Wiener criteria for existence of large solutions of nonlinear parabolic equations with absorption in a non-cylindrical domain

Hung Nguyen-Quoc *
Laurent Véron†

Laboratoire de Mathématiques et Physique Théorique,
Université François Rabelais, Tours, FRANCE

Abstract

We obtain a necessary condition and a sufficient condition, both expressed in terms of Wiener type tests involving the parabolic $W^{2,1}_{q'}$- capacity, where $q' = \frac{q}{q-1}$, for the existence of large solutions to equation $\partial_t u - \Delta u + u^q = 0$ in non-cylindrical domain, where $q > 1$. Also, we provide a sufficient condition associated with equation $\partial_t u - \Delta u + e^u - 1 = 0$. Besides, we apply our results to equation: $\partial_t u - \Delta u + a|\nabla u|^p + bu^q = 0$ for $a, b > 0$, $1 < p < 2$ and $q > 1$.

Keywords. Bessel capacities; Hausdorff capacities; parabolic boundary; Riesz potential; maximal solutions.

2010 Mathematics Subject Classification. 35K58, 28A12, 46E35.

1 Introduction

The aim of this paper is to study the problem of existence of large solutions to nonlinear parabolic equations with superlinear absorption in an *arbitrary* bounded open set $O \subset \mathbb{R}^{N+1}$, $N \geq 2$. These are solutions $u \in C^{2,1}(O)$ of equations

$$\begin{aligned} &\partial_t u - \Delta u + |u|^{q-1}u = 0 && \text{in } O, \\ &\lim_{\delta \to 0} \inf_{O \cap Q_\delta(x,t)} u = \infty && \text{for all } (x,t) \in \partial_p O, \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} &\partial_t u - \Delta u + sign(u)(e^{|u|} - 1) = 0 && \text{in } O, \\ &\lim_{\delta \to 0} \inf_{O \cap Q_\delta(x,t)} u = \infty && \text{for all } (x,t) \in \partial_p O, \end{aligned} \tag{1.2}$$

where $q > 1$ and $\partial_p O$ is the parabolic boundary of O, i.e, the set all points $X = (x,t) \in \partial O$ such that the intersection of the cylinder $Q_\delta(x,t) := B_\delta(x) \times (t - \delta^2, t)$ with O^c is not empty for any $\delta > 0$. By the maximal principle for parabolic equations we can assume that all solutions of (1.1) and (1.2) are positive. Henceforth we consider only positive solutions of the preceding equations.
In [22], we studied the existence and the uniqueness of solution of general equations in a cylindrical domain,

$$\begin{aligned} \partial_t u - \Delta u + f(u) = 0 \quad & \text{in } \Omega \times (0,\infty), \\ u = \infty \quad & \text{in } \partial_p (\Omega \times (0,\infty)), \end{aligned} \tag{1.3}$$

*E-mail address: Hung.Nguyen-Quoc@lmpt.univ-tours.fr
†E-mail address: Laurent.Veron@lmpt.univ-tours.fr

where Ω is a bounded open set in $\mathbb{R}^N$ and f is a continuous real-valued function, nondecreasing on $\mathbb{R}$ such that $f(0) \geq 0$ and $f(a) > 0$ for some $a > 0$. In order to obtain the existence of a maximal solution of $\partial_t u - \Delta u + f(u) = 0$ in $\Omega \times (0, \infty)$ there is need to assume

$$\begin{aligned} &(i) \qquad \int_a^\infty \left(\int_0^s f(\tau) d\tau \right)^{-\frac{1}{2}} ds < \infty, \\ &(ii) \qquad \int_a^\infty (f(s))^{-1} ds < \infty. \end{aligned} \tag{1.4}$$

Condition (i), due to Keller and Osserman, is a necessary and sufficient for the existence of a maximal solution to

$$-\Delta u + f(u) = 0 \ \text{ in } \Omega. \tag{1.5}$$

Condition (ii) is a necessary and sufficient for the existence of a maximal solution of the differential equation

$$\varphi' + f(\varphi) = 0 \qquad \text{in } (0, \infty), \tag{1.6}$$

and this solution tends to ∞ at 0. In [22], it is shown that if for any $m \in \mathbb{R}$ there exists $L = L(m) > 0$ such that

$$\text{for any } x, y \geq m \Rightarrow f(x + y) \geq f(x) + f(y) - L,$$

and if (1.5) has a large solution, then (1.3) admits a solution.

It is not alway true that the maximal solution to (1.5) is a large solution. However, if f satisfies

$$\int_1^\infty s^{-2(N-1)/(N-2)} f(s) ds < \infty \quad \text{if } N \geq 3,$$

or

$$\inf \left\{ a \geq 0 : \int_0^\infty f(s) e^{-as} ds < \infty \right\} < \infty \ \text{ if } N = 2,$$

then (1.5) has a large solution for any bounded domain Ω, see [16].

When $f(u) = u^q$, $q > 1$ and $N \geq 3$, the first above condition is satisfied if and only if $q < q_c := \frac{N}{N-2}$, this is called *the sub-critical case*. When $q \geq q_c$, a necessary and sufficient condition for the existence of a large solution to

$$-\Delta u + u^q = 0 \ \text{ in } \Omega; \tag{1.7}$$

is expressed in term of a Wiener-type test,

$$\int_0^1 \frac{\mathrm{Cap}_{2,q'}(\Omega^c \cap B_r(x))}{r^{N-2}} \frac{dr}{r} = \infty \quad \text{for all } \ x \in \partial\Omega. \tag{1.8}$$

In the case $q = 2$ it is obtained by probabilistic methods involving the Brownian snake by Dhersin and Le Gall [5], also see [13, 14]; this method can be extended for $1 < q \leq 2$ by using ideas from [7, 8]. In the general case the result is proved by Labutin, by using purely analytic methods [12]. Here, $q' = \frac{q}{q-1}$ and $\mathrm{Cap}_{2,q'}$ is the capacity associated to the Sobolev space $W^{2,q'}(\mathbb{R}^N)$.

In [19] we obtain sufficient conditions when $f(u) = e^u - 1$, involving the Hausdorff $\mathcal{H}_1^{N-2}-$capacity in $\mathbb{R}^N$, namely,

$$\int_0^1 \frac{\mathcal{H}_1^{N-2}(\Omega^c \cap B_r(x))}{r^{N-2}} \frac{dr}{r} = \infty \quad \text{for all } \ x \in \partial\Omega. \tag{1.9}$$

We refer to [17] for investigation of the initial trace theory of (1.3). In [9], Evans and Gariepy establish a Wiener criterion for the regularity of a boundary point (in the sense of potential theory) for the heat operator $L = \partial_t - \Delta$ in an arbitrary bounded set of $\mathbb{R}^{N+1}$. We denote by $\mathfrak{M}(\mathbb{R}^{N+1})$ the set of Radon measures in $\mathbb{R}^{N+1}$ and, for any compact set $K \subset \mathbb{R}^{N+1}$, by $\mathfrak{M}_K(\mathbb{R}^{N+1})$ the subset of $\mathfrak{M}(\mathbb{R}^{N+1})$ of measures with support in K. Their positive cones are respectively denoted by $\mathfrak{M}^+(\mathbb{R}^{N+1})$ and $\mathfrak{M}^+_K(\mathbb{R}^{N+1})$. The capacity used in this criterion is the thermal capacity defined by

$$\mathrm{Cap}_{\mathbb{H}}(K) = \sup\{\mu(K) : \mu \in \mathfrak{M}^+_K(\mathbb{R}^{N+1}), \mathbb{H} * \mu \leq 1\},$$

for any $K \subset \mathbb{R}^{N+1}$ compact, where $\mathbb{H}$ is the heat kernel in $\mathbb{R}^{N+1}$. It coincides with the parabolic Bessel $\mathcal{G}_1-$capacity $\mathrm{Cap}_{\mathcal{G}_1,2}$,

$$\mathrm{Cap}_{\mathcal{G}_1,2}(K) = \sup\left\{\int_{\mathbb{R}^{N+1}} |f|^2 dxdt : f \in L^2_+(\mathbb{R}^{N+1}),\ \mathcal{G}_1 * f \geq \chi_K\right\},$$

here $\mathcal{G}_1$ is the parabolic Bessel kernel of first order, see [20, Remark 4.12]. Garofalo and Lanconelli [10] extend this result to the parabolic operator $L = \partial_t - \mathrm{div}(A(x,t)\nabla)$, where $A(x,t) = (a_{i,j}(x,t))$, $i,j = 1,2,...,N$ is a real, symmetric, matrix-valued function on $\mathbb{R}^{N+1}$ with C^∞ entries for which there holds

$$C^{-1}|\xi|^2 \leq \sum_{i,j=1}^{N} a_{i,j}(x,t)\xi_i\xi_j \leq C|\xi|^2 \quad \forall (x,t) \in \mathbb{R}^{N+1}, \forall \xi \in \mathbb{R}^N,$$

for some constant $C > 0$.

Less is known concerning the equation

$$\partial_t u - \Delta u + f(u) = 0 \tag{1.10}$$

in a bounded open set $O \subset$ of $\mathbb{R}^{N+1}$, where f is a continuous function in $\mathbb{R}$, Gariepy and Ziemer [11, 23] prove that if there are $(x_0,t_0) \in \partial_p O$, $l \in \mathbb{R}$ and a weak solution $u \in W^{1,2}(O) \cap L^\infty(O)$ of (1.10) such that $\eta(-l-\varepsilon+u)^+, \eta(l-\varepsilon-u)^+ \in W^{1,2}_0(O)$ for any $\varepsilon > 0$ and $\eta \in C_c^\infty(B_r(x_0) \times (-r^2+t_0, r^2+t_0))$ for some $r > 0$ and if

$$\int_0^1 \frac{\mathrm{Cap}_{\mathbb{H}}\left(O^c \cap \left(B_\rho(x_0) \times (t_0 - \frac{9}{4}\alpha\rho^2, t_0 - \frac{5}{4}\alpha\rho^2)\right)\right)}{\rho^N}\frac{d\rho}{\rho} = \infty \text{ for some } \alpha > 0$$

then $\lim_{(x,t)\to(x_0,t_0)} u(x,t) = l$. This result is not easy to use because it is not clear whether (1.10) has a weak solution $u \in W^{1,2}(O)$. In this article we show that (1.10) admits a maximal solution $u \in C^{2,1}(O)$ in an arbitrary bounded open set O, by approximation by dyadic parabolic cubes from inside O, provided that f is as in (1.3) and satisfies (1.4).

Our main purpose of this article is to extend the result of Labutin [12] to nonlinear parabolic equation (1.1). Namely, we give a necessary and a sufficient condition for the existence of solutions to (1.1) in a bounded non-cylindrical domain $O \subset \mathbb{R}^{N+1}$, expressed in terms of a Wiener test based upon the parabolic $W^{2,1}_{q'}$-capacity in $\mathbb{R}^{N+1}$. We also give a sufficient condition associated (1.2) where the parabolic $W^{2,1}_{q'}$-capacity is replaced the parabolic Hausdorff $\mathcal{PH}^N_\rho$-capacity. These capacities are defined as follows: if $K \subset \mathbb{R}^{N+1}$ is compact set, we set

$$\mathrm{Cap}_{2,1,q'}(K) = \inf\{||\varphi||^{q'}_{W^{2,1}_{q'}(\mathbb{R}^{N+1})} : \varphi \in S(\mathbb{R}^{N+1}), \varphi \geq 1 \text{ in a neighborhood of } K\},$$

where

$$||\varphi||_{W^{2,1}_{q'}(\mathbb{R}^{N+1})} = ||\varphi||_{L^{q'}(\mathbb{R}^{N+1})} + ||\frac{\partial\varphi}{\partial t}||_{L^{q'}(\mathbb{R}^{N+1})} + ||\nabla\varphi||_{L^{q'}(\mathbb{R}^{N+1})} + \sum_{i,j}||\frac{\partial^2\varphi}{\partial x_i\partial x_j}||_{L^{q'}(\mathbb{R}^{N+1})}.$$

and for Suslin set $E \subset \mathbb{R}^{N+1}$,

$$\mathrm{Cap}_{2,1,q'}(E) = \sup\{\mathrm{Cap}_{2,1,q'}(D) : D \subset E, D \text{ compact}\}.$$

This capacity has been used in order to obtain potential theory estimates that are most helpful for studying quasilinear parabolic equations (see e.g. [3, 4, 20]). Thanks to a result due to Richard and Bagby [2], the capacities $\mathrm{Cap}_{2,1,p}$ and $\mathrm{Cap}_{\mathcal{G}_2,p}$ are equivalent in the sense that, for any Suslin set $K \subset \mathbb{R}^{N+1}$, there holds

$$C^{-1}\mathrm{Cap}_{2,1,q'}(K) \le \mathrm{Cap}_{\mathcal{G}_2,q'}(K) \le C\mathrm{Cap}_{2,1,p}(K),$$

for some $C = C(N,q)$, where $\mathrm{Cap}_{\mathcal{G}_2,q'}$ is the parabolic Bessel $\mathcal{G}_2-$capacity, see [20].
For $E \subset \mathbb{R}^{N+1}$, we define $\mathcal{PH}^N_\rho(E)$ by

$$\mathcal{PH}^N_\rho(E) = \inf\left\{\sum_j r_j^N : E \subset \bigcup B_{r_j}(x_j) \times (t_j - r_j^2, t_j + r_j^2),\ r_j \le \rho\right\}.$$

It is easy to see that, for $0 < \sigma \le \rho$ and $E \subset \mathbb{R}^{N+1}$, there holds

$$\mathcal{PH}^N_\rho(E) \le \mathcal{PH}^N_\sigma(E) \le C(N)\left(\frac{\rho}{\sigma}\right)^2 \mathcal{PH}^N_\rho(E). \tag{1.11}$$

With these notations, we can state the two main results of this paper.

Theorem 1.1 *Let $N \ge 2$ and $q \ge q_* := \frac{N+2}{N}$. Then*
(i) The equation

$$\partial_t u - \Delta u + u^q = 0 \ \textit{in} \ \ O \tag{1.12}$$

admits a large solution if there holds

$$\sum_{k=1}^{\infty} \frac{Cap_{2,1,q'}\left(O^c \cap \left(B_{r_k}(x) \times \left(t - 1168r_k^2, t - 1136r_k^2\right)\right)\right)}{r_k^N} = \infty, \tag{1.13}$$

for any $(x,t) \in \partial_p O$, where $r_k = 4^{-k}$, and $N \ge 3$ when $q = q_*$.
(ii) If equation (1.12) *admits a large solution, then*

$$\int_0^1 \frac{Cap_{2,1,q'}(O^c \cap Q_\rho(x,t))}{\rho^N}\frac{d\rho}{\rho} = \infty, \tag{1.14}$$

for any $(x,t) \in \partial_p O$, where $Q_\rho(x,t) = B_\rho(x) \times (t - \rho^2, t)$.

