On the Sobolev space of functions with derivative of logarithmic order

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Abstract

Two notions of "having a derivative of logarithmic order" have been studied. They come from the study of regularity of flows and renormalized solutions for the transport and continuity equation associated to weakly differentiable drifts.

Introduction

In this note we study two different notions of "having a derivative of logarithmic order". They come up naturally in the study of regular Lagrangian flows and renormalized solutions for transport and continuity equation under the Sobolev regularity of the drift [1–5]. In order to better explain this point let us present a formal computation.

Let us fix a vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ in the Euclidean space of dimension d and assume that $b \in W^{1,p}(\mathbb{R}^d; \mathbb{R}^d)$ for some $p \geq 1$. Let us consider the following problem:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} X(t,x) = b(X(t,x)), \\ X(0,x) = x, \qquad \forall x \in \mathbb{R}^d. \end{cases} \tag{ODE}$$

In this setting the (ODE) problem has been studied for a first time by DiPerna and Lions [1] and extended to the BV framework by Ambrosio in [2]. After these two pioneering works this topic has received a lot of attentions becoming a very thriving research field.

Let us now pass to the formal computation. In order to understand the regularity of the flow map X_t (this question is central in the theory since every regularity of X, even a very mild one, allows for compactness theorems, see [6] for a beautiful application to the compressible Navier-Stokes equation) it is natural to differentiate the starting equation with respect to the space variable, obtaining

$$\frac{\mathrm{d}}{\mathrm{d}t} \nabla X_t(x) = \nabla b(X_t(x)) \cdot \nabla X_t(x).$$

Passing to the modulus and integrating we get

$$\log\left(|\nabla X_t(x)|\right) \leq \int_0^t |\nabla b|(X_s(x))\,\mathrm{d}s.$$

Assuming that X_t is a measure preserving map (in the classical context this happens when $\operatorname{div} b = 0$) and exploiting the Sobolev regularity of b we can conclude that

$$\log\left(|\nabla X_t(x)|\right) \in L^p(\mathbb{R}^d), \tag{0.1}$$

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with the quantitative estimate

$$\|\log(|\nabla X_t|)\|_{L^p} \le t\,\|\nabla b\|_{L^p}\,.$$

Of course our computation is not rigorous, indeed, a priori the gradient of X_t does not exist in the distributional sense. However, from (0.1) we learned that the reasonable regularity for a flow map associated to $b \in W^{1,p}(\mathbb{R}^d;\mathbb{R}^d)$ is neither the classical Sobolev regularity nor a fractional one but something of logarithmic order. The first rigorous result in this direction has been proved in [3] where a class of functions very similar to the one we study in section 2 has been introduced.

The paper is organized as follow. In section 1 we study the class $X^{\gamma,p}$ defined by means of the Gagliardo-type semi-norm

$$[f]_{X^{\gamma,p}} := \left(\int_{B_{1/3}}\int_{\mathbb{R}^d} \frac{|f(x+h)-f(x)|^p}{|h|^d}\frac{1}{\log(1/|h|)^{1-p\gamma}}\,\mathrm{d}x\,\mathrm{d}h\right)^{1/p}. \tag{0.2}$$

This class and more general ones were considered in the literature by many authors, see for instance [7, Definition 2.1], [8, Section 2] and [9, Section 3].

Leger in [10] was the first who considered the seminorm (0.2) (in the case $p = 2$, $\gamma = 1/2$ and $\gamma = 1$ written in the phase space, see subsection 1.1) to study regularity and mixing properties of solutions to the continuity equation. In [5] the authors of the present paper considered again the semi-norm (0.2) to prove new sharp regularity estimates.

In this first section we prove the sharp Sobolev embedding inequality for the space $X^{\gamma,p}$ (see Theorem 1.5), an approximation result in Lusin's sense and the interpolation inequality between spaces L^p, $X^{\gamma,p}$ and $W^{s,p}$ (compare with [8, Theorem 3.1]).

In section 2, we consider $N^{1,p}$ another class of functions defined à la Hajlasz (see for instance [11]). We give a characterization of this space in terms of a finiteness of a suitable discrete logarithmic Dirichlet energy (see (2.3)). We also study weak differentiability properties of functions in $N^{1,p}$ and we eventually establish a link with the first introduced space $X^{\gamma,p}$.

Notation. We denote by $B_r(x)$ the ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$, where $d \ge 1$. We often write B_r instead of $B_r(0)$. Let us set

$$\fint_E f\,\mathrm{d}x = \frac{1}{\mathscr{L}^d(E)}\int_E f\,\mathrm{d}x, \qquad \forall\ E \subset \mathbb{R}^d \text{ Borel set}$$

where $\mathscr{L}^d$ is the d-dimensional Lebesgue measure. Moreover

$$Mf(x) := \sup_{r>0}\fint_{B_r(x)} |f(y)|\,\mathrm{d}y, \qquad \forall\ x \in \mathbb{R}^d,$$

denotes the Hardy-Littlewood maximal function. We shall use the expression $a \lesssim_c b$ to mean that there exists a universal constant C depending only on c such that $a \le Cb$. The same convention is adopted for $\gtrsim_c$ and $\simeq_c$.

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1 The space $X^{\gamma,p}$

Let us define the first space.

Definition 1.1. Let $p \in (0,\infty)$ and $\gamma \in (0,\infty)$ be fixed. We define

$$[f]_{X^{\gamma,p}} := \left(\int_{B_{1/3}}\int_{\mathbb{R}^d} \frac{|f(x+h)-f(x)|^p}{|h|^d}\frac{1}{\log(1/|h|)^{1-p\gamma}}\,\mathrm{d}x\,\mathrm{d}h\right)^{1/p}, \tag{1.1}$$

and we set

$$X^{\gamma,p} := \left\{ f \in L^p(\mathbb{R}^d) :\ [f]_{X^{\gamma,p}} < \infty \right\}. \tag{1.2}$$

It is immediate to verify that $X^{\gamma,p}$ endowed with

$$\|f\|^p_{X^{\gamma,p}} := \|f\|^p_{L^p} + [f]^p_{X^{\gamma,p}},$$

is a Banach space for any $p \geq 1$ and that the semi-norm $[\,\cdot\,]_{X^{\gamma,p}}$ is lower semi-continuous with respect to the strong L^p topology.

Let us point out that $X^{\gamma,p}$ is a particular case of more general classes of functions considered in [7, Definition 2.1], [8, Section 2] and [9, Section 3].

Remark 1.2. Observe that the kernel

$$K_\gamma(h) := \mathbf{1}_{B_{1/3}} \frac{1}{|h|^d \log(1/|h|)^{1-p\gamma}},$$

appearing in (1.1), is singular in $\mathbb{R}^d$ if and only if $\gamma \geq 0$, as a simple computation shows

$$\int_{\mathbb{R}^d} K_\gamma(h)\,\mathrm{d}h = \int_{\log(3)}^{\infty} t^{p\gamma-1}\,\mathrm{d}t.$$

Therefore, the semi-norm (1.1) is not trivial only when $\gamma \geq 0$.

