ON THE RELATION BETWEEN GENERALIZED MORREY SPACES AND MEASURE DATA PROBLEMS

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ABSTRACT. We consider measure data problems of p-Laplacian type. The measure on the right-hand side has the property that the total variation of a generic ball decays in terms of generic functions of the radius; we show that this condition has a natural relation with gradient integrability properties and we get, as corollary, borderline cases of classic results.

To Carlo Sbordone, mathematician and neapolitan gentleman, on the occasion of his 70th birthday.

1. INTRODUCTION

We study measure data problems of p-Laplacian type:

$$-\operatorname{div} a(x, Du) = \mu \qquad \text{in } \Omega, \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^n$, $n \geq 2$. The Carathédory vector field $a(\cdot)$ satisfies the growth and monotonicity assumptions

$$\begin{cases} \langle a(x,\xi_1) - a(x,\xi_2), \xi_1 - \xi_2 \rangle \geq \nu\big(s + |\xi_1| + |\xi_2|\big)^{p-2}|\xi_1 - \xi_2|^2, \\ |a(x,\xi)| \leq L\,\big(s + |\xi|\big)^{p-1} \end{cases} \tag{1.2}$$

for almost every $x \in \Omega$ and for all $\xi_1, \xi_2, \xi \in \mathbb{R}^n$, with $0 < \nu \leq 1 \leq L$ and $s \geq 0$. The right-hand side μ is a signed Borel measure with finite total total variation, and we suppose in general that it does not belong to the dual of the energy space naturally associated to the operator on the left-hand side. For this reason in the paper we shall always assume $p \leq n$.

The goal of this study is the analysis of some borderline cases for the integrability of the gradient of solutions to (1.1), when the measure on the right-hand side is known to have certain decay properties. In particular we suppose

$$\|\mu\|_{L^{1,\phi(\cdot)}(\Omega)} := \sup_{\substack{B_R(x_0)\subset\Omega \\ R\leq 1}} \phi(R)\frac{|\mu|(B_R(x_0))}{R^n} < \infty. \tag{1.3}$$

This is to say, we prescribe the decay of the total variation of the balls with radius R in terms of a generic increasing function $\phi : [0,1] \to [0,1]$ such that $\phi(0) = 0$; we may suppose $\phi(1) = 1$ too. Note that we are thinking to the case where $R^n/\phi(R) \to 0$ as $R \to 0$, that is the main case of interest. For simplicity, we shall assume also that the map $R \mapsto \phi(R)/R$ is strictly increasing; for instance, we can assume that the function ϕ is convex. This kind of condition on measures (and functions), natural, was first considered and studied in [12, 28]; we refer to [26, 27] too for various related functional properties. As said, we prove that this condition ensures a better degree of regularity than the one expected when no density information on the measure is known. The correct and natural

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way to encode such regularity is in terms of Marcinkiewicz spaces: this is to say, we locally estimate the decay of the measure of the super level set of Du

$$\big|\{x : |Du(x)| > \lambda\}\big|$$

in terms of functions related to ϕ. Integrability properties of the gradient of solutions to measure data problems are usually formulated in terms of Marcinkiewicz spaces, which are the optimal ones in view of the behavior of the fundamental solutions; see for instance [2, 10, 20, 22] and [4, 5]. Clearly, *generalized Marcinkiewicz spaces* must be considered here, due to the generality of the situation we are considering.

In this first part of the paper we just want to propose significant corollaries to the main estimate in generalized Marcinkiewicz spaces we are going to describe in Theorem 2.5. We think that the most interesting corollary is the following

Corollary 1.1. *Let $\vartheta \in [p, n]$. If $u \in W^{1,p}(\Omega)$ is a solution to (1.1), where $\mu \in L^\infty_{\rm loc}(\Omega)$ satisfies the assumption (1.3) with*

$$\phi(R) = R^\vartheta \log^\alpha(1/R) \qquad \textit{with} \quad \alpha > \vartheta - 1 \tag{1.4}$$

or

$$\phi(R) = R^\vartheta \log^{\vartheta-1}(1/R)\log^\varsigma(\log(1/R)) \qquad \textit{with} \quad \varsigma > \vartheta - 1 \tag{1.5}$$

for all $R \le R_0$, for some $R_0 \in (0, 1]$, then

$$|Du|^{p-1} \in L^{\frac{\vartheta}{\vartheta-1}}_{\rm loc}(\Omega) \tag{1.6}$$

and the local estimate

$$\begin{aligned}\left(\fint_{B_R(x_0)} |Du|^{\frac{\vartheta(p-1)}{\vartheta-1}}\,dx\right)^{\frac{\vartheta-1}{\vartheta(p-1)}} &\le c\fint_{B_{2R}(x_0)} \big(|Du| + s + 1\big)\,dx \\ &\quad + c\left[\frac{R}{\phi(R)}\right]^{\frac{1}{p-1}} \|\mu\|^{\frac{1}{p-1}}_{L^{1,\phi(\cdot)}(\Omega)}\end{aligned} \tag{1.7}$$

holds for every ball such that $B_{2R}(x_0) \subset \Omega$, $R \le 1$, with the constant depending on n, p, ν, L, α (or ς), $\phi(\cdot)$ and R_0.

Let us stress that in the case $\vartheta = p$, the previous result gives that if the measure decays as

$$\frac{|\mu|(B_R(x_0))}{R^n} \lesssim \frac{1}{R^p \log^\alpha(1/R)} \qquad \text{for some } \alpha > p - 1$$

then $|Du| \in L^p_{\rm loc}(\Omega)$. On the other hand, we notice that in this case a classic result by Wolff (see the forthcoming (2.1)) states that the measure belongs to the dual of the energy space, and it is therefore natural to solve (1.1) obtaining an energy solution (that are, in particular, SOLAs). Notice moreover that once fixed an exponent $q \in [n(p-1)/(n-1), p]$, the previous result, choosing

$$\vartheta = \frac{q}{q-(p-1)}$$

implies that is sufficient to set

$$\phi(R) = R^{\frac{q}{q-(p-1)}} \log^\alpha(1/R), \qquad \alpha > \frac{p-1}{q-(p-1)}$$

or

$$\phi(R) = R^{\frac{q}{q-(p-1)}} \log^{\frac{p-1}{q-(p-1)}}(1/R)\log^\varsigma(\log(1/R)), \qquad \varsigma > \frac{p-1}{q-(p-1)}$$

close to zero in order to have the gradient in L^q locally.

One could also consider conditions ensuring Orlicz regularity, in the particular class of Zygmund spaces, for the gradient:

Corollary 1.2. *Let $\vartheta \in [p,n)$ and $\gamma \in \mathbb{R}$ or $\vartheta = n$ and $\gamma \geq -1$. If $u \in W^{1,p}(\Omega)$ is as in Corollary 1.1 and $\mu \in L^{\infty}_{\rm loc}(\Omega)$ satisfies the assumption (1.3) with*

$$\phi(R) = R^{\vartheta} \log^{\alpha}(1/R) \qquad \textit{with} \quad \alpha > (\gamma+1)(\vartheta-1), \tag{1.8}$$

or

$$\phi(R) = R^{\vartheta} \log^{(\gamma+1)(\vartheta-1)}(1/R) \log^{\varsigma}(\log(1/R)) \qquad \textit{with} \quad \varsigma > \vartheta - 1, \tag{1.9}$$

for $R \leq R_0$ for some $R_0 \in (0,1]$; then

$$|Du|^{p-1} \in L^{\frac{\vartheta}{\vartheta-1}} \log^{\gamma} L \tag{1.10}$$

locally in Ω and the estimate

$$\begin{aligned} \| Du \|_{L^{\frac{\vartheta}{\vartheta-1}} \log^{\gamma} L(B_R(x_0))} &\leq c \fint_{B_{2R}(x_0)} \big(|Du| + s + 1\big)\, dx \\ &\quad + c \left[\frac{R}{\phi(R)}\right]^{\frac{1}{p-1}} \|\mu\|^{\frac{1}{p-1}}_{L^{1,\phi(\cdot)}(\Omega)} \end{aligned} \tag{1.11}$$

holds for a constant depending on n, p, ν, L, α (respectively, ς), $\phi(\cdot)$ and R_0.