Theorem 1.2 *Let $N \ge 2$. The equation*

$$\partial_t u - \Delta u + e^u - 1 = 0 \ \textit{in} \ \ O \tag{1.15}$$

admits a large solution if there holds

$$\sum_{k=1}^{\infty} \frac{\mathcal{PH}_1^N\left(O^c \cap \left(B_{r_k}(x) \times \left(t - 1168r_k^2, t - 1136r_k^2\right)\right)\right)}{r_k^N} = \infty, \tag{1.16}$$

for any $(x,t) \in \partial_p O$, with $r_k = 4^{-k}$.

From properties of the $W^{2,1}_{q'}-$capacity and the $\mathcal{PH}_1^N-$capacity, relation (1.13) holds if

$$\sum_{k=1}^{\infty} r_k^{-N}\left|O^c\cap\left(B_{r_k}(x)\times\left(t-1168r_k^2,t-1136r_k^2\right)\right)\right|^{1-\frac{2q'}{N+2}}=\infty\ \text{ when } q>q_*,$$

and

$$\sum_{k=1}^{\infty} r_k^{-N}\log_+\left(\left|O^c\cap\left(B_{r_k}(x)\times\left(t-1168r_k^2,t-1136r_k^2\right)\right)\right|^{-1}\right)^{-\frac{N}{2}}=\infty\ \text{ when } q=q_*.$$

Similarly, identity (1.16) is verified if

$$\sum_{k=1}^{\infty} r_k^{-N}\left|O^c\cap\left(B_{r_k}(x)\times\left(t-1168r_k^2,t-1136r_k^2\right)\right)\right|^{\frac{N}{N+2}}=\infty.$$

Therefore, when $O=\{(x,t)\in\mathbb{R}^{N+1}:|x|^2+\frac{|t|^2}{\lambda}<1\}$ for some $\lambda>0$, we see that $\partial O=\partial_p O$, (1.14) holds for any $(x,t)\in\partial_p O$, (1.13) and (1.16) hold for any $(x,t)\in\partial_p O\backslash\{(0,\sqrt{\lambda})\}$. However, (1.13) and (1.16) are also true at $(x,t)=(0,\sqrt{\lambda})$ if $\lambda>2272^2$ and not true if $\lambda<2272^2$.

As a consequence of Theorem 1.1 we derive a sufficient condition for the existence of large solution of some viscous Hamilton-Jacobi parabolic equations.

Theorem 1.3 *Let $q_1>1$. If there exists a large solution $v\in C^{2,1}(O)$ of*

$$\partial_t v-\Delta v+v^{q_1}=0\quad \text{ in } O,$$

then, for any $a,b>0$, $1<q<q_1$ and $1<p<\frac{2q_1}{q_1+1}$, problem

$$\begin{array}{rll}\partial_t u-\Delta u+a|\nabla u|^p+bu^q=0 & & \text{in } O,\\ u=\infty & & \text{on } \partial_p O,\end{array}\tag{1.17}$$

admits a solution $u\in C^{2,1}(O)$ which satisfies

$$u(x,t)\geq C\min\left\{a^{-\frac{1}{p-1}}R^{-\frac{2-p}{p-1}+\frac{2}{\alpha(q_1-1)}},b^{-\frac{1}{q-1}}R^{-\frac{2}{q-1}+\frac{2}{\alpha(q_1-1)}}\right\}(v(x,t))^{\frac{1}{\alpha}},$$

for all $(x,t)\in O$ where $R>0$ is such that $O\subset\tilde{Q}_R(x_0,t_0)$, $C=C(N,p,q,q_1)>0$ and $\alpha=\max\left\{\frac{2(p-1)}{(q_1-1)(2-p)},\frac{q-1}{q_1-1}\right\}\in(0,1)$.

2 Preliminaries

Throughout the paper, we denote $Q_\rho(x,t)=B_\rho(x)\times(t-\rho^2,t]$ and $\tilde{Q}_\rho(x,t)=B_\rho(x)\times(t-\rho^2,t+\rho^2)$ for $(x,t)\in\mathbb{R}^{N+1}$, $\rho>0$ and $r_k=4^{-k}$ for all $k\in\mathbb{Z}$. We also denote $A\lesssim(\gtrsim)B$ if $A\leq(\geq)CB$ for some C depending on some structural constants, $A\asymp B$ if $A\lesssim B\lesssim A$.

Definition 2.1 *Let $R\in(0,\infty]$ and $\mu\in\mathfrak{M}^+(\mathbb{R}^{N+1})$. We define $R-$truncated Riesz parabolic potential $\mathbb{I}_2$ of μ by*

$$\mathbb{I}_2^R[\mu](x,t)=\int_0^R\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^N}\frac{d\rho}{\rho}\quad \text{for all }\ (x,t)\in\mathbb{R}^{N+1},$$

and the $R-$truncated fractional maximal parabolic potential $\mathbb{M}_2$ of μ by

$$\mathbb{M}_2^R[\mu](x,t)=\sup_{0<\rho<R}\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^N}\quad \text{for all }\ (x,t)\in\mathbb{R}^{N+1}.$$

We recall two results in [20].

Theorem 2.2 *Let $q > 1, R > 0$ and K be a compact set in $\mathbb{R}^{N+1}$. There exists $\mu := \mu_K \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ with compact support in K such that*

$$\mu(K) \asymp Cap_{2,1,q'}(K) \asymp \int_{\mathbb{R}^{N+1}} \left(\mathbb{I}_2^{2R}[\mu]\right)^q dxdt$$

where the constants of equivalence depend on N, q and R. The measure μ_K is called the capacitary measure of K.

Theorem 2.3 *For any $R > 0$, there exist positive constants C_1, C_2 such that for any $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1})$ such that $||\mathbb{M}_2^R[\mu]||_{L^\infty(\mathbb{R}^{N+1})} \leq 1$, there holds*

$$⨍_Q exp(C_1 \mathbb{I}_2^R[\chi_Q \mu]) dxdt \leq C_2,$$

for all $Q = \tilde{Q}_r(y,s) \subset \mathbb{R}^{N+1}$, $r > 0$, where χ_Q is the indicator function of Q.

Frostman's Lemma in [21, Th. 3.4.27] is at the basis of the dual definition of Hausdorff capacities with doubling weight. It is easy to see that it is valid for the parabolic Hausdorff $\mathcal{PH}_\rho^N-$capacity version. As a consequence we have

Theorem 2.4 *There holds*

$$\sup\left\{\mu(K) : \mu \in \mathfrak{M}^+(\mathbb{R}^{N+1}), supp(\mu) \subset K, ||\mathbb{M}_2^\rho[\mu]||_{L^\infty(\mathbb{R}^{N+1})} \leq 1\right\} \asymp \mathcal{PH}_\rho^N(K)$$

for any compact set $K \subset \mathbb{R}^{N+1}$ and $\rho > 0$, where equivalent constant depends on N

For our purpose, we need the some results about the behavior of the capacity with respect to dilations.

Proposition 2.5 *Let $K \subset \overline{\tilde{Q}_{100}(0,0)}$ be a compact set and $1 < p < \frac{N+2}{2}$. Then*

$$Cap_{2,1,p}(K) \gtrsim |K|^{1-\frac{2p}{N+2}}, \; Cap_{2,1,\frac{N+2}{2}}(K) \gtrsim \left(\log\left(\frac{|\tilde{Q}_{200}(0,0)|}{|K|}\right)\right)^{-\frac{N}{2}}, \tag{2.1}$$

and

$$Cap_{2,1,p}(K_\rho) \asymp \rho^{N+2-2p} Cap_{2,1,p}(K), \tag{2.2}$$

$$\frac{1}{Cap_{2,1,\frac{N+2}{2}}(K_\rho)} \asymp \frac{1}{Cap_{2,1,\frac{N+2}{2}}(K)} + (\log(2/\rho))^{N/2} \tag{2.3}$$

for any $0 < \rho < 1$, where $K_\rho = \{(\rho x, \rho^2 t) : (x,t) \in K\}$.

Proposition 2.6 *Let $K \subset \overline{\tilde{Q}_1(0,0)}$ be a compact set and $1 < p \leq \frac{N+2}{2}$. Then, there exists a function $\varphi \in C_c^\infty(\tilde{Q}_{3/2}(0,0))$, $0 \leq \varphi \leq 1$ and $\varphi|_D = 1$ for some open set $D \supset K$ such that*

$$\int_{\mathbb{R}^{N+1}} \left(|D^2\varphi|^p + |\nabla\varphi|^p + |\varphi|^p + |\partial_t\varphi|^p\right) dxdt \lesssim Cap_{2,1,p}(K). \tag{2.4}$$

We will give proofs of the above two propositions in the Appendix.
It is well know that there exists a semigroup $e^{t\Delta}$ corresponding to equation

$$\begin{array}{rll} \partial_t u - \Delta u = \mu & \text{in} & \tilde{Q}_R(0,0), \\ u = 0 & \text{on} & \partial_p \tilde{Q}_R(0,0), \end{array} \tag{2.5}$$

with $\mu \in C^\infty(\tilde{Q}_R(0,0))$, i.e, we can write a solution u of (2.5) as follows

$$u(x,t) = \int_0^t \left(e^{(t-s)\Delta}\mu\right)(x,s)ds \quad \text{for all} \quad (x,t) \in \tilde{Q}_R(0,0).$$

We denote by $\mathbb{H}$ the heat kernel:

$$\mathbb{H}(x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x|^2}{4t}} \chi_{t>0}.$$

We have

$$|u(x,t)| \le (\mathbb{H} * \mu)(x,t) \quad \text{for all} \quad (x,t) \in \tilde{Q}_R(0,0).$$

In [20, Proof of Proposition 4.8] we show that

$$|(\mathbb{H} * \mu)|(x,t) \le C_1(N)\mathbb{I}_2^{2R}[|\mu|](x,t) \quad \text{for all} \quad (x,t) \in \tilde{Q}_R(0,0).$$

Here μ is extended by 0 in $(\tilde{Q}_R(0,0))^c$. Thus,

$$|\int_0^t \left(e^{(t-s)\Delta}\mu\right)(x,s)ds| \le C_1(N)\mathbb{I}_2^{2R}[|\mu|](x,t) \quad \text{for all} \quad (x,t) \in \tilde{Q}_R(0,0). \tag{2.6}$$

Moreover, we also prove in [20], that if $\mu \ge 0$ then for $(x,t) \in \tilde{Q}_R(0,0)$ and $B_\rho(x) \subset B_R(0)$,

$$\int_0^t \left(e^{(t-s)\Delta}\mu\right)(x,s)ds \ge C_2(N) \sum_{k=0}^{\infty} \frac{\mu(Q_{\frac{\rho_k}{8}}(x, t - \frac{35}{128}\rho_k^2))}{\rho_k^N}, \tag{2.7}$$

with $\rho_k = 4^{-k}\rho$.
It is easy to see that estimates (2.6) and (2.7) also holds for any bounded Radon measure μ in $\tilde{Q}_R(0,0)$. The following result is proved in [3] and [18], and also in [20] in a more general framework.

Theorem 2.7 *Let $q > 1$, $R > 0$ and μ be bounded Radon measure in $\tilde{Q}_R(0,0)$.*
(i) If μ is absolutely continuous with respect to $Cap_{2,1,q'}$ in $\tilde{Q}_R(0,0)$, then there exists a unique weak solution u to equation

$$\begin{aligned} \partial_t u - \Delta u + |u|^{q-1}u = \mu \qquad & \text{in} \ \ \tilde{Q}_R(0,0), \\ u = 0 \qquad & \text{on} \ \ \partial_p\tilde{Q}_R(0,0). \end{aligned}$$

(ii) If $\exp\left(C_1(N)\mathbb{I}_2^{2R}[|\mu|]\right) \in L^1(\tilde{Q}_R(0,0))$ then there exists a unique weak solution v to equation

$$\begin{aligned} \partial_t v - \Delta v + sign(v)(e^{|v|} - 1) = \mu \qquad & \text{in} \ \ \tilde{Q}_R(0,0), \\ v = 0 \qquad & \text{on} \ \ \partial_p\tilde{Q}_R(0,0), \end{aligned}$$

where the constant $C_1(N)$ is the one of inequality (2.6).

From estimates (2.6) and (2.7) and using comparison principle we get the estimates from below of the solutions u and v obtained in Theorem 2.7.

Proposition 2.8 *If $\mu \ge 0$ then the functions u and v of the previous theorem are nonnegative and satisfy*

$$u(x,t) \ge C_2(N) \sum_{k=0}^{\infty} \frac{\mu(Q_{\frac{\rho_k}{8}}(x, t - \frac{35}{128}\rho_k^2))}{\rho_k^N} - C_1(N)^{q+1}\mathbb{I}_2^{2R}\left[\left(\mathbb{I}_2^{2R}[\mu]\right)^q\right](x,t) \tag{2.8}$$

and

$$v(x,t) \ge C_2(N) \sum_{k=0}^{\infty} \frac{\mu(Q_{\frac{\rho_k}{8}}(x, t - \frac{35}{128}\rho_k^2))}{\rho_k^N} - C_1(N)\mathbb{I}_2^{2R}\left[\exp\left(C_1(N)\mathbb{I}_2^{2R}[\mu]\right) - 1\right](x,t). \tag{2.9}$$

for any $(x,t) \in \tilde{Q}_R(0,0)$ and $B_\rho(x) \subset B_R(0)$ and $\rho_k = 4^{-k}\rho$.

3 Maximal solutions

In this section we assume that O is an arbitrary non-cylindrical and bounded open set in $\mathbb{R}^{N+1}$ and $q > 1$. We will prove the existence of a maximal solution of

$$\partial_t u - \Delta u + u^q = 0 \tag{3.1}$$

in O. We also get analogous result where u^q is replaced by $e^u - 1$.
It is easy to see that if u satisfies (3.1) in $\tilde{Q}_r(0,0)$ ($Q_r(0,0)$) then $u_a(x,t) = a^{-2/(q-1)}u(ax, a^2t)$ satisfies (3.1) in $\tilde{Q}_{r/a}(0,0)$ ($Q_{r/a}(0,0)$) for any $a > 0$.
If $X = (x,t) \in O$, the parabolic distance from X to the parabolic boundary $\partial_p O$ of O is defined by

$$d(X, \partial_p O) = \inf_{\substack{(y,s)\in\partial_p O \\ s \le t}} \max\{|x-y|, (t-s)^{\frac{1}{2}}\}.$$

It is easy to see that there exists $C = C(N,q) > 0$ such that the function V defined by

$$V(x,t) = C\left((\rho^2+t)^{-\frac{1}{q-1}} + \left(\frac{\rho^2-|x|^2}{\rho}\right)^{-\frac{2}{q-1}}\right) \quad \text{in } B_\rho(0)\times(-\rho^2,0)$$

satisfies

$$\partial_t V - \Delta V + V^q \ge 0 \quad \text{in} \quad B_\rho(0)\times(-\rho^2,0). \tag{3.2}$$

Proposition 3.1 *There exists a maximal solution $u \in C^{2,1}(O)$ of (3.1) and it satisfies*

$$u(x,t) \le C(d((x,t),\partial_p O))^{-\frac{2}{q-1}} \quad \textit{for all } (x,t)\in O \tag{3.3}$$

for some $C = C(N,q)$.

