Good-λ and Muckenhoupt-Wheeden type bounds in quasilinear measure datum problems, with applications

Quoc-Hung Nguyen* **and Nguyen Cong Phuc**†

Abstract

Weighted good-λ type inequalities and Muckenhoupt-Wheeden type bounds are obtained for gradients of solutions to a class of quasilinear elliptic equations with measure data. Such results are obtained globally over sufficiently flat domains in $\mathbb{R}^n$ in the sense of Reifenberg. The principal operator here is modeled after the p-Laplacian, where for the first time singular case $\frac{3n-2}{2n-1} < p \leq 2-\frac{1}{n}$ is considered. Those bounds lead to useful compactness criteria for solution sets of quasilinear elliptic equations with measure data. As an application, sharp existence results and sharp bounds on the size of removable singular sets are deduced for a quasilinear Riccati type equation having a gradient source term with linear or super-linear power growth.

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*E-mail address: quochung.nguyen@sns.it, Scuola Normale Superiore, Centro Ennio de Giorgi, Piazza dei Cavalieri 3, I-56100 Pisa, Italy.

†E-mail address: pcnguyen@math.lsu.edu, Department of Mathematics, Louisiana State University, 303 Lockett Hall, Baton Rouge, LA 70803, USA.

1 Introduction and main results

In this article, we are concerned with global weighted gradient estimates for quasilinear elliptic equations with measure data. Such estimates are then applied to address the question of sharp existence and removable singularities for a quasilinear equation with strong power growth in the gradient known as an equation of Riccati type.

In particular, our first goal is to obtain 'good-λ' type bounds and nonlinear Muckenhoupt-Wheeden type inequalities for gradients of solutions to quasilinear elliptic equations with measure data:

$$\begin{cases} -\mathrm{div}(A(x,\nabla u)) &= \mu \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here Ω is a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, and μ is a finite signed Radon in Ω.

Our second goal is to employ those estimates to study a quasilinear Riccati type equation with measure data:

$$\begin{cases} -\mathrm{div}(A(x,\nabla u)) &= |\nabla u|^q + \mu \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad\quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

and removable singularities for related 'homogeneous' equations $-\mathrm{div}(A(x,\nabla u)) = |\nabla u|^q$, $q > 0$. In particular, we address a question of sharp existence for (1.2) posed by Igor E. Verbitsky (personal communication), which has also been stated as an open problem in [11], pages 13–14.

In (1.1)-(1.2) and throughout the paper, the nonlinearity $A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory vector valued function, i.e., $A(x,\xi)$ is measurable in x and continuous with respect to ξ for a.e. x. Moreover, for a.e. x, $A(x,\xi)$ is continuously differentiable in ξ away from the origin and satisfies

$$|A(x,\xi)| \leq \Lambda|\xi|^{p-1}, \quad |\nabla_\xi A(x,\xi)| \leq \Lambda|\xi|^{p-2}, \tag{1.3}$$

$$\langle \nabla_\xi A(x,\xi)\eta, \eta\rangle \geq \Lambda^{-1}|\eta|^2|\xi|^{p-2}, \tag{1.4}$$

for every $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \backslash \{(0,0)\}$ and a.e. $x \in \mathbb{R}^n$, where Λ is a positive constant.

As for p in (1.3)-(1.4), in this paper, we shall restrict ourselves to the 'singular' case:

$$\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}.$$

However, as we remark later, all of the results obtained in this paper also hold in the 'regular' case $2 - \frac{1}{n} < p \leq n$. It also makes sense to consider the case $1 < p \leq \frac{3n-2}{2n-1}$. Unfortunately, our method breaks down in this case. However, some useful partial results could be obtained for this range of p, and they will be presented elsewhere.

For our purpose we also require that the nonlinearity A satisfy a smallness condition of BMO type in the x-variable. We call such a condition the (δ, R_0)-BMO condition.

Definition 1.1 *We say that $A(x,\zeta)$ satisfies a (δ, R_0)-BMO condition for some $\delta, R_0 > 0$ if*

$$[A]_{R_0} := \sup_{y\in\mathbb{R}^n, 0<r\leq R_0} \fint_{B_r(y)} \Theta(A, B_r(y))(x)dx \leq \delta,$$

where

$$\Theta(A, B_r(y))(x) := \sup_{\zeta\in\mathbb{R}^n\setminus\{0\}} \frac{|A(x,\zeta) - \overline{A}_{B_r(y)}(\zeta)|}{|\zeta|^{p-1}},$$

and $\overline{A}_{B_r(y)}(\zeta)$ denotes the average of $A(\cdot,\zeta)$ over the ball $B_r(y)$, i.e.,

$$\overline{A}_{B_r(y)}(\zeta) := \fint_{B_r(y)} A(x,\zeta)dx = \frac{1}{|B_r(y)|}\int_{B_r(y)} A(x,\zeta)dx.$$

A typical example of such a nonlinearity A is given by $A(x,\xi) = |\xi|^{p-2}\xi$ which gives rise to the standard p-Laplacian $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$.

In the case $p = 2$, the above (δ, R_0)-BMO condition was introduced in [13], whereas such a condition for general $p \in (1,\infty)$ first appeared in the paper [40] (see also [54]). We remark that the (δ, R_0)-BMO condition allows $A(x,\xi)$ has discontinuity in x and it can be used as an appropriate substitute for the Sarason [58] VMO condition.

Due to the global nature of our gradient estimates, we also require certain regularity on the ground domain Ω. Namely, at each boundary point and every scale, we ask that the boundary of Ω be trapped between two hyperplanes separated by a distance that depends on the scale. The following defines the relevant geometry precisely.

Definition 1.2 *Given $\delta \in (0,1)$ and $R_0 > 0$, we say that Ω is a (δ, R_0)-Reifenberg flat domain if for every $x \in \partial\Omega$ and every $r \in (0, R_0]$, there exists a system of coordinates $\{z_1, z_2, \ldots, z_n\}$, which may depend on r and x, so that in this coordinate system $x = 0$ and that*

$$B_r(0) \cap \{z_n > \delta r\} \subset B_r(0) \cap \Omega \subset B_r(0) \cap \{z_n > -\delta r\}.$$

For more properties of Reifenberg flat domains and their many applications, we refer to the papers [32, 36, 37, 38, 57, 59]. This class of domains appeared first in a paper of Reifenberg (see [57]) in the context of the Plateau problem. Here we remark that they include C^1 domains and Lipschitz domains with sufficiently small Lipschitz constants (see [59]). Moreover, they also include certain domains with fractal boundaries and thus provide a wide range of applications.

In this paper, all solutions of (1.1) and (1.2) with a finite signed measure μ in Ω will be understood in the renormalized sense (see [16]). For $\mu \in \mathfrak{M}_b(\Omega)$ (the set of finite signed measures in Ω), we will tacitly extend it by zero to $\Omega^c := \mathbb{R}^n \setminus \Omega$. We let μ^+ and μ^- be the positive and negative parts, respectively, of a measure $\mu \in \mathfrak{M}_b(\Omega)$. We denote by $\mathfrak{M}_0(\Omega)$ the space of finite signed measures in Ω which are absolutely continuous with respect to the capacity $c^{\Omega}_{1,p}$. Here $c^{\Omega}_{1,p}$ is the p-capacity defined for each compact set $K \subset \Omega$ by

$$c^{\Omega}_{1,p}(K) = \inf\left\{\int_\Omega |\nabla\varphi|^p dx : \varphi \geq \chi_K, \varphi \in C_c^\infty(\Omega)\right\},$$

where χ_K is the characteristic function of the set K. We also denote by $\mathfrak{M}_s(\Omega)$ the space of finite signed measures in Ω with support on a set of zero $c^{\Omega}_{1,p}$-capacity. It is known that any $\mu \in \mathfrak{M}_b(\Omega)$ can be written uniquely in the form $\mu = \mu_0 + \mu_s$ where $\mu_0 \in \mathfrak{M}_0(\Omega)$ and $\mu_s \in \mathfrak{M}_s(\Omega)$ (see [26]). It is also known that any $\mu_0 \in \mathfrak{M}_0(\Omega)$ can be written in the form $\mu_0 = f - \mathrm{div}(F)$ where $f \in L^1(\Omega)$ and $F \in L^{\frac{p}{p-1}}(\Omega, \mathbb{R}^n)$.

For $k > 0$, we define the usual two-sided truncation operator T_k by

$$T_k(s) = \max\{\min\{s, k\}, -k\}, \qquad s \in \mathbb{R}.$$

For our purpose, the following notion of gradient is needed. If u is a measurable function defined in Ω, finite a.e., such that $T_k(u) \in W^{1,p}_{loc}(\Omega)$ for any $k > 0$, then there exists a measurable function $v : \Omega \to \mathbb{R}^n$ such that $\nabla T_k(u) = v\chi_{\{|u|<k\}}$ a.e. in Ω for all $k > 0$ (see [5, Lemma 2.1]). In this case, we define the gradient ∇u of u by $\nabla u := v$. It is known that $v \in L^1_{loc}(\Omega, \mathbb{R}^n)$ if and only if $u \in W^{1,1}_{loc}(\Omega)$ and then v is the usual weak gradient of u. On the other hand, for $1 < p \leq 2 - \frac{1}{n}$, by looking at the fundamental solution we see that in general distributional solutions of (1.1) may not even belong to $u \in W^{1,1}_{loc}(\Omega)$.

The notion of renormalized solutions is a generalization of that of entropy solutions introduced in [5] and [8], where the right-hand side is assumed to be in $L^1(\Omega)$ or in $\mathfrak{M}_0(\Omega)$. Several equivalent definitions of renormalized solutions were given in [16]. Here we use the following one:

Definition 1.3 *Let $\mu = \mu_0 + \mu_s \in \mathfrak{M}_b(\Omega)$, with $\mu_0 \in \mathfrak{M}_0(\Omega)$ and $\mu_s \in \mathfrak{M}_s(\Omega)$. A measurable function u defined in Ω and finite a.e. is called a renormalized solution of* (1.1) *if $T_k(u) \in W^{1,p}_0(\Omega)$ for any $k > 0$, $|\nabla u|^{p-1} \in L^r(\Omega)$ for any $0 < r < \frac{n}{n-1}$, and u has the following additional property. For any $k > 0$ there exist nonnegative Radon measures $\lambda^+_k, \lambda^-_k \in \mathfrak{M}_0(\Omega)$ concentrated on the sets $\{u = k\}$ and $\{u = -k\}$, respectively, such that $\mu^+_k \to \mu^+_s$, $\mu^-_k \to \mu^-_s$ in the narrow topology of measures and that*

$$\int_{\{|u|<k\}} \langle A(x, \nabla u), \nabla\varphi\rangle dx = \int_{\{|u|<k\}} \varphi d\mu_0 + \int_{\Omega} \varphi d\lambda^+_k - \int_{\Omega} \varphi d\lambda^-_k,$$

for every $\varphi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$.

Here we recall that a sequence $\{\mu_k\} \subset \mathfrak{M}_b(\Omega)$ is said to converge in the narrow topology of measures to $\mu \in \mathfrak{M}_b(\Omega)$ if

$$\lim_{k\to\infty} \int_{\Omega} \varphi \, d\mu_k = \int_{\Omega} \varphi \, d\mu,$$

for every bounded and continuous function φ on Ω.

It is known that if $\mu \in \mathfrak{M}_0(\Omega)$ then there is one and only one renormalized solution of (1.1) (see [8, 16]). However, to the best of our knowledge, for a general $\mu \in \mathfrak{M}_b(\Omega)$ the uniqueness of renormalized solutions of (1.1) is still an open problem.

In the first main result of the paper, we are concerned with a nonlinear Muckenhoupt and Wheeden type bound for gradients of solutions of (1.1) that involves the class of $\mathbf{A}_\infty$ weights. We recall that a positive function $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is said to be an $\mathbf{A}_\infty$ weight if there are two positive constants C and ν such that

$$w(E) \leq C \left(\frac{|E|}{|B|}\right)^{\nu} w(B),$$

for all balls $B = B_\rho(x)$ and all measurable subsets E of B. The pair (C, ν) is called the $\mathbf{A}_\infty$ constants of w and is denoted by $[w]_{\mathbf{A}_\infty}$.

