A MAXIMAL FUNCTION CHARACTERIZATION OF ABSOLUTELY CONTINUOUS MEASURES AND SOBOLEV FUNCTIONS

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Abstract. In this note, we give a new characterization of Sobolev $W^{1,1}$ functions among BV functions via Hardy–Littlewood maximal function. Exploiting some ideas coming from the proof of this result, we are also able to give a new characterization of absolutely continuous measures via a weakened version of Hardy–Littlewood maximal function. Finally, we show that the approach adopted in [3,8] to establish existence and uniqueness of regular Lagrangian flows associated to Sobolev vector fields cannot be further extended to the case of BV vector fields.

1. Introduction

Let μ be a Borel measure in $\mathbb{R}^d$. We let

$$\mathsf{M}\mu(x) := \sup_{r>0} \frac{|\mu|(B(x,r))}{\mathscr{L}^d(B(x,r))} \in [0,+\infty] \tag{1.1}$$

be the *(Hardy–Littlewood) maximal function of* μ at $x \in \mathbb{R}^d$, see [9, Chapter 1]. If the measure μ is absolutely continuous with respect to the Lebesgue measure with density $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, then we can rewrite (1.1) as

$$\mathsf{M}f(x) := \sup_{r>0} \fint_{B(x,r)} |f(y)| \, dy \in [0,+\infty] \tag{1.2}$$

for all $x \in \mathbb{R}^d$. It is well known that if $f \in L^p(\mathbb{R}^d)$ for some $1 < p \le +\infty$, then the maximal function in (1.2) satisfies the following *strong* (p,p)*-type estimate*

$$\|\mathsf{M}f\|_{L^p(\mathbb{R}^d)} \lesssim_d \|f\|_{L^p(\mathbb{R}^d)}. \tag{1.3}$$

Here and in the following, given two quantities A and B, we write $A \lesssim_d B$ (resp. $A \gtrsim_d B$) if there exists a dimensional constant $C > 0$ such that $A \le CB$ (resp. $A \ge CB$). If $f \in L^1(\mathbb{R}^d)$, then the maximal function in (1.2) satisfies the following *weak* $(1,1)$*-type estimate*

$$\sup_{\lambda>0} \lambda \, \mathscr{L}^d(\{x \in \mathbb{R}^d : \mathsf{M}f(x) > \lambda\}) \lesssim_d \|f\|_{L^1(\mathbb{R}^d)}. \tag{1.4}$$

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For a proof of inequalities (1.3) and (1.4), we refer the interested reader to [9, Theorem 1]. Actually, if $f \in L^1(\mathbb{R}^d)$, writing $f = f\chi_{\{|f|>\frac{\lambda}{2}\}} + f\chi_{\{|f|\le\frac{\lambda}{2}\}}$ and combining (1.3) and (1.4), we can improve (1.4) as

$$\lambda\,\mathscr{L}^d(\{x \in \mathbb{R}^d : \mathsf{M}f(x) > \lambda\}) \lesssim_d \int_{|f|>\frac{\lambda}{2}} |f(y)|\,dy,$$

so that

$$\limsup_{\lambda\to+\infty} \lambda\,\mathscr{L}^d(\{x \in \mathbb{R}^d : \mathsf{M}f(x) > \lambda\}) = 0.$$

With a similar reasoning, we also get that

$$\limsup_{\lambda\to+\infty} \lambda\,\mathscr{L}^d(\{x \in \mathbb{R}^d : \mathsf{M}\mu(x) > \lambda\} \lesssim_d |\mu^s|(\mathbb{R}^d), \tag{1.5}$$

for all finite Borel measures μ (see [7, Section 2] for more details). Here and in the following, μ^s denotes the *singular part* of the measure μ with respect to the Lebesgue measure in $\mathbb{R}^d$.

As remarked in [6, Problem 3.5], it is also possible to establish a reverse version of inequality (1.5), see Proposition 1.1 below.

Proposition 1.1. *Let μ be a finite Borel measure in $\mathbb{R}^d$. Then*

$$\inf_{\lambda>0} \lambda\,\mathscr{L}^d(\{x \in \mathbb{R}^d : \mathsf{M}\mu(x) > \lambda\}) \gtrsim_d |\mu^s|(\mathbb{R}^d). \tag{1.6}$$

More in general, given a cube $Q \subset \mathbb{R}^d$, it holds

$$\inf\left\{\lambda\,\mathscr{L}^d(\{x \in Q : \mathsf{M}\mu(x) > \lambda\}) : \lambda > \tfrac{|\mu|(Q)}{\mathscr{L}^d(Q)}\right\} \gtrsim_d |\mu^s|(Q). \tag{1.7}$$

For the reader's convenience, we give a proof of Proposition 1.1 in Appendix A.

Combining inequalities (1.5) and (1.6), we immediately deduce the following characterization of absolutely continuous measures in $\mathbb{R}^d$.

Corollary 1.2. *Let μ be a finite Borel measure in $\mathbb{R}^d$. Then $\mu \ll \mathscr{L}^d$ if and only if*

$$\lim_{\lambda\to+\infty} \lambda\,\mathscr{L}^d(\{x \in \mathbb{R}^d : \mathsf{M}\mu(x) > \lambda\}) = 0. \tag{1.8}$$

Inspired by Corollary 1.2, we give a new characterization of Sobolev $W^{1,1}$ among BV functions in term of the behaviour of a suitable maximal function. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, we define

$$\mathsf{A}f(x) := \sup_{r>0} \frac{1}{r} \fint_{B(x,r)} |f - (f)_{x,r}|\,dy \in [0, +\infty] \tag{1.9}$$

for all $x \in \mathbb{R}^d$, where $(f)_{x,r} = \fint_{B(x,r)} f\,dy$. Note that, by Poincaré's inequality and by inequality (1.5) applied to the measure $\mu = Df$, we have that

$$\limsup_{\lambda\to+\infty} \lambda\,\mathscr{L}^d\big(\{x \in \mathbb{R}^d : \mathsf{A}f(x) > \lambda\}\big) \lesssim_d |D^s f|(\mathbb{R}^d).$$

Our main result is the following, see Section 3 for the proof.

