An elliptic semilinear equation with source term and boundary measure data: the supercritical case

Marie-Françoise Bidaut-Véron*
Giang Hoang†
Quoc-Hung Nguyen‡
Laurent Véron§

Laboratoire de Mathématiques et Physique Théorique,
Université François Rabelais, Tours, FRANCE

Abstract

We give new criteria for the existence of weak solutions to equation with source term

$$-\Delta u = u^q \text{ in } \Omega, \quad u = \sigma \text{ on } \partial\Omega$$

where $q > 1$, Ω is a either a bounded smooth domain or $\mathbb{R}^N_+$ and $\sigma \in \mathfrak{M}^+(\partial\Omega)$ is a nonnegative Radon measure on $\partial\Omega$. In particular, one of the criteria is expressed in terms of some Bessel capacities on $\partial\Omega$. We also give a sufficient condition for the existence of weak solutions to equation with source mixed term.

$$-\Delta u = |u|^{q_1-1}u|\nabla u|^{q_2} \text{ in } \Omega, \quad u = \sigma \text{ on } \partial\Omega$$

where $q_1, q_2 \geq 0, q_1 + q_2 > 1, q_2 < 2$, $\sigma \in \mathfrak{M}(\partial\Omega)$ is a Radon measure on $\partial\Omega$.

1 Introduction and main results

Let Ω be a bounded smooth domain in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N_+ := \mathbb{R}^{N-1} \times (0, \infty)$, $N \geq 3$, and $g : \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ be a continuous function. In this paper, we study the solvability problem for

$$\begin{cases} -\Delta u = g(u, \nabla u) & \text{in } \Omega, \\ u = \sigma & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\sigma \in \mathfrak{M}^+(\partial\Omega)$ is a nonnegative Radon measure on $\partial\Omega$. All solutions are understood in the usual very weak sense: $u \in L^1(\Omega)$, $g(u, \nabla u) \in L^1_\rho(\Omega)$, where $\rho(x)$ is the distance from x to $\partial\Omega$ when Ω is bounded, or $u \in L^1(\mathbb{R}^N_+ \cap B)$, $g(u, \nabla u) \in L^1_\rho(\mathbb{R}^N_+ \cap B)$ for any ball B if $\Omega = \mathbb{R}^N_+$, and

$$\int_\Omega u(-\Delta\xi)dx = \int_\Omega g(u, \nabla u)\xi dx - \int_{\partial\Omega} \frac{\partial \xi}{\partial n} d\sigma \tag{1.2}$$

for any $\xi \in C^2(\overline{\Omega}) \cap C_c(\mathbb{R}^N)$ with $\xi = 0$ on $\partial\Omega$. It is well-known that such a solution u satisfies

$$u = \mathbf{G}[g(u, \nabla u)] + \mathbf{P}[\sigma] \quad \text{a. e. in } \Omega.$$

*E-mail address: veronmf@univ-tours.fr
†E-mail address: hgiangbk@gmail.com
‡E-mail address: Hung.Nguyen-Quoc@lmpt.univ-tours.fr
§E-mail address: Laurent.Veron@lmpt.univ-tours.fr

where $\mathbf{G}[.], \mathbf{P}[.]$, respectively the Green and the Poisson potentials of $-\Delta$ in Ω, are defined from the Green and the Poisson kernels by

$$\mathbf{P}[\sigma](y) = \int_{\partial\Omega} P(y,z)d\sigma(z), \quad G[g(u,\nabla u)](y) = \int_{\Omega} \mathbf{G}(y,x)g(u,\nabla u)(x)dx,$$

see [14]. The case $g(u,\nabla u) = |u|^{q-1}u$, $q>1$ has been studied by Bidaut-Véron and Vivier in [3] in case the subcritical case, i.e. $1<q<\frac{N+1}{N-1}$, and Ω is bounded. They proved that problem (1.1) admits a solution provided that $|\sigma|(\Omega)$ is small enough. Furthermore, they also proved that for any $\sigma \in \mathfrak{M}^+(\partial\Omega)$ there holds

$$\mathbf{G}[(\mathbf{P}[\sigma])^q] \leq C\sigma(\partial\Omega)\mathbf{P}[\sigma] \text{ in } \Omega. \tag{1.3}$$

for some constant $C = C(N,q,\Omega) > 0$.

Our main goal is to establish necessary and sufficient conditions for the existence of weak solution to (1.1) with nonnegative boundary measure date, in the supercritical case, together with sharp pointwise estimates of solutions. The absorption case, i.e $g(u,\nabla u) = -|u|^{q-1}u$ has been studied by Gmira and Véron in the subcritical case [7] and by Marcus and Véron in the supercritical case [12, 13, 14]. The case $g(u,\nabla u) = -|\nabla u|^q$ has been studied by Nguyen Phuoc and Véron [15] and extended to the case $g(u,\nabla u) = -|u|^p|\nabla u|^q$ by Marcus and Nguyen Phuoc [10].

We consider first (1.1) with $\Omega = \mathbb{R}^{N-1}\times(0,\infty)$ and as a consequence of our results when $g(u,\nabla u) = |u|^{q-1}u$, $q>1$ (see Theorem 1.2), we prove that if (1.1) has a nonnegative solution u with $\sigma \in \mathfrak{M}_+(\mathbb{R}^{N-1})$, then

$$\sigma(B_r^{'}(y')) \leq Cr^{N-\frac{q+1}{q-1}} \tag{1.4}$$

for any ball $B_r^{'}(y')$ in $\mathbb{R}^{N-1}$ where $C = C(q,N)$ and $q > \frac{N+1}{N-1}$; if $1<q\leq\frac{N+1}{N-1}$, then $\sigma \equiv 0$. Conversely, if $q>\frac{N+1}{N-1}$, $d\sigma = fdz$ for some $f\geq 0$ which satisfies

$$\int_{B_r'(y')} f^{1+\varepsilon}dz \leq r^{N-1-\frac{2(\varepsilon+1)}{q-1}} \tag{1.5}$$

for some $\varepsilon>0$. Then, there exists a constant $C_0 = C_0(N,q)$ such that (1.1) has a nonnegative solution if $C \leq C_0$. The above inequality is an analogue of the classical Fefferman-Phong condition [5]. In particular, (1.5) holds if f belongs to the Marcinkiewicz space $L^{\frac{(N-1)(q-1)}{2},\infty}(\mathbb{R}^{N-1})$.
Moreover, we give sufficient conditions for the existence of weak solutions to (1.1) when $g(u,\nabla u) = |u|^{q_1-1}u|\nabla u|^{q_2}$, $q_1,q_2\geq 0$, $q_1+q_2>1$ and $q_2<2$.

To state our results, let us introduce some notations. Various capacities will be used throughout the paper. Among them are the Reisz and Bessel capacities in $\mathbb{R}^{N-1}$ defined respectively by

$$\text{Cap}_{I_\gamma,s}(O) = \inf\left\{\int_{\mathbb{R}^{N-1}} f^s dy : f\geq 0, I_\gamma * f \geq \chi_O\right\},$$

$$\text{Cap}_{G_\gamma,s}(O) = \inf\left\{\int_{\mathbb{R}^{N-1}} f^s dy : f\geq 0, G_\gamma * f \geq \chi_O\right\},$$

for any Borel set $O\subset\mathbb{R}^{N-1}$, where I_γ, G_γ are the Riesz and the Bessel kernels in $\mathbb{R}^{N-1}$ with order $\gamma\in(0,N-1)$. We remark that

$$\text{Cap}_{G_\gamma,s}(O) \geq \text{Cap}_{I_\gamma,s}(O) \geq C|O|^{1-\frac{\gamma s}{N-1}} \tag{1.6}$$

for any Borel set $O \subset \mathbb{R}^{N-1}$ where $\gamma s < N-1$ and C is a positive constant. When we consider equations in a bounded smooth domain Ω in $\mathbb{R}^N$ we use a specific capacity that we define as follows: there exist open sets $O_1, ..., O_m$ in $\mathbb{R}^N$, diffeomorphisms $T_i : O_i \mapsto B_1(0)$ and compact sets $K_1, ..., K_m$ in $\partial\Omega$ such that

a. $K_i \subset O_i$, $\partial\Omega \subset \bigcup_{i=1}^{m} K_i$

b. $T_i(O_i \cap \partial\Omega) = B_1(0) \cap \{x_N = 0\}$, $T_i(O_i \cap \Omega) = B_1(0) \cap \{x_N > 0\}$.

c. for any $x \in O_i \cap \partial\Omega$, $\exists y \in O_i \cap \partial\Omega$, $d(x, \partial\Omega) = |x - y|$.

Clearly, $\rho(T_i^{-1}(z)) \asymp |z_N|$ for any $z = (z', z_N) \in B_1(0) \cap \{x_N > 0\}$ and $|\mathbf{J}_{T_i}(x)| \asymp 1$ for any $x \in O_i \cap \Omega$, here $\mathbf{J}_{T_i}$ is the Hessian matrix of T_i.

Definition 1.1 *Let $\gamma \in (0, N-1)$, $s > 0$. We denote the $\mathrm{Cap}^{\partial\Omega}_{\gamma,s}$-capacity of a compact set $E \subset \partial\Omega$ by*

$$\mathrm{Cap}^{\partial\Omega}_{\gamma,s}(E) = \sum_{i=1}^{m} \mathrm{Cap}_{G_\gamma,s}(\tilde{T}_i(E \cap K_i)),$$

where $T_i(E \cap K_i) = \tilde{T}_i(E \cap K_i) \times \{x_N = 0\}$.

Notice that, if $\gamma s > N-1$ then there exists $C = C(N, \gamma, s, \Omega) > 0$ such that

$$\mathrm{Cap}^{\partial\Omega}_{\gamma,s}(\{x\}) \geq C \tag{1.7}$$

for all $x \in \partial\Omega$.

Our first two theorems give criteria for the solvability of problem (1.1) in $\mathbb{R}^N_+$.

Theorem 1.2 *Let $q > 1$ and $\sigma \in \mathfrak{M}^+(\mathbb{R}^{N-1})$. Then, the following statements are equivalent*

1. *The inequality*

$$\sigma(K) \leq C\, \mathrm{Cap}_{I_{\frac{2}{q}},q'}(K) \tag{1.8}$$

holds for any compact set $K \subset \mathbb{R}^{N-1}$.

