RIGIDITY OF POSITIVELY CURVED SHRINKING RICCI SOLITONS IN DIMENSION FOUR

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ABSTRACT. We classify four-dimensional shrinking Ricci solitons satisfying $Sec \geq \frac{1}{24}R$, where Sec and R denote the sectional and the scalar curvature, respectively. They are isometric to either $\mathbb{R}^4$ (and quotients), $\mathbb{S}^4$, $\mathbb{RP}^4$ or $\mathbb{CP}^2$ with their standard metrics.

Key Words: Ricci solitons, Einstein metrics, positive sectional curvature

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1. INTRODUCTION

In this paper we investigate gradient shrinking Ricci solitons with positive sectional curvature. We recall that a Riemannian manifold (M^n, g) of dimension $n \geq 3$ is a gradient Ricci soliton if there exists a smooth function f on M^n such that

$$Ric + \nabla^2 f = \lambda\, g$$

for some constant λ. If ∇f is parallel, then (M^n, g) is Einstein. The Ricci soliton is called shrinking if $\lambda > 0$, steady if $\lambda = 0$ and expanding if $\lambda < 0$. Ricci solitons generate self-similar solutions of the Ricci flow, play a fundamental role in the formation of singularities and have been studied by many authors (see H.-D. Cao [5] for an overview).

It is well known that (compact) Einstein manifolds can be classified, if they are enough positively curved. Sufficient conditions are non-negative curvature operator (S. Tachibana [18]), non-negative isotropic curvature (M. J. Micallef and Y. Wang [13] in dimension four and S. Brendle [3] in every dimension) and weakly $\frac{1}{4}$-pinched sectional curvature [1] (if Sec and R denote the sectional and the scalar curvature, respectively, this condition in dimension four is implied by $Sec \geq \frac{1}{24}R$). Moreover, in dimension four, it is proved by D. Yang [19]) that four-dimensional Einstein manifolds satisfying $Sec \geq \varepsilon R$ are isometric to either $\mathbb{S}^4$, $\mathbb{RP}^4$ or $\mathbb{CP}^2$ with their standard metrics, if $\varepsilon = \frac{\sqrt{1249}-23}{480}$. The lower bound has been improved to $\varepsilon = \frac{2-\sqrt{2}}{24}$ by E. Costa [8] and, more recently, to $\varepsilon = \frac{1}{48}$ by E. Ribeiro [16] (see also X. Cao and P. Wu [6]). It is conjectured in [19] that the result should be true assuming positive sectional curvature.

In dimension $n \leq 3$, complete shrinking Ricci solitons are classified. In the last years there have been a lot of interesting results concerning the classification of shrinking Ricci

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solitons which are positively curved. For instance, it follows by the work of C. Böhm and B. Wilking [2] that the only compact shrinking Ricci solitons with positive (two-positive) curvature operator are quotients of $\mathbb{S}^n$. In dimension four, A. Naber [14] classified complete shrinkers with non-negative curvature operator. Four dimensional shrinkers with non-negative isotropic curvature were classified by X. Li, L. Ni and K. Wang [12].

Recently, O. Munteanu and J.P. Wang [17] showed that every complete shrinking Ricci solitons with positive sectional curvature are compact. It is natural to ask the following question: *given $\varepsilon > 0$, are there four dimensional non-Einstein shrinking Ricci solitons satisfying $Sec \geq \varepsilon R$?.*

In this paper we give an answer to this question proving the following

Theorem 1.1. *Let (M^4, g) be a four-dimensional complete gradient shrinking Ricci soliton with $Sec \geq \frac{1}{24}R$. Then (M^4, g) is necessarily Einstein, thus isometric to either $\mathbb{R}^4$ (and quotients), $\mathbb{S}^4$, $\mathbb{RP}^4$ or $\mathbb{CP}^2$ with their standard metrics.*

Note that, by the work of S. Brendle and R. Schoen [4], using the Ricci flow, one can show that compact Ricci shrinkers with weakly $\frac{1}{4}$-pinched sectional curvature are isometric to $\mathbb{S}^4$, $\mathbb{RP}^4$ or $\mathbb{CP}^2$ with their standard metrics. The condition $Sec \geq \frac{1}{24}R$ is a little stronger, but the proof of Theorem 1.1 that we present is completely "elliptic".