We recall the reader that a measurable function $g : A \subset \mathbb{R}^n \to \mathbb{R}$ belongs to the Orlicz space $L^{\frac{\vartheta}{\vartheta-1}} \log^{\gamma} L(A) = L^{\vartheta'} \log^{\gamma} L(A)$ for $q > 1, \gamma \in \mathbb{R}$, if

$$\int_A |g|^{\vartheta'} \log^{\gamma}(e + |g|)\, dx < \infty.$$

The (averaged) Luxemburg norm employed in (1.11) is defined as follows: for $0 < |A| < \infty$,

$$\| g \|_{L^{\vartheta'} \log^{\gamma} L(A)} = \inf \left\{ \lambda > 0 : \frac{1}{\lambda^{\vartheta'}} \fint_A |g|^{\vartheta'} \log^{\gamma}\left(e + \frac{|g|}{\lambda}\right) dx \leq 1 \right\}.$$

Remark 1.3. Note that the previous results are stated as a priori estimates for energy solutions; it is standard to extend them to SOLAs, that is, solutions of (1.1) obtained as a limit of energy solutions with regularized data; see [3, 8, 15, 18] for details and for the precise form of the local estimates in this case.

The result contained in Corollary 1.1 is the borderline (in terms of integrability of Du) case of the results in [20, 23] stating the following:

$$\mu \in L^{1,\vartheta}(\Omega), \qquad \vartheta \in [p,n] \qquad \Longrightarrow \qquad |Du|^{p-1} \in \mathcal{M}^{\frac{\vartheta}{\vartheta-1}}_{\rm loc}(\Omega). \tag{1.12}$$

The space $L^{1,\vartheta}(\Omega)$ is a classic Morrey space, that in our case corresponds to the simple choice $\phi(R) = R^{\vartheta}$; on the other hand, the space $\mathcal{M}^{\frac{\vartheta}{\vartheta-1}}_{\rm loc}(\Omega)$ is the localized version of the classic Marcinkiewicz, or weak Lebesgue, space (see (2.7)-(2.8) for $\Phi(\lambda) = \lambda^{\vartheta'}$). Note that for $\vartheta < p$ the aforementioned classic result of Wolff (see (2.1)) states that $L^{1,\vartheta}$ embeds into the dual space of $W^{1,p}$. In the case $\vartheta = n$, the results in [20] give back the classic, sharp result

$$\mu \in L^{1,n}(\Omega) \equiv \mathcal{M}_b(\Omega) \qquad \Longrightarrow \qquad |Du| \in \mathcal{M}^{\frac{n(p-1)}{n-1}}(\Omega),$$

for which we refer to [11]; notice that the class of measures satisfying (1.3) for $\vartheta = n$ is nothing else than the full space of signed Borel measures with finite total total variation. We stress that a different assumption implying the regularity in (1.10), dealing with better integrability instead of better decay properties of the datum, is contained in [21, Theorems 3 & 12]:

$$\mu \in L^{1,\vartheta}(\Omega) \cap L \log L(\Omega) \qquad \text{or} \qquad \mu \in L \log L^{\vartheta}(\Omega)$$

imply

$$|Du|^{p-1} \in L^{\frac{\vartheta}{\vartheta-1}}_{\rm loc}(\Omega);$$

note that again the improvement should be of "logarithmic type". We recall the reader that $L\log L^{\vartheta}$ is the subspace of the Orlicz-Zygmund space $L\log L$ made of the functions μ such that their $L\log L$ norm on balls decays in terms of powers of the radius:

$$\|\mu\|_{L\log L(B_R)} \lesssim R^{-\vartheta}.$$

As a final remark in the elliptic setting, we stress that we are able to reproduce another subtle phenomenon typical of elliptic equations with measure data: density information on the measure transfer into density information for the solutions. In particular, a more refined version of the result depicted in (1.12), which can still be found in [20, 23], states that not only $Du \in \mathcal{M}^{\vartheta'}(B_R(x_0))$ for every ball $B_{2R}(x_0) \subset \Omega$, but also gives a description of the decay of this norm in a way perfectly consistent with the decay of the measure. Indeed one has, for every ball as before

$$|\mu|(B_R(x_0)) \lesssim R^{n-\vartheta}, \qquad \vartheta \in [p,n] \quad \Longrightarrow \quad \||Du|^{p-1}\|^{\vartheta'}_{\mathcal{M}^{\vartheta'}(B_R(x_0))} \lesssim R^{n-\vartheta},$$

see [23, Theorem 4.3]. We refer to the forthcoming Remark 2.7 for a suitable version in our setting.

Result in the setting of classic Morrey spaces are also available in the more difficult parabolic setting: here the condition to be considered involves standard parabolic cylinders

$$\sup_{Q_R(z_0)\subset\Omega\times(0,T)} \frac{|\mu|(Q_R(z_0))}{R^{N-\vartheta}} < \infty, \tag{1.13}$$

where $Q_R(z_0) = B_R(x_0)\times(t_0-R^2, t_0+R^2)$. Note that $|Q_R| = c(n)R^N = c(n)R^{n+2}$.

The improved integrability of the gradient is again formulated in terms of Marcinkiewicz spaces: summarizing the results of [7, 4, 5], we have that

$$\mu \text{ satisfies (1.13) for some } 2 \le \vartheta \le N \quad \Longrightarrow \quad |Du| \in \mathcal{M}^{p-1+\frac{1}{\vartheta-1}}_{\rm loc}\big(\Omega\times(0,T)\big),$$

where u is a solution to the evolutionary analogue of (1.1) under structure of parabolic p-Laplacian type in the style of (1.2). Some results are also available in the case of splitting measures, and these are quite surprising in the singular case $p < 2$ (see [5] for more details); global results are also available, at least for $p = 2$: [6].

Note that many of the results of this paper, if we strengthen our assumptions (in particular, if we impose more regularity on the map $x \mapsto a(x,\xi)$) follow as corollary of recent potential estimates, see [16, 17, 18, 19]; in any case many arguments developed here are necessary to deduce the final form of the estimates, even starting from the pointwise estimates for the gradient. The adaptation of our paper to the parabolic degenerate and singular setting is also not trivial, due mainly to the fact that the lack of scaling of the equation forces to vary the techniques [4, 5] and to obtain results in a different form.

2. Notation, Theorem 2.5 and Technical tools

2.1. **Notation.** This section is devoted to fix the notation we will use in the rest of the paper. c will denote a generic constant larger than one, possibly varying from line to line. Constants we need to recall will be denoted with special symbols, such as $c_1, c_2, \tilde{c}, C$. Relevant dependencies will be highlighted between parentheses or after the equations; when non essential, the dependence on a parameter will be suppressed. We denote by

$$B_r(x_0) := \{x \in \mathbb{R}^n \,:\, |x-x_0| < r\}$$

the open ball with center x_0 and radius $r > 0$; when not important, or clear from the context, we shall omit denoting the center as follows: $B_r \equiv B_r(x_0)$. Very often, when not otherwise stated, different balls in the same context will share the same center. We shall also denote $B_1 = B_1(0)$ if not differently specified. Finally, with B being a given ball with radius r and γ being a positive number, we denote by γB the concentric ball with radius γr. With $\mathcal{B} \subset \mathbb{R}^n$ being a measurable subset with finite and positive measure $|\mathcal{B}| > 0$, and with $g\colon \mathcal{B} \to \mathbb{R}^k$, $k \geq 1$, being a measurable map, we shall denote by

$$(g)_{\mathcal{B}} \equiv \fint_{\mathcal{B}} g(x)\,dx := \frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} g(x)\,dx$$

its integral average. We set

$$p^- := \min\{p-1, 1\}.$$

and often we shall denote in short $L^{1,\phi(\cdot)}$ for $L^{1,\phi(\cdot)}(\Omega)$. For $\sigma \in \mathbb{R}$, we shall also use the notation $[\log t]^\sigma$, when $\log t$ is nonnegative, for $\log^\sigma t$. $\mathbb{N}$ is the set $\{1, 2, \dots\}$ while $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2.2. **Properties of ϕ.** A classic result by Wolff (see [14, Corollary of Theorem 1] and [1] too) implies that the a measure μ belongs to the dual space of $W^{1,p}(\mathbb{R}^n)$ if and only if its Wolff potential $\mathbf{W}^{\mu}_{1,p}$ belongs to $L^1(\mathbb{R}^n, d\mu)$. The local version of this results says that if the measure satisfies

