BV-maps with values into S^1: graphs, minimal connections and optimal lifting

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The aim of this paper is to extend to the higher dimension $n \geq 2$ the results from [11] about *minimal connections* and *optimal lifting* of maps of bounded variation with values into $\mathbb{S}^1$. More precisely, we first outline the link between lifting and connections of maps in $BV(B^n, \mathbb{S}^1)$, Theorem 4.4. Secondly, we write in an explicit way the energy of the optimal lifting of BV-maps, Theorem 4.8. Finally, we show that the minimal connection $L(u)$ can be seen as the distance from gradient maps, Theorem 4.9. The case of $W^{1,1}$-mappings from B^n into $\mathbb{S}^1$ has already been treated in [2] [9]. To prove our results, we will make use of the measure theoretical geometric approach in [6] [7] [8].

1 Currents carried by graphs of BV-maps

Let B^n be the n-dimensional unit ball and $\mathbb{S}^1 \subset \mathbb{R}^2$ the unit sphere. Let

$$BV(B^n, \mathbb{S}^1) := \{u \in BV(B^n, \mathbb{R}^2) \mid |u(x)| = 1 \quad \text{for } \mathcal{L}^n\text{-a.e. } x \in B^n\}.$$

Also, $\pi : B^n \times \mathbb{S}^1 \to B^n$ and $\widehat{\pi} : B^n \times \mathbb{S}^1 \to \mathbb{S}^1$ will denote the orthogonal projections onto the first and second factor, respectively. Finally, we denote by $\omega_{\mathbb{S}^1}$ the *volume* 1*-form on* $\mathbb{S}^1 \subset \mathbb{R}^2$

$$\omega_{\mathbb{S}^1} := y^1 dy^2 - y^2 dy^1 .$$

We recall, see [5], that the space $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ of n-dimensional currents in $B^n \times \mathbb{S}^1$ is the dual of $\mathcal{D}^n(B^n \times \mathbb{S}^1)$, the space of all the compactly supported smooth n-form ω in $B^n \times \mathbb{S}^1$. Following [8], to every function $u \in BV(B^n, \mathbb{S}^1)$ we associate an n-current G_u in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ "carried" by the "graph" of u and defined as follows. We decompose G_u into its absolutely continuous, Cantor, and Jump parts

$$G_u := G_u^a + G_u^C + G_u^J .$$

Every n-form $\omega \in \mathcal{D}^n(B^n \times \mathbb{S}^1)$ splits as $\omega^{(0)} + \omega^{(1)}$ according to the number of "vertical" differentials. Writing $\omega^{(0)} = \phi(x,y)\, dx$ for some $\phi \in C_0^\infty(B^n \times \mathbb{S}^1)$, where $dx := dx^1 \wedge \cdots \wedge dx^n$, we set

$$G_u^C(\phi(x,y)\, dx) = G_u^J(\phi(x,y)\, dx) = 0$$

and

$$G_u(\phi(x,y)\, dx) = G_u^a(\phi(x,y)\, dx) := \int_{B^n} \phi(x, u(x))\, dx . \tag{1.1}$$

Setting $\widehat{dx^i} := dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n$, we may write

$$\omega^{(1)} = \sum_{i=1}^n \sum_{j=1}^2 (-1)^{n-i} \phi_i^j(x,y)\, \widehat{dx^i} \wedge dy^j \tag{1.2}$$

for some $\phi_i^j \in C_0^\infty(B^n \times \mathbb{S}^1)$ and we set $\phi^j := (\phi_1^j, \ldots, \phi_n^j)$. We then define

$$\begin{aligned}
G_u^a(\omega^{(1)}) &:= \sum_{j=1}^2 \int_{B^n} \langle \nabla u^j(x), \phi^j(x, u(x)) \rangle\, dx = \sum_{j=1}^2 \sum_{i=1}^n \int_{B^n} \nabla_i u^j(x) \phi_i^j(x, u(x))\, dx \\
G_u^C(\omega^{(1)}) &:= \sum_{j=1}^2 \int_{B^n} \phi^j(x, u(x))\, dD^C u^j \\
G_u^J(\omega^{(1)}) &:= \sum_{i=1}^n \sum_{j=1}^2 \int_{J_u} \left(\int_{l_x} \phi_i^j(x,y)\, dy^j \right) \nu_u^i\, d\mathcal{H}^{n-1}(x) .
\end{aligned}$$

In the previous formula, J_u is the jump set of u and $\nu_u = (\nu_u^1, \dots, \nu_u^n)$ is an unit normal to J_u. Moreover, $u^-(x)$ and $u^+(x)$ are the one-sided limits of u at $x \in J_u$, with respect to the orientation ν_u, see e.g. [1] for the notation on BV-functions. Finally, for $x \in J_u$, we denote by l_x the oriented simple arc of $\mathbb{S}^1$ connecting $u^-(x)$ and $u^+(x)$ and satisfying the following properties:

i) l_x is constantly the point $u^+(x)$ if $u^+(x) = u^-(x)$;

ii) l_x does not contain the point $(-1,0)$ if $u^+(x) \neq (-1,0)$ and $u^-(x) \neq (-1,0)$;

iii) l_x is oriented in the counterclockwise sense if $u^+(x) = (-1,0)$, and in the clockwise sense if $u^-(x) = (-1,0)$.

Notice that we have

$$\int_{l_x} \omega_{\mathbb{S}^1} = \rho(u^+, u^-)\,, \tag{1.3}$$

where ρ is the signed distance on $\mathbb{S}^1$ defined in [11]. More precisely, $\rho : \mathbb{S}^1 \times \mathbb{S}^1 \to]-\pi, \pi]$ is defined by

$$\rho(\theta_1, \theta_2) := \begin{cases} \operatorname{Arg}\big(\theta_1/\theta_2\big) & \text{if } \theta_1/\theta_2 \neq -1\,, \\ \operatorname{Arg}\big(\theta_1\big) - \operatorname{Arg}(\theta_2) & \text{if } \theta_1/\theta_2 = -1\,, \end{cases} \qquad \forall\, \theta_1,\, \theta_2 \in \mathbb{S}^1\,, \tag{1.4}$$

where $\operatorname{Arg}(\theta) \in]-\pi, \pi]$ stands for the *argument* of the unit complex number $\theta \in \mathbb{S}^1 \subset \mathbb{C}$.

Remark 1.1 If $u \in W^{1,1}(B^n, \mathbb{S}^1)$, then $G_u^C = G_u^J = 0$ and the current G_u agrees with the image current $(Id \bowtie u)_\# [\![B^n]\!]$, where $(Id \bowtie u)(x) := (x, u(x))$, defined by

$$(Id \bowtie u)_\# [\![B^n]\!](\omega) := \int_{B^n} (Id \bowtie u)^\# \omega\,, \qquad \omega \in \mathcal{D}^n(B^n \times \mathbb{S}^1)\,.$$

Here the pull-back makes sense in terms of the a.e. approximate differentiability of u, i.e., of the distributional derivative of u. More precisely, since

$$(Id \bowtie u)^\# (-1)^{n-i} \phi_i^j(x,y)\, \widehat{dx^i} \wedge dy^j = (-1)^{n-i} \phi_i^j(x,u)\, \widehat{dx^i} \wedge du^j = \phi_i^j(x,u)\, \nabla_i u^j(x)\, dx$$

we readily infer that the absolutely continuous part

$$G_u^a = (Id \bowtie u)_\# [\![B^n]\!]\,. \tag{1.5}$$

Remark 1.2 Notice that the mapping $u \mapsto G_u$ from $BV(B^n, \mathbb{S}^1)$ to $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ is not continuous. More precisely, at the end of Sec. 2 we will show that there exist sequences of functions $\{u_k\} \subset BV(B^2, \mathbb{S}^1)$ such that $u_k \rightharpoonup u \in BV(B^2, \mathbb{S}^1)$ weakly in the BV-sense, with $|Du_k|(B^2) \to |Du|(B^2)$, but for which G_{u_k} does not converge to G_u weakly as currents in $\mathcal{D}_2(B^2 \times \mathbb{S}^1)$.