2. *The inequality*

$$\mathbf{G}\left[(\mathbf{P}[\sigma])^q\right] \leq C\mathbf{P}[\sigma] < \infty \quad a.e \text{ in } \quad \mathbb{R}^{N-1} \times (0, \infty) \tag{1.9}$$

holds.

3. *The problem*

$$\begin{aligned} -\Delta u &= u^q && \text{in } \mathbb{R}^{N-1} \times (0,\infty), \\ u(.,0) &= \varepsilon\sigma && \text{in } \mathbb{R}^{N-1}, \end{aligned} \tag{1.10}$$

has a positive solution for some $\varepsilon > 0$.

Moreover, there is a constant $C_0 > 0$ such that if any one of the two statement 1 and 2 holds with $C \leq C_0$, then equation (1.10) admits a solution u with $\varepsilon = 1$ which satisfies

$$u \asymp \mathbf{P}[\sigma]. \tag{1.11}$$

Conversely, if (1.10) has a solution u with $\varepsilon = 1$, then the two statement 1. and 2. hold for some $C > 0$.

Theorem 1.3 *Let $q_1, q_2 \geq 0, q_1 + q_2 > 1, q_2 < 2$ and $\sigma \in \mathfrak{M}(\mathbb{R}^{N-1})$. There exists $\delta > 0$ such that if the inequality*

$$|\sigma|(K) \leq \delta \operatorname{Cap}_{I_{\frac{2-q_2}{q_1+q_2}},(q_1+q_2)'}(K) \tag{1.12}$$

holds for any Borel set $K \subset \mathbb{R}^{N-1}$, then the problem

$$\begin{aligned} -\Delta u &= |u|^{q_1-1}u|\nabla u|^{q_2} && \text{in } \mathbb{R}^{N-1} \times (0, \infty), \\ u &= \sigma && \text{in } \mathbb{R}^{N-1}, \end{aligned} \tag{1.13}$$

has a solution and satisfies

$$|u| \leq C\mathbf{P}[|\sigma|], \quad |\nabla u| \leq C(\rho(.))^{-1}\mathbf{P}[|\sigma|]. \tag{1.14}$$

for some a constant $C > 0$.

In view of (1.6) and assuming $d\sigma = fdz$, we see that if $f \in L^{\frac{(N-1)(q-1)}{2},\infty}(\mathbb{R}^{N-1})$ and $\frac{(N-1)(q-1)}{2} > 1$, then (1.8) holds for some $C > 0$, if $f \in L^{\frac{(N-1)(q_1+q_2-1)}{2-q_2},\infty}(\mathbb{R}^{N-1})$ and $\frac{(N-1)(q_1+q_2-1)}{2-q_2} > 1$ then (1.12) holds for some $C > 0$.

In a bounded smooth domain Ω we obtain existence results analogous to Theorem (1.2) and 1.3 provided the specific capacities on $\partial\Omega$ are used instead of the Riesz capacities.

Theorem 1.4 *Let $q > 1$, Ω be a bounded open set in $\mathbb{R}^N$ with $\partial\Omega \in C^2$ and $\sigma \in \mathfrak{M}^+(\partial\Omega)$. Then, the following statements are equivalent:*

1. *The inequality*

$$\sigma(K) \leq C \operatorname{Cap}^{\partial\Omega}_{\frac{2}{q},q'}(K) \tag{1.15}$$

holds for any Borel set $K \subset \partial\Omega$.

2. *The inequalities*

$$\mathbf{G}\left[(\mathbf{P}[\sigma])^q\right] \leq C\mathbf{P}[\sigma] < \infty \quad a.e \text{ in } \quad \Omega. \tag{1.16}$$

holds.

3. *The problem*

$$\begin{aligned} -\Delta u &= u^q && \text{in } \Omega, \\ u &= \varepsilon\sigma && \text{on } \partial\Omega, \end{aligned} \tag{1.17}$$

admits a positive solution for some $\varepsilon > 0$.

Moreover, there is a constant $C_0 > 0$ such that if any one of the two statement 1. and 2. holds with $C \leq C_0$, then equation 1.17 has a solution u with $\varepsilon = 1$ which satisfies

$$u \asymp \mathbf{P}[\sigma]. \tag{1.18}$$

Conversely, if (1.17) has a solution u with $\varepsilon = 1$, then the two statements 1. and 2. hold for some $C > 0$.

Theorem 1.5 *Let $q_1, q_2 \geq 0, q_1 + q_2 > 1, q_2 < 2$, Ω be a bounded open set in $\mathbb{R}^N$ with $\partial\Omega \in C^2$ and $\sigma \in \mathfrak{M}_b(\partial\Omega)$. There exists $\delta > 0$ such that if*

$$|\sigma|(K) \leq \delta \operatorname{Cap}^{\partial\Omega}_{\frac{2-q_2}{q_1+q_2},(q_1+q_2)'}(K) \tag{1.19}$$

holds for any Borel set $K \subset \partial\Omega$ when Ω is a bounded, then problem

$$\left\{ \begin{array}{l} -\Delta u = |u|^{q_1-1}u|\nabla u|^{q_2} \quad \text{in } \Omega, \\ u = \sigma \quad \text{on } \partial\Omega, \end{array} \right. \tag{1.20}$$

has a solution and satisfies (1.14).

2 Integral equations

Let X be a metric space and $\nu \in \mathfrak{M}^+(X)$. Let $\mathbf{K}$ be a Borel positive kernel function $\mathbf{K} : X \times X \mapsto (0, \infty]$ such that $\mathbf{K}$ is symmetric and satisfies a quasi-metric inequality, i.e there is a constant $C \geq 1$ such that for all x, y we have

$$\frac{1}{\mathbf{K}(x,y)} \leq C \left(\frac{1}{\mathbf{K}(x,z)} + \frac{1}{\mathbf{K}(z,y)} \right).$$

We define the quasi-metric d by

$$d(x,y) = \frac{1}{\mathbf{K}(x,y)},$$

and by $\mathbb{B}_r(x) = \{y \in X \colon d(x,y) < r\}$ the open d-ball of radius $r > 0$ and center x.

For $\omega \in \mathfrak{M}^+(X)$, we define the potentials $\mathbf{K}\omega$ and $\mathbf{K}^\nu f$ by

$$\mathbf{K}\omega(x) = \int_X \mathbf{K}(x,y) d\omega(y), \quad \mathbf{K}^\nu f(x) = \int_X \mathbf{K}(x,y) f(y) d\nu(y),$$

and for $q > 1$, the capacity $\mathrm{Cap}^{\nu}_{\mathbf{K},q'}$ in X by

$$\mathrm{Cap}^{\nu}_{\mathbf{K},q'}(E) = \inf \left\{ \int_X g^{q'} d\nu : g \geq 0, \mathbf{K}^\nu g \geq \chi_E \right\}$$

for any Borel set $E \subset X$. The following result is proved by Kalton and Verbitsky in [9].

Theorem 2.1 *Let $q > 1$ and $\nu, \omega \in \mathfrak{M}^+(X)$ such that*

$$\int_0^{2r} \frac{\nu(\mathbb{B}_s(x))}{s} \frac{ds}{s} \leq C \int_0^r \frac{\nu(\mathbb{B}_s(x))}{s} \frac{ds}{s}, \tag{2.1}$$

$$\sup_{y \in \mathbb{B}_r(x)} \int_0^r \frac{\nu(\mathbb{B}_s(y))}{s} \frac{ds}{s} \leq C \int_0^r \frac{\nu(\mathbb{B}_s(x))}{s} \frac{ds}{s}, \tag{2.2}$$

for any $r > 0, x \in X$, where $C > 0$ is a constant. Then the following statements are equivalent:

1. *The equation $u = \mathbf{K}^\nu u^q + \varepsilon \mathbf{K}\omega$ has a solution for some $\varepsilon > 0$.*

2. *The inequality*

$$\int_E (\mathbf{K}\omega_E)^q d\sigma \leq C_3 \omega(E) \tag{2.3}$$

holds for any Borel set $E \subset X$, $\omega_E = \chi_E \omega$.

3. *For any Borel set $E \subset X$, there holds*

$$\omega(E) \leq C_1 \, \mathrm{Cap}^{\nu}_{\mathbf{K},q'}(E). \tag{2.4}$$

4. *The inequality*

$$\mathbf{K}^\nu \left(\mathbf{K}\omega\right)^q \leq C_4 \mathbf{K}\omega < \infty \quad \nu - a.e. \tag{2.5}$$

holds.

Let Ω be either $\mathbb{R}^{N-1} \times (0, \infty)$ or Ω a bounded domain in $\mathbb{R}^N$ with a C^2 boundary $\partial\Omega$. We set $X = \overline{\Omega}$. For $0 \leq \alpha \leq \beta < N$, we denote

$$\mathbf{N}_{\alpha,\beta}(x,y) = \frac{1}{|x-y|^{N-\beta} \max\{|x-y|, \rho(x), \rho(y)\}^\alpha} \quad \forall (x,y) \in \overline{\Omega} \times \overline{\Omega}. \tag{2.6}$$

Lemma 2.2 $\mathbf{N}_{\alpha,\beta}$ *is symmetric and satisfies the quasi-metric inequality.*

Proof. Clearly, $\mathbf{N}_{\alpha,\beta}$ is symmetric. Now we check the quasi-metric inequality associated to $\mathbf{N}_{\alpha,\beta}$ and $X = \overline{\Omega}$. For any $x, z, y \in \overline{\Omega}$ such that $x \neq y \neq z$, we have

$$\begin{aligned} |x-y|^{N-\beta+\alpha} &\lesssim |x-z|^{N-\beta+\alpha} + |z-y|^{N-\beta+\alpha} \\ &\lesssim \frac{1}{\mathbf{N}_{\alpha,\beta}(x,z)} + \frac{1}{\mathbf{N}_{\alpha,\beta}(z,y)}. \end{aligned}$$

