U))\rangle_\nu\right)\big]_{l_1}\left(\langle T_k(\overline{H_m}(U))\rangle_\nu\right)_t$$

$$+\int_\Omega \xi_{1,n_1}(0)\big[T_{\ell+k}\left(\overline{H_m}(U_n)-\langle T_k(\overline{H_m}(U))\rangle_\nu\right)\big]_{l_1}(0)\langle T_k(\overline{H_m}(U))\rangle_\nu(0)dx$$

$$=\nu\int_Q (1-\Phi_{\delta_1,\delta_2})T_{\ell+k}\left(U_n-\langle T_k(U)\rangle_\nu\right)\left(T_k(U)-\langle T_k(U)\rangle_\nu\right)+\int_\Omega T_{\ell+k}\left(u_{0,n}-z_\nu\right)z_\nu dx+\omega(l_1,l_2,n_1,n_2). \tag{3.27}$$

We decompose L_1 into $L_1 = K_1 + K_2 + K_3$, where

$$K_1 = -\int_Q (\xi_{1,n_1})_t(1-\Phi_{\delta_1,\delta_2})\big[T_{\ell+k}\left(\overline{H_m}(U_n)-\langle T_k(\overline{H_m}(U))\rangle_\nu\right)\big]_{l_1}\left(\overline{H_m}(U_n)-\langle T_k(\overline{H_m}(U))\rangle_\nu\right)$$

$$K_2 = \int_Q \xi_{1,n_1}(\Phi_{\delta_1,\delta_2})_t\big[T_{\ell+k}\left(\overline{H_m}(U_n)-\langle T_k(\overline{H_m}(U))\rangle_\nu\right)\big]_{l_1}\left(\overline{H_m}(U_n)-\langle T_k(\overline{H_m}(U))\rangle_\nu\right)$$

$$K_3 = -\int_Q \xi_{1,n_1}(1-\Phi_{\delta_1,\delta_2})\Big(\big[T_{\ell+k}\left(\overline{H_m}(U_n)-\langle T_k(\overline{H_m}(U))\rangle_\nu\right)\big]_{l_1}\Big)_t\left(\overline{H_m}(U_n)-\langle T_k(\overline{H_m}(U)\rangle_\nu\right).$$

Then we check easily that

$$K_1 = \int_\Omega T_{\ell+k}\left(U_n-\langle T_k(U)\rangle_\nu\right)(T)\left(U_n-\langle T_k(U)\rangle_\nu\right)(T)dx+\omega(l_1,l_2,n_1,n_2,m),$$

$$K_2 = \int_Q (\Phi_{\delta_1,\delta_2})_t T_{\ell+k}\left(U_n-\langle T_k(U)\rangle_\nu\right)\left(U_n-\langle T_k(U)\rangle_\nu\right)+\omega(l_1,l_2,n_1,n_2,m).$$

Next consider K_3. Here we use the function $\mathcal{T}_k$ defined at (2.13). We set $b=\overline{H_m}(U_n)-\langle T_k(\overline{H_m}(U))\rangle_\nu$. Hence from (2.12),

$$(([T_{\ell+k}(b)]_{l_1})_t b)(.,t) = \frac{b(.,t)}{l_1}(T_{\ell+k}(b)(.,t+l_1)-T_{\ell+k}(b)(.,t))$$

$$\le \frac{1}{l_1}(\mathcal{T}_{\ell+k}(b)((.,t+l_1))-\mathcal{T}_{\ell+k}(b)(.,t)) = ([\mathcal{T}_{\ell+k}(b)]_{l_1})_t.$$

Thus

$$\left(\big[T_{\ell+k}\left(\overline{H_m}(U_n)-\langle T_k(\overline{H_m}(U))\rangle_\nu\right)\big]_{l_1}\right)_t\left(\overline{H_m}(U_n)-\langle T_k(\overline{H_m}(U))\rangle_\nu\right) \le \left([\mathcal{T}_{\ell+k}(U_n-\langle T_k(U)\rangle_\nu]_{l_1}\right)_t.$$