We finally remark that if $n \geq 2$ in general the current G_u has a non zero boundary in $B^n \times \mathbb{S}^1$, even if $u \in W^{1,1}(B^n, \mathbb{S}^1)$, i.e., if $G_u = G_u^a$. Take for example $n = 2$ and $u(x) = x/|x|$, so that

$$\partial G_u \llcorner B^2 \times \mathbb{S}^1 = -\delta_0 \times [\![\mathbb{S}^1]\!]\,,$$

where δ_0 is the unit Dirac mass at the origin. However, ∂G_u is null on every $(n-1)$-form $\widetilde{\omega}$ in $B^n \times \mathbb{S}^1$ which has no "vertical" differentials. To this purpose, we observe that any smooth $(n-1)$-form $\widetilde{\omega} \in \mathcal{D}^{n-1}(B^n \times \mathbb{S}^1)$ as above can be written as $\widetilde{\omega} := \omega_\varphi \wedge \eta$ for some $\eta \in C^\infty(\mathbb{S}^1)$ and $\varphi = (\varphi^1, \dots, \varphi^n) \in C_0^\infty(B^n, \mathbb{R}^n)$, where $\omega_\varphi \in \mathcal{D}^{n-1}(B^n)$ is given by

$$\omega_\varphi := \sum_{i=1}^n (-1)^{i-1} \varphi^i(x)\, \widehat{dx^i}\,, \tag{1.6}$$

so that clearly $d\omega_\varphi = \operatorname{div} \varphi\, dx^1 \wedge \cdots \wedge dx^n$. Splitting $d = d_x + d_y$, we notice that $d_x \widetilde{\omega} = d\omega_\varphi \wedge \eta = \operatorname{div} \varphi(x)\, \eta(y)\, dx$, so that by (1.1)

$$\partial_x G_u(\widetilde{\omega}) := G_u(d_x \widetilde{\omega}) = G_u(\operatorname{div} \varphi(x)\, \eta(y)\, dx) = \int_{B^n} \operatorname{div}\varphi(x) \cdot \eta(u(x))\, dx\,.$$

Moreover, see [8], by the chain rule for the derivative of BV-functions we obtain:

Proposition 1.3 *We have*

$$\partial_y G_u(\omega_\varphi \wedge \eta) := G_u(d_y(\omega_\varphi \wedge \eta)) = -\int_{B^n} \mathrm{div}\varphi(x) \cdot \eta(u(x))\, dx =: \langle D(\eta \circ u), \varphi \rangle\,.$$

This yields that $\partial G_u(\widetilde{\omega}) = \partial_x G_u(\widetilde{\omega}) + \partial_y G_u(\widetilde{\omega}) = 0$, as required.

2 The singular set of BV-maps

In this section, using arguments from [7, Vol. II], we introduce the $(n-2)$-dimensional current $\mathbb{P}(u)$ in B^n that represents the *singular set* of a BV-map u. We then extend to higher dimension $n \geq 2$ the definition of the distribution $T(u)$ introduced in [11] in the case $n = 2$, see [2] and [9] for the case of Sobolev maps $u \in W^{1,1}(B^n, \mathbb{S}^1)$. We shall then show that the two definitions agree. We shall finally explain, in terms of currents, the discontinuity property of the map $u \mapsto T(u)$, as already observed in [11].

Definition 2.1 *To any* $u \in BV(B^n, \mathbb{S}^1)$ *we associate the* $(n-2)$*-current* $\mathbb{P}(u) \in \mathcal{D}_{n-2}(B^n)$ *given by* $2\pi \cdot \mathbb{P}(u) := -\pi_\#((\partial G_u) \llcorner \widehat{\pi}^\# \omega_{\mathbb{S}^1})$*, so that for every* $\xi \in \mathcal{D}^{n-2}(B^n)$

$$\mathbb{P}(u)(\xi) = -\frac{1}{2\pi} \partial G_u(\widehat{\pi}^\# \omega_{\mathbb{S}^1} \wedge \pi^\# \xi) = \frac{1}{2\pi} G_u(\widehat{\pi}^\# \omega_{\mathbb{S}^1} \wedge \pi^\# d\xi)\,. \tag{2.1}$$

If $u \in W^{1,1}(B^n, \mathbb{S}^1)$, by (1.5) we readily infer

$$\mathbb{P}(u)(\xi) = \frac{1}{2\pi} \int_{B^n} u^\# \omega_{\mathbb{S}^1} \wedge d\xi\,,$$

compare [9]. Moreover, $\mathbb{P}(u)$ is always a boundaryless current, $\partial\, \mathbb{P}(u) \llcorner B^n = 0$, but in general $\mathbb{P}(u) \neq 0$.

FORMS WITH LIPSCHITZ COEFFICIENTS. For $k = 0, \ldots, n$, we will denote by $\mathrm{Lip}(B^n, \Lambda^k TB^n)$ the class of k-forms in B^n with coefficients in $\mathrm{Lip}(B^n)$. Every $(n-2)$-form $\zeta \in \mathrm{Lip}(B^n, \Lambda^{n-2} TB^n)$ will be written as

$$\zeta = \sum_{1 \leq i < j \leq n} \zeta^{i,j}\, \widehat{dx^{i,j}}\,, \tag{2.2}$$

where $\zeta^{i,j} \in \mathrm{Lip}(B^n, \mathbb{R})$ and

$$\widehat{dx^{i,j}} := dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n\,.$$

If ζ is given by (2.2), since for every $i < j$

$$dx^i \wedge \widehat{dx^{i,j}} = (-1)^{i-1}\, \widehat{dx^j} \quad \text{and} \quad dx^j \wedge \widehat{dx^{i,j}} = (-1)^j\, \widehat{dx^i}\,,$$

we have

$$d\zeta = \sum_{1 \leq i < j \leq n} \big((-1)^{i-1} \zeta^{i,j}_{x_i}\, \widehat{dx^j} + (-1)^j \zeta^{i,j}_{x_j}\, \widehat{dx^i} \big) \tag{2.3}$$

and hence

$$d\zeta = \sum_{i=1}^n A^i(\zeta)\, \widehat{dx^i}$$

where for every fixed i

$$A^i(\zeta) := \sum_{1 \leq h < i} (-1)^{h-1} \zeta^{h,i}_{x_h} + \sum_{i < h \leq n} (-1)^h \zeta^{i,h}_{x_h}\,.$$

Denoting $y^{\overline{1}} := y^2$ and $y^{\overline{2}} := y^1$, this yields that

$$\begin{aligned}
\omega_{\mathbb{S}^1} \wedge d\zeta &= (-1)^{n-1} d\zeta \wedge \omega_{\mathbb{S}^1} \\
&= (-1)^{n-1} \Big(\sum_{i=1}^n A^i(\zeta)\, \widehat{dx^i} \Big) \wedge \Big(\sum_{j=1}^2 (-1)^j y^{\overline{j}} dy^j \Big) \\
&= \sum_{i=1}^n \sum_{j=1}^2 (-1)^{n+j-1} A_i(\zeta)\, y^{\overline{j}} \widehat{dx^i} \wedge dy^j\,.
\end{aligned}$$

Therefore, may write $\omega_{\mathbb{S}^1} \wedge d\zeta = \omega^{(1)}$ in (1.2) if $(-1)^{n-i}\phi_i^j := (-1)^{n+j-1} A_i(\zeta)\, y^{\bar{j}}$. Setting then $F^i(\zeta) := (-1)^{i+1} A_i(\zeta)$, we have

$$\phi_i^j(x,y) := (-1)^j F^i(\zeta(x))\, y^{\bar{j}}\,, \qquad F^i(\zeta) := \sum_{1\le h<i} (-1)^{h+i} \zeta^{h,i}_{x_h} - \sum_{i<h\le n} (-1)^{h+i} \zeta^{i,h}_{x_h}\,. \tag{2.4}$$

We finally set $F(\zeta) := (F^1(\zeta), \ldots, F^n(\zeta))$, and notice that if $n = 2$ and $\zeta \in \mathrm{Lip}(B^2, \mathbb{R})$

$$F(\zeta) = \nabla^\perp \zeta := (\zeta_{x_2}, -\zeta_{x_1})\,.$$

THE SINGULAR SET AS A DISTRIBUTION. To any $u \in BV(B^n, \mathbb{S}^1)$ we now associate the distribution $T(u)$ of order $(n-2)$, that is decomposed into its absolutely continuous, Cantor and Jump part

$$T(u) := T^a(u) + T^C(u) + T^J(u)\,.$$

As we shall see, in the case of dimension $n = 2$ it agrees with the definition of $T(u)$ from [11].

THE ABSOLUTELY CONTINUOUS PART. For any $1 \le i < j \le n$ consider the distribution $T^a_{i,j}(u) \in \mathcal{D}'(B^n, \mathbb{R})$ given by

$$T^a_{i,j}(u) := -(u \times u_{x_i})_{x_j} + (u \times u_{x_j})_{x_i} \tag{2.5}$$

where for every i

$$u \times u_{x_i} := u^1 u^2_{x_i} - u^2 u^1_{x_i}\,,$$

$u^h_{x_i}$ being the i^{th} component of the approximate gradient ∇u^h, that is,

$$\langle T^a_{i,j}(u), \zeta^{i,j}\rangle = \int_{B^n} \big((u \times u_{x_i})\, \zeta^{i,j}_{x_j} - (u \times u_{x_j})\, \zeta^{i,j}_{x_i} \big)\, dx \qquad \forall\, \zeta^{i,j} \in \mathrm{Lip}(B^n, \mathbb{R})\,.$$