Let us briefly discuss the analogies between $X^{\gamma,p}$ and the Sobolev spaces of fractional order $W^{s,p}$ (see [12] for a reference on this topic). Let $p > 0$ and $s \in (0,1)$ be fixed, the space $W^{s,p}$ consists of functions $f \in L^p$ such that

$$[f]_{W^{s,p}} := \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x+h)-f(x)|^p}{|h|^{d+ps}}\,\mathrm{d}x\,\mathrm{d}h\right)^{1/p} < \infty, \tag{1.3}$$

and it is endowed with

$$\|f\|^p_{W^{s,p}} := \|f\|^p_{L^p} + [f]^p_{W^{p,s}}.$$

Understanding $X^{\gamma,p}$ as the space of functions in L^p with derivative of logarithmic order γ in L^p it is natural to expect the continuous inclusions $X^{\gamma,p} \subset X^{\gamma',p} \subset W^{s,p}$ when $0 \leq \gamma \leq \gamma'$ and $s \in (0,1)$. This is, indeed, the case as the following proposition says. Its proof is a simple exercise.

Proposition 1.3. *Let $p \in (0,\infty)$ be fixed. For any $0 \leq \gamma \leq \gamma'$ and $s \in (0,1)$ there holds*

$$[f]_{X^{\gamma,p}} \leq [f]_{X^{\gamma',p}}, \qquad \|f\|_{X^{\gamma,p}} \lesssim_{s,p,\gamma} \|f\|_{W^{s,p}}.$$

1.1 The case $p = 2$

In this section we characterize the space $X^{\gamma,2}$ by means of the Fourier transform.

Theorem 1.4. *Let $\gamma > 0$ be fixed. For every $f \in L^2(\mathbb{R}^d)$ it holds*

$$\|f\|^2_{X^{\gamma,2}} \simeq_{d,\gamma} \|f\|^2_{L^2} + \int_{|\xi|>1} \log(|\xi|)^{2\gamma} |\hat{f}(\xi)|^2\,\mathrm{d}\xi.$$

Proof. Using Plancherel's formula we get,

$$\int_{\mathbb{R}^d} |f(x+h)-f(x)|^2\,\mathrm{d}x = \int_{\mathbb{R}^d} |e^{ih\cdot\xi}-1|^2 |\hat{f}(\xi)|^2\,\mathrm{d}\xi = 2\int_{\mathbb{R}^d} (1-\cos(h\cdot\xi))\,|\hat{f}(\xi)|^2\,\mathrm{d}\xi,$$

for any $h \in \mathbb{R}^d$. Thus,

$$\int_{B_{1/2}}\int_{\mathbb{R}^d} \frac{|f(x+h)-f(x)|^2}{|h|^d \log(1/|h|)^{1-2\gamma}}\,\mathrm{d}x\,\mathrm{d}h = 2\int_{\mathbb{R}^d}\left[\int_{B_{1/2}} \frac{1-\cos(h\cdot\xi)}{|h|^d\log(1/|h|)^{1-2\gamma}}\,\mathrm{d}h\right]|\hat{f}(\xi)|^2\,\mathrm{d}\xi. \tag{1.4}$$

It is enough to show that

$$\int_{B_{1/2}} \frac{1-\cos(h\cdot\xi)}{|h|^d}\frac{1}{\log(1/|h|)^{1-2\gamma}}\,\mathrm{d}h \simeq_{d,\gamma} |\xi|^2, \qquad \text{for every } |\xi| \leq 10, \tag{1.5}$$

and

$$\int_{B_{1/2}} \frac{1-\cos(h\cdot\xi)}{|h|^d}\frac{1}{\log(1/|h|)^{1-2\gamma}}\,\mathrm{d}h \simeq_{d,\gamma} \log(|\xi|)^{2\gamma}, \qquad \text{for every } |\xi|>10. \tag{1.6}$$

In order to prove (1.5) we use the elementary inequality $1-\cos(a)\simeq a^2$ for any $a\in(-2,2)$ obtaining

$$\int_{B_{1/2}} \frac{1-\cos(h\cdot\xi)}{|h|^d}\frac{1}{\log(1/|h|)^{1-2\gamma}}\,\mathrm{d}h \simeq \int_{B_{1/2}} \frac{(h\cdot\xi)^2}{|h|^{d-1}\log(1/|h|)^{1-2\gamma}}\,\mathrm{d}h \simeq_{d,\gamma} |\xi|^2.$$

Let us now pass to the proof of (1.6). First observe that

$$\int_{B_{1/2}\setminus B_{1/(10|\xi|)}} \frac{1-\cos(h\cdot\xi)}{|h|^d\log(1/|h|)^{1-2\gamma}}\,\mathrm{d}h \le 2\int_{B_{1/2}\setminus B_{1/(10|\xi|)}} \frac{1}{|h|^d\log(1/|h|)^{1-2\gamma}}\,\mathrm{d}h \lesssim_d \log(|\xi|)^{2\gamma},$$

and

$$\begin{aligned}\int_{B_{1/(10|\xi|)}} \frac{1-\cos(h\cdot\xi)}{|h|^d\log(1/|h|)^{1-2\gamma}}\,\mathrm{d}h &\lesssim \int_{B_{1/(10|\xi|)}} \frac{(h\cdot\xi)^2}{|h|^d\log(1/|h|)^{1-2\gamma}}\,\mathrm{d}h\\ &\lesssim |\xi|^2\int_{B_{1/(10|\xi|)}} \frac{1}{|h|^{d-2}\log(10|\xi|)^{1-2\gamma}}\,\mathrm{d}h\\ &\lesssim_{d,\gamma} \log(10|\xi|)^{2\gamma-1} \lesssim_d \log(|\xi|)^{2\gamma},\end{aligned}$$

where we used the elementary inequality $1-\cos(a)\lesssim a^2$ for any $a\in\mathbb{R}$ and the assumption $|\xi|>10$.

Let us now show the converse inequality. Using the Coarea formula we can write

$$\int_{B_{1/2}} \frac{1-\cos(h\cdot\xi)}{|h|^d\log(1/|h|)^{1-2\gamma}}\,\mathrm{d}h = \int_0^{1/2}\int_{S^{d-1}} \frac{1-\cos(|\xi|r\theta_1)}{r|\log(r)|^{1-2\gamma}}\,\mathrm{d}\mathcal{H}^{d-1}(\theta)\,\mathrm{d}r,$$

where $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure. It is elementary to see that, for every $\xi\in\mathbb{R}^d$ and $r>0$ satisfying $r|\xi|\ge 1$, it holds

$$\int_{S^{d-1}} (1-\cos(|\xi|r\theta_1))\,\mathrm{d}\mathcal{H}^{d-1}(\theta) \gtrsim_d 1,$$

thus,

$$\begin{aligned}\int_{B_{1/2}} \frac{1-\cos(h\cdot\xi)}{|h|^d}\frac{1}{\log(1/|h|)^{1-2\gamma}}\,\mathrm{d}h &\ge \int_{1/|\xi|}^{1/2}\int_{S^{d-1}} \frac{1-\cos(|\xi|r\theta_1)}{r|\log(r)|^{1-2\gamma}}\,\mathrm{d}\mathcal{H}^{d-1}(\theta)\,\mathrm{d}r\\ &\gtrsim_d \int_{1/|\xi|}^{1/2} \frac{1}{r|\log(r)|^{1-2\gamma}}\,\mathrm{d}r \gtrsim_{d,\gamma} \log(|\xi|)^{1-\gamma}.\end{aligned}$$