Then

$$K_3 \ge -\int_Q \xi_{1,n_1}(1-\Phi_{\delta_1,\delta_2})([\mathcal{T}_{\ell+k}\left(U_n-\langle T_k(U)\rangle_\nu\right)]_{l_1})_t$$

$$=\int_Q (\xi_{1,n_1})_t(1-\Phi_{\delta_1,\delta_2})[\mathcal{T}_{\ell+k}\left(U_n-\langle T_k(U)\rangle_\nu\right)]_{l_1}-\int_Q \xi_{1,n_1}(\Phi_{\delta_1,\delta_2})_t[\mathcal{T}_{\ell+k}\left(U_n-\langle T_k(U)\rangle_\nu\right)]_{l_1}$$

$$+\int_\Omega \xi_{1,n_1}(0)[\mathcal{T}_{\ell+k}\left(U_n-\langle T_k(U)\rangle_\nu\right)]_{l_1}(0)dx$$

$$=-\int_\Omega \mathcal{T}_{\ell+k}\left(U_n(T)-\langle T_k(U)\rangle_\nu(T)\right)dx-\int_Q (\Phi_{\delta_1,\delta_2})_t\mathcal{T}_{\ell+k}\left(U_n-\langle T_k(U)\rangle_\nu\right)$$

$$+\int_\Omega \mathcal{T}_{\ell+k}\left(u_{0,n}-z_\nu\right)dx+\omega(l_1,l_2,n_1,n_2).$$

We find by addition, since $T_{\ell+k}(r) - \mathcal{T}_{\ell+k}(r) = \overline{T}_{\ell+k}(r)$ for any $r \in \mathbb{R}$,

$$L_1 \geq \int_\Omega \mathcal{T}_{\ell+k}\left(u_{0,n} - z_\nu\right) dx + \int_\Omega \overline{T}_{\ell+k}\left(U_n(T) - \langle T_k(U)\rangle_\nu(T)\right) dx$$
$$+ \int_Q \left(\Phi_{\delta_1,\delta_2}\right)_t \overline{T}_{\ell+k}\left(U_n - \langle T_k(U)\rangle_\nu\right) + \omega(l_1, l_2, n_1, n_2, m). \tag{3.28}$$

We deduce from (3.28), (3.27), (3.26),

$$A_3 \geq \int_\Omega \overline{J}(u_{0,n}) dx + \int_\Omega \mathcal{T}_{\ell+k}\left(u_{0,n} - z_\nu\right) dx + \int_\Omega T_{\ell+k}\left(u_{0,n} - z_\nu\right) z_\nu dx \tag{3.29}$$
$$+ \int_\Omega \overline{T}_{\ell+k}\left(U_n(T) - \langle T_k(U)\rangle_\nu(T)\right) dx + \int_\Omega \mathcal{J}(U_n(T)) dx + \int_Q \left(\Phi_{\delta_1,\delta_2}\right)_t \left(\overline{T}_{\ell+k}\left(U_n - \langle T_k(U)\rangle_\nu\right) - \overline{J}(U_n)\right)$$
$$+ \nu \int_Q (1 - \Phi_{\delta_1,\delta_2}) T_{\ell+k}\left(U_n - \langle T_k(U)\rangle_\nu\right)\left(T_k(U) - \langle T_k(U)\rangle_\nu\right) + \omega(l_1, l_2, n_1, n_2, m).$$

Next we add (3.25) and (3.29). Note that $\mathcal{J}(U_n(T)) - J(U_n(T))U_n(T) = -\overline{J}(U_n(T))$, and also

$$\mathcal{T}_{\ell+k}\left(u_{0,n} - z_\nu\right) - T_{\ell+k}\left(u_{0,n} - z_\nu\right)\left(z_\nu - u_{0,n}\right) = -\overline{T}_{\ell+k}\left(u_{0,n} - z_\nu\right).$$

Then we find

$$A_1 + A_2 + A_3 \geq \int_\Omega \left(\overline{J}(u_{0,n}) - \overline{T}_{\ell+k}\left(u_{0,n} - z_\nu\right)\right) dx + \int_\Omega \left(\overline{T}_{\ell+k}\left(U_n(T) - \langle T_k(U)\rangle_\nu(T)\right) - \overline{J}(U_n(T))\right) dx$$
$$+ \int_Q \left(\Phi_{\delta_1,\delta_2}\right)_t \left(\overline{T}_{\ell+k}\left(U_n - \langle T_k(U)\rangle_\nu\right) - \overline{J}(U_n)\right)$$
$$+ \nu \int_Q (1 - \Phi_{\delta_1,\delta_2}) T_{\ell+k}\left(U_n - \langle T_k(U)\rangle_\nu\right)\left(T_k(U) - \langle T_k(U)\rangle_\nu\right) + \omega(l_1, l_2, n_1, n_2, m).$$