The distribution $T^a(u)$ is defined by

$$\langle T^a(u), \zeta\rangle := \sum_{1\le i<j\le n} (-1)^{i+j-1} \langle T^a_{i,j}(u), \zeta^{i,j}\rangle \qquad \forall\, \zeta \in \mathrm{Lip}(B^n, \Lambda^{n-2} T B^n)\,,$$

where ζ is decomposed as in (2.2). Since

$$\begin{aligned}\sum_{1\le i<j\le n} &(-1)^{i+j-1} \big((u \times u_{x_i})\, \zeta^{i,j}_{x_j} - (u \times u_{x_j})\, \zeta^{i,j}_{x_i} \big) = \\ &= \sum_{i=1}^{n} \Big\{ \sum_{1\le h<i} (-1)^{h+i} \zeta^{h,i}_{x_h} - \sum_{i<h\le n} (-1)^{h+i} \zeta^{i,h}_{x_h} \Big\} (u \times u_{x_i})\,,\end{aligned}$$

by (2.4) we obtain

$$\langle T^a(u), \zeta\rangle = \int_{B^n} \Big(\sum_{i=1}^{n} (u \times u_{x_i}) F^i(\zeta) \Big)\, dx \qquad \forall\, \zeta \in \mathrm{Lip}(B^n, \Lambda^{n-2} T B^n)\,,$$

and hence

$$\langle T^a(u), \zeta\rangle = \int_{B^n} (u \times \nabla u) \cdot F(\zeta)\, dx \qquad \forall\, \zeta \in \mathrm{Lip}(B^n, \Lambda^{n-2} T B^n)\,,$$

where

$$u \times \nabla u := (u \times u_{x_1}, \ldots, u \times u_{x_n})\,.$$

In particular, if $n = 2$ we get

$$\langle T^a(u), \zeta\rangle = \int_{B^2} (u \times \nabla u) \cdot \nabla^\perp \zeta\, dx \qquad \forall\, \zeta \in \mathrm{Lip}(B^n, \mathbb{R})\,.$$

THE CANTOR PART. In a similar way, let $D^C u = (D^C u^1, D^C u^2)$, where in components

$$D^C u^j = ((D^C u^j)_1, \dots, (D^C u^j)_n)\,.$$

Let $u \times (D^C u)_i := u^1 (D^C u^2)_i - u^2 (D^C u^1)_i$ and

$$(u \times D^C u) := (u \times (D^C u)_1, \dots, u \times (D^C u)_n)\,.$$

For any $i < j$ we introduce the distribution $T^C_{i,j}(u) \in \mathcal{D}'(B^n, \mathbb{R})$ given by

$$\langle T^C_{i,j}(u), \zeta^{i,j} \rangle = \langle \zeta^{i,j}_{x_j}, d\,(u \times (D^C u)_i) \rangle - \langle \zeta^{i,j}_{x_i}, d\,(u \times (D^C u)_j) \rangle \qquad \forall\, \zeta^{i,j} \in \mathrm{Lip}(B^n, \mathbb{R})\,. \tag{2.6}$$

The distribution $T^C(u)$ of order $(n-2)$ is then defined by

$$\langle T^C(u), \zeta \rangle := \sum_{1 \le i < j \le n} (-1)^{i+j-1} \langle T_{i,j}(u), \zeta^{i,j} \rangle \qquad \forall\, \zeta \in \mathrm{Lip}(B^n, \Lambda^{n-2} T B^n)\,,$$

where ζ is decomposed as in (2.2). Arguing as above, we readily obtain that

$$\langle T^C(u), \zeta \rangle = \int_{B^n} F(\zeta)\, d(u \times D^C u) \qquad \forall\, \zeta \in \mathrm{Lip}(B^n, \Lambda^{n-2} T B^n)\,.$$

Notice that in the case $n = 2$ this yields

$$\langle T^C(u), \zeta \rangle = \int_{B^n} \nabla^\perp \zeta\, d(u \times D^C u) \qquad \forall\, \zeta \in \mathrm{Lip}(B^n, \mathbb{R})\,.$$

THE JUMP PART. As to the jump part, the distribution $T^J(u)$ of order $(n-2)$ is defined by

$$\langle T^J(u), \zeta \rangle := \int_{J_u} \rho(u^+, u^-)\, \nu_u \cdot F(\zeta)\, d\mathcal{H}^{n-1} \qquad \forall\, \zeta \in \mathrm{Lip}(B^n, \Lambda^{n-2} T B^n)\,,$$

where $F(\zeta)$ is given by (2.4) and ρ is the signed distance on $\mathbb{S}^1$, see (1.4). If $n = 2$ we thus have

$$\langle T^J(u), \zeta \rangle := \int_{J_u} \rho(u^+, u^-)\, \nu_u \cdot \nabla^\perp \zeta\, d\mathcal{H}^1 \qquad \forall\, \zeta \in \mathrm{Lip}(B^n, \mathbb{R})\,.$$

THE LINK BETWEEN $T(u)$ AND $\mathbb{P}(u)$. By the boundedness of the BV-norm of u, the action of the current G_u extends e.g. to forms $\omega := \widehat{\pi}^\# \omega_{\mathbb{S}^1} \wedge \pi^\# d\zeta$, where $\zeta \in \mathrm{Lip}(B^n, \Lambda^{n-2} T B^n)$. Moreover, we have:

Proposition 2.2 *For every* $u \in BV(B^n, \mathbb{S}^1)$

$$2\pi\, \mathbb{P}(u)(\zeta) = \langle T(u), \zeta \rangle \qquad \forall\, \zeta \in \mathrm{Lip}(B^n, \Lambda^{n-2} T B^n)\,,$$

where $\mathbb{P}(u)$ *is the singular set of definition* (2.1)*.*

PROOF: Since by (1.5)

$$G^a_u(\widehat{\pi}^\# \omega_{\mathbb{S}^1} \wedge \pi^\# d\zeta) = \int_{B^n} u^\# \omega_{\mathbb{S}^1} \wedge d\zeta\,,$$

and for every $\zeta \in \mathrm{Lip}(B^n, \Lambda^{n-2} T B^n)$ satisfying (2.2), on account of (2.3),

$$\begin{aligned} u^\# \omega_{\mathbb{S}^1} \wedge d\zeta &= (u^1 du^2 - u^2 du^1) \wedge d\zeta \\ &= \sum_{1 \le i < j \le n} (-1)^{i+j-1} \big((u \times u_{x_i})\, \zeta^{i,j}_{x_j} - (u \times u_{x_j})\, \zeta^{i,j}_{x_i} \big)\, dx^1 \wedge \dots \wedge dx^n\,, \end{aligned}$$

we deduce that

$$G^a_u(\widehat{\pi}^\# \omega_{\mathbb{S}^1} \wedge \pi^\# d\zeta) = \sum_{1 \le i < j \le n} (-1)^{i+j-1} \langle T^a_{i,j}(u), \zeta^{i,j} \rangle = \langle T^a(u), \zeta \rangle\,.$$

Moreover, by (2.4) we have

$$\phi_i^j(x,u(x)) = (-1)^j F^i(\zeta(x))\, u^{\bar{j}}(x)\,,$$

whereas

$$\sum_{j=1}^{2} (-1)^j u^{\bar{j}} (D^C u^j)_i = (u \times D^C u)_i\,.$$

By the definition of G_u^C we then obtain

$$\begin{aligned} G_u^C(\widehat{\pi}^\# \omega_{\mathbb{S}^1} \wedge \pi^\# d\zeta) &= \sum_{j=1}^{2} \int_{B^n} (-1)^j F(\zeta)\, u^{\bar{j}}\, dD^C u^j \\ &= \int_{B^n} F(\zeta)\, d(u \times D^C u) = \langle T^C(u), \zeta \rangle\,. \end{aligned}$$

Finally, again by (2.4), and by the definition of G_u^J, we find that

$$\begin{aligned} G_u^J(\widehat{\pi}^\# \omega_{\mathbb{S}^1} \wedge \pi^\# d\zeta) &= \sum_{i=1}^{n} \sum_{j=1}^{2} \int_{J_u} \left(\int_{l_x} (-1)^j F^i(\zeta)\, y^{\bar{j}}\, dy^j \right) \nu_u^i\, d\mathcal{H}^{n-1} \\ &= \sum_{i=1}^{n} \int_{J_u} \left(\int_{l_x} \sum_{j=1}^{2} (-1)^j y^{\bar{j}}\, dy^j \right) \nu_u^i\, F^i(\zeta)\, d\mathcal{H}^{n-1}\,. \end{aligned}$$

Since $\sum_{j=1}^2 (-1)^j y^{\bar{j}}\, dy^j = \omega_{\mathbb{S}^1}$, by (1.3) we find that

$$G_u^J(\widehat{\pi}^\# \omega_{\mathbb{S}^1} \wedge \pi^\# d\zeta) = \sum_{i=1}^{n} \int_{J_u} \rho(u^+, u^-)\, \nu_u^i \cdot F^i(\zeta)\, d\mathcal{H}^{n-1} = \langle T^J(u), \zeta \rangle\,.$$

In conclusion, we have shown that

$$2\pi\ \mathbb{P}(u)(\zeta) := G_u(\widehat{\pi}^\# \omega_{\mathbb{S}^1} \wedge \pi^\# d\zeta) = \langle T(u), \zeta \rangle$$

for every $\zeta \in \mathrm{Lip}(B^n, \Lambda^{n-2} T B^n)$, as required. □

Lack of continuity. In [11] it is shown that $T : BV(B^2, \mathbb{S}^1) \to \mathcal{D}'(B^2, \mathbb{R})$ is not continuous, i.e., that there exists a sequence of functions $\{u_k\} \subset BV(B^2, \mathbb{S}^1)$ such that $u_k \rightharpoonup u \in BV(B^2, \mathbb{S}^1)$ weakly in the BV-sense, with $|Du_k|(B^2) \to |Du|(B^2)$, but for which the distribution $T(u_k)$ does not converge to $T(u)$. As we shall see below, in terms of currents, G_{u_k} does not converge to G_u weakly in $\mathcal{D}_2(B^2 \times \mathbb{S}^1)$, see Remark 1.2. In fact, in order to have an anti-symmetric distance function ρ in (1.4), it turns out that the definition of G_u cannot be continuous in the above mentioned sense. Notice that this does not hold if we restrict to Sobolev maps $u \in W^{1,1}(B^n, \mathbb{S}^1)$, compare [2] [9].