Theorem 1.4 *Let $\mu \in \mathfrak{M}_b(\Omega)$ and $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$. Let $\Phi : [0,\infty) \to [0,\infty)$ be a strictly increasing function such that $\Phi(0) = 0$, $\lim_{t\to\infty} \Phi(t) = \infty$, and Φ is of moderate growth, i.e., $\Phi(2t) \leq c\,\Phi(t)$ for all $t \geq 0$ with a constant $c > 1$. For any $w \in \mathbf{A}_\infty$, we can find $\delta = \delta(n, p, \Lambda, \Phi, [w]_{\mathbf{A}_\infty}) \in (0,1)$ such that if Ω is (δ, R_0)-Reifenberg flat and $[A]_{R_0} \leq \delta$ with some $R_0 > 0$ then for any renormalized solution u of* (1.1) *we have*

$$\int_\Omega \Phi(|\nabla u|)w(x)dx \leq C \int_\Omega \Phi([\mathbf{M}_1(\mu)]^{\frac{1}{p-1}})w(x)dx. \tag{1.5}$$

Here C depends only on $n, p, \Lambda, \Phi, [w]_{\mathbf{A}_\infty}$, and $diam(\Omega)/R_0$.

In (1.5) and in what follows the operator $\mathbf{M}_1$ is the first order fractional maximal function defined by

$$\mathbf{M}_1(\mu)(x) := \sup_{\rho>0} \frac{|\mu|(B_\rho(x))}{\rho^{n-1}} \quad \forall x \in \mathbb{R}^n.$$

We shall also use the Hardy-Littlewood maximal function $\mathbf{M}$ defined for each locally integrable function f in $\mathbb{R}^n$ by

$$\mathbf{M}(f)(x) = \sup_{\rho>0} \fint_{B_\rho(x)} |f(y)|dy \quad \forall x \in \mathbb{R}^n.$$

The proof of Theorem 1.4 above is in fact a consequence of the following good-λ type inequality involving both $\mathbf{M}_1$ and $\mathbf{M}$, which is interesting in its own right.

Theorem 1.5 *Let $w \in \mathbf{A}_\infty$, $\mu \in \mathfrak{M}_b(\Omega)$, and $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$. For any $\varepsilon > 0, R_0 > 0$ one can find constants $\delta_1 = \delta_1(n, p, \Lambda, \varepsilon, [w]_{\mathbf{A}_\infty}) \in (0,1)$, $\delta_2 = \delta_2(n, p, \Lambda, \varepsilon, [w]_{\mathbf{A}_\infty}, diam(\Omega)/R_0) \in (0,1)$ and $\Lambda_0 = \Lambda_0(n, p, \Lambda) > 1$ such that if Ω is a (δ_1, R_0)-Reifenberg flat domain and $[A]_{R_0} \leq \delta_1$ then for any renormalized solution u to* (1.1)*, we have*

$$\begin{aligned} &w(\{(\mathbf{M}(|\nabla u|^{\gamma_0}))^{1/\gamma_0} > \Lambda_0\lambda, (\mathbf{M}_1(\mu))^{\frac{1}{p-1}} \leq \delta_2\lambda\} \cap \Omega) \\ &\quad \leq C\varepsilon w(\{(\mathbf{M}(|\nabla u|^{\gamma_0}))^{1/\gamma_0} > \lambda\} \cap \Omega), \end{aligned}$$

for any $\lambda > 0$. Here γ_0 is a number in $\left(\frac{2-p}{2}, \frac{(p-1)n}{n-1}\right)$, and the constant C depends only on $n, p, \Lambda, diam(\Omega)/R_0$, and $[w]_{\mathbf{A}_\infty}$.

We now have some comments on the proof of Theorem 1.5. It is based on various tools developed for quasilinear equations with measure data and linear or nonlinear potential and Calderón-Zygmund theories (see, e.g., [5, 12, 16, 18, 19, 42, 45, 46, 49, 53, 54]). The key ingredients in this work which make it possible for us to apply those tools are some new local comparison estimates obtained in the singular case $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$; see Lemmas 2.2 and 2.5 below. Earlier those comparison estimates were known in the case $p > 2 - \frac{1}{n}$ (see [19, 45]), and thus in fact one can follow the method of this paper to prove Theorem 1.5 in the case $p > 2 - \frac{1}{n}$ (with $\gamma_0 = 1$). With this remark, Theorem 1.4 also holds for $p > 2 - \frac{1}{n}$ and so does its Corollaries 1.7-1.8 below. It is worth mentioning that the comparison estimates obtained in Lemma 2.2 can also be used to extend the recent gradient pointwise estimates by potentials obtained in [19] (see also [18, 39]) to the case $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$. This will be pursued in our forthcoming work. We should also mention that, at least for interior estimates and for the case $p \geq 2$, an unweighted version of Theorem 1.5 was obtained in [46].

We remark that the function Φ in Theorem 1.4 is quite general. In particular, we do not ask Φ to be convex or to satisfy the so-called ∇_2 condition: $\Phi(t) \geq \frac{1}{2a}\Phi(at)$ for some $a > 1$ and for all $t \geq 0$. As such one can take, e.g., $\Phi(t) = t^q$ for any $q > 0$, or even $\Phi(t) = [\log(1+t)]^\alpha$, $\alpha > 0$, etc. We emphasize that the introduction of Φ in Theorem 1.4 is not just for the sake of generality. In fact, such Φ will serve as an indispensable tool in our study of the Riccati type equation (1.2). In particular, Theorem 1.4 with such general Φ is needed to obtain a useful criterion for compactness of solution sets of equation (1.1); see Corollary 1.7 below.

In the case $\Phi(t) = t^q$, $q > 0$, estimates of the form (1.5) were obtained for (linear) fractional integral operators by Muckenhoupt-Wheeden in the pioneering work [47]. It is worth mentioning that for quasilinear problems the fractional maximal operator approach has been introduced in Mingione [46]. Also, for $\Phi(t) = t^q$, $q > 0$, and for $p > 2 - \frac{1}{n}$ estimate (1.5) was obtained in [54]. Thus Theorem 1.4 is new at least in the case $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$ considered in this paper. Moreover, using Theorem 1.5 one can also obtain a weighted Lorentz space estimate in the spirit of [54] but now for the singular case $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$. For a weight function w, the weighted Lorentz space $L^{q,s}_w(\Omega)$, $q \in (0,\infty)$, $s \in (0,\infty]$, is the space of measurable functions g on Ω such that

$$\|g\|_{L^{q,s}_w(E)} := \begin{cases} \left(q\int_0^\infty \left(\rho^q w\left(\{x \in \Omega : |g(x)| > \rho\}\right)\right)^{\frac{s}{q}} \frac{d\rho}{\rho}\right)^{1/s} < \infty & \text{if } s < \infty, \\ \sup_{\rho>0} \rho\left(w\left(\{x \in \Omega : |g(x)| > \rho\}\right)\right)^{1/q} < \infty & \text{if } s = \infty. \end{cases}$$

Here we write $w(E) = \int_E w(x)dx$ for a measurable set $E \subset \mathbb{R}^n$. Obviously, $\|g\|_{L^{q,q}_w(\Omega)} = \|g\|_{L^q_w(\Omega)}$, thus $L^{q,q}_w(\Omega) = L^q_w(\Omega)$. As usual, when $w \equiv 1$ we write $L^{q,s}(\Omega)$ instead of $L^{q,s}_w(\Omega)$.

Theorem 1.6 *Let $\mu \in \mathfrak{M}_b(\Omega)$ and $\frac{3n-2}{2n-1} < p \leq 2-\frac{1}{n}$. For any $w \in \mathbf{A}_\infty$, $0 < q < \infty$, $0 < s \leq \infty$ we can find $\delta = \delta(n,p,\Lambda,q,s,[w]_{\mathbf{A}_\infty}) \in (0,1)$ such that if Ω is (δ,R_0)-Reifenberg flat and $[A]_{R_0} \leq \delta$ for some $R_0 > 0$, then for any renormalized solution u of (1.1), we have*

$$\|\nabla u\|_{L^{q,s}_w(\Omega)} \leq C\|[\mathbf{M}_1(\mu)]^{\frac{1}{p-1}}\|_{L^{q,s}_w(\Omega)}.$$

Here the constant C depends only on $n,p,\Lambda,q,s,[w]_{\mathbf{A}_\infty}$ and $diam(\Omega)/R_0$.

Theorem 1.4 implies the following compactness criterion for solution sets of equation (1.1). This result will be needed in the proof of Theorem 1.9 below.

Corollary 1.7 *Suppose that* $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$. *For each* $j > 0$, *let* $\mu_j \in \mathfrak{M}_b(\Omega)$. *Let* u_j *be a solution of* (1.1) *with datum* $\mu = \mu_j$ *in* Ω. *Assume that* $\{[\mathbf{M}_1(\mu_j)]^{\frac{q}{p-1}}\}_j$, $q > 0$, *is a bounded and equi-integrable subset of* $L^1_w(\Omega)$ *for some* $w \in \mathbf{A}_\infty$. *Then, there exists* $\delta = \delta(n, p, \Lambda, q, [w]_{\mathbf{A}_\infty}) \in (0, 1)$ *such that if* Ω *is* (δ, R_0)*-Reifenberg flat and* $[A]_{R_0} \leq \delta$ *for some* $R_0 > 0$, *then there exist a subsequence* $\{u_{j'}\}_{j'}$ *and a finite a.e. function* u *with the property that* $T_k(u) \in W^{1,p}_0(\Omega)$ *for all* $k > 0$, $u_{j'} \to u$ *a.e., and*

$$\nabla u_{j'} \to \nabla u \quad \textit{strongly in} \quad L^q_w(\Omega, \mathbb{R}^n). \tag{1.6}$$

One can also combine Theorem 1.4 (or Theorem 1.6) with a classical result of Muckenhoupt and Wheeden [47, Theorem 3] to obtain the following gradient regularity result. This result was shown to be sharp for fractional integrals (Riesz's potentials) of order 1 (see [47, Theorem 4]).

Corollary 1.8 *Suppose that* $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$. *For each* $f \in L^1(\Omega)$, *we denote by* $u(f)$ *the (unique) renormalized solution of* (1.1) *with datum* $\mu = f$ *in* Ω. *Assume that* $1 < s < n$, $q = \frac{ns}{n-s}$, *and* $V(x)$ *is a nonnegative function in* $\mathbb{R}^n$ *such that*

$$K := \sup_{Balls\, B \subset \mathbb{R}^n} \left(\fint_B [V(x)]^q dx \right)^{\frac{1}{q}} \left(\fint_B [V(x)]^{\frac{-s}{s-1}} \right)^{\frac{s-1}{s}} < +\infty.$$

Then there exists $\delta = \delta(n, p, s, \Lambda, K) \in (0, 1)$ *such that if* Ω *is* (δ, R_0)*-Reifenberg flat and* $[A]_{R_0} \leq \delta$ *with some* $R_0 > 0$ *then we have*

$$\int_\Omega |\nabla(u(f))|^{(p-1)q}\, V(x)^q dx \leq C \int_\Omega |f(x)|^s\, V(x)^s dx,$$

where the constant C *depends only on* n, p, s, Λ, K *and* $diam(\Omega)/R_0$.

We next describe our results in regard to equation (1.2). For this, we shall need the notion of capacity associated to the Sobolev space $W^{1,s}(\mathbb{R}^n)$, $1 < s < +\infty$. For a compact set $K \subset \mathbb{R}^n$, we define

$$\mathrm{Cap}_{1,s}(K) = \inf \Big\{ \int_{\mathbb{R}^n} (|\nabla\varphi|^s + \varphi^s) dx : \varphi \in C_0^\infty(\mathbb{R}^n), \varphi \geq \chi_K \Big\}.$$

Note that $\mathrm{Cap}_{1,s}$ can be extended to all sets $E \subset \mathbb{R}^n$ by letting

$$\mathrm{Cap}_{1,s}(E) = \inf_{\substack{O \supset E \\ O\ open}} \Big\{ \sup_{\substack{K \subset O \\ K\ compact}} \mathrm{Cap}_{1,s}(K) \Big\}.$$

Moreover, by the capacitability of Borel sets (see, e.g., [1, Theorem 2.3.11]) we have

$$\mathrm{Cap}_{1,s}(E) = \sup_{\substack{K \subset E \\ K\ compact}} \mathrm{Cap}_{1,s}(K)$$

for any Borel set $E \subset \mathbb{R}^n$.

Theorem 1.9 *Let* $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$ *and* $q \geq 1$. *There exists a constant* $\delta = \delta(n, p, \Lambda, q) \in (0, 1)$ *such that the following holds. Suppose that* $[A]_{R_0} \leq \delta$ *and* Ω *is* (δ, R_0)*-Reifenberg flat for some* $R_0 > 0$. *Then there exists a constant* $c_0 = c_0(n, p, \Lambda, q, \mathrm{diam}(\Omega), R_0) > 0$ *such that if* μ *is a finite signed measure in* Ω *with*

$$|\mu|(K) \leq c_0 \, \mathrm{Cap}_{1, \frac{q}{q-p+1}}(K) \tag{1.7}$$

for all compact sets $K \subset \Omega$, *then there exists a renormalized solution* $u \in W_0^{1,q}(\Omega)$ *to the Riccati type equation* (1.2) *such that*

$$\int_K |\nabla u|^q \leq C \, \mathrm{Cap}_{1, \frac{q}{q-p+1}}(K)$$

for all compact sets $K \subset \Omega$. *Here the constant* C *depends only on* $n, p, \Lambda, q, \mathrm{diam}(\Omega)$, *and* R_0.