$$\int_\Omega \int_0^1 \left(\frac{\mu(B_\rho(x))}{\rho^{n-p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho}\, d\mu(x) = +\infty \tag{2.1}$$

then it does not belong to the dual of $W^{1,p}(\Omega)$. Since in our case

$$\int_\Omega \int_0^1 \left(\frac{|\mu|(B_\rho(x))}{\rho^{n-p}}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho}\, d|\mu|(x) \leq \|\mu\|^{\frac{1}{p-1}}_{L^{1,\phi(\cdot)}(\Omega)} |\mu|(\Omega) \int_0^1 \left(\frac{\rho^p}{\phi(\rho)}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho}$$

once we want to treat the measure data setting, it is natural to suppose that

$$\int_0^1 \left(\frac{\rho^p}{\phi(\rho)}\right)^{\frac{1}{p-1}} \frac{d\rho}{\rho} = \int_0^1 \left(\frac{\rho}{\phi(\rho)}\right)^{\frac{1}{p-1}} d\rho = +\infty. \tag{2.2}$$

We shall use the notation

$$f(\rho) = \left(\frac{\rho}{\phi(\rho)}\right)^{\frac{1}{p-1}} \tag{2.3}$$

and we take $\Psi : [1, \infty) \to (0, 1]$ as

$$\Psi(\lambda) := f^{-1}(\lambda);$$

recall that $f(1) = 1$ and notice that Ψ is decreasing. Our main assumption on Ψ will be the following one: There exist constants H_0 and λ_0, both larger than one, and a function $h_\Psi : [H_0, \infty) \to [0, \infty)$ such that

$$\Psi(H\lambda) \geq h_\Psi(H)\Psi(\lambda) \quad \forall \lambda \geq \lambda_0, H \geq H_0$$
$$\text{with} \qquad \lim_{H\to+\infty} h_\Psi(H) H^\alpha = +\infty \tag{2.4}$$

for every $\alpha > 1$. If a constant will depend on $h_\Psi(\cdot)$, H_0, or on the limit in (2.4), we shall simply say it depends on $\phi(\cdot)$.

For some results, moreover, we will need the following assumption: the existence of $\widetilde{H}_0, \tilde{\lambda}_0 \gg 1$ and a function such that $\phi(H\lambda) \geq h_\phi(H)\phi(\lambda)$ for all $H \geq \widetilde{H}_0$, $\lambda \geq \tilde{\lambda}_0$ and, for every $\beta \in (0, p)$,

$$\int_0^1 \left[\frac{1}{s^\beta h_\phi(1/s)}\right]^{\frac{n}{n-1}} \frac{ds}{s} < \infty. \tag{2.5}$$

Remark 2.1. We stress that if ϕ satisfies a ∇_2 condition of the type

$$\phi(2\rho) \geq \tilde{L}\phi(\rho) \qquad \tilde{L} > 2, \qquad \text{for every } \rho \text{ sufficiently small,}$$

then simple computations show that (2.4) holds for

$$h_\Psi(H) = H^{-\frac{\log_2(\tilde{L}/2)}{p-1}}, \qquad H_0 = \Big(\frac{\tilde{L}}{2}\Big)^{\frac{1}{p-1}}.$$

In particular, if $\tilde{L} = 2^p$, (2.4) holds with $h_\Psi(H) = 1/H$ and λ sufficiently large. Clearly also (2.5) would hold.

Remark 2.2. Note that when f is not monotone, then, since from (2.2) we have

$$\limsup_{\rho\to 0} f(\rho) = \limsup_{\rho\to 0} \left[\frac{\rho}{\phi(\rho)}\right]^{\frac{1}{p-1}} = +\infty,$$

we could define $\Psi : (1,\infty) \to (0,1)$ in the following way:

$$\Psi(\lambda) := \inf \Big\{\bar{\rho} \in (0,1] : f(\rho) < \lambda \quad \text{for all } \rho \in [\bar{\rho}, 1]\Big\}.$$

Ψ is again nonincreasing and such that

$$\lim_{\lambda\to\infty} \Psi(\lambda) = 0, \qquad f(\Psi(\lambda)) = \lambda, \qquad \Psi(f(\rho)) \geq \rho \tag{2.6}$$

in their domains. This is sufficient to extend many of our results to the setting where the function $R \mapsto \phi(R)/R$ is not necessarily monotone; we will not focus, however, on this case in order not to overload the paper with technical complications.

Before stating the general result we are aiming at, let us introduce the generalized Marcinkiewicz spaces. For an increasing function Φ such that $\Phi(0) = 0$ and $\Phi(\lambda) \to \infty$ as $\lambda \to \infty$ and a measurable set $A \subset \mathbb{R}^n$, we define the generalized Marcinkiewicz spaces $\mathcal{M}^{\Phi(\cdot)}(A)$ as the set of functions such that the following decay condition on level sets holds:

$$\sup_{\lambda>0} \Phi(\lambda/c)\big|\{x \in A : |f(x)| > \lambda\}\big| < \infty \qquad \text{for some } c > 0; \tag{2.7}$$

compare with [9, 10, 24, 25]. Their local variant is defined in the usual way. We set

$$\|f\|_{\mathcal{M}^{\Phi(\cdot)}(A)} := \inf \Big\{c > 0 : \sup_{\lambda>0} \Phi\Big(\frac{\lambda}{c}\Big)\big|\{x \in A : |f(x)| > \lambda\}\big| \leq 1\Big\} \tag{2.8}$$

and we notice that the quantity, despite the notation we employ, is not a norm since it does not, in general, satisfies the triangle's inequality. Note that is often possible to endow $\mathcal{M}^{\Phi(\cdot)}(A)$ with a true norm, involving the maximal rearrangement of f, equivalent to $\|f\|_{\mathcal{M}^{\Phi(\cdot)}(A)}$. It will also be useful to have an averaged, rescaling invariant norms. We define for $0 < |A| < \infty$

$$⫲ f\|_{\mathcal{M}^{\Phi(\cdot)}(A)} := \inf \Big\{c > 0 : \frac{1}{|A|}\sup_{\lambda>0} \Phi\Big(\frac{\lambda}{c}\Big)\big|\{x \in A : |f(x)| > \lambda\}\big| \leq 1\Big\}. \tag{2.9}$$

Remark 2.3. In the literature the spaces we denote with $\mathcal{M}^{\Phi(\cdot)}$ are usually called *weak* generalized Marcinkiewicz spaces, or *weak* Orlicz-Marcinkiewicz spaces in order to distinguish them from so-called Orlicz-Marcinkiewicz spaces, equipped with the norm expressed by the means of the maximal rearrangement mentioned before. In general, this norm is just larger than $\|f\|_{\mathcal{M}^{\Phi(\cdot)}(\Omega)}$ as defined in (2.8); it turns out to be equivalent only if Φ grows slowly enough, in the sense that the following condition must hold:

$$⨍_0^t \Phi^{-1}\Big(\frac{1}{s}\Big)\, ds \leq C\, \Phi^{-1}\Big(\frac{1}{t}\Big)$$

for some constant C and all $t > 0$. For more details, we refer to [25, Section 7.10] or to [24].

Remark 2.4. Notice that in the classic case of Morrey spaces, where $\phi(R) = R^{\vartheta}$, we have

$$f(\rho) = \rho^{-\frac{\vartheta-1}{p-1}}, \qquad \Psi(\lambda) = f^{-1}(\lambda) = \lambda^{-\frac{p-1}{\vartheta-1}}, \qquad \frac{\lambda^{p-1}}{\Psi(\lambda)} = \lambda^{\frac{\vartheta(p-1)}{\vartheta-1}}$$

and (2.4) is implied by the fact that $\vartheta \geq p$.