Example 2.3 Take for simplicity $\Omega =]0, 2\pi[\times]0, \pi[$, which is bilipschitz homeomorphic to B^2. Following [11], we define for $(\theta, \alpha) \in \Omega$

$$\psi(\theta,\alpha) := \begin{cases} -2\theta & \text{if} \quad \theta \in]0, \pi/2[\,,\ \alpha \in]0, \pi/2[\\ -\pi & \text{if} \quad \theta \in]\pi/2, 3\pi/2[\,,\ \alpha \in]0, \pi/2[\\ 2\,(\theta - 2\pi) & \text{if} \quad \theta \in]3\pi/2, 2\pi[\,,\ \alpha \in]0, \pi/2[\\ 0 & \text{if} \quad \theta \in]0, 2\pi[\,,\ \alpha \in]\pi/2, \pi[\,, \end{cases} \qquad u := e^{i\psi}\,.$$

Clearly $u \in BV(\Omega, \mathbb{S}^1)$, with $D^C u = 0$ and $J_u = \{\pi\} \times]0, \pi[$. Taking $\nu_u := (0, -1)$, hence $u^-(x) \equiv (1, 0)$, it turns out that the arc l_x in the definition of G_u^J from Sec. 1 is oriented in the clockwise sense if $x \in \{\pi\} \times]0, \pi/2[\cup]3\pi/2, 2\pi[$, and in the counterclockwise sense if $x \in \{\pi\} \times [\pi/2, 3\pi/2]$. As a consequence, $\partial G_u \llcorner \Omega \times [\![\mathbb{S}^1]\!] = (\delta_p - \delta_n) \times [\![\mathbb{S}^1]\!]$, where $p := (\pi/2, \pi/2)$ and $n := (\pi/2, 3\pi/2)$, so that $T(u) = 2\pi\ \mathbb{P}(u) = 2\pi\,(\delta_p - \delta_n)$. Setting now, for $\varepsilon > 0$ small,

$$\psi_\varepsilon(\theta,\alpha) := \begin{cases} -2\theta & \text{if} \quad \theta \in]0, (\pi-\varepsilon)/2[\,,\ \alpha \in]0, \pi/2[\\ -\pi + \varepsilon & \text{if} \quad \theta \in](\pi-\varepsilon)/2, (3\pi+\varepsilon)/2[\,,\ \alpha \in]0, \pi/2[\\ 2\,(\theta - 2\pi) & \text{if} \quad \theta \in](3\pi+\varepsilon)/2, 2\pi[\,,\ \alpha \in]0, \pi/2[\\ 0 & \text{if} \quad \theta \in]0, 2\pi[\,,\ \alpha \in]\pi/2, \pi[\,, \end{cases} \qquad u_\varepsilon := e^{i\psi_\varepsilon}\,,$$

and $u_k := u_{\varepsilon_k}$, where $\varepsilon_k \searrow 0$, clearly $\{u_k\} \subset BV(\Omega, \mathbb{S}^1)$ with $u_k \rightharpoonup u$ and $|Du_k|(\Omega) \to |Du|(\Omega)$. Moreover, for every $\varepsilon > 0$ we readily check that $u_\varepsilon \in BV(\Omega, \mathbb{S}^1)$, with $D^C u_\varepsilon = 0$ and $J_{u_\varepsilon} = \{\pi\} \times]0, \pi[$. However, taking again $\nu_{u_\varepsilon} := (0, -1)$, hence $u_\varepsilon^-(x) \equiv (1, 0)$, this time the arc l_x^ε corresponding to $G^J_{u_\varepsilon}$ is oriented in the clockwise sense for every $x \in J_{u_\varepsilon}$. This yields that $\partial G_{u_\varepsilon} \llcorner \Omega \times [\![\mathbb{S}^1]\!] = 0$, so that $T(u_\varepsilon) = 2\pi\, \mathbb{P}(u_\varepsilon) = 0$. On the other hand, G_{u_k} weakly converges in $\mathcal{D}_n(\Omega \times \mathbb{S}^1)$ to the Cartesian current $T := G_u + [\![I]\!] \times [\![\mathbb{S}^1]\!]$, where $[\![I]\!]$ is the 1-current integration over the line segment $I :=]\pi/2, 3\pi/2[\times \{\pi\}$, equipped with the natural orientation $e_1 := (1, 0)$, so that $\partial [\![I]\!] = \delta_n - \delta_p$. In particular, as noticed in [11], $T(u_k)$ does not converge weakly as a distribution to $T(u)$.

3 Minimal connections and relaxed energy

In this section we report some well-known features on *minimal connections*. We then collect some facts from [7] about the class of *Cartesian currents* in $B^n \times \mathbb{S}^1$. We finally recall some results from [8] about the *relaxed energy* of BV-functions.

THE FLAT NORM. Let $\Gamma \in \mathcal{D}_k(B^n)$, and suppose that Γ is the boundary in B^n of a $(k+1)$-current $D \in \mathcal{D}_{k+1}(B^n)$, i.e., $(\partial D) \llcorner B^n = \Gamma$, with finite mass, $\mathbf{M}(D) < \infty$. The *flat norm* of Γ is defined by

$$F_{B^n}(\Gamma) := \sup\{\Gamma(\xi) \mid \xi \in \mathcal{D}^k(B^n)\,,\ \|d\xi\| \le 1\}\,.$$

Moreover, we denote respectively by

$$\begin{array}{rcl} m_{i,B^n}(\Gamma) & := & \inf\{\mathbf{M}(L) \mid L \in \mathcal{R}_{k+1}(B^n)\,, \quad (\partial L) \llcorner B^n = \Gamma\} \\ m_{r,B^n}(\Gamma) & := & \inf\{\mathbf{M}(D) \mid D \in \mathcal{D}_{k+1}(B^n)\,, \quad (\partial D) \llcorner B^n = \Gamma\} \end{array}$$

the *integral* and *real mass* of Γ *in* B^n. Also, in case $m_{i,B^n}(\Gamma) < \infty$, we say that an integer multiplicity (say i.m.) rectifiable current $L \in \mathcal{R}_{k+1}(B^n)$ is an *integral minimal connection* of Γ *allowing connections to the boundary* if $(\partial L) \llcorner B^n = \Gamma$ and $\mathbf{M}(L) = m_{i,B^n}(\Gamma)$. We have, see Federer [4]:

$$F_{B^n}(\Gamma) = m_{r,B^n}(\Gamma)\,. \tag{3.1}$$

Taking $k = n-2$, for every $u \in BV(B^n, \mathbb{S}^1)$ we now define the $(n-1)$-current $\mathbb{D}(u) \in \mathcal{D}_{n-1}(B^n)$ by $2\pi \cdot \mathbb{D}(u) := \pi_\#(G_u \llcorner \widehat{\pi}^\# \omega_{\mathbb{S}^1})$, so that for every $\gamma \in \mathcal{D}^{n-1}(B^n)$

$$\mathbb{D}(u)(\gamma) := \frac{1}{2\pi} G_u(\widehat{\pi}^\# \omega_{\mathbb{S}^1} \wedge \pi^\# \gamma)\,.$$

Since $\mathbb{P}(u) = \partial\, \mathbb{D}(u) \llcorner B^n$, and $\mathbf{M}(\mathbb{D}(u)) < \infty$, we now define for any $n \ge 2$

$$L(u) := F_{B^n}(\mathbb{P}(u))\,, \qquad u \in BV(B^n, \mathbb{S}^1)\,.$$

On account of Proposition 2.2 we obtain

$$L(u) = \frac{1}{2\pi} \sup\{\langle T(u), \zeta\rangle \mid \zeta \in \mathrm{Lip}(B^n, \Lambda^{n-2} TB^n)\,, \quad \|\nabla \zeta\|_\infty \le 1\}\,,$$

which is the *length of the minimal connection of the singularities*, see [11] [2] [9]. Since $m_i(\mathbb{P}(u)) < \infty$, see Proposition 3.2 below, by Hardt-Pitt's result [10] in any dimension $n \ge 2$ we have

$$m_{r,B^n}(\mathbb{P}(u)) = m_{i,B^n}(\mathbb{P}(u)) \qquad \forall\, u \in BV(B^n, \mathbb{S}^1)\,.$$