Notice that $\overline{T}_{\ell+k}\left(r - s\right) - \overline{J}(r) \geq 0$ for any $r, s \in \mathbb{R}$ such that $|s| \leq k$; thus

$$\int_\Omega \left(\overline{T}_{\ell+k}\left(U_n(T) - \langle T_k(U)\rangle_\nu(T)\right) - \overline{J}(U_n(T))\right) dx \geq 0.$$

And $\{u_{0,n}\}$ converges to u_0 in $L^1(\Omega)$ and $\{U_n\}$ converges to U in $L^1(Q)$ from Proposition 2.10. Thus we obtain

$$A_1 + A_2 + A_3 \geq \int_\Omega \left(\overline{J}(u_0) - \overline{T}_{\ell+k}\left(u_0 - z_\nu\right)\right) dx + \int_Q \left(\Phi_{\delta_1,\delta_2}\right)_t \left(\overline{T}_{\ell+k}\left(U - \langle T_k(U)\rangle_\nu\right) - \overline{J}(U)\right)$$
$$+ \nu \int_Q (1 - \Phi_{\delta_1,\delta_2}) T_{\ell+k}\left(U - \langle T_k(U)\rangle_\nu\right)\left(T_k(U) - \langle T_k(U)\rangle_\nu\right) + \omega(l_1, l_2, n_1, n_2, m, n).$$

Moreover $T_{\ell+k}\left(r - s\right)\left(T_k(r) - s\right) \geq 0$ for any $r, s \in \mathbb{R}$ such that $|s| \leq k$, hence

$$A_1 + A_2 + A_3 \geq \int_\Omega \left(\overline{J}(u_0) - \overline{T}_{\ell+k}\left(u_0 - z_\nu\right)\right) dx + \int_Q \left(\Phi_{\delta_1,\delta_2}\right)_t \left(\overline{T}_{\ell+k}\left(U - \langle T_k(U)\rangle_\nu\right) - \overline{J}(U)\right)$$
$$+ \omega(l_1, l_2, n_1, n_2, m, n).$$

As $\nu \to \infty$, $\{z_\nu\}$ converges to $T_k(u_0)$, *a.e.* in Ω, thus we get

$$A_1 + A_2 + A_3 \geq \int_\Omega \left(\overline{J}(u_0) - \overline{T}_{\ell+k}\left(u_0 - T_k(u_0)\right)\right) dx + \int_Q \left(\Phi_{\delta_1,\delta_2}\right)_t \left(\overline{T}_{\ell+k}\left(U - T_k(U)\right) - \overline{J}(U)\right)$$
$$+ \omega(l_1, l_2, n_1, n_2, m, n, \nu).$$

Finally $\left|\overline{T}_{\ell+k}\left(r - T_k(r)\right) - \overline{J}(r)\right| \leq 2k|r|\chi_{\{|r|\geq\ell\}}$ for any $r \in \mathbb{R}$, thus

$$A_1 + A_2 + A_3 \geq \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell).$$

Combining all the estimates, we obtain $I_2 \leq \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2)$, which implies (3.8), since I_2 does not depend on $l_1, l_2, n_1, n_2, m, \ell$. ∎

Next we conclude the proof of Theorem 1.1:

Lemma 3.6 *The function u is a R-solution of (1.1).*

Proof. (i) First show that u satisfies (2.2). Here we proceed as in [22]. Let $\varphi \in X \cap L^\infty(Q)$ such $\varphi_t \in X' + L^1(Q)$, $\varphi(.,T) = 0$, and $S \in W^{2,\infty}(\mathbb{R})$, such that S' has compact support on $\mathbb{R}$, $S(0) = 0$. Let $M > 0$ such that $\mathrm{supp} S' \subset [-M, M]$. Taking successively (φ, S) and $(\varphi\psi_\delta^\pm, S)$ as test functions in (2.2) applied to u_n, we can write

$$A_1 + A_2 + A_3 + A_4 = A_5 + A_6 + A_7, \qquad A_{2,\delta,\pm} + A_{3,\delta,\pm} + A_{4,\delta,\pm} = A_{5,\delta,\pm} + A_{6,\delta,\pm} + A_{7,\delta,\pm},$$

where

$$A_1 = -\int_\Omega \varphi(0)S(u_{0,n})dx, \quad A_2 = -\int_Q \varphi_t S(U_n), \quad A_{2,\delta,\pm} = -\int_Q (\varphi\psi_\delta^\pm)_t S(U_n),$$
$$A_3 = \int_Q S'(U_n)A(x,t,\nabla u_n).\nabla\varphi, \quad A_{3,\delta,\pm} = \int_Q S'(U_n)A(x,t,\nabla u_n).\nabla(\varphi\psi_\delta^\pm),$$