After introducing the generalized Marcinkiewicz spaces, we can state our main result.

Theorem 2.5. *Let $u \in W^{1,p}(\Omega)$ be a solution to (1.1), where $\mu \in L^{\infty}_{\mathrm{loc}}(\Omega)$ satisfies the assumption (1.3). Assume that $R \mapsto \phi(R)/R$ is strictly increasing and the technical assumption (2.4) holds. Then*

$$|Du| \in \mathcal{M}^{\Phi(\cdot)}_{\mathrm{loc}}(\Omega) \qquad \textit{with} \qquad \Phi(s) = \frac{s^{p-1}}{\Psi(s)}, \qquad s > 0,$$

and the local estimate

$$\sup_{\lambda>0} \Phi(\lambda) \frac{|\{x \in B_R(x_0) : |Du(x)| > \lambda\}|}{|B_R(x_0)|} \leq \Phi\left(c \fint_{B_{2R}(x_0)} \big(|Du| + s\big)\, dx + c \|\mu\|^{\frac{1}{p-1}}_{L^{1,\phi(\cdot)}(\Omega)} \left[\frac{R}{\phi(R)}\right]^{\frac{1}{p-1}} \right) \tag{2.10}$$

holds for every ball $B_{2R}(x_0) \subset \Omega$ and for a constant depending on n, p, ν, L and $\phi(\cdot)$.

Note that, in view of Remark 2.4, this result gives back the result described in (1.12) when one considers usual Morrey spaces. For the following two result we shall assume that (2.4)-(2.5) both hold for every $\lambda > 0$, $H \geq 1$ with h_{Ψ}, h_{ϕ} defined over $[1, \infty)$.

Remark 2.6. An estimate for the Marcinkiewicz norm as defined in (2.8) can be deduced too:

$$\|Du\|_{\mathcal{M}^{\tilde{\Phi}(\cdot)}(B_R(x_0))} \leq c \fint_{B_{2R}(x_0)} \big(|Du| + s + 1\big)\, dx + c\, \|\mu\|^{\frac{1}{p-1}}_{L^{1,\phi(\cdot)}(\Omega)} \left[\frac{R}{\phi(R)}\right]^{\frac{1}{p-1}}, \tag{2.11}$$

where

$$\widetilde{\Phi}(s) := \frac{s^{p-1}}{c\Psi(s)} \tag{2.12}$$

and the constants depend on n, p, ν, L and $\phi(\cdot)$.

Observe that in the standard case described in Remark 2.4 it holds

$$\Phi(\lambda) = \lambda^{\frac{\vartheta(p-1)}{\vartheta-1}}. \tag{2.13}$$

Remark 2.7. If we assume (2.5) too, we can also describe the decay of the averaged Marcinkiewicz semi-norm:

$$\left[\frac{\phi(R)}{R}\right]^{\frac{1}{p-1}} \|Du\|_{\mathcal{M}^{\tilde{\Phi}(\cdot)}(B_R(x_0))} \leq c\Big[|\mu|(\Omega) + \|\mu\|_{L^{1,\phi(\cdot)}(\Omega)} + 1\Big]^{\frac{1}{p-1}}$$

for a constant depending on n, p, ν, L and $\phi(\cdot)$, for R sufficiently small. This is the natural analogue of the result encoded in [20, Theorem 1.8] for the standard setting: after some algebraic manipulations, this result states

$$R^{\frac{\vartheta-1}{p-1}} \|Du\|_{\mathcal{M}^{\frac{\vartheta(p-1)}{\vartheta-1}}(B_R(x_0))} \leq c\Big[|\mu|(\Omega) + \|\mu\|_{L^{1,\phi(\cdot)}(\Omega)}\Big]^{\frac{1}{p-1}}.$$

The averaged norm in the display above is nothing else than, up to universal constants, the standard averaged Marcinkiewicz norm, that is the norm defined in (2.9) for the choice in (2.13).

2.3. **Technical results.** We collect here two variants of classic iteration results, both fitting our purposes.

Lemma 2.8. *Let $R > 0$ and $f : [R, 2R] \to [0, \infty)$ be bounded and satisfy the relation*

$$f(r_1) \leq \frac{1}{2} f(r_2) + \Phi\Big(\frac{\mathcal{A}}{(r_2 - r_1)^\gamma}\Big) + \mathcal{B} \tag{2.14}$$

for all $R \leq r_1 < r_2 \leq 2R$ and for constants $\mathcal{A}, \mathcal{B} \geq 1$ and $\gamma > 0$. Then there exists a constant c depending on p, γ and $\phi(\cdot)$ such that

$$f(R) \leq c\Big[\Phi\Big(\frac{\mathcal{A}}{R^\gamma}\Big) + \mathcal{B}\Big].$$

Proof. We consider, as in [13, Lemma 6.1] the sequence of points R_k with $R_0 = R$ and

$$R_{k+1} - R_k = \lambda(1-\lambda)^k R,$$

$\lambda \in (0, 1)$ to be chosen. Iterating k times the relation in (2.14), $k \in \mathbb{N}$, we get

$$f(R) = f(R_0) \leq \frac{1}{2^k} f(R_k) + \sum_{i=0}^{k-1} \frac{1}{2^i} G\Big(\frac{\mathcal{A}}{[\lambda(1-\lambda)^i R]^\gamma}\Big) + \mathcal{B} \sum_{i=0}^{k-1} \frac{1}{2^i}$$

and letting $k \to \infty$ we conclude, since for i sufficiently large we have, using (2.4) for $\beta = 2$

$$\begin{aligned}
\Phi\Big(\frac{\mathcal{A}}{[\lambda(1-\lambda)^i R]^\gamma}\Big) &= \Big(\frac{\mathcal{A}}{[\lambda(1-\lambda)^i R]^\gamma}\Big)^{p-1} \Big[\Psi\Big(\frac{\mathcal{A}}{[\lambda(1-\lambda)^i R]^\gamma}\Big)\Big]^{-1} \\
&\leq \Big(\frac{\mathcal{A}}{[\lambda(1-\lambda)^i R]^\gamma}\Big)^{p-1} \Big[\Psi\Big(\frac{\mathcal{A}}{R^\gamma}\Big) h_\Psi\Big(\frac{1}{[\lambda(1-\lambda)^i]^\gamma}\Big)\Big]^{-1} \\
&\leq \Big(\frac{\mathcal{A}}{[\lambda(1-\lambda)^i R]^\gamma}\Big)^{p-1} \Big[\Psi\Big(\frac{\mathcal{A}}{R^\gamma}\Big) (\lambda(1-\lambda)^i)^{2\gamma}\Big]^{-1} \\
&= \frac{1}{\lambda^{\gamma(p+1)}} \Big(\frac{\mathcal{A}}{R^\gamma}\Big)^{p-1} \Big[\Psi\Big(\frac{\mathcal{A}}{R^\gamma}\Big)\Big]^{-1} \frac{1}{(1-\lambda)^{\gamma(p+1)i}}.
\end{aligned}$$

The proof is concluded, since a choice of λ such that $2(1-\lambda)^{\gamma(p+1)} > 1$ implies the convergence of the series

$$\sum_{i=0}^{\infty} \frac{1}{2^i} G\Big(\frac{\mathcal{A}}{[\lambda^i(1-\lambda)R]^\gamma}\Big).$$

□

Lemma 2.9. *Let $f, g : [0, 2R] \to \mathbb{R}$ be positive, non-decreasing functions such that*

$$f(\rho) \leq c_0\Big[\varepsilon f(2r) + \Big(\frac{\rho}{r}\Big)^\gamma f(r)\Big] + g(r) \qquad \textit{for all } 0 < \rho \leq r \leq R, \tag{2.15}$$

being $c_0 \geq 1$, $\varepsilon > 0$ and $\gamma > 0$. Suppose that there exist $H_0 \geq 1$ and a (non-decreasing) function $\tilde{h} : [H_0, \infty) \to [1, \infty)$ such that

$$g(Hs) \leq \tilde{h}(H) g(s) \;\; \forall s > 0, \forall H \geq H_0 \quad \textit{with} \quad \int_0^1 s^\sigma \tilde{h}(1/s) \frac{ds}{s} < \infty \tag{2.16}$$

for some $0 < \sigma < \gamma$. Then there exists a constant ε_0, depending on γ, σ, c_0 and a constant c_1, depending only on $c_0, \gamma, \sigma, \tilde{h}(\cdot)$ such that if $\varepsilon \leq \varepsilon_0$, it holds that

$$f(r_1) \leq c_1\Big[\Big(\frac{r_1}{r_2}\Big)^\sigma f(r_2) + g(r_1)\Big] \qquad \textit{for all } 0 < r_1 \leq r_2 \leq 2R. \tag{2.17}$$