Proof. Let $\mathcal{D}_k$, $k \in \mathbb{Z}$ be the collection of all the dyadic parabolic cubes (abridged p-cubes) of the form

$$\{(x_1,...,x_N,t) : m_j 2^{-k} \le x_j \le (m_j+1)2^{-k}, j=1,...,N, m_{N+1}4^{-k} \le t \le (m_{N+1}+1)4^{-k}\}$$

where $m_j \in \mathbb{Z}$. The following properties hold,

a. for each integer k, $\mathcal{D}_k$ is a partition of $\mathbb{R}^{N+1}$ and all p-cubes in $\mathcal{D}_k$ have the same sidelengths.

b. if the interiors of two p-cubes Q in $\mathcal{D}_{k_1}$ and P in $\mathcal{D}_{k_2}$, denoted $\overset{\circ}{Q}, \overset{\circ}{P}$, have nonempty intersection then either Q is contained in R or Q contains R.

c. Each Q in $\mathcal{D}_k$ is union of 2^{N+2} p-cubes in $\mathcal{D}_{k+1}$ with disjoint interiors.

Let $k_0 \in \mathbb{N}$ be such that $Q \subset O$ for some $Q \in \mathcal{D}_{k_0}$. Set $O_k = \bigcup_{\substack{Q\in\mathcal{D}_k \\ Q\subset O}} Q \quad \forall k \ge k_0$, we have $O_k \subset O_{k+1}$ and $O = \bigcup_{k\ge k_0} O_k = \bigcup_{k\ge k_0} \overset{\circ}{O}_k$. More precisely, there exist real numbers $a_1, a_2, ..., a_{n(k)}$ and open sets $\Omega_1, \Omega_2, .., \Omega_{n(k)}$ in $\mathbb{R}^N$ such that

$$a_i < a_i + 4^{-k} \le a_{i+1} < a_{i+1} + 4^k \quad \text{for} \quad i = 1,...,n(k)-1$$

and

$$\overset{\circ}{O}_k = \bigcup_{i=1}^{n(k)-1} \left(\Omega_i \times (a_i, a_i+4^{-k}]\right) \bigcup \left(\Omega_{n(k)} \times (a_{n(k)}, a_{n(k)}+4^{-k})\right).$$

For $k \geq k_0$, we claim that there exists a solution $u_k \in C^{2,1}(\overset{\circ}{O}_k)$ to problem

$$\begin{aligned} \partial_t u_k - \Delta u_k + u_k^q = 0 \qquad & \text{in } \overset{\circ}{O}_k, \\ u_k(x,t) \to \infty \qquad & \text{as } d((x,t), \partial_p \overset{\circ}{O}_k) \to 0. \end{aligned} \tag{3.4}$$

Indeed, by [6, 15] for $m > 0$, one can find nonnegative solutions $v_i \in C^{2,1}(\Omega_i \times (a_i, a_i + 4^{-k}]) \cap C(\overline{\Omega}_i \times [a_i, a_i + 4^{-k}])$ for $i = 1, .., n(k)$ to equations

$$\begin{aligned} \partial_t v_1 - \Delta v_1 + v_1^q = 0 \qquad & \text{in } \Omega_1 \times (a_1, a_1 + 4^{-k}), \\ v_1(x,t) = m \qquad & \text{on } \partial\Omega_1 \times (a_1, a_1 + 4^{-k}), \\ v_1(x,t_1) = m \qquad & \text{in } \Omega_1, \end{aligned}$$

and

$$\begin{aligned} \partial_t v_i - \Delta v_i + v_i^q = 0 \qquad & \text{in } \Omega_i \times (a_i, a_i + 4^{-k}), \\ v_i(x,t) = m \qquad & \text{on } \partial\Omega_i \times (a_i, a_i + 4^{-k}), \end{aligned}$$

$$v_i(x, a_i) = \begin{cases} m \text{ in } \Omega_i & \text{if } a_i > a_{i-1} + 4^{-k}, \\ m\chi_{\Omega_i \setminus \Omega_{i-1}}(x) + v_{i-1}(x, a_{i-1} + 4^{-k})\chi_{\Omega_{i-1}}(x) & \text{otherwise .} \end{cases}$$

Clearly,

$$u_{k,m} = v_i \text{ in } \Omega_i \times (a_i, a_i + 4^{-k}] \quad \text{for } i = 1, ..., n(k)$$

is a solution in $C^{2,1}(\overset{\circ}{O}_k) \cap C(O_k)$ to equation

$$\begin{cases} \partial_t u_{k,m} - \Delta u_{k,m} + u_{k,m}^q = 0 \text{ in } \overset{\circ}{O}_k, \\ u_{k,m} = m \text{ on } \partial_p \overset{\circ}{O}_k. \end{cases}$$

Moreover, for $(x,t) \in \overset{\circ}{O}_k$, we see that $B_{\frac{d}{2}}(x) \times (t - \frac{d^2}{4}, t) \subset \overset{\circ}{O}_k$ where $d = d((x,t), \partial_p \overset{\circ}{O}_k)$. From (3.2), we verify that

$$U(y,s) := V(y - x, s - t) = C\left((\rho^2 + s - t)^{-\frac{1}{q-1}} + \left(\frac{\rho^2 - |x-y|^2}{\rho}\right)^{-\frac{2}{q-1}}\right)$$

with $\rho = d/2$, satisfies

$$\partial_t U - \Delta U + U^q \geq 0 \quad \text{in } B_{\frac{d}{2}}(x) \times (t - \frac{d^2}{4}, t). \tag{3.5}$$

Applying the comparison principle we get

$$u_{k,m}(y,s) \leq U(y,s) \text{ in } B_{\frac{d}{2}}(x) \times (t - \frac{d^2}{4}, t],$$

which implies

$$u_{k,m}(x,t) \leq C\left(d((x,t), \partial_p \overset{\circ}{O}_k)\right)^{-\frac{2}{q-1}} \quad \text{for all } (x,t) \in \overset{\circ}{O}_k. \tag{3.6}$$

From this, we also obtain uniform local bounds for $\{u_{k,m}\}_m$. By standard regularity theory see [6, 15], $\{u_{k,m}\}_m$ is uniformly locally bounded in $C^{2,1}$. Hence, up to a subsequence, $u_{k,m} \to u_k$ $C^{1,0}_{\text{loc}}(\overset{\circ}{O}_k)$. Passing the limit, we derive that u_k is a weak solution of (3.4) in $\overset{\circ}{O}_k$, which satisfies $u_k(x,t) \to \infty$ as $d((x,t), \partial_p \overset{\circ}{O}_k) \to 0$ and

$$u_k(x,t) \leq C\left(d((x,t), \partial_p \overset{\circ}{O}_k)\right)^{-\frac{2}{q-1}} \quad \text{for all } (x,t) \in \overset{\circ}{O}_k.$$

Let $m > 0$ and $k \geq k_0$. Since $u_{k+1,m} \leq m$ in O_k and $O_k \subset O_{k+1}$, it follows by the comparison principle applied to $u_{k+1,m}$ and $u_{k,m}$ in the sub-domains $\Omega_1 \times (a_1, a_1 + 4^{-k})$, $\Omega_2 \times (a_2, a_2 + 4^{-k})$,..., $\Omega_{n(k)} \times (a_{n(k)}, a_{n(k)} + 4^{-k})$ of $\overset{\circ}{O}_k$ to obtain at end that $u_{k+1,m} \leq u_{k,m}$ in $\overset{\circ}{O}_k$, and thus $u_{k+1} \leq u_k$ in $\overset{\circ}{O}_k$. In particular, $\{u_k\}_k$ is uniformly locally bounded in L^∞_{loc}. We use the same compactness property as above to obtain that $u_k \to u$ where u is a solution of (3.1) and satisfies (3.3). By construction u is the maximal solution. ∎

Remark 3.2 *Let $R \geq 2r \geq 2$, K be a compact subset in $\overline{\tilde{Q}_r(0,0)}$. Arguing as one can easily it is clear that there exists a maximal solution of*

$$\begin{aligned} \partial_t u - \Delta u + u^q = 0 \qquad & \text{in } \ \tilde{Q}_R(0,0)\backslash K, \\ u = 0 \qquad & \text{on } \ \partial_p \tilde{Q}_R(0,0), \end{aligned} \tag{3.7}$$

which satisfies

$$u(x,t) \leq C(d((x,t), \partial_p(\tilde{Q}_R(0,0)\backslash K))^{-\frac{2}{q-1}} \quad \forall \ (x,t) \in \tilde{Q}_R(0,0)\backslash K, \tag{3.8}$$

for some $C = C(N,q)$. Furthermore, assume $K_1, K_2,,,, K_m$ are compact subsets in $\overline{\tilde{Q}_r(0,0)}$ and $K = K_1 \cup ... \cup K_m$. Let $u, u_1, ..., u_m$ be the maximal solutions of (3.7) in $\tilde{Q}_R(0,0)\backslash K$, $\tilde{Q}_R(0,0)\backslash K_1$, $\tilde{Q}_R(0,0)\backslash K_2,,,, \tilde{Q}_R(0,0)\backslash K_m$, respectively, then

$$u \leq \sum_{j=1}^{m} u_j \quad \text{in} \ \ \tilde{Q}_R(0,0)\backslash K. \tag{3.9}$$

Remark 3.3 *If the equation (3.1) admits a large solution for some $q > 1$ then for any $1 < q_1 < q$, equation*

$$\partial_t u - \Delta u + u^{q_1} = 0 \ \text{in} \ \ O \tag{3.10}$$

admits also a large solution.
Indeed, assume that u is a large solution of (3.1) and v is the maximal solution of (3.10). Take $R > 0$ such that $O \subset B_R(0) \times (-R^2, R^2)$, then the function V defined by

$$V(x,t) = (q-1)^{-\frac{1}{q-1}}(2R^2 + t)^{-\frac{1}{q-1}},$$

satisfies (3.1). It follows for all $(x,t) \in O$

$$u(x,t) \geq \inf_{(y,s)\in O} V(x,t) \geq (q-1)^{-\frac{1}{q-1}} R^{-\frac{2}{q-1}} =: a_0.$$

Thus, $\tilde{u} = a_0^{\frac{q-q_1}{q_1-1}} u$ is a subsolution of (3.10). Therefore $v \geq a_0^{\frac{q-q_1}{q_1-1}} u$ in O, thus v is a large solution.

Remark 3.4 (Sub-critical case) *Assume that $1 < q < q_*$. One easily see that the function*

$$U(x,t) = \frac{C}{t^{\frac{1}{q-1}}} e^{-\frac{|x|^2}{4t}} \chi_{t>0} \tag{3.11}$$

is a subsolution of (3.1) in $\mathbb{R}^{N+1}\backslash\{(0,0)\}$, where $C = \left(\frac{2}{q-1} - \frac{N}{2}\right)^{\frac{1}{q-1}}$.
Therefore, the maximal solutions u of (3.1) in O verify

$$u(x,t) \geq C \frac{1}{(t-s)^{\frac{1}{q-1}}} e^{-\frac{|x-y|^2}{4(t-s)}} \chi_{t>s}, \tag{3.12}$$

for all $(x,t)\in O$ *and* $(y,s)\in O^c$.
If for any $(x,t)\in\partial_p O$ *there exist* $\varepsilon\in(0,1)$ *and a decreasing sequence* $\{\delta_n\}\subset(0,1)$ *converging to* 0 *as* $n\to\infty$ *such that* $\left(B_{\delta_n}(x)\times(-\delta_n^2+t,-\varepsilon\delta_n^2+t)\right)\cap O^c\neq\emptyset$ *for any* $n\in\mathbb{N}$, *then* u *is a large solution. For proving this, we need to show that* $\lim_{\rho\to 0}\inf_{O\cap(B_\rho(x)\times(-\rho^2+t,\rho^2+t))}u=\infty$.
Let $0<\rho<\sqrt{\frac{\varepsilon}{2}}\delta_1$, *and* $n\in\mathbb{N}$ *such that* $\sqrt{\frac{\varepsilon}{2}}\delta_{n+1}\le\rho<\sqrt{\frac{\varepsilon}{2}}\delta_n$.
Since $\left(B_{\delta_n}(x)\times(-\delta_n^2+t,-\varepsilon\delta_n^2+t)\right)\cap O^c\neq\emptyset$, *there is* $(x_n,t_n)\in O^c$ *such that* $|x_n-x|<\delta_n$ *and* $-\delta_n^2+t<t_n<-\varepsilon\delta_n^2+t$. *So if* $(y,s)\in O\cap(B_\rho(x)\times(-\rho^2+t,\rho^2+t))$ *then* $|y-x_n|<(\sqrt{\varepsilon}+1)\delta_n$ *and* $\frac{\varepsilon}{2}\delta_n^2<s-t_n<(\varepsilon+1)\delta_n^2$. *Hence, thanks to* (3.12) *we have for any* $(y,s)\in O\cap(B_\rho(x)\times(-\rho^2+t,\rho^2+t))$

$$u(y,s)\ge C\frac{1}{(s-t_n)^{\frac{1}{q-1}}}e^{-\frac{|y-x_n|^2}{4(s-t_n)}}\ge C(\varepsilon+1)^{-\frac{1}{q-1}}e^{-\frac{(\sqrt{\varepsilon}+1)^2}{2\varepsilon}}\delta_n^{-\frac{2}{q-1}},$$

which implies

$$\inf_{O\cap(B_\rho(x)\times(-\rho^2+t,\rho^2+t))}u\ge C(\varepsilon+1)^{-\frac{1}{q-1}}e^{-\frac{(\sqrt{\varepsilon}+1)^2}{2\varepsilon}}\delta_n^{-\frac{2}{q-1}}\to\infty\quad as\quad\rho\to 0.$$

Remark 3.5 *Note that if* $u\in C^{2,1}(O)$ *is a solution of* (3.1) *for some* $q>1$ *then, for* $a,b>0$ *and* $1<p\le 2$, $v=b^{-\frac{1}{q-1}}u$ *is a super-solution of*

$$\partial_t v-\Delta v+a|\nabla v|^p+bv^q=0\quad in\quad O.\tag{3.13}$$

Thus, we can apply the argument of the previous proof, with equation (3.1) *replaced by* (3.13), *and deduce that there exists a maximal solution* $v\in C^{2,1}(O)$ *of* (3.13) *satisfying*

$$v(x,t)\le Cb^{-\frac{1}{q-1}}(d((x,t),\partial_p O))^{-\frac{2}{q-1}}\quad for\ all\ (x,t)\in O.$$

Furthermore, if $1<q<q_*$, $q=\frac{2p}{p+1}$, $a,b>0$ *then the function* U *in Remark 3.4 is a subsolution of* (3.13) *in* $\mathbb{R}^{N+1}\backslash\{(0,0)\}$, *for some* $C=C(N,p,q,a,b)$. *Therefore, we conclude that every maximal solution of* $v\in C^{2,1}(O)$ *of* (3.13) *satisfy*

$$v(x,t)\ge C\frac{1}{(t-s)^{\frac{1}{q-1}}}e^{-\frac{|x-y|^2}{4(t-s)}}\chi_{t>s}\tag{3.14}$$

for all $(x,t)\in O$ *and* $(y,s)\in\partial_p O$.
As in Remark 3.4, if for any $(x,t)\in\partial_p O$ *there exist* $\varepsilon\in(0,1)$ *and a decreasing sequence* $\{\delta_n\}\subset(0,1)$ *converging to* 0 *as* $n\to\infty$ *such that* $\left(B_{\delta_n}(x)\times(-\delta_n^2+t,-\varepsilon\delta_n^2+t)\right)\cap O^c\neq\emptyset$ *for any* $n\in\mathbb{N}$, *then* v *is a large solution.*

Next, we consider the following equation

$$\partial_t u-\Delta u+e^u-1=0.\tag{3.15}$$

It is easy to see that the two functions

$$V_1(t)=-\log\left(\frac{t+\rho^2}{1+\rho^2}\right)\quad\text{and}\quad V_2(x)=C-2\log\left(\frac{\rho^2-|x|^2}{\rho}\right)$$

satisfy

$$V_1'+e^{V_1}-1\ge 0\qquad\text{in}\ \ (-\rho^2,0]$$

and

$$-\Delta V_2+e^{V_2}-1\ge 0\qquad\text{in}\ \ B_\rho(0)$$

for some $C=C(N)$. Using $e^a+e^b\le e^{a+b}-1$ for $a,b\ge 0$, we obtain that V_1+V_2 is a supersolution of equation (3.15) in $B_\rho(0)\times(-\rho^2,0]$. By the same argument as in Proposition 3.1 and the estimate of the above supersolution, we obtain

Proposition 3.6 *There exists a maximal solution $u \in C^{2,1}(O)$ of*

$$\partial_t u - \Delta u + e^u - 1 = 0 \text{ in } O, \tag{3.16}$$

and it satisfies

$$u(x,t) \le C - \log\left(\frac{(d((x,t),\partial_p O))^3}{4 + (d((x,t),\partial_p O))^2}\right) \quad \text{for all } (x,t) \in O, \tag{3.17}$$

for some $C = C(N)$.