Theorem 1.3. *Let $f \in BV(\mathbb{R}^d)$. Then*

$$\liminf_{\lambda\to+\infty} \lambda\,\mathscr{L}^d\big(\{x \in \mathbb{R}^d : \mathsf{A}f(x) > \lambda\}\big) \gtrsim_d |D^s f|(\mathbb{R}^d).$$

In particular, $f \in W^{1,1}(\mathbb{R}^d)$ *if and only if*

$$\liminf_{\lambda\to+\infty} \lambda\,\mathscr{L}^d\big(\big\{x \in \mathbb{R}^d : \mathsf{A}f(x) > \lambda\big\}\big) = 0.$$

Exploiting some ideas coming from the proof of the aforementioned Theorem 1.3, we are able to improve Proposition 1.1 as follows. For a finite Borel measure μ on $\mathbb{R}^d$ (possibly with sign), we define

$$\overline{\mathsf{M}}\mu(x) := \sup_{r>0} \frac{|\mu(B(x,r))|}{\mathscr{L}^d(B(x,r))} \in [0,+\infty] \tag{1.10}$$

for all $x \in \mathbb{R}^d$. Note that the maximal function defined in (1.10) is weaker than the one recalled in (1.1), in the sense that $\overline{\mathsf{M}}\mu(x) \le \mathsf{M}\mu(x)$ for all $x \in \mathbb{R}^d$. Then the following result holds, see Section 2 for the proof.

Theorem 1.4. *Let* μ *be a finite Borel measure on* $\mathbb{R}^d$ *(possibly with sign). Then*

$$\liminf_{\lambda\to+\infty} \lambda\,\mathscr{L}^d(\{x \in \mathbb{R}^d : \overline{\mathsf{M}}\mu(x) > \lambda\}) \gtrsim_d |\mu^s|(\mathbb{R}^d). \tag{1.11}$$

Hence Corollary 1.2 still holds for any (possibly signed) finite Borel measure μ in $\mathbb{R}^d$ with (1.8) replaced by

$$\liminf_{\lambda\to+\infty} \lambda\,\mathscr{L}^d(\big\{x \in \mathbb{R}^d : \overline{\mathsf{M}}\mu(x) > \lambda\big\}) = 0.$$

The last goal of this note — which was our starting motivation for the study of inequality (1.6) — comes from the theory of ordinary differential equations (ODEs) with weakly differentiable vector fields.

The study of this theory was started by DiPerna and Lions in their seminal paper [4], in which they proved existence and uniqueness of solutions of ODEs with Sobolev vector fields. The extension of the results obtained in [4] to vector fields with BV regularity was established by Ambrosio in the groundbreaking paper [1], where the notion of *regular Lagrangian flow* was introduced as a generalization of the classical definition of flow (see [3, Definition 1.1] for a precise definition). More in detail, the main result of [1] reads as follows. For a time $T \in (0,+\infty]$, let $b\colon (0,T)\times\mathbb{R}^d \to \mathbb{R}^d$ be a bounded time-dependent vector field such that

$$b \in L^1((0,T); BV(\mathbb{R}^d;\mathbb{R}^d)), \quad \operatorname{div} b \in L^1((0,T); L^\infty(\mathbb{R}^d;\mathbb{R}^d)). \tag{1.12}$$

Consider the associated Cauchy problem

$$\begin{cases} \dfrac{dX_t}{dt}(x) = b_t(X_t(x)), & (t,x) \in (0,T)\times\mathbb{R}^d, \\ X_0(x) = x \in \mathbb{R}^d. \end{cases} \tag{1.13}$$

Then there exists a unique regular Lagrangian flow $X\colon [0,T)\times\mathbb{R}^d \to \mathbb{R}^d$ solving (1.13).

Both the results by DiPerna–Lions and Ambrosio rely on the so-called *Eulerian approach*, meaning that the problem (1.13) is studied indirectly via the closely linked *transport equation*. A more direct approach, the so-called *Lagrangian approach*, was proposed by Crippa and De Lellis in [3], where simple *a priori* estimates were exploited in order to get existence, uniqueness, compactness and even mild regularity properties of the regular Lagrangian flow associated to a Sobolev $W^{1,p}$ vector field for every $p > 1$. Their approach has been extended to the case $p = 1$ by Jabin in [8], where it was observed that if the quantity

$$\mathcal{Q}(B;\delta) := |\log\delta|^{-1}\int_0^T\int_B \min\big\{\delta^{-1}, \mathsf{M}|Db_t|(x)\big\}\ dx\,dt \tag{1.14}$$

satisfies the decay property

$$\limsup_{\delta\to 0^+}\mathcal{Q}(B;\delta) = 0 \tag{1.15}$$

for all balls $B \subset \mathbb{R}^d$, then there exists a unique regular Lagrangian flow associated to b (for a more detailed exposition of these results, see [7, Section 1]).

Using Proposition 1.1, we can prove that (1.15) holds true if and only if

$$b \in L^1((0,T); W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)),$$

so that the approach by Crippa–De Lellis and Jabin cannot be further extended to the case of BV vector fields. Our result reads as follows, see Section 4 for the proof.