$$A_4 = \int_Q S''(U_n)\varphi A(x,t,\nabla u_n).\nabla U_n, \quad A_{4,\delta,\pm} = \int_Q S''(U_n)\varphi\psi_\delta^\pm A(x,t,\nabla u_n).\nabla U_n,$$
$$A_5 = \int_Q S'(U_n)\varphi d\widehat{\lambda_{n,0}}, \quad A_6 = \int_Q S'(U_n)\varphi d\rho_{n,0}, \quad A_7 = -\int_Q S'(U_n)\varphi d\eta_{n,0},$$
$$A_{5,\delta,\pm} = \int_Q S'(U_n)\varphi\psi_\delta^\pm d\widehat{\lambda_{n,0}}, \quad A_{6,\delta,\pm} = \int_Q S'(U_n)\varphi\psi_\delta^\pm d\rho_{n,0}, \quad A_{7,\delta,\pm} = -\int_Q S'(U_n)\varphi\psi_\delta^\pm d\eta_{n,0}.$$

Since $\{u_{0,n}\}$ converges to u_0 in $L^1(\Omega)$, and $\{S(U_n)\}$ converges to $S(U)$, strongly in X and weak-* in $L^\infty(Q)$, there holds, from (3.2),

$$A_1 = -\int_\Omega \varphi(0)S(u_0)dx + \omega(n), \quad A_2 = -\int_Q \varphi_t S(U) + \omega(n), \quad A_{2,\delta,\psi_\delta^\pm} = \omega(n,\delta).$$

Moreover $T_M(U_n)$ converges to $T_M(U)$, then $T_M(U_n) + h_n$ converges to $T_k(U) + h$ strongly in X, thus

$$A_3 = \int_Q S'(U_n)A(x,t,\nabla\left(T_M\left(U_n\right) + h_n\right)).\nabla\varphi = \int_Q S'(U)A(x,t,\nabla\left(T_M\left(U\right) + h\right)).\nabla\varphi + \omega(n)$$
$$= \int_Q S'(U)A(x,t,\nabla u).\nabla\varphi + \omega(n);$$

and

$$A_4 = \int_Q S''(U_n)\varphi A(x,t,\nabla(T_M(U_n)+h_n)).\nabla T_M(U_n)$$
$$= \int_Q S''(U)\varphi A(x,t,\nabla(T_M(U)+h)).\nabla T_M(U) + \omega(n) = \int_Q S''(U)\varphi A(x,t,\nabla u).\nabla U + \omega(n).$$

In the same way, since $\psi_\delta^\pm$ converges to 0 in X,

$$A_{3,\delta,\pm} = \int_Q S'(U)A(x,t,\nabla u).\nabla(\varphi\psi_\delta^\pm) + \omega(n) = \omega(n,\delta),$$
$$A_{4,\delta,\pm} = \int_Q S''(U)\varphi\psi_\delta^\pm A(x,t,\nabla u).\nabla U + \omega(n) = \omega(n,\delta).$$

And $\{g_n\}$ strongly converges to g in $(L^{p'}(\Omega))^N$, thus

$$A_5 = \int_Q S'(U_n)\varphi f_n + \int_Q S'(U_n)g_n.\nabla\varphi + \int_Q S''(U_n)\varphi g_n.\nabla T_M(U_n)$$
$$= \int_Q S'(U)\varphi f + \int_Q S'(U)g.\nabla\varphi + \int_Q S''(U)\varphi g.\nabla T_M(U) + \omega(n)$$
$$= \int_Q S'(U)\varphi d\widehat{\mu_0} + \omega(n).$$

Now $A_{5,\delta,\pm} = \int_Q S'(U)\varphi\psi_\delta^\pm d\widehat{\lambda_{n,0}} + \omega(n) = \omega(n,\delta)$. Then $A_{6,\delta,\pm} + A_{7,\delta,\pm} = \omega(n,\delta)$. From (3.2) we verify that $A_{7,\delta,+} = \omega(n,\delta)$ and $A_{6,\delta,-} = \omega(n,\delta)$. Moreover, from (3.6) and (3.2), we find

$$|A_6 - A_{6,\delta,+}| \le \int_Q |S'(U_n)\varphi|\,(1-\psi_\delta^+)d\rho_{n,0} \le \|S\|_{W^{2,\infty}(\mathbb{R})}\|\varphi\|_{L^\infty(Q)}\int_Q (1-\psi_\delta^+)d\rho_n = \omega(n,\delta).$$