The next three propositions will be useful to prove Theorem 1.1-(ii).

Proposition 3.7 *Let $K \subset \overline{\tilde{Q}_1(0,0)}$ be a compact set and $q > 1$, $R \ge 100$. Let u be a solution of (3.7) in $\tilde{Q}_R(0,0)\backslash K$ and φ as in Proposition 2.6 with $p = q'$. Set $\xi = (1-\varphi)^{2q'}$. Then,*

$$\int_{\tilde{Q}_R(0,0)} u\left(|\Delta\xi| + |\nabla\xi| + |\partial_t\xi|\right) dxdt \lesssim Cap_{2,1,q'}(K), \tag{3.18}$$

$$u(x,t) \lesssim Cap_{2,1,q'}(K) + R^{-\frac{2}{q-1}} \quad \text{for any } (x,t) \in \tilde{Q}_{R/5}(0,0)\backslash\tilde{Q}_2(0,0), \tag{3.19}$$

and

$$\int_{\tilde{Q}_2(0,0)} u\xi dxdt \lesssim Cap_{2,1,q'}(K) + R^{-\frac{2}{q-1}}, \tag{3.20}$$

where the constants in above inequalities depend only on N, q.

Proof. *Step 1.* We claim that

$$\int_{\tilde{Q}_R(0,0)} u^q\xi dxdt \lesssim \mathrm{Cap}_{2,1,q'}(K). \tag{3.21}$$

Actually, using by parts integration and the Green formula, one has

$$\begin{aligned}
\int_{\tilde{Q}_R(0,0)} u^q\xi dxdt &= -\int_{\tilde{Q}_R(0,0)} \partial_t u\xi dxdt + \int_{\tilde{Q}_R(0,0)} (\Delta u)\xi dxdt \\
&= \int_{\tilde{Q}_R(0,0)} u\partial_t\xi dxdt + \int_{\tilde{Q}_R(0,0)} u\Delta\xi dxdt + \int_{-R^2}^{R^2}\int_{\partial B_R(0)} \left(\xi\frac{\partial u}{\partial\nu} - u\frac{\partial\xi}{\partial\nu}\right) dSdt
\end{aligned}$$

where ν is the outer normal unit vector on $\partial B_R(0)$. Clearly,

$$\frac{\partial u}{\partial\nu} \le 0 \quad \text{and} \quad \frac{\partial\xi}{\partial\nu} = 0 \quad \text{on} \quad \partial B_R(0).$$

Thus,

$$\begin{aligned}
&\int_{\tilde{Q}_R(0,0)} u^q\xi dxdt \le \int_{\tilde{Q}_R(0,0)} u|\partial_t\xi| dxdt + \int_{\tilde{Q}_R(0,0)} u|\Delta\xi| dxdt \\
&\le 2q'\int_{\tilde{Q}_R(0,0)} u(1-\varphi)^{2q'-1}|\partial_t\varphi| dxdt + 2q'(2q'-1)\int_{\tilde{Q}_R(0,0)} u(1-\varphi)^{2q'-2}|\nabla\varphi|^2 dxdt \\
&\qquad + 2q'\int_{\tilde{Q}_R(0,0)} u(1-\varphi)^{2q'-1}|\Delta\varphi| dxdt \\
&\le 2q'\int_{\tilde{Q}_R(0,0)} u\xi^{1/q}|\partial_t\varphi| dxdt + 2q'(2q'-1)\int_{\tilde{Q}_R(0,0)} u\xi^{1/q}|\nabla\varphi|^2 dxdt \\
&\qquad + 2q'\int_{\tilde{Q}_R(0,0)} u\xi^{1/q}|\Delta\varphi| dxdt.
\end{aligned} \tag{3.22}$$

In the last inequality, we have used the fact that $(1-\phi)^{2q'-1} \leq (1-\phi)^{2q'-2} = \xi^{1/q}$. Hence, by Hölder's inequality,

$$\int_{\tilde{Q}_R(0,0)} u^q \xi dxdt \lesssim \int_{\tilde{Q}_R(0,0)} |\partial_t \varphi|^{q'} dxdt + \int_{\tilde{Q}_R(0,0)} |\nabla \varphi|^{2q'} dxdt$$
$$+ \int_{\tilde{Q}_R(0,0)} |\Delta \varphi|^{q'} dxdt.$$

By the Gagliardo-Nirenberg inequality,

$$\int_{\tilde{Q}_R(0,0)} |\nabla \varphi|^{2q'} dxdt \lesssim ||\varphi||^{q'}_{L^\infty(\tilde{Q}_R(0,0))} \int_{\tilde{Q}_R(0,0)} |D^2 \varphi|^{q'} dxdt$$
$$\lesssim \int_{\tilde{Q}_R(0,0)} |D^2 \varphi|^{q'} dxdt.$$

Hence, we find

$$\int_{\tilde{Q}_R(0,0)} u^q \xi dxdt \lesssim \int_{\tilde{Q}_R(0,0)} (|\partial_t \varphi|^{q'} + |D^2 \varphi|^{q'}) dxdt,$$

and derive (3.21) from (2.4). In view of (3.22), we also obtain

$$\int_{\tilde{Q}_R(0,0)} u(|\Delta \xi| + |\partial_t \xi|) dxdt \lesssim \text{Cap}_{2,1,q'}(K)$$

and

$$\int_{\tilde{Q}_R(0,0)} u|\nabla \xi| dxdt \lesssim \text{Cap}_{2,1,q'}(K),$$

since

$$\int_{\tilde{Q}_R(0,0)} u|\nabla \xi| dxdt = 2q' \int_{\tilde{Q}_R(0,0)} u \xi^{(2q'-1)/2q'} |\nabla \varphi| dxdt$$
$$\leq 2q' \int_{\tilde{Q}_R(0,0)} u \xi^{1/q} |\nabla \varphi| dxdt$$
$$\lesssim \int_{\tilde{Q}_R(0,0)} u^q \xi dxdt + \int_{\tilde{Q}_R(0,0)} |\nabla \varphi|^{q'} dxdt.$$

It yields (3.18).
Step 2. Relation (3.19) holds. Let η be a cut off function on $\tilde{Q}_{R/4}(0,0)$ with respect to $\tilde{Q}_{R/3}(0,0)$ such that $|\partial_t \eta| + |D^2 \eta| \lesssim R^{-2}$ and $|\nabla \eta| \lesssim R^{-1}$. We have

$$\partial_t(\eta \xi u) - \Delta(\eta \xi u) = F \in C_c(\tilde{Q}_{R/3}(0,0)).$$

Hence, we can write

$$(\eta \xi u)(x,t) = \int_{\mathbb{R}^N} \int_{-\infty}^{t} \frac{1}{(4\pi(t-s))^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} F(y,s) dsdy \quad \forall (x,t) \in \mathbb{R}^{N+1}.$$

Now, we fix $(x,t) \in \tilde{Q}_{R/5}(0,0) \backslash \tilde{Q}_2(0,0)$. Since $\text{supp}\{|\nabla \eta|\} \cap \text{supp}\{|\nabla \xi|\} = \emptyset$ and

$$F = \eta \xi \left(\partial_t u - \Delta u\right) - 2\left(\eta \nabla \xi + \xi \nabla \eta\right) \nabla u + \left(\xi \partial_t \eta + \eta \partial_t \xi - 2\nabla \eta \nabla \xi - \Delta \eta \xi - \eta \Delta \xi\right) u$$
$$\leq -2\left(\eta \nabla \xi + \xi \nabla \eta\right) \nabla u + \left(\xi \partial_t \eta + \eta \partial_t \xi - \xi \Delta \eta - \eta \Delta \xi\right) u,$$

there holds

$$
\begin{aligned}
u(x,t)=(\eta\xi u)(x,t) \leq & -2\int_{\mathbb{R}^N}\int_{-\infty}^{t} \frac{1}{(4\pi(t-s))^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} \left(\eta\nabla\xi+\xi\nabla\eta\right)\nabla u ds dy \\
& +\int_{\mathbb{R}^N}\int_{-\infty}^{t} \frac{1}{(4\pi(t-s))^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} \left(\eta\partial_t\xi-\eta\Delta\xi\right) u ds dy \\
& +\int_{\mathbb{R}^N}\int_{-\infty}^{t} \frac{1}{(4\pi(t-s))^{\frac{N}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} \left(\partial_t\eta\xi-\xi\Delta\eta\right) u ds dy. \\
& = I_1+I_2+I_3.
\end{aligned}
$$

By parts integration

$$
\begin{aligned}
I_1 = & \, 2(4\pi)^{-N/2}\int_{-\infty}^{t}\int_{\mathbb{R}^N} \frac{(x-y)}{2(t-s)^{(N+2)/2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left(\eta\nabla\xi+\xi\nabla\eta\right) u dy ds \\
& + 2(4\pi)^{-N/2}\int_{-\infty}^{t}\int_{\mathbb{R}^N} \frac{1}{(t-s)^{N/2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left(\xi\Delta\eta+\eta\Delta\xi\right) u \, dy ds.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\frac{1}{(t-s)^{N/2}} e^{-\frac{|x-y|^2}{4(t-s)}} & \lesssim \left(\max\{|x-y|, |t-s|^{1/2}\}\right)^{-N}, \\
\left|\frac{(x-y)}{2(t-s)^{(N+2)/2}} e^{-\frac{|x-y|^2}{4(t-s)}}\right| & \lesssim \left(\max\{|x-y|, |t-s|^{1/2}\}\right)^{-N-1},
\end{aligned}
$$

and

$$
\begin{aligned}
& \max\{|x-y|, |t-s|^{1/2}\} \gtrsim 1 \quad \forall (y,s)\in \operatorname{supp}\{|D^\alpha\xi|\}\cup\operatorname{supp}\{|\partial_t\xi|\}, \\
& \max\{|x-y|, |t-s|^{1/2}\} \gtrsim R \quad \forall (y,s)\in \operatorname{supp}\{|D^\alpha\eta|\}\cup\operatorname{supp}\{|\partial_t\eta|\} \ \ \forall |\alpha|\geq 1.
\end{aligned}
$$

We deduce

$$
\begin{aligned}
I_1 \lesssim & \int_{\mathbb{R}^{N+1}} \left(\max\{|x-y|, |t-s|^{1/2}\}\right)^{-N-1} (\eta|\nabla\xi|+\xi|\nabla\eta|) u \, dy ds \\
& + \int_{\mathbb{R}^{N+1}} \left(\max\{|x-y|, |t-s|^{1/2}\}\right)^{-N} (\xi|\Delta\eta|+\eta|\Delta\xi|) \, u \, dy ds \\
& \lesssim \int_{\mathbb{R}^{N+1}} (|\nabla\xi|+|\Delta\xi|) u \, dy ds + \int_{\tilde{Q}_{R/3}(0,0)\setminus\tilde{Q}_{R/4}(0,0)} (R^{-N-1}|\nabla\eta|+R^{-N}|\Delta\eta|) u \, dy ds \\
& \lesssim \int_{\mathbb{R}^{N+1}} (|\nabla\xi|+|\Delta\xi|) u \, dy ds + \sup_{\tilde{Q}_{R/3}(0,0)\setminus\tilde{Q}_{R/4}(0,0)} u,
\end{aligned}
$$

$$
\begin{aligned}
I_2 & \lesssim \int_{\mathbb{R}^{N+1}} \left(\max\{|x-y|, |t-s|^{1/2}\}\right)^{-N} (|\partial_t\xi|+|\Delta\xi|) u \, dy ds \\
& \lesssim \int_{\mathbb{R}^{N+1}} (|\partial_t\xi|+|\Delta\xi|) u \, dy ds,
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 & \lesssim \int_{\mathbb{R}^{N+1}} \left(\max\{|x-y|, |t-s|^{1/2}\}\right)^{-N} (|\partial_t\eta|+|\Delta\eta|) u \, dy ds \\
& \lesssim \int_{\tilde{Q}_{R/3}(0,0)\setminus\tilde{Q}_{R/4}(0,0)} R^{-N}(|\partial_t\eta|+|\Delta\eta|) u \, dy ds \\
& \lesssim \sup_{\tilde{Q}_{R/3}(0,0)\setminus\tilde{Q}_{R/4}(0,0)} u.
\end{aligned}
$$

Hence,

$$u(x,t) \le I_1 + I_2 + I_3 \lesssim \int_{\mathbb{R}^{N+1}} (|\partial_t \xi| + |\nabla \xi| + |\Delta \xi|) u \, dyds + \sup_{\tilde{Q}_{R/3}(0,0) \setminus \tilde{Q}_{R/4}(0,0)} u.$$

Combining this with (3.18) and (3.8), we obtain (3.19).