Proposition 1.5. *Let $b\colon (0,T)\times\mathbb{R}^d \to \mathbb{R}^d$ be a vector field satisfying* (1.12). *Then*

$$\int_0^T |D^s b_t|(B)\ dt \lesssim_d \liminf_{\delta\to 0^+}\mathcal{Q}(B;\delta) \le \limsup_{\delta\to 0^+}\mathcal{Q}(B;\delta) \lesssim_d \int_0^T |D^s b_t|(\overline{B})\ dt \tag{1.16}$$

for all balls $B \subset \mathbb{R}^d$, where $\mathcal{Q}(B;\delta)$ is as in (1.14). *In particular, the decay property* (1.15) *is satisfied if and only if $b \in L^1((0,T); W^{1,1}_{loc}(\mathbb{R}^d;\mathbb{R}^d))$.*

2. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. Write $\mu = \eta|\mu|$, where $\eta\colon \mathbb{R}^d \to \mathbb{R}$ satisfies $|\eta(x)| = 1$ for $|\mu|$-a.e. $x \in \mathbb{R}^d$. For each $\varepsilon > 0$ let $\eta_\varepsilon \in \mathrm{Lip}(\mathbb{R}^d)$ be a Lipschitz function with Lipschitz constant $C_\varepsilon > 0$ such that

$$\int_{\mathbb{R}^d} |\eta(x) - \eta_\varepsilon(x)|\ d|\mu|(x) < \varepsilon. \tag{2.1}$$

We claim that

$$\frac{1}{\omega_d r^d}\left|\int_{B(x,r)} \eta_\varepsilon\ d|\mu|\right| + 2C_\varepsilon r\mathsf{M}|\mu|(x) \ge \frac{1}{\omega_d r^d}\int_{B(x,r)} |\eta_\varepsilon|\ d|\mu|, \tag{2.2}$$

for all $x \in \mathbb{R}^d$ and all $r > 0$. Indeed, we can estimate

$$\begin{aligned}
\left|\int_{B(x,r)} \eta_\varepsilon(y)\ d|\mu|(y)\right| &\ge \left|\int_{B(x,r)} \eta_\varepsilon(x)\ d|\mu|(y)\right| - \int_{B(x,r)} |\eta_\varepsilon(y) - \eta_\varepsilon(x)|\ d|\mu|(y)\\
&\ge \int_{B(x,r)} |\eta_\varepsilon(x)|\ d|\mu|(y) - C_\varepsilon r\int_{B(x,r)} d|\mu|(y)\\
&\ge \int_{B(x,r)} |\eta_\varepsilon(y)|\ d|\mu|(y) - \int_{B(x,r)} \big||\eta_\varepsilon(y)| - |\eta_\varepsilon(x)|\big|\ d|\mu|(y) - C_\varepsilon r\int_{B(x,r)} d|\mu|(y)\\
&\ge \int_{B(x,r)} |\eta_\varepsilon(y)|\ d|\mu|(y) - 2C_\varepsilon r\int_{B(x,r)} d|\mu|(y)
\end{aligned}$$

for all $r > 0$, from which (2.2) follows.

Let us now consider the measures $\nu_\varepsilon := |\eta - \eta_\varepsilon|\,|\mu|$ for $\varepsilon > 0$. We claim that

$$\frac{1}{\omega_d r^d}\left|\int_{B(x,r)} \eta\, d|\mu|\right| + 2C_\varepsilon r\mathsf{M}\mu(x) + 2\mathsf{M}\nu_\varepsilon(x) \geq \frac{1}{\omega_d r^d}\int_{B(x,r)} |\eta|\, d|\mu| \tag{2.3}$$

for all $x \in \mathbb{R}^d$ and all $r > 0$. Indeed, assuming $\mathsf{M}\mu(x) < +\infty$ and $\mathsf{M}\nu_\varepsilon(x) < +\infty$ without loss of generality, by (2.2) we can estimate

$$\begin{aligned}\frac{1}{\omega_d r^d}\left|\int_{B(x,r)} \eta\, d|\mu|\right| &\geq \frac{1}{\omega_d r^d}\left|\int_{B(x,r)} \eta_\varepsilon\, d|\mu|\right| - \mathsf{M}\nu_\varepsilon(x)\\ &\geq \frac{1}{\omega_d r^d}\int_{B(x,r)} |\eta_\varepsilon|\, d|\mu| - 2C_\varepsilon r\mathsf{M}\mu(x) - \mathsf{M}\nu_\varepsilon(x)\\ &\geq \frac{1}{\omega_d r^d}\int_{B(x,r)} |\eta|\, d|\mu| - 2C_\varepsilon r\mathsf{M}\mu(x) - 2\mathsf{M}\nu_\varepsilon(x).\end{aligned}$$

For $\tau > 0$ define

$$\mathsf{M}^\tau\mu(x) := \sup_{0<r<\tau} \frac{|\mu|(B(x,r))}{\mathscr{L}^d(B(x,r))} \in [0,+\infty], \tag{2.4}$$

and similarly

$$\overline{\mathsf{M}}^\tau\mu(x) := \sup_{0<r<\tau} \frac{|\mu(B(x,r))|}{\mathscr{L}^d(B(x,r))} \in [0,+\infty],$$

for all $x \in \mathbb{R}^d$. Taking the supremum with respect to $r \in (0,\tau)$ in (2.3), we find

$$\overline{\mathsf{M}}^\tau\mu(x) + 2C_\varepsilon\tau\mathsf{M}\mu(x) + 2\mathsf{M}\nu_\varepsilon(x) \geq \mathsf{M}^\tau\mu(x), \tag{2.5}$$

for all $x \in \mathbb{R}^d$.

