Differential structure associated to axiomatic Sobolev spaces

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Abstract

The aim of this note is to explain in which sense an axiomatic Sobolev space over a general metric measure space (à la Gol'dshtein-Troyanov) induces – under suitable locality assumptions – a first-order differential structure.

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Contents

Introduction

An axiomatic approach to the theory of Sobolev spaces over abstract metric measure spaces has been proposed by V. Gol'dshtein and M. Troyanov in [6]. Their construction covers many important notions: the weighted Sobolev space on a Riemannian manifold, the Hajłasz Sobolev space [7] and the Sobolev space based on the concept of upper gradient [2,3,8,9].

A key concept in [6] is the so-called *D-structure*: given a metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ and an exponent $p\in(1,\infty)$, we associate to any function $u\in L^p_{loc}(\mathrm{X})$ a family $D[u]$ of non-negative Borel functions called *pseudo-gradients*, which exert some control from above on the variation of u. The pseudo-gradients are not explicitly specified, but they are rather supposed

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to fulfil a list of axioms. Then the space $W^{1,p}(\mathrm{X},\mathsf{d},\mathfrak{m},D)$ is defined as the set of all functions in $L^p(\mathfrak{m})$ admitting a pseudo-gradient in $L^p(\mathfrak{m})$. By means of standard functional analytic techniques, it is possible to associate to any Sobolev function $u\in W^{1,p}(\mathrm{X},\mathsf{d},\mathfrak{m},D)$ a uniquely determined minimal object $\underline{D}u\in D[u]\cap L^p(\mathfrak{m})$, called *minimal pseudo-gradient* of u.

More recently, the first author of the present paper introduced a differential structure on general metric measure spaces (cf. [4, 5]). The purpose was to develop a second-order differential calculus on spaces satisfying lower Ricci curvature bounds (or briefly, RCD spaces; we refer to [1, 12, 13] for a presentation of this class of spaces). The fundamental tools for this theory are the L^p-normed L^∞-modules, among which a special role is played by the *cotangent module*, denoted by $L^2(T^*\mathrm{X})$. Its elements can be thought of as 'measurable 1-forms on X'.

The main result of this paper – namely Theorem 3.2 – says that any D-structure (satisfying suitable locality properties) gives rise to a natural notion of cotangent module $L^p(T^*\mathrm{X};D)$, whose properties are analogous to the ones of the cotangent module $L^2(T^*\mathrm{X})$ described in [4]. Roughly speaking, the cotangent module allows us to represent minimal pseudo-gradients as pointwise norms of suitable linear objects. More precisely, this theory provides the existence of an abstract differential $\mathrm{d}:\,W^{1,p}(\mathrm{X},\mathsf{d},\mathfrak{m},D)\to L^p(T^*\mathrm{X};D)$, which is a linear operator such that the pointwise norm $|\mathrm{d}u|\in L^p(\mathfrak{m})$ of $\mathrm{d}u$ coincides with $\underline{D}u$ in the $\mathfrak{m}$-a.e. sense for any function $u\in W^{1,p}(\mathrm{X},\mathsf{d},\mathfrak{m},D)$.

1 General notation

For the purpose of the present paper, a *metric measure space* is a triple $(\mathrm{X},\mathsf{d},\mathfrak{m})$, where

$$
\begin{array}{ll}
(\mathrm{X},\mathsf{d}) & \text{is a complete and separable metric space,}\\
\mathfrak{m}\neq 0 & \text{is a non-negative Borel measure on X, finite on balls.}
\end{array}
\tag{1.1}
$$

Fix $p\in[1,\infty)$. Several functional spaces over X will be used in the forthcoming discussion:

$$
\begin{array}{rl}
L^0(\mathfrak{m}): & \text{the Borel functions } u:\,\mathrm{X}\to\mathbb{R}, \text{ considered up to } \mathfrak{m}\text{-a.e. equality.}\\
L^p(\mathfrak{m}): & \text{the functions } u\in L^0(\mathfrak{m}) \text{ for which } |u|^p \text{ is integrable.}\\
L^p_{loc}(\mathfrak{m}): & \text{the functions } u\in L^0(\mathfrak{m}) \text{ with } u_{|B}\in L^p\big(\mathfrak{m}_{|B}\big) \text{ for any } B\subseteq \mathrm{X} \text{ bounded Borel.}\\
L^\infty(\mathfrak{m}): & \text{the functions } u\in L^0(\mathfrak{m}) \text{ that are essentially bounded.}\\
L^0(\mathfrak{m})^+: & \text{the Borel functions } u:\,\mathrm{X}\to[0,+\infty], \text{ considered up to } \mathfrak{m}\text{-a.e. equality.}\\
L^p(\mathfrak{m})^+: & \text{the functions } u\in L^0(\mathfrak{m})^+ \text{ for which } |u|^p \text{ is integrable.}\\
L^p_{loc}(\mathfrak{m})^+: & \text{the functions } u\in L^0(\mathfrak{m})^+ \text{ with } u_{|B}\in L^p\big(\mathfrak{m}_{|B}\big)^+ \text{ for any } B\subseteq \mathrm{X} \text{ bounded Borel.}\\
\mathrm{LIP}(\mathrm{X}): & \text{the Lipschitz functions } u:\mathrm{X}\to\mathbb{R}, \text{ with Lipschitz constant denoted by } \mathrm{Lip}(u).\\
\mathsf{Sf}(\mathrm{X}): & \text{the functions } u\in L^0(\mathfrak{m}) \text{ that are simple, i.e. with a finite essential image.}
\end{array}
$$

Observe that for any $u\in L^p_{loc}(\mathfrak{m})^+$ it holds that $u(x)<+\infty$ for $\mathfrak{m}$-a.e. $x\in\mathrm{X}$. We also recall that the space $\mathsf{Sf}(\mathrm{X})$ is strongly dense in $L^p(\mathfrak{m})$ for every $p\in[1,\infty]$.

Remark 1.1 In [6, Section 1.1] a more general notion of $L^p_{loc}(\mathfrak{m})$ is considered, based upon the concept of $\mathcal{K}$*-set*. We chose the present approach for simplicity, but the following discussion would remain unaltered if we replaced our definition of $L^p_{loc}(\mathfrak{m})$ with the one of [6]. ■

2 Axiomatic theory of Sobolev spaces

We begin by briefly recalling the axiomatic notion of Sobolev space that has been introduced by V. Gol'dshtein and M. Troyanov in [6, Section 1.2]:

Definition 2.1 (D-structure) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. Let* $p \in [1, \infty)$ *be fixed. Then a* D-structure *on* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is any map* D *associating to each function* $u \in L^p_{loc}(\mathfrak{m})$ *a family* $D[u] \subseteq L^0(\mathfrak{m})^+$ *of* pseudo-gradients *of* u*, which satisfies the following axioms:*

- **A1 (Non triviality)** *It holds that* $\mathrm{Lip}(u)\,\chi_{\{u>0\}} \in D[u]$ *for every* $u \in L^p_{loc}(\mathfrak{m})^+ \cap \mathrm{LIP}(\mathrm{X})$.
- **A2 (Upper linearity)** *Let* $u_1, u_2 \in L^p_{loc}(\mathfrak{m})$ *be fixed. Consider* $g_1 \in D[u_1]$ *and* $g_2 \in D[u_2]$*. Suppose that the inequality* $g \geq |\alpha_1|\,g_1 + |\alpha_2|\,g_2$ *holds* $\mathfrak{m}$*-a.e. in* X *for some* $g \in L^0(\mathfrak{m})^+$ *and* $\alpha_1, \alpha_2 \in \mathbb{R}$*. Then* $g \in D[\alpha_1\,u_1 + \alpha_2\,u_2]$.
- **A3 (Leibniz rule)** *Fix a function* $u \in L^p_{loc}(\mathfrak{m})$ *and a pseudo-gradient* $g \in D[u]$ *of* u*. Then for every* $\varphi \in \mathrm{LIP}(\mathrm{X})$ *bounded it holds that* $g\,\sup_{\mathrm{X}}|\varphi| + \mathrm{Lip}(\varphi)\,|u| \in D[\varphi\,u]$.
- **A4 (Lattice property)** *Fix* $u_1, u_2 \in L^p_{loc}(\mathfrak{m})$*. Given any* $g_1 \in D[u_1]$ *and* $g_2 \in D[u_2]$*, one has that* $\max\{g_1, g_2\} \in D\big[\max\{u_1, u_2\}\big] \cap D\big[\min\{u_1, u_2\}\big]$.
- **A5 (Completeness)** *Consider two sequences* $(u_n)_n \subseteq L^p_{loc}(\mathfrak{m})$ *and* $(g_n)_n \subseteq L^p(\mathfrak{m})$ *that satisfy* $g_n \in D[u_n]$ *for every* $n \in \mathbb{N}$*. Suppose that there exist* $u \in L^p_{loc}(\mathfrak{m})$ *and* $g \in L^p(\mathfrak{m})$ *such that* $u_n \to u$ *in* $L^p_{loc}(\mathfrak{m})$ *and* $g_n \to g$ *in* $L^p(\mathfrak{m})$*. Then* $g \in D[u]$.