Proof. We define $\tau \equiv \tau(\gamma, \sigma, c_0) < 1/2$ as

$$\tau = \frac{1}{(2^{\gamma+1}c_0)^{1/(\gamma-\sigma)}} \qquad \Longleftrightarrow \qquad 2^\gamma c_0 \tau^{\gamma-\sigma} \leq \frac{1}{2};$$

moreover we set $\varepsilon_0 \equiv \varepsilon_0(\gamma, \sigma, c_0) = \tau^\sigma/[2c_0]$ and we take $r_2 \leq 2R$. Using (2.15) with $\rho = \tau^{\ell+1}r_2$ and $2r = \tau^\ell r_2$, $\ell \in \mathbb{N}_0$ and assuming $\varepsilon \leq \varepsilon_0$, we get

$$f(\tau^{\ell+1}r_2) \leq c_0\Big[\varepsilon_0 + 2^\gamma\tau^\sigma\tau^{\gamma-\sigma}\Big]f(\tau^\ell r_2) + g(\tau^\ell r_2) = \tau^\sigma f(\tau^\ell r_2) + g(\tau^\ell r_2);$$

iterating k times, $k \in \mathbb{N}$, we then infer

$$f(\tau^k r_2) \leq \tau^{k\sigma} f(r_2) + \sum_{i=0}^{k-1} \tau^{i\sigma} g(\tau^{k-1-i} r_2). \tag{2.18}$$

We split

$$\sum_{i=0}^{k-1} \tau^{i\sigma} g(\tau^{k-1-i} r_2) = \sum_{i=0}^{i_0-1}(\dots) + \sum_{i=i_0}^{k-1}(\dots)$$

where $i_0 \equiv i_0(\gamma, \sigma, c_0, H_0)$ is the smallest integer such that $\tau^{-i_0-2} \geq H_0$; clearly we shall only consider $k > i_0$. For the first sum we simply have

$$\sum_{i=0}^{i_0-1} \tau^{i\sigma} g(\tau^{k-1-i} r_2) \leq \sum_{i=0}^{i_0-1} g\Big(\frac{\tau^{k+1}r_2}{\tau^{i_0+2}}\Big) \leq i_0\, \tilde{h}(\tau^{-i_0-2}) g(\tau^{k+1}r_2) = c\, g(\tau^{k+1}r_2)$$

with $c \equiv c(\gamma, \sigma, c_0, \tilde{h}(\cdot))$. For the second sum, since g is increasing, using (2.16), we estimate

$$\begin{aligned}
\sum_{i=i_0}^{k-1} \tau^{i\sigma} g(\tau^{k-1-i} r_2) &= -\frac{1}{\log \tau} \sum_{i=i_0}^{k-1} \tau^{i\sigma} g(\tau^{k-1-i} r_2) \int_{\tau^{i+3}}^{\tau^{i+2}} \frac{d\rho}{\rho} \\
&\leq c(\tau, \gamma, \sigma)\, g(\tau^{k+1}r_2) \sum_{i=i_0}^{k-1} \tilde{h}(1/\tau^{i+2}) \tau^{(i+3)\sigma} \int_{\tau^{i+3}}^{\tau^{i+2}} \frac{d\rho}{\rho} \\
&\leq c\, g(\tau^{k+1}r_2) \sum_{i=i_0}^{k-1} \int_{\tau^{i+3}}^{\tau^{i+2}} \rho^\sigma \tilde{h}(1/\rho)\, \frac{d\rho}{\rho} \\
&\leq c\, g(\tau^{k+1}r_2) \int_0^1 \rho^\sigma \tilde{h}(1/\rho)\, \frac{d\rho}{\rho},
\end{aligned}$$

with $c \equiv c(\gamma, \sigma, c_0)$; the last integral is finite thanks to (2.16). Thus, merging the two estimates into (2.19), we obtain

$$f(\tau^k r_2) \leq \tau^{k\sigma} f(r_2) + c(\gamma, \sigma, c_0, \tilde{h}(\cdot)) g(\tau^{k+1}r_2) \tag{2.19}$$

for $k > i_0$ and now the conclusion of the proof is standard. In particular for $\varrho \leq \tau^{i_0+1}r_2$ we take $k \geq i_0 + 1$ such that $\tau^{k+1}r_2 < \varrho \leq \tau^k r_2$ and we have

$$f(r_1) < f(\tau^k r_2) \leq \frac{1}{\tau^\sigma}\Big(\frac{r_1}{r_2}\Big)^\sigma f(r_2) + c\, g(r_1) \leq c_1\Big[\Big(\frac{r_1}{r_2}\Big)^\sigma f(r_2) + g(r_1)\Big].$$

If $\rho \in (\tau^{i_0+1}r_2, r_2]$, on the other hand, the estimate above is trivial if we further enlarge the constant c_1 by a factor depending on σ, τ, i_0. □

3. Miscellanea of classic results

We collect in this section several results already present in the literature. For this, for a given ball $B_R(x_0) \subset \Omega$, we consider the so-called $a(\cdot)$-harmonic lifting

$$\begin{cases} \operatorname{div} a(x, Dv) = 0 & \text{in } B_R(x_0), \\ v = u & \text{on } \partial B_R(x_0); \end{cases} \tag{3.1}$$

existence follows from standard monotonicity methods, since the boundary datum belongs to $W^{1,p}(B_R)$. The first result we need is a simple comparison estimate.

Lemma 3.1 (Comparison estimate)**.** *Let $v \in u + W_0^{1,p}(B_R(x_0))$ be the solution to (3.1). Then for every*

$$q \in \left[1, \min\left\{\frac{n(p-1)}{n-1}, p\right\}\right), \tag{3.2}$$

there exists a constant $c \equiv c(n,p,\nu,q)$ such that

$$\begin{aligned} \fint_{B_R(x_0)} |Du - Dv|^q \, dx &\le c \left[\frac{|\mu|(B_{2R}(x_0))}{R^{n-1}}\right]^{\frac{q}{p-1}} \\ &\quad + c\,\chi_{\{p<2\}} \left(\fint_{B_{2R}(x_0)} (|Du| + s)\, dx \right)^{q(2-p)} \left[\frac{|\mu|(B_{2R}(x_0))}{R^{n-1}}\right]^q. \end{aligned} \tag{3.3}$$

Proof. For the comparison estimate

$$\begin{aligned} \fint_{B_R(x_0)} |Du - Dv|^q \, dx &\le c \left[\frac{|\mu|(B_R(x_0))}{R^{n-1}}\right]^{\frac{q}{p-1}} \\ &\quad + c\,\chi_{\{p<2\}} \left(\fint_{B_R(x_0)} (|Du| + s)^q\, dx \right)^{2-p} \left[\frac{|\mu|(B_R(x_0))}{R^{n-1}}\right]^q, \end{aligned} \tag{3.4}$$

for q as in (3.2) and $c \equiv c(n,p,\nu,q)$, we refer to [20, Lemma 4.1] or [18, Lemma 2] for the case $p \ge 2$ and [23, Lemma 9.2] for the subquadratic case $p < 2$. In the case $p < 2$ we must use the standard reverse-Hölder's inequality that can be found in [3, Lemma 3.1] (see also [5, Proposition 4.3]): with q and c as above, one has

$$\fint_{B_R(x_0)} (|Du| + s)^q \, dx \le c \left(\fint_{B_{2R}(x_0)} (|Du| + s)\, dx \right)^q + c \left[\frac{|\mu|(B_{2R}(x_0))}{R^{n-1}}\right]^{\frac{q}{p-1}}.$$

□

Since v solves an equation of p-Laplacian type, the following gradient higher integrability result holds (see [13, Remark 6.12]):

Proposition 3.2. *Let $v \in u + W_0^{1,p}(B_R(x_0))$ be the solution of (3.1). Then there exist an exponent $\chi \equiv \chi(n,p,\nu,L) > 1$ and a constant $c \equiv c(n,p,\nu,L)$ such that*

$$\fint_{B_{R/2}(x_0)} (|Dv| + s)^{p\chi} \, dx \le c \left(\fint_{B_R(x_0)} (|Dv| + s)\, dx \right)^{p\chi} \tag{3.5}$$

holds.