This gives (1.6). The proof is complete. □

1.2 Sobolev embedding and interpolation inequality

The first result of this section is a Sobolev embedding theorem for $X^{\gamma,p}$:

Theorem 1.5. *Let $p>0$ and $\gamma>0$ be fixed. For any $f\in L^p(\mathbb{R}^d)$ it holds*

$$|f(x)|^p\log\left(\frac{|f(x)|}{\|f\|_{L^p(\mathbb{R}^d)}}+2\right)^{p\gamma} \lesssim_{p,\gamma} |f(x)|^p + \int_{B_{1/3}} \frac{|f(x+h)-f(x)|^p}{|h|^d}\frac{1}{\log(1/|h|)^{1-p\gamma}}\,\mathrm{d}h, \tag{1.7}$$

for $\mathscr{L}^d$-a.e. every $x\in\mathbb{R}^d$.

In particular the following log-Sobolev inequality holds true for any $f\in X^{\gamma,p}$,

$$\int_{\mathbb{R}^d} |f(x)|^p\log\left(\frac{|f(x)|}{\|f\|_{L^p(\mathbb{R}^d)}}+2\right)^{p\gamma}\mathrm{d}x \lesssim_{p,\gamma} \|f\|^p_{X^{\gamma,p}}. \tag{1.8}$$

Let us point out that the inequality (1.8) can be obtained as a consequence of the embedding theorems studied in [7]. Nevertheless, our strategy is completely different and based on (1.7), that, as far as we know, is a new result. It is also worth mentioning that our proof of Theorem 1.5 is really elementary and short.

Proof. We may assume without loss of generality that $\|f\|_{L^p} = 1$.

Clearly it is enough to show

$$|f(x)|^p \log(|f(x)|)^{p\gamma} \lesssim_{p,\gamma} |f(x)|^p + \int_{B_{1/3}} \frac{|f(x+h) - f(x)|^p}{|h|^d} \frac{1}{\log(1/|h|)^{1-p\gamma}} \,\mathrm{d}h, \tag{1.9}$$

for $\mathscr{L}^d$-a.e. $x \in \{ z \in \mathbb{R}^d : |f(z)| > C_{p,\gamma} \}$, where $C_{p,\gamma} \geq 1$ is a fixed constant depending only on p and γ.

For any $t \in (0, 1/6)$, using the assumption $\|f\|_{L^p} = 1$ we get

$$\begin{aligned} |f(x)|^p &= \fint_{t^{1/d}<|h|<(2t)^{1/d}} |f(x)|^p \,\mathrm{d}h \\ &\lesssim_p \fint_{t^{1/d}<|h|<(2t)^{1/d}} |f(x+h)|^p \,\mathrm{d}h + \fint_{t^{1/d}<|h|<(2t)^{1/d}} |f(x+h) - f(x)|^p \,\mathrm{d}h \\ &\lesssim_p \frac{1}{t} + \frac{1}{t} \int_{t^{1/d}<|h|<(2t)^{1/d}} |f(x+h) - f(x)|^p \,\mathrm{d}h. \end{aligned}$$

Here the constant does not depend on d since

$$\mathscr{L}^d(B_{2^{1/d}}) - \mathscr{L}^d(B_1) \geq \mathscr{L}^d(B_1) \geq \mathscr{L}^d([0,1]^d) = 1.$$

Let us now fix $0 < \lambda < 1/6$. We integrate the inequality above against $\frac{1}{t\log(1/t)^{1-p\gamma}}$ with respect to $t \in (\lambda, 1/6)$ obtaining

$$\begin{aligned} &|f(x)|^p \frac{1}{p\gamma} \left(\log(1/\lambda)^{p\gamma} - \log(6)^{p\gamma}\right) \\ &\quad = |f(x)|^p \int_\lambda^{1/6} \frac{1}{t\log(1/t)^{1-p\gamma}} \,\mathrm{d}t \\ &\quad \lesssim_p \int_\lambda^{1/2} \frac{1}{t^2 \log(1/t)^{1-p\gamma}} \,\mathrm{d}t + \int_\lambda^{1/6} \int_{t^{1/d}<|h|<(2t)^{1/d}} \frac{|f(x+h) - f(x)|^p}{t|\log(t)|^{1-p\gamma}} \,\mathrm{d}h \frac{\mathrm{d}t}{t} \\ &\quad \lesssim_p \lambda^{-1} \log(1/\lambda)^{p\gamma} + \int_{B_{1/3}} \frac{|f(x+h) - f(x)|^p}{|h|^d} \frac{1}{\log(1/|h|)^{1-p\gamma}} \,\mathrm{d}h, \end{aligned}$$

so, rearranging the terms, we end up with

$$(|f(x)|^p - C_{p,\gamma}\lambda^{-1}) \log(1/\lambda)^{p\gamma} \lesssim_{p,\gamma} |f(x)|^p + \int_{B_{1/3}} \frac{|f(x+h) - f(x)|^p}{|h|^d} \frac{1}{\log(1/|h|)^{1-p\gamma}} \,\mathrm{d}h,$$

for any $0 < \lambda < 1/6$ and for $\mathscr{L}^d$ a.e. $x \in \mathbb{R}^d$. Eventually we can choose $\lambda = 2C_{p,\gamma}/|f(x)|^p$ when $x \in \{ z \in \mathbb{R}^d : |f(z)|^p > 12C_{p,\gamma} \}$ and (1.9) immediately follows. The proof is complete. □

The just explained strategy can be used also to obtain a very short proof of the fractional Sobolev embedding theorem. In the fractional context, a very similar argument already appears in the literature (see [13, pag. 241]).

Proposition 1.6. *Let us fix $s \in (0,1)$ and $p \in (0, d/s)$. We set $p^* := \frac{dp}{d-sp}$. For any $f \in W^{s,p}(\mathbb{R}^d)$ the following point-wise inequality holds true*

$$|f(x)|^{p^*} \lesssim_{p,d,s} \|f\|_{L^{p^*}}^{p^*-p} \int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^p}{|x-y|^{d+ps}} \,\mathrm{d}y, \qquad \textit{for } \mathscr{L}^d\textit{-a.e. } x \in \mathbb{R}^d. \tag{1.10}$$

In particular we deduce the well-known Sobolev inequality

$$\|f\|_{L^{p^*}} \lesssim_{p,d,s} [f]_{W^{s,p}}. \tag{1.11}$$

Proof. For any $x \in \mathbb{R}^d$ and $t > 0$ we can write

$$\begin{aligned} |f(x)|^p &= \fint_{B_t(0)} |f(x)|^p \, \mathrm{d}h \\ &\lesssim_p \fint_{B_t(0)} |f(x+h) - f(x)|^p \, \mathrm{d}h + \fint_{B_t(x)} |f(h)|^p \, \mathrm{d}h. \end{aligned}$$