Similarly we also have $|A_7 - A_{7,\delta,-}| \le \omega(n,\delta)$. Hence $A_6 = \omega(n)$ and $A_7 = \omega(n)$. Therefore, we finally obtain (2.2):

$$-\int_\Omega \varphi(0)S(u_0)dx - \int_Q \varphi_t S(U) + \int_Q S'(U)A(x,t,\nabla u).\nabla\varphi + \int_Q S''(U)\varphi A(x,t,\nabla u).\nabla U = \int_Q S'(U)\varphi d\widehat{\mu_0}. \tag{3.30}$$

(ii) Next, we prove (2.3) and (2.4). We take $\varphi \in C_c^\infty(Q)$ and take $((1-\psi_\delta^-)\varphi, \overline{H_m})$ as test functions in (3.30), with $\overline{H_m}$ as in (2.14). We can write $D_{1,m} + D_{2,m} = D_{3,m} + D_{4,m} + D_{5,m}$, where

$$D_{1,m} = -\int_Q \left((1-\psi_\delta^-)\varphi\right)_t \overline{H_m}(U), \qquad D_{2,m} = \int_Q H_m(U)A(x,t,\nabla u).\nabla\left((1-\psi_\delta^-)\varphi\right),$$
$$D_{3,m} = \int_Q H_m(U)(1-\psi_\delta^-)\varphi d\widehat{\mu_0}, \qquad D_{4,m} = \frac{1}{m}\int_{m\le U\le 2m} (1-\psi_\delta^-)\varphi A(x,t,\nabla u).\nabla U, \tag{3.31}$$
$$D_{5,m} = -\frac{1}{m}\int_{-2m\le U\le -m} (1-\psi_\delta^-)\varphi A(x,t,\nabla u)\nabla U.$$

Taking the same test functions in (2.2) applied to u_n, there holds $D^n_{1,m}+D^n_{2,m}=D^n_{3,m}+D^n_{4,m}+D^n_{5,m}$, where

$$D^n_{1,m}=-\int\limits_Q\left((1-\psi^-_\delta)\varphi\right)_t\overline{H_m}(U_n),\qquad D^n_{2,m}=\int\limits_Q H_m(U_n)A(x,t,\nabla u_n).\nabla\left((1-\psi^-_\delta)\varphi\right),$$

$$D^n_{3,m}=\int\limits_Q H_m(U_n)(1-\psi^-_\delta)\varphi d(\widehat{\lambda_{n,0}}+\rho_{n,0}-\eta_{n,0}),\quad D^n_{4,m}=\frac{1}{m}\int\limits_{m\le U\le 2m}(1-\psi^-_\delta)\varphi A(x,t,\nabla u_n).\nabla U_n,$$

$$D^n_{5,m}=-\frac{1}{m}\int\limits_{-2m\le U_n\le -m}(1-\psi^-_\delta)\varphi A(x,t,\nabla u_n).\nabla U_n \tag{3.32}$$

In (3.32), we go to the limit as $m\to\infty$. Since $\{\overline{H}_m(U_n)\}$ converges to U_n and $\{H_m(U_n)\}$ converges to 1, *a.e.* in Q, and $\{\nabla H_m(U_n)\}$ converges to 0, weakly in $(L^p(Q))^N$, we obtain the relation $D^n_1+D^n_2=D^n_3+D^n$, where

$$D^n_1=-\int_Q\left((1-\psi^-_\delta)\varphi\right)_tU_n,\quad D^n_2=\int_Q A(x,t,\nabla u_n)\nabla\left((1-\psi^-_\delta)\varphi\right),\quad D^n_3=\int_Q(1-\psi^-_\delta)\varphi d\widehat{\lambda_{n,0}}$$

$$\begin{aligned}D^n&=\int_Q(1-\psi^-_\delta)\varphi d(\rho_{n,0}-\eta_{n,0})+\int_Q(1-\psi^-_\delta)\varphi d((\rho_{n,s}-\eta_{n,s})^+-(\rho_{n,s}-\eta_{n,s})^-)\\&=\int_Q(1-\psi^-_\delta)\varphi d(\rho_n-\eta_n).\end{aligned}$$

Clearly, $D_{i,m}-D^n_i=\omega(n,m)$ for $i=1,2,3$. From Lemma (3.3) and (3.2)-(3.4), we obtain $D_{5,m}=\omega(n,m,\delta)$, and

$$\frac{1}{m}\int\limits_{\{m\le U<2m\}}\psi^-_\delta\varphi A(x,t,\nabla u).\nabla U=\omega(n,m,\delta),$$

thus,

$$D_{4,m}=\frac{1}{m}\int\limits_{\{m\le U<2m\}}\varphi A(x,t,\nabla u).\nabla U+\omega(n,m,\delta).$$