Therefore, by (3.1) we obtain that

$$L(u) = m_{i,B^n}(\mathbb{P}(u)) = \min\{\mathbf{M}(L) \mid L \in \mathcal{R}_{n-1}(B^n)\,, \quad (\partial L) \llcorner B^n = (-1)^n\, \mathbb{P}(u)\}\,. \tag{3.2}$$

CARTESIAN CURRENTS. Following [7], the class of Cartesian currents $\mathrm{cart}(B^n \times \mathbb{S}^1)$ can be characterized by the class of the i.m. rectifiable currents $T \in \mathcal{R}_n(B^n \times \mathbb{S}^1)$ with finite mass, $\mathbf{M}(T) < \infty$, no interior boundary, $\partial T = 0$ in $B^n \times \mathbb{S}^1$, and that can be decomposed as

$$T = G_{u_T} + L_T \times [\![\mathbb{S}^1]\!]\,, \tag{3.3}$$

where G_{u_T} is the current defined as in Sec. 1 in correspondence of a function $u_T \in BV(B^n, \mathbb{S}^1)$. Moreover, L_T is an $(n-1)$-dimensional i.m. rectifiable current in $\mathcal{R}_{n-1}(B^n)$. The null-boundary condition yields that

$$\partial L_T \llcorner B^n = (-1)^n \, \mathbb{P}(u_T) \,,$$

where $\mathbb{P}(u_T) \in \mathcal{D}_{n-2}(B^n)$ is given by (2.1). Set now

$$\mathcal{T}_u := \{T \in \mathrm{cart}(B^n \times \mathbb{S}^1) \mid u_T = u \ \text{ in } \ (3.3)\} \,, \qquad u \in BV(B^n, \mathbb{S}^1) \,. \tag{3.4}$$

From [6], see also [7], we have:

Proposition 3.1 *For every $u \in BV(B^n, \mathbb{S}^1)$ the class $\mathcal{T}_u$ is non-empty.*

By the definition of integral mass this yields:

Proposition 3.2 *For every $u \in BV(B^n, \mathbb{S}^1)$, the current $\mathbb{P}(u)$ is an $(n-2)$-dimensional integral flat chain, i.e., $m_{i,B^n}(\mathbb{P}(u)) < \infty$.*

Remark 3.3 In the case $n = 2$ Proposition 3.2 corresponds to [11, Thm. 1]. In fact, a 0-dimensional integral flat chain Λ in B^2 is a distribution in $\mathcal{D}'(B^2)$ given by the at most countable sum of unit Dirac masses $\Lambda = \sum_k (\delta_{p_k} - \delta_{n_k})$ for some sequences $\{p_k\}$, $\{n_k\} \subset B^2$ such that $\sum_k |p_k - n_k| < \infty$. On account of Proposition 2.2, we thus infer that for every $u \in BV(B^2, \mathbb{S}^1)$

$$T(u) = 2\pi \sum_k (\delta_{p_k} - \delta_{n_k}) \,, \quad \{p_k\}, \{n_k\} \subset B^2 \,, \quad \sum_k |p_k - n_k| < \infty \,.$$

THE BV-ENERGY OF CARTESIAN CURRENTS. Following [8], the *BV-energy* $\mathcal{E}_{1,1}(T)$ of a current $T \in \mathrm{cart}(B^n \times \mathbb{S}^1)$ is defined by

$$\mathcal{E}_{1,1}(T) = \int_{B^n} |\nabla u_T| \, dx + |D^C u_T|(B^n) + E_{Jc}(T) \,,$$

provided that T is decomposed as in (3.3). The *jump-concentration* energy term $E_{Jc}(T)$ takes into account of both the jump part $G^J_{u_T}$ and of the concentration part $L_T \times [\![\mathbb{S}^1]\!]$, and in general

$$E_{Jc}(T) < \int_{J_u} \mathcal{H}^1(l_x) \, d\mathcal{H}^{n-1}(x) + 2\pi \, \mathbf{M}(L_T) \,.$$

More precisely, we may write

$$T = G^a_{u_T} + G^C_{u_T} + T^{Jc} \,, \qquad T^{Jc} := G^J_{u_T} + L_T \times [\![\mathbb{S}^1]\!] \,.$$

It turns out that

$$T^{Jc}(\phi(x,y) \, dx) = 0 \qquad \forall \, \phi \in C^\infty_0(B^n \times \mathbb{S}^1)$$

and if $\omega = \omega^{(1)}$ is given by (1.2), we have

$$T^{Jc}(\omega^{(1)}) = \sum_{i=1}^n \sum_{j=1}^2 \int_{J_c(T)} \left(\int_{\Gamma_x} \phi^j_i(x,y) \, dy^j \right) \nu_i \, d\mathcal{H}^{n-1}(x) \,.$$

In the above formula, $J_c(T)$ is the countably $\mathcal{H}^{n-1}$-rectifiable set of B^n given by the union of the jump set J_{u_T} and of the $(n-1)$-rectifiable set $\mathcal{L}_T$ of positive density of L_T. Moreover, $\nu = (\nu_1, \ldots, \nu_n)$ is an extension to $J_c(T)$ of the unit normal ν_{u_T} to J_{u_T}. Finally, for $x \in J_c(T)$, Γ_x is an oriented arc of $\mathbb{S}^1$ connecting $u^-_T(x)$ and $u^+_T(x)$, in such a way that $[\![\Gamma_x]\!]$ is an i.m. rectifiable current in $\mathcal{D}_1(\mathbb{S}^1)$ satisfying $\partial [\![\Gamma_x]\!] = \delta_{u^+_T(x)} - \delta_{u^-_T(x)}$.

Notice that $\Gamma_x = l_x$ if $x \notin \mathcal{L}_T$, and that $\partial [\![\Gamma_x]\!] = 0$ if $x \notin J_{u_T}$, i.e., Γ_x is an integral 1-cycle. Therefore, by Federer's decomposition theorem we may write

$$[\![\Gamma_x]\!] = [\![\gamma_x]\!] + k(x) \cdot [\![\mathbb{S}^1]\!] \,, \qquad \mathbf{M}([\![\Gamma_x]\!]) = \mathcal{L}(\gamma_x) + 2\pi \, |k(x)| \,, \qquad k(x) \in \mathbb{Z} \,,$$

where $\mathcal{L}(\gamma_x)$ is the length of an oriented simple arc γ_x in $\mathbb{S}^1$, satisfying $\partial[\![\gamma_x]\!] = \delta_{u_T^+(x)} - \delta_{u_T^-(x)}$, and $k : J_c(T) \to \mathbb{Z}$ is an integer-valued $\mathcal{H}^{n-1} \llcorner J_c(T)$-summable function, with $k(x) = 0$ if $x \notin \mathcal{L}_T$. With the above notation we have

$$E_{Jc}(T) := \int_{J_{u_T}} \mathcal{L}(\gamma_x)\, d\mathcal{H}^{n-1}(x) + 2\pi \int_{\mathcal{L}_T} |k(x)|\, d\mathcal{H}^{n-1}(x)\,. \tag{3.5}$$

Relaxed energy. Let now

$$\widetilde{\mathcal{E}_{1,1}}(u) := \inf\left\{ \liminf_{h\to\infty} \int_{B^n} |Du_h|\, dx \;:\; \{u_h\} \subset C^1(B^n, \mathbb{S}^1)\,, \quad u_h \to u \quad \text{a.e.}\right\}.$$

From the density of smooth maps [6], see also [8], we have:

Proposition 3.4 *For every $u \in BV(B^n, \mathbb{S}^1)$ the relaxed energy is finite, $\widetilde{\mathcal{E}_{1,1}}(u) < \infty$. Moreover,*

$$\widetilde{\mathcal{E}_{1,1}}(u) = \inf\{\mathcal{E}_{1,1}(T) \mid T \in \mathcal{T}_u\}\,.$$

From Proposition 3.4 we then obtain:

Proposition 3.5 *Let $n \geq 2$. For every $u \in BV(B^n, \mathbb{S}^1)$ we have*

$$\widetilde{\mathcal{E}_{1,1}}(u) = \int_{B^n} |\nabla u|\, dx + |D^C u|(B^n) + \inf\{E_{Jc}(T) \mid T \in \mathcal{T}_u\}\,,$$

where $E_{Jc}(T)$ is given by (3.5) *and $\mathcal{T}_u$ by* (3.4).

Energy estimate. Finally, let

$$|u|_{BV\,\mathbb{S}^1} := \int_{B^n} |\nabla u|\, dx + |D^C u|(B^n) + \int_{J_u} d_{\mathbb{S}^1}(u^+, u^-)\, d\mathcal{H}^{n-1}\,,$$

where $d_{\mathbb{S}^1}$ stands for the *geodesic* distance in $\mathbb{S}^1$. In [8] it is proved:

Proposition 3.6 *Let $n \geq 2$. For every $u \in BV(B^n, \mathbb{S}^1)$ we have $\widetilde{\mathcal{E}_{1,1}}(u) \leq 2\,|u|_{BV\,\mathbb{S}^1}$.*

4 Optimal lifting

This section contains new results. Firstly, we outline the link between liftings and connections of maps in $BV(B^n, \mathbb{S}^1)$, Theorem 4.4. Secondly, we write in an explicit way the energy of the *optimal lifting* of BV-maps, Theorem 4.8. Finally, we show that the minimal connection $L(u)$ can be seen as the distance from gradient maps, Theorem 4.9. These results have been proved in [11] in the case $n = 2$, see [2] and [9] for the case $W^{1,1}(B^n, \mathbb{S}^1)$. Analogous results for BV-maps with prescribed boundary data can be obtained in a similar way. For the sake of clearness, we postpone the proofs to the next section.