By means of Hölder's inequality we get

$$\fint_{B_t(x)} |f(h)|^p \, \mathrm{d}h \leq C_d \, \|f\|_{L^{p*}}^p \, \frac{1}{t^{d-sp}},$$

and we end up with

$$\begin{aligned} |f(x)|^p \frac{1}{t^{sp}} - C_d \, \|f\|_{L^{p*}}^p \, \frac{1}{t^d} &\lesssim_{p,d} \fint_{B_t(0)} |f(x+h) - f(x)|^p \frac{1}{t^{sp}} \, \mathrm{d}h \\ &\lesssim_{p,d} \int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^p}{|x-y|^{d+ps}} \, \mathrm{d}y. \end{aligned}$$

Choosing t such that $t^{d-sp} = 2C_d \left(\frac{\|f\|_{L^{p*}}}{|f(x)|} \right)^p$ we get (1.10). Integrating (1.10) with respect x over $\mathbb{R}^d$ we get (1.11). □

Remark 1.7. The estimate (1.7) could be improved in the following way: for any $\gamma > 0$, $p > 0$ and every $f \in L^{p,\infty}(\mathbb{R}^d)$ it holds

$$|f(x)|^p \log \left(\frac{|f(x)|}{\|f\|_{L^{p,\infty}}} + 2 \right)^{p\gamma} \lesssim_{p,\gamma} |f(x)|^p + \int_{B_{1/3}} \frac{|f(x+h) - f(x)|^p}{|h|^d} \frac{1}{\log(1/|h|)^{1-p\gamma}} \, \mathrm{d}h, \qquad (1.12)$$

for $\mathscr{L}^d$-a.e. every $x \in \mathbb{R}^d$.

Let us explain how to modify the proof of (1.7) to get (1.12). We first assume $p > 1$ and we use the inequality

$$\int_E |f(x)| \, \mathrm{d}x \lesssim_p \mathscr{L}^d(E)^{1-1/p} \, \|f\|_{L^{p,\infty}}, \qquad \forall E \subset \mathbb{R}^d \text{ Borel},$$

obtaining

$$\begin{aligned} |f(x)| &\leq \fint_{t^{1/d} < |h| < (2t)^{1/d}} |f(x+h)| \, \mathrm{d}h + \fint_{t^{1/d} < |h| < (2t)^{1/d}} |f(x+h) - f(x)| \, \mathrm{d}h \\ &\lesssim_p \left(\frac{1}{t} \, \|f\|_{L^{p,\infty}} \right)^{1/p} + \left(\frac{1}{t} \int_{t^{1/d} < |h| < (2t)^{1/d}} |f(x+h) - f(x)|^p \, \mathrm{d}h \right)^{1/p}. \end{aligned}$$

The proof of (1.12) is achieved arguing exactly as in the proof of (1.7).

We finally extend (1.12) to every $p \in (0, 1]$ by means of the elementary inequality

$$|a^\alpha - b^\alpha| \lesssim_\alpha |a - b|^\alpha,$$

for any $a, b > 0$ and $\alpha \in (0, 1]$, see [14, Theorem 2.4].

Remark 1.8. It is worth remarking that (1.8) and (1.7) are dimension free, namely the constant in front of the right hand side does not depends on d.

Let us now show that (1.8) (and thus (1.7)) is sharp. We prove that

$$[\mathbf{1}_{B_r(0)}]_{X^{\gamma,p}} \simeq_{p,\gamma} r^d \log(1/r)^{p\gamma}, \qquad \text{for any } 0 < r < 1/6. \qquad (1.13)$$

Thus the function $\mathbf{1}_{B_r}$ saturates (1.8). It is enough to show the inequality $\lesssim$ in (1.13), since the converse is guaranteed by (1.8).

Observe that

$$[\mathbf{1}_{B_r(0)}]_{X^{\gamma,p}} = 2\int_{B_r(0)}\int_{\mathbb{R}^d\setminus B_r(0)} \mathbf{1}_{|x-y|<1/3}\frac{1}{|x-y|^d}\frac{1}{\log(1/|x-y|)^{1-p\gamma}}\,\mathrm{d}y\,\mathrm{d}x$$
$$= 2\int_{B_r(0)}\int_{B_{1/3}(0)\setminus B_r(x)} \frac{1}{|y|^d}\frac{1}{\log(1/|y|)^{1-p\gamma}}\,\mathrm{d}y\,\mathrm{d}x.$$

Take $x \in B_r(0)$, exploiting the inclusion $B(0, r-|x|) \subset B(x,r)$ we deduce

$$\int_{B_{1/3}(0)\setminus B_r(x)} \frac{1}{|y|^d}\frac{1}{\log(1/|y|)^{1-p\gamma}}\,\mathrm{d}y \le \int_{B_{1/3}(0)\setminus B_{r-|x|}(0)} \frac{1}{|y|^d}\frac{1}{\log(1/|y|)^{1-p\gamma}}\,\mathrm{d}y$$
$$\simeq_{p,\gamma} \log(1/(r-|x|))^{p\gamma},$$

thus

$$[\mathbf{1}_{B_r(0)}]_{X^{\gamma,p}} \lesssim_{p,\gamma} \int_{B_r(0)} \log(1/(r-|x|))^{p\gamma}\,\mathrm{d}x \simeq_{d,p,\gamma} \int_0^r \varepsilon^{d-1}\log(1/\varepsilon)^{p\gamma}\,\mathrm{d}\varepsilon \simeq_{d,p,\gamma} r^d \log(1/r)^{p\gamma}.$$

The function $\mathbf{1}_{B_r(0)}$ is a very natural candidate to show the sharpness of (1.8), since, in general, Sobolev embeddings are related to the isoperimetric problem and balls are minimizers, at least in the classical context. However, it is worth to mention that a notion of logarithmic perimeter of order $\gamma > 0$ associated to a set can be obtained writing

$$P^L_\gamma(E) := [\mathbf{1}_E]_{X^{\gamma,1}}.$$

We expect that balls are the only minimizers of P^L_γ as it happens in the classical and fractional case (see for instance [15]), but we do not investigate this problem here.

The last result we present in this section is an interpolation inequality between spaces $L^p(\mathbb{R}^d)$, $X^{\gamma,p}$ and $W^{s,p}$. Even though this result can be deduced from [8, Theorem 3.1] we prefer to present a very simple and direct proof.