Remark 2.2 It follows from axioms **A1** and **A2** that $0 \in D[c]$ for every constant map $c \in \mathbb{R}$. Moreover, axiom **A2** grants that the set $D[u] \cap L^p(\mathfrak{m})$ is convex and that $D[\alpha\,u] = |\alpha|\,D[u]$ for every $u \in L^p_{loc}(\mathfrak{m})$ and $\alpha \in \mathbb{R} \setminus \{0\}$, while axiom **A5** implies that each set $D[u] \cap L^p(\mathfrak{m})$ is closed in the space $L^p(\mathfrak{m})$. ■

Given any Borel set $B \subseteq \mathrm{X}$, we define the *p-Dirichlet energy* of a map $u \in L^p(\mathfrak{m})$ on B as

$$\mathcal{E}_p(u|B) := \inf\left\{\int_B g^p\,\mathrm{d}\mathfrak{m}\;\middle|\; g \in D[u]\right\} \in [0, +\infty]. \tag{2.1}$$

For the sake of brevity, we shall use the notation $\mathcal{E}_p(u)$ to indicate $\mathcal{E}_p(u|\mathrm{X})$.

Definition 2.3 (Sobolev space) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. Let* $p \in [1, \infty)$ *be fixed. Given a* D*-structure on* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$*, we define the* Sobolev class *associated to* D *as*

$$\mathrm{S}^p(\mathrm{X}) = \mathrm{S}^p(\mathrm{X}, \mathsf{d}, \mathfrak{m}, D) := \big\{u \in L^p_{loc}(\mathfrak{m}) \;:\; \mathcal{E}_p(u) < +\infty\big\}. \tag{2.2}$$

Moreover, the Sobolev space *associated to* D *is defined as*

$$W^{1,p}(\mathrm{X}) = W^{1,p}(\mathrm{X}, \mathsf{d}, \mathfrak{m}, D) := L^p(\mathfrak{m}) \cap \mathrm{S}^p(\mathrm{X}, \mathsf{d}, \mathfrak{m}, D). \tag{2.3}$$

Theorem 2.4 *The space $W^{1,p}(\mathrm{X},\mathsf{d},\mathfrak{m},D)$ is a Banach space if endowed with the norm*

$$\|u\|_{W^{1,p}(\mathrm{X})} := \Big(\|u\|^p_{L^p(\mathfrak{m})} + \mathcal{E}_p(u)\Big)^{1/p} \qquad \text{for every } u \in W^{1,p}(\mathrm{X}). \tag{2.4}$$

For a proof of the previous result, we refer to [6, Theorem 1.5].

Proposition 2.5 (Minimal pseudo-gradient) *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space and let $p \in (1,\infty)$. Consider any D-structure on $(\mathrm{X},\mathsf{d},\mathfrak{m})$. Let $u \in \mathrm{S}^p(\mathrm{X})$ be given. Then there exists a unique element $\underline{D}u \in D[u]$, which is called the* minimal pseudo-gradient *of u, such that $\mathcal{E}_p(u) = \|\underline{D}u\|^p_{L^p(\mathfrak{m})}$.*

Both existence and uniqueness of the minimal pseudo-gradient follow from the fact that the set $D[u]\cap L^p(\mathfrak{m})$ is convex and closed by Remark 2.2 and that the space $L^p(\mathfrak{m})$ is uniformly convex; see [6, Proposition 1.22] for the details.

In order to associate a differential structure to an axiomatic Sobolev space, we need to be sure that the pseudo-gradients of a function depend only on the local behaviour of the function itself, in a suitable sense. For this reason, we propose various notions of locality:

Definition 2.6 (Locality) *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Fix $p \in (1,\infty)$. Then we define five notions of locality for D-structures on $(\mathrm{X},\mathsf{d},\mathfrak{m})$:*

L1 *If $B \subseteq \mathrm{X}$ is Borel and $u \in \mathrm{S}^p(\mathrm{X})$ is $\mathfrak{m}$-a.e. constant in B, then $\mathcal{E}_p(u|B) = 0$.*

L2 *If $B \subseteq \mathrm{X}$ is Borel and $u \in \mathrm{S}^p(\mathrm{X})$ is $\mathfrak{m}$-a.e. constant in B, then $\underline{D}u = 0$ $\mathfrak{m}$-a.e. in B.*

L3 *If $u \in \mathrm{S}^p(\mathrm{X})$ and $g \in D[u]$, then $\chi_{\{u>0\}}\, g \in D[u^+]$.*

L4 *If $u \in \mathrm{S}^p(\mathrm{X})$ and $g_1, g_2 \in D[u]$, then $\min\{g_1,g_2\} \in D[u]$.*

L5 *If $u \in \mathrm{S}^p(\mathrm{X})$ then $\underline{D}u \le g$ holds $\mathfrak{m}$-a.e. in X for every $g \in D[u]$.*

Remark 2.7 In the language of [6, Definition 1.11], the properties **L1** and **L3** correspond to *locality* and *strict locality*, respectively. ■

We now discuss the relations among the several notions of locality:

Proposition 2.8 *Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Let $p \in (1,\infty)$. Fix a D-structure on $(\mathrm{X},\mathsf{d},\mathfrak{m})$. Then the following implications hold:*

$$\begin{aligned} \mathbf{L3} \quad &\Longrightarrow \quad \mathbf{L2} \quad \Longrightarrow \quad \mathbf{L1}, \\ \mathbf{L4} \quad &\Longleftrightarrow \quad \mathbf{L5} \\ \mathbf{L1}+\mathbf{L5} \quad &\Longrightarrow \quad \mathbf{L2}+\mathbf{L3}. \end{aligned} \tag{2.5}$$

Proof.
L2 $\Longrightarrow$ **L1**. Simply notice that $\mathcal{E}_p(u|B) \le \int_B (\underline{D}u)^p \, \mathrm{d}\mathfrak{m} = 0$.

L3 $\Longrightarrow$ **L2**. Take a constant $c \in \mathbb{R}$ such that the equality $u = c$ holds $\mathfrak{m}$-a.e. in B. Given that $\underline{D}u \in D[u-c] \cap D[c-u]$ by axiom **A2** and Remark 2.2, we deduce from **L3** that

$$\chi_{\{u>c\}}\, \underline{D}u \in D\big[(u-c)^+\big],$$
$$\chi_{\{u<c\}}\, \underline{D}u \in D\big[(c-u)^+\big].$$

Given that $u - c = (u-c)^+ - (c-u)^+$, by applying again axiom **A2** we see that

$$\chi_{\{u\neq c\}}\, \underline{D}u = \chi_{\{u>c\}}\, \underline{D}u + \chi_{\{u<c\}}\, \underline{D}u \in D[u-c] = D[u].$$

