Quasilinear Lane-Emden equations with absorption and measure data

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Abstract We study the existence of solutions to the equation $-\Delta_p u + g(x,u) = \mu$ when $g(x,.)$ is a nondecreasing function and μ a measure. We characterize the good measures, i.e. the ones for which the problem has a renormalized solution. We study particularly the cases where $g(x,u) = |x|^{-\beta}|u|^{q-1}u$ and $g(x,u) = \text{sign }(u)(e^{\tau|u|^\lambda} - 1)$. The results state that a measure is good if it is absolutely continuous with respect to an appropriate Lorentz-Bessel capacities.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain containing 0 and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function. We assume that for almost all $x \in \Omega$, $r \mapsto g(x,r)$ is nondecreasing and odd. In this article we consider the following problem

$$\begin{aligned} -\Delta_p u + g(x,u) = \mu \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{in } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Delta_p u = \text{div}\left(|\nabla u|^{p-2}\nabla u\right)$, $(1 < p < N)$, is the p-Laplacian and μ a bounded measure. A measure for which the problem admits a solution, in an appropriate class, is called a *good measure*. When $p = 2$ and $g(x,u) = g(u)$ the problem has been considered by Benilan and Brezis [4] in the subcritical case that is when any bounded measure is good. They prove that such is the case if $N \geq 3$ and g satisfies

$$\int_1^\infty g(s)s^{-\frac{N-1}{N-2}}ds < \infty.$$

The supercritical case, always with $p = 2$, has been considered by Baras and Pierre [3] when $g(u) = |u|^{q-1}u$ and $q > 1$. They prove that the corresponding problem to (1.1) admits a solution (always unique in that case) if and only if the measure μ is absolutely continuous with respect to the Bessel capacity $\text{Cap}_{2,q'}$ $(q' = q/(q-1))$. In the case $p \neq 2$ it is shown by Bidaut-Véron [6] that if problem (1.1) with $\beta = 0$ and $g(s) = |s|^{q-1}s$ $(q > p-1 > 0)$ admits a solution, then μ is absolutely continuous with respect to any capacity $\text{Cap}_{p,\frac{q}{q+1-p}+\varepsilon}$ for any $\varepsilon > 0$.

In this article we introduce a new class of Bessel capacities which are modeled on Lorentz spaces $L^{s,q}$ instead of L^q spaces. If G_α is the Bessel kernel of order $\alpha > 0$, we denote by $L^{\alpha,s,q}(\mathbb{R}^N)$ the Besov space which is the space of functions $\phi = G_\alpha * f$ for some $f \in L^{s,q}(\mathbb{R}^N)$ and we set $||\phi||_{\alpha,s,q} = \|f\|_{s,q}$ (a norm which is defined by using rearrangements). Then we set

$$\text{Cap}_{\alpha,s,q}(E) = \inf\{||f||_{s,q} : \ f \geq 0, \ G_\alpha * f \geq 1 \quad \text{on } E\}$$

for any Borel set E. We say that a measure μ in Ω is absolutely continuous with respect to the capacity $\text{Cap}_{\alpha,s,q}$ if ,

$$\forall E \subset \Omega,\ E \text{ Borel },\ \text{Cap}_{\alpha,s,q}(E) = 0 \Longrightarrow |\mu|(E) = 0.$$

We also introduce the Wolff potential of a measure $\mu \in \mathfrak{M}^+(\mathbb{R}^N)$ by

$$\mathbf{W}_{\alpha,s}[\mu](x) = \int_0^\infty \left(\frac{\mu(B_t(x))}{t^{N-\alpha s}}\right)^{\frac{1}{s-1}} \frac{dt}{t}$$

if $\alpha > 0$, $1 < s < \alpha^{-1}N$. When we are dealing with bounded domains $\Omega \subset B_R$ and $\mu \in \mathfrak{M}^+(\Omega)$, it is useful to introduce truncated Wolff potentials.

$$\mathbf{W}^R_{\alpha,s}[\mu](x) = \int_0^R \left(\frac{\mu(B_t(x))}{t^{N-\alpha s}} \right)^{\frac{1}{s-1}} \frac{dt}{t}.$$

We prove the following existence results concerning

$$\begin{aligned} -\Delta_p u + |x|^{-\beta} g(u) &= \mu \qquad &\text{in } \Omega \\ u &= 0 \qquad &\text{in } \partial\Omega \end{aligned} \tag{1.2}$$

Theorem 1.1 *Assume $1 < p < N$, $q > p-1$ and $0 \leq \beta < N$ and μ is a bounded Radon measure in Ω.*
1- If $g(s) = |s|^{q-1}s$, then (1.2) admits a renormalized solution if μ is absolutely continuous with respect to the capacity $Cap_{p,\frac{Nq}{Nq-(p-1)(N-\beta))},\frac{q}{q+1-p}}$.
2- If g satisfies

$$\int_1^\infty g(s)s^{-q-1}ds < \infty \tag{1.3}$$

then (1.2) admits a renormalized solution if μ is absolutely continuous with respect to the capacity $C_{p,\frac{Nq}{Nq-(p-1)(N-\beta))},1}$.
Furthermore, in both case there holds

$$-c\mathbf{W}^{2diam(\Omega)}_{1,p}[\mu^-](x) \leq u(x) \leq c\mathbf{W}^{2diam(\Omega)}_{1,p}[\mu^+](x) \qquad \text{for almost all } x \in \Omega, \tag{1.4}$$

where c is a positive constant depending on p and N.

In order to deal with exponential nonlinearities we introduce for $0 < \alpha < N$ the fractional maximal operator (resp. the truncated fractional maximal operator), defined for a positive measure μ by

$$\mathbf{M}_\alpha[\mu](x) = \sup_{t>0} \frac{\mu(B_t(x))}{t^{N-\alpha}}, \quad \left(\text{resp } \mathbf{M}_{\alpha,R}[\mu](x) = \sup_{0<t<R} \frac{\mu(B_t(x))}{t^{N-\alpha}} \right), \tag{1.5}$$

and the η-fractional maximal operator (resp. the truncated η-fractional maximal operator)

$$\mathbf{M}^\eta_\alpha[\mu](x) = \sup_{t>0} \frac{\mu(B_t(x))}{t^{N-\alpha}h_\eta(t)}, \quad \left(\text{resp } \mathbf{M}^\eta_{\alpha,R}[\mu](x) = \sup_{0<t<R} \frac{\mu(B_t(x))}{t^{N-\alpha}h_\eta(t)} \right), \tag{1.6}$$

where $\eta \geq 0$ and

$$h_\eta(t) = \begin{cases} (-\ln t)^{-\eta} & \text{if } 0 < t < \frac{1}{2} \\ (\ln 2)^{-\eta} & \text{if } t \geq \frac{1}{2} \end{cases} \tag{1.7}$$

Theorem 1.2 *Assume $1 < p < N$, $\tau > 0$ and $\lambda \geq 1$. Then there exists $M > 0$ depending on N, p, τ and λ such that if a measure in Ω, $\mu = \mu^+ - \mu^-$ can be decomposed as follows*

$$\mu^+ = f_1 + \nu_1 \qquad \textit{and } \; \mu^- = f_2 + \nu_2, \tag{1.8}$$

where $f_j \in L^1_+(\Omega)$ *and* $\nu_j \in \mathfrak{M}^b_+(\Omega)$ *(*$j = 1,2$*), and if*

$$\left\| \mathbf{M}^{\frac{(p-1)(\lambda-1)}{\lambda}}_{p,2\mathrm{diam}\,(\Omega)}[\nu_j] \right\|_{L^\infty(\Omega)} < M, \tag{1.9}$$

there exists a renormalized solution to

$$\begin{aligned} -\Delta_p u + \mathrm{sign(u)}\left(\mathrm{e}^{\tau|\mathrm{u}|^\lambda} - 1\right) &= \mu && \text{in } \Omega \\ u &= 0 && \text{in } \partial\Omega. \end{aligned} \tag{1.10}$$

and satisfies (1.4).

Our study is based upon delicate estimates on Wolff potentials and η-fractional maximal operators which are developed in the first part of this paper.

2 Lorentz spaces and capacities

2.1 Lorentz spaces

Let (X, Σ, α) be a measured space. If $f : X \to \mathbb{R}$ is a measurable function, we set $S_f(t) := \{x \in X : |f|(x) > t\}$ and $\lambda_f(t) = \alpha(S_f(t))$. The decreasing rearrangement f^* of f is defined by

$$f^*(t) = \inf\{s > 0 : \lambda_f(s) \le t\}.$$

It is well known that $(\Phi(f))^* = \Phi(f^*)$ for any continuous and nondecreasing function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$. We set

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(\tau)d\tau \qquad \forall t > 0.$$

and, for $1 \le s < \infty$ and $1 < q \le \infty$,

$$\|f\|_{L^{s,q}} = \begin{cases} \left(\displaystyle\int_0^\infty t^{\frac{q}{s}}(f^{**}(t))^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } q < \infty \\ \operatorname*{supess}_{t>0} t^{\frac{1}{s}} f^{**}(t) & \text{if } q = \infty \end{cases} \tag{2.1}$$

It is known that $L^{s,q}(X, \alpha)$ is a Banach space when endowed with the norm $\|.\|_{L^{s,q}}$. Furthermore there holds (see e.g. [12])

$$\left\| t^{\frac{1}{s}} f^* \right\|_{L^q(\mathbb{R}^+, \frac{dt}{t})} \le \|f\|_{L^{s,q}} \le \frac{s}{s-1} \left\| t^{\frac{1}{s}} f^* \right\|_{L^q(\mathbb{R}^+, \frac{dt}{t})}, \tag{2.2}$$

the left-hand side inequality being valid only if $s > 1$. Finally, if $f \in L^{s,q}(\mathbb{R}^N)$ (with $1 \le q, s < \infty$ and α being the Lebesgue measure) and if $\{\rho_n\} \subset C^\infty_c(\mathbb{R}^N)$ is a sequence of mollifiers, $f * \rho_n \to f$ and $(f\chi_{B_n}) * \rho_n \to f$ in $L^{s,q}(\mathbb{R}^N)$, where χ_{B_n} is the indicator function of the ball B_n centered at the origin of radius n. In particular $C^\infty_c(\mathbb{R}^N)$ is dense in $L^{s,q}(\mathbb{R}^N)$.

2.2 Wolff potentials, fractional and η-fractional maximal operators

If D is either a bounded domain or whole $\mathbb{R}^N$, we denote by $\mathfrak{M}(D)$ (resp $\mathfrak{M}^b(D)$) the set of Radon measure (resp. bounded Radon measures) in D. Their positive cones are $\mathfrak{M}_+(D)$ and $\mathfrak{M}^b_+(D)$ respectively. If $0 < R \leq \infty$ and $\mu \in \mathfrak{M}_+(D)$ and $R \geq \operatorname{diam}(D)$, we define, for $\alpha > 0$ and $1 < s < \alpha^{-1}N$, the R-truncated Wolff-potential by

$$\mathbf{W}^R_{\alpha,s}[\mu](x) = \int_0^R \left(\frac{\mu(B_t(x))}{t^{N-\alpha s}}\right)^{\frac{1}{s-1}} \frac{dt}{t} \quad \text{for a.e. } x \in \mathbb{R}^N. \tag{2.3}$$

If $h_\eta(t) = \min\{(-\ln t)^{-\eta}, (\ln 2)^{-\eta}\}$ and $0 < \alpha < N$, the truncated η-fractional maximal operator is

$$\mathbf{M}^\eta_{\alpha,R}[\mu](x) = \sup_{0<t<R} \frac{\mu(B_t(x))}{t^{N-\alpha}h_\eta(t)} \quad \text{for a.e. } x \in \mathbb{R}^N. \tag{2.4}$$

If $R = \infty$, we drop it in expressions (2.3) and (2.4). In particular

$$\mu(B_t(x)) \leq t^{N-\alpha}h_\eta(t)\mathbf{M}^\eta_{\alpha,R}[\mu](x). \tag{2.5}$$

We also define $\mathbf{G}_\alpha$ the Bessel potential of a measure μ by

$$\mathbf{G}_\alpha[\mu](x) = \int_{\mathbb{R}^N} G_\alpha(x-y)d\mu(y) \qquad \forall x \in \mathbb{R}^N, \tag{2.6}$$

where G_α is the Bessel kernel of order α in $\mathbb{R}^N$.

Definition 2.1 *We denote by $L^{\alpha,s,q}(\mathbb{R}^N)$ the Besov space the space of functions $\phi = G_\alpha * f$ for some $f \in L^{s,q}(\mathbb{R}^N)$ and we set $\|\phi\|_{\alpha,s,q} = \|f\|_{s,q}$. If we set*

$$C_{\alpha,s,q}(E) = \inf\{\|f\|_{s,q} : \ f \geq 0, \ G_\alpha * f \geq 1 \quad on \ E\}, \tag{2.7}$$

then $C_{\alpha,s,q}$ is a capacity, see [1].

2.3 Estimates on potentials

In the sequel, we denote by $|A|$ the N-dimensional Lebesgue measure of a measurable set A and, if F, G are functions defined in $\mathbb{R}^N$, we set $\{F > a\} := \{x \in \mathbb{R}^N : F(x) > a\}$, $\{G \leq b\} := \{x \in \mathbb{R}^N : G(x) \leq b\}$ and $\{F > a, G \leq b\} := \{F > a\} \cap \{G \leq b\}$. The following result is an extension of [14, Th 1.1]

Proposition 2.2 *Let $0 \leq \eta < p-1$, $0 < \alpha p < N$ and $r > 0$. There exist $c_0 > 0$ depending on N, α, p, η and $\epsilon_0 > 0$ depending on N, α, p, η, r such that, for all $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$ with diam(supp(μ)) $\leq r$ and $R \in (0, \infty]$, $\epsilon \in (0, \epsilon_0]$, $\lambda > \left(\mu(\mathbb{R}^N)\right)^{\frac{1}{p-1}} l(r,R)$ there holds,*

$$\begin{aligned} &\left|\left\{\mathbf{W}^R_{\alpha,p}[\mu] > 3\lambda, (\mathbf{M}^\eta_{\alpha p,R}[\mu])^{\frac{1}{p-1}} \leq \epsilon\lambda\right\}\right| \\ &\qquad \leq c_0 \exp\left(-\left(\tfrac{p-1-\eta}{4(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \alpha p \ln 2\, \epsilon^{-\frac{p-1}{p-1-\eta}}\right) \left|\{\mathbf{W}^R_{\alpha,p}[\mu] > \lambda\}\right|. \end{aligned} \tag{2.8}$$

where $l(r,R) = \frac{N-\alpha p}{p-1}\left(\min\{r,R\}^{-\frac{N-\alpha p}{p-1}} - R^{-\frac{N-\alpha p}{p-1}}\right)$ if $R < \infty$, $l(r,R) = \frac{N-\alpha p}{p-1} r^{-\frac{N-\alpha p}{p-1}}$ if $R = \infty$. Furthermore, if $\eta = 0$, ϵ_0 is independent of r and (2.8) holds for all $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$ with compact support in $\mathbb{R}^N$ and $R \in (0, \infty]$, $\epsilon \in (0, \epsilon_0]$, $\lambda > 0$.