Proposition 1.9. *Let $p > 0$, $s \in (0,1)$ and $\gamma > 0$ be fixed. For any $f \in L^p(\mathbb{R}^d)$ we have*

$$[f]_{X^{\gamma,p}} \lesssim_{p,s,\gamma} \|f\|_{L^p} \log\left(2 + \frac{\|f\|_{W^{s,p}}}{\|f\|_{L^p}}\right)^\gamma. \tag{1.14}$$

Proof. Assume without loss of generality $\|f\|_{L^p} = 1$. Let $\lambda \in (0, 1/3)$ be fixed, we have

$$[f]^p_{X^{\gamma,p}} = \int_{B_\lambda(0)}\int_{\mathbb{R}^d} \frac{|f(x+h)-f(x)|^p}{|h|^d\log(1/|h|)^{1-p\gamma}}\,\mathrm{d}x\,\mathrm{d}h + \int_{B_{1/3}(0)\setminus B_\lambda(0)}\int_{\mathbb{R}^d} \frac{|f(x+h)-f(x)|^p}{|h|^d\log(1/|h|)^{1-p\gamma}}\,\mathrm{d}x\,\mathrm{d}h$$
$$\le \frac{\lambda^{ps}}{\log(1/\lambda)^{1-p\gamma}}[f]^p_{W^{s,p}} + 2^p\|f\|^p_{L^p}\int_{B_{1/3}(0)\setminus B_\lambda(0)} \frac{1}{|h|^d\log(1/|h|)^{1-p\gamma}}\,\mathrm{d}h$$
$$\lesssim_{d,p,\gamma} \frac{\lambda^{ps}}{\log(1/\lambda)^{1-p\gamma}}[f]^p_{W^{s,p}} + \frac{1}{\log(1/\lambda)^{-p\gamma}} \lesssim_{p,\gamma} \log(1/\lambda)^{p\gamma}(\lambda^{ps}[f]^p_{W^{s,p}} + 1).$$

When $[f]_{W^{s,p}} \ge 3^s$ we can plug $\lambda = [f]^{-1/s}_{W^{s,p}}$ to the previous expression, otherwise we set $\lambda = 1/3$, obtaining

$$[f]_{L^p} \lesssim_{p,s,\gamma} \log\left(2 + \|f\|_{W^{s,p}}\right)^\gamma,$$

that implies our conclusion. □

Remark 1.10. In the particular case $p = 2$ the just stated result (1.14) could be achieved using the Jensen inequality and the characterization of $X^{\gamma,2}$ by means of Fourier transform (see Theorem 1.4).

1.3 Lusin's estimate

It is well-known that a quantitative Lusin's approximation property (see [16], [18]) characterizes Sobolev spaces, even in the very abstract setting of measure metric spaces (see for instance [19]). In this section we study this approximation property for $X^{\gamma,p}$ functions.

For any $f \in X^{\gamma,p}$ we define

$$L_{\gamma,p}f(x) := \left(\int_{B_{1/3}} \frac{|f(x+h)-f(x)|^p}{|h|^d} \frac{1}{\log(1/|h|)^{1-p\gamma}} \, \mathrm{d}h \right)^{1/p} \quad \forall\, x \in \mathbb{R}^d,$$

it follows that $L_{\gamma,p}f \in L^p$ with $\|f\|_{L^p} = [f]_{X^{\gamma,p}}$. The main result of the section is the following.

Theorem 1.11. *Let $p > 0$ and $\gamma > 0$ be fixed. For any $f \in X^{\gamma,p}$ it holds*

$$|f(x) - f(y)| \lesssim_{d,p,\gamma} \log(1/|x-y|)^{-\gamma} \left(L_{\gamma,p}f(x) + L_{\gamma,p}f(y) \right), \tag{1.15}$$

for any $x, y \in \mathbb{R}^d$ such that $|x-y| < \frac{1}{36}$.

Let us begin with a simple lemma.

Lemma 1.12. *Let $p > 0$, $x, y \in \mathbb{R}^d$ be fixed. For any $f \in L^p$ it holds*

$$|f(x)-f(y)|^p \lesssim_{d,p} \fint_{B_{3r}(0)\setminus B_r(0)} |f(x+h)-f(x)|^p \, \mathrm{d}h + \fint_{B_{3r}(0)\setminus B_r(0)} |f(y+h)-f(y)|^p \, \mathrm{d}h, \tag{1.16}$$

for any $r \geq 2|x-y|$.

Proof. Let us estimate

$$\begin{aligned} |f(x)-f(y)|^p &= \fint_{B_{5r/2}(0)\setminus B_{3r/2}(0)} |f(x)-f(y)|^p \, \mathrm{d}z \\ &\lesssim_p \fint_{B_{5r/2}(x)\setminus B_{3r/2}(x)} |f(x)-f(z)|^p \, \mathrm{d}z + \fint_{B_{5r/2}(x)\setminus B_{3r/2}(x)} |f(z)-f(y)|^p \, \mathrm{d}z \\ &\lesssim_d \fint_{B_{3r}(0)\setminus B_r(0)} |f(x+h)-f(x)|^p \, \mathrm{d}h + \fint_{B_{5r/2}(x)\setminus B_{3r/2}(x)} |f(z)-f(y)|^p \, \mathrm{d}z. \end{aligned}$$

Observe that $B_{5r/2}(x) \setminus B_{3r/2}(x) \subset B_{3r}(y) \setminus B_r(y)$ for any $r \geq 2|x-y|$, thus

$$\fint_{B_{5r/2}(x)\setminus B_{3r/2}(x)} |f(z)-f(y)|^p \, \mathrm{d}z \lesssim_d \fint_{B_{3r}(0)\setminus B_r(0)} |f(y+h)-f(y)|^p \, \mathrm{d}h,$$

the proof is complete. □

We are now ready to prove Theorem 1.11.

Proof of Theorem 1.11. We integrate both sides of (1.16) with respect to $r \in (1/3, 2|x-y|)$ against $\frac{1}{r\log(1/r)^{1-p\gamma}}$ getting

$$\begin{aligned} |f(x)-f(y)|^p \int_{2|x-y|}^{1/3} \frac{1}{r\log(1/r)^{1-p\gamma}} \, \mathrm{d}r \qquad & \\ \lesssim_{d,p} \int_{2|x-y|}^{1/3} \fint_{B_{3r}(0)\setminus B_r(0)} |f(x+h)-f(x)|^p \, \mathrm{d}h \frac{\mathrm{d}r}{r\log(1/r)^{1-p\gamma}} & \\ + \int_{2|x-y|}^{1/3} \fint_{B_{3r}(0)\setminus B_r(0)} |f(y+h)-f(y)|^p \, \mathrm{d}h \frac{\mathrm{d}r}{r\log(1/r)^{1-p\gamma}}. & \end{aligned}$$

Observe that

$$\int_{2|x-y|}^{1/3} \fint_{B_{3r}(0)\setminus B_r(0)} |f(x+h)-f(x)|^p \, \mathrm{d}h \frac{\mathrm{d}r}{r\log(1/r)^{1-p\gamma}} \lesssim_{d,p,\gamma} (L_{\gamma,p}f(x))^p,$$

and that

$$\int_{2|x-y|}^{1/3} \frac{1}{r\log(1/r)^{1-p\gamma}}\,\mathrm{d}r = \frac{1}{p\gamma}\left(\log\left(\frac{1}{2|x-y|}\right)^{p\gamma} - \log(3)^{p\gamma}\right)$$
$$\gtrsim_{p\gamma} \log\left(\frac{1}{6|x-y|}\right)^{p\gamma} \geq 2^{p\gamma}\log\left(\frac{1}{|x-y|}\right)^{p\gamma},$$

where in the last step we used $|x-y| \leq \frac{1}{36}$. The proof is complete. □

The just described strategy leads to a very simple proof the standard Lusin approximation result for $W^{s,p}$ functions with Hölder functions.