We claim that

$$\liminf_{\lambda\to+\infty} \lambda\,\mathscr{L}^d(\{x \in \mathbb{R}^d : \overline{\mathsf{M}}\mu(x) > \lambda\}) + C_\varepsilon\tau|\mu^s|(\mathbb{R}^d) + \varepsilon \gtrsim_d |\mu^s|(\mathbb{R}^d), \tag{2.6}$$

for all $\varepsilon > 0$ and $\tau > 0$. Indeed, since $|\mu|(\mathbb{R}^d) < +\infty$, given $\tau > 0$, for all $\lambda > 0$ sufficiently large it holds

$$\{x \in \mathbb{R}^d : \mathsf{M}^\tau\mu(x) > \lambda\} = \{x \in \mathbb{R}^d : \mathsf{M}\mu(x) > \lambda\}. \tag{2.7}$$

Thus, on the one hand, by (1.6), we have

$$\begin{aligned}&\liminf_{\lambda\to+\infty} \lambda\,\mathscr{L}^d(\{x \in \mathbb{R}^d : \mathsf{M}^\tau\mu(x) > \lambda\})\\ &\qquad = \liminf_{\lambda\to+\infty} \lambda\,\mathscr{L}^d(\{x \in \mathbb{R}^d : \mathsf{M}\mu(x) > \lambda\}) \gtrsim_d |\mu^s|(\mathbb{R}^d).\end{aligned}$$

On the other hand, by (1.5), (2.1) and (2.5) we can estimate

$$\begin{aligned}&\liminf_{\lambda\to+\infty} \lambda\,\mathscr{L}^d(\{x \in \mathbb{R}^d : \mathsf{M}^\tau\mu(x) > \lambda\})\\ &\quad\leq \liminf_{\lambda\to+\infty} \lambda\,\mathscr{L}^d(\{x \in \mathbb{R}^d : \overline{\mathsf{M}}^\tau\mu(x) + 2C_\varepsilon\tau\mathsf{M}\mu(x) + 2\mathsf{M}\nu_\varepsilon(x) > \lambda\})\\ &\quad\lesssim_d \liminf_{\lambda\to+\infty} \lambda\,\mathscr{L}^d(\{x \in \mathbb{R}^d : \overline{\mathsf{M}}^\tau\mu(x) > \lambda\})\\ &\qquad + C_\varepsilon\tau \limsup_{\lambda\to+\infty} \lambda\,\mathscr{L}^d(\{x \in \mathbb{R}^d : \mathsf{M}\mu(x) > \lambda\})\\ &\qquad + \limsup_{\lambda\to+\infty} \lambda\,\mathscr{L}^d(\{x \in \mathbb{R}^d : \mathsf{M}\nu_\varepsilon(x) > \lambda\})\end{aligned}$$

$$
\begin{aligned}
&\leq \liminf_{\lambda\to+\infty} \lambda\,\mathscr{L}^d(\{x\in\mathbb{R}^d : \overline{\mathsf{M}}^\tau \mu(x)>\lambda\}) + C_\varepsilon\tau|\mu^s|(\mathbb{R}^d) + \int_{\mathbb{R}^d} |\eta-\eta_\varepsilon|\ d|\mu| \\
&\leq \liminf_{\lambda\to+\infty} \lambda\,\mathscr{L}^d(\{x\in\mathbb{R}^d : \overline{\mathsf{M}}^\tau \mu(x)>\lambda\}) + C_\varepsilon\tau|\mu^s|(\mathbb{R}^d) + \varepsilon \\
&= \liminf_{\lambda\to+\infty} \lambda\,\mathscr{L}^d(\{x\in\mathbb{R}^d : \overline{\mathsf{M}} \mu(x)>\lambda\}) + C_\varepsilon\tau|\mu^s|(\mathbb{R}^d) + \varepsilon,
\end{aligned}
$$

for all $\varepsilon > 0$ and $\tau > 0$. Inequality (2.6) thus follows. Therefore, passing to the limit in (2.6) first as $\tau \to 0$ and then as $\varepsilon \to 0$, we get (1.11). This concludes the proof.

3. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. The idea of the proof is to estimate the quantity $\mathsf{A}f$ defined in (1.9) from below with the integral average of $|Df|$, in the spirit of the reverse Poincaré's inequality. Obviously, it is not possible to get such an estimate for arbitrary BV functions (with a constant that does not depend on the function itself).

However, it is simple to see that a reverse Poincaré's inequality is true for one-variable monotone functions, so that one would expect that a sort of reverse Poincaré's inequality may hold for arbitrary BV functions if first one specifies a direction $\nu \in \mathbb{R}^d$, $|\nu| = 1$, and then adds a suitable correction term measuring how far is f from being dependent only on the direction ν and monotone.

Lemma 3.1. *There exist two dimensional constants $C_1, C_2 > 0$ such that*

$$
\frac{1}{r}\int_{B(x,C_2r)} |f-(f)_{x,C_2r}|\ dy + \int_{B(x,C_2r)} (1-\langle\nu,\eta\rangle)\ d|Df| \geq C_1 \int_{B(x,r)} d|Df| \tag{3.1}
$$

for all $\nu \in \mathbb{R}^d$ with $|\nu| = 1$, $x \in \mathbb{R}^d$, $r > 0$ and $f \in BV(\mathbb{R}^d)$, where $\eta\colon \mathbb{R}^d \to \mathbb{R}^d$ satisfies $Df = \eta\,|Df|$ and $|\eta| = 1$ $|Df|$-a.e. in $\mathbb{R}^d$.