Since $\left|\int_Q(1-\psi^-_\delta)\varphi d\eta_n\right|\le\|\varphi\|_{L^\infty}\int_Q(1-\psi^-_\delta)d\eta_n$, it follows that $\int_Q(1-\psi^-_\delta)\varphi d\eta_n=\omega(n,m,\delta)$ from (3.4). And $\left|\int_Q\psi^-_\delta\varphi d\rho_n\right|\le\|\varphi\|_{L^\infty}\int_Q\psi^-_\delta d\rho_n$, thus, from (3.2), $\int_Q(1-\psi^-_\delta)\varphi d\rho_n=\int_Q\varphi d\mu^+_s+\omega(n,m,\delta)$. Then $D^n=\int_Q\varphi d\mu^+_s+\omega(n,m,\delta)$. Therefore by subtraction, we get successively

$$\frac{1}{m}\int\limits_{\{m\le U<2m\}}\varphi A(x,t,\nabla u).\nabla U=\int_Q\varphi d\mu^+_s+\omega(n,m,\delta),$$

$$\lim_{m\to\infty}\frac{1}{m}\int\limits_{\{m\le U<2m\}}\varphi A(x,t,\nabla u).\nabla U=\int_Q\varphi d\mu^+_s, \tag{3.33}$$

which proves (2.3) when $\varphi\in C^\infty_c(Q)$. Next assume only $\varphi\in C^\infty(\overline{Q})$. Then

$$\begin{aligned}&\lim_{m\to\infty}\frac{1}{m}\int\limits_{\{m\le U<2m\}}\varphi A(x,t,\nabla u).\nabla U\\&=\lim_{m\to\infty}\frac{1}{m}\int\limits_{\{m\le U<2m\}}\varphi\psi^+_\delta A(x,t,\nabla u)\nabla U+\lim_{m\to\infty}\frac{1}{m}\int\limits_{\{m\le U<2m\}}\varphi(1-\psi^+_\delta)A(x,t,\nabla u).\nabla U\\&=\int_Q\varphi\psi^+_\delta d\mu^+_s+\lim_{m\to\infty}\frac{1}{m}\int\limits_{\{m\le U<2m\}}\varphi(1-\psi^+_\delta)A(x,t,\nabla u).\nabla U=\int_Q\varphi d\mu^+_s+D,\end{aligned}$$

where

$$D=\int_Q \varphi(1-\psi_\delta^+)d\mu_s^+ + \lim_{n\to\infty}\frac{1}{m}\int_{\{m\leq U<2m\}} \varphi(1-\psi_\delta^+)A(x,t,\nabla u).\nabla U=\omega(\delta).$$

Therefore, (3.33) still holds for $\varphi\in C^\infty(\overline{Q})$, and we deduce (2.3) by density, and similarly, (2.4). This completes the proof of Theorem 1.1. ∎

4 Approximations of measures

Corollary 1.2 is a direct consequence of Theorem 1.1 and the following approximation property:

Proposition 4.1 *Let $\mu=\mu_0+\mu_s\in\mathcal{M}_b^+(Q)$ with $\mu_0\in\mathcal{M}_0^+(Q)$ and $\mu_s\in\mathcal{M}_s^+(Q)$.*
(i) Then, we can find a decomposition $\mu_0=(f,g,h)$ with $f\in L^1(Q), g\in(L^{p'}(Q))^N, h\in X$ such that

$$||f||_{1,Q}+\|g\|_{p',Q}+||h||_X+\mu_s(\Omega)\leq 2\mu(Q) \tag{4.1}$$

(ii) Furthermore, there exists sequences of measures $\mu_{0,n}=(f_n,g_n,h_n),\mu_{s,n}$ such that $f_n,g_n,h_n\in C_c^\infty(Q)$ strongly converge to f,g,h in $L^1(Q),(L^{p'}(Q))^N$ and X respectively, and $\mu_{s,n}\in(C_c^\infty(Q))^+$ converges to μ_s and $\mu_n:=\mu_{0,n}+\mu_{s,n}$ converges to μ in the narrow topology, and satisfying $|\mu_n|(Q)\leq\mu(Q)$,