Proof. *Case* $R = \infty$. Let $\lambda > 0$; since $\mathbf{W}_{\alpha,p}[\mu]$ is lower semicontinuous, the set

$$D_\lambda := \{\mathbf{W}_{\alpha,p}[\mu] > \lambda\}$$

is open. By Whitney covering lemma, there exists a countable set of closed cubes $\{Q_i\}_i$ such that $D_\lambda = \cup_i Q_i$, $\overset{o}{Q}_i \cap \overset{o}{Q}_j = \emptyset$ for $i \neq j$ and

$$\mathrm{diam(Q_i)} \leq \mathrm{dist}\,(\mathrm{Q_i}, \mathrm{D}^c_\lambda) \leq 4\,\mathrm{diam(Q_i)}.$$

Let $\epsilon > 0$ and $F_{\epsilon,\lambda} = \left\{\mathbf{W}_{\alpha,p}[\mu] > 3\lambda, (\mathbf{M}^\eta_{\alpha p}[\mu])^{\frac{1}{p-1}} \leq \epsilon\lambda\right\}$. We claim that there exist $c_0 = c_0(N,\alpha,p,\eta) > 0$ and $\epsilon_0 = \epsilon_0(N,\alpha,p,\eta,r) > 0$ such that for any $Q \in \{Q_i\}_i$, $\epsilon \in (0,\epsilon_0]$ and $\lambda > \left(\mu(\mathbb{R}^N)\right)^{\frac{1}{p-1}} l(r,\infty)$ there holds

$$|F_{\epsilon,\lambda} \cap Q| \leq c_0 \exp\left(-\left(\frac{p-1-\eta}{4(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \epsilon^{-\frac{p-1}{p-1-\eta}} \alpha p \ln 2\right)|Q|\,. \tag{2.9}$$

The first we show that there exists $c_1 > 0$ depending on N, α, p and η such that for any $Q \in \{Q_i\}_i$ there holds

$$F_{\epsilon,\lambda} \cap Q \subset E_{\epsilon,\lambda}\ \forall \epsilon \in (0, c_1], \lambda > 0 \tag{2.10}$$

where

$$E_{\epsilon,\lambda} = \left\{x \in Q : \mathbf{W}^{5\,\mathrm{diam(Q)}}_{\alpha,p}[\mu](x) > \lambda, (M^\eta_{\alpha p}[\mu](x))^{\frac{1}{p-1}} \leq \epsilon\lambda\right\}. \tag{2.11}$$

Infact, take $Q \in \{Q_i\}_i$ such that $Q \cap F_{\epsilon,\lambda} \neq \emptyset$ and let $x_Q \in D^c_\lambda$ such that $\mathrm{dist}\,(x_Q, Q) \leq 4\,\mathrm{diam(Q)}$ and $\mathbf{W}_{\alpha,p}[\mu](x_Q) \leq \lambda$. For $k \in \mathbb{N}$, $r_0 = 5\,\mathrm{diam(Q)}$ and $x \in F_{\epsilon,\lambda} \cap Q$, we have

$$\int_{2^k r_0}^{2^{k+1} r_0} \left(\tfrac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \tfrac{dt}{t} = A + B$$

where

$$A = \int_{2^k r_0}^{2^k \frac{1+2^{k+1}}{1+2^k} r_0} \left(\tfrac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \tfrac{dt}{t} \text{ and } B = \int_{2^k \frac{1+2^{k+1}}{1+2^k} r_0}^{2^{k+1} r_0} \left(\tfrac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \tfrac{dt}{t}.$$

Since

$$\mu(B_t(x)) \leq t^{N-\alpha p} h_\eta(t) M^\eta_{\alpha p}[\mu](x) \leq t^{N-\alpha p} h_\eta(t) (\epsilon\lambda)^{p-1}. \tag{2.12}$$

Then

$$B \leq \int_{2^k \frac{1+2^{k+1}}{1+2^k} r_0}^{2^{k+1} r_0} \left(\frac{t^{N-\alpha p} h_\eta(t)(\epsilon\lambda)^{p-1}}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} = \epsilon\lambda \int_{2^k \frac{1+2^{k+1}}{1+2^k} r_0}^{2^{k+1} r_0} (h_\eta(t))^{\frac{1}{p-1}} \frac{dt}{t}$$

Replacing $h_\eta(t)$ by its value we obtain $B \leq c_2 \epsilon\lambda 2^{-k}$ after a lengthy computation where c_2 depends only on p and η. Since $\delta := (\frac{2^k}{2^k+1})^{\frac{N-\alpha p}{p-1}}$, then $1 - \delta \leq c_3 2^{-k}$ where c_3 depends only on $\frac{N-\alpha p}{p-1}$, thus

$$\begin{aligned}
(1-\delta)A &\leq c_3 2^{-k} \int_{2^k r_0}^{2^{k+1} r_0} \left(\tfrac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \tfrac{dt}{t} \\
&\leq c_3 2^{-k} \epsilon\lambda \int_{2^k r_0}^{2^{k+1} r_0} (h_\eta(t))^{\frac{1}{p-1}} \tfrac{dt}{t} \\
&\leq c_4 2^{-k} \epsilon\lambda,
\end{aligned}$$

where $c_4 = c_4(N,\alpha,p,\eta) > 0$.
By a change of variables and using that for any $x \in F_{\epsilon,\lambda} \cap Q$ and $t \in [r_0(1+2^k), r_0(1+2^{k+1})]$, $B_{\frac{2^k t}{1+2^k}}(x) \subset B_t(x_Q)$, we get

$$\delta A = \int_{r_0(1+2^k)}^{r_0(1+2^{k+1})} \left(\frac{\mu(B_{\frac{2^k t}{1+2^k}}(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \leq \int_{r_0(1+2^k)}^{r_0(1+2^{k+1})} \left(\frac{\mu(B_t(x_Q))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t}.$$

Therefore

$$\int_{2^k r_0}^{2^{k+1} r_0} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \leq c_5 2^{-k}\epsilon\lambda + \int_{r_0(1+2^k)}^{r_0(1+2^{k+1})} \left(\frac{\mu(B_t(x_Q))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t},$$

with $c_5 = c_5(N,\alpha,p,\eta) > 0$. This implies

$$\int_{r_0}^{\infty} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \leq 2c_5\epsilon\lambda + \int_{2r_0}^{\infty} \left(\frac{\mu(B_t(x_Q))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \leq (1+2c_5\epsilon)\lambda, \tag{2.13}$$

since $\mathbf{W}_{\alpha,p}[\mu](x_Q) \leq \lambda$. If $\epsilon \in (0, c_1]$ with $c_1 = (2c_5)^{-1}$ then

$$\int_{r_0}^{\infty} \left(\frac{\mu(B_t(x)}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \leq 2\lambda$$

which implies (2.10).
Now, we let $\lambda > \left(\mu(\mathbb{R}^N)\right)^{\frac{1}{p-1}} l(r,\infty)$. Let B_1 be a ball with radius r such that $supp(\mu) \subset B_1$. We denote B_2 by the ball concentric to B_1 with radius $2r$. Since $x \notin B_2$,

$$\mathbf{W}_{\alpha,p}[\mu](x) = \int_r^{\infty} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \leq \left(\mu(\mathbb{R}^N)\right)^{\frac{1}{p-1}} l(r,\infty).$$

Thus, we obtain $D_\lambda \subset B_2$. In particular, $r_0 = 5\,\mathrm{diam(Q)} \leq 20\mathrm{r}$.
Next we set $m_0 = \frac{\max(1,\ln(40r))}{\ln 2}$, so that $2^{-m} r_0 \leq 2^{-1}$ if $m \geq m_0$. Then for any $x \in E_{\epsilon,\lambda}$

$$\begin{aligned}
\int_{2^{-m}r_0}^{r_0} \left(\tfrac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \tfrac{dt}{t} &\leq \epsilon\lambda \int_{2^{-m}r_0}^{r_0} (h_\eta(t))^{\frac{1}{p-1}} \tfrac{dt}{t} \\
&\leq \epsilon\lambda \int_{2^{-m}r_0}^{2^{-m_0}r_0} (-\ln t)^{\frac{-\eta}{p-1}} \tfrac{dt}{t} + \epsilon\lambda \int_{2^{-m_0}r_0}^{r_0} (\ln 2)^{\frac{-\eta}{p-1}} \tfrac{dt}{t} \\
&\leq m_0\epsilon\lambda + \tfrac{(p-1)((m-m_0)\ln 2)^{1-\frac{\eta}{p-1}}}{p-1-\eta}\epsilon\lambda.
\end{aligned}$$

For the last inequality we have used $a^{1-\frac{\eta}{p-1}} - b^{1-\frac{\eta}{p-1}} \leq (a-b)^{1-\frac{\eta}{p-1}}$ valid for any $a \geq b \geq 0$. Therefore,

$$\int_{2^{-m}r_0}^{r_0} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \leq \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}}\epsilon\lambda \qquad \forall m \in \mathbb{N}, m > m_0^{\frac{p-1}{p-1-\eta}}. \tag{2.14}$$

Set

$$g_i(x) = \int_{2^{-i}r_0}^{2^{-i+1}r_0} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t},$$

then

$$\begin{aligned}\mathbf{W}^{r_0}_{\alpha,p}[\mu](x) &\leq \tfrac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}}\epsilon\lambda + \mathbf{W}^{2^{-m}r_0}_{\alpha,p}[\mu](x)\\ &\leq \tfrac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}}\epsilon\lambda + \sum_{i=m+1}^{\infty} g_i(x)\end{aligned}$$

for all $m > m_0^{\frac{p-1}{p-1-\eta}}$. We deduce that, for $\beta > 0$,

$$\begin{aligned}|E_{\epsilon,\lambda}| &\leq \left|\left\{x \in Q : \sum_{i=m+1}^{\infty} g_i(x) > \left(1 - \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}}\epsilon\right)\lambda\right\}\right|\\ &\leq \left|\left\{x \in Q : \sum_{i=m+1}^{\infty} g_i(x) > \sum_{i=m+1}^{\infty} 2^{-\beta(i-m-1)}(1-2^{-\beta})\left(1 - \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}}\epsilon\right)\lambda\right\}\right|\\ &\leq \sum_{i=m+1}^{\infty} \left|\left\{x \in Q : g_i(x) > 2^{-\beta(i-m-1)}(1-2^{-\beta})\left(1 - \frac{2(p-1)}{p-1-\eta} m^{1-\frac{\eta}{p-1}}\epsilon\right)\lambda\right\}\right|.\end{aligned} \tag{2.15}$$

Next we claim that

$$|\{x \in Q : g_i(x) > s\}| \leq \frac{c_6(N,\eta)}{s^{p-1}} 2^{-i\alpha p} |Q| (\epsilon\lambda)^{p-1}. \tag{2.16}$$

To see that, we pick $x_0 \in E_{\epsilon,\lambda}$ and we use the Chebyshev's inequality

$$\begin{aligned}|\{x \in Q : g_i(x) > s\}| &\leq \tfrac{1}{s^{p-1}} \int_Q |g_i|^{p-1} dx\\ &= \tfrac{1}{s^{p-1}} \int_Q \left(\int_{r_0 2^{-i}}^{r_0 2^{-i+1}} \left(\tfrac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \tfrac{dt}{t}\right)^{p-1} dx\\ &\leq \tfrac{1}{s^{p-1}} \int_Q \tfrac{\mu(B_{r_0 2^{-i+1}}(x))}{(r_0 2^{-i})^{N-\alpha p}} := A.\end{aligned}$$

Thanks to Fubini's theorem, the last term A of the above inequality can be rewritten as

$$\begin{aligned}A &= \tfrac{1}{s^{p-1}} \tfrac{1}{(r_0 2^{-i})^{N-\alpha p}} \int_Q \int_{\mathbb{R}^N} \chi_{B_{r_0 2^{-i+1}}(x)}(y) d\mu(y) dx\\ &= \tfrac{1}{s^{p-1}} \tfrac{1}{(r_0 2^{-i})^{N-\alpha p}} \int_{Q+B_{r_0 2^{-i+1}}(0)} \int_Q \chi_{B_{r_0 2^{-i+1}}(y)}(x) dx d\mu(y)\\ &\leq \tfrac{1}{s^{p-1}} \tfrac{1}{(r_0 2^{-i})^{N-\alpha p}} \int_{Q+B_{r_0 2^{-i+1}}(0)} |B_{r_0 2^{-i+1}}(y)| d\mu(y)\\ &\leq c_7(N) \tfrac{1}{s^{p-1}} 2^{-i\alpha p} r_0^{\alpha p} \mu(Q + B_{r_0 2^{-i+1}}(0))\\ &\leq c_7(N) \tfrac{1}{s^{p-1}} 2^{-i\alpha p} r_0^{\alpha p} \mu(B_{r_0(1+2^{-i+1})}(x_0)),\end{aligned}$$

since $Q+B_{r_0 2^{-i+1}}(0)\subset B_{r_0(1+2^{-i+1})}(x_0)$. Using the fact that $\mu(B_t(x_0))\leq (ln\,2)^{-\eta}t^{N-\alpha p}(\epsilon\lambda)^{p-1}$ for all $t>0$ and $r_0=5\,\mathrm{diam(Q)}$, we obtain

$$A\leq c_8(N,\eta)\frac{1}{s^{p-1}}2^{-i\alpha p}r_0^{\alpha p}(r_0(1+2^{-i+1}))^{N-\alpha p}(\epsilon\lambda)^{p-1}\leq c_9(N,\eta)\frac{1}{s^{p-1}}2^{-i\alpha p}\left|Q\right|(\epsilon\lambda)^{p-1},$$

which is (2.16). Consequently, (2.15) can be rewritten as

$$\begin{aligned}|E_{\epsilon,\lambda}|&\leq\sum_{i=m+1}^{\infty}\frac{c_6(N,\eta)}{\left(2^{-\beta(i-m-1)}(1-2^{-\beta})\left(1-\frac{2(p-1)}{p-1-\eta}m^{1-\frac{\eta}{p-1}}\epsilon\right)\lambda\right)^{p-1}}2^{-i\alpha p}(\epsilon\lambda)^{p-1}\left|Q\right|\\&\leq c_6(N,\eta)2^{-(m+1)\alpha p}\left(\frac{\epsilon}{1-\frac{2(p-1)}{p-1-\eta}m^{1-\frac{\eta}{p-1}}\epsilon}\right)^{p-1}\left|Q\right|\left(1-2^{-\beta}\right)^{-p+1}\sum_{i=m+1}^{\infty}2^{(\beta(p-1)-\alpha p)(i-m-1)}.\end{aligned}\tag{2.17}$$