Hence the minimality of $\underline{D}u$ grants that

$$\int_{\mathrm{X}} (\underline{D}u)^p \,\mathrm{d}\mathfrak{m} \leq \int_{\{u\neq c\}} (\underline{D}u)^p \,\mathrm{d}\mathfrak{m},$$

which implies that $\underline{D}u = 0$ holds $\mathfrak{m}$-a.e. in $\{u = c\}$, thus also $\mathfrak{m}$-a.e. in B. This means that the D-structure satisfies the property **L2**, as required.
L4 $\Longrightarrow$ **L5**. We argue by contradiction: suppose the existence of $u \in \mathrm{S}^p(\mathrm{X})$ and $g \in D[u]$ such that $\mathfrak{m}\big(\{\underline{D}u > g\}\big) > 0$, whence $h := \min\{\underline{D}u, g\} \in L^p(\mathfrak{m})$ satisfies $\int h^p \,\mathrm{d}\mathfrak{m} < \int (\underline{D}u)^p \,\mathrm{d}\mathfrak{m}$. Since $h \in D[u]$ by **L4**, we deduce that $\mathcal{E}_p(u) < \int (\underline{D}u)^p \,\mathrm{d}\mathfrak{m}$, getting a contradiction.
L5 $\Longrightarrow$ **L4**. Since $\underline{D}u \leq g_1$ and $\underline{D}u \leq g_2$ hold $\mathfrak{m}$-a.e., we see that $\underline{D}u \leq \min\{g_1, g_2\}$ holds $\mathfrak{m}$-a.e. as well. Therefore $\min\{g_1, g_2\} \in D[u]$ by **A2**.
L1+L5 $\Longrightarrow$ **L2+L3**. Property **L1** grants the existence of $(g_n)_n \subseteq D[u]$ with $\int_B (g_n)^p \,\mathrm{d}\mathfrak{m} \to 0$. Hence **L5** tells us that $\int_B (\underline{D}u)^p \,\mathrm{d}\mathfrak{m} \leq \lim_n \int_B (g_n)^p \,\mathrm{d}\mathfrak{m} = 0$, which implies that $\underline{D}u = 0$ holds $\mathfrak{m}$-a.e. in B, yielding **L2**. We now prove the validity of **L3**: it holds that $D[u] \subseteq D[u^+]$, because we know that $h = \max\{h, 0\} \in D\big[\max\{u, 0\}\big] = D[u^+]$ for every $h \in D[u]$ by **A4** and $0 \in D[0]$, in particular $u^+ \in \mathrm{S}^p(\mathrm{X})$. Given that $u^+ = 0$ $\mathfrak{m}$-a.e. in the set $\{u \leq 0\}$, one has that $\underline{D}u^+ = 0$ holds $\mathfrak{m}$-a.e. in $\{u \leq 0\}$ by **L2**. Hence for any $g \in D[u]$ we have $\underline{D}u^+ \leq \chi_{\{u>0\}}\, g$ by **L5**, which implies that $\chi_{\{u>0\}}\, g \in D[u^+]$ by **A2**. Therefore **L3** is proved. $\square$

Definition 2.9 (Pointwise local) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and* $p \in (1, \infty)$*. Then a* D*-structure on* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *is said to be* pointwise local *provided it satisfies* **L1** *and* **L5** *(thus also* **L2**, **L3** *and* **L4** *by Proposition 2.8).*

We now recall other two notions of locality for D-structures that appeared in the literature:

Definition 2.10 (Strong locality) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and* $p \in (1, \infty)$*. Consider a* D*-structure on* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$*. Then we give the following definitions:*

i) *We say that* D *is* strongly local in the sense of Timoshin *provided*

$$\chi_{\{u_1<u_2\}}\, g_1 + \chi_{\{u_2<u_1\}}\, g_2 + \chi_{\{u_1=u_2\}}\, (g_1 \wedge g_2) \in D[u_1 \wedge u_2] \tag{2.6}$$

whenever $u_1, u_2 \in \mathrm{S}^p(\mathrm{X})$*,* $g_1 \in D[u_1]$ *and* $g_2 \in D[u_2]$*.*

ii) *We say that D is* strongly local in the sense of Shanmugalingam *provided*

$$\chi_B\, g_1 + \chi_{\mathrm{X}\setminus B}\, g_2 \in D[u_2] \quad \text{for every } g_1 \in D[u_1] \text{ and } g_2 \in D[u_2] \tag{2.7}$$

whenever $u_1, u_2 \in \mathrm{S}^p(\mathrm{X})$ satisfy $u_1 = u_2$ $\mathfrak{m}$-a.e. on some Borel set $B \subseteq \mathrm{X}$.

The above two notions of strong locality have been proposed in [11] and [10], respectively. We now prove that they are actually both equivalent to our pointwise locality property:

Lemma 2.11 *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space and $p \in (1, \infty)$. Fix any D-structure on $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$. Then the following are equivalent:*

i) *D is pointwise local.*

ii) *D is strongly local in the sense of Shanmugalingam.*

iii) *D is strongly local in the sense of Timoshin.*

Proof.
i) $\Longrightarrow$ ii) Fix $u_1, u_2 \in \mathrm{S}^p(\mathrm{X})$ such that $u_1 = u_2$ $\mathfrak{m}$-a.e. on some $E \subseteq \mathrm{X}$ Borel. Pick $g_1 \in D[u_1]$ and $g_2 \in D[u_2]$. Observe that $\underline{D}(u_2 - u_1) + g_1 \in D\big[(u_2 - u_1) + u_1\big] = D[u_2]$ by **A2**, so that we have $\big(\underline{D}(u_2 - u_1) + g_1\big) \wedge g_2 \in D[u_2]$ by **L4**. Since $\underline{D}(u_2 - u_1) = 0$ $\mathfrak{m}$-a.e. on B by **L2**, we see that $\chi_B\, g_1 + \chi_{\mathrm{X}\setminus B}\, g_2 \geq \big(\underline{D}(u_2 - u_1) + g_1\big) \wedge g_2$ holds $\mathfrak{m}$-a.e. in X, whence accordingly we conclude that $\chi_B\, g_1 + \chi_{\mathrm{X}\setminus B}\, g_2 \in D[u_2]$ by **A2**. This shows the validity of ii).
ii) $\Longrightarrow$ i) First of all, let us prove **L1**. Let $u \in \mathrm{S}^p(\mathrm{X})$ and $c \in \mathbb{R}$ satisfy $u = c$ $\mathfrak{m}$-a.e. on some Borel set $B \subseteq \mathrm{X}$. Given any $g \in D[u]$, we deduce from ii) that $\chi_{\mathrm{X}\setminus B}\, g \in D[u]$, thus accordingly $\mathcal{E}_p(u|B) \leq \int_B (\chi_{\mathrm{X}\setminus B}\, g)^p \,\mathrm{d}\mathfrak{m} = 0$. This proves the property **L1**.

To show property **L4**, fix $u \in \mathrm{S}^p(\mathrm{X})$ and $g_1, g_2 \in D[u]$. Let us denote $B := \{g_1 \leq g_2\}$. Therefore ii) grants that $g_1 \wedge g_2 = \chi_B\, g_1 + \chi_{\mathrm{X}\setminus B}\, g_2 \in D[u]$, thus obtaining **L4**. By recalling Proposition 2.8, we conclude that D is pointwise local.
i) + ii) $\Longrightarrow$ iii) Fix $u_1, u_2 \in \mathrm{S}^p(\mathrm{X})$, $g_1 \in D[u_1]$ and $g_2 \in D[u_2]$. Recall that $g_1 \vee g_2 \in D[u_1 \wedge u_2]$ by axiom **A4**. Hence by using property ii) twice we obtain that

$$\begin{aligned} &\chi_{\{u_1 \leq u_2\}}\, g_1 + \chi_{\{u_1 > u_2\}}\, (g_1 \vee g_2) \in D[u_1 \wedge u_2], \\ &\chi_{\{u_2 \leq u_1\}}\, g_2 + \chi_{\{u_2 > u_1\}}\, (g_1 \vee g_2) \in D[u_1 \wedge u_2]. \end{aligned} \tag{2.8}$$

The pointwise minimum between the two functions that are written in (2.8) – namely given by $\chi_{\{u_1 < u_2\}}\, g_1 + \chi_{\{u_2 < u_1\}}\, g_2 + \chi_{\{u_1 = u_2\}}\, (g_1 \wedge g_2)$ – belongs to the class $D[u_1 \wedge u_2]$ as well by property **L4**, thus showing iii).
iii) $\Longrightarrow$ i) First of all, let us prove **L1**. Fix a function $u \in \mathrm{S}^p(\mathrm{X})$ that is $\mathfrak{m}$-a.e. equal to some constant $c \in \mathbb{R}$ on a Borel set $B \subseteq \mathrm{X}$. By using iii) and the fact that $0 \in D[0]$, we have that

$$\begin{aligned} \chi_{\{u < c\}}\, g &\in D\big[(u - c) \wedge 0\big] = D\big[-(u - c)^+\big] = D\big[(u - c)^+\big], \\ \chi_{\{u > c\}}\, g &\in D\big[(c - u) \wedge 0\big] = D\big[-(c - u)^+\big] = D\big[(c - u)^+\big]. \end{aligned} \tag{2.9}$$

Since $u - c = (u-c)^+ - (c-u)^+$, we know from **A2** and (2.9) that

$$\chi_{\{u\neq c\}}\, g = \chi_{\{u<c\}}\, g + \chi_{\{u>c\}}\, g \in D[u-c] = D[u],$$

whence $\mathcal{E}_p(u|B) \leq \int_B (\chi_{\{u\neq c\}}\, g)^p \, \mathrm{d}\mathfrak{m} = 0$. This proves the property **L1**.