2. Estimates on manifolds with positive sectional curvature

To fix the notation we recall that the Riemann curvature operator of a Riemannian manifold (M^n, g) is defined as in [10] by

$$Rm(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z \,.$$

In a local coordinate system the components of the $(3,1)$–Riemann curvature tensor are given by $R^l_{ijk}\frac{\partial}{\partial x^l} = Rm\big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\big)\frac{\partial}{\partial x^k}$ and we denote by $R_{ijkl} = g_{lp}R^p_{ijk}$ its $(4,0)$–version. Throughout the paper the Einstein convention of summing over the repeated indices will be adopted. The Ricci tensor Ric is obtained by the contraction $(Ric)_{ik} = R_{ik} = g^{jl}R_{ijkl}$, $R = g^{ik}R_{ik}$ will denote the scalar curvature and $(\overset{\circ}{Ric})_{ik} = R_{ik} - \frac{1}{n}R\, g_{ik}$ the traceless Ricci tensor. The Riemannian metric induces norms on all the tensor bundles, in coordinates this norm is given, for a tensor $T = T^{j_1\ldots j_l}_{i_1\ldots i_k}$, by

$$|T|^2_g = g^{i_1 m_1}\cdots g^{i_k m_k} g_{j_1 n_1}\cdots g_{j_l n_l} T^{j_1\ldots j_l}_{i_1\ldots i_k} T^{n_1\ldots n_l}_{m_1\ldots m_k} \,.$$

The first key observation are the following pointwise estimates which are satisfied by every metric with $Sec \geq \varepsilon R$ for some $\varepsilon \in \mathbb{R}$.

Proposition 2.1. *Let (M^n, g) be a Riemannian manifold of dimension $n \geq 3$. If the sectional curvature satisfies $Sec \geq \varepsilon R$ for some $\varepsilon \in \mathbb{R}$, then the following two estimates hold*

$$R_{ijkl}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jl} \leq \frac{1-n^2\varepsilon}{n} R|\overset{\circ}{Ric}|^2 + \overset{\circ}{R}_{ij}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jk}$$

and

$$R_{ijkl}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jl}\leq\frac{n^2-4n+2-n^2(n-2)(n-3)\varepsilon}{2n}R|\overset{\circ}{Ric}|^2-(n-1)\overset{\circ}{R}_{ij}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jk}\,.$$

In particular, in dimension four

$$R_{ijkl}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jl}\leq\frac{1-16\varepsilon}{4}R|\overset{\circ}{Ric}|^2+\overset{\circ}{R}_{ij}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jk}\,,$$

$$R_{ijkl}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jl}\leq\frac{1-16\varepsilon}{4}R|\overset{\circ}{Ric}|^2-3\overset{\circ}{R}_{ij}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jk}\,.$$

Proof. Let $\{e_i\}$, $i=1,\dots,n$, be the eigenvectors of $\overset{\circ}{Ric}$ and let λ_i be the corresponding eigenvalues. Moreover, let σ_{ij} be the sectional curvature defined by the two-plane spanned by e_i and e_j. Since the sectional curvature satisfy $Sec\geq\varepsilon R$, it is natural to define the tensor

$$\overline{Rm}=Rm-\frac{\varepsilon}{2}R\,g\owedge g\,.$$

In particular

$$\overline{Ric}=Ric-(n-1)\varepsilon R\,g\,,\qquad \overline{R}=\big(1-n(n-1)\varepsilon\big)R\qquad\text{and}\qquad\overline{\sigma}_{ij}=\sigma_{ij}-\varepsilon R\geq 0\,.$$

Moreover, if μ_k and $\overline{\mu}_k$ are the eigenvalues with eigenvector e_k of Ric and $\overline{Ric}$, respectively, one has

$$\mu_k=\sum_{i\neq k}\sigma_{ik}\qquad\text{and}\qquad\overline{\mu}_k=\sum_{i\neq k}\overline{\sigma}_{ik}\,.$$