Since $|\rho(x) - \rho(y)| \leq |x-y|$, so

$$\begin{aligned} &|x-y|^{N-\beta}(\rho(x))^{\alpha} + |x-y|^{N-\beta}(\rho(y))^{\alpha} \lesssim |x-y|^{N-\beta}(\min\{\rho(x),\rho(y)\})^{\alpha} + |x-y|^{N-\beta+\alpha} \\ &\quad \lesssim \left(|x-z|^{N-\beta} + |z-y|^{N-\beta}\right)(\min\{\rho(x),\rho(y)\})^{\alpha} + |x-z|^{N-\beta+\alpha} + |z-y|^{N-\beta+\alpha} \\ &\quad = \left((\rho(x))^{\alpha}|x-z|^{N-\beta} + |x-z|^{N-\beta+\alpha}\right) + \left((\rho(y))^{\alpha}|z-y|^{N-\beta} + |z-y|^{N-\beta+\alpha}\right) \\ &\quad \lesssim \frac{1}{\mathbf{N}_{\alpha,\beta}(x,z)} + \frac{1}{\mathbf{N}_{\alpha,\beta}(z,y)}, \end{aligned}$$

Thus,

$$\frac{1}{\mathbf{N}_{\alpha,\beta}(x,y)} \lesssim \frac{1}{\mathbf{N}_{\alpha,\beta}(x,z)} + \frac{1}{\mathbf{N}_{\alpha,\beta}(z,y)}.$$

■

Lemma 2.3 *If* $d\nu(x) = \chi_{\Omega}(\rho(x))^{\alpha_0}dx$ *with* $\alpha_0 \geq 0$, *then* (2.1) *and* (2.2) *hold.*

Proof. It is easy to see that for any $x \in \overline{\Omega}$, $s > 0$

$$B_{2^{-\frac{\alpha+1}{N-\beta}}S}(x) \cap \overline{\Omega} \subset \mathbb{B}_s(x) \subset B_S(x) \cap \overline{\Omega}, \tag{2.7}$$

with $S = \min\{s^{\frac{1}{N-\beta+\alpha}}, s^{\frac{1}{N-\beta}}(\rho(x))^{-\frac{\alpha}{N-\beta}}\}$ and $\mathbb{B}_s(x) = \overline{\Omega}$ when $s > 2^{\frac{\alpha N}{N-\alpha}}(diam(\Omega))^N$.
We show that for any $0 \leq s < 8diam(\Omega)$, $x \in \overline{\Omega}$

$$\nu(B_s(x)) \asymp (\max\{\rho(x), s\})^{\alpha_0}s^N. \tag{2.8}$$

Indeed, take $0 \leq s < 8diam(\Omega)$, $x \in \overline{\Omega}$. There exists $\varepsilon = \varepsilon(\Omega) \in (0,1)$ and $x_s \in \Omega$ such that $B_{\varepsilon s}(x_s) \subset B_s(x) \cap \Omega$ and $d(x_s, \partial\Omega) > \varepsilon s$.
(a) If $0 \leq s \leq \frac{\rho(x)}{4}$, so for any $y \in B_s(x)$, $\rho(y) \asymp \rho(x)$. Thus, $\nu(B_s(x)) \asymp (\rho(x))^{\alpha_0}|B_s(x) \cap \Omega| \asymp (d(x,\partial\Omega))^{\alpha_0}s^N$.
(b) If $s > \frac{\rho(x)}{4}$. Since $\rho(y) \leq \rho(x) + |x-y| < 5s$ for any $y \in B_s(x)$, $\nu(B_s(x)) \lesssim s^{N+\alpha_0}$
(b.1) If $s \leq 4\rho(x)$, we have

$$\nu(B_s(x)) \gtrsim \nu(B_{\frac{\rho(x)}{4}}(x)) \asymp (\rho(x))^{\alpha_0+N} \gtrsim s^{N+\alpha_0}$$

.

(b.2) If $s \geq 4\rho(x)$. We have for any $y \in B_{\varepsilon s/2}(x_s)$, $\rho(y) \geq -|y-x_s| + \rho(x_s) > \varepsilon s/2$. It follows

$$\nu(B_s(x)) \gtrsim \nu(B_{\varepsilon s/2}(x_s)) \gtrsim s^{N+\alpha_0}.$$

Thus, for any $0 \leq s < 2^{\frac{(\alpha+1)(N-\beta+\alpha)}{N-\beta}}(diam(\Omega))^{N-\beta+\alpha}$, $x \in \overline{\Omega}$ we have

$$\begin{aligned} \nu(\mathbb{B}_s(x)) &\asymp (\max\{\rho(x), \min\{s^{\frac{1}{N-\beta+\alpha}}, s^{\frac{1}{N-\beta}}(\rho(x))^{-\frac{\alpha}{N-\beta}}\}\})^{\alpha_0} \\ &\quad \times \left(\min\{s^{\frac{1}{N-\beta+\alpha}}, s^{\frac{1}{N-\beta}}(\rho(x))^{-\frac{\alpha}{N-\beta}}\}\right)^N \\ &\asymp \begin{cases} s^{\frac{\alpha_0+N}{N-\beta+\alpha}} & \text{if } \rho(x) \leq s^{\frac{1}{N-\beta+\alpha}}, \\ (\rho(x))^{\alpha_0 - \frac{\alpha N}{N-\beta}} s^{\frac{N}{N-\beta}} & \text{if } \rho(x) \geq s^{\frac{1}{N-\beta+\alpha}}, \end{cases} \end{aligned}$$

and $\nu(\mathbb{B}_s(x))=\nu(\overline{\Omega})\asymp(diam(\Omega))^{\alpha_0+N}$ if $s>2^{\frac{(\alpha+1)(N-\beta+\alpha)}{N-\beta}}(diam(\Omega))^{N-\beta+\alpha}$. We get,

$$\int_0^r\frac{\nu(\mathbb{B}_s(x))}{s}\frac{ds}{s}\asymp\begin{cases}(diam(\Omega))^{\alpha_0+\beta-\alpha} & \text{if } r>(diam(\Omega))^{N-\beta+\alpha},\\ r^{\frac{\alpha_0+\beta-\alpha}{N-\beta+\alpha}} & \text{if } r\in((\rho(x))^{N-\beta+\alpha},(diam(\Omega))^{N-\beta+\alpha}],\\ (\rho(x))^{\alpha_0-\frac{\alpha N}{N-\beta}}r^{\frac{\beta}{N-\beta}} & \text{if } \ r\in(0,(\rho(x))^{N-\beta+\alpha}].\end{cases}$$

So, (2.1) holds. It remains to check (2.2). For any $x\in\overline{\Omega}$ and $r>0$, clearly, if $r>\frac{1}{2}(\rho(x))^{N-\beta+\alpha}$ we have

$$\sup_{y\in\mathbb{B}_r(x)}\int_0^r\frac{\nu(\mathbb{B}_s(y))}{s}\frac{ds}{s}\lesssim\min\{r^{\frac{\alpha_0+\beta-\alpha}{N-\beta+\alpha}},(diam(\Omega))^{\alpha_0+\beta-\alpha}\},$$

we obtain

$$\sup_{y\in\mathbb{B}_r(x)}\int_0^r\frac{\nu(\mathbb{B}_s(y))}{s}\frac{ds}{s}\lesssim\int_0^r\frac{\nu(\mathbb{B}_s(x))}{s}\frac{ds}{s}.$$

If $0<r\leq\frac{1}{2}(\rho(x))^{N-\beta+\alpha}$, we have $\mathbb{B}_r(x)\subset B_{r^{\frac{1}{N-\beta}}(\rho(x))^{-\frac{\alpha}{N-\beta}}}(x)$ and $\rho(x)\asymp\rho(y)$ for all $y\in B_{r^{\frac{1}{N-\beta}}(\rho(x))^{-\frac{\alpha}{N-\beta}}}(x)$, thus

$$\begin{aligned}\sup_{y\in\mathbb{B}_r(x)}\int_0^r\frac{\nu(\mathbb{B}_s(y))}{s}\frac{ds}{s}&\leq\sup_{|y-x|<r^{\frac{1}{N-\beta}}(\rho(x))^{-\frac{\alpha}{N-\beta}}}\int_0^r\frac{\nu(\mathbb{B}_s(y))}{s}\frac{ds}{s}\\&\asymp\sup_{|y-x|<r^{\frac{1}{N-\beta}}(\rho(x))^{-\frac{\alpha}{N-\beta}}}(\rho(y))^{\alpha_0-\frac{\alpha N}{N-\beta}}r^{\frac{\beta}{N-\beta}}\\&\asymp(\rho(x))^{\alpha_0-\frac{\alpha N}{N-\beta}}r^{\frac{\beta}{N-\beta}}\\&\asymp\int_0^r\frac{\nu(\mathbb{B}_s(x))}{s}\frac{ds}{s}.\end{aligned}$$

Therefore, (2.2) holds. ■

Definition 2.4 *For $\alpha_0\geq0,0\leq\alpha\leq\beta<N$ and $s>1$, we define $\mathrm{Cap}^{\alpha_0}_{\mathbf{N}_{\alpha,\beta},s}$ by*

$$\mathrm{Cap}^{\alpha_0}_{\mathbf{N}_{\alpha,\beta},s}(E)=\inf\left\{\int_{\overline{\Omega}}g^s(\rho(x))^{\alpha_0}dx:g\geq0,\mathbf{N}_{\alpha,\beta}[g(\rho(.))^{\alpha_0}]\geq\chi_E\right\}$$

for any Borel set $E\subset\overline{\Omega}$.

Clearly, we have

$$\mathrm{Cap}^{\alpha_0}_{\mathbf{N}_{\alpha,\beta},s}(E)=\inf\left\{\int_{\overline{\Omega}}g^s(\rho(x))^{-\alpha_0(s-1)}dx:g\geq0,\mathbf{N}_{\alpha,\beta}[g]\geq\chi_E\right\}$$

for any Borel set $E\subset\overline{\Omega}$. Furthermore we have by [1, Theorem 2.5.1],

$$\left(\mathrm{Cap}^{\alpha_0}_{\mathbf{N}_{\alpha,\beta},s}(E)\right)^{1/s}=\sup\left\{\omega(E):\omega\in\mathfrak{M}^+_b(E),||\mathbf{N}_{\alpha,\beta}[\omega]||_{L^{s'}(\Omega,(\rho(.))^{\alpha_0}dx)}\leq1\right\}\qquad(2.9)$$

for any compact set $E\subset\overline{\Omega}$, where s' is the conjugate exponent of s.
Thanks to Lemma 2.2 and 2.3 , we can apply Theorem 2.1 to obtain.