Step 3. End of the proof. Let θ be a cut off function on $\tilde{Q}_3(0,0)$ with respect to $\tilde{Q}_4(0,0)$. As above, we have for any $(x,t) \in \mathbb{R}^{N+1}$

$$\begin{aligned}(\theta\xi u)(x,t) &\lesssim \int_{\mathbb{R}^{N+1}} (\max\{|x-y|, |t-s|^{1/2}\})^{-N-1} (\theta|\nabla\xi| + \xi|\nabla\theta|) u \, dyds \\ &+ \int_{\mathbb{R}^{N+1}} (\max\{|x-y|, |t-s|^{1/2}\})^{-N} (\theta|\Delta\xi| + \xi|\Delta\theta|) u \, dyds \\ &+ \int_{\mathbb{R}^{N+1}} (\max\{|x-y|, |t-s|^{1/2}\})^{-N} (\theta|\partial_t\xi| + \theta|\Delta\xi|) u \, dyds \\ &+ \int_{\mathbb{R}^{N+1}} (\max\{|x-y|, |t-s|^{1/2}\})^{-N} (\xi|\partial_t\theta| + \xi|\Delta\theta|) u \, dyds.\end{aligned}$$

Hence, by Fubini theorem,

$$\begin{aligned}\int_{\tilde{Q}_2(0,0)} \eta u dx dt &= \int_{\tilde{Q}_2(0,0)} \theta\eta u dx dt \\ &\lesssim A \int_{\mathbb{R}^{N+1}} (\theta|\nabla\xi| + \xi|\nabla\theta| + \theta|\Delta\xi| + \xi|\Delta\theta| + \theta|\partial_t\xi| + \xi|\partial_t\theta|) \, u \, dyds \\ &\lesssim \int_{\mathbb{R}^{N+1}} (|\partial_t\xi| + |\nabla\xi| + |\Delta\xi|) u \, dyds + \sup_{\tilde{Q}_4(0,0) \setminus \tilde{Q}_3(0,0)} u\end{aligned}$$

where

$$A = \sup_{(y,s) \in \tilde{Q}_4(0,0)} \int_{\tilde{Q}_2(0,0)} ((\max\{|x-y|, |t-s|^{1/2}\})^{-N} + (\max\{|x-y|, |t-s|^{1/2}\})^{-N-1}) dx dt.$$

Therefore we obtain (3.20) from (3.18) and (3.19). ∎

Proposition 3.8 *Let* $K \subset \{(x,t) : \varepsilon < \max\{|x|, |t|^{1/2}\} < 1\}$ *be a compact set,* $0 < \varepsilon < 1$ *and* u *be the maximal solution of* (3.7) *in* $\tilde{Q}_R(0,0) \setminus K$ *with* $R \ge 100$. *Then*

$$\sup_{\tilde{Q}_{\varepsilon/4}(0,0)} u \lesssim \sum_{j=-2}^{j_\varepsilon - 2} \frac{Cap_{2,1,q'}(K \cap \tilde{Q}_{\rho_j}(0,0))}{\rho_j^N} + j_\varepsilon R^{-\frac{2}{q-1}} \quad \text{if} \quad q > q_*, \tag{3.23}$$

and

$$\sup_{\tilde{Q}_{\varepsilon/4}(0,0)} u \lesssim \sum_{j=0}^{j_\varepsilon} \frac{Cap_{2,1,q'}(K_j)}{\rho_j^N} + j_\varepsilon R^{-\frac{2}{q-1}} \quad \text{if} \quad q = q_*, \tag{3.24}$$

where $\rho_j = 2^{-j}$, $K_j = \{(x/\rho_{j+3}, t/\rho_{j+3}^2) : (x,t) \in K \cap \tilde{Q}_{\rho_{j-2}}(0,0)\}$ *and* $j_\varepsilon \in \mathbb{N}$ *is such that* $\rho_{j_\varepsilon} \le \varepsilon < \rho_{j_\varepsilon - 1}$.

Proof. For $j \in N$, we define $S_j = \{x : \rho_j \le \max\{|x|, |t|^{1/2}\} \le \rho_{j-1}\}$.
Fix any $1 \le j \le j_\varepsilon$. We cover S_j by $L = L(N) \in \mathbb{N}^*$ closed cylinders

$$\overline{\tilde{Q}_{\rho_{j+3}}(x_{k,j}, t_{k,j})}, \quad k = 1, ..., L(N)$$

where $(x_{k,j}, t_{k,j}) \in S_j$.
For $k = 1, ..., L(N)$, let $u_j, u_{k,j}$ be the maximal solutions of (3.7) where K is replaced by $K \cap S_j$ and $K \cap \overline{\tilde{Q}_{\rho_{j+3}}(x_{k,j}, t_{k,j})}$, respectively. Clearly the function $\tilde{u}_{k,j}$ defined by

$$\tilde{u}_{k,j}(x,t) = \rho_{j+3}^{\frac{2}{q-1}} u_{k,j}(\rho_{j+3} x + x_{k,j}, \rho_{j+3}^2 t + t_{k,j})$$

is the maximal solution of (3.7) when $(K_{k,j}, \tilde{Q}_{R/\rho_{j+3}}(-x_{k,j}/\rho_{j+3}, -t_{k,j}/\rho_{j+3}^2))$ is replacing $(K, \tilde{Q}_R(0,0))$, with

$$K_{k,j} = \{(y/\rho_{j+3}, s/\rho_{j+3}^2) : (y,s) \in -(x_{k,j}, t_{k,j}) + K \cap \overline{\tilde{Q}_{\rho_{j+3}}(x_{k,j}, t_{k,j})}\} \subset \overline{\tilde{Q}_1(0,0)}.$$

Let $\overline{u}_{k,j}$ be the maximal solution of (3.7) with $(K, \tilde{Q}_R(0,0))$ replaced by $(K_{k,j}, \tilde{Q}_{2R/\rho_{j+3}}(0,0))$. Since $\tilde{Q}_{R/\rho_{j+3}}(-x_{k,j}/\rho_{j+3}, -t_{k,j}/\rho_{j+3}^2) \subset \tilde{Q}_{2R/\rho_{j+3}}(0,0)$, then, by the comparison principle as in the proof of Proposition 3.1 we get $\tilde{u}_{k,j} \le \overline{u}_{k,j}$ in $\tilde{Q}_{R/\rho_{j+3}}(-x_{k,j}/\rho_{j+3}, -t_{k,j}/\rho_{j+3}^2) \backslash K_{k,j}$ and thus

$$\tilde{u}_{k,j}(x,t) \lesssim \mathrm{Cap}_{2,1,q'}(K_{k,j}) + (R/\rho_{j+3})^{-\frac{2}{q-1}},$$

for any $(x,t) \in \left(\tilde{Q}_{2R/(5\rho_{j+3})}(0,0) \cap \tilde{Q}_{R/\rho_{j+3}}(-x_{k,j}/\rho_{j+3}, -t_{k,j}/\rho_{j+3}^2)\right) \backslash \tilde{Q}_2(0,0) = D$. Fix $(x_0,t_0) \in \tilde{Q}_{\varepsilon/4}(0,0)$. Clearly, $((x_0 - x_{k,j})/\rho_{j+3}, (t_0 - t_{k,j})/\rho_{j+3}) \in D$, hence

$$\begin{aligned} u_{k,j}(x_0,t_0) &= \rho_{j+3}^{-\frac{2}{q-1}} \tilde{u}_{k,j}((x_0 - x_{k,j})/\rho_{j+3}, (t_0 - t_{k,j})/\rho_{j+3}^2) \\ &\lesssim \frac{\mathrm{Cap}_{2,1,q'}(K_{k,j})}{\rho_j^{\frac{2}{q-1}}} + R^{-\frac{2}{q-1}}. \end{aligned}$$

Therefore, using (3.9) in Remark 3.2 and the fact that

$$\mathrm{Cap}_{2,1,q'}(K_{k,j}) = \mathrm{Cap}_{2,1,q'}(K_{k,j} + (x_{k,j}/\rho_{j+3}, t_{k,j}/\rho_{j+3}^2)) \le \mathrm{Cap}_{2,1,q'}(K_j),$$

we derive

$$\begin{aligned} u(x_0,t_0) \le \sum_{j=1}^{j_\varepsilon} u_j(x_0,t_0) &\le \sum_{j=1}^{j_\varepsilon} \sum_{k=1}^{L(N)} u_{k,j}(x_0,t_0) \\ &\lesssim \sum_{j=0}^{j_\varepsilon} \frac{\mathrm{Cap}_{2,1,q'}(K_j)}{\rho_j^{\frac{2}{q-1}}} + j_\varepsilon R^{-\frac{2}{q-1}}, \end{aligned}$$

which yields (3.24). If $q > q_*$, then by (2.2) in Proposition 2.5, we have

$$\mathrm{Cap}_{2,1,q'}(K_j) \lesssim \rho_{j+3}^{-N-2+2q'} \mathrm{Cap}_{2,1,q'}(K \cap \tilde{Q}_{\rho_{j-2}}(0,0)),$$

which implies (3.23). ∎

Proposition 3.9 *Let K, u, ξ be as in Proposition 3.7. For any compact set K_0 in $\overline{\tilde{Q}_1(0,0)}$ with positive measure $|K_0|$, there exists $\varepsilon = \varepsilon(N,q,|K_0|) > 0$ such that*

$$Cap_{2,1,q'}(K) \le \varepsilon \Rightarrow \inf_{K_0} u \lesssim \int_{\tilde{Q}_2(0,0)} u\xi dxdt,$$

where the constant in the inequality $\lesssim$ depends on K_0. In particular,

$$Cap_{2,1,q'}(K) \le \varepsilon \Rightarrow \inf_{K_0} u \lesssim Cap_{2,1,q'}(K) + R^{-\frac{2}{q-1}}. \tag{3.25}$$

Proof. It is enough to prove that there exists $\varepsilon > 0$ such that

$$\mathrm{Cap}_{2,1,q'}(K) \le \varepsilon \Rightarrow |K_1| \ge 1/2|K_0| \tag{3.26}$$

where $K_1 = \{(x,t) \in K_0 : \xi(x,t) \ge 1/2\}$. By (2.1) in Proposition 2.5, we have the following estimates

$$|K_0 \backslash K_1|^{1-\frac{2q'}{N+2}} \lesssim \mathrm{Cap}_{2,1,q'}(K_0 \backslash K_1)$$

if $q > q_*$, and

$$\left(\log\left(\frac{|\tilde{Q}_{200}(0,0)|}{|K_0\backslash K_1|}\right)\right)^{-\frac{N}{2}} \lesssim \mathrm{Cap}_{2,1,q'}(K_0\backslash K_1)$$

if $q = q_*$. On the other hand,

$$\begin{aligned}\mathrm{Cap}_{2,1,q'}(K_0\backslash K_1) &= \mathrm{Cap}_{2,1,q'}(\{K_0 : \varphi > 1-(1/2)^{1/(2q')}\}) \\ &\leq (1-(1/2)^{1/(2q')})^{-q'} \int_{\mathbb{R}^{N+1}} \left(|D^2\varphi|^{q'} + |\nabla\varphi|^{q'} + |\varphi|^{q'} + |\partial_t\varphi|^{q'}\right) dxdt \\ &\lesssim \mathrm{Cap}_{2,1,q'}(K)\end{aligned}$$

where φ is in Proposition 3.7. Henceforth, one can find $\varepsilon = \varepsilon(N, q, |K_0|) > 0$ such that

$$\mathrm{Cap}_{2,1,q'}(K) \leq \varepsilon \Rightarrow |K_0\backslash K_1| \leq 1/2\ |K_0|.$$

This implies (3.26). ■

4 Large solutions

In the first part of this section, we prove theorem 1.1-(ii), then we prove theorems 1.1-(i) and 1.2, at end we consider a parabolic viscous Hamilton-Jacobi equation.

4.1 Proof of Theorem 1.1-(ii)

Let $R_0 \geq 4$ such that $O \subset\subset \tilde{Q}_{R_0}(0,0)$. Assume that the equation (1.12) has a large solution u. Take any $(x,t) \in \partial_p O$. We will to prove that (1.14) holds. We can assume $(x,t) = (0,0)$. Set $K = \tilde{Q}_{2R_0}(0,0)\backslash O$ and define

$$\begin{aligned}T_j &= \{x : \rho_{j+1} \leq \max\{|x|, |t|^{1/2}\} \leq \rho_j, t \leq 0\}, \\ \tilde{T}_j &= \{x : \rho_{j+3} \leq \max\{|x|, |t|^{1/2}\} \leq \rho_{j-2}, t \leq 0\}.\end{aligned}$$

Here $\rho_j = 2^{-j}$. For $j \geq 3$, let u_1, u_2, u_3, u_4 be the maximal solutions of (3.7) when K is replaced by $K \cap \overline{Q_{\rho_{j+3}}(0,0)}$, $K \cap \tilde{T}_j$, $\left(K \cap \overline{Q_1(0,0)}\right) \backslash Q_{\rho_{j-2}}(0,0)$ and $K\backslash Q_1(0,0)$ respectively and $R \geq 100R_0$. From (3.9) in Remark 3.2, we can assert that

$$u \leq u_1 + u_2 + u_3 + u_4 \quad \text{in} \;\; O \cap \{(x,t) \in \mathbb{R}^{N+1} : t \leq 0\}.$$

Thus,

$$\inf_{T_j} u \leq ||u_1||_{L^\infty(T_j)} + ||u_3||_{L^\infty(T_j)} + ||u_4||_{L^\infty(T_j)} + \inf_{T_j} u_2. \tag{4.1}$$

Case 1: $q > q_*$. By (3.8) in Remark 3.2,

$$||u_4||_{L^\infty(T_j)} \lesssim 1. \tag{4.2}$$

By (3.23) in Proposition 3.8,

$$||u_3||_{L^\infty(T_j)} \lesssim \sum_{i=-2}^{j-4} \frac{\mathrm{Cap}_{2,1,q'}(K \cap Q_{\rho_i}(0,0))}{\rho_i^N} + jR^{-\frac{2}{q-1}}. \tag{4.3}$$

Since $(x,t)\mapsto \overline{u}_1(x,t)=\rho_{j+3}^{2/(q-1)}u_1(\rho_{j+3}x,\rho_{j+3}^2t)$ is the maximal solution of (3.7) when $(K,\tilde{Q}_R(0,0))$ is replaced by $(\{(y/\rho_{j+3},s/\rho_{j+3}^2):(y,s)\in K\cap\overline{Q_{\rho_{j+3}}(0,0)}\},\tilde{Q}_{R/\rho_{j+3}}(0,0))$, we derive, thanks to (3.19) in Proposition 3.7 and (2.2) in Proposition 2.5,

$$||\overline{u}_1||_{L^\infty(T_{-3})}\lesssim\frac{\mathrm{Cap}_{2,1,q'}(K\cap Q_{\rho_{j+2}}(0,0))}{\rho_j^{N+2-2q'}}+(R/\rho_{j+3})^{-\frac{2}{q-1}},$$

from which follows

$$||u_1||_{L^\infty(T_j)}\lesssim\frac{\mathrm{Cap}_{2,1,q'}(K\cap Q_{\rho_{j+2}}(0,0))}{\rho_j^{N}}+R^{-\frac{2}{q-1}}. \tag{4.4}$$