$$||f_n||_{1,Q}+\|g_n\|_{p',Q}+||h_n||_X+\mu_{s,n}(Q)\leq 2\mu(Q). \tag{4.2}$$

Proof. (i) Step 1. Case where μ has a compact support in Q. By [15], we can find a decomposition $\mu_0=(f,g,h)$ with f,g,h have a compact support in Q. Let $\{\varphi_n\}$ be sequence of mollifiers in $\mathbb{R}^{N+1}$. Then $\mu_{0,n}=\varphi_n*\mu_0\in C_c^\infty(Q)$ for n large enough. We see that $\mu_{0,n}(Q)=\mu_0(Q)$ and $\mu_{0,n}$ admits the decomposition $\mu_{0,n}=(f_n,g_n,h_n)=(\varphi_n*f,\varphi_n*g,\varphi_n*h)$. Since $\{f_n\},\{g_n\},\{h_n\}$ strongly converge to f,g,h in $L^1(Q),(L^{p'}(Q))^N$ and X respectively, we have for n_0 large enough,

$$||f-f_{n_0}||_{1,Q}+||g-g_{n_0}||_{p',Q}+||h-h_{n_0}||_{L^p((0,T);W_0^{1,p}(\Omega))}\leq\frac{1}{2}\mu_0(Q).$$

Then we obtain a decomposition $\mu=(\hat{f},\hat{g},\hat{h})=(\mu_{n_0}+f-f_{n_0},g-g_{n_0},h-h_{n_0})$, such that

$$||\hat{f}||_{1,Q}+||\hat{g}||_{p',Q}+||\hat{h}||_X+\mu_s(Q)\leq\frac{3}{2}\mu(Q) \tag{4.3}$$

Step 2. General case. Let $\{\theta_n\}$ be a nonnegative, nondecreasing sequence in $C_c^\infty(Q)$ which converges to 1, *a.e.* in Q. Set $\tilde{\mu}_0=\theta_0\mu$, and $\tilde{\mu}_n=(\theta_n-\theta_{n-1})\mu$, for any $n\geq1$. Since $\tilde{\mu}_n=\tilde{\mu}_{0,n}+\tilde{\mu}_{s,n}\in\mathcal{M}_0(Q)\cap\mathcal{M}_b^+(Q)$ has compact support with $\tilde{\mu}_{0,n}\in\mathcal{M}_0(Q),\tilde{\mu}_{s,n}\in\mathcal{M}_s(Q)$, by Step 1, we can find a decomposition $\tilde{\mu}_{0,n}=(\tilde{f}_n,\tilde{g}_n,\tilde{h}_n)$ such that

$$||\tilde{f}_n||_{1,Q}+\|\tilde{g}_n\|_{p',Q}+||\tilde{h}_n||_X+\tilde{\mu}_{s,n}(\Omega)\leq\frac{3}{2}\tilde{\mu}_n(Q).$$

Let $\overline{f}_n=\sum_{k=0}^{n}\tilde{f}_k$, $\overline{g}_n=\sum_{k=0}^{n}\tilde{g}_k$, $\bar{h}_n=\sum_{k=0}^{n}\tilde{h}_k$ and $\bar{\mu}_{s,n}=\sum_{k=0}^{n}\tilde{\mu}_{s,k}$. Clearly, $\theta_n\mu_0=(\overline{f}_n,\overline{g}_n,\bar{h}_n)$, $\theta_n\mu_s=\bar{\mu}_{s,n}$ and $\{\overline{f}_n\},\{\overline{g}_n\},\{\bar{h}_n\}$ and $\{\bar{\mu}_{s,n}\}$ converge strongly to some f,g,h, and μ_s respectively in $L^1(Q)$,$(L^{p'}(Q))^N$, X and $\mathcal{M}_b^+(Q)$, and

$$||\overline{f}_n||_{1,Q}+||\overline{g}_n||_{p',Q}+||\bar{h}_n||_X+\bar{\mu}_{s,n}(Q)\leq\frac{3}{2}\mu(Q).$$

Therefore, $\mu_0=(f,g,h)$, and (4.1) holds.