Moreover the next Morrey-type estimate encodes the Hölder character of solutions to (3.1): see [20] for the result in this form and several applications in the setting of measure data.

Proposition 3.3. *Let $v \in u + W^{1,p}_0(B_R(x_0))$ be as in Proposition 3.2 and let $q \in (0,p]$. There exist an exponent $\beta_0 \equiv \beta_0(n,p,L/\nu) \in (0,1]$ and a constant $c \equiv c(n,p,L/\nu,q)$ such that*

$$\fint_{B_\rho(x_0)} (|Dv|+s)^q\,dx \le c\Big(\frac{\rho}{R}\Big)^{q(\beta_0-1)} \fint_{B_R(x_0)} (|Dv|+s)^q\,dx$$

holds for every radius $\rho \in (0,R]$.

The previous result, together with our assumption (1.3), allows us to prove a decay estimate that generalize the one available for solutions to measure data equations with Morrey data, see [23, Proposition 10.1].

Lemma 3.4. *Let $u \in C^1_{\rm loc}(\Omega)$ be a solution to (1.1) under the assumption (1.3) and (2.5) on $\mu \in L^\infty(\Omega)$ and let $B_{2R}(x_0) \subset \Omega$ with $R \le 1/[2\widetilde{H}_0]$; suppose moreover that $R \mapsto \phi(R)/R$ is monotone increasing. For every exponent q as in (3.2), there exists a constant c depending on n,p,ν,L,q and $\phi(\cdot)$ such that*

$$\fint_{B_R(x_0)} (|Du|+s)^q\,dx \le c\left[|\mu|(\Omega)+\|\mu\|_{L^{1,\phi}(\Omega)}\right]^{\frac{q}{p-1}}\left[\frac{R}{\phi(R)}\right]^{\frac{q}{p-1}}. \tag{3.6}$$

Proof. Fix $\rho, r \in (0,R]$ with $\rho < r$ and define the p-harmonic lifting $v \in u+W^{1,p}_0(B_r(x_0))$ as in (3.1) over $B_r(x_0)$, being q fixed as in the statement, we have

$$\begin{aligned}
\fint_{B_\rho(x_0)} (|Du|+s)^q\,dx &\le c\fint_{B_\rho(x_0)} (|Dv|+s)^q\,dx + c\fint_{B_\rho(x_0)} |Du-Dv|^q\,dx \\
&\le c\Big(\frac{\rho}{r}\Big)^{q(\beta_0-1)} \fint_{B_r(x_0)} (|Dv|+s)^q\,dx \\
&\qquad + c\Big(\frac{r}{\rho}\Big)^{n} \fint_{B_r(x_0)} |Du-Dv|^q\,dx \\
&\le c\Big(\frac{\rho}{r}\Big)^{q(\beta_0-1)} \fint_{B_r(x_0)} (|Du|+s)^q\,dx \\
&\qquad + c\Big(\frac{r}{\rho}\Big)^{n} \fint_{B_r(x_0)} |Du-Dv|^q\,dx.
\end{aligned}$$

At this point we estimate, using Lemma 3.1, Young's inequality only in the case $p<2$ and (1.3)

$$\begin{aligned}
\fint_{B_r(x_0)} |Du-Dv|^q\,dx &\le c\left[\frac{|\mu|(B_{2r}(x_0))}{r^{n-1}}\right]^{\frac{q}{p-1}} \\
&\quad + c\,\chi_{\{p<2\}}\Big(\fint_{B_{2r}(x_0)} (|Du|+s)\,dx\Big)^{q(2-p)}\left[\frac{|\mu|(B_{2r}(x_0))}{R^{n-1}}\right]^{q} \\
&\le c_\varepsilon\left[\frac{|\mu|(B_{2r}(x_0))}{R^{n-1}}\right]^{\frac{q}{p-1}} + \varepsilon\Big(\fint_{B_{2r}(x_0)} (|Du|+s)\,dx\Big)^q \\
&\le c_\varepsilon \|\mu\|_{L^{1,\phi}}^{\frac{q}{p-1}}\left[\frac{r}{\phi(4r)}\right]^{\frac{q}{p-1}} + \varepsilon\Big(\fint_{B_{2r}(x_0)} (|Du|+s)\,dx\Big)^q \\
&\le c_\varepsilon \|\mu\|_{L^{1,\phi}}^{\frac{q}{p-1}}\left[\frac{r}{\phi(r)}\right]^{\frac{q}{p-1}} + \varepsilon\fint_{B_{2r}(x_0)} (|Du|+s)^q\,dx;
\end{aligned}$$

c_ε is a constant depending on n,p,ν,L and ε; thus

$$\int_{B_\rho(x_0)} (|Du|+s)^q\,dx \le \bar{c}\left[\Big(\frac{\rho}{r}\Big)^{n-q(1-\beta_0)} \int_{B_r(x_0)} (|Du|+s)^q\,dx\right.$$

$$+\,\varepsilon \int_{B_{2r}(x_0)} (|Du|+s)^q\,dx\Big] + c_\varepsilon \|\mu\|_{L^{1,\phi}}^{\frac{q}{p-1}} r^n \left[\frac{r}{\phi(r)}\right]^{\frac{q}{p-1}},$$

after renaming ε; $\bar c$ depends only on $n, p, \nu/L$.

We can now apply the variant of a classic iteration result in Lemma 2.9 to deduce (3.6): being ε_0 the constant appearing in Lemma 2.9 for the choice $\gamma = n - q(1-\beta_0), \sigma = n - q(1-\beta_0/2), c_0 = \bar c$, we take in (2.15) $\varepsilon = \varepsilon_0$ (and $c_\varepsilon = c_{\varepsilon_0}$ accordingly) and

$$f(\rho) = \int_{B_\rho(x_0)} (|Du|+s)^q\,dx, \qquad g(\rho) = c_{\varepsilon_0} \|\mu\|_{L^{1,\phi}}^{\frac{q}{p-1}} \rho^n \left[\frac{\rho}{\phi(\rho)}\right]^{\frac{q}{p-1}},$$

we have, using our assumption (2.5), for every $H \ge \widetilde H_0$,

$$g(H\rho) \le H^n \left[\frac{H}{h_\phi(H)}\right]^{\frac{q}{p-1}} g(\rho)$$

and, calling $\beta = 1 + (p-1)(1-\beta_0/2) < p$ and $\epsilon = (p-\beta)/2$, using Hölder's inequality

$$\begin{aligned}
&\int_0^1 s^{n-q(1-\beta_0/2)} \left(\frac{1}{s}\right)^n \left[\frac{1}{s h_\phi(1/s)}\right]^{\frac{q}{p-1}} \frac{ds}{s} \\
&\quad = \int_0^1 \left[\frac{1}{s^\beta h_\phi(1/s)}\right]^{\frac{q}{p-1}} \frac{ds}{s} = \int_0^1 s^{\epsilon\frac{q}{p-1}} \left[\frac{1}{s^{\frac{\beta+p}{2}} h_\phi(1/s)}\right]^{\frac{q}{p-1}} \frac{ds}{s} \\
&\quad \le \left(\int_0^1 s^{\tilde\epsilon} \frac{ds}{s}\right)^{1-\frac{(n-1)q}{n(p-1)}} \left(\int_0^1 \left[\frac{1}{s^{\frac{\beta+p}{2}} h_\phi(1/s)}\right]^{\frac{n}{n-1}} \frac{ds}{s}\right)^{\frac{(n-1)q}{n(p-1)}}
\end{aligned}$$