If we choose $\beta=\beta(\alpha,p)$ so that $\beta(p-1)-\alpha p<0$, we obtain

$$|E_{\epsilon,\lambda}|\leq c_{10}2^{-m\alpha p}\left(\frac{\epsilon}{1-\frac{2(p-1)}{p-1-\eta}m^{1-\frac{\eta}{p-1}}\epsilon}\right)^{p-1}|Q|\qquad\forall m>m_0^{\frac{p-1}{p-1-\eta}}\tag{2.18}$$

where $c_{10}=c_{10}(N,\alpha,p,\eta)>0$. Put $\epsilon_0=\min\left\{\frac{1}{\frac{4(p-1)}{p-1-\eta}m_0+1},c_1\right\}$. For any $\epsilon\in(0,\epsilon_0]$ we choose $m\in\mathbb{N}$ such that

$$\left(\frac{p-1-\eta}{2(p-1)}\right)^{\frac{p-1}{p-1-\eta}}\left(\frac{1}{\epsilon}-1\right)^{\frac{p-1}{p-1-\eta}}-1<m\leq\left(\frac{p-1-\eta}{2(p-1)}\right)^{\frac{p-1}{p-1-\eta}}\left(\frac{1}{\epsilon}-1\right)^{\frac{p-1}{p-1-\eta}}.$$

Then

$$\left(\frac{\epsilon}{1-\frac{2(p-1)}{p-1-\eta}m^{1-\frac{\eta}{p-1}}\epsilon}\right)^{p-1}\leq 1$$

and

$$2^{-m\alpha p}\leq 2^{\alpha p-\alpha p\left(\frac{p-1-\eta}{2(p-1)}\right)^{\frac{p-1}{p-1-\eta}}\left(\frac{1}{\epsilon}-1\right)^{\frac{p-1}{p-1-\eta}}}\leq 2^{\alpha p}\exp\left(-\alpha p\ln 2\left(\frac{p-1-\eta}{4(p-1)}\right)^{\frac{p-1}{p-1-\eta}}\epsilon^{-\frac{p-1}{p-1-\eta}}\right).$$

Combining these inequalities with (2.18) and (2.10), we get (2.9).

In the case $\eta=0$ we still have for any $m\in\mathbb{N}$, $\lambda,\epsilon>0$ and $x\in E_{\epsilon,\lambda}$

$$\mathbf{W}_{\alpha,p}^{r_0}[\mu](x)\leq m\epsilon\lambda+\sum_{i=m+1}^{\infty}g_i(x)$$

Accordingly (2.18) reads as

$$|E_{\epsilon,\lambda}|\leq c_{10}2^{-m\alpha p}\left(\frac{\epsilon}{1-m\epsilon}\right)^{p-1}|Q|\quad\forall m\in\mathbb{N},\lambda,\epsilon>0\text{ with }m\epsilon<1.$$

Put $\epsilon_0 = \min\{\frac{1}{2}, c_1\}$. For any $\epsilon \in (0, \epsilon_0]$ and $m \in \mathbb{N}$ satisfies $\epsilon^{-1} - 2 < m \leq \epsilon^{-1} - 1$, we finally get from (2.10)

$$|F_{\epsilon,\lambda} \cap Q| \leq |E_{\epsilon,\lambda}| \leq c_{10} 2^{2\alpha p} \exp\left(-\alpha p \epsilon^{-1} \ln 2\right) |Q|, \tag{2.19}$$

which ends the proof in the case $R = \infty$.

Case $R < \infty$. For $\lambda > 0$, $D_\lambda = \{\mathbf{W}^R_{\alpha,p} > \lambda\}$ is open. Using again Whitney covering lemma, there exists a countable set of closed cubes $\mathcal{Q} := \{Q_i\}$ such that $\cup_i Q_i = D_\lambda$, $\overset{o}{Q}_i \cap \overset{o}{Q}_j = \emptyset$ for $i \neq j$ and $\operatorname{dist}(Q_i, D^c_\lambda) \leq 4\,\mathrm{diam}(\mathrm{Q_i})$. If $Q \in \mathcal{Q}$: is such that $\mathrm{diam}\,(\mathrm{Q}) > \frac{\mathrm{R}}{8}$, there exists a finite number n_Q of closed dyadic cubes $\{P_{j,Q}\}_{j=1}^{n_Q}$ such that $\cup_{j=1}^{n_Q} P_{j,Q} = Q$, $\overset{o}{P}_{i,Q} \cap \overset{o}{P}_{j,Q} = \emptyset$ if $i \neq j$ and $\frac{R}{16} < \mathrm{diam}\,(\mathrm{P_{j,Q}}) \leq \frac{\mathrm{R}}{8}$. We set $\mathcal{Q}' = \left\{Q \in \mathcal{Q} : \mathrm{diam}\,(\mathrm{Q}) \leq \frac{\mathrm{R}}{8}\right\}$, $\mathcal{Q}'' = \left\{P_{i,Q} : 1 \leq i \leq n_Q, Q \in \mathcal{Q}, \mathrm{diam}\,(\mathrm{Q}) > \frac{\mathrm{R}}{8}\right\}$ and $\mathcal{F} = \mathcal{Q}' \cup \mathcal{Q}''$.

For $\epsilon > 0$ we denote again $F_{\epsilon,\lambda} = \left\{\mathbf{W}^R_{\alpha,p}[\mu] > 3\lambda, (\mathbf{M}^\eta_{\alpha p,R}[\mu])^{\frac{1}{p-1}} \leq \epsilon\lambda\right\}$. Let $Q \in \mathcal{F}$ such that $F_{\epsilon,\lambda} \cap Q \neq \emptyset$ and $r_0 = 5\,\mathrm{diam}\,(\mathrm{Q})$.

If $\operatorname{dist}(D^c_\lambda, Q) \leq 4\,\mathrm{diam}\,(\mathrm{Q})$, that is if there exists $x_Q \in D^c_\lambda$ such that $\operatorname{dist}(x_Q, Q) \leq 4\,\mathrm{diam}\,(\mathrm{Q})$ and $\mathbf{W}^R_{\alpha,p}[\mu](x_Q) \leq \lambda$, we find, by the same argument as in the case $R = \infty$, (2.13), that for any $x \in F_{\epsilon,\lambda} \cap Q$ there holds

$$\int_{r_0}^{R} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \leq (1 + c_{11}\epsilon)\lambda. \tag{2.20}$$

where $c_{11} = c_{11}(N, \alpha, p, \eta) > 0$.

If $\operatorname{dist}(D^c_\lambda, Q) > 4\,\mathrm{diam}\,(\mathrm{Q})$, we have $\frac{R}{16} < \mathrm{diam}\,(\mathrm{Q}) \leq \frac{\mathrm{R}}{8}$ since $Q \in \mathcal{Q}''$. Then, for all $x \in F_{\epsilon,\lambda} \cap Q$, there holds

$$\begin{aligned}\int_{r_0}^{R} \left(\tfrac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \tfrac{dt}{t} &\leq \int_{\frac{5R}{16}}^{R} \left(\tfrac{t^{N-\alpha p}(\ln 2)^{-\eta}(\epsilon\lambda)^{p-1}}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \tfrac{dt}{t} \\ &= (\ln 2)^{-\frac{\eta}{p-1}} \ln \tfrac{16}{5}\ \epsilon\lambda \\ &\leq 2\epsilon\lambda.\end{aligned} \tag{2.21}$$

Thus, if we take $\epsilon \in (0, c_{12}]$ with $c_{12} = \min\{1, c_{11}^{-1}\}$, we derive

$$F_{\epsilon,\lambda} \cap Q \subset E_{\epsilon,\lambda}, \tag{2.22}$$

where

$$E_{\epsilon,\lambda} = \left\{\mathbf{W}^{r_0}_{\alpha,p}[\mu] > \lambda, \left(\mathbf{M}^\eta_{\alpha p,R}[\mu]\right)^{\frac{1}{p-1}} \leq \epsilon\lambda\right\}.$$

Furthermore, since $x \notin B_2$,

$$\mathbf{W}^R_{\alpha,p}[\mu](x) = \int_{\min\{r,R\}}^{R} \left(\frac{\mu(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}} \frac{dt}{t} \leq \left(\mu(\mathbb{R}^N)\right)^{\frac{1}{p-1}} l(r, R).$$

Thus, if $\lambda > \left(\mu(\mathbb{R}^N)\right)^{\frac{1}{p-1}} l(r,R)$ then $D_\lambda \subset B_2$ which implies $r_0 = 5\,\mathrm{diam(Q)} \leq 20\mathrm{r}$. The end of the proof is as in the case $R = \infty$. $\square$

In the next result we list a series of equivalent norms concerning Radon measures.

Theorem 2.3 *Assume $\alpha > 0$, $0 < p-1 < q < \infty$, $0 < \alpha p < N$ and $0 < s \leq \infty$. Then there exists a constant $c_{13} = c_{13}(N,\alpha,p,q,s) > 0$ such that for any $R \in (0,\infty]$ and $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$, there holds*

$$c_{13}^{-1} \left\| \mathbf{W}_{\alpha,p}^R[\mu] \right\|_{L^{q,s}(\mathbb{R}^N)} \leq \| \mathbf{M}_{\alpha p,R}[\mu] \|_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)}^{\frac{1}{p-1}} \leq c_{13} \left\| \mathbf{W}_{\alpha,p}^R[\mu] \right\|_{L^{q,s}(\mathbb{R}^N)}. \tag{2.23}$$

For any $R > 0$, there exists $c_{14} = c_{14}(N,\alpha,p,q,s,R) > 0$ such that for any $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$,

$$c_{14}^{-1} \left\| \mathbf{W}_{\alpha,p}^R[\mu] \right\|_{L^{q,s}(\mathbb{R}^N)} \leq \| \mathbf{G}_{\alpha p}[\mu] \|_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)}^{\frac{1}{p-1}} \leq c_{14} \left\| \mathbf{W}_{\alpha,p}^R[\mu] \right\|_{L^{q,s}(\mathbb{R}^N)}. \tag{2.24}$$

In (2.24), $\left\| \mathbf{W}_{\alpha,p}^R[\mu] \right\|_{L^{q,s}(\mathbb{R}^N)}$ can be replaced by $\| \mathbf{M}_{\alpha p,R}[\mu] \|_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)}^{\frac{1}{p-1}}$.

Proof. We denote μ_n by $\chi_{B_n}\mu$ for $n \in \mathbb{N}^*$.
Step 1 We claim that

$$\left\| \mathbf{W}_{\alpha,p}^R[\mu] \right\|_{L^{q,s}(\mathbb{R}^N)} \leq c'_{13} \| \mathbf{M}_{\alpha p,R}[\mu] \|_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)}^{\frac{1}{p-1}}. \tag{2.25}$$

From Proposition 2.2 there exist positive constants $c_0 = c_0(N,\alpha,p), a = a(\alpha,p)$ and $\epsilon_0 = \epsilon_0(N,\alpha,p)$ such that for all $n \in \mathbb{N}^*$, $t > 0$, $0 < R \leq \infty$ and $0 < \epsilon \leq \epsilon_0$, there holds

$$\left| \left\{ \mathbf{W}_{\alpha,p}^R[\mu_n] > 3t, (\mathbf{M}_{\alpha p,R}^\eta[\mu_n])^{\frac{1}{p-1}} \leq \epsilon t \right\} \right| \leq c_0 \exp\left(-a\epsilon^{-1}\right) \left| \{ \mathbf{W}_{\alpha,p}^R[\mu_n] > t \} \right|. \tag{2.26}$$

In the case $0 < s < \infty$ and $0 < q < \infty$, we have

$$\left| \{ \mathbf{W}_{\alpha,p}^R[\mu_n] > 3t \} \right|^{\frac{s}{q}} \leq c_{15} \exp\left(-\frac{s}{q} a\epsilon^{-1}\right) \left| \{ \mathbf{W}_{\alpha,p}^R[\mu_n] > t \} \right|^{\frac{s}{q}} + c_{15} \left| \left\{ (\mathbf{M}_{\alpha p,R}^\eta[\mu_n])^{\frac{1}{p-1}} > \epsilon t \right\} \right|^{\frac{s}{q}}.$$

with $c_{15} = c_{15}(N,\alpha,p,q,s) > 0$.
Multiplying by t^{s-1} and integrating over $(0,\infty)$, we obtain

$$\int_0^\infty t^s \left| \{ \mathbf{W}_{\alpha,p}^R[\mu_n] > 3t \} \right|^{\frac{s}{q}} \frac{dt}{t} \leq c_{15} \exp\left(-\tfrac{s}{q} a\epsilon^{-1}\right) \int_0^\infty t^s \left| \{ \mathbf{W}_{\alpha,p}^R[\mu_n] > t \} \right|^{\frac{s}{q}} \frac{dt}{t}$$
$$+ c_{15} \int_0^\infty t^s \left| \left\{ \mathbf{M}_{\alpha p,R}^\eta[\mu_n] > (\epsilon t)^{p-1} \right\} \right|^{\frac{s}{q}} \frac{dt}{t}.$$

By a change of variable, we derive

$$\left(3^{-s} - c_{15} \exp\left(-\tfrac{s}{q} a\epsilon^{-1}\right)\right) \int_0^\infty t^s \left| \{ \mathbf{W}_{\alpha,p}^R[\mu_n] > t \} \right|^{\frac{s}{q}} \frac{dt}{t}$$
$$\leq \tfrac{c_{15}\epsilon^{-s}}{p-1} \int_0^\infty t^{\frac{s}{p-1}} \left| \left\{ \mathbf{M}_{\alpha p,R}^\eta[\mu_n] > t \right\} \right|^{\frac{s}{q}} \frac{dt}{t}.$$