Theorem 2.5 *Let $\omega\in\mathfrak{M}^+(\overline{\Omega})$, $\alpha_0\geq0,0\leq\alpha\leq\beta<N$ and $q>1$. We set*

$$\mathbf{N}_{\alpha,\beta}[\gamma](x)=\int_{\overline{\Omega}}\mathbf{N}_{\alpha,\beta}(x,y)d\gamma(y)$$

and $\mathbf{N}_{\alpha,\beta}[f]:=\mathbf{N}_{\alpha,\beta}[fdx]$ if $f\in L^1_{loc}(\Omega)$, $f\geq0$. Then the following statements are equivalent:

1. *The equation* $u = \mathbf{N}_{\alpha,\beta}[u^q(\rho(.))^{\alpha_0}] + \varepsilon\mathbf{N}_{\alpha,\beta}[\omega]$ *has a solution for some* $\varepsilon > 0$.

2. *The inequality*

$$\int_{E\cap\Omega} (\mathbf{N}_{\alpha,\beta}[\omega_E])^q(\rho(x))^{\alpha_0}dx \leq C_3\omega(E) \tag{2.10}$$

holds for any Borel set $E \subset \overline{\Omega}$, $\omega_E = \omega\chi_E$.

3. *The inequality*

$$\omega(K) \leq C_1 \operatorname{Cap}^{\alpha_0}_{\mathbf{N}_{\alpha,\beta},q'}(K) \tag{2.11}$$

holds for any compact set $K \subset \overline{\Omega}$

4. *The inequality*

$$\mathbf{N}_{\alpha,\beta}\left[(\mathbf{N}_{\alpha,\beta}[\omega])^q\,(\rho(.))^{\alpha_0}\right] \leq C_4\mathbf{N}_{\alpha,\beta}[\omega] < \infty \quad a.e \ in \quad \Omega \tag{2.12}$$

holds.

To apply the previous theorem we need the following result.

Proposition 2.6 *Let* $q > 1$, $\nu, \omega \in \mathfrak{M}^+(X)$. *Suppose that* $A_1, A_2, B_1, B_2 : X\times X \mapsto [0, +\infty)$ *are Borel positive Kernel functions with* $A_1 \asymp A_2, B_1 \asymp B_2$. *Then, the following statements are equivalent:*

1. *The problem* $u = A_1^\nu u^q + \varepsilon B_1\omega \ \ \nu-a.e$ *has a position for some* $\varepsilon > 0$.

2. *The problem* $u = A_2^\nu u^q + \varepsilon B_2\omega \ \nu-a.e$ *has a position for some* $\varepsilon > 0$.

3. *The problem* $u \asymp A_1^\nu u^q + \varepsilon B_1\omega \ \nu-a.e$ *has a position for some* $\varepsilon > 0$.

4. *The problem* $u \gtrsim A_1^\nu u^q + \varepsilon B_1\omega \ \nu-a.e$ *has a position for some* $\varepsilon > 0$.

Proof. We only prove that 4 implies 2. Suppose that there exist $c_1 > 0, \varepsilon_0 > 0$ and a position Borel function u such that

$$A_1^\nu u^q + \varepsilon_0 B_1\omega \leq c_1 u.$$

Taken $c_2 > 0$ with $A_2 \leq c_2A_1, B_2 \leq c_2B$. We consider $u_{n+1} = A_2^\nu u_n^q + \varepsilon_0(c_1c_2)^{-\frac{q}{q-1}}B_2\omega$ and $u_0 = 0$ for any $n \geq 0$. Clearly, $u_n \leq (c_1c_2)^{-\frac{1}{q-1}}u$ for any n and $\{u_n\}$ is nondecreasing. Thus, $U = \lim_{n\to\infty} u_n$ is a solution of $U = A_2^\nu U^q + \varepsilon_0(c_1c_2)^{-\frac{q}{q-1}}B_2\omega$. ■

The following results provide some relations between the capacities $\operatorname{Cap}^{\alpha_0}_{\mathbf{N}_{\alpha,\beta},s}$ and the Riesz capacities on $\mathbb{R}^{N-1}$ which allow to define the capacities on $\partial\Omega$.

Proposition 2.7 *Assume that* $\Omega = \mathbb{R}^{N-1} \times (0, \infty)$ *and let* $\alpha_0 \geq 0$ *such that* $-1 + s'(1 + \alpha - \beta) < \alpha_0 < -1 + s'(N - \beta + \alpha)$. *There holds*

$$\operatorname{Cap}^{\alpha_0}_{\mathbf{N}_{\alpha,\beta},s}(K \times \{0\}) \asymp \operatorname{Cap}_{I_{\beta-\alpha+\frac{\alpha_0+1}{s'}-1},s'}(K) \tag{2.13}$$

for any compact set $K \subset \mathbb{R}^{N-1}$,

Proof. Thanks to [1, Theorem 2.5.1] and (2.9), we get (2.13) from the following estimate: for any $\mu \in \mathfrak{M}^+(\mathbb{R}^{N-1})$

$$||\mathbf{N}_{\alpha,\beta}[\mu \otimes \delta_{\{x_N=0\}}]||_{L^{s'}(\Omega,(\rho(.)))^{\alpha_0}dx)} \asymp ||I_{\beta-\alpha+\frac{\alpha_0+1}{s'}-1}[\mu]||_{L^{s'}(\mathbb{R}^{N-1})} \tag{2.14}$$

where $I_\gamma[\mu]$ is the Riesz potential of μ in $\mathbb{R}^{N-1}$, i.e

$$I_\gamma[\mu](y) = \int_0^\infty \frac{\mu(B'_r(y))}{r^{N-1-\gamma}} \frac{dr}{r} \quad \forall\, y \in \mathbb{R}^{N-1},$$

with $B'_r(y)$ is a ball in $\mathbb{R}^{N-1}$. We have

$$\begin{aligned}
||\mathbf{N}_{\alpha,\beta}[\mu \otimes \delta_{\{x_N=0\}}]||^{s'}_{L^{s'}(\Omega,(\rho(.))^{\alpha_0}dx)} &= \int_{\mathbb{R}^{N-1}} \int_0^\infty \left(\int_{\mathbb{R}^{N-1}} \frac{d\mu(z)}{(|x'-z|^2 + x_N^2)^{\frac{N-\beta+\alpha}{2}}} \right)^{s'} x_N^{\alpha_0} dx_N dx' \\
&\asymp \int_{\mathbb{R}^{N-1}} \int_0^\infty \left(\int_{x_N}^\infty \frac{\mu(B'_r(x'))}{r^{N-\beta+\alpha}} \frac{dr}{r} \right)^{s'} x_N^{\alpha_0} dx_N dx'.
\end{aligned}$$

Notice that

$$\begin{aligned}
\int_0^\infty \left(\int_{x_N}^\infty \frac{\mu(B'_r(x'))}{r^{N-\beta+\alpha}} \frac{dr}{r} \right)^{s'} x_N^{\alpha_0} dx_N &\geq \int_0^\infty \left(\int_{x_N}^{2x_N} \frac{\mu(B'_r(x'))}{r^{N-\beta+\alpha}} \frac{dr}{r} \right)^{s'} x_N^{\alpha_0} dx_N \\
&\gtrsim \int_0^\infty \left(\frac{\mu(B'_{x_N}(x'))}{x_N^{N-\beta+\alpha-\frac{\alpha_0+1}{s'}}} \right)^{s'} \frac{dx_N}{x_N}.
\end{aligned}$$

Thus, using Hölder's inequality, we obtain

$$\begin{aligned}
\int_0^\infty \left(\int_{x_N}^\infty \frac{\mu(B'_r(x'))}{r^{N-\beta+\alpha}} \frac{dr}{r} \right)^{s'} x_N^{\alpha_0} dx_N &\leq \int_0^\infty \left(\int_{x_N}^\infty r^{-\frac{s}{2s'}} \frac{dr}{r} \right)^{\frac{s'}{s}} \int_{x_N}^\infty \left(\frac{\mu(B'_r(x'))}{r^{N-\beta+\alpha-\frac{1}{2s'}}} \right)^{s'} \frac{dr}{r} x_N^{\alpha_0} dx_N \\
&= C \int_0^\infty \int_{x_N}^\infty \left(\frac{\mu(B'_r(x'))}{r^{N-\beta+\alpha-\frac{1}{2s'}}} \right)^{s'} \frac{dr}{r} x_N^{\alpha_0-\frac{1}{2}} dx_N \\
&= C \int_0^\infty \int_0^r x_N^{\alpha_0-\frac{1}{2}} dx_N \left(\frac{\mu(B'_r(x'))}{r^{N-\beta+\alpha-\frac{1}{2s'}}} \right)^{s'} \frac{dr}{r} \\
&= C \int_0^\infty \left(\frac{\mu(B'_r(x'))}{r^{N-\beta+\alpha-\frac{\alpha_0+1}{s'}}} \right)^{s'} \frac{dr}{r}.
\end{aligned}$$

Thus,

$$||\mathbf{N}_{\alpha,\beta}[\mu \otimes \delta_{\{x_N=0\}}]||_{L^{s'}(\Omega,(\rho(.)))^{\alpha_0}dx)} \asymp \left(\int_{\mathbb{R}^{N-1}} \int_0^\infty \left(\frac{\mu(B'_r(y))}{r^{N-\beta+\alpha-\frac{\alpha_0+1}{s'}}} \right)^{s'} \frac{dr}{r} dy \right)^{1/s'}. \tag{2.15}$$

It implies (2.14) from [4, Theorem 2.3]. ■

Proposition 2.8 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain a C^2 boundary. Assume $\alpha_0 \geq 0$ and $-1 + s'(1+\alpha-\beta) < \alpha_0 < -1 + s'(N-\beta+\alpha)$. Then there holds*

$$\mathrm{Cap}^{\alpha_0}_{\mathbf{N}_{\alpha,\beta},s}(E) \asymp \mathrm{Cap}^{\partial\Omega}_{\beta-\alpha+\frac{\alpha_0+1}{s'}-1,s}(E) \tag{2.16}$$

for any compact set $E \subset \partial\Omega \subset \mathbb{R}^N$.

