A SHORT PROOF OF A NONEXISTENCE RESULT

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ABSTRACT. In this paper we provide a short proof of a non existence result for an isoperimetric type problem. Precisely, we prove the existence of a critical mass m_c such that the minimum problem with prescribed volume m does not admit solutions for $m > m_c$. Moreover, we found an explicit lower bound for the critical mass.

1. Introduction

The aim of this paper is to study the non existence of minimizers of the functional

$$I_K(E) = P(E) + V(E) - KR(E) \tag{1.1}$$

under the volume constraint $|E| = m$, where $E \subset \mathbb{R}^3$ is a measurable set. Here

$$V(E) = \frac{1}{2}\int_E \int_E \frac{1}{|x-y|} dxdy$$

is a Coulombic repulsive potential of the set with itself,

$$R(E) = \int_E \frac{1}{|x|} dx,$$

$P(E)$ stands for the standard Euclidean perimeter in the De Giorgi sense and $K > 0$. If one considers each term of (1.1) separately, it is well known that the ball is an extremal for all of them: precisely it minimizes the perimeter and maximizes both $V(E)$ and $R(E)$ under volume constraint. Indeed it is the competition among these three terms that makes the problem mathematically challenging. Therefore, while it can be proved that for $m < K$ the ball is the unique minimizer of (1.1), (see [5], [4] and [3], [2] and [6] for related results), it is natural to expect that minimizers do not occur when m is large enough. To see this, assume that $|E| = 1$, consider the rescaled set λE and observe that

$$I_K(\lambda E) = \lambda^2 P(E) + \lambda^4 V(E) - K\lambda^2 R(E). \tag{1.2}$$

When λ is large enough, the leading term in (1.2) is $V(\lambda E) = \lambda^4 V(E)$. Therefore, in order to minimize the energy (1.2) it would be convenient to lower as much as possible the value of $V(E)$. However this is not feasible since the functional V does not admit minimizers. In [5], Lu and Otto proved the existence of a critical mass m_c such that if $m > m_c$ the constrained minimum problem for I_K has no minimizer, see also [3]. In this paper we give a short proof of the non existence result of [5]. Our result reads as follows:

Theorem 1.1. *If $m > 8 + 2K$ the problem*

$$\min\{I_K(E) : E \subset \mathbb{R}^3, \quad |E| = m\}$$

has no sulutions.

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Note that the above theorem gives also a lower bound for the critical threshold m_c.

2. Non existence of minimizers

We define the quantity

$$\mathcal{I}_K[m] := \inf_{|E|=m} I_K(E)$$

Since the functional is not invariant under translation, we can not expect I_K to be subadditive. However the following weak form of subadditivity was stated in [5, Lemma 4].

Lemma 2.1. *Let A, B real positive numbers. Then it holds*

$$\mathcal{I}_K[A+B] \le \mathcal{I}_K[A] + \mathcal{I}_0[B].$$

Proof of Theorem 1.1. We use a strategy introduced in [1]. For $\omega \in \mathbb{S}^2$ and $l \in \mathbb{R}$ we set

$$H_{\omega,l} = \{x \in \mathbb{R}^3 : x \cdot \omega = l\}, \quad H^+_{\omega,l} = \{x \in \mathbb{R}^3 : x \cdot \omega \ge l\}, \quad H^-_{\omega,l} = \mathbb{R}^3 \setminus H^+_{\omega,l}.$$

Then, if $\Omega \subset \mathbb{R}^3$.

$$\Omega^+_{\omega,l} = \Omega \cap H^+_{\omega,l}, \quad \Omega^-_{\omega,l} = \Omega \cap H^-_{\omega,l}.$$

Given $m > 0$, let E be a minimizer of I_K under the constraint $|E| = m$. Using Lemma 2.1 and the minimizing property of E, we have

$$I_K(E) = \mathcal{I}_K[m] \le \mathcal{I}_K[|E^-_{\omega,l}|] + \mathcal{I}_0[|E^+_{\omega,l}|] \le I_K(|E^-_{\omega,l}) + I_0(|E^+_{\omega,l}).$$

The above inequality can be rewritten as

$$P(E) + V(E) - KR(E) \le P(E^-_{\omega,l}) + V(E^-_{\omega,l}) - KR(E^-_{\omega,l}) + P(E^+_{\omega,l}) + V(E^+_{\omega,l}). \tag{2.1}$$