for some $\tilde\epsilon \equiv \tilde\epsilon(n, p, q, \nu/L) > 0$ (note that $q/(p-1) < n/(n-1)$); hence

$$\begin{aligned}
\int_0^1 s^{n-q(1-\beta_0/2)} \left(\frac{1}{s}\right)^n \left[\frac{1}{s h_\phi(1/s)}\right]^{\frac{q}{p-1}} \frac{ds}{s} &\le c \left(\int_0^1 \left[\frac{1}{s^{\frac{\beta+p}{2}} h_\phi(1/s)}\right]^{\frac{n}{n-1}} \frac{ds}{s}\right)^{\frac{(n-1)q}{n(p-1)}} \\
&\le c(n, p, q, \nu/L, \phi(\cdot))
\end{aligned}$$

and (2.16) is satisfied. Now we can suppose

$$\left(\frac{\rho}{r}\right)^{n-q(1-\beta_0/2)} \le \left(\frac{\rho}{r}\right)^n \left[\frac{\rho/r}{\phi(\rho/r)}\right]^{\frac{q}{p-1}} = \left(\frac{\rho}{r}\right)^n f(\rho/r)$$

for ρ sufficiently small from (2.4); (2.17) in this situation gives, for all $0 < \rho \le r \le 2R$

$$\begin{aligned}
\fint_{B_\rho(x_0)} (|Du|+s)^q\,dx &\le c\left[\left[\frac{\rho/r}{\phi(\rho/r)}\right]^{\frac{q}{p-1}} \fint_{B_r(x_0)} (|Du|+s)^q\,dx + \|\mu\|_{L^{1,\phi}}^{\frac{q}{p-1}} \left[\frac{\rho}{\phi(\rho)}\right]^{\frac{q}{p-1}}\right] \\
&\le c\left[\frac{\rho}{\phi(\rho)}\right]^{\frac{q}{p-1}} \left[\left[\frac{1}{r h_\phi(1/r)}\right]^{\frac{q}{p-1}} \fint_{B_r(x_0)} (|Du|+s)^q\,dx + \|\mu\|_{L^{1,\phi}}^{\frac{q}{p-1}}\right] \\
&\le c\left[\frac{\rho}{\phi(\rho)}\right]^{\frac{q}{p-1}} \left[\fint_{B_r(x_0)} (|Du|+s)^q\,dx + \|\mu\|_{L^{1,\phi}}^{\frac{q}{p-1}}\right]
\end{aligned}$$

if $r \le 1/\widetilde H_0$. (3.6) follows, since

$$\int_{B_r(x_0)} (|Du|+s)^q\,dx \le c(n, p, \nu, q)|\mu|(\Omega),$$

see [8, 20]. □

4. THE PROOF OF THEOREM 2.5

We start fixing a ball $B_{2R} \equiv B_{2R}(x_0) \subset \Omega$, $R \le 1/2$ and arbitrarily an exponent q such that

$$1 < q < \frac{n(p-1)}{n-1} < p \tag{4.1}$$

so that it only depends on n and p: for instance, we can choose the midpoint of the interval $[1, n(p-1)/(n-1)]$. For two intermediate radii $R \le r_1 < r_2 \le 2R$ we define

$$\lambda_{\text{ref}} = \fint_{B_{r_2}} |Du|\, dx + M\left[|B_{r_2}|^{\frac{1}{n}} \frac{|\mu|(B_{r_2})}{|B_{r_2}|}\right]^{\frac{1}{p-1}} + \lambda_0, \tag{4.2}$$

with $B_{r_2} \equiv B_{r_2}(x_0)$, where λ_0 appears in (2.4). Moreover we take λ satisfying $\lambda > B\lambda_0$ with $B \ge 1$ defined as

$$B := \left(\frac{40r_2}{r_2 - r_1}\right)^n \tag{4.3}$$

In a standard way one can prove that for every $\lambda > B\lambda_{\text{ref}}$ and for almost every point $\bar{x}$ of the super-level set

$$E(\lambda, B_{r_1}) := \left\{x \in B_{r_1}(x_0) : |Du(x)| > \lambda\right\}$$

there exists a radius $r_{\bar{x}} < (r_2 - r_1)/40$ such that

$$\frac{\lambda}{40^{n/p^-}} \le \fint_{B_{jr_{\bar{x}}}(\bar{x})} |Du|\, dx + M\left[|B_{jr_{\bar{x}}}(\bar{x})|^{\frac{1}{n}} \frac{|\mu|(B_{jr_{\bar{x}}}(\bar{x}))}{|B_{jr_{\bar{x}}}(\bar{x})|}\right]^{\frac{1}{p-1}} \le \lambda \tag{4.4}$$

for every $j \in \{1, \ldots, 40\}$; note that $B_{jr_{\bar{x}}}(\bar{x}) \subset B_{r_2}$. The procedure is nowadays quite standard, see for instance [4, 5, 20, 23].

We fix now one of these balls $B_{r_{\bar{x}}}(\bar{x})$, $r_{\bar{x}}$ being the radius defined above, and we note that one of the following two inequalities must hold:

$$\frac{\lambda}{80^{n/p^-}} \le \fint_{B_{2r_{\bar{x}}}(\bar{x})} |Du|\, dx \qquad \text{or} \qquad \left(\frac{\lambda}{80^{n/p^-}}\right)^{p-1} \le M^{p-1} \frac{|\mu|(B_{2r_{\bar{x}}}(\bar{x}))}{|B_{2r_{\bar{x}}}(\bar{x})|^{\frac{n-1}{n}}}. \tag{4.5}$$

The first case. Suppose that the first inequality in (4.5) is in force: for $\varsigma \in (0,1)$ to be chosen, we estimate

$$\begin{aligned}\int_{B_{2r_{\bar{x}}}(\bar{x})} |Du|\, dx &= \int_{B_{2r_{\bar{x}}}(\bar{x}) \cap E(\varsigma\lambda, B_{r_2})} |Du|\, dx + \int_{B_{2r_{\bar{x}}}(\bar{x}) \setminus E(\varsigma\lambda, B_{r_2})} |Du|\, dx \\ &\le \varsigma\lambda\big|B_{2r_{\bar{x}}}(\bar{x})\big| + \int_{E(\varsigma\lambda, B_{2r_{\bar{x}}}(\bar{x}))} |Du|\, dx \\ &\le \varsigma\lambda\big|B_{2r_{\bar{x}}}(\bar{x})\big| + \big|E(\varsigma\lambda, B_{2r_{\bar{x}}}(\bar{x}))\big|^{\frac{1}{q'}} \left(\int_{E(\varsigma\lambda, B_{2r_{\bar{x}}}(\bar{x}))} |Du|^q\, dx\right)^{\frac{1}{q}},\end{aligned}$$

q as in (4.1); enlarging the domain of integration, taking averages and using $(4.5)_1$ yields

$$\frac{\lambda}{80^{n/p^-}} \le \varsigma\lambda + \left(\frac{|E(\varsigma\lambda, B_{2r_{\bar{x}}}(\bar{x}))|}{|B_{2r_{\bar{x}}}(\bar{x})|}\right)^{\frac{1}{q'}} \left(\fint_{B_{2r_{\bar{x}}}(\bar{x})} |Du|^q\, dx\right)^{\frac{1}{q}}. \tag{4.6}$$