Denoting by $\overline{R}_{ijkl}$ the components of $\overline{Rm}$, we get

$$\begin{aligned}\overline{R}_{ijkl}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jl}-\overline{R}_{ij}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jk}&=\sum_{i,j=1}^{n}\lambda_i\lambda_j\overline{\sigma}_{ij}-\sum_{k=1}^{n}\mu_k\lambda_k^2\\&=2\sum_{i<j}\lambda_i\lambda_j\overline{\sigma}_{ij}-\sum_{i<j}\big(\lambda_i^2+\lambda_j^2\big)\overline{\sigma}_{ij}\\&=-\sum_{i<j}\big(\lambda_i-\lambda_j\big)^2\overline{\sigma}_{ij}\leq 0\,.\end{aligned}\tag{2.1}$$

Using the definition of $\overline{Rm}$ and $\overline{Ric}$, we obtain

$$R_{ijkl}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jl}+\varepsilon R|\overset{\circ}{Ric}|^2\leq R_{ij}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jk}-(n-1)\varepsilon R|\overset{\circ}{Ric}|^2=\overset{\circ}{R}_{ij}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jk}+\frac{1-n(n-1)\varepsilon}{n}R|\overset{\circ}{Ric}|^2$$

and this proves the first inequality of this proposition.

In order to show the second one, we will follow the proof of [7, Proposition 3.1]. We observe that

$$\overline{R}_{ikjl}\overset{\circ}{R}_{ij}\overset{\circ}{R}_{kl}-\frac{n-2}{2n}\overline{R}|\overset{\circ}{Ric}|^2=\sum_{i,j=1}^{n}\lambda_i\lambda_j\overline{\sigma}_{ij}-\frac{n-2}{2n}\overline{R}\sum_{k=1}^{n}\lambda_k^2\,.$$

Since the modified scalar curvature $\overline{R}$ can be written as

$$\overline{R} \,=\, g^{ij}g^{kl}\overline{R}_{ikjl} \,=\, \sum_{i,j=1}^{n}\overline{\sigma}_{ij} \,=\, 2\sum_{i<j}\overline{\sigma}_{ij}\,,$$

one has the following

$$\sum_{i,j=1}^{n}\lambda_i\lambda_j\overline{\sigma}_{ij}-\frac{n-2}{2n}\overline{R}\sum_{k=1}^{n}\lambda_k^2 = 2\sum_{i<j}\lambda_i\lambda_j\overline{\sigma}_{ij}-\frac{n-2}{n}\sum_{i<j}\overline{\sigma}_{ij}\sum_{k=1}^{n}\lambda_k^2$$
$$= \sum_{i<j}\Big(2\lambda_i\lambda_j-\frac{n-2}{n}\sum_{k=1}^{n}\lambda_k^2\Big)\overline{\sigma}_{ij}\,.$$

On the other hand, one has

$$\sum_{k=1}^{n}\lambda_k^2 \,=\, \lambda_i^2+\lambda_j^2+\sum_{k\neq i,j}\lambda_k^2\,.$$

Moreover, using the Cauchy-Schwarz inequality and the fact that $\sum_{k=1}^{n}\lambda_k=0$, we obtain

$$\sum_{k\neq i,j}\lambda_k^2 \,\geq\, \frac{1}{n-2}\Big(\sum_{k\neq i,j}\lambda_k\Big)^2 \,=\, \frac{1}{n-2}\big(\lambda_i+\lambda_j\big)^2$$

with equality if and only if $\lambda_k=\lambda_{k'}$ for every $k,k'\neq i,j$. Hence, the following estimate holds

$$\sum_{k=1}^{n}\lambda_k^2 \,\geq\, \frac{n-1}{n-2}\big(\lambda_i^2+\lambda_j^2\big)+\frac{2}{n-2}\lambda_i\lambda_j\,.$$