Proposition 1.13. *Let $p \geq 1$ and $s \in (0,1)$ be fixed. For any $f \in W^{s,p}$ it holds*

$$|f(x) - f(y)| \lesssim_{d,s,p} |x-y|^s (D_{s,p}f(x) + D_{s,p}f(y)), \qquad \forall x, y \in \mathbb{R}^d,$$

where

$$D_{s,p}f(x) := \left(\int_{\mathbb{R}^d} \frac{|f(x+h) - f(x)|^p}{|h|^{d+ps}}\right)^{1/p} \in L^p(\mathbb{R}^d).$$

Proof. As in the proof of Theorem 1.11, we integrate both sides of (1.16) with respect to r against a suitable kernel $K(r)$, in this case we should consider $K(r) = \frac{\mathbf{1}_{r>2|x-y|}}{r^{d+ps}}$. □

Let us finally prove a partial converse of Theorem 1.11.

Proposition 1.14. *Let $p > 0$ and $\gamma > 0$ be fixed. Let $f \in L^p(\mathbb{R}^d)$ satisfy*

$$|f(x) - f(y)| \leq \log(1/|x-y|)^{-\gamma}(g(x) + g(y)), \qquad \forall x, y \in \mathbb{R}^d, \tag{1.17}$$

for some $g \in L^p(\mathbb{R}^d)$. Then $f \in X^{\alpha,p}$ for any $\alpha < \gamma$ with estimate

$$[f]^p_{X^{\alpha,p}} \lesssim_d \frac{1}{p(\gamma - \alpha)} \|g\|^p_{L^p}.$$

Proof. Let us fix $0 < \alpha < \gamma$, we estimate

$$[f]^p_{X^{\alpha,p}} = \int_{B_{1/3}} \frac{1}{|h|^d \log(1/|h|)^{1-p\alpha}} \int_0^\infty p\lambda^{p-1}\mathscr{L}^d(\{\, x : \ |\, f(x+h) - f(x)| > \lambda \,\})\,\mathrm{d}\lambda\,\mathrm{d}h$$
$$\leq \int_{B_{1/3}} \frac{1}{|h|^d \log(1/|h|)^{1-p\alpha}} \int_0^\infty p\lambda^{p-1}\mathscr{L}^d(\{\, x : \ g(x) + g(x+h) > \lambda\log(1/\,|\,h|)^\gamma \,\})\,\mathrm{d}\lambda\,\mathrm{d}h,$$

changing variables according to $\lambda\log(1/|h|)^\gamma = t$ we get

$$[f]_{X^{\alpha,p}} \leq \int_{B_{1/3}} \frac{1}{|h|^d \log(1/|h|)^{1-p(\alpha-\gamma)}} \int_0^\infty pt^{p-1}\mathscr{L}^d(\{\, x : \ g(x) + g(x+h) > t \,\})\,\mathrm{d}t\,\mathrm{d}h$$
$$\lesssim \int_{B_{1/3}} \frac{1}{|h|^d \log(1/|h|)^{1-p(\alpha-\gamma)}}\,\mathrm{d}h\, \|g\|^p_{L^p} \simeq_d \frac{1}{p(\gamma-\alpha)} \|g\|^p_{L^p}.$$

The proof is complete. □

2 The space $N^{s,p}$

The aim of this section is to study another class of functions with derivative of logarithmic order. It comes up naturally in the study of regularity for Lagrangian flows associated to Sobolev vector fields, see [3].

Definition 2.1. Let $p \geq 1$ and $s \in (0,1]$ be fixed. We say that $f \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ belongs to $N^{s,p}$ if there exists a positive function $g \in L^p(\mathbb{R}^d)$ such that

$$|f(x) - f(y)| \leq |x-y|^s \left(\exp\{g(x) + g(y)\} - 1\right) \qquad \forall x, y \in \mathbb{R}^d. \tag{2.1}$$

We set $[f]_{N^p} := \inf\{\|g\|_{L^p}\}$, where the infimum runs over all possible g satisfying (2.1).

$[\,\cdot\,]_{N^{1,p}}$ in general is not a semi-norm. It satisfies the triangle inequality and

$$[\lambda f]_{N^{s,p}} \leq |\lambda| [f]_{N^{s,p}}, \tag{2.2}$$

as it can be seen using the elementary identity $e^t - 1 = \sum_{n\geq 1} \frac{t^n}{n!}$, for any $t \in \mathbb{R}$. But in general in (2.2) the inequality is strict.

For any $f \in L^p_{\text{loc}}(\mathbb{R}^d)$ let us define the functional

$$\Phi_s f(x) := \sup_{r>0} \fint_{B_r(x)} \log\left(1 + \frac{|f(x) - f(y)|}{r^s}\right) \mathrm{d}y. \tag{2.3}$$

Roughly speaking it can be seen as a discrete fractional logarithmic Dirichlet's energy. The aim of the next proposition is to link the condition (2.1) to integrability properties of the function $\Phi^*_s f$.

Proposition 2.2. *Let $p > 1$ and $f \in L^p_{\text{loc}}(\mathbb{R}^d)$ be fixed. Then, $f \in N^{s,p}$ if and only if $\Phi_s f \in L^p(\mathbb{R}^d)$ and it holds*

$$[f]_{N^{s,p}} \simeq_{d,p} \|\Phi_s f\|_{L^p}. \tag{2.4}$$

Proof. For any $f \in N^{s,p}$ it is immediate to see that $\Phi_s f \leq 2Mg$, where M is the Hardy-Littlewood maximal function. Thus we get

$$\|\Phi_s f\|_{L^p} \lesssim_{d,p} \|g\|_{L^p}, \tag{2.5}$$

thanks to the boundedness of M in L^p (see [20]). In order to achieve the proof of (2.4) it remains to show the converse of (2.5).

For any $x, y \in \mathbb{R}^d$ let us set $r = |x - y|$, we get

$$\begin{aligned}
\log\left(1 + \frac{|f(x) - f(y)|}{r^s}\right) &= \fint_{B_r(x)} \log\left(1 + \frac{|f(x) - f(y)|}{r^s}\right) \mathrm{d}z \\
&\leq \fint_{B_r(x)} \log\left(1 + \frac{|f(x) - f(z)|}{r^s}\right) \mathrm{d}z + \fint_{B_r(x)} \log\left(1 + \frac{|f(z) - f(y)|}{r^s}\right) \mathrm{d}z \\
&\leq \Phi_s f(x) + \fint_{B_r(x)} \log\left(1 + \frac{|f(z) - f(y)|}{r^s}\right) \mathrm{d}z,
\end{aligned}$$

to estimate the last term it is enough to observe that $B(x, r) \subset B(y, 2r)$, obtaining

$$\fint_{B_r(x)} \log\left(1 + \frac{|f(z) - f(y)|}{r^s}\right) \mathrm{d}z \lesssim_d \Phi_s f(y).$$

Thus, we end up with

$$\log\left(1 + \frac{|f(x) - f(y)|}{|x - y|^s}\right) \lesssim_d \Phi_s f(x) + \Phi_s f(y),$$

that implies $[f]_{N^{s,p}} \lesssim_d \|\Phi_s f(y)\|_{L^p}$, and thus (2.4). The proof is complete. $\square$

Let us point out that the implication $\Phi_s f \in L^p(\mathbb{R}^d) \implies f \in N^{s,p}$ in the case $s = 1$ was used in [3] to prove a regularity result for Lagrangian flows. Two remarks are in order.