Proof. We claim that there exist two dimensional constants $C_1, C_2 > 0$ such that

$$
\frac{1}{r}\int_{B(x,C_2r)} |f-(f)_{x,C_2r}|\ dy + \int_{B(x,C_2r)} (|\nabla f|-\langle\nu,\nabla f\rangle)\ dy \geq C_1 \int_{B(x,r)} |\nabla f|\ dy \tag{3.2}
$$

for all $\nu \in \mathbb{R}^d$ with $|\nu| = 1$, $x \in \mathbb{R}^d$, $r > 0$ and $f \in C^\infty(\mathbb{R}^d)$. By a standard rescaling argument, we just need to prove that there exists a dimensional constant $C > 0$ such that

$$
\int_{2Q} |f-(f)_{2Q}|\ dx + \int_{2Q} (|\nabla f|-\langle \mathrm{e}_1,\nabla f\rangle)\ dy \geq C\int_Q |\nabla f|\ dy \tag{3.3}
$$

for all $f \in C^\infty(\mathbb{R}^d)$, where $Q = [-1,1]^d$ and $(f)_{2Q} := \fint_{2Q} f\,dy$. We prove (3.3) by contradiction. For all $n \in \mathbb{N}$, assume there exists $f_n \in C^\infty(\mathbb{R}^d)$ such that

$$
\int_Q |\nabla f_n|\ dy = 1, \quad (f_n)_{2Q} = 0, \quad \int_{2Q} |f_n|\ dy + \int_{2Q} (|\nabla f_n| - \langle \mathrm{e}_1,\nabla f_n\rangle)\ dy < \frac{1}{n}.
$$

Then consider $g_n\colon 2I \to \mathbb{R}$, $I = [-1,1]$, defined as

$$
g_n(t) := \int_{[-1,1]^{d-1}} g_n(t,y')\ dy', \qquad t \in 2I.
$$

For all $s \in [0,1]$, we have

$$\begin{aligned}\frac{1}{n} &\geq \int_{2Q} \left(|\nabla f_n| - \langle \mathrm{e}_1, \nabla f_n\rangle\right)\, dy \\ &\geq \int_{-1-s}^{1+s} \int_{[-1,1]^{d-1}} |\nabla f_n(y_1,y')| - \langle \mathrm{e}_1, \nabla f_n(y_1,y')\rangle \, dy'\, dy_1 \\ &\geq 1 - \int_{-1-s}^{1+s} g_n'(y_1)\, dy_1 = 1 - \big(g_n(1+s) - g_n(-1-s)\big),\end{aligned}$$

so that

$$g_n(1+s) - g_n(-1-s) \geq 1 - \frac{1}{n}, \qquad n \in \mathbb{N}.$$

This contradicts the fact that $g_n \to 0$ in $L^1(2I)$, since $\|g_n\|_{L^1(2I)} \leq \|f_n\|_{L^1(2Q)} \leq \frac{1}{n}$. This concludes the proof of (3.3) and thus inequality (3.2) follows.

We can now conclude the proof by a standard approximation argument. Given $f \in BV(\mathbb{R}^d)$, by [2, Theorem 3.9 and Proposition 3.13] we can find $f_n \in BV(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ such that $f_n \to f$ in $L^1(\mathbb{R}^d)$, $\|\nabla f_n\|_{L^1(\mathbb{R}^d)} \to |Df|(\mathbb{R}^d)$ and $|\nabla f_n|\mathscr{L}^d \overset{*}{\rightharpoonup} |Df|$ in $\mathbb{R}^d$ as $n \to +\infty$. Therefore, inequality (3.1) follows by Reshetnyak's continuity Theorem, see [2, Theorem 2.39]. □

Remark 3.2. We must have $C_2 > 1$ in Lemma 3.1, as the following example shows. For $n \in \mathbb{N}$, consider $f_n\colon I \to \mathbb{R}$, $I = [-1,1]$, defined as

$$f_n(x) = \begin{cases} nx + n - 1 & -1 \leq x < -\left(1 - \frac{1}{n}\right) \\ 0 & -\left(1-\frac{1}{n}\right) \leq x \leq 1 - \frac{1}{n} \\ nx + 1 - n & 1 - \frac{1}{n} < x \leq 1. \end{cases}$$

Then $(f_n) = 0$, $\|f_n\|_{L^1(I)} = \frac{1}{n}$ and $\|f_n'\|_{L^1(I)} = 2$, so that inequality (3.6) with $C_2 = 1$, $v = 1$, $x = 0$ and $r = 1$ would imply $C_1 \geq 2n$, a contradiction.

We will not apply Lemma 3.1 directly, but we will use the following easy consequence of it. There exist two dimensional constants $C_1, C_2 > 0$ such that

$$\frac{1}{r}\int_{B(x,C_2r)} |f - (f)_{x,C_2r}|\, dy + 2\int_{B(x,C_2r)} |\eta - v|\, d|Df| \geq C_1 \int_{B(x,r)} d|Df| \tag{3.4}$$

for all $v \in \mathbb{R}^d$, $x \in \mathbb{R}^d$, $r > 0$ and $f \in BV(\mathbb{R}^d)$, where $\eta\colon \mathbb{R}^d \to \mathbb{R}^d$ is as in Lemma 3.1. The proof of (3.4) is immediate. Indeed, since we can assume $v \neq 0$ without loss of generality, we just need to notice that

$$2|\eta - v| \geq 1 - \left\langle \frac{v}{|v|}, \eta \right\rangle \qquad |Df|\text{-a.e. in } \mathbb{R}^d,$$

and apply Lemma 3.1 with $\nu = v/|v|$.

Having inequality (3.4) at our disposal, we are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Fix $f \in BV(\mathbb{R}^d)$ and write $Df = \eta\,|Df|$, where $\eta\colon \mathbb{R}^d \to \mathbb{R}^d$ is as in Lemma 3.1. For each $\varepsilon > 0$, let $\eta_\varepsilon \in \mathrm{Lip}(\mathbb{R}^d)$ be a Lipschitz function with Lipschitz constant $C_\varepsilon > 0$ such that the measure $\nu_\varepsilon := |\eta - \eta_\varepsilon|\,|Df|$ satisfies

$$\nu_\varepsilon(\mathbb{R}^d) = \int_{\mathbb{R}^d} |\eta - \eta_\varepsilon|\, d|Df| < \varepsilon. \tag{3.5}$$