Using this, since $\overline{\sigma}_{ij}\geq 0$, it follows that

$$\sum_{i,j=1}^{n}\lambda_i\lambda_j\overline{\sigma}_{ij}-\frac{n-2}{2n}\overline{R}\sum_{k=1}^{n}\lambda_k^2 \leq \frac{n-1}{n}\sum_{i<j}\Big(2\lambda_i\lambda_j-\big(\lambda_i^2+\lambda_j^2\big)\Big)\overline{\sigma}_{ij}$$
$$= -\frac{n-1}{n}\sum_{i<j}(\lambda_i-\lambda_j)^2\overline{\sigma}_{ij}$$
$$= \frac{n-1}{n}\Big(\overline{R}_{ijkl}\mathring{R}_{ik}\mathring{R}_{jl}-\overline{R}_{ij}\mathring{R}_{ik}\mathring{R}_{jk}\Big)\,,$$

where in the last equality we have used equation (2.1). Hence, we proved

$$\overline{R}_{ikjl}\mathring{R}_{ij}\mathring{R}_{kl}-\frac{n-2}{2n}\overline{R}|\mathring{Ric}|^2 \leq \frac{n-1}{n}\Big(\overline{R}_{ijkl}\mathring{R}_{ik}\mathring{R}_{jl}-\overline{R}_{ij}\mathring{R}_{ik}\mathring{R}_{jk}\Big)\,,$$

i.e.

$$\overline{R}_{ikjl}\mathring{R}_{ij}\mathring{R}_{kl} \leq \frac{n-2}{2}\overline{R}|\mathring{Ric}|^2-(n-1)\overline{R}_{ij}\mathring{R}_{ik}\mathring{R}_{jk}\,.$$

Finally, substituting $\overline{Rm}$, $\overline{Ric}$ and $\overline{R}$ we obtain the the second inequality of this proposition. □

Taking the convex combination of the two previous estimates we obtain the following.

Corollary 2.2. *Let (M^n, g) be a Riemannian manifold of dimension $n \geq 3$. If the sectional curvature satisfies $Sec \geq \varepsilon R$ for some $\varepsilon \in \mathbb{R}$, then, for every $s \in [0,1]$, one has*

$$R_{ijkl}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jl} \leq \left(\frac{n^2-4n+2-n^2(n-2)(n-3)\varepsilon}{2n} - \frac{n-4}{2}\big(1-n(n-1)\varepsilon\big)s\right) R|\overset{\circ}{Ric}|^2 \\ -(n-1-ns)\overset{\circ}{R}_{ij}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jk}\,.$$

In particular, in dimension four, for every $s \in [0,1]$, one has

$$R_{ijkl}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jl} \leq \frac{1-16\varepsilon}{4} R|\overset{\circ}{Ric}|^2 - (3-4s)\overset{\circ}{R}_{ij}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jk}\,,$$

Remark 2.3. Taking $\varepsilon = 0$ and $s = \frac{n-1}{n}$, we recover the estimate on manifolds with non-negative sectional curvature which was proved in [7].

3. Some formulas for Ricci solitons

Let (M^n, g) be a n-dimensional complete gradient shrinking Ricci solitons

$$Ric + \nabla^2 f \;=\; \lambda g$$

for some smooth function f and some positive constant $\lambda > 0$. First of all we recall the following well known formulas (for the proof see [9])

Lemma 3.1. *Let (M^n, g) be a gradient Ricci soliton. Then the following formulas hold*

$$\Delta f \;=\; n\lambda - R$$

$$\Delta_f R \;=\; 2\lambda R - 2|Ric|^2$$

$$\Delta_f R_{ik} \;=\; 2\lambda R_{ik} - 2R_{ijkl}R_{jl}$$

where the Δ_f denotes the f-Laplacian, $\Delta_f = \Delta - \nabla_{\nabla f}$.