It is worth mentioning that the capacitary condition (1.7) is sharp. Namely, if (1.2) has a solution with ω being nonnegative and compactly supported in Ω then (1.7) holds with a different constant c_0 (see [33, 51]). Moreover, it is also practically useful. In particular, it implies that the Marcinkiewicz space condition $\mu \in L^{\frac{n(q-p+1)}{q}, \infty}(\Omega)$, $q > \frac{n(p-1)}{n-1}$, (with a small norm) is sufficient for the solvability of (1.2). Other sufficient conditions of Fefferman-Phong type involving Morrey spaces can also be deduced from (1.7) (see Corollaries 3.5 and 3.6 in [51]). See also Theorem 1.10 below in which (1.7) is used in the study of removable singularities for the homogeneous Riccati type equation $-\mathrm{div}(A(x, \nabla u)) = |\nabla u|^q$.

Theorem 1.9 extends similar existence results obtained earlier for $2 - \frac{1}{n} < p \leq n$ in [54, 55]. See also [51, 52] or [2, 3] where the case $q > p$ or $q = p$ is studied, respectively. In particular, Theorem 1.9 solves an open problem in [11, page 13] at least for compactly supported measures and for $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$, $q \geq 1$. It is natural to expect that Theorem 1.9 should also hold for $p-1 < q < 1$ but we are not able to prove it here due to the lack of convexity. It is also worth mentioning that the 'linear' case $p = 2$ was first considered in the pioneering work [33]. There is a vast literature on equations of the form (1.2) (but mostly for $0 < q \leq p$). We refer to [4, 6, 9, 10, 11, 15, 17, 21, 24, 28, 29, 43, 48] and to [7, 22, 23, 34, 35, 25, 30, 31, 56] for various contributions.

Finally, as mentioned above Theorem 1.9 can be used to give sharp bound on the size of removable singular sets for homogeneous Riccati type equations. We recall that a Borel set $E \subset \Omega$ is a said to be a removable singular set for the equation $-\mathrm{div}(A(x, \nabla u)) = |\nabla u|^q$ in Ω if any solution u to

$$\begin{cases} u \in W^{1,q}_{loc}(\Omega \setminus E), \text{ and} \\ -\mathrm{div}(A(x, \nabla u)) = |\nabla u|^q \text{ in } \mathcal{D}'(\Omega \setminus E) \end{cases}$$

can be extended to be a solution to

$$\begin{cases} u \in W^{1,q}_{loc}(\Omega), \text{ and} \\ -\mathrm{div}(A(x, \nabla u)) = |\nabla u|^q \text{ in } \mathcal{D}'(\Omega). \end{cases}$$

Theorem 1.10 *Let $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$ and $q \geq 1$. There exists a constant $\delta = \delta(n, p, \Lambda, q) \in (0, 1)$ such that the following holds. Suppose that $[A]_{R_0} \leq \delta$ and Ω is (δ, R_0)-Reifenberg flat for some $R_0 > 0$. If a Borel set $E \subset \Omega$ is a removable set for the equation $-\mathrm{div}(A(x, \nabla u)) = |\nabla u|^q$ in Ω, then it must hold that*

$$\mathrm{Cap}_{1, \frac{q}{q-p+1}}(E) = 0.$$

The proof of Theorem 1.10 is based on Theorem 1.9 and is similar to that of [51, Theorem 3.9].

Remark 1.11 *By [51, Theorem 3.8], Theorem 1.10 is sharp at least in the natural class of A-superharmonic functions in Ω. Namely, if K is a compact set in Ω with $\mathrm{Cap}_{1, \frac{q}{q-p+1}}(K) = 0$ then any solution u to*

$$\begin{cases} -\mathrm{div}(A(x, \nabla u)) \geq 0 \ \textit{in } \mathcal{D}'(\Omega), \\ u \in W^{1,q}_{loc}(\Omega \setminus K), \ \textit{and} \\ -\mathrm{div}(A(x, \nabla u)) = |\nabla u|^q \ \textit{in } \mathcal{D}'(\Omega \setminus K), \end{cases}$$

is also a solution to

$$\begin{cases} u \in W^{1,q}_{loc}(\Omega), \ \textit{and} \\ -\mathrm{div}(A(x, \nabla u)) = |\nabla u|^q \ \textit{in } \mathcal{D}'(\Omega). \end{cases}$$

The paper is organized as follows. In Section 2 we obtain some important comparison estimates that are needed for the proof of Theorem 1.5. The proof of good-λ type bounds, Theorem 1.5, is given in Section 3. Then in Section 4, we prove Theorem 1.4 and Corollary 1.7. Finally, we obtain existence results for the Riccati type equation (1.2), Theorem 1.9, in Section 5.

2 Local interior and boundary estimates

In this section, we obtain certain local interior and boundary comparison estimates that are essential to our development later. First let us consider the interior ones. With $u \in W^{1,p}_{loc}(\Omega)$ and for each ball $B_{2R} = B_{2R}(x_0) \subset\subset \Omega$, we consider the unique solution $w \in W^{1,p}_0(B_{2R}) + u$ to the equation

$$\begin{cases} -\mathrm{div}\,(A(x, \nabla w)) & = & 0 & \textit{in} & B_{2R}, \\ w & = & u & \text{on} & \partial B_{2R}. \end{cases} \tag{2.1}$$

We first recall the following version of interior Gehring's lemma that was proved in [20, Theorem 6.7].

Lemma 2.1 *Let w be as in (2.1). There exist constants $\theta_1 > p$ and $C > 0$ depending only on n, p, Λ such that the following estimate*

$$\left(\fint_{B_{\rho/2}(y)} |\nabla w|^{\theta_1} dxdt \right)^{\frac{1}{\theta_1}} \leq C \left(\fint_{B_\rho(y)} |\nabla w|^{p-1} dx \right)^{\frac{1}{p-1}} \tag{2.2}$$

holds for all $B_\rho(y) \subset B_{2R}$.

The next lemma gives an estimate for the difference $\nabla u - \nabla w$. This is one of the key estimates of this paper. We remark that earlier this kind of comparison estimates is known only in the case $p > 2 - \frac{1}{n}$ (see [45, 19]). Here we are able to obtain it for $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$.

Lemma 2.2 *Let w be in* (2.1). *Assume that $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$. Then it holds that*

$$\left(\fint_{B_{2R}} |\nabla u - \nabla w|^{\gamma_0} dx\right)^{\frac{1}{\gamma_0}} \leq C\left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right]^{\frac{1}{p-1}} + \\ + C\frac{|\mu|(B_{2R})}{R^{n-1}}\left(\fint_{B_{2R}} |\nabla u|^{\gamma_0} dx\right)^{\frac{2-p}{\gamma_0}}, \tag{2.3}$$

for some $\frac{2-p}{2} \leq \gamma_0 < \frac{(p-1)n}{n-1} \leq 1$. In particular, for any $\varepsilon > 0$ one can find $C_\varepsilon > 0$ such that

$$\left(\fint_{B_{2R}} |\nabla u - \nabla w|^{\gamma_0} dx\right)^{\frac{1}{\gamma_0}} \leq C_\varepsilon\left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right]^{\frac{1}{p-1}} + \varepsilon\left(\fint_{B_{2R}} |\nabla u|^{\gamma_0} dx\right)^{\frac{1}{\gamma_0}}. \tag{2.4}$$

Proof. For any $\varphi \in W_0^{1,p}(B_{2R})$, we have

$$\int_{B_{2R}} \langle A(x,\nabla u) - A(x,\nabla w), \nabla\varphi\rangle dx = \int_{B_{2R}} \varphi d\mu. \tag{2.5}$$

We now set

$$T_{h,m}(s) = \begin{cases} T_m(s) & \text{if } |s| \geq 2h, \\ 2\,\mathrm{sgn}(s)(|s| - h) & \text{if } h < |s| < 2h, \\ 0 & \text{if } |s| \leq h, \end{cases}$$

for $m > 2h > 0$. It is easy to see that we can take $\varphi = T_{h,k^{1-\alpha}}(|u-w|^{-\alpha}(u-w))$ with $\alpha \in (-\infty, 1)$ and $0 < h < \frac{k^{1-\alpha}}{2}$ as a test function in (2.5). This gives

$$\int_{B_{2R}\cap\{x:(2h)^{\frac{1}{1-\alpha}} < |u-w| < k\}} |u-w|^{-\alpha} g(u,w) dx \leq Ck^{1-\alpha}|\mu|(B_{2R}),$$

where

$$g(u,w) = \frac{|\nabla(u-w)|^2}{(|\nabla w| + |\nabla u|)^{2-p}}. \tag{2.6}$$

Thus sending $h \to 0$ we get

$$\int_{B_{2R}\cap\{x:|u-w|<k\}} |u-w|^{-\alpha} g(u,w) dx \leq Ck^{1-\alpha}|\mu|(B_{2R}).$$

We now estimate $|u-w|^{-\alpha} g(u,w)$ in $L^\gamma(B_{2R})$ for some appropriate γ. To do so we employ the method of [5] (see also [53]). For $k, \lambda \geq 0$, we let

$$\Phi(k,\lambda) = |\{x : |u-w| > k, |u-w|^{-\alpha} g(u,w) > \lambda\} \cap B_{2R}|.$$

As $\lambda \mapsto \Phi(k,\lambda)$ is non-increasing, we find

$$\begin{aligned}
\Phi(0,\lambda) &\leq \frac{1}{\lambda}\int_0^\lambda \Phi(0,s)ds \leq \Phi(k,0) + \frac{1}{\lambda}\int_0^\lambda \Phi(0,s) - \Phi(k,s)ds \\
&= |\{x : |u-w| > k\} \cap B_{2R}| + \\
&\quad + \frac{1}{\lambda}\int_0^\lambda |\{x : |u-w| \leq k, |u-w|^{-\alpha} g(u,w) > s\} \cap B_{2R}| ds \\
&\leq k^{-\beta}\|u-w\|^\beta_{L^\beta(B_{2R})} + \frac{1}{\lambda}\int_{B_{2R}\cap\{x:|u-w|\leq k\}} |u-w|^{-\alpha} g(u,w)dx \\
&\leq k^{-\beta}\|u-w\|^\beta_{L^\beta(B_{2R})} + \frac{Ck^{1-\alpha}}{\lambda}|\mu|(B_{2R}),
\end{aligned}$$

for any $\beta > 0$. Then choosing

$$k = \left[\frac{\lambda\|u-w\|^\beta_{L^\beta(B_{2R})}}{|\mu|(B_{2R})}\right]^{\frac{1}{1-\alpha+\beta}},$$

we obtain

$$\lambda^{\frac{\beta}{1-\alpha+\beta}} |\{x : |u-w|^{-\alpha} g(u,w) > \lambda\} \cap B_{2R}| \leq C|\mu|(B_{2R})^{\frac{\beta}{1-\alpha+\beta}} \|u-w\|^{\frac{\beta(1-\alpha)}{1-\alpha+\beta}}_{L^\beta(B_{2R})}$$

for all $\lambda > 0$. Thus by Holder's inequality, for $0 < \gamma < \frac{\beta}{1-\alpha+\beta}$, we get

$$\begin{aligned}
&\int_{B_{2R}} |u-w|^{-\alpha\gamma} g(u,w)^\gamma dx \\
&\leq C|B_{2R}|^{1-\frac{\gamma(1-\alpha+\beta)}{\beta}} \|[|u-w|^{-\alpha} g(u,w)]^\gamma\|_{L^{\frac{\beta}{\gamma(1-\alpha+\beta)},\infty}(B_{2R})} \\
&\leq CR^{n-\frac{n\gamma(1-\alpha+\beta)}{\beta}} |\mu|(B_{2R})^\gamma \|u-w\|^{\gamma(1-\alpha)}_{L^\beta(B_{2R})}.
\end{aligned} \tag{2.7}$$

We next define a quantity

$$M := \int_{B_{2R}} |\nabla(u-w)|\, |u-w|^{-\frac{\alpha}{p}}.$$

Applying Sobolev's inequality for the function $|u-w|^{\frac{p-\alpha}{p}}$, we have

$$\int_{B_{2R}} |u-w|^{\frac{(p-\alpha)n}{p(n-1)}} \leq C\left(\int_{B_{2R}} |\nabla|u-w|^{1-\frac{\alpha}{p}}|dx\right)^{\frac{n}{n-1}} = CM^{\frac{n}{n-1}}. \tag{2.8}$$