We choose ϵ small enough so that $3^{-s} - c_{15}\exp\left(-\frac{s}{q}a\epsilon^{-1}\right) > 0$, we derive from (2.2) and $\left\|t^{1/s_1}f^*\right\|_{L^{s_2}\left(\mathbb{R},\frac{dt}{t}\right)} = s_1^{1/s_2}\left\|\lambda_f^{1/s_1}t\right\|_{L^{s_2}\left(\mathbb{R},\frac{dt}{t}\right)}$ for any $f \in L^{s_1,s_2}(\mathbb{R}^N)$ with $0 < s_1 < \infty, 0 < s_2 \leq \infty$

$$\left\|\mathbf{W}_{\alpha,p}^R[\mu_n]\right\|_{L^{q,s}(\mathbb{R}^N)} \leq c'_{13}\left\|\mathbf{M}_{\alpha p,R}[\mu_n]\right\|_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)}^{\frac{1}{p-1}},$$

and (2.25) follows by Fatou's lemma. Similarly, we can prove (2.25) in the case $s = \infty$.
Step 2 We claim that

$$\left\|\mathbf{W}_{\alpha,p}^R[\mu]\right\|_{L^{q,s}(\mathbb{R}^N)} \geq c''_{13}\left\|\mathbf{M}_{\alpha p,R}[\mu]\right\|_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)}^{\frac{1}{p-1}}. \tag{2.27}$$

For $R > 0$ we have

$$\begin{aligned}\mathbf{W}_{\alpha,p}^{2R}[\mu_n](x) &= \mathbf{W}_{\alpha,p}^R[\mu_n](x) + \int_R^{2R}\left(\frac{\mu_n(B_t(x))}{t^{N-\alpha p}}\right)^{\frac{1}{p-1}}\frac{dt}{t} \\ &\leq \mathbf{W}_{\alpha,p}^R[\mu_n](x) + \left(\frac{\mu_n(B_{2R}(x))}{R^{N-\alpha p}}\right)^{\frac{1}{p-1}}.\end{aligned} \tag{2.28}$$

Thus

$$\left|\left\{x : \mathbf{W}_{\alpha,p}^{2R}[\mu_n](x) > 2t\right\}\right| \leq \left|\left\{x : \mathbf{W}_{\alpha,p}^R[\mu_n](x) > t\right\}\right| + \left|\left\{x : \frac{\mu_n(B_{2R}(x))}{R^{N-\alpha p}} > t^{p-1}\right\}\right|,$$

Consider $\{z_j\}_{i=1}^m \subset B_2$ such that $B_2 \subset \bigcup_{i=1}^m B_{\frac{1}{2}}(z_i)$. Thus $B_{2R}(x) \subset \bigcup_{i=1}^m B_{\frac{R}{2}}(x + Rz_i)$ for any $x \in \mathbb{R}^N$ and $R > 0$. Then

$$\begin{aligned}\left|\left\{x : \frac{\mu_n(B_{2R}(x))}{R^{N-\alpha p}} > t^{p-1}\right\}\right| &\leq \left|\left\{x : \sum_{i=1}^m \frac{\mu_n(B_{\frac{R}{2}}(x + Rz_i))}{R^{N-\alpha p}} > t^{p-1}\right\}\right| \\ &\leq \sum_{i=1}^m \left|\left\{x : \frac{\mu_n(B_{\frac{R}{2}}(x + Rz_i))}{R^{N-\alpha p}} > \frac{1}{m}t^{p-1}\right\}\right| \\ &\leq \sum_{i=1}^m \left|\left\{x - Rz_i : \frac{\mu_n(B_{\frac{R}{2}}(x))}{R^{N-\alpha p}} > \frac{1}{m}t^{p-1}\right\}\right| \\ &= m\left|\left\{x : \frac{\mu_n(B_{\frac{R}{2}}(x))}{R^{N-\alpha p}} > \frac{1}{m}t^{p-1}\right\}\right|.\end{aligned}$$

Moreover from (2.28)

$$\left(\frac{\mu_n(B_{\frac{R}{2}}(x))}{R^{N-\alpha p}}\right)^{\frac{1}{p-1}} \leq 2\mathbf{W}_{\alpha,p}^R[\mu_n](x),$$

thus

$$\left|\left\{x : \frac{\mu_n(B_{2R}(x))}{R^{N-\alpha p}} > t^{p-1}\right\}\right| \leq m\left|\left\{x : \mathbf{W}_{\alpha,p}^R[\mu_n](x) > \frac{1}{2m^{\frac{1}{p-1}}}t\right\}\right|.$$

This leads to

$$\left|\left\{x : \mathbf{W}_{\alpha,p}^{2R}[\mu_n](x) > 2t\right\}\right| \leq (m+1)\left|\left\{x : \mathbf{W}_{\alpha,p}^R[\mu_n](x) > \frac{1}{2m^{\frac{1}{p-1}}}t\right\}\right| \quad \forall t > 0$$

This implies

$$\left\|\mathbf{W}^{2R}_{\alpha,p}[\mu_n]\right\|_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)} \leq c_{16}\left\|\mathbf{W}^{R}_{\alpha,p}[\mu_n]\right\|_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)}.$$

with $c_{16}=c_{16}(N,\alpha,p,q,s)>0$. By Fatou's lemma, we get

$$\left\|\mathbf{W}^{2R}_{\alpha,p}[\mu]\right\|_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)} \leq c_{16}\left\|\mathbf{W}^{R}_{\alpha,p}[\mu]\right\|_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)}. \tag{2.29}$$

On the other hand, from the identity in (2.28) we derive that for any $\rho\in(0,R)$,

$$\mathbf{W}^{2R}_{\alpha,p}[\mu](x)\geq \mathbf{W}^{2\rho}_{\alpha,p}[\mu](x)\geq c_{17}\sup_{0<\rho\leq R}\left(\frac{\mu(B_\rho(x))}{\rho^{N-\alpha p}}\right)^{\frac{1}{p-1}},$$

with $c_{17}=c_{17}(N,\alpha,p)>0$, from which follows

$$\mathbf{W}^{2R}_{\alpha,p}[\mu](x)\geq c_{17}\left(\mathbf{M}_{\alpha p,R}[\mu](x)\right)^{\frac{1}{p-1}}. \tag{2.30}$$

Combining (2.29) and (2.30) we obtain (2.27) and then (2.23). Notice that the estimates are independent of R and thus valid if $R=\infty$.

Step 3 We claim that (2.24) holds. By the previous result we have also

$$c_{18}^{-1}\left\|\mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu]\right\|_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)} \leq \left\|\mathbf{M}_{\alpha p,R}[\mu]\right\|_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)} \leq c_{18}\left\|\mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu]\right\|_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)}. \tag{2.31}$$

where $c_{18}=c_{18}(N,\alpha,p,q,s)>0$. For $R>0$, the Bessel kernel satisfies[18, V-3-1]

$$c_{19}^{-1}\left(\frac{\chi_{B_R}(x)}{|x|^{N-\alpha p}}\right)\leq G_{\alpha p}(x)\leq c_{19}\left(\frac{\chi_{B_{\frac{R}{2}}}(x)}{|x|^{N-\alpha p}}\right)+c_{19}e^{-\frac{|x|}{2}}\qquad \forall x\in\mathbb{R}^N,$$

where $c_{19}=c_{19}(N,\alpha,p,R)>0$. Therefore

$$c_{19}^{-1}\left(\frac{\chi_{B_R}}{|.|^{N-\alpha p}}\right)*\mu\leq \mathbf{G}_{\alpha p}[\mu]\leq c_{19}\left(\frac{\chi_{B_{\frac{R}{2}}}}{|.|^{N-\alpha p}}\right)*\mu+c_{19}e^{-\frac{|.|}{2}}*\mu. \tag{2.32}$$

By integration by parts, we get

$$\left(\frac{\chi_{B_R}}{|.|^{N-\alpha p}}\right)*\mu(x)=(N-\alpha p)\mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu](x)+\frac{\mu(B_R(x))}{R^{N-\alpha p}}\geq (N-\alpha p)\mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu](x),$$

which implies

$$c_{20}\left\|\mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu]\right\|_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)} \leq \left\|\mathbf{G}_{\alpha p}[\mu]\right\|_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)}. \tag{2.33}$$

where $c_{20}=c_{20}(N,\alpha,p,q,s)>0$. Furthermore $e^{-\frac{|x|}{2}}\leq c_{21}\chi_{B_{\frac{R}{2}}}*e^{-\frac{|.|}{2}}(x)$ where $c_{21}=c_{21}(N,R)>0$, thus

$$e^{-\frac{|.|}{2}}*\mu\leq c_{21}\left(\chi_{B_{\frac{R}{2}}}*e^{-\frac{|.|}{2}}\right)*\mu=c_{21}e^{-\frac{|.|}{2}}*\left(\chi_{B_{\frac{R}{2}}}*\mu\right).$$

Since

$$\chi_{B_{\frac{R}{2}}} * \mu(x) = \mu(B_{\frac{R}{2}}(x)) \leq c_{22}\mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu](x)$$

where $c_{22} = c_{22}(N,\alpha,p,R) > 0$, we derive with $c_{23} = c_{21}c_{22}$

$$e^{-\frac{|.|}{2}} * \mu \leq c_{23} e^{-\frac{|.|}{2}} * \mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu].$$

Using Young inequality, we obtain

$$\begin{aligned}\left\| e^{-\frac{|.|}{2}} * \mu \right\|_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)} &\leq c_{23} \left\| e^{-\frac{|.|}{2}} * \mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu] \right\|_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)} \\ &\leq c_{24} \left\| \mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu] \right\|_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)} \left\| e^{-\frac{|.|}{2}} \right\|_{L^{1,\infty}(\mathbb{R}^N)} \\ &\leq c_{25} \left\| \mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu] \right\|_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)}.\end{aligned} \tag{2.34}$$

where $c_{25} = c_{25}(N,\alpha,p,R) > 0$.
Since by integration by parts there holds as above

$$\left(\frac{\chi_{B_{\frac{R}{2}}}}{|.|^{N-\alpha p}}\right) * \mu(x) = (N-\alpha p)\mathbf{W}^{\frac{R}{2}}_{\frac{\alpha p}{2},2}[\mu](x) + 2^{N-\alpha p}\frac{\mu(B_{\frac{R}{2}}(x))}{R^{N-\alpha p}} \leq c_{26}\mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu](x),$$

where $c_{26} = c_{26}(N,\alpha,p) > 0$ we obtain

$$\left\| \left(\frac{\chi_{B_R}}{|.|^{N-\alpha p}}\right) * \mu \right\|_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)} \leq c_{27} \left\| \mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu] \right\|_{L^{\frac{q}{p-1},\frac{s}{p-1}}}. \tag{2.35}$$

where $c_{27} = c_{27}(N,\alpha,p,q,s) > 0$. Thus

$$\|\mathbf{G}_{\alpha p}[\mu]\|_{L^{\frac{q}{p-1},\frac{s}{p-1}}(\mathbb{R}^N)} \leq c_{28} \left\| \mathbf{W}^{R}_{\frac{\alpha p}{2},2}[\mu] \right\|_{L^{\frac{q}{p-1},\frac{s}{p-1}}}. \tag{2.36}$$

where $c_{28} = c_{28}(N,\alpha,p,q,s,R) > 0$.
follows by combining (2.32), (2.34) and (2.35). Then, combining (2.33), (2.36) and using (2.31), (2.23) we obtain (2.24). □

Remark. Proposition 5.1 in [17] is a particular case of the previous result.

Theorem 2.4 *Let $\alpha > 0$, $p > 1$, $0 \leq \eta < p-1$, $0 < \alpha p < N$ and $r > 0$. Set $\delta_0 = \left(\frac{p-1-\eta}{12(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \alpha p \ln 2$. Then there exists $c_{29} > 0$, depending on N, α, p, η and r such that for any $R \in (0,\infty]$, $\delta \in (0,\delta_0)$, $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$, any ball $B_1 \subset \mathbb{R}^N$ with radius $\leq r$ and ball B_2 concentric to B_1 with radius double B_1's radius, there holds*

$$\frac{1}{|B_2|}\int_{B_2} \exp\left(\delta \frac{\left(\mathbf{W}^{R}_{\alpha,p}[\mu_{B_1}](x)\right)^{\frac{p-1}{p-1-\eta}}}{\|\mathbf{M}^{\eta}_{\alpha p,R}[\mu_{B_1}]\|^{\frac{1}{p-1-\eta}}_{L^\infty(B_1)}}\right) dx \leq \frac{c_{29}}{\delta_0 - \delta} \tag{2.37}$$

where $\mu_{B_1} = \chi_{B_1}\mu$. Furthermore, if $\eta = 0$, c_{29} is independent of r.