In particular, defining $\overset{\circ}{R}_{ij} = R_{ij} - \frac{1}{n}R\, g_{ij}$, a simple computation shows the following equation for the f-Laplacian of the squared norm of the trece-less Ricci tensor $\overset{\circ}{Ric}$

Lemma 3.2. *Let (M^n, g) be a gradient Ricci soliton. Then the following formula holds*

$$\frac{1}{2}\Delta_f|\overset{\circ}{Ric}|^2 \;=\; |\nabla\overset{\circ}{Ric}|^2 + 2\lambda|\overset{\circ}{Ric}|^2 - 2R_{ijkl}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jl} - \frac{2}{n}R|\overset{\circ}{Ric}|^2\,.$$

Moreover we have the following scalar curvature estimate [15].

Lemma 3.3. *Let (M^n, g) be a complete gradient shrinking Ricci soliton. Then either g is flat or its scalar curvature is positive $R > 0$.*

Finally, we show this simple identity.

Lemma 3.4. *Let (M^n, g) be a compact gradient Ricci soliton. Then the following formula holds*

$$\int |\nabla R|^2 \, dV = \frac{n-4}{2}\,\lambda \int R^2 \, dV - \frac{n-4}{2n} \int R^3 \, dV + 2 \int R |\overset{\circ}{Ric}|^2 \, dV \,.$$

In particular, in dimension four

$$\int |\nabla R|^2 \, dV = 2 \int R |\overset{\circ}{Ric}|^2 \, dV \,.$$

Proof. Integrating by parts and using Lemma 3.1 we obtain

$$\begin{aligned}
\int |\nabla R|^2 \, dV &= - \int R \Delta R \, dV \\
&= -\frac{1}{2} \int \langle \nabla R^2, \nabla f \rangle \, dV - 2\lambda \int R^2 \, dV + 2 \int R |\overset{\circ}{Ric}|^2 \, dV + \frac{2}{n} \int R^3 \, dV \\
&= \frac{1}{2} \int R^2 \Delta f \, dV - 2\lambda \int R^2 \, dV + 2 \int R |\overset{\circ}{Ric}|^2 \, dV + \frac{2}{n} \int R^3 \, dV \\
&= \frac{n-4}{2}\,\lambda \int R^2 \, dV - \frac{n-4}{2n} \int R^3 \, dV + 2 \int R |\overset{\circ}{Ric}|^2 \, dV \,.
\end{aligned}$$

□

4. Proof of Theorem 1.1

Let (M^4, g) be a complete gradient shrinking Ricci soliton of dimension four and assume that $Sec \geq \varepsilon R$ on M^4 for some $\varepsilon > 0$. By Lemma 3.3 either g is flat or $R > 0$. In this second case, by the result in [17] we know that M^4 must be compact. From now on we can assume that (M^4, g) is compact with $Sec \geq \varepsilon R > 0$. Lemma 3.2 gives

$$\frac{1}{2} \Delta_f |\overset{\circ}{Ric}|^2 = |\nabla \overset{\circ}{Ric}|^2 + 2\lambda |\overset{\circ}{Ric}|^2 - 2 R_{ijkl} \overset{\circ}{R}_{ik} \overset{\circ}{R}_{jl} - \frac{1}{2} R |\overset{\circ}{Ric}|^2 \,.$$

Integrate over M^4 and using equation (3.1) we obtain

$$\begin{aligned}
0 &= \frac{1}{2} \int \langle \nabla |\overset{\circ}{Ric}|^2, \nabla f \rangle \, dV + \int |\nabla \overset{\circ}{Ric}|^2 \, dV + 2 \int \lambda |\overset{\circ}{Ric}|^2 \, dV \\
&\quad - 2 \int R_{ijkl} \overset{\circ}{R}_{ik} \overset{\circ}{R}_{jl} \, dV - \frac{1}{2} \int R |\overset{\circ}{Ric}|^2 \, dV \\
&= \int |\nabla \overset{\circ}{Ric}|^2 \, dV - 2 \int R_{ijkl} \overset{\circ}{R}_{ik} \overset{\circ}{R}_{jl} \, dV \,. \qquad (4.1)
\end{aligned}$$