(ii) We take a sequence $\{m_n\}$ in $\mathbb{N}$ such that $f_n = \varphi_{m_n} * \overline{f}_n, g_n = \varphi_{m_n} * \overline{g}_n, h_n = \varphi_{m_n} * \bar{h}_n, \varphi_{m_n} * \bar{\mu}_{s,n} \in (C_c^\infty(Q))^+$, $\int_Q \varphi_{m_n} * \bar{\mu}_{s,n} dxdt = \bar{\mu}_{s,n}(Q)$ and

$$||f_n - \overline{f}_n||_{1,Q} + ||g_n - \overline{g}_n||_{p',Q} + ||h_n - \bar{h}_n||_X \leq \frac{1}{n+2}\mu(Q).$$

Let $\mu_{0,n} = \varphi_{m_n} * (\theta_n \mu_0) = (f_n, g_n, h_n)$, $\mu_{s,n} = \varphi_{m_n} * \bar{\mu}_{s,n}$ and $\mu_n = \mu_{0,n} + \mu_{s,n}$. Therefore, $\{f_n\}, \{g_n\}, \{h_n\}$ strongly converge to f, g, h in $L^1(Q), (L^{p'}(Q))^N$ and X respectively. And (4.2) holds. Furthermore, $\{\mu_{s,n}\}, \{\mu_n\}$ converge to μ_s, μ in the weak topology of measures, and $\mu_{s,n}(Q) = \int_Q \theta_n d\mu_s, \mu_n(Q) = \int_Q \theta_n d\mu$ converges to $\mu_s(Q), \mu(Q)$, thus $\{\mu_{s,n}\}, \{\mu_n\}$ converges to μ_s, μ in the narrow topology and $|\mu_n|(Q) \leq \mu(Q)$. ∎

Observe that part (i) of Proposition 4.1 was used in [22], even if there was no explicit proof. Otherwise part (ii) is a *key point* for finding applications to the stability Theorem. Note also a very useful consequence for approximations by *nondecreasing* sequences:

Proposition 4.2 *Let $\mu \in \mathcal{M}_b^+(Q)$ and $\varepsilon > 0$. Let $\{\mu_n\}$ be a nondecreasing sequence in $\mathcal{M}_b^+(Q)$ converging to μ in $\mathcal{M}_b(Q)$. Then, there exist $f_n, f \in L^1(Q)$, $g_n, g \in (L^{p'}(Q))^N$ and $h_n, h \in X$, $\mu_{n,s}, \mu_s \in \mathcal{M}_s^+(Q)$ such that*

$$\mu = f - \operatorname{div} g + h_t + \mu_s, \qquad \mu_n = f_n - \operatorname{div} g_n + (h_n)_t + \mu_{n,s},$$

and $\{f_n\}, \{g_n\}, \{h_n\}$ strongly converge to f, g, h in $L^1(Q), (L^{p'}(Q))^N$ and X respectively, and $\{\mu_{n,s}\}$ converges to μ_s (strongly) in $\mathcal{M}_b(Q)$ and

$$||f_n||_{1,Q} + ||g_n||_{p',Q} + ||h_n||_X + \mu_{n,s}(\Omega) \leq 2\mu(Q). \tag{4.4}$$

Proof. Since $\{\mu_n\}$ is nondecreasing, then $\{\mu_{n,0}\}$, $\{\mu_{n,s}\}$ are nondecreasing too. Clearly, $\|\mu - \mu_n\|_{\mathcal{M}_b(Q)} = \|\mu_0 - \mu_{n,0}\|_{\mathcal{M}_b(Q)} + \|\mu_s - \mu_{n,s}\|_{\mathcal{M}_b(Q)}$. Hence, $\{\mu_{n,s}\}$ converges to μ_s and $\{\mu_{n,0}\}$ converges to μ_0 (strongly) in $\mathcal{M}_b(Q)$. Set $\widetilde{\mu}_{0,0} = \mu_{0,0}$, and $\widetilde{\mu}_{n,0} = \mu_{n,0} - \mu_{n-1,0}$ for any $n \geq 1$. By Proposition 4.1, (i), we can find $\tilde{f}_n \in L^1(Q)$, $\tilde{g}_n \in (L^{p'}(Q))^N$ and $\tilde{h}_n \in X$ such that $\tilde{\mu}_{n,0} = (\tilde{f}_n, \tilde{g}_n, \tilde{h}_n)$ and

$$||\tilde{f}_n||_{1,Q} + ||\tilde{g}_n||_{p',Q} + ||\tilde{h}_n||_X \leq 2\tilde{\mu}_{n,0}(Q)$$

Let $f_n = \sum_{k=0}^{n} \tilde{f}_k$, $G_n = \sum_{k=0}^{n} \tilde{g}_k$ and $h_n = \sum_{k=0}^{n} \tilde{h}_k$. Clearly, $\mu_{n,0} = (f_n, g_n, h_n)$ and the convergence properties hold with (4.4), since

$$||f_n||_{1,Q} + ||g_n||_{p',Q} + ||h_n||_X \leq 2\mu_0(Q).$$

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