Connections as triplets. Using the notation from Sec. 3, if $u \in BV(B^n, \mathbb{S}^1)$ and $T \in \mathcal{T}_u$ we have

$$T = G_u + L_T \times [\![\mathbb{S}^1]\!]$$

where $G_u \in \mathcal{D}_n(B^n \times \mathbb{S}^1)$ is the current carried by the graph of u, see Sec. 1, and $L_T \in \mathcal{R}_{n-1}(B^n)$ satisfies

$$\partial L_T \llcorner B^n = (-1)^n\, \mathbb{P}(u)\,. \tag{4.1}$$

Since $L_T \in \mathcal{R}_{n-1}(B^n)$, following [5] we may also write $L_T = \tau(\mathcal{L}_T, \theta_T, \overrightarrow{\xi_T})$, where $\mathcal{L}_T$ is an $(n-1)$-rectifiable set of B^n, $\theta_T : \mathcal{L}_T \to \mathbb{N}^+$ is a positive integer-valued $\mathcal{H}^{n-1} \llcorner \mathcal{L}_T$-summable function, the *multiplicity*, and $\overrightarrow{\xi_T} : \mathcal{L}_T \to \Lambda_{n-1}\mathbb{R}^n$ is an $\mathcal{H}^{n-1} \llcorner \mathcal{L}_T$-measurable map with values in the space of the $(n-1)$-vectors of $\mathbb{R}^n$, such that $|\overrightarrow{\xi_T}| \equiv 1$ and $\overrightarrow{\xi_T}(x)$ provides an orientation to the approximate tangent $(n-1)$-space at $\mathcal{H}^{n-1}$-a.e. point $x \in \mathcal{L}_T$. This means that

$$L_T(\omega) = \int_{\mathcal{L}_T} \theta_T \, \langle \omega, \overrightarrow{\xi_T} \rangle \, d\mathcal{H}^{n-1} \qquad \forall\, \omega \in \mathcal{D}^{n-1}(B^n)\,.$$

Let now $\nu_T : \mathcal{L}_T \to \mathbb{S}^{n-1}$ be such that $\nu_T(x)$ defines the unit normal to $\mathcal{L}_T$ at x, oriented in such a way that $\nu_T \wedge \overrightarrow{\xi_T} = e_1 \wedge \cdots \wedge e_n$ for $\mathcal{H}^{n-1}$-a.e. $x \in \mathcal{L}_T$, where $(e_1, \ldots, e_n)$ is the canonical basis in $\mathbb{R}^n$. Also, let $f_T : \mathcal{L}_T \to 2\pi\,\mathbb{N}^+$ be given by $f_T(x) := 2\pi\,\theta_T(x)$. We have:

Proposition 4.1 *Let $L_T \in \mathcal{R}_{n-2}(B^n)$ be such that* (4.1) *holds true. For every $\zeta \in \mathrm{Lip}(B^n, \Lambda^{n-2} T B^n)$*

$$\langle T(u), \zeta \rangle = (-1)^n \int_{\mathcal{L}_T} f_T \langle F(\zeta), \nu_T \rangle \, d\mathcal{H}^{n-1} \tag{4.2}$$

provided that ζ is decomposed as in (2.2) *and $F(\zeta)$ is the vector field associated to ζ, see* (2.4)*.*

Conversely, let $(\mathcal{L}, f, \nu)$ be any triplet such that:

i) $\mathcal{L} \subset B^n$ is $(n-1)$-rectifiable;

ii) $f : \mathcal{L}_T \to 2\pi\,\mathbb{N}^+$ is $\mathcal{H}^{n-1} \llcorner \mathcal{L}$-summable;

iii) $\nu : \mathcal{L} \to \mathbb{S}^{n-1}$ is $\mathcal{H}^{n-1} \llcorner \mathcal{L}_T$-measurable with $\nu(x)$ orthogonal to the approximate tangent $(n-1)$-space at $\mathcal{H}^{n-1}$-a.e. point $x \in \mathcal{L}$.

To $(\mathcal{L}, f, \nu)$ it corresponds the i.m. rectifiable $(n-1)$-current $L \in \mathcal{R}_{n-1}(B^n)$ such that, writing $L = \tau(\widetilde{\mathcal{L}}, \theta, \overrightarrow{\xi})$, then $\widetilde{\mathcal{L}} = \mathcal{L}$, $\theta = f/2\pi$ and $\nu \wedge \overrightarrow{\xi} \equiv e_1 \wedge \cdots \wedge e_n$. Therefore, in the sequel the following identification will be assumed:

$$\mathcal{R}_{n-1}(B^n) \simeq \{ (\mathcal{L}, f, \nu) \mid \text{i), ii) and iii) hold true} \}\,.$$

These facts lead us to give the following:

Definition 4.2 *To any $u \in BV(B^n, \mathbb{S}^1)$ we associate the set $\mathcal{J}(T(u))$ of triplets $(\mathcal{L}, f, \nu)$ satisfying the properties* i), ii) *and* iii) *above and such that for every $\zeta \in \mathrm{Lip}(B^n, \Lambda^{n-2} T B^n)$*

$$\langle T(u), \zeta \rangle = (-1)^n \int_{\mathcal{L}} f \langle F(\zeta), \nu \rangle \, d\mathcal{H}^{n-1}$$

provided that ζ is decomposed as in (2.2) *and $F(\zeta)$ is the vector field associated to ζ, see* (2.4)*.*

Remark 4.3 By the proof of Proposition 4.1, on account of Proposition 2.2, we readily infer that (4.2) implies (4.1). Therefore, we obtain that

$$\mathcal{J}(T(u)) = \{ L \in \mathcal{R}_{n-1}(B^n) \mid \partial L \llcorner B^n = (-1)^n \, \mathbb{P}(u) \}\,.$$

This yields that $\mathcal{J}(T(u))$ *identifies the Cartesian currents $T \in \mathrm{cart}(B^n \times \mathbb{S}^1)$ with underlying function* $u_T = u$, i.e., the class $\mathcal{T}_u$ in (3.4).

The link between connections and liftings. We have:

Theorem 4.4 *Let $n \geq 2$ and $u \in BV(B^n, \mathbb{S}^1)$. For every triplet $(\mathcal{L}, f, \nu) \in \mathcal{J}(T(u))$ there exists a lifting $\psi \in BV(B^n, \mathbb{R})$ of u such that*

$$D\psi = (u \times \nabla u)\, dx + d\,(u \times D^C u) + \rho(u^+, u^-)\, \nu_u \, \mathcal{H}^{n-1} \llcorner J_u + (-1)^{n-1} f\, \nu\, \mathcal{H}^{n-1} \llcorner \mathcal{L}\,. \tag{4.3}$$

Conversely, for every lifting $\psi \in BV(B^n, \mathbb{R})$ of u there exists a triplet $(\mathcal{L}, f, \nu) \in \mathcal{J}(T(u))$ such that (4.3) *holds true.*

Existence of liftings. The proof of Theorem 4.4 relies on the following result from [6] stating the existence of a lifting of currents in $\mathrm{cart}(B^n \times \mathbb{S}^1)$. We recall, see [7], that the current *subgraph* of an L^1-function $\psi \in L^1(B^n, \mathbb{R})$ is the $(n+1)$-dimensional current in $\mathcal{D}_{n+1}(B^n \times \mathbb{R})$ defined by

$$SG_\psi(\phi(x,t)dx \wedge dt) := \int_{B^n} \left(\int_0^{\psi(x)} \phi(x,t)\, dt \right) dx\,, \qquad \phi \in C_c^\infty(B^n \times \mathbb{R})\,. \tag{4.4}$$

Notice that the boundary current ∂SG_ψ has finite mass in $B^n \times \mathbb{R}$ if and only if ψ belongs to the class $BV(B^n, \mathbb{R})$. Finally, in the sequel we will denote by $i : B^n \times \mathbb{R} \to B^n \times \mathbb{S}^1$ the map

$$i(x,t) := (x, \cos t, \sin t)\,,$$

and by G_{q_0} the current in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ integration over the graph of the constant map $q_0(x) \equiv (1,0)$.

The following existence result was proved in [6], see also [7, Vol. II, Sec. 6.2.2].