On the other hand, given $a_1, a_2, b_1, b_2, b_3 \in \mathbb{R}$ we define the three tensor

$$F_{ijk} := \nabla_k \overset{\circ}{R}_{ij} + a_1 \nabla_j \overset{\circ}{R}_{ik} + a_2 \nabla_i \overset{\circ}{R}_{jk} + b_1 \nabla_k R g_{ij} + b_2 \nabla_j R g_{ik} + b_3 \nabla_i R g_{jk} \,.$$

Using the Bianchi identity $\nabla_i \overset{\circ}{R}_{ij} = \frac{1}{4}\nabla_j R$, a computation gives

$$|F|^2 = (1+a_1^2+a_2^2)|\nabla \overset{\circ}{Ric}|^2 + 2(a_1+a_2+a_1a_2)\nabla_k \overset{\circ}{R}_{ij}\nabla_j \overset{\circ}{R}_{ik}$$
$$+\frac{1}{2}\left(a_1(b_1+b_3)+a_2(b_1+b_2)+b_2+b_3+8(b_1^2+b_2^2+b_3^2)+4(b_1b_2+b_1b_3+b_2b_3)\right)|\nabla R|^2 .$$

In particular,

$$\int |\nabla \overset{\circ}{Ric}|^2\,dV = \frac{1}{1+a_1^2+a_2^2}\int |F|^2\,dV - \frac{2(a_1+a_2+a_1a_2)}{1+a_1^2+a_2^2}\int \nabla_k \overset{\circ}{R}_{ij}\nabla_j \overset{\circ}{R}_{ik}\,dV$$
$$-\frac{a_1(b_1+b_3)+a_2(b_1+b_2)+b_2+b_3+8(b_1^2+b_2^2+b_3^2)+4(b_1b_2+b_1b_3+b_2b_3)}{2(1+a_1^2+a_2^2)}\int |\nabla R|^2\,dV$$
$$= \frac{1}{1+a_1^2+a_2^2}\int |F|^2\,dV - \frac{2(a_1+a_2+a_1a_2)}{1+a_1^2+a_2^2}\int \nabla_k \overset{\circ}{R}_{ij}\nabla_j \overset{\circ}{R}_{ik}\,dV \tag{4.2}$$
$$-\frac{a_1(b_1+b_3)+a_2(b_1+b_2)+b_2+b_3+8(b_1^2+b_2^2+b_3^2)+4(b_1b_2+b_1b_3+b_2b_3)}{1+a_1^2+a_2^2}\int R|\overset{\circ}{Ric}|^2\,dV ,$$

where, in the last equality we have used Lemma 3.4. On the other hand, integrating by parts and commuting the covariant derivatives, one has

$$\begin{aligned}
\int \nabla_k \overset{\circ}{R}_{ij}\nabla_j \overset{\circ}{R}_{ik}\,dV &= -\int \overset{\circ}{R}_{ij}\nabla_k\nabla_j \overset{\circ}{R}_{ik}\,dV\\
&= -\int \left(\overset{\circ}{R}_{ij}\nabla_j\nabla_k \overset{\circ}{R}_{ik} + R_{kjil}\overset{\circ}{R}_{ij}\overset{\circ}{R}_{kl} + R_{ij}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jl}\right)dV\\
&= -\int \left(\frac{1}{4}\overset{\circ}{R}_{ij}\nabla_i\nabla_j R - R_{ijkl}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jl} + \overset{\circ}{R}_{ij}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jl} + \frac{1}{4}R|\overset{\circ}{Ric}|^2\right)dV\\
&= \int \left(\frac{1}{16}|\nabla R|^2 + R_{ijkl}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jl} - \overset{\circ}{R}_{ij}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jl} - \frac{1}{4}R|\overset{\circ}{Ric}|^2\right)dV\\
&= \int \left(R_{ijkl}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jl} - \overset{\circ}{R}_{ij}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jl} - \frac{1}{8}R|\overset{\circ}{Ric}|^2\right)dV .
\end{aligned} \tag{4.3}$$