Now fix $x \in \mathbb{R}^d$. Applying Lemma 3.1 with $v = \eta_\varepsilon(x)$, we get

$$\frac{1}{r}\int_{B(x,C_2r)} |f(y) - (f)_{x,C_2r}|\, dy + 2\int_{B(x,C_2r)} |\eta(y) - \eta_\varepsilon(x)|\, d|Df|(y) \geq C_1 \int_{B(x,r)} d|Df|(y). \tag{3.6}$$

Note that

$$\begin{aligned}\int_{B(x,C_2r)} &|\eta(y) - \eta_\varepsilon(x)|\, d|Df|(y) \\ &\leq \int_{B(x,C_2r)} |\eta(y) - \eta_\varepsilon(y)|\, d|Df|(y) + \int_{B(x,C_2r)} |\eta_\varepsilon(y) - \eta_\varepsilon(x)|\, d|Df|(y) \\ &\leq \int_{B(x,C_2r)} d\nu_\varepsilon(y) + C_2 C_\varepsilon r \int_{B(x,C_2r)} d|Df|(y).\end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), we get that

$$\mathsf{A}f(x) + \mathsf{M}\nu_\varepsilon(x) + \frac{C_\varepsilon r}{\omega_d (C_2 r)^d}\int_{B(x,C_2r)} d|Df| \gtrsim_d \frac{1}{\omega_d r^d}\int_{B(x,r)} d|Df|, \tag{3.8}$$

for all $x \in \mathbb{R}^d$ and $r > 0$, where $\mathsf{A}f$ is the function defined in (1.9).

Now fix $\tau > 0$. Taking the supremum for $r \in (0, \tau)$ in (3.8) and recalling the definition in (2.4), we get that

$$\mathsf{A}f(x) + \mathsf{M}\nu_\varepsilon(x) + C_\varepsilon \tau\, \mathsf{M}^{C_2\tau}(Df)(x) \gtrsim_d \mathsf{M}^\tau(Df)(x).$$

Thus, by the observation made in (2.7), inequalities (3.5), (1.5) and Proposition 1.1, we conclude that

$$\begin{aligned}|D^s f|(\mathbb{R}^d) &\lesssim_d \liminf_{\lambda\to+\infty} \lambda\, \mathscr{L}^d\big(\big\{x \in \mathbb{R}^d : \mathsf{M}(Df)(x) > \lambda\big\}\big) \\ &= \liminf_{\lambda\to+\infty} \lambda\, \mathscr{L}^d\big(\big\{x \in \mathbb{R}^d : \mathsf{M}^\tau(Df)(x) > \lambda\big\}\big) \\ &\lesssim_d \liminf_{\lambda\to+\infty} \lambda\, \mathscr{L}^d\big(\big\{x \in \mathbb{R}^d : \mathsf{A}f(x) > \lambda\big\}\big) \\ &\quad + \limsup_{\lambda\to+\infty} \lambda\, \mathscr{L}^d\big(\big\{x \in \mathbb{R}^d : \mathsf{M}\nu_\varepsilon(x) > \lambda\big\}\big) \\ &\quad + C_\varepsilon\tau \limsup_{\lambda\to+\infty} \lambda\, \mathscr{L}^d\big(\big\{x \in \mathbb{R}^d : \mathsf{M}^{C_2\tau}(Df)(x) > \lambda\big\}\big) \\ &\lesssim_d \liminf_{\lambda\to+\infty} \lambda\, \mathscr{L}^d\big(\big\{x \in \mathbb{R}^d : \mathsf{A}f(x) > \lambda\big\}\big) + \nu_\varepsilon(\mathbb{R}^d) + C_\varepsilon\tau |D^s f|(\mathbb{R}^d) \\ &\lesssim_d \liminf_{\lambda\to+\infty} \lambda\, \mathscr{L}^d\big(\big\{x \in \mathbb{R}^d : \mathsf{A}f(x) > \lambda\big\}\big) + \varepsilon + C_\varepsilon\tau |D^s f|(\mathbb{R}^d),\end{aligned}$$

for all $\varepsilon > 0$ and $\tau > 0$. Passing to the limit first as $\tau \to 0$ and then as $\varepsilon \to 0$, we get

$$\liminf_{\lambda\to+\infty} \lambda\, \mathscr{L}^d\big(\big\{x \in \mathbb{R}^d : \mathsf{A}f(x) > \lambda\big\}\big) \gtrsim_d |D^s f|(\mathbb{R}^d).$$

This concludes the proof. □

4. Proof of Proposition 1.5

In this section, we prove Proposition 1.5. Let $b\colon (0,T)\times\mathbb{R}^d\to\mathbb{R}^d$ be a vector field satisfying (1.12). By [7, Remark 10], the quantity

$$\mathscr{Q}(B;\delta) = |\log\delta|^{-1}\int_0^T\int_B \min\bigl\{\delta^{-1},\mathsf{M}|Db_t|(x)\bigr\}\,dx\,dt$$

defined in (1.14) satisfies

$$\limsup_{\delta\to0^+}\mathscr{Q}(B;\delta)\lesssim_d \int_0^T |D^s b_t|(\overline{B})\,dt$$

for all balls $B\subset\mathbb{R}^d$. This proves the second part of (1.16). To prove the first part of (1.16), fix a ball $B=B_r\subset\mathbb{R}^d$ of radius $r>0$. We claim that

$$\liminf_{\delta\to0^+}|\log\delta|^{-1}\int_B \min\bigl\{\delta^{-1},\mathsf{M}|Db_t|(x)\bigr\}\,dx \gtrsim_d |D^s b_t|(B_r) \tag{4.1}$$

holds for a.e. $t\in(0,T)$, so that the conclusion follows by Fatou's Lemma. Indeed, for any $\varepsilon\in(0,r/2)$ and for a.e. $t\in(0,T)$, we can estimate

$$\int_{B_r}\min\bigl\{\delta^{-1},\mathsf{M}|Db_t|(x)\bigr\}\,dx \ge \int_{\delta^{-1/2}}^{\delta^{-1}} \mathscr{L}^d\left(\bigl\{x\in B_r : \mathsf{M}\bigl(\mathbf{1}_{B_{r-\varepsilon}}|Db_t|\bigr)(x)>\lambda\bigr\}\right)d\lambda.$$