Since, $(x,t)\mapsto \overline{u}_2(x,t)=\rho_{j-2}^{2/(q-1)}u_2(\rho_{j-2}x,\rho_{j-2}^2t)$ is the maximal solution of (3.7) when the couple $(K,\tilde{Q}_R(0,0))$ is replaced by $(\{(y/\rho_{j-2},s/\rho_{j-2}^2):(y,s)\in K\cap\tilde{T}_j\},\tilde{Q}_{R/\rho_{j-2}}(0,0))$, Proposition 3.9 and relation (2.2) in Proposition 2.5 yield

$$\frac{\mathrm{Cap}_{2,1,q'}(K\cap\tilde{T}_j)}{\rho_{j-2}^{N+2-2q'}}\leq\varepsilon\Rightarrow\inf_{T_2}\overline{u}_2\lesssim\frac{\mathrm{Cap}_{2,1,q'}(K\cap\tilde{T}_j)}{\rho_{j-2}^{N+2-2q'}}+(R/\rho_{j-2})^{-\frac{2}{q-1}},$$

which implies

$$\frac{\mathrm{Cap}_{2,1,q'}(K\cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-2}^{N+2-2q'}}\leq\varepsilon\Rightarrow\inf_{T_j}u_2\lesssim\frac{\mathrm{Cap}_{2,1,q'}(K\cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-2}^{N}}+R^{-\frac{2}{q-1}}, \tag{4.5}$$

for some $\varepsilon=\varepsilon(N,q)>0$.
First, we assume that there exists $J\in\mathbb{N}$, $J\geq 10$ such that

$$\frac{\mathrm{Cap}_{2,1,q'}(K\cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-2}^{N+2-2q'}}\leq\varepsilon\quad\forall\, j\geq J.$$

Then, from (4.1) and (4.2), (4.3), (4.4), (4.5), we have

$$\inf_{T_j}u\lesssim\sum_{i=-2}^{j+2}\frac{\mathrm{Cap}_{2,1,q'}(K\cap Q_{\rho_i}(0,0))}{\rho_i^N}+jR^{-\frac{2}{q-1}}+1,$$

for any $j\geq J$. Letting $R\to\infty$,

$$\inf_{T_j}u\lesssim\sum_{i=-2}^{j+2}\frac{\mathrm{Cap}_{2,1,q'}(K\cap Q_{\rho_i}(0,0))}{\rho_i^N}+1.$$

Since $\inf_{T_j}u\to\infty$ as $j\to\infty$, we get

$$\sum_{i=0}^{\infty}\frac{\mathrm{Cap}_{2,1,q'}(K\cap Q_{\rho_i}(0,0))}{\rho_i^N}=\infty,$$

which implies that (1.14) holds with $(x,t)=(0,0)$.
Alternatively, assume that for infinitely many j

$$\frac{\mathrm{Cap}_{2,1,q'}(K\cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-2}^{N+2-2q'}}>\varepsilon$$

Then,

$$\frac{\mathrm{Cap}_{2,1,q'}(K\cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-2}^{N}}>\rho_{j-2}^{2-2q'}\varepsilon\to\infty\quad\text{when}\quad j\to\infty.$$

We also derive that (1.14) holds with $(x,t)=(0,0)$. This proves the case $q>q_*$.

Case 2: $q=q_*$. Similarly to Case 1, we have: for $j\geq 6$

$$||u_4||_{L^\infty(T_j)} \lesssim 1, \tag{4.6}$$

$$||u_3||_{L^\infty(T_j)} \lesssim \sum_{i=0}^{j-2} \frac{\mathrm{Cap}_{2,1,q'}(K_j)}{\rho_i^N} + jR^{-\frac{2}{q-1}}, \tag{4.7}$$

$$||u_1||_{L^\infty(T_j)} \lesssim \frac{\mathrm{Cap}_{2,1,q'}(K_j)}{\rho_j^N} + R^{-\frac{2}{q-1}}, \tag{4.8}$$

$$\mathrm{Cap}_{2,1,q'}(K_{j-5}) \leq \varepsilon \Rightarrow \inf_{T_j} u_2 \lesssim \frac{\mathrm{Cap}_{2,1,q'}(K_{j-5})}{\rho_j^N} + R^{-\frac{2}{q-1}}, \tag{4.9}$$

where $K_j=\{(x/\rho_{j+3}, t/\rho_{j+3}^2):(x,t)\in K\cap Q_{\rho_{j-3}}(0,0)\}$ and $\varepsilon=\varepsilon(N)>0$.
From (2.2) in Proposition 2.5, we have

$$\frac{1}{\mathrm{Cap}_{2,1,q'}(K\cap Q_{\rho_{j-3}}(0,0))} \leq \frac{c}{\mathrm{Cap}_{2,1,q'}(K_j)} + cj^{N/2}$$

for any $j\geq 4$ where $c=c(N)$. If there are infinitely many $j\geq 4$ such that

$$\mathrm{Cap}_{2,1,q'}(K\cap Q_{\rho_{j-3}}(0,0)) > \frac{1}{2cj^{N/2}},$$

then (1.14) holds with $(x,t)=(0,0)$ since

$$\frac{\mathrm{Cap}_{2,1,q'}(K\cap Q_{\rho_{j-3}}(0,0))}{\rho_{j-3}^N} > \frac{2^{j-3}}{2cj^{N/2}} \to \infty \quad \text{when } j\to\infty.$$

Now, we assume that there exists $J\geq 6$ such that

$$\mathrm{Cap}_{2,1,q'}(K\cap Q_{\rho_{j-3}}(0,0)) \leq \frac{1}{2cj^{N/2}}.$$

Then,

$$\mathrm{Cap}_{2,1,q'}(K_j) \leq 2c\mathrm{Cap}_{2,1,q'}(K\cap Q_{\rho_{j-3}}(0,0)) \ \ \forall\, j\geq J.$$

This leads to

$$\mathrm{Cap}_{2,1,q'}(K_j) \leq 2c\mathrm{Cap}_{2,1,q'}(K\cap Q_{\rho_{j-3}}(0,0)) \leq \varepsilon \quad \forall\, j\geq J'+J,$$

for some $J'=J'(N)$. Hence, from (4.6)-(4.9) we have, for any $j\geq J'+J+3$,

$$\begin{aligned}
&||u_4||_{L^\infty(T_j)} \lesssim 1,\\
&||u_3||_{L^\infty(T_j)} \lesssim \sum_{i=J'+J+1}^{j-2} \frac{\mathrm{Cap}_{2,1,q'}(K\cap Q_{\rho_{i-3}}(0,0))}{\rho_i^N} + C(J'+J) + jR^{-\frac{2}{q-1}},\\
&||u_1||_{L^\infty(T_j)} \lesssim \frac{\mathrm{Cap}_{2,1,q'}(K\cap Q_{\rho_{j-3}}(0,0))}{\rho_j^N} + R^{-\frac{2}{q-1}},\\
&\inf_{T_j} u_2 \lesssim \frac{\mathrm{Cap}_{2,1,q'}(K\cap Q_{\rho_{j-8}}(0,0))}{\rho_j^N} + R^{-\frac{2}{q-1}},
\end{aligned}$$

where $C(J'+J)=\sum_{i=0}^{J'+J} \frac{\mathrm{Cap}_{2,1,q'}(K_j)}{\rho_i^N}$.
Consequently we derive

$$\inf_{T_j} u \lesssim \sum_{i=0}^{j} \frac{\mathrm{Cap}_{2,1,q'}(K\cap Q_{\rho_i}(0,0))}{\rho_i^N} + C(J'+J) + 1 + jR^{-\frac{2}{q-1}} \ \ \forall\, j\geq J'+J+3$$

from (4.1). Letting $R\to\infty$ and $j\to\infty$ we obtain

$$\sum_{i=0}^{\infty}\frac{\mathrm{Cap}_{2,1,q'}(K\cap Q_{\rho_i}(0,0))}{\rho_i^N}=\infty,$$

i.e (1.14) holds with $(x,t)=(0,0)$. This completes the proof of Theorem 1.1-(ii).

4.2 Proof of Theorem 1.1-(i) and Theorem 1.2

Fix $(x_0,t_0)\in\partial_pO$. We can assume that $(x_0,t_0)=0$. Let $\delta\in(0,1/100)$. For $(y_0,s_0)\in(B_\delta(0)\times(-\delta^2,\delta^2))\cap O$, we set

$$M_k=O^c\cap\left(\overline{B_{r_{k+2}}(y_0)}\times[s_0-(73+\frac{1}{2})r_{k+2}^2,s_0-(70+\frac{1}{2})r_{k+2}^2]\right)$$

and

$$S_k=\{(x,t):r_{k+1}\le\max\{|x-y_0|,|t-s_0|^{\frac{1}{2}}\}<r_k\}\ \text{for}\ \ k=1,2,...,$$

where $r_k=4^{-k}$. Note that $M_k=\emptyset$ for k large enough and $M_k\subset S_k$ for all k. Let $R_0\ge4$ such that $O\subset\subset\tilde{Q}_{R_0}(0,0)$. By Theorem 2.2 and 2.4 and estimate (1.11) there exist two sequences $\{\mu_k\}_k$ and $\{\nu_k\}_k$ of nonnegative Radon measures such that

$$\mathrm{supp}(\mu_k)\subset M_k,\ \mathrm{supp}(\nu_k)\subset M_k,\tag{4.10}$$

$$\mu_k(M_k)\asymp\mathrm{Cap}_{2,1,q'}(M_k)\asymp\int_{\mathbb{R}^{N+1}}\left(\mathbb{I}_2^{2R_0}[\mu_k]\right)^q dxdt\tag{4.11}$$

and

$$\nu_k(M_k)\asymp\mathcal{PH}_1^N(M_k),\ \ ||\mathbb{M}_1^{2R_0}[\nu_k]||_{L^\infty(\mathbb{R}^{N+1})}\le1\quad\text{for}\ k=1,2,...,\tag{4.12}$$

where the constants of equivalence depend on N,q,R_0.
Take $\varepsilon>0$ such that $\exp\left(C_1\varepsilon\mathbb{I}_2^{2R_0}[\sum_{k=1}^\infty\nu_k]\right)\in L^1(\tilde{Q}_{R_0}(0,0))$ where the constant $C_1=C_1(N)$ is the one of inequality (2.6). By Theorem 2.7 and Proposition 2.8, there exist two nonnegative solutions U_1,U_2 of problems

$$\begin{aligned}\partial_tU_1-\Delta U_1+U_1^q&=\varepsilon\sum_{k=1}^\infty\mu_k&&\text{in }\tilde{Q}_{R_0}(0,0),\\ U_1&=0&&\text{on }\ \partial_p\tilde{Q}_{R_0}(0,0).\end{aligned}$$

and

$$\begin{aligned}\partial_tU_2-\Delta U_2+e^{U_2}-1&=\varepsilon\sum_{k=1}^\infty\nu_k&&\text{in }\tilde{Q}_{R_0}(0,0),\\ U_2&=0&&\text{on }\ \partial_p\tilde{Q}_{R_0}(0,0),\end{aligned}$$

respectively which satisfy

$$\begin{aligned}U_1(y_0,z_0)\gtrsim&\sum_{i=0}^\infty\sum_{k=1}^\infty\varepsilon\frac{\mu_k(B_{\frac{r_i}{8}}(y_0)\times(s_0-\frac{37}{128}r_i^2,s_0-\frac{35}{128}r_i^2))}{r_i^N}\\&-\mathbb{I}_2^{2R_0}\left[\left(\mathbb{I}_2^{2R_0}[\varepsilon\sum_{k=1}^\infty\mu_k]\right)^q\right](y_0,s_0)=:A\end{aligned}\tag{4.13}$$

and

$$\begin{aligned}U_2(y_0,z_0)\gtrsim&\sum_{i=0}^\infty\sum_{k=1}^\infty\varepsilon\frac{\nu_k(B_{\frac{r_i}{8}}(y_0)\times(s_0-\frac{37}{128}r_i^2,s_0-\frac{35}{128}r_i^2))}{r_i^N}\\&-\mathbb{I}_2^{2R_0}\left[\exp\left(C_1\mathbb{I}_2^{2R_0}[\varepsilon\sum_{k=1}^\infty\nu_k]\right)-1\right](y_0,s_0)=:B\end{aligned}\tag{4.14}$$

and $U_1, U_2 \in C^{2,1}(O)$.
Let u_1, u_2 be the maximal solutions of equations (3.1) and (3.16) respectively.
We have $u_1(y_0,s_0) \geq U_1(y_0,s_0)$ and $u_2(y_0,s_0) \geq U_2(y_0,s_0)$. Now, we claim that

$$A \gtrsim \sum_{k=1}^{\infty} \frac{\mathrm{Cap}_{2,1,q'}(M_k)}{r_k^N} \tag{4.15}$$

and

$$B \gtrsim -c_1(R_0) + \sum_{k=1}^{\infty} \frac{\mathcal{PH}_1^N(M_k)}{r_k^N}. \tag{4.16}$$

Proof of assertion (4.15)**.** From (4.11) we have

$$A \gtrsim \varepsilon \sum_{k=1}^{\infty} \frac{\mathrm{Cap}_{2,1q'}(M_k)}{r_k^N} - \varepsilon^q A_0 \tag{4.17}$$

with

$$A_0 = \mathbb{I}_2^{2R_0}\left[\left(\mathbb{I}_2^{2R_0}[\sum_{k=1}^{\infty}\mu_k]\right)^q\right](y_0,s_0).$$

Take $i_0 \in \mathbb{Z}$ such that $r_{i_0+1} < \max\{2R_0, 1\} \leq r_{i_0}$. Then

$$\begin{aligned}
A_0 &\lesssim \sum_{i=i_0}^{\infty} r_i^{-N} \int_{\tilde{Q}_{r_i}(y_0,s_0)} \left(\mathbb{I}_2^{2R_0}[\sum_{k=1}^{\infty}\mu_k]\right)^q dxdt \\
&= \sum_{i=i_0}^{\infty}\sum_{j=i}^{\infty} r_i^{-N} \int_{S_j} \left(\mathbb{I}_2^{2R_0}[\sum_{k=1}^{\infty}\mu_k]\right)^q dxdt \\
&= \sum_{j=k_0}^{\infty}\sum_{i=i_0}^{j} r_i^{-N} \int_{S_j} \left(\mathbb{I}_2^{2R_0}[\sum_{k=1}^{\infty}\mu_k]\right)^q dxdt \\
&\lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \left(\mathbb{I}_2^{2R_0}[\sum_{k=1}^{\infty}\mu_k]\right)^q dxdt.
\end{aligned}$$

Here we have used the fact that $\sum_{i=i_0}^{j} r_i^{-N} \leq \frac{4}{3} r_j^{-N}$ for all j.
Setting $\mu_k \equiv 0$ for all $i_0 - 1 \leq k \leq 0$, the previous inequality becomes

$$\begin{aligned}
A_0 &\lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \left(\mathbb{I}_2^{2R_0}[\mu_j + \sum_{k=i_0-1}^{j-1}\mu_k + \sum_{k=j+1}^{\infty}\mu_k]\right)^q dxdt \\
&\lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \left(\mathbb{I}_2^{2R_0}[\mu_j]\right)^q dxdt \\
&+ \sum_{j=i_0}^{\infty} r_j^2 \left(\sum_{k=i_0-1}^{j-1} ||\mathbb{I}_2^{2R_0}[\mu_k]||_{L^\infty(S_j)}\right)^q \\
&+ \sum_{j=i_0}^{\infty} r_j^2 \left(\sum_{k=j+1}^{\infty} ||\mathbb{I}_2^{2R_0}[\mu_k]||_{L^\infty(S_j)}\right)^q \\
&= A_1 + A_2 + A_3.
\end{aligned} \tag{4.18}$$