Proof. Let $\mu \in \mathfrak{M}_+(\mathbb{R}^N)$ such that $M := \left\|\mathbf{M}^{\eta}_{\alpha p,R}[\mu_{B_1}]\right\|_{L^\infty(B_1)} < \infty$. By Proposition 2.2-(2.8) with $\mu = \mu_{B_1}$, there exist $c_0 > 0$ depending on N, α, p, η and $\epsilon_0 > 0$ depending on N, α, p, η and r such that, for all $R \in (0, \infty]$, $\epsilon \in (0, \epsilon_0]$, $t > \left(\mu_{B_1}(\mathbb{R}^N)\right)^{\frac{1}{p-1}} l(r', R)$ where r' is radius of B_1 there holds,

$$\begin{aligned}&\left|\left\{\mathbf{W}^R_{\alpha,p}[\mu_{B_1}] > 3t, (\mathbf{M}^{\eta}_{\alpha p,R}[\mu_{B_1}])^{\frac{1}{p-1}} \le \epsilon t\right\}\right| \\ &\qquad \le c_0 \exp\left(-\left(\tfrac{p-1-\eta}{4(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \alpha p \ln 2\, \epsilon^{-\frac{p-1}{p-1-\eta}}\right) \left|\{\mathbf{W}^R_{\alpha,p}[\mu_{B_1}] > t\}\right|.\end{aligned} \tag{2.38}$$

Since $\left(\mu_{B_1}(\mathbb{R}^N)\right)^{\frac{1}{p-1}} l(r', R) \le \frac{N-\alpha p}{p-1}(\ln 2)^{-\frac{\eta}{p-1}} M^{\frac{1}{p-1}}$, thus in (2.8) we can choose

$$\epsilon = t^{-1}\left\|\mathbf{M}^{\eta}_{\alpha p,R}[\mu_{B_1}]\right\|^{\frac{1}{p-1}}_{L^\infty(\mathbb{R}^N)} = t^{-1} M^{\frac{1}{p-1}} \quad \forall t > \max\{\epsilon_0^{-1}, \frac{N-\alpha p}{p-1}(\ln 2)^{-\frac{\eta}{p-1}}\} M^{\frac{1}{p-1}}$$

and as in the proof of Proposition 2.2, $\left\{\mathbf{W}^R_{\alpha,p}[\mu_{B_1}] > t\right\} \subset B_2$.
Then

$$\left|\left\{\mathbf{W}^R_{\alpha,p}[\mu_{B_1}] > 3t\right\} \cap B_2\right| \le c_0 \exp\left(-\left(\tfrac{p-1-\eta}{4(p-1)}\right)^{\frac{p-1}{p-1-\eta}} \alpha p \ln 2 M^{-\frac{1}{p-1-\eta}} t^{\frac{p-1}{p-1-\eta}}\right) |B_2|. \tag{2.39}$$

This can be written under the form

$$|\{F > t\} \cap B_2| \le |B_2|\, \chi_{(0,t_0]} + c_0 \exp(-\delta_0 t) |B_2|\, \chi_{(t_0,\infty)}(t). \tag{2.40}$$

where $F = M^{-\frac{1}{p-1-\eta}}\left(\mathbf{W}^R_{\alpha,p}[\mu_{B_1}]\right)^{\frac{p-1}{p-1-\eta}}$ and $t_0 = \left(3\max\{\epsilon_0^{-1}, \frac{N-\alpha p}{p-1}(\ln 2)^{-\frac{\eta}{p-1}}\}\right)^{\frac{p-1}{p-1-\eta}}$.
Take $\delta \in (0, \delta_0)$, by Fubini's theorem

$$\int_{B_2} \exp(\delta F(x))\, dx = \delta \int_0^\infty \exp(\delta t) |\{F > t\} \cap B_2|\, dt$$

Thus,

$$\begin{aligned}\int_{B_2} \exp(\delta F(x))\, dx &\le \delta \int_0^{t_0} \exp(\delta t)\, dt\, |B_2| + c_0\delta \int_{t_0}^\infty \exp(-(\delta_0 - \delta)t)\, dt\, |B_2| \\ &\le (\exp(\delta t_0) - 1)|B_2| + \tfrac{c_0 \delta}{\delta_0 - \delta}|B_2|\end{aligned}$$

which is the desired inequality. □

Remark. By the proof of Proposition 2.2, we see that $\epsilon_0 \ge \frac{c_{30}}{\max(1, \ln 40r)}$ where $c_{30} = c_{30}(N, \alpha, p, \eta) > 0$. Thus, $t_0 \le c_{31}(\max(1, \ln 40r))^{\frac{p-1}{p-1-\eta}}$. Therefore $c_{29} \le c_{32}\exp\left(c_{33}(\max(1, \ln 40r))^{\frac{p-1}{p-1-\eta}}\right)$ where c_{32} and c_{33} depend on N, α, p and η.

2.4 Approximation of measures

The next result is an extension of a classical result of Feyel and de la Pradelle [11]. This type of result has been intensively used in the framework of Sobolev spaces since the pioneering work of Baras and Pierre [3], but apparently it is new in the case of Bessel-Lorentz spaces. We recall that a sequence of bounded measures $\{\mu_n\}$ in Ω converges to some bounded measure μ in Ω in the *narrow topology* of $\mathfrak{M}^b(\Omega)$ if

$$\lim_{n\to\infty}\int_\Omega \phi d\mu_n = \int_\Omega \phi d\mu \qquad \forall \phi \in C_b(\Omega) := C(\Omega)\cap L^\infty(\Omega). \tag{2.41}$$

Theorem 2.5 *Assume Ω is an open subset of $\mathbb{R}^N$. Let $\alpha > 0$, $1 < s < \infty$, $1 \leq q < \infty$ and $\mu \in \mathfrak{M}_+(\Omega)$. If μ is absolutely continuous with respect to $C_{\alpha,s,q}$ in Ω, there exists a nondecreasing sequence $\{\mu_n\} \subset \mathfrak{M}^b_+(\Omega)\cap(L^{\alpha,s,q}(\mathbb{R}^N))'$, with compact support in Ω which converges to μ weakly in the sense of measures. Furthermore, if $\mu \in \mathfrak{M}^b_+(\Omega)$, then $\mu_n \rightharpoonup \mu$ in the narrow topology.*

Proof. Step 1. Assume that μ has compact support. Let $\phi \in L^{\alpha,s,q}(\mathbb{R}^N)$ and $\tilde{\phi}$ its $C_{\alpha,s,q}$-quasicontinuous representative. Since μ is abolutely continuous with respect to $C_{\alpha,s,q}$, we can define the mapping

$$\phi \mapsto P(\phi) = \int_{\mathbb{R}^N} \tilde{\phi}^+ d\mu\lfloor_\Omega$$

where $\mu\lfloor_\Omega$ is the extension of μ by 0 in Ω^c. By Fatou's lemma, P is lower semicontinuous on $L^{\alpha,s,q}(\mathbb{R}^N)$. Furthermore it is convex and potitively homogeneous of degree 1. If $Epi(P)$ denotes the epigraph of P, i.e.

$$Epi(P) = \{(\phi,t) \in L^{\alpha,s,q}(\mathbb{R}^N)\times\mathbb{R} : t \geq P(\phi)\},$$

it is a closed convex cone. Let $\epsilon > 0$ and $\phi_0 \in C^\infty_c$, $\phi_0 \geq 0$. Since $(\phi_0, P(\phi_0)-\epsilon) \notin Epi(P)$, there exist $\ell \in (L^{\alpha,s,q}(\mathbb{R}^N))'$, a and b in $\mathbb{R}$ such that

$$a + bt + \ell(\phi) \leq 0 \qquad \forall (\phi,t) \in Epi(P), \tag{2.42}$$

$$a + b(P(\phi_0)-\epsilon) + \ell(\phi_0) > 0. \tag{2.43}$$

Since $(0,0) \in Epi(P)$, $a \leq 0$. Since $(s\phi, st) \in Epi(P)$ for all $s > 0$, $s^{-1}a + bt + \ell(\phi) \leq 0$, which implies

$$bt + \ell(\phi) \leq 0 \qquad \forall (\phi,t) \in Epi(P).$$

Finally, since $(0,1) \in Epi(P)$, $b \leq 0$. But if $b = 0$ we would have $\ell(\phi) \leq -a$ for all $\phi \in L^{\alpha,s,q}(\mathbb{R}^N)$. which would lead to $\ell = 0$ and $a > 0$ from (2.43), a contradiction. Therefore $b < 0$. Then, we put $\theta(\phi) = -\frac{\ell(\phi)}{b}$ and derive that, for any $(\phi,t) \in Epi(P)$, there holds $\theta(\phi) \leq t$, and in particular

$$\theta(\phi) \leq P(\phi) \qquad \forall \phi \in L^{\alpha,s,q}(\mathbb{R}^N). \tag{2.44}$$

Since $\phi \leq 0 \Longrightarrow P(\phi) = 0$, θ is a positive linear functional on $L^{\alpha,s,q}(\mathbb{R}^N)$. Furthermore

$$\sup_{\substack{\phi \in C^\infty_c(\mathbb{R}^N)\\ \|\phi\|_{L^\infty}\leq 1}} |\theta(\phi)| = \sup_{\substack{\phi \in C^\infty_c(\mathbb{R}^N)\\ \|\phi\|_{L^\infty}\leq 1}} \theta(\phi) \leq \sup_{\substack{\phi \in C^\infty_c(\mathbb{R}^N)\\ \|\phi\|_{L^\infty}\leq 1}} P(\phi) = P(1) = \mu(\Omega).$$

By the Riesz representation theorem, there exists $\sigma \in \mathfrak{M}_+(\mathbb{R}^N)$ such that

$$\theta(\phi) = \int_{\mathbb{R}^N} \phi d\sigma \qquad \forall \phi \in C_c^\infty(\mathbb{R}^N). \tag{2.45}$$

Inequality (2.44) implies $0 \leq \sigma \leq \mu\lfloor_\Omega$. Thus $supp(\sigma) \subset supp(\mu\lfloor_\Omega) = supp(\mu)$ and σ vanishes on Borel subsets of $C_{\alpha,s,q}$ capacity zero, as μ does it, besides (2.45) also values for all $\phi \in C^\infty(\mathbb{R}^N)$. From (2.43), we have

$$\int_{\mathbb{R}^N} \tilde{\phi}_0 d\sigma = \theta(\phi_0) > P(\phi_0) - \epsilon + \frac{a}{b} \geq \int_{\mathbb{R}^N} \tilde{\phi}_0 d\mu\lfloor_\Omega - \epsilon.$$

This implies

$$0 \leq \int_{\mathbb{R}^N} \tilde{\phi}_0 d(\mu\lfloor_\Omega - \sigma) \leq \epsilon. \tag{2.46}$$

It remains to prove that $\sigma \in (L^{\alpha,s,q}(\mathbb{R}^N))'$. For all $f \in C_c^\infty(\mathbb{R}^N)$, $f \geq 0$, there holds

$$\int_{\mathbb{R}^N} \mathbf{G}_\alpha[f] d\sigma = \theta(\mathbf{G}_\alpha[f]) \leq \|\theta\|_{(L^{\alpha,s,q}(\mathbb{R}^N))'} \|\mathbf{G}_\alpha[f]\|_{L^{\alpha,s,q}(\mathbb{R}^N)}, \tag{2.47}$$

since $\theta = -b^{-1}\ell$ and $\ell \in (L^{\alpha,s,q}(\mathbb{R}^N))'$. Now, given $f \in L^{s,q}(\mathbb{R}^N)$, $f \geq 0$ and a sequence of molifiers $\{\rho_n\}$, $(\chi_{B_n} f) * \rho_n \in C_c^\infty(\mathbb{R}^N)$ and $(\chi_{B_n} f) * \rho_n \to f$ in $L^{s,q}(\mathbb{R}^N)$, where χ_{B_n} is the indicator function of the ball B_n centered at the origin of radius n. Furthermore, there is a subsequence $\{n_k\}$ such that $\lim_{n_k \to \infty} \mathbf{G}_\alpha[(\chi_{B_{n_k}} f) * \rho_{n_k}](x) \to \mathbf{G}_\alpha[f](x)$, $C_{\alpha,s,q}$-quasi everywhere. Using Fatou's lemma and lower semicontinuity of the norm

$$\begin{aligned}
\int_{\mathbb{R}^N} \mathbf{G}_\alpha[f] d\sigma &\leq \liminf_{n_k \to \infty} \int_{\mathbb{R}^N} \mathbf{G}_\alpha[(\chi_{B_{n_k}} f) * \rho_{n_k}] d\sigma \\
&\leq \liminf_{n_k \to \infty} \|\theta\|_{(L^{\alpha,s,q}(\mathbb{R}^N))'} \left\| \mathbf{G}_\alpha[(\chi_{B_{n_k}} f) * \rho_{n_k}] \right\|_{L^{\alpha,s,q}(\mathbb{R}^N)} \\
&\leq \|\theta\|_{(L^{\alpha,s,q}(\mathbb{R}^N))'} \|\mathbf{G}_\alpha[f]\|_{L^{\alpha,s,q}(\mathbb{R}^N)}.
\end{aligned}$$

Therefore (2.47) also holds for all $f \in L^{s,q}(\mathbb{R}^N), f \geq 0$. Consequently $\sigma \in \mathfrak{M}_+^b(\mathbb{R}^N) \cap (L^{\alpha,s,q}(\mathbb{R}^N))'$ satisfies

$$\left| \int_{\mathbb{R}^N} \mathbf{G}_\alpha[f] d\sigma \right| \leq \|\theta\|_{(L^{\alpha,s,q}(\mathbb{R}^N))'} \|\mathbf{G}_\alpha[f]\|_{L^{\alpha,s,q}(\mathbb{R}^N)} \qquad \forall f \in L^{s,q}(\mathbb{R}^N). \tag{2.48}$$

Step 2. We assume that μ has no longer compact support. Set $\Omega_n = \{x \in \Omega : \mathrm{dist}\,(x, \Omega^c) \geq n^{-1}, |x| \leq n\}$, then $\Omega_n \subset \overline{\Omega_n} \subset \Omega_{n+1} \subset \Omega$ for $n \geq n_0$ such that $\Omega_{n_0} \neq \emptyset$. Let $\{\phi_n\} \subset C_c^\infty(\mathbb{R}^N)$ be an increasing sequence such that $0 \leq \phi_n \leq 1$, $\phi_n = 1$ in a neighborhood of $\overline{\Omega_n}$ and $supp(\phi_n) \subset \Omega_{n+1}$. and let $\nu_n = \phi_n \mu$. For $n \geq n_0$ there is $\sigma_n \in \mathfrak{M}_+^b(\mathbb{R}^N) \cap (L^{\alpha,s,q}(\mathbb{R}^N))'$ with $0 \leq \sigma_n \leq \nu_n$ and

$$\frac{1}{n} > \int_\Omega \phi_n d(\nu_n - \sigma_n) \geq \int_{\Omega_n} d(\nu_n - \sigma_n) = \int_{\Omega_n} d(\mu - \sigma_n).$$

We set $\mu_n = \sup\{\sigma_1, \sigma_2, ..., \sigma_n\}$, then $\{\mu_n\}$ is nondecreasing and $supp(\mu_n) \subset \Omega_{n+1}$, and $\mu_n \in \mathfrak{M}_+^b(\mathbb{R}^N) \cap (L^{\alpha,s,q}(\mathbb{R}^N))'$. Finally, let $\phi \in C_c(\Omega)$ and $m \in \mathbb{N}^*$ such that $supp(\phi) \subset \Omega_m$. For all $n \geq m$, we have

$$\left| \int_\Omega \phi d\mu_n - \int_\Omega \phi d\mu \right| \leq \left| \int_{\Omega_n} d(\mu - \mu_n) \right| \|\phi\|_{L^\infty(\mathbb{R}^N)} \leq \frac{1}{n} \|\phi\|_{L^\infty(\mathbb{R}^N)}.$$

Thus $\mu_n \rightharpoonup \mu$ weakly in the sense of measures.