Proposition 4.5 *Let $T \in \mathrm{cart}(B^n \times \mathbb{S}^1)$. The following facts hold:*

i) *There exists a current $\Sigma \in \mathcal{D}_{n+1}(B^n \times \mathbb{S}^1)$ such that*

$$T - G_{q_0} = (-1)^n \partial \Sigma\,.$$

ii) *There exists a function $\psi \in BV(B^n, \mathbb{R})$ such that $\Sigma = i_\# SG_\psi$, i.e.,*

$$T - G_{q_0} = (-1)^n i_\# \partial SG_\psi\,. \tag{4.5}$$

iii) *If $u_T \in BV(B^n, \mathbb{S}^1)$ is the BV-function corresponding to T, then*

$$u_T = e^{i\psi} \qquad \mathcal{L}^n\text{-a.e. on } B^n\,.$$

Remark 4.6 From Theorem 4.4 we readily infer that the converse of Proposition 4.5 holds true: *for every $u \in BV(B^n, \mathbb{S}^1)$ and every lifting $\psi \in BV(B^n, \mathbb{R})$ of u there exists a current $T \in \mathrm{cart}(B^n \times \mathbb{S}^1)$ such that $u_T = u$, i.e., $T \in \mathcal{T}_u$, and* (4.5) *holds true.*

Optimal lifting. Following [2] [11], we consider for every $u \in BV(B^n, \mathbb{S}^1)$ the energy

$$\widehat{\mathcal{E}_{1,1}}(u) := \inf\{\, |D\psi|(B^n) \;\mid\; \psi \in BV(B^n, \mathbb{R})\,,\; u = e^{i\psi} \text{ a.e. on } B^n \}\,.$$

Since B^n is simply connected, arguing as in [2, Prop. 2], see also [11, Rmk. 4], we obtain that

$$\widehat{\mathcal{E}_{1,1}}(u) = \widetilde{\mathcal{E}_{1,1}}(u) \qquad \forall\, u \in BV(B^n, \mathbb{S}^1)\,.$$

Remark 4.7 Arguing as in [11], we also infer that the infimum in the above formula is achieved. Moreover, by Proposition 3.6 we immediately obtain that

$$\widehat{\mathcal{E}_{1,1}}(u) \le 2\, |u|_{BV\, \mathbb{S}^1} \qquad \forall\, u \in BV(B^n, \mathbb{S}^1)\,,$$

compare [3] for the case $n = 2$.

Denoting by χ_A the characteristic function of a set A, we have:

Theorem 4.8 *Let $n \ge 2$. For every $u \in BV(B^n, \mathbb{S}^1)$*

$$\widehat{\mathcal{E}_{1,1}}(u) = \int_{B^n} |\nabla u|\, dx + |D^C u|(B^n) + \min_{(\mathcal{L}, f, \nu) \in \mathcal{J}(T(u))} \left\{ \int_{\mathcal{L} \cup J_u} |f\, \nu\, \chi_{\mathcal{L}} + (-1)^{n-1} \rho(u^+, u^-)\, \nu_u\, \chi_{J_u}|\, d\mathcal{H}^{n-1} \right\}.$$

Minimal connection as a distance. As a consequence of Proposition 4.5, we finally prove in any dimension $n \ge 2$ the following:

Theorem 4.9 *For any $u \in BV(B^n, \mathbb{S}^1)$ we have*

$$L(u) = \frac{1}{2\pi} \min_{\psi \in BV(B^n, \mathbb{R})} |(u \times \nabla u)\, dx + d\,(u \times D^C u) + \rho(u^+, u^-)\, \nu_u\, d\mathcal{H}^{n-1} \llcorner J_u - D\psi|(B^n)\,.$$

5 Proofs

PROOF OF PROPOSITION 4.1: By Proposition 2.2 and (4.1) we know that

$$\langle T(u),\zeta\rangle = (-1)^n\, 2\pi\, L_T(d\zeta)\,.$$

Moreover, since $L_T = \tau(\mathcal{L}_T,\theta_T,\overrightarrow{\xi_T})$, we have

$$L_T(d\zeta) = \int_{\mathcal{L}_T} \theta_T\langle d\zeta,\overrightarrow{\xi_T}\rangle\, d\mathcal{H}^{n-1}\,.$$

From Sec. 1 we also know that

$$d\zeta = \sum_{i=1}^n (-1)^{i-1}F^i(\zeta)\,\widehat{dx^i}\,.$$

Now, since $\nu_T\wedge\overrightarrow{\xi_T} = e_1\wedge\cdots\wedge e_n$ and $|\overrightarrow{\xi_T}| = 1$, writing

$$\overrightarrow{\xi_T} = \sum_{i=1}^n (-1)^{i-1}\xi^i\widehat{e_i}\,,\qquad \widehat{e_i} := e_1\wedge\cdots\wedge e_{i-1}\wedge e_{i+1}\wedge\cdots\wedge e_n\,,$$

we infer that $\nu_T = \sum_{i=1}^n \xi^i e_i$, so that for $\mathcal{H}^{n-1}$-a.e. $x\in\mathcal{L}_T$ we obtain

$$\langle d\zeta,\overrightarrow{\xi_T}\rangle = \sum_{i=1}^n F^i(\zeta)\,\xi^i = \langle F(\zeta),\nu_T\rangle$$

and hence the assertion. $\square$

PROOF OF THEOREM 4.4: Let $(\mathcal{L},f,\nu)\in\mathcal{J}(T(u))$. By Remark 4.3, there exists a current $T\in\mathcal{T}_u$ such that $T = G_u + L_T\times[\![\mathbb{S}^1]\!]$ and $L_T\in\mathcal{R}_{n-1}(B^n)$ corresponds to $(\mathcal{L},f,\nu)$, according to Sec. 4. By Proposition 4.5, in correspondence to $T\in\mathcal{T}_u$ there exists a function $\psi_T\in BV(B^n,\mathbb{R})$ such that

$$G_u + L_T\times[\![\mathbb{S}^1]\!] - G_{q_0} = (-1)^n i_\#\partial SG_{\psi_T} \qquad \text{on}\quad \mathcal{D}^n(B^n\times\mathbb{S}^1)\,. \tag{5.1}$$

In the sequel we omit to write the action of the projection maps π and $\widehat{\pi}$. For any $\varphi\in C_c^\infty(B^n,\mathbb{R}^n)$, let $\omega_\varphi\in\mathcal{D}^{n-1}(B^n)$ be given by (1.6), so that

$$\begin{aligned}\omega_\varphi\wedge u^\#\omega_{\mathbb{S}^1} &= \sum_{i=1}^n (-1)^{i-1}\varphi^i\widehat{dx^i}\wedge(u^1du^2-u^2du^1)\\ &= (-1)^{n-1}\sum_{i=1}^n \varphi^i\cdot(u\times u_{x_i})\,dx\,.\end{aligned}$$

By (1.5) we thus have

$$G_u^a(\omega_\varphi\wedge\omega_{\mathbb{S}^1}) = \int_{B^n}\omega_\varphi\wedge u^\#\omega_{\mathbb{S}^1} = (-1)^{n-1}\int_{B^n}\langle u\times\nabla u,\varphi\rangle\,dx\,. \tag{5.2}$$

Moreover, since

$$\omega_\varphi\wedge\omega_{\mathbb{S}^1} = \left(\sum_{i=1}^n(-1)^{i-1}\varphi^i\widehat{dx^i}\right)\wedge\left(\sum_{j=1}^2(-1)^j y^{\overline{j}}dy^j\right) = \sum_{i=1}^n\sum_{j=1}^2(-1)^{i-1+j}\varphi^i\, y^{\overline{j}}\,\widehat{dx^i}\wedge dy^j\,,$$

we may write $\omega_\varphi\wedge\omega_{\mathbb{S}^1} = \omega^{(1)}\in\mathcal{D}^n(B^n\times\mathbb{S}^1)$ in (1.2) by taking $\phi_i^j := (-1)^{n-1+j}\varphi^i\, y^{\overline{j}}$, so that

$$\sum_{j=1}^2\phi^j(x,u)\,dD^Cu^j = (-1)^{n-1}\varphi\sum_{j=1}^2(-1)^j u^{\overline{j}}\,dD^Cu^j = (-1)^{n-1}\varphi\, d(u\times D^Cu)\,.$$

By the definition of G_u^C we then infer

$$G_u^C(\omega_\varphi \wedge \omega_{\mathbb{S}^1}) = (-1)^{n-1} \int_{B^n} \varphi \, d(u \times D^C u)\,. \tag{5.3}$$

Also, by the definition of G_u^J, on account of (1.3) we have

$$\begin{aligned} G_u^J(\omega_\varphi \wedge \omega_{\mathbb{S}^1}) &= (-1)^{n-1} \sum_{i=1}^n \sum_{j=1}^2 \int_{J_u} \left(\int_{l_x} (-1)^j \varphi^i(x)\, y^{\bar{j}}\, dy^j \right) \nu_u^i(x)\, d\mathcal{H}^{n-1}(x) \\ &= (-1)^{n-1} \sum_{i=1}^n \int_{J_u} \left(\int_{l_x} \omega_{\mathbb{S}^1} \right) \varphi^i(x)\, \nu_u^i(x)\, d\mathcal{H}^{n-1}(x) \\ &= (-1)^{n-1} \int_{J_u} \rho(u^+, u^-)\, \varphi \cdot \nu_u \, d\mathcal{H}^{n-1}\,. \end{aligned} \tag{5.4}$$