Then using Holder's inequality and (2.8), we get

$$\begin{aligned}
&\int_{B_{2R}} |\nabla(u-w)|^{\frac{(p-\alpha)n}{pn-\alpha}} \\
&\leq \left(\int_{B_{2R}} |u-w|^{\frac{(p-\alpha)n}{p(n-1)}}\right)^{\frac{\alpha(n-1)}{pn-\alpha}} \left(\int_{B_{2R}} |\nabla(u-w)|\, |u-w|^{-\frac{\alpha}{p}} dx\right)^{\frac{n(p-\alpha)}{pn-\alpha}} \\
&\leq CM^{\frac{pn}{pn-\alpha}}.
\end{aligned} \tag{2.9}$$

Our next goal is to bound M. To this end, using $1 < p < 2$, we have

$$|\nabla(u-w)| \le C\left(g(u,w)^{1/p} + g(u,w)^{1/2}|\nabla u|^{\frac{2-p}{2}}\right),$$

and thus

$$M \le C\int_{B_{2R}} \left[|u-w|^{-\frac{\alpha}{p}} g(u,w)^{1/p} + |u-w|^{-\frac{\alpha}{p}} g(u,w)^{\frac{1}{2}}|\nabla u|^{\frac{2-p}{2}}\right]. \tag{2.10}$$

We now assume that

$$1/p < \frac{\beta}{1-\alpha+\beta}, \quad \beta = \frac{(p-\alpha)n}{p(n-1)}. \tag{2.11}$$

Thus, we can apply (2.7) to $\gamma = 1/p$, to get

$$\begin{aligned}\int_{B_{2R}} |u-w|^{-\frac{\alpha}{p}} g(u,w)^{1/p} &\le CR^{n-\frac{n(1-\alpha+\beta)}{p\beta}} |\mu|(B_{2R})^{1/p} \|u-w\|_{L^\beta(B_{2R})}^{(1-\alpha)/p} \\ &\le CR^{n-\frac{n(1-\alpha+\beta)}{p\beta}} |\mu|(B_{2R})^{1/p} M^{\frac{1-\alpha}{p-\alpha}}, \end{aligned} \tag{2.12}$$

where we used (2.8) in the last inequality.

Assume also that

$$\gamma_0 := \frac{(p-\alpha)n}{pn-\alpha} > \frac{2-p}{2}. \tag{2.13}$$

Then by Holder's inequality with exponents $\frac{2\gamma_0}{2\gamma_0+p-2}$ and $\frac{2\gamma_0}{2-p}$,

$$\begin{aligned}&\int_{B_{2R}} |u-w|^{-\frac{\alpha}{p}} g(u,w)^{\frac{1}{2}}|\nabla u|^{\frac{2-p}{2}} \\ &\le \left(\int_{B_{2R}} |u-w|^{-\frac{2\alpha\gamma_0}{p(2\gamma_0+p-2)}} g(u,w)^{\frac{\gamma_0}{2\gamma_0+p-2}}\right)^{\frac{2\gamma_0+p-2}{2\gamma_0}} \left(\int_{B_{2R}} |\nabla u|^{\gamma_0}\right)^{\frac{2-p}{2\gamma_0}}. \end{aligned} \tag{2.14}$$

We further restrict that

$$\alpha < \frac{p}{2}, \quad \frac{\gamma_0}{2\gamma_0+p-2} < \frac{\beta}{1-\frac{2\alpha}{p}+\beta}, \tag{2.15}$$

which then by (2.7) gives

$$\begin{aligned}&\int_{B_{2R}} |u-w|^{-\frac{2\alpha\gamma_0}{p(2\gamma_0+p-2)}} g(u,w)^{\frac{\gamma_0}{2\gamma_0+p-2}} \\ &\le CR^{n-\frac{n\gamma_0(1-\frac{2\alpha}{p}+\beta)}{\beta(2\gamma_0+p-2)}} |\mu|(B_{2R})^{\frac{\gamma_0}{2\gamma_0+p-2}} \|u-w\|_{L^\beta(B_{2R})}^{\frac{\gamma_0(1-\frac{2\alpha}{p})}{2\gamma_0+p-2}} \\ &\le CR^{n-\frac{n\gamma_0(1-\frac{2\alpha}{p}+\beta)}{\beta(2\gamma_0+p-2)}} |\mu|(B_{2R})^{\frac{\gamma_0}{2\gamma_0+p-2}} M^{\frac{\gamma_0(p-2\alpha)}{(p-\alpha)(2\gamma_0+p-2)}}. \end{aligned} \tag{2.16}$$

Hence, combining (2.10), (2.12), (2.14), and (2.16) we have

$$M \le C R^{n-\frac{n(1-\alpha+\beta)}{p\beta}} |\mu|(B_{2R})^{1/p} M^{\frac{1-\alpha}{p-\alpha}} + \\ + \left(R^{n-\frac{n\gamma_0(1-\frac{2\alpha}{p}+\beta)}{\beta(2\gamma_0+p-2)}} |\mu|(B_{2R})^{\frac{\gamma_0}{2\gamma_0+p-2}} M^{\frac{\gamma_0(p-2\alpha)}{(p-\alpha)(2\gamma_0+p-2)}} \right)^{\frac{2\gamma_0+p-2}{2\gamma_0}} \times \\ \times \left(\int_{B_{2R}} |\nabla u|^{\gamma_0} \right)^{\frac{2-p}{2\gamma_0}} \tag{2.17}$$

provided that (2.11), (2.13), and (2.15) are satisfied.

Let $\alpha_0 = \frac{\alpha}{p} < 1/2$ so that $\beta = \frac{(1-\alpha_0)n}{n-1}$. We have

$$1/p < \frac{\beta}{1-\alpha+\beta} \iff p > \frac{n(2-\alpha_0)-1}{n-\alpha_0},$$

$$\frac{\gamma_0}{2\gamma_0+p-2} < \frac{\beta}{1-\frac{2\alpha}{p}+\beta} \iff p > \frac{n(2-\alpha_0)-1}{n-\alpha_0},$$

and

$$\gamma_0 = \frac{(p-\alpha)n}{pn-\alpha} > \frac{2-p}{2} \iff p > \frac{2\alpha_0(n-1)}{n-\alpha_0}.$$

Therefore, if

$$p > \frac{3n-2}{2n-1},$$

then (2.11), (2.13), and (2.15) hold for any

$$1/2 > \alpha_0 > \frac{-1+(2-p)n}{n-p}.$$

With this, using Holder's inequality, we get from (2.17) that

$$M \le C \left[R^{n-\frac{n(1-\alpha+\beta)}{p\beta}} |\mu|(B_{2R})^{1/p} \right]^{\frac{p-\alpha}{p-1}} + \\ + \left(R^{n-\frac{n\gamma_0(1-\frac{2\alpha}{p}+\beta)}{\beta(2\gamma_0+p-2)}} |\mu|(B_{2R})^{\frac{\gamma_0}{2\gamma_0+p-2}} \right)^{\frac{(2\gamma_0+p-2)(p-\alpha)}{p\gamma_0}} \times \\ \times \left(\int_{B_{2R}} |\nabla u|^{\gamma_0} \right)^{\frac{(p-\alpha)(2-p)}{p\gamma_0}}. \tag{2.18}$$

Thus it follows from (2.9) and (2.18) that

$$\left(\int_{B_{2R}} |\nabla(u-w)|^{\gamma_0} \right)^{\frac{p-\alpha}{p\gamma_0}} \le C \left[R^{n-\frac{n(1-\alpha+\beta)}{p\beta}} |\mu|(B_{2R})^{1/p} \right]^{\frac{p-\alpha}{p-1}} + \\ + \left(R^{n-\frac{n\gamma_0(1-\frac{2\alpha}{p}+\beta)}{\beta(2\gamma_0+p-2)}} |\mu|(B_{2R})^{\frac{\gamma_0}{2\gamma_0+p-2}} \right)^{\frac{(2\gamma_0+p-2)(p-\alpha)}{p\gamma_0}} \left(\int_{B_{2R}} |\nabla u|^{\gamma_0} \right)^{\frac{(p-\alpha)(2-p)}{p\gamma_0}}.$$

That is, we obtain (2.3) with $\frac{2-p}{2} < \gamma_0 < \frac{(p-1)n}{n-1} \leq 1$ as desired.

Finally, using Young's inequality, we get the bound (2.4) which completes the proof of the lemma. ■

The following proposition provides a useful estimate for the difference $\nabla u - \nabla v$ for a well-controlled locally Lipschitz function v.

Proposition 2.3 *Let $\mu \in \mathfrak{M}_b(\Omega)$ and $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$. Let γ_0 be as in Lemma 2.2. There exists $v \in W^{1,p}(B_R) \cap W^{1,\infty}(B_{R/2})$ such that for any $\varepsilon > 0$,*

$$\|\nabla v\|_{L^\infty(B_{R/2})} \leq C\left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right]^{\frac{1}{p-1}} + C\left(\fint_{B_{2R}} |\nabla u|^{\gamma_0}\right)^{1/\gamma_0},$$

and

$$\left(\fint_{B_R} |\nabla u - \nabla v|^{\gamma_0} dx\right)^{\frac{1}{\gamma_0}} \leq C_\varepsilon \left[\frac{|\mu|(B_{2R})}{R^{n-1}}\right]^{\frac{1}{p-1}} + \\ C(([A]_{R_0})^\kappa + \varepsilon)\left(\fint_{B_{2R}} |\nabla u|^{\gamma_0}\right)^{1/\gamma_0}.$$

for some $C_\varepsilon = C(n, p, \Lambda, \varepsilon) > 0$. Here κ is a constant in $(0, 1)$.

Proof. By [54, Lemma 2.3 and Corollary 2.4], there exists $v \in W^{1,p}(B_R) \cap W^{1,\infty}(B_{R/2})$ such that

$$\|\nabla v\|_{L^\infty(B_{R/2})} \leq C\left(\fint_{B_R} |\nabla w|^p\right)^{1/p},$$

and

$$\fint_{B_R} |\nabla w - \nabla v| dx \leq C([A]_{R_0})^\kappa \left(\fint_{B_R} |\nabla w|^p\right)^{1/p},$$

for some $\kappa \in (0, 1)$. Combining these with (2.2) in Lemma 2.1, (2.4) in Lemma 2.2, we get the desired results. ■

Next, we focus on the corresponding estimates near the boundary. We recall that Ω is (δ_0, R_0)-Reifenberg flat with $\delta_0 < 1/2$. Fix $x_0 \in \Omega$ and $0 < R < R_0/10$. With $u \in W_0^{1,p}(\Omega)$ being a solution to (1.1), we now consider the unique solution $w \in W_0^{1,p}(\Omega_{10R}(x_0)) + u$ to the following equation

$$\begin{cases} -\operatorname{div}(A(x, \nabla w)) = 0 & \text{in} \quad \Omega_{10R}(x_0), \\ w = u & \text{on} \quad \partial\Omega_{10R}(x_0). \end{cases} \tag{2.19}$$

Hereafter, the notation $\Omega_r(x)$ indicates the set $\Omega \cap B_r(x)$. By [53, Lemma 2.5], we have the following boundary counterpart of Lemma 2.1.

Lemma 2.4 *Let w be as in* (2.19). *There exist constants $\theta_1 > p$ and $C > 0$ depending only on n, p, δ_0, Λ such that the following estimate*

$$\left(\fint_{B_{\rho/2}(y)} |\nabla w|^{\theta_1} dxdt\right)^{\frac{1}{\theta_1}} \leq C \left(\fint_{B_{3\rho}(y)} |\nabla w|^{p-1} dx\right)^{\frac{1}{p-1}}, \tag{2.20}$$

holds for all $B_{3\rho}(y) \subset B_{10R}(x_0)$.

We also have the following analogues of Lemmas 2.2.

Lemma 2.5 *Assume that $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$. Let w be as in* (2.19) *and γ_0 be as in Lemma 2.2. Then we have*

$$\left(\fint_{B_{10R}(x_0)} |\nabla(u-w)|^{\gamma_0} dx\right)^{\frac{1}{\gamma_0}} \leq C \left[\frac{|\mu|(B_{10R}(x_0))}{R^{n-1}}\right]^{\frac{1}{p-1}} + \\ + C\frac{|\mu|(B_{10R}(x_0))}{R^{n-1}} \left(\fint_{B_{10R}(x_0)} |\nabla u|^{\gamma_0} dx\right)^{\frac{2-p}{\gamma_0}}.$$

In particular, for any $\varepsilon > 0$,

$$\left(\fint_{B_{10R}(x_0)} |\nabla(u-w)|^{\gamma_0} dx\right)^{\frac{1}{\gamma_0}} \leq C_\varepsilon \left[\frac{|\mu|(B_{10R}(x_0))}{R^{n-1}}\right]^{\frac{1}{p-1}} + \\ \varepsilon \left(\fint_{B_{10R}(x_0)} |\nabla u|^{\gamma_0} dx\right)^{\frac{1}{\gamma_0}}. \tag{2.21}$$

Using Lemma 2.5 we derive the following boundary version of Proposition 2.3.