Step 3. Assume that $\mu \in \mathfrak{M}_+^b(\Omega)$. Then $\mu_n(\Omega) \leq \mu(\Omega)$. Thus

$$\mu_n(\Omega) = \mu_n(\Omega_{n_0}) + \sum_{k=n_0}^{\infty} \mu_n(\overline{\Omega}_{k+1} \setminus \Omega_k)$$

Since the sequence $\{\mu_n\}$ is nondecreasing and $\lim_{k\to\infty} \mu_n(\overline{\Omega}_{k+1} \setminus \Omega_k) = \mu(\overline{\Omega}_{k+1} \setminus \Omega_k)$by the previous construction, we obtain by monotone convergence

$$\lim_{n\to\infty} \mu_n(\Omega) = \mu(\Omega_{n_0}) + \sum_{k=n_0}^{\infty} \mu(\overline{\Omega}_{k+1} \setminus \Omega_k) = \mu(\Omega)$$

Next we consider $\phi \in C_b(\Omega) := C(\Omega) \cap L^\infty(\Omega)$, then

$$\left| \int_\Omega \phi d\mu_n - \int_\Omega \phi d\mu \right| \leq \left| \int_\Omega d(\mu - \mu_n) \right| \|\phi\|_{L^\infty(\Omega)} \leq (\mu(\Omega) - \mu_n(\Omega)) \|\phi\|_{L^\infty(\Omega)} \to 0.$$

Thus $\mu_n \rightharpoonup \mu$ in the narrow topology of measures. □

As a consequence of Theorem 2.5 and Theorem 2.3 we obtain the following.

Theorem 2.6 *Let $p-1 < s_1 < \infty$, $p-1 < s_2 \leq \infty$, $0 < \alpha p < N$, $R > 0$ and $\mu \in \mathfrak{M}_+(\Omega)$. If μ is absolutely continuous with respect to the capacity $C_{\alpha p, \frac{s_1}{s_1-p+1}, \frac{s_2}{s_2-p+1}}$, there exists a nondecreasing sequence $\{\mu_n\} \subset \mathfrak{M}_+(\Omega)$ with compact support in Ω which converges to μ in the weak sense of measures and such that $\mathbf{W}_{\alpha,p}^R[\mu_n] \in L^{s_1,s_2}(\mathbb{R}^N)$, for all n. Furthermore, if $\mu \in \mathfrak{M}_+^b(\Omega)$, μ_n converges to to μ in the narrow topology.*

Proof. By Theorem 2.5 there exists a nondecreasing sequence $\{\mu_n\}$ of nonnegative measures with compact support in Ω, all elements of $(L^{\alpha p, \frac{s_1}{s_1-p+1}, \frac{s_2}{s_2-p+1}}(\mathbb{R}^N))'$, which converges weakly to μ. If $\mu \in \mathfrak{M}_+^b(\Omega)$, the convergence holds in the narrow topology. Noting that for a positive measure σ in $\mathbb{R}^N$,

$$\mathbf{G}_{\alpha p}[\sigma] \in L^{\frac{s_1}{p-1}, \frac{s_2}{p-1}}(\mathbb{R}^N) \Longleftrightarrow \sigma \in (L^{\alpha p, \frac{s_1}{s_1-p+1}, \frac{s_2}{s_2-p+1}}(\mathbb{R}^N))',$$

it implies $\mathbf{G}_{\alpha p}[\mu_n] \in L^{\frac{s_1}{p-1}, \frac{s_2}{p-1}}(\mathbb{R}^N)$. Then, by Theorem 2.3, $\mathbf{W}_{\alpha,p}^R[\mu_n] \in L^{s_1,s_2}(\mathbb{R}^N)$. □

3 Renormalized solutions

3.1 Classical results

Although the notion of renormalized solutions is becoming more and more present in the theory of quasilinear equations with measure data, it has not yet acquainted a popularity which could avoid us to present some of its main aspects. Let Ω be a bounded domain in $\mathbb{R}^N$. If $\mu \in \mathfrak{M}^b(\Omega)$, we denote by μ^+ and μ^- respectively its positive and negative part. We denote by $\mathfrak{M}_0(\Omega)$ the space of measures in Ω which are absolutely continuous with respect to the $c^{\Omega}_{1,p}$-capacity defined on a compact set $K \subset \Omega$ by

$$c^{\Omega}_{1,p}(K) = \inf\left\{\int_{\Omega} |\nabla\phi|^p\,dx : \phi \geq \chi_K, \phi \in C^{\infty}_c(\Omega)\right\}. \tag{3.1}$$

We also denote $\mathfrak{M}_s(\Omega)$ the space of measures in Ω with support on a set of zero $c^{\Omega}_{1,p}$-capacity. Classically, any $\mu \in \mathfrak{M}^b(\Omega)$ can be written in a unique way under the form $\mu = \mu_0 + \mu_s$ where $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}^b(\Omega)$ and $\mu_s \in \mathfrak{M}_s(\Omega)$. We recall that any $\mu_0 \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}^b(\Omega)$ can be written under the form $\mu_0 = f - div\, g$ where $f \in L^1(\Omega)$ and $g \in L^{p'}(\Omega)$.

For $k > 0$ and $s \in \mathbb{R}$ we set $T_k(s) = \max\{\min\{s,k\}, -k\}$. We recall that if u is a measurable function defined and finite a.e. in Ω, such that $T_k(u) \in W^{1,p}_0(\Omega)$ for any $k > 0$, there exists a measurable function $v : \Omega \to \mathbb{R}^N$ such that $\nabla T_k(u) = \chi_{|u|\leq k} v$ a.e. in Ω and for all $k > 0$. We define the gradient ∇u of u by $v = \nabla u$. We recall the definition of a renormalized solution given in [10].

Definition 3.1 *Let $\mu = \mu_0 + \mu_s \in \mathfrak{M}^b(\Omega)$. A measurable function u defined in Ω and finite a.e. is called a renormalized solution of*

$$\begin{aligned} -\Delta_p u &= \mu && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.2}$$

if $T_k(u) \in W^{1,p}_0(\Omega)$ for any $k > 0$, $|\nabla u|^{p-1} \in L^r(\Omega)$ for any $0 < r < \frac{N}{N-1}$, and u has the property that for any $k > 0$ there exist $\lambda^+_k, \lambda^-_k \in \mathfrak{M}^b_+(\Omega) \cap \mathfrak{M}_0(\Omega)$, respectively concentrated on the sets $u = k$ and $u = -k$, with the property that $\lambda^+_k \rightharpoonup \mu^+_s$, $\lambda^-_k \rightharpoonup \mu^-_s$ in the narrow topology of measures, such that

$$\int_{\{|u|<k\}} |\nabla u|^{p-2}\,\nabla u \nabla\phi\,dx = \int_{\{|u|<k\}} \phi d\mu_0 + \int_{\Omega} \phi d\lambda^+_k - \int_{\Omega} \phi d\lambda^-_k, \tag{3.3}$$

for every $\phi \in W^{1,p}_0(\Omega) \cap L^{\infty}(\Omega)$.

Remark. If u is a renormalized solution of problem (3.2) and $\mu \in \mathfrak{M}^b_+(\Omega)$, then $u \geq 0$ in Ω. Indeed, taking $k > m > 0$ and $\phi = T_m(\max\{-u,0\})$, then $0 \leq \phi \leq m$ and we have

$$\begin{aligned} \int_{\{|u|<k\}} |\nabla u|^{p-2}\,\nabla u \nabla\phi dx &= \int_{\{|u|<k\}} T_m(\max\{-u,0\}) d\mu_0 + \int_{\Omega} T_m(\max\{-u,0\}) d\lambda^+_k \\ &\quad - \int_{\Omega} T_m(\max\{-u,0\}) d\lambda^-_k \\ &\geq -m\lambda^-_k(\Omega). \end{aligned}$$

Thus

$$\int_\Omega |\nabla T_m(\max\{-u,0\})|^p \leq m\lambda_k^-(\Omega)$$

Letting $k \to \infty$, we obtain $\nabla T_m(\max\{-u,0\}) = 0$ a.e., thus $u \geq 0$ a.e. in Ω.

We recall the following important results, see [10, Th 4.1, Sec 5.1].

Theorem 3.2 *Let $\{\mu_n\} \subset \mathfrak{M}^b(\Omega)$ be a sequence such that $\sup_n |\mu_n|(\Omega) < \infty$ and let $\{u_n\}$ be renormalized solutions of*

$$\begin{aligned} -\Delta_p u_n &= \mu_n \qquad && \text{in } \Omega \\ u_n &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.4}$$

Then, up to a subsequence, $\{u_n\}$ converges a.e. to a solution u of $-\Delta_p u = \mu$ in the sense of distributions in Ω, for some measure $\mu \in \mathfrak{M}^b(\Omega)$, and for every $k > 0$, $k^{-1}\int_\Omega |\nabla T_k(u)|^p \leq M$ for some $M > 0$.

Finally we recall the following fundamental stability result of [10] which extends Theorem 3.2.

Theorem 3.3 *Let $\mu = \mu_0 + \mu_s^+ - \mu_s^- \in \mathfrak{M}^b(\Omega)$, with $\mu_0 = f - \operatorname{div} g \in \mathfrak{M}_0(\Omega)$, $\mu_s^+, \mu_s^- \in \mathfrak{M}_s^+(\Omega)$. Assume there are sequences $\{f_n\} \subset L^1(\Omega)$, $\{g_n\} \subset (L^{p'}(\Omega))^N$, $\{\eta_n^1\}, \{\eta_n^2\} \subset \mathfrak{M}_+^b(\Omega)$ such that $f_n \rightharpoonup f$ weakly in $L^1(\Omega)$, $g_n \to g$ in $L^{p'}(\Omega)$ and $\operatorname{div} g_n$ is bounded in $\mathfrak{M}^b(\Omega)$, $\eta_n^1 \rightharpoonup \mu_s^+$ and $\eta_n^2 \rightharpoonup \mu_s^-$ in the narrow topology. If $\mu_n = f_n - \operatorname{div} g_n + \eta_n^1 - \eta_n^2$ and u_n is a renormalized solution of (3.4), then, up to a subsequence, u_n converges a.e. to a renormalized solution u of (3.2). Furthermore $T_k(u_n) \to T_k(u)$ in $W_0^{1,p}(\Omega)$.*

3.2 Applications

We present below some interesting consequences of the above theorem.

Corollary 3.4 *Let $\mu \in \mathfrak{M}^b(\Omega)$ with compact support in Ω and $\omega \in \mathfrak{M}^b(\Omega)$. Let $\{f_n\} \subset L^1(\Omega)$ which converges weakly to $f \in L^1(\Omega)$ and $\mu_n = \rho_n * \mu$ where $\{\rho_n\}$ is a sequence of mollifiers. If u_n is a renormalized solution of*

$$\begin{aligned} -\Delta_p u_n &= f_n + \mu_n + \omega \qquad && \text{in } \Omega \\ u_n &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.5}$$

then, up to a subsequence, u_n converges to a renormalized solution of

$$\begin{aligned} -\Delta_p u &= f + \mu + \omega \qquad && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.6}$$

Proof. We write $\omega = \tilde{h} - \operatorname{div}\tilde{g} + \omega_s^+ - \omega_s^-$ and $\mu = h - \operatorname{div} g + \mu_s^+ - \mu_s^-$, with $h, \tilde{h} \in L^1(\Omega)$, $g, \tilde{g} \in (L^{p'}(\Omega))^N$, h, g, μ_s^+ and μ_s^- with support in a compact set $K \subset \Omega$. For n_0 large enough, $\rho_n * h$, $\rho_n * g$, $\rho_n * \mu_s^+$and $\rho_n * \mu_s^-$ have also their support in a fixed compact subset of Ω for all $n \geq n_0$. Moreover $\rho_n * h \to h$ and $\rho_n * g \to g$ in $L^1(\Omega)$ and $(L^{p'}(\Omega))^N$ respectively and $div\, \rho_n * g \to div\, g$ in $W^{-1,p'}(\Omega)$. Therefore

$$f_n + \mu_n + \omega = f_n + \tilde{h} + \rho_n * h - div\,(\tilde{g} + \rho_n * g) + \omega_s^+ + \rho_n * \mu_s^+ - \omega_s^- - \rho_n * \mu_s^-$$

is an approximation of the measure $f + \mu + \omega$ in the sense of Theorem 3.3. This implies the claim. □

Corollary 3.5 *Let $\mu_i \in \mathfrak{M}_+^b(\Omega)$, $i = 1, 2$, and $\{\mu_{i,n}\} \subset \mathfrak{M}_+^b(\Omega)$ be a nondecreasing and converging to μ_i in $\mathfrak{M}_+^b(\Omega)$. Let $\{f_n\} \subset L^1(\Omega)$ which converges to some f weakly in $L^1(\Omega)$. Let $\{\vartheta_n\} \subset \mathfrak{M}^b(\Omega)$ which converges to some $\vartheta \in \mathfrak{M}_s(\Omega)$ in the narrow topology. For any $n \in \mathbb{N}$ let u_n be a renormalized solution of*

$$\begin{aligned} -\Delta_p u_n &= f_n + \mu_{1,n} - \mu_{2,n} + \vartheta_n \qquad && \text{in } \Omega \\ u_n &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.7}$$

Then, up to a subsequence, u_n converges a.e. to a renormalized solution of problem

$$\begin{aligned} -\Delta_p u &= f + \mu_1 - \mu_2 + \vartheta \qquad && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{3.8}$$

The proof of this results is based upon two lemmas

Lemma 3.6 *For any $\mu \in \mathfrak{M}_0(\Omega) \cap \mathfrak{M}_+^b(\Omega)$ there exists $f \in L^1(\Omega)$ and $h \in W^{-1,p'}(\Omega)$ such that $\mu = f + h$ and*

$$\|f\|_{L^1(\Omega)} + \|h\|_{W^{-1,p'}(\Omega)} + \|h\|_{\mathfrak{M}^b(\Omega)} \leq 5\mu(\Omega). \tag{3.9}$$

Proof. Following [9] and the proof of [7, Th 2.1], one can write $\mu = \phi\gamma$ where $\gamma \in W^{-1,p'}(\Omega) \cap \mathfrak{M}_+^b(\Omega)$ and $0 \leq \phi \in L^1(\Omega, \gamma)$. Let $\{\Omega_n\}_{n \in \mathbb{N}_*}$ be an increasing sequence of compact subsets of Ω such that $\cup_n \Omega_n = \Omega$. We define the sequence of measures $\{\nu_n\}_{n \in \mathbb{N}_*}$ by

$$\begin{aligned} &\nu_n = T_n(\chi_{\Omega_n}\phi)\gamma - T_{n-1}(\chi_{\Omega_{n-1}}\phi)\gamma \quad \text{for } n \geq 2 \\ &\nu_1 = T_1(\chi_{\Omega_1}\phi)\gamma. \end{aligned}$$