Given $\omega \in \mathbb{S}^2$, for a.e. $l \in \mathbb{R}$ we have $P(E^-_{\omega,l}) = P(E; H^-_{\omega,l}) + \mathcal{H}^2(E \cap H_{\omega,l})$ and

$$V(E) = V(E^-_{\omega,l}) + V(E^+_{\omega,l}) + \int_{E^-_{\omega,l}} \int_{E^+_{\omega,l}} \frac{1}{|x-y|} dxdy.$$

Therefore, from (2.1) we obtain

$$\int_{E^-_{\omega,l}} \int_{E^+_{\omega,l}} \frac{1}{|x-y|} dxdy \le 2\mathcal{H}^2(E \cap H_{\omega,l}) + K \int_{E^+_{\omega,l}} \frac{1}{|x|} dx.$$

Integrating this inequality with respect to l from 0 to ∞, we have

$$\int_0^\infty \int_{E^-_{\omega,l}} \int_{E^+_{\omega,l}} \frac{1}{|x-y|} dxdydl \le 2|E^+_{\omega,0}| + K \int_0^\infty \int_{E^+_{\omega,l}} \frac{1}{|x|} dx.$$

In order to estimate the last integral we observe, using the layer cake formula and Fubini's theorem, that

$$\int_{E^+_{\omega,0}} \frac{x \cdot \omega}{|x|} dx = \int_{E^+_{\omega,0}} \frac{1}{|x|} \int_0^\infty \chi_{(0, x\cdot\omega)}(t) dtdx = \int_0^\infty \int_{E^+_{\omega,0}} \frac{1}{|x|} \chi_{(t,\infty)}(x \cdot \omega) dtdx = \int_0^\infty \int_{E^+_{\omega,l}} \frac{1}{|x|} dxdl.$$

Thus, we have

$$\int_0^\infty \int_{E^-_{\omega,l}} \int_{E^+_{\omega,l}} \frac{1}{|x-y|} dxdydl \le 2 \int_0^\infty \mathcal{H}^2(E \cap H_{\omega,l}) dl + K \int_{E^+_{\omega,0}} \frac{|x \cdot \omega|}{|x|} dx. \tag{2.2}$$

Interchanging the role of $E^-_{\omega,l}$ and $E^+_{\omega,l}$ in the above formula, we have

$$I_K(E) = \mathcal{I}_K[m] \le \mathcal{I}_K[|E^+_{\omega,l}|] + \mathcal{I}_0[|E^-_{\omega,l}|] \le I_K(|E^+_{\omega,l}) + I_0(|E^-_{\omega,l}).$$

From which, arguing as in the proof of (2.2), we obtain

$$\int_{-\infty}^0 \int_{E^-_{\omega,l}} \int_{E^+_{\omega,l}} \frac{1}{|x-y|} dxdydl \le 2 \int_{-\infty}^0 \mathcal{H}^2(E \cap H_{\omega,l}) dl + K \int_{E^-_{\omega,l}} \frac{|x \cdot \omega|}{|x|} dxdl.$$

Summing this inequality with (2.2) we have

$$\int_{-\infty}^{\infty}\int_{E^-_{\omega,l}}\int_{E^+_{\omega,l}}\frac{1}{|x-y|}dxdydl \le 2|E| + K\int_E \frac{|x\cdot\omega|}{|x|}dx. \tag{2.3}$$

Using Fubini's theorem,

$$\int_{-\infty}^{\infty}\int_{E^-_{\omega,l}}\int_{E^+_{\omega,l}}\frac{1}{|x-y|}dxdydl = \int_E\int_E\int_{-\infty}^{\infty}\frac{\chi_{\{y\cdot\omega<l<x\cdot\omega\}}(y)}{|x-y|}dxdydl = \int_E\int_E\frac{(\omega\cdot(x-y))_+}{|x-y|}dxdy.$$

Since for $a\in\mathbb{R}^3$

$$\int_{\mathbb{S}^2}|\omega\cdot a|d\omega = 2\int_{\mathbb{S}^2}(\omega\cdot a)_+d\omega = 2\pi|a|,$$

averaging over $\omega\in\mathbb{S}^2$ and using Fubini once again, we obtain

$$\frac{1}{4\pi}\int_E\int_E\int_{\mathbb{S}^{n-1}}\frac{(\omega\cdot(x-y))_+}{|x-y|}dxdyd\omega = \frac{1}{4}|E|^2, \qquad \frac{1}{4\pi}\int_E\int_{\mathbb{S}^{n-1}}\frac{|x\cdot\omega|}{|x|}dxd\omega = \frac{1}{2}|E|$$

and thus (2.3) yields

$$\frac{m^2}{4} < \left(2+\frac{K}{2}\right)m.$$

From this inequality the result follows. □

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