Proposition 2.6 *Let $\mu \in \mathfrak{M}_b(\Omega)$ and $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$. Let γ_0 be as in Lemma 2.2. For any $\varepsilon > 0$, there exists $\delta_0 = \delta_0(n, p, \Lambda, \varepsilon) \in (0, 1)$ such that the following holds. If Ω is (δ_0, R_0)-Reifenberg flat and $u \in W_0^{1,p}(\Omega)$, $x_0 \in \partial\Omega$, and $0 < R < R_0/10$, then there exists a function $V \in W^{1,\infty}(B_{R/10}(x_0))$ such that*

$$\|\nabla V\|_{L^\infty(B_{R/10}(x_0))} \leq C \left[\frac{|\mu|(B_{10R}(x_0))}{R^{n-1}}\right]^{\frac{1}{p-1}} + C \left(\fint_{B_{10R}} |\nabla u|^{\gamma_0}\right)^{1/\gamma_0},$$

and

$$\left(\fint_{B_{R/10}(x_0)} |\nabla(u-V)|^{\gamma_0} dx\right)^{\frac{1}{\gamma_0}} \\ \leq C_\varepsilon \left[\frac{|\mu|(B_{10R}(x_0))}{R^{n-1}}\right]^{\frac{1}{p-1}} + C(([A]_{R_0})^\kappa + \varepsilon) \left(\fint_{B_{10R}(x_0)} |\nabla u|^{\gamma_0}\right)^{1/\gamma_0}.$$

for some $C_\varepsilon = C(n, p, \Lambda, \varepsilon) > 0$. Here κ is a constant in $(0, 1)$.

Proof. By [54, Corollary 2.13], for any $\varepsilon > 0$, there exists $\delta_0 = \delta_0(n, p, \Lambda, \varepsilon) \in (0,1)$ such that if Ω is a (δ_0, R_0)-Reifenberg flat domain then we can find $V \in W^{1,\infty}(B_{R/10}(x_0))$ satisfying

$$\|\nabla V\|_{L^\infty(B_{R/10}(x_0))} \leq C \left(\fint_{B_R(x_0)} |\nabla w|^p \right)^{1/p},$$

and

$$\fint_{B_{R/10}(x_0)} |\nabla w - \nabla V| dx \leq C(([A]_{R_0})^\kappa + \varepsilon) \left(\fint_{B_R(x_0)} |\nabla w|^p \right)^{1/p},$$

for some $\kappa \in (0,1)$. Combining these with (2.20) in Lemma 2.4, (2.21) in Lemma 2.5, we arrive at the conclusion. ■

3 Good-λ type bounds on Reifenberg flat domains

The purpose of this section is to prove Theorem 1.5. Our main tools here are Propositions 2.3 and 2.6 and Lemma 3.1 below. This lemma can be viewed as a substitution for the Calderón-Zygmund-Krylov-Safonov decomposition. The weighted version that is used here was obtained in [42]. See also [60, 12, 14] for the case the weight $w \equiv 1$.

Lemma 3.1 *Let Ω be a (δ, R_0)-Reifenberg flat domain with $\delta < 1/4$ and let w be an $\mathbf{A}_\infty$ weight. Suppose that the sequence of balls $\{B_r(y_i)\}_{i=1}^L$ with centers $y_i \in \overline{\Omega}$ and radius $r \leq R_0/4$ covers Ω. Let $E \subset F \subset \Omega$ be measurable sets for which there exists $0 < \varepsilon < 1$ such that*

1. *$w(E) < \varepsilon w(B_r(y_i))$ for all $i = 1, ..., L$, and*

2. *for all $x \in \Omega$, $\rho \in (0, 2r]$, we have $w(E \cap B_\rho(x)) \geq \varepsilon w(B_\rho(x)) \Longrightarrow B_\rho(x) \cap \Omega \subset F$.*

Then $w(E) \leq C\varepsilon w(F)$ for a constant C depending only on n and $[w]_{\mathbf{A}_\infty}$.

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. We shall use some of the ideas in the proofs of [54, Theorem 1.4] and [49, Theorem 8.4] (see also [50, Theorem 3.1]).

Let γ_0 be as in Lemma 2.2 and let u be a renormalized solution of (1.1). We first recall from [16, Theorem 4.1] that

$$\|\nabla u\|_{L^{\frac{(p-1)n}{n-1},\infty}(\Omega)} \leq C \left[|\mu|(\Omega)\right]^{\frac{1}{p-1}},$$

which implies that

$$\left(\frac{1}{R^n} \int_\Omega |\nabla u|^\gamma \right)^{1/\gamma} \leq C_\gamma \left[\frac{|\mu|(\Omega)}{R^{n-1}} \right]^{\frac{1}{p-1}}, \qquad \text{with} \quad R = diam(\Omega), \tag{3.1}$$

for any $\gamma \in \left(0, \frac{(p-1)n}{n-1}\right)$.

For $k > 0$, let $\mu_0, \lambda_k^+, \lambda_k^-$ be as in Definition 1.3. Let $u_k \in W_0^{1,p}(\Omega)$ be the unique solution of the equation

$$\begin{cases} -\mathrm{div}(A(x, \nabla u_k)) &= \mu_k \quad \text{in} \quad \Omega, \\ u_k &= 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where we set $\mu_k = \chi_{\{|u|<k\}}\mu_0 + \lambda_k^+ - \lambda_k^-$.

Note that we have $u_k = T_k(u)$ and $\mu_k \to \mu$ in the narrow topology of measures (see [16, Remark 2.32]). Thus,

$$\nabla u_k \to \nabla u \quad \text{in} \quad L^\gamma(\Omega) \quad \forall \gamma \in \left(0, \frac{(p-1)n}{n-1}\right). \tag{3.2}$$

Let us set

$$E_{\lambda,\delta_2} = \{(\mathbf{M}(|\nabla u|^{\gamma_0}))^{1/\gamma_0} > \Lambda_0\lambda, (\mathbf{M}_1(\mu))^{\frac{1}{p-1}} \leq \delta_2\lambda\} \cap \Omega,$$

and

$$F_\lambda = \{(\mathbf{M}(|\nabla u|^{\gamma_0}))^{1/\gamma_0} > \lambda\} \cap \Omega,$$

for $\delta_2 \in (0,1)$ and $\lambda > 0$. Here Λ_0 is a constant depending only on n, p, γ_0, Λ and is to be chosen later. Also, let $\{y_i\}_{i=1}^L \subset \Omega$ and a ball B_0 with radius $2R$ such that

$$\Omega \subset \bigcup_{i=1}^L B_{r_0}(y_i) \subset B_0,$$

where $r_0 = \min\{R_0/1000, R\}$.

We now claim that

$$w(E_{\lambda,\delta_2}) \leq \varepsilon w(B_{r_0}(y_i)) \quad \forall \lambda > 0, \forall i = 1, 2, \ldots, L, \tag{3.3}$$

provided $\delta_2 = \delta_2(n, p, \Lambda, \epsilon, [w]_{\mathbf{A}_\infty}, R/R_0) > 0$ is small enough.

Indeed, we may assume that $E_{\lambda,\delta_2} \neq \emptyset$ and thus

$$|\mu|(\Omega) \leq R^{n-1}(\delta_2\lambda)^{p-1}.$$

Since $\mathbf{M}$ is a bounded operator from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$, in view of (3.1) with $\gamma = \gamma_0$ we find

$$|E_{\lambda,\delta_2}| \leq \frac{C}{(\Lambda_0\lambda)^{\gamma_0}} \int_\Omega |\nabla u|^{\gamma_0} dx \leq \frac{CR^n}{(\Lambda_0\lambda)^{\gamma_0}} \left[\frac{|\mu|(\Omega)}{R^{n-1}}\right]^{\frac{\gamma_0}{p-1}}.$$

Thus we obtain

$$|E_{\lambda,\delta_2}| \leq \frac{CR^n}{(\Lambda_0\lambda)^{\gamma_0}} \left[\frac{R^{n-1}(\delta_2\lambda)^{p-1}}{R^{n-1}}\right]^{\frac{\gamma_0}{p-1}} = C\delta_2^{\gamma_0}|B_0|.$$

Hence using the property of $\mathbf{A}_\infty$ weights we have

$$w(E_{\lambda,\delta_2}) \leq c\left(\frac{|E_{\lambda,\delta_2}|}{|B_0|}\right)^\nu w(B_0) \leq C\delta_2^{\nu\gamma_0} w(B_0),$$

where (c,ν) is a pair of $\mathbf{A}_\infty$ constants of w. It is known that (see, e.g., [27]) there exist $c_1 = c_1(n,c,\nu)$ and $\nu_1 = \nu_1(n,c,\nu)$ such that

$$\frac{w(B_0)}{w(B_{r_0}(y_i))} \le c_1 \left(\frac{|B_0|}{|B_{r_0}(y_i)|}\right)^{\nu_1} \quad \forall i = 1,2,\dots,L.$$

Thus we obtain

$$w(E_{\lambda,\delta_2}) \le C\delta_2^{\nu\gamma_0}\left(\frac{|B_0|}{|B_{r_0}(y_i)|}\right)^{\nu_1} w(B_{r_0}(y_i)) < \varepsilon w(B_{r_0}(y_i)) \ \ \forall i = 1,2,\dots,L,$$

provided δ_2 is small enough depending on $n,p,\gamma_0,\epsilon,[w]_{\mathbf{A}_\infty},R/R_0$. This proves (3.3).

Next we verify that for all $x \in \Omega$, $r \in (0,2r_0]$, and $\lambda > 0$ we have

$$w(E_{\lambda,\delta_2} \cap B_r(x)) \ge \varepsilon w(B_r(x)) \Longrightarrow B_r(x) \cap \Omega \subset F_\lambda, \tag{3.4}$$

provided δ_2 is small enough depending on $n,p,\Lambda,\gamma_0,\epsilon,[w]_{\mathbf{A}_\infty},R/R_0$.

Indeed, take $x \in \Omega$ and $0 < r \le 2r_0$. By contraposition, assume that $B_r(x) \cap \Omega \cap F_\lambda^c \ne \emptyset$ and $E_{\lambda,\delta_2} \cap B_r(x) \ne \emptyset$ i.e., there exist $x_1,x_2 \in B_r(x)\cap\Omega$ such that $[\mathbf{M}(|\nabla u|^{\gamma_0})(x_1)]^{1/\gamma_0} \le \lambda$ and $\mathbf{M}_1(\mu)(x_2) \le (\delta_2\lambda)^{p-1}$. We need to prove that

$$w(E_{\lambda,\delta_2} \cap B_r(x)) < \varepsilon w(B_r(x)). \tag{3.5}$$

Clearly,

$$\mathbf{M}(|\nabla u|)(y) \le \max\{\left[\mathbf{M}\left(\chi_{B_{2r}(x)}|\nabla u|^{\gamma_0}\right)(y)\right]^{\frac{1}{\gamma_0}}, 3^n\lambda\} \ \ \forall y \in B_r(x).$$

Therefore, for all $\lambda > 0$ and $\Lambda_0 \ge 3^n$,

$$E_{\lambda,\delta_2} \cap B_r(x) = \{\mathbf{M}\left(\chi_{B_{2r}(x)}|\nabla u|^{\gamma_0}\right)^{\frac{1}{\gamma_0}} > \Lambda_0\lambda, (\mathbf{M}_1(\mu))^{\frac{1}{p-1}} \le \delta_2\lambda\} \cap \Omega \cap B_r(x).$$

To prove (3.5) we separately consider the case $B_{8r}(x) \subset\subset \Omega$ and the case $B_{8r}(x)\cap \Omega^c \ne \emptyset$.

1. The case $B_{8r}(x) \subset\subset \Omega$**:** Applying Proposition 2.3 to $u = u_k \in W_0^{1,p}(\Omega), \mu = \mu_k$ and $B_{2R} = B_{8r}(x)$, there is a function $v_k \in W^{1,p}(B_{4r}(x)) \cap W^{1,\infty}(B_{2r}(x))$ such that for any $\eta > 0$,

$$\|\nabla v_k\|_{L^\infty(B_{2r}(x))} \le C\left[\frac{|\mu_k|(B_{8r}(x))}{r^{n-1}}\right]^{\frac{1}{p-1}} + C\left(\fint_{B_{8r}(x)} |\nabla u_k|^{\gamma_0}\right)^{1/\gamma_0},$$

and

$$\left(\fint_{B_{4r}} |\nabla u_k - \nabla v_k|^{\gamma_0} dx\right)^{\frac{1}{\gamma_0}} \le C_\eta\left[\frac{|\mu_k|(B_{8r}(x))}{r^{n-1}}\right]^{\frac{1}{p-1}} + C(([A]_{R_0})^\kappa + \eta)\left(\fint_{B_{8r}} |\nabla u_k|^{\gamma_0}\right)^{1/\gamma_0},$$

for some $\kappa \in (0,1)$.