On the other hand, we have

$$L_T \times [\![\mathbb{S}^1]\!](\omega_\varphi \wedge \omega_{\mathbb{S}^1}) = L_T(\omega_\varphi) \cdot [\![\mathbb{S}^1]\!](\omega_{\mathbb{S}^1}) = 2\pi \, L_T(\omega_\varphi) \tag{5.5}$$

where, we recall, $L_T = \tau(\mathcal{L}, f/2\pi, \overrightarrow{\xi})$ with $\nu \wedge \overrightarrow{\xi} = e_1 \wedge \cdots \wedge e_n$, so that, as in the proof of Proposition 4.1,

$$2\pi \, L_T(\omega_\varphi) = \int_{\mathcal{L}} f \langle \omega_\varphi, \overrightarrow{\xi} \rangle \, d\mathcal{H}^{n-1} = \int_{\mathcal{L}} f \langle \varphi, \nu \rangle \, d\mathcal{H}^{n-1}\,,$$

whereas clearly

$$G_{q_0}(\omega_\varphi \wedge \omega_{\mathbb{S}^1}) = 0\,. \tag{5.6}$$

Finally, since $d\omega_\varphi = \mathrm{div}\varphi \, dx$ and

$$i^\# d(\omega_\varphi \wedge \omega_{\mathbb{S}^1}) = i^\# (d\omega_\varphi \wedge \omega_{\mathbb{S}^1}) = i^\# (\mathrm{div}\varphi \, dx \wedge \omega_{\mathbb{S}^1}) = \mathrm{div}\varphi \, dx \wedge dt\,,$$

on account of (4.4) we have

$$\begin{aligned} i_\# \partial SG_{\psi_T}(\omega_\varphi \wedge \omega_{\mathbb{S}^1}) &= SG_{\psi_T}(i_\# d(\omega_\varphi \wedge \omega_{\mathbb{S}^1})) \\ &= SG_{\psi_T}(\mathrm{div}\varphi(x) dx \wedge dt) \\ &= \int_{B^n} \mathrm{div}\varphi(x)\, (\psi_T(x) - 0)\, dx = -\langle D\psi_T, \varphi \rangle\,. \end{aligned} \tag{5.7}$$

In conclusion, by (5.1) we have obtained for any $\varphi \in C_c^\infty(B^n, \mathbb{R}^n)$

$$\langle D\psi_T, \varphi \rangle = \int_{B^n} \langle u \times \nabla u, \varphi \rangle \, dx + \int_{B^n} \varphi \, d(u \times D^C u) + \int_{J_u} \rho(u^+, u^-)\, \varphi \cdot \nu \, d\mathcal{H}^{n-1} + (-1)^{n-1} \int_{\mathcal{L}} f \langle \varphi, \nu \rangle \, d\mathcal{H}^{n-1}$$

and hence (4.3). To prove the converse, arguing as in [11], for every lifting ψ of u we have

$$\nabla \psi \, dx = (u \times \nabla u)\, dx\,, \qquad d\, D^C \psi = d\,(u \times D^C u)\,.$$

Moreover, the jump set $J_u \subset J_\psi$ and, possibly changing the orientation of the unit normal ν_ψ to J_ψ, we may assume that $\mathcal{H}^{n-1}$-a.e. on J_u

$$\nu_\psi = \nu_u\,, \quad e^{i\psi^+} = u^+\,, \quad e^{i\psi^-} = u^-\,.$$

Therefore, we have

$$\begin{aligned} &\psi^+ - \psi^- \equiv \rho(u^+, u^-) \pmod{2\pi} \quad \mathcal{H}^{n-1}\text{-a.e. in } J_u \\ &\psi^+ - \psi^- \equiv 0 \pmod{2\pi} \quad \mathcal{H}^{n-1}\text{-a.e. in } J_\psi \setminus J_u\,, \end{aligned}$$

whence there exists an $\mathcal{H}^{n-1} \llcorner J_\psi$-integrable function $f_\psi : J_\psi \to 2\pi\,\mathbb{Z}$ such that

$$D\psi = (u \times \nabla u)\, dx + d\,(u \times D^C u) + \rho(u^+, u^-)\, \nu_u \, \mathcal{H}^{n-1} \llcorner J_u + (-1)^{n-1} f_\psi \, \nu \, \mathcal{H}^{n-1} \llcorner J_\psi\,. \tag{5.8}$$

Setting

$$\mathcal{L}_\psi := \{x \in J_\psi \mid f_\psi(x) \neq 0\},$$

we clearly have $\mathcal{H}^{n-1}(\mathcal{L}_\psi) < \infty$, hence $\mathcal{L}_\psi$ is $(n-1)$-rectifiable. Moreover, possibly changing the orientation of ν_ψ, we may and do assume that $f_\psi \llcorner \mathcal{L}_\psi$ takes values in $2\pi\,\mathbb{N}^+$. Set now $L_\psi := \tau(\mathcal{L}_\psi, \theta_\psi, \overrightarrow{\xi_\psi})$, where $\theta_\psi := f_\psi/2\pi$ and $\overrightarrow{\xi_\psi}$ is such that $\nu_\psi \wedge \overrightarrow{\xi_\psi} = e_1 \wedge \cdots \wedge e_n$ for $\mathcal{H}^{n-1}$-a.e. $x \in \mathcal{L}_\psi$. It turns out that $L_\psi \in \mathcal{R}_{n-1}(B^n)$ whereas, by the property (5.8) and by the same computation as in the first part of the proof, we infer that (5.1) holds true, with $\psi_T = \psi$ and $L_T = L_\psi$. Since (5.1) yields that $\partial(G_u + L_\psi \times [\![\mathbb{S}^1]\!]) = 0$ on $\mathcal{D}^{n-1}(B^n \times \mathbb{S}^1)$, we readily obtain that $\partial L_\psi = (-1)^n\,\mathbb{P}(u)$. By Proposition 4.1 we finally conclude that $(\mathcal{L}_\psi, f_\psi, \nu_\psi)$ belongs to $\mathcal{J}(T(u))$, see Definition 4.2, whereas (4.3) is given by (5.8). $\square$

Proof of Theorem 4.8: By Theorem 4.4, for every $(\mathcal{L}, f, \nu) \in \mathcal{J}(T(u))$ there exists a lifting $\psi \in BV(B^n, \mathbb{R})$ of u such that (4.3) holds true. This clearly yields the inequality "$\leq$", as the measures $(u \times \nabla u)\,dx$, $d\,(u \times D^C u)$, and $\rho(u^+, u^-)\,\nu_u\,\mathcal{H}^{n-1} \llcorner J_u + (-1)^{n-1} f\,\nu\,\mathcal{H}^{n-1} \llcorner \mathcal{L}$ are mutually singular. Moreover, since by Remark 4.7 there exists a lifting $\psi \in BV(B^n, \mathbb{R})$ of u such that $\widehat{\mathcal{E}_{1,1}}(u) = |D\psi|(B^n)$, the equality follows again by Theorem 4.4, and the minimum is achieved. $\square$

Proof of Theorem 4.9: From Sec. 3 we know that for any $u \in BV(B^n, \mathbb{S}^1)$ the integral mass

$$m_{i,B^n}(\mathbb{P}(u)) = \inf\{\mathbf{M}(L_T) \mid G_u + L_T \times [\![\mathbb{S}^1]\!] \in \mathcal{T}_u\}. \tag{5.9}$$

By Proposition 4.5, to any $T \in \mathcal{T}_u$ it corresponds a function $\psi_T \in BV(B^n, \mathbb{R})$ such that (5.1) holds true. Moreover, Remark 4.6 yields that the converse holds true. We then apply (5.1) to $\omega \in \mathcal{D}^n(B^n \times \mathbb{S}^1)$ given by $\omega = \pi^{\#}\omega_\varphi \wedge \widehat{\pi}^{\#}\omega_{\mathbb{S}^1}$, where $\omega_\varphi \in \mathcal{D}^{n-1}(B^n)$ is defined by (1.6). On account of (5.2), (5.3), (5.4), (5.5), (5.6), and (5.7) we readily infer that

$$(-1)^n\,2\pi\,L_T(\omega_\varphi) = \int_{B^n} \langle u \times \nabla u, \varphi\rangle\,dx + \int_{B^n} \varphi\,d(u \times D^C u) + \int_{J_u} \rho(u^+, u^-)\,\varphi \cdot \nu\,d\mathcal{H}^{n-1} - \langle D\psi_T, \phi\rangle.$$

This yields

$$\begin{aligned} \mathbf{M}(L_T) &:= \sup\{L_T(\omega_\phi) \mid \|\phi\|_\infty \leq 1\} \\ &= \frac{1}{2\pi}|(u \times \nabla u)\,dx + d(u \times D^C u) + \rho(u^+, u^-)\,\nu\,d\mathcal{H}^{n-1} \llcorner J_u - D\psi_T|(B^n). \end{aligned}$$

In conclusion, by (5.9) and (3.2) we obtain the assertion. $\square$

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