Remark 2.3. The assumption $p > 1$ in Proposition 2.2 plays a role only in the implication $f \in N^{s,p} \implies \Phi_s f \in L^p(\mathbb{R}^d)$. Indeed in the case $p = 1$ only the weaker implication $f \in N^{s,1} \implies \Phi_s f \in L^{1,\infty}(\mathbb{R}^d)$ is available.

Remark 2.4. For any $f \in L^p(\mathbb{R}^d)$ and $q \geq 1$ let us consider the functional

$$\Phi_{s,q} f(x) := \sup_{r>0} \fint_{B_r(x)} \log\left(1 + \frac{|f(x) - f(y)|}{r^s}\right)^q \mathrm{d}y. \tag{2.6}$$

It is easily seen that

$$\|\Phi_{s,q} f\|_{L^{p/q}} \lesssim_{d,p} \|g\|^q_{L^p}, \qquad \text{when } 1 \leq q < p,$$

and

$$\|\Phi_{s,p} f\|_{L^{1,\infty}} \lesssim_{d,p} \|g\|^p_{L^p}.$$

Let us now recall the definition of weak differentiability.

Definition 2.5. We say that a measurable function $f:\mathbb{R}^d\to\mathbb{R}$ is *weakly differentiable* at $x\in\mathbb{R}^d$ if there exists a linear map $L:\mathbb{R}^d\to\mathbb{R}$ such that the sequence

$$\lim_{r\to 0}\frac{f(x+ry)-f(x)-L(ry)}{r|y|}\to 0,\qquad \text{locally in measure,}$$

more precisely

$$\lim_{r\to 0}\int_{B_R}\left|\frac{f(x+ry)-f(x)-L(ry)}{r|y|}\right|\wedge 1\,\mathrm{d}y=0,$$

for any $0<R<\infty$. We set $\nabla f(x)\cdot y:=L(y)$.

It is well-known that a function f is weakly differentiable at $\mathscr{L}^d$-a.e. $x\in\mathbb{R}^d$ if and only if it can be approximated with Lipschitz functions in Lusin's sense. Namely, for any $\varepsilon>0$ there exists a Lipschitz function $g:\mathbb{R}^d\to\mathbb{R}$ such that $\mathscr{L}^d(\{f\neq g\})<\varepsilon$. See [21] for a good reference on this topic.

The aim of our next proposition is to study, in a quantitative manner, weakly differentiability properties of functions $f\in N^{1,p}$. Precisely we have the following.

Proposition 2.6. *Every $f\in N^{1,p}$ is weakly differentiable at $\mathscr{L}^d$-a.e. point. Denoting by ∇f its weak differential we have the following*

(i) $\int_{\mathbb{R}^d}\log\left(1+|\nabla f|\right)^p\mathrm{d}x\lesssim [f]^p_{N^{1,p}}$*;*

(ii) for $\mathscr{L}^d$-a.e. $x\in\mathbb{R}^d$ there holds

$$\lim_{r\to 0}\fint_{B_r(0)}\log\left(1+\frac{|f(x+y)-f(x)-\nabla f(x)\cdot y|}{|y|}\right)^p\mathrm{d}y=0.$$

Proof. It is straightforward to see that f is weakly differentiable (see discussion below). For any constant $M>0$ we have

$$\fint_{B_1(0)}\log\left(1+\frac{|f(x+ry)-f(x)|}{r|y|}\wedge M\right)\mathrm{d}y\lesssim \Phi_1^*f(x),$$

recalling that $y\to\frac{f(x+ry)-f(x)}{r|y|}$ converges locally in measure to $\nabla f(x)\cdot\frac{y}{|y|}$ for $\mathscr{L}^d$-a.e. $x\in\mathbb{R}^d$ we deduce

$$\fint_{B_1(0)}\log\left(1+\left|\nabla f(x)\cdot\frac{y}{|y|}\right|\wedge M\right)\mathrm{d}y=\lim_{r\to 0}\fint_{B_1(0)}\log\left(1+\frac{|f(x+ry)-f(x)|}{r|y|}\wedge M\right)\mathrm{d}y\lesssim\Phi_1^*f(x), \tag{2.7}$$

for $\mathscr{L}^d$-a.e. $x\in\mathbb{R}^d$. It is immediate to deduce (i) from (2.7) and Proposition 2.2. Let us pass to the proof of (ii). First of all let us consider an increasing convex function Ψ such that

$$\lim_{t\to\infty}\frac{\Psi(t)}{t}=\infty,$$

and

$$\int_{\mathbb{R}^d}\Psi(g(x)^p)\,\mathrm{d}x<\infty,$$

it exists thanks to the Dunford-Pettis lemma, see [18]. We can also assume that $t\mapsto\frac{\Psi(t)}{t}$ is increasing. Setting

$$f_r(y)=\frac{|f(x+ry)-f(x)-\nabla f(x)\cdot ry|}{r|y|},$$

and using the very definition of $N^{1,p}$ we get

$$\begin{aligned}\sup_{r>0}\int_{B_1(0)}\Psi(\log(1+f_r(y))^p)\,\mathrm{d}y&\lesssim_\Psi\int_{B_1(0)}\Psi((g(x+ry)+g(x))^p)\,\mathrm{d}y+\Psi(\log(1+|\nabla f(x)|)^p)\\&\lesssim_\Psi M\left(\Psi(g^p)\right)(x)+\Psi(\log(1+|\nabla f(x)|)^p),\end{aligned}$$

thus it is finite for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$ thanks to the (1,1) weak estimate for the maximal function. Therefore, it is enough to prove that (ii) holds for every $x \in \mathbb{R}^d$ such that

$$\sup_{r>0} \int_{B_1(0)} \Psi(\log(1+f_r(y))^p)\, \mathrm{d}y =: T < \infty. \tag{2.8}$$

Let us fix a parameter $0 < \lambda < 1/2$, using the Jensen inequality we get

$$\begin{aligned}
&\fint_{B_r(0)} \log\left(1 + \frac{|f(x+y) - f(x) - \nabla f(x)\cdot y|}{|y|}\right)^p \mathrm{d}y \\
&= \fint_{B_1(0)} \log(1+f_r(y))^p\, \mathrm{d}y \\
&\simeq_d \int_{B_1 \cap \{ f_r > \lambda^{-1} \}} \log(1+f_r(y))^p\, \mathrm{d}y + \int_{B_1 \cap \{ \lambda < f_r < \lambda^{-1} \}} \log(1+f_r(y))^p\, \mathrm{d}y \\
&+ \int_{B_1 \cap \{ f_r \le \lambda \}} \log(1+f_r(y))^p\, \mathrm{d}y \\
&\lesssim_d \frac{\log(1+\lambda^{-1})^p}{\Psi\left(\log(1+\lambda^{-1})^p\right)} T + \log(1+\lambda^{-1})^p \mathscr{L}^d(B_1 \cap \{f_r > \lambda\}) + \log(1+\lambda)^p.
\end{aligned}$$

Taking first the limit for $r \to 0$ and after $\lambda \to 0$ we get the sought conclusion since $f_r \to 0$ locally in measure. $\square$

Remark 2.7. It is natural to wonder if the statement of Proposition 2.6 has a converse, or if some quantitative version of the weakly differentiability at $\mathscr{L}^d$-a.e. almost every point could guarantee the property (2.1). The answer is negative, indeed for any $p \in [1,\infty)$ it is possible to built a function $f \in L^p(\mathbb{R}^d)$ that is weakly differentiable almost everywhere with $\nabla f = 0$ but does not belong to $N^{1,q}$ for any q.