Note that $\mathsf{M}\bigl(\mathbf{1}_{B_{r-\varepsilon}}|Db_t|\bigr)(x)\lesssim_d |Db_t|(B_r)\varepsilon^{-d}$ for every $x\in\mathbb{R}^d\setminus B_r$ and for a.e. $t\in(0,T)$. Indeed, if $|x|>r$ then $B_{r-\varepsilon}\cap B(x,s)=\varnothing$ for all $s\in[0,\varepsilon]$, so that

$$\mathsf{M}\bigl(\mathbf{1}_{B_{r-\varepsilon}}|Db_t|\bigr)(x) = \sup_{s>0}\frac{|Db_t|(B_{r-\varepsilon}\cap B(x,s))}{\mathscr{L}^d(B(x,s))} \le \frac{|Db_t|(B_{r-\varepsilon})}{\mathscr{L}^d(B(x,\varepsilon))}\lesssim_d |Db_t|(B_r)\varepsilon^{-d}.$$

Now, using the decomposition

$$\begin{aligned}\mathscr{L}^d\left(\bigl\{x\in B_r : \mathsf{M}\bigl(\mathbf{1}_{B_{r-\varepsilon}}|Db_t|\bigr)(x)>\lambda\bigr\}\right) =&\,\mathscr{L}^d\left(\bigl\{x\in \mathbb{R}^d : \mathsf{M}\bigl(\mathbf{1}_{B_{r-\varepsilon}}|Db_t|\bigr)(x)>\lambda\bigr\}\right)\\ &-\mathscr{L}^d\left(\bigl\{x\in \mathbb{R}^d\setminus B_r : \mathsf{M}\bigl(\mathbf{1}_{B_{r-\varepsilon}}|Db_t|\bigr)(x)>\lambda\bigr\}\right)\end{aligned}$$

and Proposition 1.1, we get that

$$\inf_{\lambda\in(\delta^{-1/2},\delta^{-1})}\lambda\,\mathscr{L}^d\left(\bigl\{x\in B_r : \mathsf{M}\bigl(\mathbf{1}_{B_{r-\varepsilon}}|Db_t|\bigr)(x)>\lambda\bigr\}\right)\gtrsim_d |D^s b_t|(B_{r-\varepsilon})$$

for all $\delta\lesssim_d |Db_t|(B_r)^{-1/2}\varepsilon^{d/2}$ and for a.e. $t\in(0,T)$. Hence, for $\delta>0$ sufficiently small, we obtain that

$$|\log\delta|^{-1}\int_{B_r}\min\bigl\{\delta^{-1},\mathsf{M}|Db_t|(x)\bigr\}\,dx\gtrsim_d |\log\delta|^{-1}\int_{\delta^{-1/2}}^{\delta^{-1}}|D^s b_t|(B_{r-\varepsilon})\,\frac{d\lambda}{\lambda}\gtrsim_d |D^s b_t|(B_{r-\varepsilon}).$$

Therefore

$$\liminf_{\delta\to0^+}|\log\delta|^{-1}\int_B \min\bigl\{\delta^{-1},\mathsf{M}|Db_t|(x)\bigr\}\,dx \gtrsim_d |D^s b_t|(B_{r-\varepsilon}),$$

so that claim (4.1) follows by letting $\varepsilon\to0^+$.

Appendix A. Proof of Proposition 1.1

In this section, we prove Proposition 1.1. The main ingredient of the argument is the following well-known reverse weak $(1,1)$-type inequality for the maximal function in (1.2). For the proof, which uses Calderon–Zygmund decomposition, we refer to [9, Chapter 1, Section 5].

Lemma A.1. *There exists a dimensional constant $C>0$ such that*

$$t\,\mathscr{L}^d(\{x\in\mathbb{R}^d : Mf(x)>Ct\})\gtrsim_d \int_{\{f>t\}} f\,d\mathscr{L}^d \tag{A.1}$$

for all $t>0$ and all non-negative $f\in L^1(\mathbb{R}^d)$. More in general, there exists a dimensional constant $C>0$ such that

$$t\,\mathscr{L}^d(\{x\in Q : Mf(x)>Ct\})\gtrsim_d \int_{\{f>t\}\cap Q} f\,d\mathscr{L}^d \tag{A.2}$$

for all cube $Q\subset\mathbb{R}^d$, for all $t>\fint_Q f\,d\mathscr{L}^d$ and all non-negative $f\in L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

We are now ready to prove Proposition 1.1.

Proof of Proposition 1.1. Without loss of generality, we can assume that μ is non-negative and singular with respect to $\mathscr{L}^d$.

For all $\varepsilon>0$ define $f_\varepsilon\in L^1(\mathbb{R}^d)$ as

$$f_\varepsilon(x):=\frac{\mu(B(x,\varepsilon))}{\mathscr{L}^d(B(x,\varepsilon))}=\frac{\mu(B(x,\varepsilon))}{\omega_d\,\varepsilon^d},\qquad x\in\mathbb{R}^d,$$

where $\omega_d=\mathscr{L}^d(B(0,1))$.