Proof. Let $K_1, ..., K_m$ be as in definition 1.1. We have

$$\mathrm{Cap}^{\alpha_0}_{\mathbf{N}_{\alpha,\beta},s}(E) \asymp \sum_{i=1}^m \mathrm{Cap}^{\alpha_0}_{\mathbf{N}_{\alpha,\beta},s}(E \cap K_i)$$

for any compact set $E \subset \partial\Omega$. By definition 1.1, we need to prove that

$$\mathrm{Cap}^{\alpha_0}_{\mathbf{N}_{\alpha,\beta},s}(E \cap K_i) \asymp \mathrm{Cap}_{G_{\beta-\alpha+\frac{\alpha_0+1}{s'}-1},s}(\tilde{T}_i(E \cap K_i)) \;\; \forall\, i = 1, 2, ..., m. \tag{2.17}$$

We can show that for any $\omega \in \mathfrak{M}^+_b(\partial\Omega)$ and $i = 1, ..., m$, there exists $\omega_i \in \mathfrak{M}^+_b(\tilde{T}_i(K_i))$ with $T_i(K_i) = \tilde{T}_i(K_i) \times \{x_N = 0\}$ such that

$$\omega_i(O) = \omega(T_i^{-1}(O \times \{0\}))$$

for all Borel set $O \subset \tilde{T}_i(K_i)$, its proof can be found in [1, Proof of Lemma 5.2.2]. Thanks to [1, Theorem 2.5.1], it is enough to show that for any $i \in \{1, 2, ..., m\}$ there holds

$$||\mathbf{N}_{\alpha,\beta}[\chi_{K_i}\omega]||_{L^{s'}(\Omega,(\rho(.)))^{\alpha_0}dx)} \asymp ||G_{\beta-\alpha+\frac{\alpha_0+1}{s'}-1}[\omega_i]||_{L^{s'}(\mathbb{R}^{N-1})}, \tag{2.18}$$

where $G_\gamma[\omega_i]$ $(0 < \gamma < N-1)$ is the Bessel potential of ω_i in $\mathbb{R}^{N-1}$, i.e

$$G_\gamma[\omega_i](x) = \int_{\mathbb{R}^{N-1}} G_\gamma(x-y)d\omega_i(y).$$

Indeed, we have

$$\begin{aligned}
||\mathbf{N}_{\alpha,\beta}[\omega\chi_{K_i}]||^{s'}_{L^{s'}(\Omega,(\rho(.)))^{\alpha_0}dx)} &= \int_\Omega \left(\int_{K_i} \frac{d\omega(z)}{|x-z|^{N-\beta+\alpha}}\right)^{s'} (\rho(x))^{\alpha_0}dx \\
= \int_{O_i\cap\Omega} &\left(\int_{K_i} \frac{d\omega(z)}{|x-z|^{N-\beta+\alpha}}\right)^{s'} (\rho(x))^{\alpha_0}dx + \int_{\Omega\setminus O_i} \left(\int_{K_i} \frac{d\omega(z)}{|x-z|^{N-\beta+\alpha}}\right)^{s'} (\rho(x))^{\alpha_0}dx \\
\asymp \int_{O_i\cap\Omega} &\left(\int_{K_i} \frac{d\omega(z)}{|x-z|^{N-\beta+\alpha}}\right)^{s'} (\rho(x))^{\alpha_0}dx + (\omega(K_i))^{s'}.
\end{aligned}$$

Here we used $|x-z| \asymp 1$ for any $x \in \Omega\setminus O_i, z \in K_i$.
We get by using a change of variable

$$\begin{aligned}
&\int_{O_i\cap\Omega} \left(\int_{K_i} \frac{d\omega(z)}{|x-z|^{N-\beta+\alpha}}\right)^{s'} (\rho(x))^{\alpha_0}dx + (\omega(K_i))^{s'} \\
&\quad = \int_{T_i(O_i\cap\Omega)} \left(\int_{K_i} \frac{d\omega(z)}{|T_i^{-1}(y)-z|^{N-\beta+\alpha}}\right)^{s'} (\rho(T_i^{-1}(y)))^{\alpha_0}|\mathbf{J}_{T_i}(T_i^{-1}(y))|^{-1}dy + (\omega(K_i))^{s'} \\
&\quad \asymp \int_{B_1(0)\cap\{x_N>0\}} \left(\int_{K_i} \frac{d\omega(z)}{|y-T_i(z)|^{N-\beta+\alpha}}\right)^{s'} y_N^{\alpha_0}dy + (\omega(K_i))^{s'} \quad \text{with } y = (y', y_N),
\end{aligned}$$

since $|T_i^{-1}(y) - z| \asymp |y - T_i(z)|$, $|\mathbf{J}_{T_i}(T_i^{-1}(y))| \asymp 1$ and $\rho(T_i^{-1}(y)) \asymp y_N$ for all $(y, z) \in T_i(O_i \cap \Omega) \times K_i$. From the definition of ω_i, we have

$$\begin{aligned}
&\int_{B_1(0)\cap\{x_N>0\}} \left(\int_{K_i} \frac{1}{|y-T_i(z)|^{N-\beta+\alpha}}d\omega(z)\right)^{s'} y_n^{\alpha_0}dy + (\omega(K_i))^{s'} \\
&\quad = \int_{B_1(0)\cap\{x_N>0\}} \left(\int_{\tilde{T}_i(K_i)} \frac{1}{(|y'-\xi|^2+y_N^2)^{\frac{N-\beta+\alpha}{2}}}d\omega_i(\xi)\right)^{s'} y_N^{\alpha_0}dy_Ndy' + (\omega(K_i))^{s'} \\
&\quad \asymp \int_{\mathbb{R}^{N-1}} \int_0^\infty \left(\int_{\min\{y_N,R\}}^{2R} \frac{\omega_i(B'_r(y'))}{r^{N-\beta+\alpha}}\frac{dr}{r}\right)^{s'} y_N^{\alpha_0}dy_Ndy' \quad \text{with } R = diam(\Omega).
\end{aligned}$$

As in the proof of Proposition 2.7, we also obtain

$$\int_{\mathbb{R}^{N-1}}\int_0^\infty\left(\int_{\min\{y_N,R\}}^{2R}\frac{\omega_i(B'_r(y'))}{r^{N-\beta+\alpha}}\frac{dr}{r}\right)^{s'}y_N^{\alpha_0}dy_Ndy'$$
$$\asymp\int_{\mathbb{R}^{N-1}}\int_0^{2R}\left(\frac{\omega_i(B'_r(y'))}{r^{N-\beta+\alpha-\frac{\alpha_0+1}{s'}}}\right)^{s'}\frac{dr}{r}dy'.$$

Therefore, we get (2.18) from [4, Theorem 2.3]. ■

To end this section, we give the following property which will be usefull in the sequel.

Proposition 2.9 *Let $0\le\alpha\le\beta<N, s>1$ and $\alpha\ge\frac{\alpha_0}{s'}\ge0$. For any $\mu\in\mathfrak{M}^+(\Omega)$ there holds*

$$\int_K\rho(x)^{s\alpha-(s-1)\alpha_0}d\mu(x)\le\mathrm{Cap}^{\alpha_0}_{\mathbf{N}_{\alpha,\beta},s}(K)\tag{2.19}$$

for any compact set $K\subset\Omega$ provided that

$$\mu(E)\le C\,\mathrm{Cap}_{\mathbf{G}_\beta,s}(E)\tag{2.20}$$

if Ω is a bounded domain, and

$$\mu(E)\le C\,\mathrm{Cap}_{\mathbf{I}_\beta,s}(E)\tag{2.21}$$

if $\Omega=\mathbb{R}^{N-1}\times(0,\infty)$, for any compact set $E\subset\Omega$, here C is a constant. Where $\mathbf{G}_\beta,\mathbf{I}_\beta$ are the Bessel kernel, Riesz kernel of order β in $\mathbb{R}^N$ and $\mathrm{Cap}_{\mathbf{G}_\beta,s},\mathrm{Cap}_{\mathbf{I}_\beta,s}$ are $(\mathbf{G}_\beta,s),(\mathbf{I}_\beta,s)$-capacities, see [1].

Proof. For a compact set $E\subset\Omega$ we have

$$\mathrm{Cap}_{\mathbf{G}_\beta,s}(E)=\sup\left\{(\omega(E))^s:\omega\in\mathfrak{M}_b^+(E),||\mathbf{G}_\beta[\omega]||_{L^{s'}(\mathbb{R}^N)}\le1\right\},$$
$$\mathrm{Cap}_{\mathbf{I}_\beta,s}(E)=\sup\left\{(\omega(E))^s:\omega\in\mathfrak{M}_b^+(E),||\mathbf{I}_\beta[\omega]||_{L^{s'}(\mathbb{R}^N)}\le1\right\}.$$

Thanks to [1, Theorem 3.6.2] or [4, Theorem 2.3], we have for any $\omega\in\mathfrak{M}_b(\Omega)$,

$$||\mathbf{G}_\beta[\omega]||^{s'}_{L^{s'}(\mathbb{R}^N)}\gtrsim\int_\Omega\left(\int_\Omega\frac{d\omega(y)}{|x-y|^{N-\beta}}\right)^{s'}dx,$$

if Ω is a bounded domain, and

$$||\mathbf{I}_\beta[\omega]||^{s'}_{L^{s'}(\mathbb{R}^N)}\gtrsim\int_\Omega\left(\int_\Omega\frac{d\omega(y)}{|x-y|^{N-\beta}}\right)^{s'}dx,$$

if $\Omega=\mathbb{R}^{N-1}\times(0,\infty)$. Thus, taking a compact set $K\subset\Omega$, from (2.20) and (2.21) we get for any $\varepsilon>0$

$$\mu(K\cap\{x:\rho(x)^{s\alpha-(s-1)\alpha_0}\ge\varepsilon\})\lesssim(\omega(K\cap\{x:\rho(x)^{s\alpha-(s-1)\alpha_0}\ge\varepsilon\}))^s,\tag{2.22}$$

for any $\omega\in\mathfrak{M}^+(K\cap\{x:\rho(x)^{s\alpha-(s-1)\alpha_0}\ge\varepsilon\})$ such that

$$\int_\Omega\left(\int_\Omega\frac{d\omega(y)}{|x-y|^{N-\beta}}\right)^{s'}dx\lesssim1.\tag{2.23}$$