Let us illustrate how to built such example. Let us fix an integer $M > 0$. It is enough to built a function f supported in $[0,1]$ that is weakly differentiable with $f' = 0$ $\mathscr{L}^1$-a.e., $\|f\|_{L^\infty} = 1$ and $[f]_{N^{1,1}} \ge \frac{1}{2}\log(1+M)$. Let us define

$$f(x) := \sum_{k=0}^{M-1} (-1)^k \mathbf{1}_{(K/M,(K+1)/M]}(x) \qquad \forall\, x \in [0,1].$$

It is trivial to see that $\|f\|_{L^\infty} = 1$ and f is differentiable at every point outside a finite set with derivative equal to zero. Let us show that $[f]_{N^{1,1}} \ge \frac{1}{2}\log(1+M)$. Take any g realizing the inequality in (2.1), $x \in (i/M,(i+1)/M]$ and $y \in ((i+1)/M,(i+2)/M]$. We have

$$g(x) + g(y) \ge \log(1 + 2/|x-y|) \ge \log(1+M), \tag{2.9}$$

we integrate $x \in (i/M,(i+1)/M]$ and $y \in ((i+1)/M,(i+2)/M]$ with respect to the Lebesgue measure obtaining

$$\int_{i/M}^{(i+2)/M} g \ge \frac{1}{M}\log(1+M),$$

that trivially implies our conclusion.

The last result of this paper concerns with the link between the spaces $X^{\gamma,p}$ and $N^{s,p}$. A version of this result was crucial in our paper [5] to study a sharp regularity for the continuity equation associated to a divergence-free Sobolev drift.

Theorem 2.8. *Let $p \ge 1$ and $s \in (0,1]$ be fixed. For any f satisfying (2.1) the following estimate holds true*

$$\int_{B_{1/3}} \int_{\mathbb{R}^d} \frac{1 \wedge |f(x+h) - f(x)|^q}{|h|^d} \frac{1}{\log(1/|h|)^{1-p}}\, \mathrm{d}x\, \mathrm{d}h \lesssim_{s,p,d} \|g\|_{L^p}^p + \|g\|_{L^q}^q. \tag{2.10}$$

In particular we have the embedding

$$N^{s,p} \cap L^\infty(\mathbb{R}^d) \hookrightarrow X^{1,p}, \tag{2.11}$$

for any $s \in (0,1]$ and $p \ge 1$.

Proof. By (2.1), we can write

$$\begin{aligned}
&\int_{B_{1/e}}\int_{\mathbb{R}^d}\frac{1\wedge|f(x+h)-f(x)|^q}{|h|^d\log(1/|h|)^{1-p}}\,\mathrm{d}x\,\mathrm{d}h\\
&\le\int_{B_{1/e}}\int_0^1 q\lambda^{q-1}\mathscr{L}^d(\{x:|h|^s\left(\exp\{g(x+h)+g(x)\}-1\right)>\lambda\})\,\mathrm{d}\lambda\frac{1}{|h|^d\log(1/|h|)^{1-p}}\,\mathrm{d}h\\
&=\int_{B_{1/e}}\int_0^{1/|h|^s} q\lambda^{q-1}\mathscr{L}^d(\{x:\left(\exp\{g(x+h)+g(x)\}-1\right)>\lambda\})\,\mathrm{d}\lambda\frac{1}{|h|^{d-sq}\log(1/|h|)^{1-p}}\,\mathrm{d}h.
\end{aligned}$$

Note that for $0<\lambda<2$, one has

$$\begin{aligned}
\mathscr{L}^d(\{x:\exp\{g(x+h)+g(x)\}-1>\lambda\})&\le\mathscr{L}^d(\{x:4(g(x+h)+g(x))>\lambda\})\\
&\le 2\mathscr{L}^d(\{x:8g(x)>\lambda\}).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{B_{1/e}}\int_{\mathbb{R}^d}\frac{1\wedge|f(x+h)-f(x)|^q}{|h|^d\log(1/|h|)^{1-p}}\,\mathrm{d}x\,\mathrm{d}h\\
&\le\int_{B_{1/e}}\int_0^2 2q\lambda^{q-1}\mathscr{L}^d(\{x:8g(x)>\lambda\})\,\mathrm{d}\lambda\frac{1}{|h|^{d-sq}\log(1/|h|)^{1-p}}\,\mathrm{d}h\\
&+\int_{B_{1/e}}\int_2^{1/|h|^s} q\lambda^{q-1}\mathscr{L}^d(\{x:\exp\{g(x+h)+g(x)\}>\lambda\})\,\mathrm{d}\lambda\frac{1}{|h|^{d-sq}\log(1/|h|)^{1-p}}\,\mathrm{d}h\\
&\lesssim_{p,d,s,q}\int_0^1 q\lambda^{q-1}\mathscr{L}^d(\{g>\lambda\})\,\mathrm{d}\lambda+\int_{B_{1/e}}\int_2^{1/|h|^s}\lambda^{q-1}\mathscr{L}^d(\{g>\frac{1}{2}\log(\lambda)\})\frac{\mathrm{d}\lambda\,\mathrm{d}h}{|h|^{d-sq}\log(1/|h|)^{1-p}}\\
&\lesssim_{p,d,s,q}\|g\|_{L^q}^q+\int_2^\infty\left[\int_{|h|^s<1/\lambda}\frac{1}{|h|^{d-sq}\log(1/|h|)^{1-p}}\,\mathrm{d}h\right]\lambda^{q-1}\mathscr{L}^q(\{g>\frac{1}{2}\log(\lambda)\})\,\mathrm{d}\lambda.
\end{aligned}$$

Since for $\lambda>2$,

$$\int_{|h|^s<1/\lambda}\frac{1}{|h|^{d-sq}\log(1/|h|)^{1-p}}\,\mathrm{d}h\lesssim_{p,d,s}\lambda^{-q}\log(\lambda)^{p-1},$$

we deduce

$$\begin{aligned}
&\int_{B_{1/e}}\int_{\mathbb{R}^d}\frac{1\wedge|f(x+h)-f(x)|}{|h|^d}\frac{1}{\log(1/|h|)^{1-p}}\,\mathrm{d}x\,\mathrm{d}h\\
&\lesssim_{p,d,s,q}\|g\|_{L^q}^q+\int_1^\infty\lambda^{-1}\log(\lambda)^{p-1}\mathscr{L}^d(\{g>\frac{1}{2}\log(\lambda)\})\,\mathrm{d}\lambda\\
&\lesssim_{p,d,s,q}\|g\|_{L^q}^q+\int_1^\infty\lambda^{p-1}\mathscr{L}^d(\{g>\lambda\})\,\mathrm{d}\lambda,
\end{aligned}$$

which implies (2.10). The proof is complete. □

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