From equation (4.2), we obtain

$$\int |\nabla \overset{\circ}{Ric}|^2\,dV = \frac{1}{1+a_1^2+a_2^2}\int |F|^2 dV - \frac{2(a_1+a_2+a_1a_2)}{1+a_1^2+a_2^2}\int \left(R_{ijkl}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jl} - \overset{\circ}{R}_{ij}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jl}\right)dV$$
$$+ Q_1\int R|\overset{\circ}{Ric}|^2\,dV$$

with

$$Q_1 := \frac{a_1+a_2+a_1a_2}{4(1+a_1^2+a_2^2)}$$
$$-\frac{a_1(b_1+b_3)+a_2(b_1+b_2)+b_2+b_3+8(b_1^2+b_2^2+b_3^2)+4(b_1b_2+b_1b_3+b_2b_3)}{1+a_1^2+a_2^2}.$$

Using this inequality in (4.1), we obtain that

$$0=\frac{1}{1+a_1^2+a_2^2}\int|F|^2dV-\frac{2(1+a_1^2+a_2^2+a_1+a_2+a_1a_2)}{1+a_1^2+a_2^2}\int R_{ijkl}\mathring{R}_{ik}\mathring{R}_{jl}\,dV \tag{4.4}$$
$$+\frac{2(a_1+a_2+a_1a_2)}{1+a_1^2+a_2^2}\int\mathring{R}_{ij}\mathring{R}_{ik}\mathring{R}_{jl}\,dV+Q_1\int R|\mathring{Ric}|^2\,dV\,.$$

From Corollary 2.2 we have

$$R_{ijkl}\mathring{R}_{ik}\mathring{R}_{jl}\leq\frac{1-16\varepsilon}{4}R|\mathring{Ric}|^2-(3-4s)\mathring{R}_{ij}\mathring{R}_{ik}\mathring{R}_{jk} \tag{4.5}$$

for every $s\in[0,1]$. Thus, if $a_1+a_2+a_1a_2\geq 0$, for every $s\in[0,1]$, estimate (4.4) gives

$$0\geq\frac{1}{1+a_1^2+a_2^2}\int|F|^2dV$$
$$+\frac{2\Big((3-4s)(1+a_1^2+a_2^2)+4(1-s)(a_1+a_2+a_1a_2)\Big)}{1+a_1^2+a_2^2}\int\mathring{R}_{ij}\mathring{R}_{ik}\mathring{R}_{jl}\,dV \tag{4.6}$$
$$+Q_2\int R|\mathring{Ric}|^2\,dV \tag{4.7}$$

with

$$\begin{aligned}Q_2&:=Q_1-\frac{(1-16\varepsilon)(1+a_1^2+a_2^2+a_1+a_2+a_1a_2)}{2(1+a_1^2+a_2^2)}\\&=\frac{a_1+a_2+a_1a_2}{4(1+a_1^2+a_2^2)}-\frac{(1-16\varepsilon)(1+a_1^2+a_2^2+a_1+a_2+a_1a_2)}{2(1+a_1^2+a_2^2)}\\&\quad-\frac{a_1(b_1+b_3)+a_2(b_1+b_2)+b_2+b_3+8(b_1^2+b_2^2+b_3^2)+4(b_1b_2+b_1b_3+b_2b_3)}{1+a_1^2+a_2^2}\,.\end{aligned}$$

Now, choose $a_1=a_2=1$ and $b_1=b_2=b_3=:b$. Then

$$Q_2=-12b^2-2b+16\varepsilon-\frac{3}{4}\,.$$

In particular, the maximum is attained at $b=-1/12$ and is given by

$$Q_2=\frac{48\varepsilon-2}{3}\,. \tag{4.8}$$

Actually a (long) computation gives that the maximum of the function Q_2 defined for general variables (a_1,a_2,b_1,b_2,b_3) is attained at the point

$$(a_1,a_2,b_1,b_2,b_3)=\Big(1,1,-\frac{1}{12},-\frac{1}{12},-\frac{1}{12}\Big) \tag{4.9}$$

and is given by the value (4.8). Moreover, under the choice (4.9), one has

$$\frac{2\Big((3-4s)(1+a_1^2+a_2^2)+4(1-s)(a_1+a_2+a_1a_2)\Big)}{1+a_1^2+a_2^2}=2(7-8s)\,.$$

In particular, choosing

$$s = \frac{7}{8}\,,$$

from (4.6) we obtain

$$0 \geq \frac{1}{3}\int |F|^2 dV + \frac{48\varepsilon - 2}{3}\int R|\overset{\circ}{Ric}|^2\, dV\,.$$

Thus, if $\varepsilon > 1/24$, then $\overset{\circ}{Ric} \equiv 0$, i.e. (M^4, g) is Einstein. By Berger classification result [1] we conclude the proof of Theorem 1.1 in this case.