To show property **L4**, fix $u \in \mathrm{S}^p(\mathrm{X})$ and $g_1, g_2 \in D[u]$. Hence (2.6) with $u_1 = u_2 := u$ simply reads as $g_1 \wedge g_2 \in D[u]$, which gives **L4**. This proves that D is pointwise local. $\square$

Remark 2.12 (L1 does not imply L2) In general, as we are going to show in the following example, it can happen that a D-structure satisfies **L1** but not **L2**.

Let $G = (V, E)$ be a locally finite connected graph. The distance $\mathsf{d}(x, y)$ between two vertices $x, y \in V$ is defined as the minimum length of a path joining x to y, while as a reference measure $\mathfrak{m}$ on V we choose the counting measure. Notice that any function $u : V \to \mathbb{R}$ is locally Lipschitz and that any bounded subset of V is finite. We define a D-structure on the metric measure space $(V, \mathsf{d}, \mathfrak{m})$ in the following way:

$$D[u] := \Big\{ g : V \to [0, +\infty] \;\Big|\; \big|u(x) - u(y)\big| \leq g(x) + g(y) \text{ for any } x, y \in V \text{ with } x \sim y \Big\} \tag{2.10}$$

for every $u : V \to \mathbb{R}$, where the notation $x \sim y$ indicates that x and y are adjacent vertices, i.e. that there exists an edge in E joining x to y.

We claim that D fulfills **L1**. To prove it, suppose that some function $u : \mathrm{X} \to \mathbb{R}$ is constant on some set $B \subseteq V$, say $u(x) = c$ for every $x \in B$. Define the function $g : V \to [0, +\infty)$ as

$$g(x) := \begin{cases} 0 & \text{if } x \in B, \\ |c| + \big|u(x)\big| & \text{if } x \in V \setminus B. \end{cases}$$

Hence $g \in D[u]$ and $\int_B g^p \, \mathrm{d}\mathfrak{m} = 0$, so that $\mathcal{E}_p(u|B) = 0$. This proves the validity of **L1**.

On the other hand, if V contains more than one vertex, then **L2** is not satisfied. Indeed, consider any non-constant function $u : V \to \mathbb{R}$. Clearly any pseudo-gradient $g \in D[u]$ of u is not identically zero, thus there exists $x \in V$ such that $\underline{D}u(x) > 0$. Since u is trivially constant on the set $\{x\}$, we then conclude that property **L2** does not hold. ■

Hereafter, we shall focus our attention on the pointwise local D-structures. Under these locality assumptions, one can show the following calculus rules for minimal pseudo-gradients, whose proof is suitably adapted from analogous results that have been proved in [2].

Proposition 2.13 (Calculus rules for $\underline{D}u$) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and let* $p \in (1, \infty)$. *Consider a pointwise local D-structure on* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$. *Then the following hold:*

i) *Let* $u \in \mathrm{S}^p(\mathrm{X})$ *and let* $N \subseteq \mathbb{R}$ *be a Borel set with* $\mathcal{L}^1(N) = 0$. *Then the equality* $\underline{D}u = 0$ *holds* $\mathfrak{m}$*-a.e. in* $u^{-1}(N)$.

ii) Chain rule. *Let* $u \in \mathrm{S}^p(\mathrm{X})$ *and* $\varphi \in \mathrm{LIP}(\mathbb{R})$. *Then* $|\varphi'| \circ u \, \underline{D}u \in D[\varphi \circ u]$. *More precisely,* $\varphi \circ u \in \mathrm{S}^p(\mathrm{X})$ *and* $\underline{D}(\varphi \circ u) = |\varphi'| \circ u \, \underline{D}u$ *holds* $\mathfrak{m}$*-a.e. in* X.

iii) LEIBNIZ RULE. *Let $u, v \in \mathrm{S}^p(\mathrm{X}) \cap L^\infty(\mathfrak{m})$. Then $|u|\,\underline{D}v + |v|\,\underline{D}u \in D[uv]$. In other words, $uv \in \mathrm{S}^p(\mathrm{X}) \cap L^\infty(\mathfrak{m})$ and $\underline{D}(uv) \leq |u|\,\underline{D}v + |v|\,\underline{D}u$ holds $\mathfrak{m}$-a.e. in X.*

Proof.
STEP 1. First, consider φ affine, say $\varphi(t) = \alpha\, t + \beta$. Then $|\varphi'| \circ u\, \underline{D}u = |\alpha|\, \underline{D}u \in D[\varphi \circ u]$ by Remark 2.2 and **A2**. Now suppose that the function φ is piecewise affine, i.e. there exists a sequence $(a_k)_{k\in\mathbb{Z}} \subseteq \mathbb{R}$, with $a_k < a_{k+1}$ for all $k \in \mathbb{Z}$ and $a_0 = 0$, such that each $\varphi|_{[a_k,a_{k+1}]}$ is an affine function. Let us denote $A_k := u^{-1}\big([a_k, a_{k+1})\big)$ and $u_k := (u \vee a_k) \wedge a_{k+1}$ for every index $k \in \mathbb{Z}$. By combining **L3** with the axioms **A2** and **A5**, we can see that $\chi_{A_k}\, \underline{D}u \in D[u_k]$ for every $k \in \mathbb{Z}$. Called $\varphi_k : \mathbb{R} \to \mathbb{R}$ that affine function coinciding with φ on $[a_k, a_{k+1})$, we deduce from the previous case that $|\varphi_k'| \circ u_k\, \underline{D}u_k \in D[\varphi_k \circ u_k] = D[\varphi \circ u_k]$, whence we have that $|\varphi'| \circ u_k\, \chi_{A_k}\, \underline{D}u \in D[\varphi \circ u_k]$ by **L5**, **A2** and **L2**. Let us define $(v_n)_n \subseteq \mathrm{S}^p(\mathrm{X})$ as

$$v_n := \varphi(0) + \sum_{k=0}^{n} \big(\varphi \circ u_k - \varphi(a_k)\big) + \sum_{k=-n}^{-1} \big(\varphi \circ u_k - \varphi(a_{k+1})\big) \quad \text{for every } n \in \mathbb{N}.$$

Hence $g_n := \sum_{k=-n}^{n} |\varphi'| \circ u_k\, \chi_{A_k}\, \underline{D}u \in D[v_n]$ for all $n \in \mathbb{N}$ by **A2** and Remark 2.2. Given that one has $v_n \to \varphi \circ u$ in $L^p_{loc}(\mathfrak{m})$ and $g_n \to |\varphi'| \circ u\, \underline{D}u$ in $L^p(\mathfrak{m})$ as $n \to \infty$, we finally conclude that $|\varphi'| \circ u\, \underline{D}u \in D[\varphi \circ u]$, as required.
STEP 2. We aim to prove the chain rule for $\varphi \in C^1(\mathbb{R}) \cap \mathrm{LIP}(\mathbb{R})$. For any $n \in \mathbb{N}$, let us denote by φ_n the piecewise affine function interpolating the points $\big(k/2^n, \varphi(k/2^n)\big)$ with $k \in \mathbb{Z}$. We call $D \subseteq \mathbb{R}$ the countable set $\big\{k/2^n \,:\, k \in \mathbb{Z},\, n \in \mathbb{N}\big\}$. Therefore φ_n uniformly converges to φ and $\varphi_n'(t) \to \varphi'(t)$ for all $t \in \mathbb{R} \setminus D$. In particular, the functions $g_n := |\varphi_n'| \circ u\, \underline{D}u$ converge $\mathfrak{m}$-a.e. to $|\varphi'| \circ u\, \underline{D}u$ by **L2**. Moreover, $\mathrm{Lip}(\varphi_n) \leq \mathrm{Lip}(\varphi)$ for every $n \in \mathbb{N}$ by construction, so that $(g_n)_n$ is a bounded sequence in $L^p(\mathfrak{m})$. This implies that (up to a not relabeled subsequence) $g_n \rightharpoonup |\varphi'| \circ u\, \underline{D}u$ weakly in $L^p(\mathfrak{m})$. Now apply Mazur lemma: for any $n \in \mathbb{N}$, there exists $(\alpha_i^n)_{i=n}^{N_n} \subseteq [0,1]$ such that $\sum_{i=n}^{N_n} \alpha_i^n = 1$ and $h_n := \sum_{i=n}^{N_n} \alpha_i^n\, g_i \overset{n}{\to} |\varphi'| \circ u\, \underline{D}u$ strongly in $L^p(\mathfrak{m})$. Given that $g_n \in D[\varphi_n \circ u]$ for every $n \in \mathbb{N}$ by STEP 1, we deduce from axiom **A2** that $h_n \in D[\psi_n \circ u]$ for every $n \in \mathbb{N}$, where $\psi_n := \sum_{i=n}^{N_n} \alpha_i^n\, \varphi_i$. Finally, it clearly holds that $\psi_n \circ u \to \varphi \circ u$ in $L^p_{loc}(\mathfrak{m})$, whence $|\varphi'| \circ u\, \underline{D}u \in D[\varphi \circ u]$ by **A5**.
STEP 3. We claim that