From (2.22), and using the Minkowski inequality we get

$$\begin{aligned}\int_K \rho(x)^{s\alpha-(s-1)\alpha_0} d\mu(x) &= \int_K \int_0^\infty \chi_{\rho(x)^{s\alpha-(s-1)\alpha_0}\geq\varepsilon} d\varepsilon d\mu(x) \\ &= \int_0^\infty \mu(K\cap\{x:\rho(x)^{s\alpha-(s-1)\alpha_0}\geq\varepsilon\})d\varepsilon \\ &\lesssim \int_0^\infty (\omega(K\cap\{x:\rho(x)^{s\alpha-(s-1)\alpha_0}\geq\varepsilon\}))^s d\varepsilon \\ &\leq \left(\int_K \rho(x)^{\alpha-\frac{\alpha_0}{s'}} d\omega(x)\right)^s,\end{aligned}$$

from which it follows

$$\int_K \rho(x)^{s\alpha-(s-1)\alpha_0} d\mu(x) \lesssim (\omega_0(K))^s \ \text{ with } \ d\omega_0(x) = \rho(x)^{\alpha-\frac{\alpha_0}{s'}} d\omega(x).$$

Moreover, from (2.23) we have

$$\begin{aligned}1 &\gtrsim \int_\Omega \left(\int_\Omega \frac{1}{|x-y|^{N-\beta}\rho(y)^{\alpha-\frac{\alpha_0}{s'}}} d\omega_0(y)\right)^{s'} dx \\ &\geq \int_\Omega \left(\int_\Omega \frac{1}{|x-y|^{N-\beta}\max\{|x-y|,\rho(x),\rho(y)\}^{\alpha-\frac{\alpha_0}{s'}}} d\omega_0(y)\right)^{s'} dx \\ &\geq \int_\Omega \left(\int_\Omega \frac{\rho(x)^{\frac{\alpha_0}{s'}}}{|x-y|^{N-\beta}\max\{|x-y|,\rho(x),\rho(y)\}^{\alpha}} d\omega_0(y)\right)^{s'} dx \\ &= ||\mathbf{N}_{\alpha,\beta}[\omega_0]||^{s'}_{L^{s'}(\Omega,(\rho(.))^{\alpha_0}dx)}.\end{aligned}$$

Therefore, we get (2.19).

■

3 Proof of the main results

We denote

$$\mathbf{P}[\sigma](x) = \int_{\partial\Omega} \mathbf{P}(x,z)d\sigma(z), \quad \mathbf{G}[\sigma](x) = \int_\Omega \mathbf{G}(x,y)d\mu(y)$$

for any $\sigma\in\mathfrak{M}(\partial\Omega), \mu\in\mathfrak{M}(\Omega)$. Then the unique weak solution of

$$\begin{aligned}-\Delta u &= \mu && \text{in } \Omega \\ u &= \sigma && \text{on } \partial\Omega,\end{aligned}$$

can be represented by

$$u(x) = \mathbf{G}[\sigma](x) + \mathbf{P}[\sigma](x) \ \ \forall\, x\in\Omega. \tag{3.1}$$

We recall below some well-know estimates for the Green and the Poisson kernels.

$$\mathbf{G}(x,y) \asymp \min\left\{\frac{1}{|x-y|^{N-2}}, \frac{\rho(x)\rho(y)}{|x-y|^N}\right\} \tag{3.2}$$

$$\mathbf{P}(x,z) \asymp \frac{\rho(x)}{|x-z|^N} \tag{3.3}$$

and

$$|\nabla_x \mathbf{G}(x,y)| \lesssim \frac{\rho(y)}{|x-y|^N} \min\left\{1, \frac{|x-y|}{\sqrt{\rho(x)\rho(y)}}\right\}, \quad |\nabla_x \mathbf{P}(x,z)| \lesssim \frac{1}{|x-z|^N}$$

for any $(x,y,z) \in \Omega \times \Omega \times \partial\Omega$, see [3]. Since $|\rho(x) - \rho(y)| \leq |x-y|$ we have

$$\max\left\{\rho(x)\rho(y), |x-y|^2\right\} \asymp \max\left\{|x-y|, \rho(x), \rho(y)\right\}^2.$$

Thus,

$$\min\left\{1, \left(\frac{|x-y|}{\sqrt{\rho(x)\rho(y)}}\right)^\gamma\right\} \asymp \frac{|x-y|^\gamma}{\left(\max\left\{|x-y|, \rho(x), \rho(y)\right\}\right)^\gamma} \quad \text{for} \quad \gamma > 0. \tag{3.4}$$

Therefore,

$$\mathbf{G}(x,y) \asymp \rho(x)\rho(y)\mathbf{N}_{2,2}(x,y), \quad \mathbf{P}(x,z) \asymp \rho(x)\mathbf{N}_{\alpha,\alpha}(x,z) \tag{3.5}$$

and

$$|\nabla_x \mathbf{G}(x,y)| \lesssim \rho(y)\mathbf{N}_{1,1}(x,y), \quad |\nabla_x \mathbf{P}(x,z)| \lesssim \mathbf{N}_{\alpha,\alpha}(x,z) \tag{3.6}$$

for all $(x,y,z) \in \overline{\Omega} \times \overline{\Omega} \times \partial\Omega$, $\alpha \geq 0$.

Proof of Theorem 1.2 and Theorem 1.4. By (3.5), we have

$$\begin{aligned}
&\mathbf{G}\left[(\mathbf{P}[\sigma])^q\right] \lesssim \mathbf{P}[\sigma] < \infty \quad a.e \text{ in} \quad \Omega. \\
&\Longleftrightarrow \\
&\mathbf{N}_{2,2}\left[(\mathbf{N}_{2,2}[\sigma])^q (\rho(.))^{q+1}\right] \lesssim \mathbf{N}_{2,2}[\sigma] < \infty \quad a.e \text{ in} \quad \Omega,
\end{aligned}$$

and

$$\begin{aligned}
U \asymp \mathbf{G}[U^q] + \mathbf{P}[\sigma] &\Longleftrightarrow U \asymp \rho(.)\mathbf{N}_{2,2}[\rho(.)U^q] + \rho(.)\mathbf{N}_{2,2}[\sigma] \\
&\Longleftrightarrow V \asymp \mathbf{N}_{2,2}[(\rho(.))^{q+1}V^q] + \mathbf{N}_{2,2}[\sigma], V = U(\rho(.))^{-1}.
\end{aligned}$$

By Proposition 2.7 and 2.8 we have:

$$\text{Cap}_{I_{\frac{2}{q}},q'}(K) \asymp \text{Cap}^{q+1}_{\mathbf{N}_{2,2},q'}(K \times \{0\}) \ \ \forall \ \text{ compact set } \ K \subset \mathbb{R}^{N-1}$$

if $\Omega = \mathbb{R}^{N-1} \times (0,\infty)$, and

$$\text{Cap}^{\partial\Omega}_{\frac{2}{q},q'}(O) \asymp \text{Cap}^{q+1}_{\mathbf{N}_{2,2},q'}(O) \ \ \forall \text{ compact set } \ O \subset \partial\Omega$$

if Ω is a bounded domain. Thanks to Theorem (2.5) with $\alpha = 2, \beta = 2, \alpha_0 = q+1$ and proposition 2.6, we get the results.

■

Proof of Theorem 1.3 and 1.5. Thanks to Theorem 2.5 and Proposition 2.7, 2.8, we have only to prove that there exist $\delta > 0$ such that, if

$$\mathbf{N}_{1,1}\left[(\mathbf{N}_{1,1}[|\sigma|])^{q_1+q_2}(\rho(.))^{q_1+1}\right] \leq \delta\mathbf{N}_{1,1}[|\sigma|] < \infty \quad a.e \text{ in} \quad \Omega, \tag{3.7}$$

then problem (1.20) has a solution u and it satisfies

$$|u| \leq C\rho(.)\mathbf{N}_{1,1}[|\sigma|], \quad |\nabla u| \leq C\mathbf{N}_{1,1}[|\sigma|].$$

By (3.5) and (3.6), we have

$$\mathbf{G}_0(x,y) \leq C\rho(x)\rho(y)\mathbf{N}_{1,1}(x,y), \quad |\nabla_x \mathbf{G}_0(x,y)| \leq C\rho(y)\mathbf{N}_{1,1}(x,y) \tag{3.8}$$
$$\mathbf{P}_0(x,y) \leq C\rho(y)\mathbf{N}_{1,1}(x,z), \quad |\nabla_x \mathbf{P}_0(x,z)| \leq C\mathbf{N}_{1,1}(x,z) \tag{3.9}$$

for all $(x,y,z) \in \Omega \times \Omega \times \partial\Omega$ for some constant $C > 0$.
Suppose that (3.7) holds with $\delta = \frac{(q_1+q_2-1)^{q_1+q_2-1}}{(C(q_1+q_2))^{q_1+q_2}}$. Let $\mathcal{S}$ be the subspace of functions $f \in \mathbf{W}^{1,1}_{loc}(\Omega)$ with norm

$$||f||_{\mathcal{S}} = ||f||_{L^{q_1+q_2}(\Omega,(\rho(.))^{1-q_2}dx)} + |||\nabla f|||_{L^{q_1+q_2}(\Omega,(\rho(.))^{1+q_2}dx)} < \infty.$$

Put

$$\mathbf{E} = \left\{u \in \mathcal{S} : |u| \leq \lambda\rho(.)\mathbf{N}_{1,1}[|\sigma|], \ |\nabla u| \leq \lambda \mathbf{N}_{1,1}[|\sigma|] \quad \text{a.e in } \ \Omega\right\},$$

with $\lambda = \frac{C(q_1+q_2)}{q_1+q_2-1}$.
Clearly, $\mathbf{E}$ is closed under the strong topology of $\mathcal{S}$ and convex.
For any $u \in \mathbf{E}$, set

$$\mathbf{F}(u)(x) = \int_\Omega \mathbf{G}(x,y)|u(y)|^{q_1-1}u(y)|\nabla u(y)|^{q_2}dy + \int_{\partial\Omega} \mathbf{P}(x,z)d\sigma(z).$$