Using (4.11) we obtain

$$A_1 \le \sum_{k=1}^{\infty} \frac{\mathrm{Cap}_{2,1,q'}(M_k)}{r_k^N}. \tag{4.19}$$

Next, using (4.10) we have for any $(x,t) \in S_j$ if $k \ge j+1$,

$$\mathbb{I}_2^{2R_0}[\mu_k](x,t) = \int_{r_{j+1}}^{2R_0} \frac{\mu_k(\tilde{Q}_\rho(x,t))}{\rho^N} \frac{d\rho}{\rho} \lesssim \frac{\mu_k(\mathbb{R}^{N+1})}{r_j^N} \tag{4.20}$$

and if $k \le j-1$

$$\mathbb{I}_2^{2R_0}[\mu_k](x,t) = \int_{r_{k+1}}^{2R_0} \frac{\mu_k(\tilde{Q}_\rho(x,t))}{\rho^N} \frac{d\rho}{\rho} \lesssim \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N}. \tag{4.21}$$

Thus,

$$A_2 \lesssim \sum_{j=i_0}^{\infty} r_j^2 \left(\sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^q$$

and

$$A_3 \lesssim \sum_{j=i_0}^{\infty} r_j^{2-Nq} \left(\sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^q.$$

Noticing that $(a+b)^q - a^q \le q(a+b)^{q-1}b$ for any $a,b \ge 0$, we get

$$\begin{aligned}
(1-4^{-2}) &\sum_{j=i_0}^{\infty} r_j^2 \left(\sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^q \\
&= \sum_{j=i_0}^{\infty} r_j^2 \left(\sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^q - \sum_{j=i_0+1}^{\infty} r_j^2 \left(\sum_{k=i_0-1}^{j-2} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^q \\
&\le \sum_{j=i_0}^{\infty} q r_j^2 \left(\sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^{q-1} \frac{\mu_{j-1}(\mathbb{R}^{N+1})}{r_{j-1}^N}.
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
(1-4^{2-Nq}) &\sum_{j=i_0}^{\infty} r_j^{2-Nq} \left(\sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^q \\
&\le \sum_{j=i_0}^{\infty} q r_j^{2-Nq} \left(\sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^{q-1} \mu_{j+1}(\mathbb{R}^{N+1}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
A_2 + A_3 \lesssim &\sum_{j=i_0}^{\infty} r_j^2 \left(\sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^{q-1} \frac{\mu_{j-1}(\mathbb{R}^{N+1})}{r_{j-1}^N} \\
&+ \sum_{j=i_0}^{\infty} r_j^{2-Nq} \left(\sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^{q-1} \mu_{j+1}(\mathbb{R}^{N+1}).
\end{aligned}$$

Since $\mu_k(\mathbb{R}^{N+1}) \lesssim r_k^{N+2-2q'}$ if $q > q_*$ and $\mu_k(\mathbb{R}^{N+1}) \lesssim \min\{k^{-\frac{1}{q-1}}, 1\}$ if $q = q_*$ for any k, we infer that

$$r_j^2 \left(\sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N} \right)^{q-1} \lesssim 1$$

and

$$r_j^{2-Nq} \left(\sum_{k=j+1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \right)^{q-1} \lesssim r_{j+1}^{-N} \quad \text{for any} \quad j.$$

In the case $q = q_*$ we assume $N \geq 3$ in order to ensure that

$$\sum_{j=1}^{\infty} \mu_k(\mathbb{R}^{N+1}) \lesssim \sum_{k=1}^{\infty} k^{-\frac{1}{q-1}} < \infty.$$

This leads to

$$A_2 + A_3 \lesssim \sum_{k=1}^{\infty} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N}.$$

Combining this with (4.19) and (4.18), we deduce

$$A_0 \lesssim \sum_{k=1}^{\infty} \frac{\mathrm{Cap}_{2,1,q'}(M_k)}{r_k^N}.$$

Consequently, we obtain (4.15) from (4.17), for ε small enough.

Proof of assertion (4.16)**.** From (4.12) we get

$$B \gtrsim \varepsilon \sum_{k=1}^{\infty} \frac{\mathcal{PH}_1^N(M_k)}{r_k^N} - B_0,$$

where

$$B_0 = \mathbb{I}_2^{2R_0} \left[\exp\left(C_1 \mathbb{I}_2^{2R_0}[\varepsilon \sum_{k=1}^{\infty} \nu_k] \right) - 1 \right] (y_0, s_0).$$

We show that

$$B_0 \leq c(N, q, R_0) \quad \text{for } \varepsilon \text{ small enough.} \tag{4.22}$$

In fact, as above we have

$$B_0 \lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \exp\left(C_1 \varepsilon \mathbb{I}_2^{2R_0}[\sum_{k=1}^{\infty} \nu_k] \right) dxdt.$$

Consequently,

$$\begin{aligned} B_0 &\lesssim \sum_{j=i_0}^{\infty} r_j^{-N} \int_{S_j} \exp\left(3C_1 \varepsilon \mathbb{I}_2^{2R_0}[\nu_j]\right) dxdt \\ &+ \sum_{j=i_0}^{\infty} r_j^2 \exp\left(3C_1 \varepsilon \sum_{k=i_0-1}^{j-1} ||\mathbb{I}_2^{2R_0}[\nu_k]||_{L^\infty(S_j)} \right) \\ &+ \sum_{j=i_0}^{\infty} r_j^2 \exp\left(3C_1 \varepsilon \sum_{k=j+1}^{\infty} ||\mathbb{I}_2^{2R_0}[\nu_k]||_{L^\infty(S_j)} \right) \\ &= B_1 + B_2 + B_3. \end{aligned} \tag{4.23}$$

Here we have used the inequality $\exp(a+b+c) \le \exp(3a)+\exp(3b)+\exp(3c)$ for all a,b,c. By Theorem 2.3, we have

$$\int_{S_j} \exp\left(3C_1\varepsilon \mathbb{I}_2^{2R_0}[\nu_j]\right) dxdt \lesssim r_j^{N+2} \quad \text{for all } j,$$

for $\varepsilon > 0$ small enough. Hence,

$$B_1 \lesssim \sum_{j=i_0}^{\infty} r_j^2 \lesssim (\max\{2R_0, 1\})^2. \tag{4.24}$$

Note that estimates (4.20) and (4.21) are also true with ν_k; we deduce

$$\begin{aligned} B_2 + B_3 \lesssim & \sum_{j=i_0}^{\infty} r_j^2 \exp\left(c_2\varepsilon \sum_{k=i_0-1}^{j-1} \frac{\mu_k(\mathbb{R}^{N+1})}{r_k^N}\right) \\ & + \sum_{j=i_0}^{\infty} r_j^2 \exp\left(c_2\varepsilon \sum_{k=j+1}^{\infty} \frac{\mu_k(\mathbb{R}^{N+1})}{r_j^N}\right). \end{aligned}$$

From (4.12) we have $\mu_k(\mathbb{R}^{N+1}) \lesssim r_k^N$ for all k, therefore

$$\begin{aligned} B_2 + B_3 & \lesssim \sum_{j=i_0}^{\infty} r_j^2 \exp\left(c_3\varepsilon(j-i_0)\right) + \sum_{j=i_0}^{\infty} r_j^2 \exp\left(c_3\varepsilon\right) \\ & \lesssim \sum_{j=i_0}^{\infty} \exp\left(c_3\varepsilon(j-i_0) - 4\log(2)j\right) + r_{i_0}^2 \\ & \le c_4(N,q,R_0) \quad \text{for } \varepsilon \text{ small enough.} \end{aligned}$$

Combining this with (4.24) and (4.23) we obtain (4.22).
This implies straightforwardly $\exp\left(C_1\varepsilon\mathbb{I}_2^{2R_0}[\sum_{k=1}^{\infty}\nu_k]\right) \in L^1(\tilde{Q}_{R_0}(0,0))$.
We conclude that for any $(y_0,s_0) \in (B_\delta(0) \times (-\delta^2,\delta^2)) \cap O$,

$$u_1(y_0,s_0) \gtrsim \sum_{k=1}^{\infty} \frac{\mathrm{Cap}_{2,1q'}\left(M_k(y_0,s_0)\right)}{r_k^N}$$

and

$$u_2(y_0,s_0) \gtrsim -c_1(R_0) + \sum_{k=1}^{\infty} \frac{\mathcal{PH}_1^N\left(M_k(y_0,s_0)\right)}{r_k^N},$$

where $r_k = 4^{-k}$ and

$$M_k(y_0,s_0) = O^c \cap \left(\overline{B_{r_{k+2}}(y_0)} \times [s_0 - (73+\frac{1}{2})r_{k+2}^2, s_0 - (70+\frac{1}{2})r_{k+2}^2]\right).$$

Take $r_{k_\delta+4} \le \delta < r_{k_\delta+3}$, we have for $1 \le k \le k_\delta$

$$\begin{aligned} M_k(y_0,s_0) & \supset O^c \cap \left(B_{r_{k+2}-\delta}(0) \times \left(\delta^2 - (73+\frac{1}{2})r_{k+2}^2, -\delta^2 - (70+\frac{1}{2})r_{k+2}^2\right)\right) \\ & \supset O^c \cap \left(B_{r_{k+3}}(0) \times \left(-73r_{k+2}^2, -71r_{k+2}^2\right)\right) \\ & = O^c \cap \left(B_{r_{k+3}}(0) \times \left(-1168r_{k+3}^2, -1136r_{k+3}^2\right)\right). \end{aligned}$$

Finally

$$\begin{aligned} & \inf_{(y_0,s_0)\in(B_\delta(0)\times(-\delta^2,\delta^2))\cap O} u_1(y_0,s_0) \\ & \gtrsim \sum_{k=4}^{k_\delta+3} \frac{\mathrm{Cap}_{2,1,q'}\left(O^c \cap \left(B_{r_k}(0) \times \left(-1168r_k^2, -1136r_k^2\right)\right)\right)}{r_k^N} \to \infty \quad \text{as} \quad \delta \to 0, \end{aligned}$$

and

$$\inf_{(y_0,s_0)\in(B_\delta(0)\times(-\delta^2,\delta^2))\cap O} u_2(y_0,s_0) \gtrsim -c_1(R_0)$$
$$+\sum_{k=4}^{k_\delta+3} \frac{\mathcal{PH}_1^N\left(O^c\cap\left(B_{r_k}(0)\times\left(-1168r_k^2,-1136r_k^2\right)\right)\right)}{r_k^N} \to\infty \quad \text{as} \quad \delta\to 0.$$

This completes the proof of Theorem 1.1-(i) and Theorem 1.2.

4.3 The viscous Hamilton-Jacobi parabolic equations

In this section we apply our previous result to the question of existence of a large solution of the following type of parabolic viscous Hamilton-Jacobi equation

$$\begin{aligned} \partial_t u-\Delta u+a|\nabla u|^p+bu^q&=0 && \text{in } O,\\ u&=\infty && \text{on } \partial_p O, \end{aligned} \tag{4.25}$$

where $a>0, b>0$ and $1<p\le 2$, $q\ge 1$. First, we show that such a large solution to (4.25) does not exist when $q=1$. Equivalently namely, for $a>0$, $b>0$ and $p>1$ there exists no function $u\in C^{2,1}(O)$ satisfying

$$\begin{aligned} \partial_t u-\Delta u+a|\nabla u|^p&\ge -bu && \text{in } O,\\ u&=\infty && \text{on } \partial_p O. \end{aligned} \tag{4.26}$$

Indeed, assuming that such a function $u\in C^{2,1}(O)$, exists, we define

$$U(x,t)=u(x,t)e^{bt}-\frac{\varepsilon}{2}|x|^2,$$

for $\varepsilon>0$ and denote by $(x_0,t_0)\in O\backslash\partial_p O$ the point where U achieves it minimum in O, i.e. $U(x_0,t_0)=\inf\{U(x,t):(x,t)\in O\}$. Clearly, we have

$$\partial_t U(x_0,t_0)\le 0,\quad \Delta U(x_0,t_0)\ge 0 \quad \text{and} \quad \nabla U(x_0,t_0)=0.$$

Thus,

$$\partial_t u(x_0,t_0)\le -bu(x_0,t_0),\quad -\Delta u(x_0,t_0)\le -\varepsilon N e^{-bt_0} \ \text{ and } \ a|\nabla u(x_0,t_0)|^p=a\varepsilon^p|x_0|^p e^{-pbt_0},$$

from which follows

$$\begin{aligned} \partial_t u(x_0,t_0)-\Delta u(x_0,t_0)+a|\nabla u(x_0,t_0)|^p &\le -bu(x_0,t_0)+\varepsilon e^{-bt_0}\left(-N+a\varepsilon^{p-1}|x_0|^p e^{-(p-1)bt_0}\right)\\ &< -bu(x_0,t_0) \end{aligned}$$

for ε small enough, which is a contradiction.