Since $\nu_k \geq 0$, then $\sum_{k=1}^{\infty} \nu_k = \mu$ with strong convergence in $\mathfrak{M}^b(\Omega)$, $\|\nu_k\|_{\mathfrak{M}^b(\Omega)} = \nu_k(\Omega)$ and $\sum_{k=1}^{\infty} \|\nu_k\|_{\mathfrak{M}^b(\Omega)} = \mu(\Omega)$. Let $\{\rho_n\}$ be a sequence of mollifiers. We may assume that $\eta_n = \rho_n * \nu_n \in C_c^\infty(\Omega)$,

$$\|\eta_n - \nu_n\|_{W^{-1,p'}(\Omega)} \leq 2^{-n}\mu(\Omega)$$

Set $f_n = \sum_{k=1}^{n} \eta_k$, then $\|f_n\|_{L^1(\Omega)} \leq \sum_{k=1}^{n} \|\eta_k\|_{L^1(\Omega)} \leq \sum_{k=1}^{n} \|\nu_k\|_{\mathfrak{M}^b(\Omega)} \leq \mu(\Omega)$. If we define $f = \lim_{n\to\infty} f_n$, then $f \in L^1(\Omega)$ with $\|f\|_{L^1(\Omega)} \leq \mu(\Omega)$. Set $h_n = \sum_{k=1}^{n} (\nu_k - \eta_k)$, then $h_n \in W^{-1,p'}(\Omega) \cap \mathfrak{M}^b(\Omega)$, $\|h_n\|_{W^{-1,p'}(\Omega)} \leq 2\mu(\Omega)$ and h_n converges strongly in $W^{-1,p'}(\Omega)$ to some h which satisfies $\|h\|_{W^{-1,p'}(\Omega)} \leq 2\mu(\Omega)$. Since $\mu = f + h$ and $\|h\|_{\mathfrak{M}^b(\Omega)} \leq 2\mu(\Omega)$, the result follows. □

Lemma 3.7 *Let $\mu \in \mathfrak{M}^b_+(\Omega)$. If $\{\mu_n\} \subset \mathfrak{M}^b_+(\Omega)$ is a nondecreasing sequence which converges to μ in $\mathfrak{M}^b(\Omega)$, there exist $F_n, F \in L^1(\Omega)$, $G_n, G \in W^{-1,p'}(\Omega)$ and $\mu_{n\,s}, \mu_s \in \mathfrak{M}_s(\Omega)$ such that*

$$\mu_n = \mu_{n\,0} + \mu_{n\,s} = F_n + G_n + \mu_{n\,s} \quad \text{and} \quad \mu = \mu_0 + \mu_s = F + G + \mu_s,$$

such that $F_n \to F$ in $L^1(\Omega)$, $G_n \to G$ in $W^{-1,p'}(\Omega)$ and in $\mathfrak{M}^b(\Omega)$ and $\mu_{n\,s} \to \mu_s$ in $\mathfrak{M}^b(\Omega)$, and

$$\|F_n\|_{L^1(\Omega)} + \|G_n\|_{W^{-1,p'}(\Omega)} + \|G_n\|_{\mathfrak{M}^b(\Omega)} + \|\mu_{n\,s}\|_{\mathfrak{M}^b(\Omega)} \le 6\mu(\Omega). \tag{3.10}$$

Proof. Since $\{\mu_n\}$ is nondecreasing $\{\mu_{n\,0}\}$ and $\{\mu_{n\,s}\}$ share this property. Clearly

$$\|\mu - \mu_n\|_{\mathfrak{M}^b(\Omega)} = \|\mu_0 - \mu_{n\,0}\|_{\mathfrak{M}^b(\Omega)} + \|\mu_s - \mu_{n\,s}\|_{\mathfrak{M}^b(\Omega)},$$

thus $\mu_{n\,0} \to \mu_0$ and $\mu_{n\,s} \to \mu_s$ in $\mathfrak{M}^b(\Omega)$. Furthermore $\|\mu_{n\,s}\|_{\mathfrak{M}^b(\Omega)} \le \mu_s(\Omega) \le \mu(\Omega)$. Set $\tilde{\mu}_{0\,0} = 0$ and $\tilde{\mu}_{n\,0} = \mu_{n\,0} - \mu_{n-1\,0}$ for $n \in \mathbb{N}_*$. From Lemma 3.6, for any $n \in \mathbb{N}$, one can find $f_n \in L^1(\Omega)$, $h_n \in W^{-1,p'}(\Omega) \cap \mathfrak{M}^b(\Omega)$ such that $\tilde{\mu}_{n\,0} = f_n + h_n$ and

$$\|f_n\|_{L^1(\Omega)} + \|h_n\|_{W^{-1,p'}(\Omega)} + \|h_n\|_{\mathfrak{M}^b(\Omega)} \le 5\tilde{\mu}_{n\,0}(\Omega).$$

If we define $F_n = \sum_{k=1}^{n} f_k$ and $G_n = \sum_{k=1}^{n} h_k$, then $\mu_{n\,0} = F_n + G_n$ and

$$\|F_n\|_{L^1(\Omega)} + \|G_n\|_{W^{-1,p'}(\Omega)} + \|G_n\|_{\mathfrak{M}^b(\Omega)} \le 5\tilde{\mu}_0(\Omega).$$

Therefore the convergence statements and (3.10) hold. □

Proof of Corollary 3.5. We set $\nu_n = f_n + \mu_{n,1} - \mu_{n,2} + \vartheta_n$ and $\nu = f + \mu_1 - \mu_2 + \vartheta$. From Lemma 3.7 we can write

$$\nu_n = f_n + F_{1\,n} - F_{2\,n} + G_{1\,n} - G_{2\,n} + \mu_{1\,n\,s} - \mu_{2\,n\,s} + \vartheta_n$$

and

$$\nu = f + F_1 - F_2 + G_1 - G_2 + \mu_{1\,s} - \mu_{2\,s} + \vartheta,$$

and the convergence properties listed in the lemma hold. Therefore we can apply Theorem 3.3 and the conclusion follows. □

In the next result we prove the main pointwise estimates on renormalized solutions.

Theorem 3.8 *Let Ω be a bounded domain of $\mathbb{R}^N$. Then there exists a constant $c > 0$, dependent on p and N such that if $\mu \in \mathfrak{M}^b(\Omega)$ and u is a renormalized solution of problem (3.2) there holds*

$$-c\mathbf{W}_{1,p}^{2\,diam\,\Omega}[\mu^-] \le u(x) \le c\mathbf{W}_{1,p}^{2\,diam\,\Omega}[\mu^+] \quad a.e. \text{ in } \Omega. \tag{3.11}$$

Proof. We claim the there exist renormalized solutions u_1 and u_2 of problem (3.2) with respective data μ^+ and μ^- such that

$$-u_2 \leq u \leq u_1 \qquad \text{a.e. in } \Omega. \tag{3.12}$$

We use the decomposition $\mu = \mu^+ - \mu^- = (\mu_0^+ - \mu_s^+) - (\mu_0^- - \mu_s^-)$. We put $u_k = T_k(u)$, $\mu_k = \mathbf{1}_{\{|u|<k\}}\mu_0 + \lambda_k^+ - \lambda_k^-$, $v_k = \mathbf{1}_{\{|u|<k\}}\mu_0^+ + \lambda_k^+$. Since $\mu_k \in \mathfrak{M}_0(\Omega)$, problem (3.2) with data μ_k admits a unique renormalized solution (see [7]), and clearly u_k is such a solution. Since $v_k \in \mathfrak{M}_0(\Omega)$, problem (3.2) with data v_k admits a unique solution $u_{k,1}$ which is furthermore nonnegative and dominates u_k a.e. in Ω. From Corollary 3.5, $\{u_{k,1}\}$ converges a.e. in Ω to a renormalized solution u_1 of (3.2) with data μ^+ and $u \leq u_1$. Similarly $-u \leq u_2$ where u_2 is a renormalized solution of (3.2) with μ^-. Finally, from [17, Th 6.9] there is a positive constant c dependent only on p and N such that

$$u_1(x) \leq c\mathbf{W}_{1,p}^{2\,diam\,\Omega}[\mu^+] \quad \text{and} \quad u_2(x) \leq c\mathbf{W}_{1,p}^{2\,diam\,\Omega}[\mu^-] \quad \text{a.e. in } \Omega. \tag{3.13}$$

This implies the claim. □

4 Equations with absorption terms

4.1 The general case

Let $g : \Omega\times\mathbb{R} \mapsto \mathbb{R}$ be a Caratheodory function such that the map $s \mapsto g(x,s)$ is nondecreasing and odd for almost all $x \in \Omega$. If U is a function defined in Ω we define the function $g \circ U$ in Ω by

$$g \circ U(x) = g(x, U(x)) \quad \text{for almost all } x \in \Omega.$$

We consider the problem

$$\begin{aligned} -\Delta_p u + g \circ u &= \mu \qquad && \text{in } \Omega \\ u &= 0 && \text{in } \partial\Omega. \end{aligned} \tag{4.14}$$

where $\mu \in \mathfrak{M}^b(\Omega)$. We say that u is a *renormalized solution* of problem (4.14) if $g\circ u \in L^1(\Omega)$ and u is a renormalized solution of

$$\begin{aligned} -\Delta_p u &= \mu - g \circ u \qquad && \text{in } \Omega \\ u &= 0 && \text{in } \partial\Omega. \end{aligned} \tag{4.15}$$

Theorem 4.1 *Let $\mu_i \in \mathfrak{M}_+^b(\Omega)$, $i = 1, 2$, such that there exists a nondecreasing sequences $\{\mu_{i,n}\} \subset \mathfrak{M}_+^b(\Omega)$, with compact support in Ω, converging to μ_i and $g \circ \left(c\mathbf{W}_{1,p}^{2\,diam\,\Omega}[\mu_{i,n}]\right) \in L^1(\Omega)$ with the same constant c as in Theorem 3.8. Then there exists a renormalized solution of*

$$\begin{aligned} -\Delta_p u + g \circ u &= \mu_1 - \mu_2 \qquad && \text{in } \Omega \\ u &= 0 && \text{in } \partial\Omega, \end{aligned} \tag{4.16}$$

such that

$$-c\mathbf{W}_{1,p}^{2\,diam\,\Omega}[\mu_2](x) \leq u(x) \leq c\mathbf{W}_{1,p}^{2\,diam\,\Omega}[\mu_1](x) \quad \text{a.e. in } \Omega. \tag{4.17}$$

Lemma 4.2 *Assume g belongs to $L^\infty(\Omega\times\mathbb{R})$, besides the assumptions of Theorem 4.1. Let $\lambda_i \in \mathfrak{M}^b_+(\Omega)$ $(i=1,2)$, with compact support in Ω. Then there exist renormalized solutions u, u_i, v_i $(i=1,2)$ to problems*

$$\begin{array}{rl} -\Delta_p u + g\circ u = \lambda_1-\lambda_2 & in\ \Omega \\ u=0 & in\ \partial\Omega, \end{array} \tag{4.18}$$

$$\begin{array}{rl} -\Delta_p u_i + g\circ u_i = \lambda_i & in\ \Omega \\ u_i=0 & in\ \partial\Omega, \end{array} \tag{4.19}$$

$$\begin{array}{rl} -\Delta_p v_i = \lambda_i & in\ \Omega \\ v_i=0 & in\ \partial\Omega, \end{array} \tag{4.20}$$

such that

$$\begin{aligned} -c\mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\lambda_2](x) \le -v_2(x) \le -u_2(x) \le u(x) \\ \le u_1(x) \le v_1(x) \le c\mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\lambda_1](x) \end{aligned} \tag{4.21}$$

for almost all $x\in\Omega$.

Proof. Let $\{\rho_n\}$ be a sequence of mollifiers, $\lambda_{i,n}=\rho_n*\lambda_i$, $(i=1,2)$ and $\lambda_n=\lambda_{1,n}-\lambda_{2,n}$. Then, for n_0 large enough, $\lambda_{1,n}$, $\lambda_{2,n}$ and λ_n are bounded with compact support in Ω for all $n\ge n_0$ and by minimization there exist unique solutions in $W^{1,p}_0(\Omega)$ to problems

$$\begin{array}{rl} -\Delta_p u_n + g\circ u_n = \lambda_n & \text{in } \Omega \\ u_n=0 & \text{in } \partial\Omega, \end{array}$$

$$\begin{array}{rl} -\Delta_p u_{i,n} + g\circ u_{i,n} = \lambda_{i,n} & \text{in } \Omega \\ u_{i,n}=0 & \text{in } \partial\Omega, \end{array}$$

$$\begin{array}{rl} -\Delta_p v_{i,n} = \lambda_{i,n} & \text{in } \Omega \\ v_{i,n}=0 & \text{in } \partial\Omega, \end{array}$$

and by the maximum principle, they satisfy

$$-v_{2,n}(x)\le -u_{2,n}(x)\le u_n(x)\le u_{1,n}(x)\le v_{1,n}(x), \quad \forall x\in\Omega,\ \forall n\ge n_0. \tag{4.22}$$

Since the λ_i are bounded measure and $g\in L^\infty(\Omega\times\mathbb{R})$ the the sequences of measures $\{\lambda_{1,n}-\lambda_{2,n}-g\circ u_n\}$, $\{\lambda_{i,n}-g\circ u_{i,n}\}$ and $\{\lambda_{i,n}\}$ are uniformly bounded in $\mathfrak{M}^b(\Omega)$. Thus, by Theorem 3.2 there exists a subsequence, still denoted by the index n such that $\{u_n\}$, $\{u_{i,n}\}$, $\{v_{i,n}\}$ converge a.e. in Ω to functions $\{u\}$, $\{u_i\}$, $\{v_i\}$ $(i=1,2)$ when $n\to\infty$. Furthermore $g\circ u_n$ and $g\circ u_{i,n}$ converge in $L^1(\Omega)$ to $g\circ u$ and $g\circ u_i$ respectively. By Corollary 3.4, we can assume that $\{u\}$, $\{u_i\}$, $\{v_i\}$ are renormalized solutions of (4.18)-(4.20), and by Theorem 3.8, $v_i(x)\le c\mathbf{W}_{1,p}^{2\,diam\,\Omega}[\lambda_i](x)$, a.e. in Ω. Thus we get (4.21). □

Lemma 4.3 *Let g satisfy the assumptions of Theorem 4.1 and let $\lambda_i\in\mathfrak{M}^b_+(\Omega)$ $(i=1,2)$, with compact support in Ω such that $g\circ\left(c\mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\lambda_i]\right)\in L^1(\Omega)$, where c is the constant* *of Theorem 4.1. Then there exist renormalized solutions u, u_i of the problems* (4.18)-(4.19) *such that*

$$-c\mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\lambda_2](x) \leq -u_2(x) \leq u(x) \leq u_1(x) \leq c\mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\lambda_1](x) \tag{4.23}$$

for almost all $x \in \Omega$. Furthermore, if ω_i, θ_i have the same properties as the λ_i and satisfy $\omega_i \leq \lambda_i \leq \theta_i$, one can find solutions u_{ω_i} and u_{θ_i} of problems (4.19) with right-hand respective side ω_i and θ_i, such that $u_{\omega_i} \leq u_i \leq u_{\theta_i}$.