We claim that $(f_\varepsilon)_{\varepsilon>0}$ satisfies the following *almost semigroup property*: for all $r>0$ and $x\in\mathbb{R}^d$, it holds

$$(f_\varepsilon)_{x,r}:=\fint_{B(x,r)} f_\varepsilon\,dy\lesssim_d f_{r+\varepsilon}(x). \tag{A.3}$$

Indeed, fix $x\in\mathbb{R}^d$ and $r>0$. By Tonelli's Theorem, we can write

$$(f_\varepsilon)_{x,r}=\frac{1}{\omega_d\varepsilon^d}\frac{1}{\omega_d r^d}\int_{\mathbb{R}^d}\mathscr{L}^d\big(B(x,r)\cap B(y,\varepsilon)\big)\,d\mu(y).$$

Since

$$\mathscr{L}^d\big(B(x,r)\cap B(y,\varepsilon)\big)\le \mathbf{1}_{B(x,r+\varepsilon)}(y)\,\min\{\omega_d\varepsilon^d,\omega_d r^d\},$$

for all $y\in\mathbb{R}^d$, we deduce that

$$(f_\varepsilon)_{x,r}\le\frac{\mu(B(x,r+\varepsilon))}{\omega_d(r+\varepsilon)^d}\,\frac{\omega_d(r+\varepsilon)^d\min\{\omega_d\varepsilon^d,\omega_d r^d\}}{\omega_d\varepsilon^d\,\omega_d r^d}\le 2^d f_{r+\varepsilon}(x).$$

This concludes the proof of (A.3).

Thanks to (A.3), we easily get

$$\mathsf{M}f_\varepsilon(x)\lesssim_d \mathsf{M}\mu(x),$$

for all $x\in\mathbb{R}^d$. Thus, by Lemma A.1, we conclude that

$$t\mathscr{L}^d(\{x\in\mathbb{R}^d : \mathsf{M}\mu(x)>C_d t\})\gtrsim_d\int_{\{f_\varepsilon>t\}} f_\varepsilon\,dx, \tag{A.4}$$

for all $t > 0$ and all $\varepsilon > 0$, where $C_d > 0$ is a dimensional constant.

We now claim that

$$\limsup_{\varepsilon\to 0^+} \int_{\{f_\varepsilon > t\}} f_\varepsilon \, dx \gtrsim_d \mu(\mathbb{R}^d), \tag{A.5}$$

for all $t > 0$, so that (1.6) follows immediately combining (A.4) and (A.5). Indeed, by Tonelli's Theorem we have

$$\int_{\{f_\varepsilon > t\}} f_\varepsilon \, dx = \int_{\mathbb{R}^d} \frac{\mathscr{L}^d(\{f_\varepsilon > t\} \cap B(x,\varepsilon))}{\omega_d \varepsilon^d} \, d\mu(x), \tag{A.6}$$

for all $\varepsilon > 0$. Hence, by Fatou's Lemma, we get that

$$\limsup_{\varepsilon\to 0^+} \int_{\{f_\varepsilon > t\}} f_\varepsilon \, dx \geq \int_{\mathbb{R}^d} \liminf_{\varepsilon\to 0^+} \frac{\mathscr{L}^d(\{f_\varepsilon > t\} \cap B(x,\varepsilon))}{\omega_d \varepsilon^d} \, d\mu(x).$$

We now claim that

$$\mathscr{L}^d(\{f_\varepsilon > t\} \cap B(x,\varepsilon)) \geq \frac{1}{2^d} \omega_d \varepsilon^d, \tag{A.7}$$

for μ-a.e. $x \in \mathbb{R}^d$ and all $\varepsilon > 0$. To prove (A.7), we need to observe two preliminary facts.

First, notice that, given $\varepsilon > 0$ and $t > 0$, we have

$$f_{\varepsilon/2}(x) > 2^d t \implies B(x,\varepsilon/2) \subset \{x \in \mathbb{R}^d : f_\varepsilon(x) > t\}. \tag{A.8}$$

Implication (A.8) follows from the trivial inclusion $B(x,\varepsilon/2) \subset B(y,\varepsilon)$ for all $y \in B(x,\varepsilon/2)$.

Second, notice that

$$\lim_{\varepsilon\to 0^+} f_\varepsilon(x) = +\infty, \tag{A.9}$$

for μ-a.e. $x \in \mathbb{R}^d$. Indeed, we have

$$\left\{ x \in \mathbb{R}^d : \liminf_{\varepsilon\to 0^+} f_\varepsilon(x) < +\infty \right\} \subset \bigcup_{n\in\mathbb{N}} A_n, \tag{A.10}$$

where

$$A_n := \left\{ x \in \mathbb{R}^d : \liminf_{\varepsilon\to 0^+} \frac{\mu(B(x,\varepsilon))}{\omega_d \varepsilon^d} \leq n \right\}.$$

By a standard covering argument (for instance apply Vitali's covering Lemma, see [9, Section 1.6]), one can prove that

$$\mu(E) \leq n \mathscr{L}^d(E) \quad \text{for all Borel sets } E \subset A_n.$$

Since μ is singular with respect to $\mathscr{L}^d$, we must have that $\mu(A_n) = 0$ for all $n \in \mathbb{N}$ and thus, by (A.10), we conclude that

$$\mu\left(\left\{ x \in \mathbb{R}^d : \liminf_{\varepsilon\to 0^+} f_\varepsilon(x) < +\infty \right\} \right) = 0.$$

We can now prove (A.7). Fix $x \in \mathbb{R}^d$ such that (A.8) holds true. Then there exists $\varepsilon_x > 0$ such that $f_{\varepsilon/2}(x) > 2^d t$ for all $\varepsilon < \varepsilon_x$. Hence $B(x,\varepsilon/2) \subset \{f_\varepsilon > t\}$ and so

$$\mathscr{L}^d(\{\mu_\varepsilon > t\} \cap B(x,\varepsilon)) \geq \frac{1}{2^d} \omega_d \varepsilon^d$$

for all $\varepsilon < \varepsilon_x$. Thus (A.7) follows and the proof of (1.6) is complete.

The proof of the local inequality (1.7) similarly follows from (A.2) and is left to the reader. □

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