$$\underline{D}u = 0 \quad \mathfrak{m}\text{-a.e. in } u^{-1}(K), \qquad \text{for every } K \subseteq \mathbb{R} \text{ compact with } \mathcal{L}^1(K) = 0. \tag{2.11}$$

For any $n \in \mathbb{N} \setminus \{0\}$, define $\psi_n := n\, \mathsf{d}(\cdot, K) \wedge 1$ and denote by φ_n the primitive of ψ_n such that $\varphi_n(0) = 0$. Since each ψ_n is continuous and bounded, any function φ_n is of class C^1 and Lipschitz. By applying the dominated convergence theorem we see that the $\mathcal{L}^1$-measure of the ε-neighbourhood of K converges to 0 as $\varepsilon \searrow 0$, thus accordingly φ_n uniformly converges to $\mathrm{id}_\mathbb{R}$ as $n \to \infty$. This implies that $\varphi_n \circ u \to u$ in $L^p_{loc}(\mathfrak{m})$. Moreover, we know from STEP 2 that $|\psi_n| \circ u\, \underline{D}u \in D[\varphi_n \circ u]$, thus also $\chi_{\mathrm{X} \setminus u^{-1}(K)}\, \underline{D}u \in D[\varphi_n \circ u]$. Hence $\chi_{\mathrm{X} \setminus u^{-1}(K)}\, \underline{D}u \in D[u]$ by **A5**, which forces the equality $\underline{D}u = 0$ to hold $\mathfrak{m}$-a.e. in $u^{-1}(K)$, proving (2.11).
STEP 4. We are in a position to prove i). Choose any $\mathfrak{m}' \in \mathscr{P}(\mathrm{X})$ such that $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$ and call $\mu := u_* \mathfrak{m}'$. Then μ is a Radon measure on $\mathbb{R}$, in particular it is inner regular. We can thus

find an increasing sequence of compact sets $K_n \subseteq N$ such that $\mu\big(N \setminus \bigcup_n K_n\big) = 0$. We already know from Step 3 that $\underline{D}u = 0$ holds $\mathfrak{m}$-a.e. in $\bigcup_n u^{-1}(K_n)$. Since $u^{-1}(N) \setminus \bigcup_n u^{-1}(K_n)$ is $\mathfrak{m}$-negligible by definition of μ, we conclude that $\underline{D}u = 0$ holds $\mathfrak{m}$-a.e. in $u^{-1}(N)$. This shows the validity of property i).

Step 5. We now prove ii). Let us fix $\varphi \in \mathrm{LIP}(\mathbb{R})$. Choose some convolution kernels $(\rho_n)_n$ and define $\varphi_n := \varphi * \rho_n$ for all $n \in \mathbb{N}$. Then $\varphi_n \to \varphi$ uniformly and $\varphi_n' \to \varphi'$ pointwise $\mathcal{L}^1$-a.e., whence accordingly $\varphi_n \circ u \to \varphi \circ u$ in $L^p_{loc}(\mathfrak{m})$ and $|\varphi_n'| \circ u \, \underline{D}u \to |\varphi'| \circ u \, \underline{D}u$ pointwise $\mathfrak{m}$-a.e. in X. Since $|\varphi_n'| \circ u \, \underline{D}u \leq \mathrm{Lip}(\varphi) \, \underline{D}u$ for all $n \in \mathbb{N}$, there exists a (not relabeled) subsequence such that $|\varphi_n'| \circ u \, \underline{D}u \rightharpoonup |\varphi'| \circ u \, \underline{D}u$ weakly in $L^p(\mathfrak{m})$. We know that $|\varphi_n'| \circ u \, \underline{D}u \in D[\varphi_n \circ u]$ for all $n \in \mathbb{N}$ because the chain rule holds for all $\varphi_n \in C^1(\mathbb{R}) \cap \mathrm{LIP}(\mathbb{R})$, hence by combining Mazur lemma and **A5** as in Step 2 we obtain that $|\varphi'| \circ u \, \underline{D}u \in D[\varphi \circ u]$, so that $\varphi \circ u \in \mathrm{S}^p(\mathrm{X})$ and the inequality $\underline{D}(\varphi \circ u) \leq |\varphi'| \circ u \, \underline{D}u$ holds $\mathfrak{m}$-a.e. in X.

Step 6. We conclude the proof of ii) by showing that one actually has $\underline{D}(\varphi \circ u) = |\varphi'| \circ u \, \underline{D}u$. We can suppose without loss of generality that $\mathrm{Lip}(\varphi) = 1$. Let us define the functions $\psi_\pm$ as $\psi_\pm(t) := \pm t - \varphi(t)$ for all $t \in \mathbb{R}$. Then it holds $\mathfrak{m}$-a.e. in $u^{-1}\big(\{\pm\varphi' \geq 0\}\big)$ that

$$\underline{D}u = \underline{D}(\pm u) \leq \underline{D}(\varphi \circ u) + \underline{D}(\psi_\pm \circ u) \leq \big(|\varphi'| \circ u + |\psi_\pm'| \circ u\big) \, \underline{D}u = \underline{D}u,$$

which forces the equality $\underline{D}(\varphi \circ u) = \pm\varphi' \circ u \, \underline{D}u$ to hold $\mathfrak{m}$-a.e. in the set $u^{-1}\big(\{\pm\varphi' \geq 0\}\big)$. This grants the validity of $\underline{D}(\varphi \circ u) = |\varphi'| \circ u \, \underline{D}u$, thus completing the proof of item ii).

Step 7. We show iii) for the case in which $u, v \geq c$ is satisfied $\mathfrak{m}$-a.e. in X, for some $c > 0$. Call $\varepsilon := \min\{c, c^2\}$ and note that the function log is Lipschitz on the interval $[\varepsilon, +\infty)$, then choose any Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$ that coincides with log on $[\varepsilon, +\infty)$. Now call C the constant $\log\big(\|uv\|_{L^\infty(\mathfrak{m})}\big)$ and choose a Lipschitz function $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi = \exp$ on the interval $[\log \varepsilon, C]$. By applying twice the chain rule ii), we thus deduce that $uv \in \mathrm{S}^p(\mathrm{X})$ and the $\mathfrak{m}$-a.e. inequalities

$$\begin{aligned} \underline{D}(uv) &\leq |\psi'| \circ \varphi \circ (uv) \, \underline{D}\big(\varphi \circ (uv)\big) \leq |uv| \, \big(\underline{D}\log u + \underline{D}\log v\big) \\ &= |uv| \left(\frac{\underline{D}u}{|u|} + \frac{\underline{D}v}{|v|} \right) = |u| \, \underline{D}v + |v| \, \underline{D}u. \end{aligned}$$

Therefore the Leibniz rule iii) is verified under the additional assumption that $u, v \geq c > 0$.