Proof of Theorem 1.3. By Remark 3.3, we have

$$\inf\{v(x,t);(x,t)\in O\}\ge (q_1-1)^{-\frac{1}{q_1-1}}R^{-\frac{2}{q_1-1}}.$$

Take $V=\lambda v^{\frac{1}{\alpha}}\in C^{2,1}(O)$ for $\lambda>0$. Thus $v=\lambda^{-\alpha}V^\alpha$,

$$\inf\{V(x,t);(x,t)\in O\}>0\}\ge \lambda(q_1-1)^{-\frac{1}{\alpha(q_1-1)}}R^{-\frac{2}{\alpha(q_1-1)}},$$

and

$$\partial_t v-\Delta v+v^{q_1}=\alpha\lambda^{-\alpha}V^{\alpha-1}\partial_t V-\alpha\lambda^{-\alpha}V^{\alpha-1}\Delta V+\alpha(1-\alpha)\lambda^{-\alpha}V^{\alpha-1}\frac{|\nabla V|^2}{V}+\lambda^{-\alpha q_1}V^{\alpha q_1}.$$

This leads to

$$\partial_t V - \Delta V + (1-\alpha)\frac{|\nabla V|^2}{V} + \alpha^{-1}\lambda^{-\alpha(q_1-1)}V^{\alpha q_1-\alpha+1} = 0 \quad \text{in} \;\; O.$$

Using Hölder's inequality,

$$\begin{aligned}(1-\alpha)\frac{|\nabla V|^2}{V} + (2\alpha)^{-1}\lambda^{-\alpha(q_1-1)}V^{\alpha q_1-\alpha+1} &\geq c_1|\nabla V|^p\lambda^{-\frac{\alpha(q_1-1)(2-p)}{2}}V^{\frac{\alpha(q_1-1)(2-p)}{2}-(p-1)} \\ &\geq c_2|\nabla V|^p\lambda^{-(p-1)}R^{-2+p+\frac{2(p-1)}{\alpha(q_1-1)}}\end{aligned}$$

and

$$(2\alpha)^{-1}\lambda^{-\alpha(q_1-1)}V^{\alpha q_1-\alpha+1} \geq c_3\lambda^{-(q-1)}R^{-2+\frac{2(q-1)}{\alpha(q_1-1)}}V^q.$$

If we choose

$$\lambda = \min\{c_2^{\frac{1}{p-1}}, c_3^{\frac{1}{q-1}}\}\min\left\{a^{-\frac{1}{p-1}}R^{-\frac{2-p}{p-1}+\frac{2}{\alpha(q_1-1)}}, b^{-\frac{1}{q-1}}R^{-\frac{2}{q-1}+\frac{2}{\alpha(q_1-1)}}\right\}$$

then

$$\begin{aligned}c_2\lambda^{-(p-1)}R^{-2+p+\frac{2(p-1)}{\alpha(q_1-1)}} &\geq a, \\ c_3\lambda^{-(q-1)}R^{-2+\frac{2(q-1)}{\alpha(q_1-1)}} &\geq b,\end{aligned}$$

from what follows

$$\partial_t V - \Delta V + a|\nabla V|^p + bV^q \leq 0 \quad \text{in} \;\; O.$$

By Remark 3.5, there exists a maximal solution $u \in C^{2,1}(O)$ of

$$\partial_t u - \Delta u + a|\nabla u|^p + bu^q = 0 \quad \text{in} \;\; O.$$

Therefore, $u \geq V = \lambda v^{\frac{1}{\alpha}}$ and u is a large solution of (4.25). This completes the proof of Theorem 1.3. ■

5 Appendix

Proof of Proposition 2.5.

Step 1. We claim that the following relation holds:

$$\int_{\mathbb{R}^{N+1}} \left(\mathbb{I}_2^1[\mu](x,t)\right)^{(N+2)/N} dxdt \asymp \int_{\mathbb{R}^{N+1}}\int_0^1 (\mu(\tilde{Q}_r(x,t)))^{2/N}\frac{dr}{r}d\mu(x,t). \tag{5.1}$$

In fact, we have for $\rho_j = 2^{-j}$, $j \in \mathbb{Z}$,

$$\begin{aligned}\sum_{j=1}^{\infty}\int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x,t)))^{2/N}d\mu(x,t) &\lesssim \int_{\mathbb{R}^{N+1}}\int_0^1 (\mu(\tilde{Q}_r(x,t)))^{2/N}\frac{dr}{r}d\mu(x,t) \\ &\lesssim \sum_{j=0}^{\infty}\int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x,t)))^{2/N}d\mu(x,t).\end{aligned}$$

Note that for any $j \in \mathbb{Z}$

$$\begin{aligned}\rho_j^{-N-2}\int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_{j+1}}(x,t)))^{(N+2)/N}dxdt &\lesssim \int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_j}(x,t)))^{2/N}d\mu(x,t) \\ &\lesssim \rho_j^{-N-2}\int_{\mathbb{R}^{N+1}} (\mu(\tilde{Q}_{\rho_{j-1}}(x,t)))^{(N+2)/N}dxdt.\end{aligned}$$

Thus,

$$\sum_{j=2}^{\infty}\rho_j^{-N}\int_{\mathbb{R}^{N+1}}(\mu(\tilde{Q}_{\rho_j}(x,t)))^{(N+2)/N}dxdt \lesssim \int_{\mathbb{R}^{N+1}}\int_0^1(\mu(\tilde{Q}_r(x,t)))^{2/N}\frac{dr}{r}d\mu(x,t)$$
$$\lesssim \sum_{j=-1}^{\infty}\rho_j^{-N}\int_{\mathbb{R}^{N+1}}(\mu(\tilde{Q}_{\rho_j}(x,t)))^{(N+2)/N}dxdt.$$

This yields

$$\int_{\mathbb{R}^{N+1}}\left(\mathbb{M}_2^{1/4}[\mu](x,t)\right)^{(N+2)/N}dxdt \lesssim \int_{\mathbb{R}^{N+1}}\int_0^1(\mu(\tilde{Q}_r(x,t)))^{2/N}\frac{dr}{r}d\mu(x,t)$$
$$\lesssim \int_{\mathbb{R}^{N+1}}\left(\mathbb{I}_2^4[\mu](x,t)\right)^{(N+2)/N}dxdt.$$

By [20, Theorem 4.2],

$$\int_{\mathbb{R}^{N+1}}\left(\mathbb{M}_2^{1/4}[\mu](x,t)\right)^{(N+2)/N}dxdt \asymp \int_{\mathbb{R}^{N+1}}\left(\mathbb{I}_2^4[\mu](x,t)\right)^{(N+2)/N}dxdt,$$

thus we obtain (5.1).

Step 2. End of the proof. The first inequality in (2.1) is proved in [20]. We now prove the second inequality. By Theorem 2.4 there is $\mu \in \mathfrak{M}^+(\mathbb{R}^{N+1}), \mathrm{supp}(\mu) \subset K$ such that

$$||\mathbb{M}_2^2[\mu]||_{L^\infty(\mathbb{R}^{N+1})} \leq 1 \text{ and } \mu(K) \asymp \mathcal{PH}_2^N(K) \gtrsim |K|^{N/(N+2)}. \tag{5.2}$$

Thanks to (5.1), we have for $\delta = \min\{1, (\mu(K))^{1/N}\}$

$$||\mathbb{I}_2^1[\mu]||_{L^{(N+2)/N}(\mathbb{R}^{N+1})}^{(N+2)/N} \asymp \int_{\mathbb{R}^{N+1}}\int_0^1(\mu(\tilde{Q}_r(x,t)))^{2/N}\frac{dr}{r}d\mu(x,t)$$
$$\asymp \int_{\mathbb{R}^{N+1}}\left(\int_0^\delta + \int_\delta^1\right)(\mu(\tilde{Q}_r(x,t)))^{2/N}\frac{dr}{r}d\mu(x,t)$$
$$\lesssim \int_0^\delta r^2\frac{dr}{r}\int_{\mathbb{R}^{N+1}}d\mu(x,t) + \int_\delta^1\frac{dr}{r}\left(\int_{\mathbb{R}^{N+1}}d\mu(x,t)\right)^{(N+2)/N}$$
$$\lesssim (\mu(K))^{(N+2)/N}\left(1+\log_+\left((\mu(K))^{-1}\right)\right)$$
$$\lesssim (\mu(K))^{(N+2)/N}\log\left(\frac{|\tilde{Q}_{200}(0,0)|}{|K|}\right).$$

Set $\tilde{\mu} = \left(\log\left(\frac{|\tilde{Q}_{200}(0,0)|}{|K|}\right)\right)^{-N/(N+2)}\mu/\mu(K)$, then $||\mathbb{I}_2^1[\tilde{\mu}]||_{L^{(N+2)/N}(\mathbb{R}^{N+1})} \lesssim 1$.
It is well known that

$$\mathrm{Cap}_{2,1,\frac{N+2}{2}}(K) \asymp \sup\{(\omega(K))^{(N+2)/2} : \omega \in \mathfrak{M}^+(K), ||\mathbb{I}_2^1[\omega]||_{L^{(N+2)/N}(\mathbb{R}^{N+1})} \lesssim 1\} \tag{5.3}$$

see [20, Section 4]. This gives the second inequality in (2.1).
It is easy to prove (2.2) from its definition. Moreover, (5.3) implies that

$$\frac{1}{\mathrm{Cap}_{2,1,\frac{N+2}{2}}(K)^{2/N}} \asymp \inf\{||\mathbb{I}_2^1[\omega]||_{L^{(N+2)/N}(\mathbb{R}^{N+1})}^{(N+2)/N} : \omega \in \mathfrak{M}^+(K), \omega(K) = 1\}.$$

We deduce from (5.1) that

$$\frac{1}{\mathrm{Cap}_{2,1,\frac{N+2}{2}}(K)^{2/N}} \asymp \inf\left\{\int_{\mathbb{R}^{N+1}}\int_0^1(\omega(\tilde{Q}_r(x,t)))^{2/N}\frac{dr}{r}d\mu(x,t) : \omega \in \mathfrak{M}^+(K), \omega(K) = 1\right\}. \tag{5.4}$$

As in [12, proof of Lemma 2.2], it is easy to derive (2.3) from (5.4). ■

Proof of Proposition 2.6. Thanks to the Poincaré inequality, it is enough to show that there exists $\varphi \in C_c^\infty(\tilde{Q}_{3/2}(0,0))$ such that $0 \le \varphi \le 1$, with $\varphi = 1$ in an open neighborhood of K and

$$\int_{\mathbb{R}^{N+1}} (|D^2\varphi|^p + |\partial_t\varphi|^p) dxdt \lesssim \mathrm{Cap}_{2,1,p}(K). \tag{5.5}$$

By definition, one can find $0 \le \phi \in S(\mathbb{R}^{N+1})$, $\phi \ge 1$ in a neighborhood of K such that

$$\int_{\mathbb{R}^{N+1}} (|D^2\phi|^p + |\nabla\phi|^p + |\phi|^p + |\partial_t\phi|^p) dxdt \le 2\mathrm{Cap}_{2,1,p}(K).$$

Let η be a cut off function on $\tilde{Q}_1(0,0)$ with respect to $\tilde{Q}_{3/2}(0,0)$ and $H \in C^\infty(\mathbb{R})$ such that

$0 \le H(t) \le t^+$, $|t||H''(t)| \lesssim 1$ for all $t \in \mathbb{R}$, $H(t) = 0$ for $t \le 1/4$ and $H(t) = 1$ for $t \ge 3/4$.

We claim that

$$\int_{\mathbb{R}^{N+1}} (|D^2\varphi|^p + |\partial_t\varphi|^p) dxdt \lesssim \int_{\mathbb{R}^{N+1}} (|D^2\phi|^p + |\nabla\phi|^p + |\phi|^p + |\partial_t\phi|^p) dxdt, \tag{5.6}$$

where $\varphi = \eta H(\phi)$. Indeed, we have

$$|D^2\varphi| \lesssim |D^2\eta| H(\phi) + |\nabla\eta||H'(\phi)||\nabla\phi| + \eta|H''(\phi)||\nabla\phi|^2 + \eta|H'(\phi)||D^2\phi|$$

and

$$|\partial_t\varphi| \lesssim |\partial_t\eta| H(\phi) + \eta|H'(\phi)||\phi_t|, \quad H(\phi) \le \phi, \quad \phi|H''(\phi)| \lesssim 1.$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}^{N+1}} (|D^2\varphi|^p + |\partial_t\varphi|^p) dxdt &\lesssim \int_{\mathbb{R}^{N+1}} (|D^2\phi|^p + |\nabla\phi|^p + |\phi|^p + |\partial_t\phi|^p) dxdt \\ &+ \int_{\mathbb{R}^{N+1}} \frac{|\nabla\phi|^{2p}}{\phi^p} dxdt.\end{aligned}$$

This implies (5.6) since, according to [1], one has

$$\int_{\mathbb{R}^N} \frac{|\nabla\phi(t)|^{2p}}{\phi(t)^p} dx \lesssim \int_{\mathbb{R}^N} |D^2\phi(t)|^p dx \quad \forall t \in \mathbb{R}.$$

■

References

[1] D. R. Adams. *On the existence of capacitary strong type estimates in* $\mathbb{R}^N$, Ark. Mat. **14**, 125-140 (1976).

[2] R.J. Bagby. *Lebesgue spaces of parabolic potentials*, Ill. J. Math. **15**, 610-634 (1971).

[3] P. Baras and M. Pierre. *Problèmes paraboliques semi-linéaires avec données mesures*, Applicable Anal. **18**, 111-149 (1984).

[4] P. Baras, M. Pierre, *Critère d'existence des solutions positives pour des équations semi-linéaires non monotones*, Ann. Inst. H. Poincaré, Anal. Non Lin. **3**, (1985), 185-212.

[5] J. S. Dhersin and J. F. Le Gall. *Wieners test for super-Brownian motion and the Brownian snake*, Probab. Theory Relat. Fields **108**, 103-29 (1997).

[6] E. DiBenedetto. *Degenerate parabolic equations.* Universitext. Springer-Verlag, New York, (1993).

[7] E. B. Dynkin, S. E. Kuznetsov, *Fine topology and fine trace on the boundary associated with a class of semilinear differential equations.* Comm. Pure Appl. Math. **51**, 897-936 (1998).

[8] E. B. Dynkin, S. E. Kuznetsov, *Superdiffusions and removable singularities for quasilinear partial differential equations.* Comm. Pure Appl. Math. **49**, 125-176 (1996).

[9] L. C. Evans, R. F. Gariepy, *Wiener criterion for the heat equation.* Arch. Rat. Mech. Anal. **78**, 293-314 (1982).

[10] N. Garofalo and E. Lanconelli, *Wiener's criterion for parabolic equations with variable coefficients and its consequences*, Trans. Amer. Math. Soc. **308**, No.2, 811-836 (1988).

[11] R. F. Gariepy and W.P. Ziemer, *Thermal capacity and boundary regularity*, J. Diff. Eq. **45**, 374-388 (1982).

[12] D. Labutin. *Wiener regularity for large solutions of nonlinear equations*, Archiv for Math. **41**, 307-339 (2003).

[13] J.-F. LeGall, *Branching processes, random trees and superprocesses.* Proceedings of the International Congress of Mathematicians (Berlin, 1998), Doc. Math., Extra Vol. III 279-289 (1998).

[14] J.-F. LeGall, *Spatial branching processes, random snakes and partial differential equations.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1999.

[15] G.M. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific press, River Edge (1996).

[16] M. Marcus and L. Véron, *Nonlinear second order elliptic equations involving measures*, Series in Nonlinear Analysis and Applications **21**, De Gruyter, Berlin/Boston (2013).

[17] M. Marcus and L. Véron. *Capacitary estimates of solutions of semilinear parabolic equations*, Calc. Var. & Part. Diff. Equ. **48**, 131-183 (2013).

[18] T. Nguyen-Phuoc *Parabolic equations with exponential nonlinearity and measure data*, J. Diff. Equ. (to appear), arXiv:1312.2509.

[19] H. Nguyen-Quoc and L. Véron., *Wiener criteria for existence of large solutions of quasilinear elliptic equations with absorption*, arXiv:1308.2956v3.

[20] H. Nguyen-Quoc, *Potential estimates and quasilinear equations with measure data* arXiv:1405.2587v1.

[21] B. O. Tureson. *Nonlinear Potential Theory and weighted Sobolev Spaces*, Lecture Notes in Mathematics, Vol. 1736, Springer-Verlag (2000).

[22] L. Véron. *A note on maximal solutions of nonlinear parabolic equations with absorption*, Asymptotic Anal. **72**, 189-200 (2011).

[23] W.P. Ziemer, *Behavior at the boundary of solution of quasilinear parabolic equations*, J. Diff. Eq. **35**, 291-305 (1980).