Using $[\mathbf{M}(|\nabla u|^{\gamma_0})(x_1)]^{1/\gamma_0} \leq \lambda$ and $[\mathbf{M}_1(\mu)(x_2)]^{\frac{1}{p-1}} \leq \delta_2\lambda$ with $x_1, x_2 \in B_r(x)$, and property (3.2), we get

$$\begin{aligned}
\limsup_{k\to\infty} \|\nabla v_k\|_{L^\infty(B_{2r}(x))} &\leq C\left[\frac{|\mu|(\overline{B_{8r}(x)})}{r^{n-1}}\right]^{\frac{1}{p-1}} + C\left(\fint_{B_{8r}(x)} |\nabla u|^{\gamma_0}\right)^{1/\gamma_0} \\
&\leq C[\mathbf{M}_1(\mu)(x_2)]^{\frac{1}{p-1}} + C\,[\mathbf{M}(|\nabla u|^{\gamma_0})(x_1)]^{1/\gamma_0} \\
&\leq C\lambda,
\end{aligned}$$

and

$$\begin{aligned}
&\limsup_{k\to\infty} \left(\fint_{B_{4r}(x)} |\nabla u_k - \nabla v_k|^{\gamma_0} dx\right)^{\frac{1}{\gamma_0}} \\
&\leq C_\eta \left[\frac{|\mu|(\overline{B_{8r}(x)})}{r^{n-1}}\right]^{\frac{1}{p-1}} + C(([A]_{R_0})^\kappa + \eta)\left(\fint_{B_{8r}(x)} |\nabla u|^{\gamma_0}\right)^{1/\gamma_0} \\
&\leq C_\eta[\mathbf{M}_1(\mu)(x_2)]^{\frac{1}{p-1}} + C(([A]_{R_0})^\kappa + \eta)\,[\mathbf{M}(|\nabla u|^{\gamma_0})(x_1)]^{1/\gamma_0} \\
&\leq C\,(C_\eta\delta_2 + \delta_1^\kappa + \eta)\,\lambda.
\end{aligned}$$

Here also we used that $\mu_k \to \mu$ in the narrow topology of measures and that $[A]_{R_0} \leq \delta_1$.

Thus there exists $k_0 > 1$ such that for all $k \geq k_0$ we have

$$\|\nabla v_k\|_{L^\infty(B_{2r}(x))} \leq C\lambda, \tag{3.6}$$

and

$$\left(\fint_{B_{4r}(x)} |\nabla u_k - \nabla v_k|^{\gamma_0} dx\right)^{\frac{1}{\gamma_0}} \leq C\,(C_\eta\delta_2 + \delta_1^\kappa + \eta)\,\lambda. \tag{3.7}$$

Since

$$(\mathbf{M}(|\sum_{j=1}^{3} f_j|^{\gamma_0}))^{1/\gamma_0} \leq 3\sum_{j=1}^{3}(\mathbf{M}(|f_j|^{\gamma_0}))^{1/\gamma_0},$$

we find

$$\begin{aligned}
|E_{\lambda,\delta_2} \cap B_r(x)| \leq\, &|\{\mathbf{M}\left(\chi_{B_{2r}(x)}|\nabla(u_k - v_k)|^{\gamma_0}\right)^{\frac{1}{\gamma_0}} > \Lambda_0\lambda/9\} \cap B_r(x)| \\
&+ |\{\mathbf{M}\left(\chi_{B_{2r}(x)}|\nabla(u - u_k)|^{\gamma_0}\right)^{\frac{1}{\gamma_0}} > \Lambda_0\lambda/9\} \cap B_r(x)| \\
&+ |\{\mathbf{M}\left(\chi_{B_{2r}(x)}|\nabla v_k|^{\gamma_0}\right)^{\frac{1}{\gamma_0}} > \Lambda_0\lambda/9\} \cap B_r(x)|.
\end{aligned} \tag{3.8}$$

In view of (3.6) we see that for $\Lambda_0 \geq \max\{3^n, 10C\}$ (C is the constant in (3.6)) and $k \geq k_0$, it holds that

$$|\{\mathbf{M}\left(\chi_{B_{2r}(x)}|\nabla v_k|^{\gamma_0}\right)^{\frac{1}{\gamma_0}} > \Lambda_0\lambda/9\} \cap B_r(x)| = 0.$$

Thus, we deduce from (3.8) and (3.7) that for $k \geq k_0$,

$$\begin{aligned}
|E_{\lambda,\delta_2} \cap B_r(x)| &\leq |\{\left[\mathbf{M}\left(\chi_{B_{2r}(x)}|\nabla(u_k - v_k)|^{\gamma_0}\right)\right]^{\frac{1}{\gamma_0}} > \Lambda_0\lambda/9\} \cap B_r(x)| \\
&\quad + |\{\left[\mathbf{M}\left(\chi_{B_{2r}(x)}|\nabla(u - u_k)|^{\gamma_0}\right)\right]^{\frac{1}{\gamma_0}} > \Lambda_0\lambda/9\} \cap B_r(x)| \\
&\leq \frac{C}{\lambda^{\gamma_0}}\left[\int_{B_{2r}(x)} |\nabla(u_k - v_k)|^{\gamma_0} + \int_{B_{2r}(x)} |\nabla(u - u_k)|^{\gamma_0}\right] \\
&\leq \frac{C}{\lambda^{\gamma_0}}\left[(C_\eta\delta_2 + \delta_1^\kappa + \eta)^{\gamma_0}\lambda^{\gamma_0}r^n + \int_{B_{2r}(x)} |\nabla(u - u_k)|^{\gamma_0}\right].
\end{aligned}$$

At this point, letting $k \to \infty$ we get

$$|E_{\lambda,\delta_2} \cap B_r(x)| \leq C\left(C_\eta\delta_2 + \delta_1^\kappa + \eta\right)^{\gamma_0} |B_r(x)|.$$

Thus,

$$\begin{aligned}
w(E_{\lambda,\delta_2} \cap B_r(x)) &\leq c\left(\frac{|E_{\lambda,\delta_2} \cap B_r(x)|}{|B_r(x)|}\right)^\nu w(B_r(x)) \\
&\leq c\left(C_\eta\delta_2 + \delta_1^\kappa + \eta\right)^{\gamma_0\nu} w(B_r(x)) \\
&< \varepsilon w(B_r(x)),
\end{aligned}$$

where $\eta, \delta_1 \leq C(n, p, \Lambda, \gamma_0, \epsilon, [w]_{\mathbf{A}_\infty})$ and $\delta_2 \leq C(n, p, \Lambda, \gamma_0, \epsilon, [w]_{\mathbf{A}_\infty}, R/R_0)$.

2. The case $B_{8r}(x) \cap \Omega^c \neq \emptyset$: Let $x_3 \in \partial\Omega$ such that $|x_3 - x| = \mathrm{dist}(x, \partial\Omega)$. We have

$$B_{2r}(x) \subset B_{10r}(x_3) \subset B_{100r}(x_3) \subset B_{108r}(x) \subset B_{109r}(x_1), \tag{3.9}$$

and

$$B_{100r}(x_3) \subset B_{108r}(x) \subset B_{109r}(x_2). \tag{3.10}$$

Applying Proposition 2.6 to $u = u_k \in W_0^{1,p}(\Omega), \mu = \mu_k$ and $B_{10R} = B_{100r}(x_3)$, for any $\eta > 0$ there exists $\delta_0 = \delta_0(n, p, \Lambda, \eta)$ such that the following holds. If Ω is a (δ_0, R_0)-Reifenberg flat domain, there exists a function $V_k \in W^{1,\infty}(B_{10r}(x_3))$ such that

$$\|\nabla V_k\|_{L^\infty(B_{10r}(x_3))} \leq C\left[\frac{|\mu_k|(B_{100r}(x_3))}{r^{n-1}}\right]^{\frac{1}{p-1}} + C\left(\fint_{B_{100r}(x_3)} |\nabla u_k|^{\gamma_0}\right)^{1/\gamma_0},$$

and

$$\begin{aligned}
&\left(\fint_{B_{10r}(x_3)} |\nabla(u_k - V_k)|^{\gamma_0} dx\right)^{\frac{1}{\gamma_0}} \\
&\quad \leq C_\eta\left[\frac{|\mu_k|(B_{100r}(x_3))}{r^{n-1}}\right]^{\frac{1}{p-1}} + C(([A]_{R_0})^\kappa + \eta)\left(\fint_{B_{100r}(x_3)} |\nabla u_k|^{\gamma_0}\right)^{1/\gamma_0},
\end{aligned}$$

for some $\kappa \in (0, 1)$.

Since $[\mathbf{M}(|\nabla u|^{\gamma_0})(x_1)]^{1/\gamma_0} \leq \lambda$ and $[\mathbf{M}_1(\mu)(x_2)]^{\frac{1}{p-1}} \leq \delta_2 \lambda$ with $x_1, x_2 \in B_r(x)$, by (3.9), (3.10), the fact that $[A]_{R_0} \leq \delta_1$, and property (3.2), we get

$$\begin{aligned}
\limsup_{k\to\infty} \|\nabla V_k\|_{L^\infty(B_{2r}(x))} &\leq C \left[\frac{|\mu|(\overline{B_{100r}(x_3)})}{r^{n-1}}\right]^{\frac{1}{p-1}} + C\left(\fint_{B_{100r}(x_3)} |\nabla u|^{\gamma_0}\right)^{1/\gamma_0} \\
&\leq C \left[\frac{|\mu|(B_{109r}(x_2))}{r^{n-1}}\right]^{\frac{1}{p-1}} + C\left(\fint_{B_{109r}(x_1)} |\nabla u|^{\gamma_0}\right)^{1/\gamma_0} \\
&\leq C\left([\mathbf{M}_1(\mu)(x_2)]^{\frac{1}{p-1}} + [\mathbf{M}(|\nabla u|^{\gamma_0})(x_1)]^{1/\gamma_0}\right) \\
&\leq C\lambda,
\end{aligned}$$

and

$$\begin{aligned}
&\limsup_{k\to\infty} \left(\fint_{B_{2r}(x)} |\nabla(u_k - V_k)|^{\gamma_0} dx\right)^{\frac{1}{\gamma_0}} \\
&\quad\leq C_\eta [\mathbf{M}_1(\mu)(x_2)]^{\frac{1}{p-1}} + C(([A]_{R_0})^\kappa + \eta)\,[\mathbf{M}(|\nabla u|^{\gamma_0})(x_1)]^{1/\gamma_0} \\
&\quad\leq C\,(C_\eta \delta_2 + \delta_1^\kappa + \eta)\,\lambda.
\end{aligned}$$

Thus we can find $k_0 > 1$ such that for all $k \geq k_0$ we have

$$\|\nabla V_k\|_{L^\infty(B_{2r}(x))} \leq C\lambda, \tag{3.11}$$

and

$$\left(\fint_{B_{2r}(x)} |\nabla(u_k - V_k)|^{\gamma_0} dx\right)^{\frac{1}{\gamma_0}} \leq C\,(C_\eta \delta_2 + \delta_1^\kappa + \eta)\,\lambda. \tag{3.12}$$

As in the interior case we also have for $k \geq k_0$,

$$\begin{aligned}
|E_{\lambda,\delta_2} \cap B_r(x)| &\leq |\{\mathbf{M}\left(\chi_{B_{2r}(x)} |\nabla(u_k - v_k)|^{\gamma_0}\right)^{\frac{1}{\gamma_0}} > \Lambda_0 \lambda/9\} \cap B_r(x)| \\
&\quad + |\{\mathbf{M}\left(\chi_{B_{2r}(x)} |\nabla(u - u_k)|^{\gamma_0}\right)^{\frac{1}{\gamma_0}} > \Lambda_0 \lambda/9\} \cap B_r(x)|,
\end{aligned}$$

for a constant $\Lambda_0 > 1$ depending only on n, p, Λ. Therefore, we deduce from (3.11) and (3.12) that, for $k \geq k_0$,

$$\begin{aligned}
|E_{\lambda,\delta_2} \cap B_r(x)| &\leq \frac{C}{\lambda^{\gamma_0}} \left(\int_{B_{2r}(x)} |\nabla(u_k - v_k)|^{\gamma_0} + \int_{B_{2r}(x)} |\nabla(u - u_k)|^{\gamma_0}\right) \\
&\leq \frac{C}{\lambda^{\gamma_0}} \left((C_\eta \delta_2 + \delta_1^\kappa + \eta)^{\gamma_0} \lambda^{\gamma_0} r^n + \int_{B_{2r}(x)} |\nabla(u - u_k)|^{\gamma_0}\right).
\end{aligned}$$

Then letting $k \to \infty$ we get

$$|E_{\lambda,\delta_2} \cap B_r(x)| \leq C\,(C_\eta \delta_2 + \delta_1^\kappa + \eta)^{\gamma_0}\, |B_r(x)|.$$

Thus we have

$$\begin{aligned}
w(E_{\lambda,\delta_2} \cap B_r(x)) &\leq c \left(\frac{|E_{\lambda,\delta_2} \cap B_r(x)|}{|B_r(x)|} \right)^{\nu} w(B_r(x)) \\
&\leq c \left(C_\eta \delta_2 + \delta_1^\kappa + \eta \right)^{\gamma_0 \nu} w(B_r(x)) \\
&< \varepsilon w(B_r(x)).
\end{aligned}$$

where $\eta, \delta_1 \leq C(n, p, \Lambda, \gamma_0, \varepsilon, [w]_{\mathbf{A}_\infty})$ and $\delta_2 \leq C(n, p, \Lambda, \gamma_0, \varepsilon, [w]_{\mathbf{A}_\infty}, R/R_0)$.