Step 8. We conclude by proving item iii) for general $u, v \in \mathrm{S}^p(\mathrm{X}) \cap L^\infty(\mathfrak{m})$. Given any $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, let us denote $I_{n,k} := \big[k/n, (k+1)/n\big)$. Call $\varphi_{n,k} : \mathbb{R} \to \mathbb{R}$ the continuous function that is the identity on $I_{n,k}$ and constant elsewhere. For any $n \in \mathbb{N}$, let us define

$$\begin{aligned} u_{n,k} &:= u - \frac{k-1}{n}, \qquad & \tilde{u}_{n,k} &:= \varphi_{n,k} \circ u - \frac{k-1}{n} \qquad & &\text{for all } k \in \mathbb{Z}, \\ v_{n,\ell} &:= v - \frac{\ell-1}{n}, \qquad & \tilde{v}_{n,\ell} &:= \varphi_{n,\ell} \circ v - \frac{\ell-1}{n} \qquad & &\text{for all } \ell \in \mathbb{Z}. \end{aligned}$$

Notice that the equalities $u_{n,k} = \tilde{u}_{n,k}$ and $v_{n,\ell} = \tilde{v}_{n,\ell}$ hold $\mathfrak{m}$-a.e. in $u^{-1}(I_{n,k})$ and $v^{-1}(I_{n,\ell})$, respectively. Hence $\underline{D}u_{n,k} = \underline{D}\tilde{u}_{n,k} = \underline{D}u$ and $\underline{D}v_{n,\ell} = \underline{D}\tilde{v}_{n,\ell} = \underline{D}v$ hold $\mathfrak{m}$-a.e. in $u^{-1}(I_{n,k})$ and $v^{-1}(I_{n,\ell})$, respectively, but we also have that

$$\underline{D}(u_{n,k} \, v_{n,\ell}) = \underline{D}(\tilde{u}_{n,k} \, \tilde{v}_{n,\ell}) \quad \text{is verified } \mathfrak{m}\text{-a.e. in } u^{-1}(I_{n,k}) \cap v^{-1}(I_{n,\ell}).$$

Moreover, we have the $\mathfrak{m}$-a.e. inequalities $1/n \leq \tilde{u}_{n,k}, \tilde{v}_{n,\ell} \leq 2/n$ by construction. Therefore for any $k, \ell \in \mathbb{Z}$ it holds $\mathfrak{m}$-a.e. in $u^{-1}(I_{n,k}) \cap v^{-1}(I_{n,\ell})$ that

$$\begin{aligned}
\underline{D}(uv) &\leq \underline{D}(\tilde{u}_{n,k}\,\tilde{v}_{n,\ell}) + \frac{|k-1|}{n}\,\underline{D}v_{n,\ell} + \frac{|\ell-1|}{n}\,\underline{D}u_{n,k} \\
&\leq |\tilde{v}_{n,\ell}|\,\underline{D}\tilde{u}_{n,k} + |\tilde{u}_{n,k}|\,\underline{D}\tilde{v}_{n,\ell} + \frac{|k-1|}{n}\,\underline{D}v_{n,\ell} + \frac{|\ell-1|}{n}\,\underline{D}u_{n,k} \\
&\leq \left(|v| + \frac{4}{n}\right)\underline{D}u + \left(|u| + \frac{4}{n}\right)\underline{D}v,
\end{aligned}$$

where the second inequality follows from the case $u, v \geq c > 0$, treated in Step 7. This implies that the inequality $\underline{D}(uv) \leq |u|\,\underline{D}v + |v|\,\underline{D}u + 4\,(\underline{D}u + \underline{D}v)/n$ holds $\mathfrak{m}$-a.e. in X. Given that $n \in \mathbb{N}$ is arbitrary, the Leibniz rule iii) follows. □

3 Cotangent module associated to a D-structure

It is shown in [4] that any metric measure space possesses a first-order differential structure, whose construction relies upon the notion of $L^p(\mathfrak{m})$*-normed* $L^\infty(\mathfrak{m})$*-module*. For completeness, we briefly recall its definition and we refer to [4,5] for a comprehensive exposition of this topic.

Definition 3.1 (Normed module) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space and* $p \in [1, \infty)$*. Then an* $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module *is any quadruplet* $\big(\mathscr{M}, \|\cdot\|_{\mathscr{M}}, \cdot\,, |\cdot|\big)$ *such that*

i) $\big(\mathscr{M}, \|\cdot\|_{\mathscr{M}}\big)$ *is a Banach space,*

ii) $(\mathscr{M}, \cdot)$ *is an algebraic module over the commutative ring* $L^\infty(\mathfrak{m})$*,*

iii) *the* pointwise norm *operator* $|\cdot| : \mathscr{M} \to L^p(\mathfrak{m})^+$ *satisfies*

$$\begin{aligned}
|f \cdot v| &= |f||v| \quad \mathfrak{m}\text{-a.e.} &&\text{for every } f \in L^\infty(\mathfrak{m}) \text{ and } v \in \mathscr{M}, \\
\|v\|_{\mathscr{M}} &= \big\||v|\big\|_{L^p(\mathfrak{m})} &&\text{for every } v \in \mathscr{M}.
\end{aligned} \tag{3.1}$$

A key role in [4] is played by the *cotangent module* $L^2(T^*\mathrm{X})$, which has a structure of $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module; see [5, Theorem/Definition 1.8] for its characterisation. The following result shows that a generalised version of such object can be actually associated to any D-structure, provided the latter is assumed to be pointwise local.

Theorem 3.2 (Cotangent module associated to a D-structure) *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be any metric measure space and let* $p \in (1, \infty)$*. Consider a pointwise local D-structure on* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$*. Then there exists a unique couple* $\big(L^p(T^*\mathrm{X}; D), \mathrm{d}\big)$*, where* $L^p(T^*\mathrm{X}; D)$ *is an* $L^p(\mathfrak{m})$*-normed* $L^\infty(\mathfrak{m})$*-module and* $\mathrm{d} : \mathrm{S}^p(\mathrm{X}) \to L^p(T^*\mathrm{X}; D)$ *is a linear map, such that the following hold:*

i) *the equality* $|\mathrm{d}u| = \underline{D}u$ *is satisfied* $\mathfrak{m}$*-a.e. in* X *for every* $u \in \mathrm{S}^p(\mathrm{X})$*,*

ii) *the vector space* $\mathcal{V}$ *of all elements of the form* $\sum_{i=1}^n \chi_{B_i}\,\mathrm{d}u_i$*, where* $(B_i)_i$ *is a Borel partition of* X *and* $(u_i)_i \subseteq \mathrm{S}^p(\mathrm{X})$*, is dense in the space* $L^p(T^*\mathrm{X}; D)$*.*

Uniqueness has to be intended up to unique isomorphism: given another such couple $(\mathscr{M},\mathrm{d}')$, there is a unique isomorphism $\Phi:\,L^p(T^*\mathrm{X};D)\to\mathscr{M}$ such that $\Phi(\mathrm{d}u)=\mathrm{d}'u$ for all $u\in S^p(\mathrm{X})$.

The space $L^p(T^*\mathrm{X};D)$ is called cotangent module*, while the map* d *is called* differential.

Proof.
UNIQUENESS. Consider any element $\omega\in\mathcal{V}$ written as $\omega=\sum_{i=1}^n\chi_{B_i}\,\mathrm{d}u_i$, with $(B_i)_i$ Borel partition of X and $u_1,\dots,u_n\in S^p(\mathrm{X})$. Notice that the requirements that Φ is $L^\infty(\mathfrak{m})$-linear and $\Phi\circ\mathrm{d}=\mathrm{d}'$ force the definition $\Phi(\omega):=\sum_{i=1}^n\chi_{B_i}\,\mathrm{d}'u_i$. The $\mathfrak{m}$-a.e. equality

$$\big|\Phi(\omega)\big|=\sum_{i=1}\chi_{B_i}\,|\mathrm{d}'u_i|=\sum_{i=1}^n\chi_{B_i}\,\underline{D}u_i=\sum_{i=1}^n\chi_{B_i}\,|\mathrm{d}u_i|=|\omega|$$

grants that $\Phi(\omega)$ is well-defined, in the sense that it does not depend on the particular way of representing ω, and that $\Phi:\,\mathcal{V}\to\mathscr{M}$ preserves the pointwise norm. In particular, one has that the map $\Phi:\,\mathcal{V}\to\mathscr{M}$ is (linear and) continuous. Since $\mathcal{V}$ is dense in $L^p(T^*\mathrm{X};D)$, we can uniquely extend Φ to a linear and continuous map $\Phi:\,L^p(T^*\mathrm{X};D)\to\mathscr{M}$, which also preserves the pointwise norm. Moreover, we deduce from the very definition of Φ that the identity $\Phi(h\,\omega)=h\,\Phi(\omega)$ holds for every $\omega\in\mathcal{V}$ and $h\in\mathsf{Sf}(\mathrm{X})$, whence the $L^\infty(\mathfrak{m})$-linearity of Φ follows by an approximation argument. Finally, the image $\Phi(\mathcal{V})$ is dense in $\mathscr{M}$, which implies that Φ is surjective. Therefore Φ is the unique isomorphism satisfying $\Phi\circ\mathrm{d}=\mathrm{d}'$.
EXISTENCE. First of all, let us define the *pre-cotangent module* as

$$\mathsf{Pcm}:=\left\{\big\{(B_i,u_i)\big\}_{i=1}^n\;\middle|\;\begin{array}{c}n\in\mathbb{N},\;u_1,\dots,u_n\in \mathrm{S}^p(\mathrm{X}),\\(B_i)_{i=1}^n\text{ Borel partition of X}\end{array}\right\}.$$