Using (3.8) and (3.9), we have

$$|\mathbf{F}(u)| \leq C\rho(.)\mathbf{N}_{1,1}\left[|u|^{q_1}|\nabla u|^{q_2}\rho(.)\right] + C\rho(.)\mathbf{N}_{1,1}[|\sigma|],$$
$$|\nabla \mathbf{F}(u)| \leq C\mathbf{N}_{1,1}\left[|u|^{q_1}|\nabla u|^{q_2}\rho(.)\right] + C\mathbf{N}_{1,1}[|\sigma|].$$

Since $|u| \leq \lambda\rho(.)\mathbf{N}_{1,1}[|\sigma|]$, $|\nabla u| \leq \lambda\mathbf{N}_{1,1}[|\sigma|]$ and (3.7) holds with $\delta = \frac{(q_1+q_2-1)^{q_1+q_2-1}}{(C(q_1+q_2))^{q_1+q_2}}$, we get

$$|\mathbf{F}(u)| \leq \lambda\rho(.)\mathbf{N}_{1,1}[|\sigma|], \ |\nabla \mathbf{F}(u)| \leq \lambda\mathbf{N}_{1,1}[|\sigma|].$$

Thus, $\mathbf{F}$ is the map from $\mathbf{E}$ to $\mathbf{E}$. It is not difficult to show that $\mathbf{F}$ is continuous and $\mathbf{F}(E)$ is precompact in $\mathcal{S}$. Consequently, by Schauder's fixed point theorem, there exist $u \in \mathbf{E}$ such that $\mathbf{F}(u) = u$. Hence, u is a solution of (1.20) and it satisfies

$$|u| \leq \lambda\rho(.)\mathbf{N}_{1,1}[|\sigma|], \ |\nabla u| \leq \lambda\mathbf{N}_{1,1}[|\sigma].$$

This completes the proof of the Theorems. ■

4 Extension to Schrödinger operators with Hardy potentials

Let $\mathbf{G}_k, \mathbf{P}_k$ be the Green kernel and Poisson kernel of $-\Delta - \frac{\kappa}{(\rho(.))^2}$ in Ω with $\kappa \in [0, \frac{1}{4}]$. We have

$$\mathbf{G}_\kappa(x,y) \asymp \min\left\{\frac{1}{|x-y|^{N-2}}, \frac{(\rho(x)\rho(y))^{\frac{1+\sqrt{1-4\kappa}}{2}}}{|x-y|^{N-1+\sqrt{1-4\kappa}}}\right\}$$
$$\mathbf{P}_\kappa(x,z) \asymp \frac{(\rho(x))^{\frac{1+\sqrt{1-4\kappa}}{2}}}{|x-z|^{N-1+\sqrt{1-4\kappa}}}$$

for all $(x,y,z) \in \overline{\Omega} \times \overline{\Omega} \times \partial\Omega$, see [6, 11, 7]. Therefore, from (3.4) we get

$$\mathbf{G}_\kappa(x,y) \asymp (\rho(x)\rho(y))^{\frac{1+\sqrt{1-4\kappa}}{2}}\mathbf{N}_{1+\sqrt{1-4\kappa},2}(x,y) \tag{4.1}$$

and

$$\mathbf{P}_\kappa(x,z) \asymp (\rho(x))^{\frac{1+\sqrt{1-4\kappa}}{2}} \mathbf{N}_{\alpha,1-\sqrt{1-4\kappa}+\alpha}(x,z), \tag{4.2}$$

for all $(x,y,z) \in \overline{\Omega} \times \overline{\Omega} \times \partial\Omega$, $\alpha \geq 0$. We denote

$$\mathbf{P}_\kappa[\sigma](x) = \int_{\partial\Omega} \mathbf{P}_k(x,z)d\sigma(z), \quad \mathbf{G}_\kappa[\mu](x) = \int_\Omega \mathbf{G}_k(x,y)d\mu(y)$$

for any $\sigma \in \mathfrak{M}(\partial\Omega), \mu \in \mathfrak{M}(\Omega)$.
Thus, as Proof of Theorem 1.2 and Theorem 1.4 we have

$$\mathbf{G}_\kappa\left[(\mathbf{P}_\kappa[\sigma])^q\right] \lesssim \mathbf{P}_\kappa[\sigma] < \infty \quad a.e \text{ in } \quad \Omega.$$
$$\Leftrightarrow \mathbf{N}_{1+\sqrt{1-4\kappa},2}\left[\left(\mathbf{N}_{1+\sqrt{1-4\kappa},2}[\sigma]\right)^q (\rho(.))^{\frac{(q+1)(1+\sqrt{1-4\kappa})}{2}}\right] \lesssim \mathbf{N}_{1+\sqrt{1-4\kappa},2}[\sigma] < \infty \quad a.e \text{ in } \quad \Omega,$$

and

$$U \asymp \mathbf{G}_\kappa[U^q] + \mathbf{P}_\kappa[\sigma]$$
$$\Leftrightarrow \quad V \asymp \mathbf{N}_{1+\sqrt{1-4\kappa},2}[(\rho(.))^{\frac{(q+1)(1+\sqrt{1-4\kappa})}{2}} V^q] + \mathbf{N}_{1+\sqrt{1-4\kappa},2}[\sigma], V = U(\rho(.))^{-\frac{1+\sqrt{1-4\kappa}}{2}}.$$

Thanks to Theorem (2.5) with $d\omega(x) = (\rho(x))^{\frac{1+\sqrt{1-4\kappa}}{2}} d\mu(x) + d\sigma(x) \in \mathfrak{M}^+(\overline{\Omega})$, $\alpha = 1 + \sqrt{1-4\kappa}, \beta = 2, \alpha_0 = \frac{(q+1)(1+\sqrt{1-4\kappa})}{2}$ and proposition 2.6, 2.7, 2.8 where $\sigma \in \mathfrak{M}^+(\partial\Omega)$ and $\mu \in \mathfrak{M}^+(\Omega)$, we obtain

Theorem 4.1 *Let $q > 1, 0 \leq \kappa \leq \frac{1}{4}$ and $\sigma \in \mathfrak{M}^+(\partial\Omega)$ and $\mu \in \mathfrak{M}^+(\Omega)$. Then, the following statements are equivalent*

1. *The inequality*

$$\int_K (\rho(x))^{\frac{1+\sqrt{1-4\kappa}}{2}} d\mu(x) \leq C \operatorname{Cap}^{\frac{(q+1)(1+\sqrt{1-4\kappa})}{2}}_{\mathbf{N}_{1+\sqrt{1-4\kappa},2},q'}(K), \tag{4.3}$$

holds for any compact set $K \subset \Omega$ and

$$\sigma(O) \leq C \operatorname{Cap}_{I_{\frac{q+3-(q-1)\sqrt{1-4\kappa}}{2q}},q'}(O) \tag{4.4}$$

holds for any Borel set $O \subset \mathbb{R}^{N-1}$ if $\Omega = \mathbb{R}^{N-1} \times (0,\infty)$; and

$$\sigma(O) \leq C \operatorname{Cap}^{\partial\Omega}_{\frac{q+3-(q-1)\sqrt{1-4\kappa}}{2q},q'}(O) \tag{4.5}$$

holds for any Borel set $O \subset \partial\Omega$ if Ω is a bounded domain.

2. *The inequalities*

$$\mathbf{G}_\kappa\left[(\mathbf{P}_\kappa[\sigma])^q\right] \leq C\mathbf{P}_\kappa[\sigma] < \infty \quad a.e \text{ in } \quad \Omega, \tag{4.6}$$
$$\mathbf{G}_\kappa\left[(\mathbf{G}_\kappa[\mu])^q\right] \leq C\mathbf{G}_\kappa[\mu] < \infty \quad a.e \text{ in } \quad \Omega, \tag{4.7}$$

holds.

3. *Problem*

$$\begin{cases} -\Delta u - \frac{\kappa}{(\rho(.))^2}u = u^q + \varepsilon\mu & \text{in } \Omega, \\ u = \varepsilon\sigma & \text{on } \partial\Omega, \end{cases} \tag{4.8}$$

has a positive solution for some $\varepsilon > 0$.

Moreover, there is a constant $C_0 > 0$ such that if any one of the two statement 1. and 2. holds with $C \leq C_0$, then equation 4.8 has a solution u with $\varepsilon = 1$ which satisfies

$$u \asymp \mathbf{G}_\kappa[\mu] + \mathbf{P}_\kappa[\sigma]. \tag{4.9}$$

Conversely, if (4.8) *has a solution u with $\varepsilon = 1$, then the two statement 1. and 2. hold for some $C > 0$.*

Remark 4.2 *By Proposition 2.9,* (4.3) *holds if*

$$\mu(E) \le C\, \mathrm{Cap}_{\mathbf{G}_2,q'}(E) \tag{4.10}$$

when Ω *is a bounded domain,*

$$\mu(E) \le C\, \mathrm{Cap}_{\mathbf{I}_2,q'}(E) \tag{4.11}$$

when $\Omega = \mathbb{R}^{N-1} \times (0,\infty)$*, for any compact set* $E \subset \Omega$*, for some small constant* C*.*

We can solve problem (4.8) with distributional data:

$$\begin{cases} -\Delta u - \frac{\kappa}{(\rho(.))^2} u = |u|^{q-1}u + \mathrm{div}(F) + \mu \;\text{ in } \Omega, \\ u = \sigma \quad \text{on } \partial\Omega. \end{cases} \tag{4.12}$$

Theorem 4.3 *Let* $q > 1, 0 \le \kappa \le \frac{1}{4}$ *and* $\sigma \in \mathfrak{M}(\partial\Omega)$ *and* $\mu \in \mathfrak{M}(\Omega)$ *and* $F \in L^1_{loc}(\Omega)$*. Set* $H(x) = \chi_\Omega(x) \left|\int_\Omega \nabla_y \mathbf{G}_\kappa(x,y).F(y)dy\right|$*. There exists* $\delta > 0$ *such that if*

$$|\mu|(K) \le \delta\, \mathrm{Cap}_{\mathbf{I}_2,q'}(K), \quad \int_K H(x)^q dx \le \delta\, \mathrm{Cap}_{\mathbf{I}_2,q'}(K), \tag{4.13}$$

$$|\sigma|(O) \le \delta\, \mathrm{Cap}_{I_{\frac{q+3-(q-1)\sqrt{1-4\kappa}}{2q}},q'}(O) \tag{4.14}$$

hold for any compact sets $K \subset \mathbb{R}^N$*,* $O \subset \mathbb{R}^{N-1}$ *when* $\Omega = \mathbb{R}^{N-1} \times (0,\infty)$*; and*

$$|\mu|(K) \le \delta\, \mathrm{Cap}_{\mathbf{G}_2,q'}(K), \quad \int_K H(x)^q dx \le \delta\, \mathrm{Cap}_{\mathbf{G}_2,q'}(K), \tag{4.15}$$

$$|\sigma|(O) \le \delta\, \mathrm{Cap}^{\partial\Omega}_{\frac{q+3-(q-1)\sqrt{1-4\kappa}}{2q},q'}(O) \tag{4.16}$$

hold for any compact set $K \subset \mathbb{R}^N$*,* $O \subset \partial\Omega$ *when* Ω *is bounded, then problem* (4.12) *has a solution and satisfies*

$$|u| \le \frac{4^{q-1}q}{q-1} \left(\mathbf{G}_\kappa[H^q] + \mathbf{G}_\kappa[|\mu|] + \mathbf{P}_\kappa[|\sigma|]\right) + H. \tag{4.17}$$