With (3.3) and (3.4) in hand, we can now apply Lemma 3.1 with $E = E_{\lambda,\delta_2}$ and $F = F_\lambda$ to complete the proof of the theorem. ■

4 Proofs of Theorem 1.4 and Corollary 1.7

In this section we prove Theorem 1.4 and Corollary 1.7. We mention here that the proof of the weighted Lorentz space bound, Theorem 1.6, can be done similarly to that of Theorem 1.4 and thus will be skipped. We now begin with the proof of Theorem 1.4 using mainly the good-λ type bound obtained in Theorem 1.5.

Proof of Theorem 1.4. By Theorem 1.5, for any $\varepsilon > 0, R_0 > 0$ one finds $\delta = \delta(n, p, \Lambda, \varepsilon, [w]_{\mathbf{A}_\infty}) \in (0, 1/2)$, $\delta_2 = \delta_2(n, p, \Lambda, \varepsilon, [w]_{\mathbf{A}_\infty}, diam(\Omega)/R_0) \in (0, 1)$, and $\Lambda_0 = \Lambda_0(n, p, \gamma_0, \Lambda) > 1$ such that if Ω is a (δ, R_0)-Reifenberg flat domain and $[A]_{R_0} \leq \delta$ then

$$\begin{aligned}
w(\{(\mathbf{M}(|\nabla u|^{\gamma_0}))^{1/\gamma_0} > \Lambda_0 \lambda, (\mathbf{M}_1(\mu))^{\frac{1}{p-1}} \leq \delta_2 \lambda\} \cap \Omega) \\
\leq C \varepsilon w(\{(\mathbf{M}(|\nabla u|^{\gamma_0}))^{1/\gamma_0} > \lambda\} \cap \Omega),
\end{aligned}$$

for all $\lambda > 0$. Here the constant γ_0 is as in Lemmas 2.2, and the constant C depends only on $n, p, \gamma_0, \Lambda, [w]_{\mathbf{A}_\infty}$, and $diam(\Omega)/R_0$. Thus, as Φ is invertible with $\Phi^{-1} : [0, \infty) \to [0, \infty)$, we find

$$\begin{aligned}
&w(\{(\mathbf{M}(|\nabla u|^{\gamma_0}))^{1/\gamma_0} > \Phi^{-1}(t)\} \cap \Omega) \\
&\quad \leq w(\{(\mathbf{M}_1(\mu))^{\frac{1}{p-1}} > \frac{\delta_2}{\Lambda_0} \Phi^{-1}(t)\} \cap \Omega) + C \varepsilon w(\{(\mathbf{M}(|\nabla u|^{\gamma_0}))^{1/\gamma_0} > \frac{\Phi^{-1}(t)}{\Lambda_0}\} \cap \Omega)
\end{aligned}$$

for all $t > 0$. This gives, for any $T > 0$,

$$\begin{aligned}
&\int_0^T w(\{x \in \Omega : \Phi[(\mathbf{M}(|\nabla u|^{\gamma_0}))^{\frac{1}{\gamma_0}}] > t\}) dt \\
&\quad \leq C \varepsilon \int_0^T w(\{x \in \Omega : \Phi[\Lambda_0 (\mathbf{M}(|\nabla u|^{\gamma_0}))^{\frac{1}{\gamma_0}}] > t\}) dt \\
&\qquad + \int_0^T w(\{x \in \Omega : \Phi[\frac{\Lambda_0}{\delta_2} (\mathbf{M}_1(\mu))^{\frac{1}{p-1}}] > t\}) dt.
\end{aligned}$$

As $\Phi(2t) \le c\,\Phi(t)$ and Φ is increasing, this yields

$$\begin{aligned}
&\int_0^T w(\{x \in \Omega : \Phi[(\mathbf{M}(|\nabla u|^{\gamma_0}))^{\frac{1}{\gamma_0}}] > t\})dt \\
&\qquad \le C\varepsilon \int_0^T w(\{x \in \Omega : H_1\Phi[(\mathbf{M}(|\nabla u|^{\gamma_0}))^{\frac{1}{\gamma_0}}] > t\})dt \\
&\qquad\quad + \int_0^T w(\{x \in \Omega : H_2\Phi[(\mathbf{M}_1(\mu))^{\frac{1}{p-1}}] > t\})dt,
\end{aligned}$$

where $H_1 = c^{[[\log_2(\Lambda_0)]]}$ and $H_2 = c^{[[\log_2(\frac{\Lambda_0}{\delta_2})]]}$. Here $[[a]]$ denotes the smallest integer greater than or equal to a. Thus by simple changes of variables we arrive at

$$\begin{aligned}
&\int_0^T w(\{x \in \Omega : \Phi[(\mathbf{M}(|\nabla u|^{\gamma_0}))^{\frac{1}{\gamma_0}}] > t\})dt \\
&\qquad \le H_1 C\varepsilon \int_0^{\frac{T}{H_1}} w(\{x \in \Omega : \Phi[(\mathbf{M}(|\nabla u|^{\gamma_0}))^{\frac{1}{\gamma_0}}] > s\})ds \\
&\qquad\quad + H_2 \int_0^{\frac{T}{H_2}} w(\{x \in \Omega : \Phi[(\mathbf{M}_1(\mu))^{\frac{1}{p-1}}] > s\})ds.
\end{aligned}$$

Now using $H_1 > 1$ and letting $\varepsilon = \frac{1}{2H_1 C}$ we can absorb the first term on the right to the left, which yields

$$\begin{aligned}
&\int_0^T w(\{x \in \Omega : \Phi[(\mathbf{M}(|\nabla u|^{\gamma_0}))^{\frac{1}{\gamma_0}}] > t\})dt \\
&\qquad \le 2H_2 \int_0^{\frac{T}{H_2}} w(\{x \in \Omega : \Phi[(\mathbf{M}_1(\mu))^{\frac{1}{p-1}}] > s\})ds.
\end{aligned}$$

Then sending $T \to \infty$ in the above bound and recalling that

$$\int_\Omega \Phi(|f|)wdx = \int_0^\infty w(\{x \in \Omega : \Phi(|f(x)|) > t\})dt,$$

we deduce

$$\int_\Omega \Phi[(\mathbf{M}(|\nabla u|^{\gamma_0}))^{\frac{1}{\gamma_0}}]wdx \le 2H_2 \int_\Omega \Phi[(\mathbf{M}_1(\mu))^{\frac{1}{p-1}}]wdx.$$

This yields (1.5) as desired and completes the proof of the theorem. ∎

We next prove Corollary 1.7 which provides a compactness criterion for solution sets of equation (1.1):

Proof of Corollary 1.7. By de la Vallée-Poussin Lemma on equi-integrability, there exists an increasing function $G : [0, \infty) \to [0, \infty)$ with $G(0) = 0$ and

$$\lim_{t\to\infty} \frac{G(t)}{t} = \infty,$$

such that

$$\sup_j \int_\Omega G([\mathbf{M}_1(|\mu_j|)]^{\frac{q}{p-1}})wdx \le C.$$

Moreover, we may assume that G satisfies a moderate growth condition (see [44]): there exists $c_1 > 1$ such that

$$G(2t) \leq c_1 \, G(t) \qquad \forall t \geq 0.$$

Then applying Theorem 1.4 with $\Phi(t) := G(t^q)$, which is also of moderate growth, we get

$$\int_\Omega G(|\nabla u_j|^q) w(x) dx \leq C \int_\Omega G([\mathbf{M}_1(\mu_j)]^{\frac{q}{p-1}}) w(x) dx \leq C.$$

Thus by de la Vallée-Poussin Lemma the set $\{|\nabla u_j|^q\}_j$ is also bounded and equi-integrable in $L^1_w(\Omega)$.

On the other hand, from the assumption we have

$$|\mu_j|(\Omega) \leq C. \tag{4.1}$$

By (4.1), it follows from the proof of Theorem 3.4 in [16] that there exists a subsequence $\{u_{j'}\}_{j'}$ converging a.e. to a function u such that $|u| < \infty$ a.e., $T_k(u) \in W_0^{1,p}(\Omega)$ for all $k > 0$, and moreover

$$\nabla u_{j'} \to \nabla u \quad \text{a.e. in } \Omega.$$

We can now apply Vitali Convergence Theorem to obtain the strong convergence (1.6). This completes the proof of the corollary. ■

5 Existence of solutions to Riccati type equations

We shall prove Theorem 1.9 in this section. To that end, we need some preliminaries.

Definition 5.1 *Given $s > 1$ we define the space $M^{1,s}(\Omega)$ to be the set of all finite signed measures μ in Ω such that the quantity $[\mu]_{M^{1,s}(\Omega)} < +\infty$, where*

$$[\mu]_{M^{1,s}(\Omega)} := \sup\left\{|\mu|(K)/\mathrm{Cap}_{1,\,s}(K) : \mathrm{Cap}_{1,\,s}(K) > 0\right\},$$

with the supremum being taken over all compact sets $K \subset \Omega$.

Due to the capacitability of Borel sets, $[\mu]_{M^{1,s}(\Omega)}$ remains unchanged in the above definition even if the supremum is taken over all Borel sets $K \subset \Omega$.

Given a nonnegative locally finite measure ν in $\mathbb{R}^n$, we define its first order Riesz's potentials by

$$\mathbf{I}_1^\rho \nu(x) = \int_0^\rho \frac{\nu(B_t(x))}{t^{n-1}} \frac{dt}{t}, \qquad x \in \mathbb{R}^n,$$

where $\rho \in (0, \infty]$. When $\rho = \infty$, we write $\mathbf{I}_1 \nu$ instead of $\mathbf{I}_1^\infty \nu$ and note that in this case we have

$$\mathbf{I}_1 \nu(x) = c(n) \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-1}} d\nu(y), \qquad x \in \mathbb{R}^n.$$

Let $M = M(n) \geq 1$ be a constant such that

$$\mathbf{M}(\mathbf{I}_1(f))(x) \leq M \, \mathbf{I}_1(f)(x), \qquad x \in \mathbb{R}^n, \tag{5.1}$$

for all $f \in L^1_{\rm loc}(\mathbb{R}^n)$, $f \geq 0$. Recall that $\mathbf{M}$ is the Hardy-Littlewood maximal function. Inequality (5.1) follows from an application of Fubini's Theorem and the fact the function $x \mapsto |x|^{1-n}$ is an $\mathbf{A}_1$ weight. By an $\mathbf{A}_1$ weight we mean a nonnegative function $w \in L^1_{\rm loc}(\mathbb{R}^n)$, $w \not\equiv 0$, such that

$$\mathbf{M}(w)(x) \leq C\, w(x), \qquad \text{a.e. } x \in \mathbb{R}^n,$$

for a constant $C > 0$. The least possible value of C will be denoted by $[w]_{\mathbf{A}_1}$ and is called the $\mathbf{A}_1$ constant of w. It is well-known that $\mathbf{A}_1 \subset \mathbf{A}_\infty$.