We define an equivalence relation on Pcm as follows: we declare that $\big\{(B_i,u_i)\big\}_i\sim\big\{(C_j,v_j)\big\}_j$ provided $\underline{D}(u_i-v_j)=0$ holds $\mathfrak{m}$-a.e. on $B_i\cap C_j$ for every i,j. The equivalence class of an element $\big\{(B_i,u_i)\big\}_i$ of Pcm will be denoted by $[B_i,u_i]_i$. We can endow the quotient $\mathsf{Pcm}/\sim$ with a vector space structure:

$$\begin{aligned}[B_i,u_i]_i+[C_j,v_j]_j&:=[B_i\cap C_j,u_i+v_j]_{i,j},\\\lambda\,[B_i,u_i]_i&:=[B_i,\lambda\,u_i]_i,\end{aligned}\tag{3.2}$$

for every $[B_i,u_i]_i,[C_j,v_j]_j\in\mathsf{Pcm}/\sim$ and $\lambda\in\mathbb{R}$. We only check that the sum operator is well-defined; the proof of the well-posedness of the multiplication by scalars follows along the same lines. Suppose that $\big\{(B_i,u_i)\big\}_i\sim\big\{(B'_k,u'_k)\big\}_k$ and $\big\{(C_j,v_j)\big\}_j\sim\big\{(C'_\ell,v'_\ell)\big\}_\ell$, in other words $\underline{D}(u_i-u'_k)=0$ $\mathfrak{m}$-a.e. on $B_i\cap B'_k$ and $\underline{D}(v_j-v'_\ell)=0$ $\mathfrak{m}$-a.e. on $C_j\cap C'_\ell$ for every i,j,k,ℓ, whence accordingly

$$\underline{D}\big((u_i+v_j)-(u'_k+v'_\ell)\big)\overset{\mathbf{L5}}{\leq}\underline{D}(u_i-u'_k)+\underline{D}(v_j-v'_\ell)=0\quad\text{ holds }\mathfrak{m}\text{-a.e. on }(B_i\cap C_j)\cap(B'_k\cap C'_\ell).$$

This shows that $\big\{(B_i\cap C_j,u_i+v_j)\big\}_{i,j}\sim\big\{(B'_k\cap C'_\ell,u'_k+v'_\ell)\big\}_{k,\ell}$, thus proving that the sum operator defined in (3.2) is well-posed. Now let us define

$$\big\|[B_i,u_i]_i\big\|_{L^p(T^*\mathrm{X};D)}:=\sum_{i=1}^n\bigg(\int_{B_i}(\underline{D}u_i)^p\,\mathrm{d}\mathfrak{m}\bigg)^{1/p}\qquad\text{for every }[B_i,u_i]_i\in\mathsf{Pcm}/\sim.\tag{3.3}$$

Such definition is well-posed: if $\{(B_i,u_i)\}_i \sim \{(C_j,v_j)\}_j$ then for all i,j it holds that

$$|\underline{D}u_i - \underline{D}v_j| \overset{\textbf{L5}}{\leq} \underline{D}(u_i - v_j) = 0 \quad \mathfrak{m}\text{-a.e. on } B_i \cap C_j,$$

i.e. that the equality $\underline{D}u_i = \underline{D}v_j$ is satisfied $\mathfrak{m}$-a.e. on $B_i \cap C_j$. Therefore one has that

$$\begin{aligned}\sum_i \left(\int_{B_i} (\underline{D}u_i)^p \, \mathrm{d}\mathfrak{m} \right)^{1/p} &= \sum_{i,j} \left(\int_{B_i \cap C_j} (\underline{D}u_i)^p \, \mathrm{d}\mathfrak{m} \right)^{1/p} = \sum_{i,j} \left(\int_{B_i \cap C_j} (\underline{D}v_j)^p \, \mathrm{d}\mathfrak{m} \right)^{1/p} \\ &= \sum_j \left(\int_{C_j} (\underline{D}v_j)^p \, \mathrm{d}\mathfrak{m} \right)^{1/p},\end{aligned}$$

which grants that $\|\cdot\|_{L^p(T^*\mathrm{X};D)}$ in (3.3) is well-defined. The fact that it is a norm on $\mathsf{Pcm}/\sim$ easily follows from standard verifications. Hence let us define

$$\begin{aligned}&L^p(T^*\mathrm{X};D) := \text{completion of } \big(\mathsf{Pcm}/\sim, \|\cdot\|_{L^p(T^*\mathrm{X};D)}\big),\\ &\mathrm{d} : \mathrm{S}^p(\mathrm{X}) \to L^p(T^*\mathrm{X};D), \quad \mathrm{d}u := [\mathrm{X},u] \text{ for every } u \in \mathrm{S}^p(\mathrm{X}).\end{aligned}$$

Observe that $L^p(T^*\mathrm{X};D)$ is a Banach space and that d is a linear operator. Furthermore, given any $[B_i,u_i]_i \in \mathsf{Pcm}/\sim$ and $h = \sum_j \lambda_j \, \chi_{C_j} \in \mathsf{Sf}(\mathrm{X})$, where $(\lambda_j)_j \subseteq \mathbb{R}$ and $(C_j)_j$ is a Borel partition of X, we set

$$\begin{aligned}\big|[B_i,u_i]_i\big| &:= \sum_i \chi_{B_i} \, \underline{D}u_i,\\ h\,[B_i,u_i]_i &:= [B_i \cap C_j, \lambda_j \, u_i]_{i,j}.\end{aligned}$$

One can readily prove that such operations, which are well-posed again by the pointwise locality of D, can be uniquely extended to a pointwise norm $|\cdot| : L^p(T^*\mathrm{X};D) \to L^p(\mathfrak{m})^+$ and to a multiplication by L^∞-functions $L^\infty(\mathfrak{m}) \times L^p(T^*\mathrm{X};D) \to L^p(T^*\mathrm{X};D)$, respectively. Therefore the space $L^p(T^*\mathrm{X};D)$ turns out to be an $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module when equipped with the operations described so far. In order to conclude, it suffices to notice that

$$|\mathrm{d}u| = \big|[\mathrm{X},u]\big| = \underline{D}u \quad \text{holds } \mathfrak{m}\text{-a.e.} \qquad \text{for every } u \in \mathrm{S}^p(\mathrm{X})$$

and that $[B_i,u_i]_i = \sum_i \chi_{B_i} \, \mathrm{d}u_i$ for all $[B_i,u_i]_i \in \mathsf{Pcm}/\sim$, giving i) and ii), respectively. $\square$

In full analogy with the properties of the cotangent module that is studied in [4], we can show that the differential d introduced in Theorem 3.2 is a closed operator, which satisfies both the chain rule and the Leibniz rule.

Theorem 3.3 (Closure of the differential) *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space and let* $p \in (1,\infty)$. *Consider a pointwise local* D*-structure on* $(\mathrm{X},\mathsf{d},\mathfrak{m})$. *Then the differential operator* d *is* closed, *i.e. if a sequence* $(u_n)_n \subseteq \mathrm{S}^p(\mathrm{X})$ *converges in* $L^p_{loc}(\mathfrak{m})$ *to some* $u \in L^p_{loc}(\mathfrak{m})$ *and* $\mathrm{d}u_n \rightharpoonup \omega$ *weakly in* $L^p(T^*\mathrm{X};D)$ *for some* $\omega \in L^p(T^*\mathrm{X};D)$, *then* $u \in \mathrm{S}^p(\mathrm{X})$ *and* $\mathrm{d}u = \omega$.