Proof of Theorem 4.3. Thanks to Theorem 4.1 and Proposition 4.2, conditions (4.13), (4.14), (4.15) and (4.16)

$$\mathbf{G}_\kappa \left[(\mathbf{P}_\kappa[|\sigma|])^q\right] \le C\mathbf{P}_\kappa[|\sigma|] < \infty \;\; a.e \text{ in } \;\Omega, \tag{4.18}$$

$$\mathbf{G}_\kappa \left[(\mathbf{G}_\kappa[|\mu|])^q\right] \le C\mathbf{G}_\kappa[|\mu|] < \infty \;\; a.e \text{ in } \;\Omega, \tag{4.19}$$

$$\mathbf{G}_\kappa \left[(\mathbf{G}_\kappa[\nu])^q\right] \le C\mathbf{G}_\kappa[\nu] < \infty \;\; a.e \text{ in } \;\Omega. \tag{4.20}$$

where $d\nu(x) = H(x)^q dx$ and $C \approx \delta^{q-1}$.Thus, we need to show that (4.12) has a solution for some small constant C. Consider the sequence $\{u_n\}_{n\ge1} \subset L^q(\Omega)$ of functions defined by $u_1 = 0$ and

$$\begin{cases} -\Delta u_{n+1} - \frac{\kappa}{(\rho(.))^2} u_{n+1} = |u_n|^{q-1}u_n + \mathrm{div}(F) + \mu \;\text{ in } \Omega, \\ u_{n+1} = \sigma \quad \text{on } \partial\Omega, \end{cases} \tag{4.21}$$

for any $n \ge 1$. We have

$$\begin{aligned} u_{n+1}(x) = &\int_\Omega \mathbf{G}_\kappa(x,y)|u_n(y)|^{q-1}u_n(y)dy - \int_\Omega \nabla_y \mathbf{G}_\kappa(x,y).F(y)dy \\ &+ \int_\Omega \mathbf{G}_\kappa(x,y)d\mu(y) + \int_{\partial\Omega} \mathbf{P}_\kappa(x,z)d\sigma(z). \end{aligned}$$

So,

$$|u_{n+1}| \leq \mathbf{G}_\kappa[|u_n|^q] + \mathbf{G}_\kappa[|\mu|] + \mathbf{P}_\kappa[|\sigma|] + H,$$
$$|u_{n+2} - u_{n+1}| \leq q\mathbf{G}_\kappa[\left(|u_{n+1}|^{q-1} + |u_n|^{q-1}\right)|u_{n+1} - u_n|] \quad n \geq 1$$

Assume (4.18), (4.19) and (4.20) hold with $C \leq \left(\frac{q-1}{4^{q-1}q}\right)^q \frac{1}{q-1}$, then

$$|u_n| \leq \frac{4^{q-1}q}{q-1}\left(\mathbf{G}_\kappa[\nu] + \mathbf{G}_\kappa[|\mu|] + \mathbf{P}_\kappa[|\sigma|]\right) + H \quad \forall\ n \geq 1. \tag{4.22}$$

Indeed, clearly, we have (4.22) when $n = 1$. Now assume that (4.22) is true with $n = m$, that is,

$$|u_m| \leq \frac{4^{q-1}q}{q-1}\left(\mathbf{G}_\kappa[\nu] + \mathbf{G}_\kappa[|\mu|] + \mathbf{P}_\kappa[|\sigma|]\right) + H.$$

So,

$$\begin{aligned}
|u_{m+1}| &\leq \mathbf{G}_\kappa[|u_m|^q] + \mathbf{G}_\kappa[|\mu|] + \mathbf{P}_\kappa[|\sigma|] + H \\
&\leq 4^{q-1}\left(\frac{4^{q-1}q}{q-1}\right)^q \left(\mathbf{G}_\kappa[(\mathbf{G}_\kappa[\nu])^q] + \mathbf{G}_\kappa[(\mathbf{G}_\kappa[|\mu|])^q] + \mathbf{G}_\kappa[(\mathbf{P}_\kappa[|\sigma|])^q]\right) \\
&\quad + 4^{q-1}\mathbf{G}_\kappa[H^q] + \mathbf{G}_\kappa[|\mu|] + \mathbf{P}_\kappa[|\sigma|] + H \\
&\leq 4^{q-1}\left(\frac{4^{q-1}q}{q-1}\right)^q C\left(\mathbf{G}_\kappa[\nu] + \mathbf{G}_\kappa[|\mu|] + \mathbf{P}_\kappa[|\sigma|]\right) + 4^{q-1}\mathbf{G}_\kappa[\nu] + \mathbf{G}_\kappa[|\mu|] + \mathbf{P}_\kappa[|\sigma|] + H \\
&\leq \frac{4^{q-1}q}{q-1}\left(\mathbf{G}_\kappa[\nu] + \mathbf{G}_\kappa[|\mu|] + \mathbf{P}_\kappa[|\sigma|]\right) + H.
\end{aligned}$$

So, (4.22) is also true with $n = m + 1$. Thus, (4.22) is true for all $n \geq 1$.
Next, arguing by induction we can show that

$$|u_{n+1} - u_n| \leq C_0 b^{n-2}\left(\mathbf{G}_\kappa[\nu] + \mathbf{G}_\kappa[|\mu|] + \mathbf{P}_\kappa[|\sigma|]\right) \quad n \geq 2 \tag{4.23}$$

with $C_0 = q4^{q-1}\left(\frac{4^{q-1}q}{q-1}\right)^q (C+1)$, $b = q3^{q-1}\left(\frac{4^{q-1}q}{q-1}\right)^{q-1} C$.
Hence, if $C \leq \left(\frac{q-1}{4^{q-1}q}\right)^q \frac{1}{3^{q-1}(q-1)}$ then $b < 1$ and (4.22) holds and u_n converges to $u = u_2 + \sum_{n=2}^\infty (u_{n+1} - u_n)$ a.e. Moreover,

$$\mathbf{G}_\kappa[||u_{m+1}|^{q-1}u_{m+1} - |u_m|^{q-1}u_m|] \to 0 \quad \text{a.e}$$

Clearly, u is solution of (4.12) and satisfies (4.17). This completes the proof of Theorem 4.3.
■

References

[1] D. R. Adams, L.I. Heberg, *Function Spaces and Potential Theory*, Grundlehren der Mathematischen Wisenschaften **31**, Springer-Verlag (1999).

[2] D. R. Adams, *On the existence of capacitary strong type estimates in* $\mathbb{R}^N$, Ark. Mat. **14** (1976), 125-140.

[3] M.F. Bidaut-Véron, L. Vivier, *An elliptic semilinear equation with source term involving boundary measures: the subcritical case* , Rev. Mat. Iberoamericana, **16** (1976), N. 3, 2000.

[4] M. F. Bidaut-Véron, H. Nguyen Quoc, L. Véron, *Quasilinear Lane-Emden equations with absorption and measure data*, J. Math. Pures Appl. **102**, 315-337 (2014).

[5] C. Fefferman, *The uncertainty principle*, Bull. Amer. Math.Soc. **9** (1983),129-206.

[6] S. Filippas, L. Moschini, A. Tertikas, *Sharp two-sided heat kernel estimates for critical Schrödinger operators on bounded domains.* Comm. Math. Phys. **273** (2007), 237-281.

[7] K. T. Gkikas, L. Véron, *Boundary singularities of solutions of semilinear elliptic equations with critical Hardy potentials*, arXiv:1410.1176v2 (2014).

[8] A. Gmira, L. Véron, *Boundary singularities of solutions of some nonlinear elliptic equations*, Duke Math. J. **64** (1991), 271-324.

[9] N.J. Kalton, I.E. Verbitsky, *Nonlinear equations and weighted norm inequality*, Trans. Amer. Math. Soc. **351** (1999) 3441-3497.

[10] M. Marcus, T. Nguyen Phuoc, *Positive solutions of quasilinear elliptic equations with subquadratic growth in the gradient*, arXiv:1311.7519v1 (2013).

[11] M. Marcus, P. T. Nguyen, *Moderate solutions of semilinear elliptic equations with Hardy potential*, arXiv:1407.3572v1 (2014).

[12] M. Marcus, L. Véron,*Removable singularities and boundary trace.* J. Math. Pures Appl. **80** (2001), 879-900.

[13] M. Marcus, L. Véron, *Boundary trace of positive solutions of semilinear elliptic equations in Lipschitz domains: the subcritical case*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **10** (2011), 913-984.

[14] M. Marcus, L. Véron, *Nonlinear second order elliptic equations involving measures.* De Gruyter Series in Nonlinear Analysis and Applications, 21. De Gruyter, Berlin (2014). xiv+248 pp. ISBN: 978-3-11-030515-9; 978-3-11-030531-9 35-0.

[15] T. Nguyen Phuoc, L. Véron, *Boundary singularities of solutions to elliptic viscous HamiltonJacobi equations*, J. Funct. An. **263** (2012) 14871538.

[16] Quoc-Hung Nguyen, *Potential estimates and quasilinear parabolic equations with measure data,* 120 pages, submitted.

[17] L. Véron, *Elliptic Equations Involving Measures*, Handbook of Differential Equations, M. Chipot, P. Quittner, eds. Elsevier: Stationary Partial Differential Equations volume 1, 593-712, 2004.