Proof. From Lemma 4.2 there exist renormalized solutions u_n, $u_{i,n}$ to problems

$$\begin{aligned} -\Delta_p u_n + T_n(g \circ u_n) &= \lambda_1 - \lambda_2 && \text{in } \Omega \\ u_n &= 0 && \text{on } \partial\Omega, \end{aligned}$$

and

$$\begin{aligned} -\Delta_p u_{i,n} + T_n(g \circ u_{i,n}) &= \lambda_i && \text{in } \Omega \\ u_{i,n} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

$i = 1, 2$, and they satisfy

$$-c\mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\lambda_2](x) \leq -u_{2,n}(x) \leq u_n(x) \leq u_{1,n}(x) \leq c\mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\lambda_1](x). \tag{4.24}$$

Since $\int_\Omega |T_n(g \circ u_n)|dx \leq \lambda_1(\Omega) + \lambda_2(\Omega)$ and $\int_\Omega T_n(g \circ u_{i,n})dx \leq \lambda_i(\Omega)$ thus as in Lemma 4.2 one can choose a subsequence, still denoted by the index n such that $\{u_n, u_{1,n}, u_{2,n}\}$ converges a.e. in Ω to $\{u, u_1, u_2\}$ for which (4.24) is satisfied a.e. in Ω. Since $g \circ \Big(c\mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\lambda_i]\Big) \in L^1(\Omega)$ we derive from (4.24) and the dominated convergence theorem that $T_n(g \circ u_n) \to g \circ u$ and $T_n(g \circ u_{i,n}) \to g \circ u_i$ in $L^1(\Omega)$. It follows from Theorem 3.3 that u and u_i are respective solutions of (4.18), (4.19). The last statement follows from the same assertion in Lemma 4.2. □

Proof of Theorem 4.1. From Lemma 4.3, there exist renormalized solutions u_n, $u_{i,n}$ to problems

$$\begin{aligned} -\Delta_p u_n + g \circ u_n &= \mu_{1,n} - \mu_{2,n} && \text{in } \Omega \\ u_n &= 0 && \text{on } \partial\Omega, \end{aligned}$$

and

$$\begin{aligned} -\Delta_p u_{i,n} + g \circ u_{i,n} &= \mu_{i,n} && \text{in } \Omega \\ u_{i,n} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

$i = 1, 2$ such that $\{u_{i,n}\}$ is nonnegative and nondecreasing and they satisfy

$$-c\mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\mu_2](x) \leq -u_{2,n}(x) \leq u_n(x) \leq u_{1,n}(x) \leq c\mathbf{W}_{1,p}^{2\,diam\,(\Omega)}[\mu_1](x) \tag{4.25}$$

a.e. in Ω. As in the proof of Lemma 4.3, up to the same subsequence, $\{u_{1,n}\}$, $\{u_{2,n}\}$ and $\{u_n\}$ converge to u_1, u_2 and u a.e. in Ω. Since $g \circ u_{i,n}$ are nondecreasing, positive and $\int_\Omega g \circ u_{i,n} dx \leq \mu_{i,n}(\Omega) \leq \mu_i(\Omega)$, it follows from the monotone convergence theorem that $\{g \circ u_{i,n}\}$ converges to $g \circ u_i$ in $L^1(\Omega)$. Finally, since $|g \circ u_n| \leq g \circ u_1 + g \circ u_2$, $\{g \circ u_n\}$ converges to $g \circ u$ in $L^1(\Omega)$ by dominated convergence. Applying Corollary 3.5 we conclude that u is a renormalized solution of (4.16) and that (4.17) holds. □

4.2 Proofs of Theorem 1.1 and Theorem 1.2

We are now in situation of proving the two theorems stated in the introduction.

Proof of Theorem 1.1. **1-** Since μ is absolutely continuous with respect to the capacity $C_{p,\frac{Nq}{Nq-(p-1)(N-\beta))},\frac{q}{q+1-p}}$, μ^+ and μ^- share this property. By Theorem 2.6 there exist two nondecreasing sequences $\{\mu_{1,n}\}$ and $\{\mu_{2,n}\}$ of positive bounded measures with compact support in Ω which converge to μ^+ and μ^- respectively and which have the property that $\mathbf{W}^R_{1,p}[\mu_{i,n}] \in L^{\frac{Nq}{N-\beta},q}(\mathbb{R}^N)$, for $i = 1,2$ and all $n \in \mathbb{N}$. Furthermore, with $R = diam\,(\Omega)$,

$$\begin{aligned}
\int_{\mathbb{R}^N} \tfrac{1}{|x|^\beta}\left(\mathbf{W}^{2R}_{1,p}[\mu_{i,n}](x)\right)^q dx &\le \int_0^\infty \left(\tfrac{1}{|.|^\beta}\right)^*(t)\left(\left(\mathbf{W}^{2R}_{1,p}[\mu_{i,n}]\right)^*(t)\right)^q dt \\
&\le c_{34}\int_0^\infty \tfrac{1}{t^{\frac{\beta}{N}}}\left(\left(\mathbf{W}^{2R}_{1,p}[\mu_{i,n}]\right)^*(t)\right)^q dt \\
&\le c_{34}\left\|\mathbf{W}^{2R}_{1,p}[\mu_{i,n}]\right\|^q_{L^{\frac{Nq}{N-\beta},q}(\mathbb{R}^N)} \\
&< \infty.
\end{aligned} \tag{4.26}$$

Then the result follows from Theorem 4.1.

2- Because μ is absolutely continuous with respect to the capacity $C_{p,\frac{Nq}{Nq-(p-1)(N-\beta))},1}$, so are μ^+ and μ^-. Applying again Theorem 2.6 there exist two nondecreasing sequences $\{\mu_{1,n}\}$ and $\{\mu_{2,n}\}$ of positive bounded measures with compact support in Ω which converge to μ^+ and μ^- respectively and such that $\mathbf{W}^R_{1,p}[\mu_{i,n}] \in L^{\frac{Nq}{N-\beta},1}(\mathbb{R}^N)$. This implies in particular

$$\left(\mathbf{W}^{2R}_{1,p}[\mu_{i,n}](.)\right)^*(t) \le c_{35}t^{-\frac{N-\beta}{Nq}}, \qquad \forall t > 0, \tag{4.27}$$

for some $c_{34} > 0$. Therefore, by Theorem 2.3

$$\begin{aligned}
\int_\Omega \tfrac{1}{|x|^\beta} g\left(c\mathbf{W}^{2R}_{1,p}[\mu_{i,n}](x)\right) dx &\le \int_0^{|\Omega|}\left(\tfrac{1}{|.|^\beta}\right)^*(t) g\left(c\left(\mathbf{W}^{2R}_{1,p}[\mu_{i,n}]\right)^*(t)\right) dt \\
&\le c_{36}\int_0^{|\Omega|}\tfrac{1}{t^{\frac{\beta}{N}}} g\left(c\left(\mathbf{W}^{2R}_{1,p}[\mu_{i,n}]\right)^*(t)\right) dt \\
&\le c_{36}\int_0^{|\Omega|}\tfrac{1}{t^{\frac{\beta}{N}}} g\left(c_{35}ct^{-\frac{N-\beta}{Nq}}\right) dt \\
&\le c_{37}\int_a^\infty g(t)t^{-q-1}dt \\
&< \infty,
\end{aligned} \tag{4.28}$$

where $a > 0$ depends on $|\Omega|$, $c_{35}c$, N, β, q. Thus the result follows by Theorem 4.1. ☐

Proof of Theorem 1.2. Again we take $R = diam\,(\Omega)$. Let $\{\Omega_n\}_{n\in\mathbb{N}_*}$ be an increasing sequence of compact subsets of Ω such that $\cup_n\Omega_n = \Omega$. We define $\mu_{i,n} = T_n(\chi_{\Omega_n}f_i) + \chi_{\Omega_n}\nu_i$ $(i = 1,2)$. Then $\{\mu_{1,n}\}$ and $\{\mu_{2,n}\}$ are nondecreasing sequences of elements of $\mathfrak{M}^b_+(\Omega)$ with compact support, and they converge to μ^+ and μ^- respectively. Since for any $\epsilon > 0$ there exists $c_\epsilon > 0$ such that

$$\left(\mathbf{W}_{1,p}^{2R}[\mu_{i,n}]\right)^\lambda \leq c_\epsilon n^{\frac{\lambda}{p-1}} + (1+\epsilon)\left(\mathbf{W}_{1,p}^{2R}[\nu_i]\right)^\lambda, \tag{4.29}$$

a.e. in Ω, it follows

$$\exp\left(\tau\left(c\mathbf{W}_{1,p}^{2R}[\mu_{i,n}]\right)^\lambda\right) \leq c_{\epsilon,n,c}\exp\left(\tau(1+\epsilon)\left(c\mathbf{W}_{1,p}^{2R}[\nu_i]\right)^\lambda\right). \tag{4.30}$$

If there holds

$$\left\|\mathbf{M}_{p,2R}^{\frac{(p-1)(\lambda-1)}{\lambda}}[\nu_i]\right\|_{L^\infty(\Omega)} < \left(\frac{p\ln 2}{\tau(12\lambda c)^\lambda}\right)^{\frac{p-1}{\lambda}}, \tag{4.31}$$

we can choose $\epsilon > 0$ small enough so that

$$\tau(1+\epsilon)c^\lambda < \frac{p\ln 2}{(12\lambda)^\lambda\left\|\mathbf{M}_{p,2R}^{\frac{(p-1)(\lambda-1)}{\lambda}}[\nu_i]\right\|_{L^\infty(\Omega)}^{\frac{\lambda}{p-1}}}.$$

Hence, by Theorem 2.4 with $\eta = \frac{(p-1)(\lambda-1)}{\lambda}$, $\exp\left(\tau(1+\epsilon)\left(c\mathbf{W}_{1,p}^{2R}[\nu_i]\right)^\lambda\right) \in L^1(\Omega)$, which implies $\exp\left(\tau\left(c\mathbf{W}_{1,p}^{2diam\,(\Omega)}[\mu_{i,n}]\right)^\lambda\right) \in L^1(\Omega)$. We conclude by Theorem 4.1.

References

[1] D. R. Adams and L. I. Hedberg: *Function Spaces and Potential Theory*, Springer, New York, (1996).

[2] N. Aronszjan, P. Mulla, P. Szeptycki: *On spaces of potentials connected with L^q classes*, **Ann. Inst. Fourier Grenoble 13**, 211-306 (1963).

[3] P. Baras, M. Pierre: *Singularités éliminables pour des équations semi linéaires*, **Ann. Inst. Fourier Grenoble 34**, 185-206 (1984).

[4] Ph. Benilan, H. Brezis: Nonlinear problems related to the Thomas-Fermi equation, **unpublished paper**, see [8]

[5] M. F. Bidaut-Veron: *Necessary conditions of existence for an elliptic equation with source term and measure data involving p-Laplacian*, Proc. 2001 Luminy Conf. on Quasilinear Elliptic and Parabolic Equations and Systems, **Elect. J. Diff. Equ. Conf. 8**, 23-34 (2002).

[6] M. F. Bidaut-Veron: *Removable singularities and existence for a quasilinear equation with absorption or source term and measure data*, **Adv. Nonlinear Stud. 3**, 2563 (2003).

[7] L. Boccardo, T. Galouet, L. Orsina: *Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data*, **Ann. Inst. H. Poincaré, Anal. Non Linéaire 13**, 539-555 (1996).

[8] H. Brezis, Some variational problems of the Thomas-Fermi type, in Variational Inequalities, eds. R.W. Cottle, F. Giannessi and J. L. Lions, Wiley, Chichester (1980), 53-73.

[9] G. Dal Maso: *On the integral representation of certain local functionals*, **Ricerche Mat. 32**, 85-113 (1983).

[10] G. Dal Maso, F. Murat, L. Orsina, A. Prignet: *Renormalized solutions of elliptic equations with general measure data*, **Ann.Sc. Norm. Sup. Pisa 28**, 741-808 (1999)

[11] D. Feyel, A. de la Pradelle: *Topologies fines et compactifications associées à certains espaces de Dirichlet*, **Ann. Inst. Fourier Grenoble 27**, 121-146 (1977).

[12] L. Grafakos: *Classical Fourier Analysis* 2nd ed., Graduate Texts in Math. **249**, Springer-Verlag (2008).

[13] J. Heinonen, T. Kilpelainen, O. Martio *Nonlinear Potential Theory*, Oxford Univ. Press, Oxford (1993).

[14] P. Honzik and B. Jaye: *On the good-λ inequality for nonlinear potentials*, **Proc. Amer. Math. Soc. 140**, 4167-4180 (2012).

[15] B. Muckenhoupt, R. Wheeden: *Weighted norm inequalities for fractional integrals*, **Trans. Amer. Math. Soc. 192**, 261-274 (1974).

[16] R. O. Neil, *Convolution operators on $L^{p,q}$ spaces*, **Duke Math. J. 30**, 129-142 (1963).

[17] N. C. Phuc, I. E. Verbitsky: *Quasilinear and Hessian equations of Lane-Emden type*,**Ann. Math. 168**, 859-914 (2008).

[18] E. M. Stein: *Singular Integrals and Differentiability of Functions*, Princeton Univ. Press, Princeton N.J. (1971).

[19] B. O. Tureson, *Nonlinear Potential Theory and Sobolev Spaces*, Springer-Verlag (2000).

[20] L. Véron: *Elliptic equations involving measures*, Stationary partial differential equations. Vol. I, 593–712, **Handb. Differ. Equ.**, North-Holland, Amsterdam, (2004).

[21] L.Véron: *Singularities of solutions of second other Quasilinear Equations*, Pitman Research Notes in Math. Series 353, Adison Wesley, Longman 1996.

[22] W. Ziemer: *Weakly Differentiable Functions*, Springer-Verlag (1989).