Proof. Since d is linear, we can assume with no loss of generality that $\mathrm{d}u_n \to \omega$ in $L^p(T^*\mathrm{X};D)$ by Mazur lemma, so that $\mathrm{d}(u_n - u_m) \to \omega - \mathrm{d}u_m$ in $L^p(T^*\mathrm{X};D)$ for any $m \in \mathbb{N}$. In particular, one has $u_n - u_m \to u - u_m$ in $L^p_{loc}(\mathfrak{m})$ and $\underline{D}(u_n - u_m) = \big|\mathrm{d}(u_n - u_m)\big| \to |\omega - \mathrm{d}u_m|$ in $L^p(\mathfrak{m})$ as $n \to \infty$ for all $m \in \mathbb{N}$, whence $u - u_m \in \mathrm{S}^p(\mathrm{X})$ and $\underline{D}(u - u_m) \le |\omega - \mathrm{d}u_m|$ holds $\mathfrak{m}$-a.e. for all $m \in \mathbb{N}$ by **A5** and **L5**. Therefore $u = (u - u_0) + u_0 \in \mathrm{S}^p(\mathrm{X})$ and

$$\begin{aligned}\varlimsup_{m\to\infty} \|\mathrm{d}u - \mathrm{d}u_m\|_{L^p(T^*\mathrm{X};D)} &= \varlimsup_{m\to\infty} \big\|\underline{D}(u - u_m)\big\|_{L^p(\mathfrak{m})} \le \varlimsup_{m\to\infty} \|\omega - \mathrm{d}u_m\|_{L^p(T^*\mathrm{X};D)} \\ &= \varlimsup_{m\to\infty} \lim_{n\to\infty} \|\mathrm{d}u_n - \mathrm{d}u_m\|_{L^p(T^*\mathrm{X};D)} = 0,\end{aligned}$$

which grants that $\mathrm{d}u_m \to \mathrm{d}u$ in $L^p(T^*\mathrm{X};D)$ as $m \to \infty$ and accordingly that $\mathrm{d}u = \omega$. □

Proposition 3.4 (Calculus rules for $\mathrm{d}u$**)** *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be any metric measure space and let* $p \in (1,\infty)$*. Consider a pointwise local* D*-structure on* $(\mathrm{X},\mathsf{d},\mathfrak{m})$*. Then the following hold:*

i) *Let* $u \in \mathrm{S}^p(\mathrm{X})$ *and let* $N \subseteq \mathbb{R}$ *be a Borel set with* $\mathcal{L}^1(N) = 0$*. Then* $\chi_{u^{-1}(N)}\,\mathrm{d}u = 0$*.*

ii) Chain rule. *Let* $u \in \mathrm{S}^p(\mathrm{X})$ *and* $\varphi \in \mathrm{LIP}(\mathbb{R})$ *be given. Recall that* $\varphi \circ u \in \mathrm{S}^p(\mathrm{X})$ *by Proposition 2.13. Then* $\mathrm{d}(\varphi \circ u) = \varphi' \circ u\,\mathrm{d}u$*.*

iii) Leibniz rule. *Let* $u, v \in \mathrm{S}^p(\mathrm{X}) \cap L^\infty(\mathfrak{m})$ *be given. Recall that* $uv \in \mathrm{S}^p(\mathrm{X}) \cap L^\infty(\mathfrak{m})$ *by Proposition 2.13. Then* $\mathrm{d}(uv) = u\,\mathrm{d}v + v\,\mathrm{d}u$*.*

Proof.
i) We have that $|\mathrm{d}u| = \underline{D}u = 0$ holds $\mathfrak{m}$-a.e. on $u^{-1}(N)$ by item i) of Proposition 2.13, thus accordingly $\chi_{u^{-1}(N)}\,\mathrm{d}u = 0$, as required.
ii) If φ is an affine function, say $\varphi(t) = \alpha\,t + \beta$, then $\mathrm{d}(\varphi \circ u) = \mathrm{d}(\alpha\,u + \beta) = \alpha\,\mathrm{d}u = \varphi' \circ u\,\mathrm{d}u$. Now suppose that φ is a piecewise affine function. Say that $(I_n)_n$ is a sequence of intervals whose union covers the whole real line $\mathbb{R}$ and that $(\psi_n)_n$ is a sequence of affine functions such that $\varphi|_{I_n} = \psi_n$ holds for every $n \in \mathbb{N}$. Since φ' and ψ_n' coincide $\mathcal{L}^1$-a.e. in the interior of I_n, we have that $\mathrm{d}(\varphi \circ f) = \mathrm{d}(\psi_n \circ f) = \psi_n' \circ f\,\mathrm{d}f = \varphi' \circ f\,\mathrm{d}f$ holds $\mathfrak{m}$-a.e. on $f^{-1}(I_n)$ for all n, so that $\mathrm{d}(\varphi \circ u) = \varphi' \circ u\,\mathrm{d}u$ is verified $\mathfrak{m}$-a.e. on $\bigcup_n u^{-1}(I_n) = \mathrm{X}$.

To prove the case of a general Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$, we want to approximate φ with a sequence of piecewise affine functions: for any $n \in \mathbb{N}$, let us denote by φ_n the function that coincides with φ at $\big\{k/2^n \,:\, k \in \mathbb{Z}\big\}$ and that is affine on the interval $\big[k/2^n, (k+1)/2^n\big]$ for every $k \in \mathbb{Z}$. It is clear that $\mathrm{Lip}(\varphi_n) \le \mathrm{Lip}(\varphi)$ for all $n \in \mathbb{N}$. Moreover, one can readily check that, up to a not relabeled subsequence, $\varphi_n \to \varphi$ uniformly on $\mathbb{R}$ and $\varphi_n' \to \varphi'$ pointwise $\mathcal{L}^1$-almost everywhere. The former grants that $\varphi_n \circ u \to \varphi \circ u$ in $L^p_{loc}(\mathfrak{m})$. Given that $|\varphi_n' - \varphi'|^p \circ u\,(\underline{D}u)^p \le 2^p\,\mathrm{Lip}(\varphi)^p\,(\underline{D}u)^p \in L^1(\mathfrak{m})$ for all $n \in \mathbb{N}$ and $|\varphi_n' - \varphi'|^p \circ u\,(\underline{D}u)^p \to 0$ pointwise $\mathfrak{m}$-a.e. by the latter above together with i), we obtain $\int |\varphi_n' - \varphi'|^p \circ u\,(\underline{D}u)^p\,\mathrm{d}\mathfrak{m} \to 0$ as $n \to \infty$ by the dominated convergence theorem. In other words, $\varphi_n' \circ u\,\mathrm{d}u \to \varphi' \circ u\,\mathrm{d}u$ in the strong topology of $L^p(T^*\mathrm{X};D)$. Hence Theorem 3.3 ensures that $\mathrm{d}(\varphi \circ u) = \varphi' \circ u\,\mathrm{d}u$, thus proving the chain rule ii) for any $\varphi \in \mathrm{LIP}(\mathbb{R})$.

iii) In the case $u, v \geq 1$, we argue as in the proof of Proposition 2.13 to deduce from ii) that

$$\frac{\mathrm{d}(uv)}{uv} = \mathrm{d}\log(uv) = \mathrm{d}\big(\log(u) + \log(v)\big) = \mathrm{d}\log(u) + \mathrm{d}\log(v) = \frac{\mathrm{d}u}{u} + \frac{\mathrm{d}v}{v},$$

whence we get $\mathrm{d}(uv) = u\,\mathrm{d}v + v\,\mathrm{d}u$ by multiplying both sides by uv.

In the general case $u, v \in L^\infty(\mathfrak{m})$, choose a constant $C > 0$ so big that $u + C, v + C \geq 1$. By the case treated above, we know that

$$\begin{aligned}\mathrm{d}\big((u+C)(v+C)\big) &= (u+C)\,\mathrm{d}(v+C) + (v+C)\,\mathrm{d}(u+C)\\ &= (u+C)\,\mathrm{d}v + (v+C)\,\mathrm{d}u \\ &= u\,\mathrm{d}v + v\,\mathrm{d}u + C\,\mathrm{d}(u+v),\end{aligned} \tag{3.4}$$

while a direct computation yields

$$\mathrm{d}\big((u+C)(v+C)\big) = \mathrm{d}\big(uv + C(u+v) + C^2\big) = \mathrm{d}(uv) + C\,\mathrm{d}(u+v). \tag{3.5}$$

By subtracting (3.5) from (3.4), we finally obtain that $\mathrm{d}(uv) = u\,\mathrm{d}v + v\,\mathrm{d}u$, as required. This completes the proof of the Lebniz rule iii). □

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