With $R = {\rm diam}(\Omega)$ and $q > p-1$, for each measure $\mu \in M^{1,\frac{q}{q-p+1}}(\Omega)$ we define the set

$$E_1(\mu) := \Big\{ v \in W^{1,q}_0(\Omega) : \int_\Omega |\nabla v|^q w dx \leq T_1 \int_\Omega \mathbf{I}_1^{2R}(|\mu|)^{\frac{q}{p-1}} w dx$$
$$\text{for all } w \in \mathbf{A}_1 \cap L^\infty(\Omega) \text{ such that } [w]_{\mathbf{A}_1} \leq M \Big\}.$$

Here $T_1 > 0$ is to be determined. Now if $q \geq 1$, then under the strong topology of $W^{1,q}_0(\Omega)$, we have that $E_1(\mu)$ is closed and convex. Note that by [51, Corollary 2.5] (see also [41, Theorem 1.2]) if $\mu \in M^{1,\frac{q}{q-p+1}}(\Omega)$ then so is $\mathbf{I}_1^{2R}(|\mu|)^{\frac{q}{p-1}}$ with

$$[\mathbf{I}_1^{2R}(|\mu|)^{\frac{q}{p-1}}]_{M^{1,\frac{q}{q-p+1}}} \leq C(n,p,q,R)\, [\mu]^{\frac{q}{p-1}}_{M^{1,\frac{q}{q-p+1}}}. \tag{5.2}$$

We will need the following lemma.

Lemma 5.2 *Let $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$, $q > p-1$, $\mu \in M^{1,\frac{q}{q-p+1}}(\Omega)$. Let $E_1(\mu)$ and $T_1 > 0$ be as above. Then for any $v \in E_1(\mu)$ we have*

$$\mathbf{I}_1^{2R}(|\nabla v|^q \chi_\Omega)(x) \leq C_1\, T_1\, [\mu]^{\frac{q-p+1}{p-1}}_{M^{1,\frac{q}{q-p+1}}(\Omega)}\, \mathbf{I}_1^{2R}(|\mu|)(x), \tag{5.3}$$

for a.e. $x \in \Omega$ and $R = {\rm diam}(\Omega)$. Here C_1 depends only on n, p, and q.

Proof. For any $v \in E_1(\mu)$, by (5.1) we have

$$\int_{\mathbb{R}^n} |\nabla v|^q \chi_\Omega \mathbf{I}_1(f) dx \leq T_1 \int_{\mathbb{R}^n} \mathbf{I}_1^{2R}(|\mu|)^{\frac{q}{p-1}} \chi_\Omega \mathbf{I}_1(f) dx,$$

for any $f \in L^\infty(\mathbb{R}^n), f \geq 0$, with compact support. Hence by Fubini's Theorem,

$$\int_{\mathbb{R}^n} \mathbf{I}_1(|\nabla v|^q \chi_\Omega) f dx \leq T_1 \int_{\mathbb{R}^n} \mathbf{I}_1[\mathbf{I}_1^{2R}(|\mu|)^{\frac{q}{p-1}} \chi_\Omega] f dx,$$

which yields

$$\mathbf{I}_1(|\nabla v|^q \chi_\Omega) \leq T_1\, \mathbf{I}_1[\mathbf{I}_1^{2R}(|\mu|)^{\frac{q}{p-1}} \chi_\Omega] \leq C\, T_1\, \mathbf{I}_1^{2R}[\mathbf{I}_1^{2R}(|\mu|)^{\frac{q}{p-1}} \chi_\Omega] \quad \text{a.e. in } \Omega.$$

On the other hand, by inequality (2.10) of [51] with $\nu = |\mu|, \rho = 4R, \alpha = 1/2, p = 2$ and with q replaced by $\frac{q}{p-1} > 1$ we find

$$\mathbf{I}_1^{2R}[\mathbf{I}_1^{2R}(|\mu|)^{\frac{q}{p-1}}](x) \leq C[\mu]^{\frac{q-p+1}{p-1}}_{M^{1,\frac{q}{q-p+1}}(\Omega)}\, \mathbf{I}_1^{4R}(|\mu|)(x) \leq C[\mu]^{\frac{q-p+1}{p-1}}_{M^{1,\frac{q}{q-p+1}}(\Omega)}\, \mathbf{I}_1^{2R}(|\mu|)(x).$$

for a.e. $x \in \Omega$. Thus for a.e. $x \in \Omega$ we have (5.3) as desired. ■

We are now ready to prove Theorem 1.9.

Proof of Theorem 1.9. Since $q \geq 1$, we have $q > p-1$. First we assume that $\mu \in \mathfrak{M}_0(\Omega)$ and let $S : E_1(\mu) \to W_0^{1,q}(\Omega)$ be defined by $S(v) = u$ where $u \in W_0^{1,q}(\Omega)$ is the unique renormalized solution of

$$\begin{cases} -\mathrm{div}(A(x, \nabla u)) &= |\nabla v|^q + \mu \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

We claim that we can find $T_1 > 0$ and $c_0 > 0$ such that if (1.7) holds with c_0 then

$$S : E_1(\mu) \to E_1(\mu). \tag{5.4}$$

Indeed, for any weight $w \in \mathbf{A}_1 \cap L^\infty(\Omega)$ with $[w]_{\mathbf{A}_1} \leq M$ and for any $v \in E_1(\mu)$, by Theorem 1.4 there exists $\delta = \delta(n, p, \Lambda, q) \in (0, 1)$ such that if $[A]_{R_0} \leq \delta$ and Ω is (δ, R_0)-Reifenberg flat for some $R_0 > 0$, then we have

$$\int_\Omega |\nabla S(v)|^q w dx \leq N \int_\Omega \left[\mathbf{I}_1^{2R}(|\nabla v|^q \chi_\Omega + |\mu|)\right]^{\frac{q}{p-1}} w dx.$$

Here $N > 0$ depends only on n, p, q, Λ, and R/R_0. We used the elementary fact that, for any $\nu \in \mathfrak{M}_b(\Omega)$,

$$\mathbf{M}_1(|\nu|) \leq C\, \mathbf{I}_1^{2R}(|\nu|) \quad \text{a.e. in } \Omega. \tag{5.5}$$

Thus by Lemma 5.2 we find

$$\int_\Omega |\nabla S(v)|^q w dx \leq N \left[C_1 T_1 [\mu]_{M^{1,\frac{q}{q-p+1}}(\Omega)}^{\frac{q-p+1}{p-1}} + 1\right]^{\frac{q}{p-1}} \int_\Omega \mathbf{I}_1^{2R}(|\mu|)^{\frac{q}{p-1}} w dx.$$

We now choose $T_1 = 2N$ and c_0 such that

$$0 < c_0 \leq c_1 := [(2^{\frac{p-1}{q}} - 1)/(2NC_1)]^{\frac{p-1}{q-p+1}}. \tag{5.6}$$

Then it follows that if condition (1.7) holds then

$$\int_\Omega |\nabla S(v)|^q w dx \leq T_1 \int_\Omega \mathbf{I}_1^{2R}(|\mu|)^{\frac{q}{p-1}} w dx.$$

This gives (5.4) for this choice of T_1 and c_0.

We next show the continuity of S on $E_1(\mu)$. Let $\{v_k\}$ be a sequence in $E_1(\mu)$ such that v_k converges strongly in $W_0^{1,q}(\Omega)$ to a function $v \in E_1(\mu)$. Set $u_k = S(v_k)$. We have

$$\begin{cases} -\mathrm{div}(A(x, \nabla u_k)) &= |\nabla v_k|^q + \mu \quad \text{in } \Omega, \\ u_k &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Now it follows from Lemma 5.2 that

$$\mathbf{I}_1^{2R}(|\nabla v_k|^q \chi_\Omega + |\mu|)(x) \leq C\, \mathbf{I}_1^{2R}(|\mu|)(x) \quad \text{a.e. } x \in \Omega.$$

Thus by (5.5) and Corollary 1.7 there exist a subsequence $\{u_{k'}\}_{k'}$ and a finite a.e. function u with the property that $T_s(u) \in W_0^{1,p}(\Omega)$ for all $s > 0$, $u_{k'} \to u$ a.e., and

$$\nabla u_{k'} \to \nabla u \quad \text{strongly in} \quad L^q(\Omega, \mathbb{R}^n). \tag{5.7}$$

Since $u_k \in W_0^{1,q}(\Omega)$, we also have $u \in W_0^{1,q}(\Omega)$. Moreover, as $|\nabla v_k|^q \to |\nabla v|^q$ in $L^1(\Omega)$, by the stability result in [8, Theorem 3.4], we have $u = S(v) \in W_0^{1,q}(\Omega)$. Thus the limit in (5.7) is independent of the subsequence, which implies that the whole sequence $u_k \to u$ strongly in $W_0^{1,q}(\Omega)$. This proves the continuity of S.

Similarly, using Corollary 1.7 we can show that $S(E_1(\mu))$ is precompact under the strong topology of $W_0^{1,q}(\Omega)$.

At this point, we can apply Schauder Fixed Point Theorem to obtain a renormalized solution $u \in E_1(\mu)$ to equation (1.2). Moreover, by Lemma 5.2, Theorem 1.4, and (5.5), we have

$$\int_\Omega |\nabla u|^q w dx \leq C \int_\Omega \mathbf{I}_1^{2R}(|\mu|)^{\frac{q}{p-1}} w dx,$$

for all weights $w \in \mathbf{A}_\infty$.Thus by (5.2) and [41, Lemma 3.1], we obtain

$$[|\nabla u|^q]_{M^{1,\frac{q}{q-p+1}}(\Omega)} \leq C[\mu]^{\frac{q}{p-1}}_{M^{1,\frac{q}{q-p+1}}(\Omega)} \leq C.$$

This completes the proof of Theorem 1.6 in the case $\mu \in \mathfrak{M}_0(\Omega)$.

We now remove the assumption $\mu \in \mathfrak{M}_0(\Omega)$. Recall that $\mu = \mu_0 + \mu_s$ where $\mu_0 \in \mathfrak{M}_0(\Omega)$ and $\mu_s \in \mathfrak{M}_s(\Omega)$. Let $\mu_k = \mu_0 + \rho_k * \mu_s$, where $\{\rho_k\}_{k>0}$ is a standard sequence of mollifiers. Then by [54, Lemma 5.7], we have $\mu_k \in \mathfrak{M}_0(\Omega) \cap M^{1,\frac{q}{q-p+1}}(\Omega)$ with

$$[\mu_k]_{M^{1,\frac{q}{q-p+1}}(\Omega)} \leq B\, [\mu]_{M^{1,\frac{q}{q-p+1}}(\Omega)},$$

for some $B > 1$. Thus if we further restrict c_0 so that $B\, c_0 \leq c_1$, where c_1 is defined in (5.6), then we have

$$[\mu_k]_{M^{1,\frac{q}{q-p+1}}(\Omega)} \leq c_1.$$

This allows us to apply the above result: for each $k > 0$ there exists a renormalized solution $u_k \in E_1(\mu_k)$ to the equation

$$\begin{cases} -\text{div}(A(x, \nabla u_k)) &= |\nabla u_k|^q + \mu_k \quad \text{in } \Omega, \\ u_k &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

such that

$$[|\nabla u_k|^q]_{M^{1,\frac{q}{q-p+1}}(\Omega)} \leq C,$$

and

$$\mathbf{I}_1^{2R}(|\nabla u_k|^q \chi_\Omega + |\mu_k|)(x) \leq C\, \mathbf{I}_1^{2R}(|\mu_k|)(x) \qquad \text{a.e. } x \in \Omega.$$

Thus by (5.5),

$$\mathbf{M}_1(|\nabla u_k|^q \chi_\Omega + |\mu_k|)(x) \leq C\, \mathbf{I}_1^{2R}(|\mu_k|)(x) \qquad \text{a.e. } x \in \Omega.$$

Now observe that we have

$$\begin{aligned}\mathbf{I}_1^{2R}(|\mu_k|) &\leq \mathbf{I}_1^{2R}(|\mu_0|) + \mathbf{I}_1^{2R}(\rho_k * |\mu_s|)\\ &= \mathbf{I}_1^{2R}(|\mu_0|) + \rho_k * (\mathbf{I}_1^{2R}(|\mu_s|))\\ &\leq 2\mathbf{M}(\mathbf{I}_1^{2R}(|\mu|)) \qquad \text{a.e. } x \in \Omega,\end{aligned}$$

where $\mathbf{M}$ is the Hardy-Littlewood maximal function.

Thus, since $\mathbf{M}(\mathbf{I}_1^{2R}(|\mu|)) \in L^{\frac{q}{p-1}}(\Omega)$, we see that the set $\{\mathbf{M}_1(|\nabla u_k|^q + \mu_k)^{\frac{q}{p-1}}\}$ is equi-integrable in $L^1(\Omega)$. Then by Corollary 1.7 and the stability result of [8, Theorem 3.4], there exists a subsequence $\{u_{k'}\}$ converging a.e. and strongly in $W_0^{1,q}(\Omega)$ to a function $u \in W_0^{1,q}(\Omega)$ such that u solves (1.2). This completes the proof of the theorem. ∎

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