If $\varepsilon = 1/24$, then $Q_1 = 1/3$, $Q_2 = 0$ and all previous inequalities become equalities. In particular, $F \equiv 0$. Moreover, from (4.5), we get

$$R_{ijkl}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jl} \equiv \frac{1}{12}R|\overset{\circ}{Ric}|^2 \quad \text{and} \quad \overset{\circ}{R}_{ij}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jk} \equiv 0\,. \tag{4.10}$$

From equation (4.3) and Lemma 3.4 we get

$$\int \nabla_k \overset{\circ}{R}_{ij}\nabla_j \overset{\circ}{R}_{ik}\, dV = -\frac{1}{24}\int R|\overset{\circ}{Ric}|^2 dV = -\frac{1}{48}\int |\nabla R|^2 dV.$$

Thus, equation (4.2) gives

$$\int |\nabla \overset{\circ}{Ric}|^2 dV = \frac{1}{12}\int |\nabla R|^2 dV. \tag{4.11}$$

Now, to conclude, we have to use the fact that $F \equiv 0$, i.e.

$$0 = \nabla_k \overset{\circ}{R}_{ij} + \nabla_j \overset{\circ}{R}_{ik} + \nabla_i \overset{\circ}{R}_{jk} - \frac{1}{12}\left(\nabla_k R g_{ij} + \nabla_j R g_{ik} + \nabla_i R g_{jk}\right)\,.$$

Taking the diverge in k and contracting with $\overset{\circ}{R}_{ij}$, we obtain

$$\begin{aligned}
0 &= \overset{\circ}{R}_{ij}\left[\Delta \overset{\circ}{R}_{ij} + \nabla_k\nabla_j \overset{\circ}{R}_{ik} + \nabla_k\nabla_i \overset{\circ}{R}_{jk} - \frac{1}{12}\left(\Delta R g_{ij} + 2\nabla_i\nabla_j R\right)\right] \\
&= \frac{1}{2}\Delta|\overset{\circ}{Ric}|^2 - |\nabla \overset{\circ}{Ric}|^2 + \overset{\circ}{R}_{ij}\left[\nabla_j\nabla_k \overset{\circ}{R}_{ik} + \nabla_i\nabla_k \overset{\circ}{R}_{jk} - \frac{1}{6}\nabla_i\nabla_j R\right] - 2R_{ijkl}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jl} \\
&= \frac{1}{2}\Delta|\overset{\circ}{Ric}|^2 - |\nabla \overset{\circ}{Ric}|^2 + \frac{1}{3}\overset{\circ}{R}_{ij}\nabla_i\nabla_j R - 2R_{ijkl}\overset{\circ}{R}_{ik}\overset{\circ}{R}_{jl} \\
&= \frac{1}{2}\Delta|\overset{\circ}{Ric}|^2 - |\nabla \overset{\circ}{Ric}|^2 + \frac{1}{3}\overset{\circ}{R}_{ij}\nabla_i\nabla_j R - \frac{1}{6}R|\overset{\circ}{Ric}|^2\,,
\end{aligned}$$

where we used (4.10). Integrating by parts over M, using (4.11), we obtain

$$0 = -\frac{1}{6}\int |\nabla R|^2 dV - \frac{1}{6}\int R|\overset{\circ}{Ric}|^2 dV$$

which implies $\overset{\circ}{Ric} \equiv 0$, i.e. (M^4, g) is Einstein and the thesis follows again by Berger result. This concludes the proof of Theorem 1.1.

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