To estimate the last averaged integral in terms of λ, we define the $a(\cdot)$-comparison lifting $v \in u + W^{1,p}_0(B_{20r_{\bar{x}}}(\bar{x}))$ as in (3.1) with $R = 20r_{\bar{x}}$ and $x_0 = \bar{x}$. Since (4.4) implies both

$$\left[\frac{|\mu|(B_{40r_{\bar{x}}}(\bar{x}))}{(40r_{\bar{x}})^{n-1}}\right]^{\frac{1}{p-1}} \le \frac{c(n,p)\lambda}{M}, \qquad \fint_{B_{40r_{\bar{x}}}(\bar{x})} |Du|\, dx \le \lambda, \tag{4.7}$$

a consequence of the comparison estimate (3.4) in this setting is

$$
\begin{aligned}
\fint_{B_{20r_{\bar x}}(\bar x)} |Du-Dv|^q\,dx &\le c\left[\frac{|\mu|(B_{40r_{\bar x}}(\bar x))}{(40r_{\bar x})^{n-1}}\right]^{\frac{q}{p-1}}\\
&\quad + c\,\chi_{\{p<2\}}\left(\fint_{B_{40r_{\bar x}}(\bar x)}(|Du|+s)\,dx\right)^{(2-p)q}\left[\frac{|\mu|(B_{40r_{\bar x}}(\bar x))}{(40r_{\bar x})^{n-1}}\right]^{q}\\
&\le c\frac{\lambda^q}{M^q}+c\,\chi_{\{p<2\}}\frac{\lambda^q}{M^{q(p-1)}}\le c\frac{\lambda^q}{M^{p^-q}}, \qquad (4.8)
\end{aligned}
$$

with c depending on n,p and ν. Thus, enlarging the domain of integration several times, using the quasi-subadditivity of the map $s\mapsto s^q$ and the higher integrability of Proposition 3.2 we get

$$
\begin{aligned}
\fint_{B_{2r_{\bar x}}(\bar x)} |Du|^q\,dx &\le c(q)10^n\fint_{B_{20r_{\bar x}}(\bar x)} |Du-Dv|^q\,dx + c(q)5^n\fint_{B_{10r_{\bar x}}(\bar x)}|Dv|^q\,dx\\
&\le c\fint_{B_{20r_{\bar x}}(\bar x)} |Du-Dv|^q\,dx + c\left(\fint_{B_{20r_{\bar x}}(\bar x)}(|Dv|+s)\,dx\right)^q\\
&\le c\fint_{B_{20r_{\bar x}}(\bar x)} |Du-Dv|^q\,dx\\
&\quad + c\left(\fint_{B_{20r_{\bar x}}(\bar x)}(|Du|+s)\,dx+\fint_{B_{20r_{\bar x}}(\bar x)}|Du-Dv|\,dx\right)^q\\
&\le c\fint_{B_{20r_{\bar x}}(\bar x)} |Du-Dv|^q\,dx + c\left(\fint_{B_{20r_{\bar x}}(\bar x)}(|Du|+s)\,dx\right)^q
\end{aligned}
$$

with $c\equiv(n,p,\nu,L)$; at this point, we use (4.8) and (4.7) to get

$$
\fint_{B_{2r_{\bar x}}(\bar x)} |Du|^q\,dx \le c\,\lambda^q
$$

for a constant depending on n,p,ν and L; in turn, inserting the last estimate into (4.6) we finally get

$$
\frac{\lambda}{80^{n/p^-}} \le \varsigma\lambda + c\left(\frac{|E(\varsigma\lambda, B_{2r_{\bar x}}(\bar x))|}{|B_{2r_{\bar x}}(\bar x)|}\right)^{\frac{1}{q'}}\lambda
$$

and dividing by λ^q and absorbing (we fix the value of ς as $160^{n/p^-}$)

$$
\frac{1}{c}\le c\left(\frac{|E(\varsigma\lambda, B_{2r_{\bar x}}(\bar x))|}{|B_{2r_{\bar x}}(\bar x)|}\right)^{\frac{1}{q'}}.
$$

Summing up, this means that there exists a constant depending on n,p,ν and L such that if the first alternative in (4.5) holds, then, fixing $\varsigma\equiv\varsigma(n,p)$ as above, we have

$$
|B_{2r_{\bar x}}(\bar x)|\le c\,|E(\varsigma\lambda, B_{2r_{\bar x}}(\bar x))|.
$$

The second case. Suppose instead that the second inequality in (4.5) holds. As a first consequence, using (1.3) we have

$$
\lambda^{p-1}\le c(n,p)\,M^{p-1}\frac{|\mu|(B_{2r_{\bar x}}(\bar x))}{|B_{2r_{\bar x}}(\bar x)|^{\frac{n-1}{n}}}\le c\,M^{p-1}\frac{r_{\bar x}^n}{r_{\bar x}^{n-1}\phi(2r_{\bar x})}=c\,M^{p-1}\frac{r_{\bar x}}{\phi(r_{\bar x})}
$$

with $c\equiv c(n,p)$ and thus, keeping into account (2.6),

$$
2r_{\bar x}\le\Psi\Big(\frac{\lambda}{c\,M}\Big);
$$

hence, as second consequence of $(4.5)_2$

$$
|B_{2r_{\bar x}}(\bar x)|\le c\left(\frac{M}{\lambda}\right)^{p-1}(2r_{\bar x})|\mu|(B_{2r_{\bar x}}(\bar x))\le\left(\frac{c\,M}{\lambda}\right)^{p-1}\Psi\Big(\frac{\lambda}{c\,M}\Big)|\mu|(B_{2r_{\bar x}}(\bar x))
$$

with $c \equiv c(n,p)$. Thus, merging the two alternatives, we have

$$
\begin{aligned}
|B_{20r_{\bar x}}(\bar x)| &\le c(n)|B_{2r_{\bar x}}(\bar x)| \\
&\le c\,|E(\varsigma\lambda, B_{2r_{\bar x}}(\bar x))| + \Big(\frac{c\,M}{\lambda}\Big)^{p-1}\Psi\Big(\frac{\lambda}{c\,M}\Big)|\mu|(B_{2r_{\bar x}}(\bar x)). \qquad (4.9)
\end{aligned}
$$

Covering and iteration. We consider the collection of balls

$$
\mathcal{E}_\lambda := \{B_{2r_{\bar x}}(\bar x)\}_{\bar x \in E(\lambda, B_{r_1})}
$$

where $r_{\bar x}$ is defined accordingly to (4.4). Using Vitali covering lemma we extract a countable sub-collection $\mathcal{F}_\lambda \subset \mathcal{E}_\lambda$ such that the 5-times enlarged balls cover almost almost all $E(\lambda, B_{r_1})$ and the balls are pairwise disjoints. This is to say, if we denote the balls of $\mathcal{F}_\lambda$ by $B_i := B_{2r_{\bar x_i}}(\bar x_i)$, for $i \in \mathcal{I}_\lambda$, being possibly $\mathcal{I}_\lambda = \mathbb{N}$, and with $\bar x_i \in E(\lambda, B_{r_1})$, we have

$$
B_i \cap B_j = \emptyset \quad \text{whenever } i \neq j \quad \text{and} \quad E(\lambda, B_{r_1}) \subset \bigcup_{i \in \mathcal{I}_\lambda} 5B_i \cup \mathcal{N}, \qquad (4.10)
$$

with $|\mathcal{N}| = 0$; remember that since $r_{\bar x_i} < (r_2 - r_1)/40$ we see that $5B_i = B_{10r_{\bar x_i}}(\bar x_i) \subset B_{r_2}$ for all $i \in \mathcal{I}_\lambda$. For $H \ge H_0$, with H_0 appearing in (2.4), to be chosen later we estimate

$$
\big|E(H\lambda, r_1)\big| \le \sum_{i \in \mathcal{I}_\lambda} \big|5B_i \cap E(2H\lambda, r_2)\big|.
$$

We split every term in the following way:

$$
\begin{aligned}
\big|5B_i \cap E(2H\lambda, r_2)\big| &\le \big|\{x \in 5B_i : |Du(x)| + s > 2H\lambda\}\big| \\
&\le \big|\{x \in 5B_i : |Du(x) - Dv_i(x)| > H\lambda\}\big| \\
&\quad + \big|\{x \in 5B_i : |Dv_i(x)| > H\lambda\}\big| =: I_i + II_i. \qquad (4.11)
\end{aligned}
$$

Here v_i is the comparison function solution to (3.1) in $10B_i \equiv B_{20r_{\bar x_i}}(\bar x_i)$. We estimate separately the two pieces: for the first one we use (4.8) and subsequently (4.9) to infer

$$
\begin{aligned}
I_i &\le \frac{1}{H\lambda}\int_{20B_i} |Du - Dv_i|\,dz \le \frac{c}{H\lambda}\cdot\frac{1}{M^{p^-}}|20B_i|\lambda \qquad (4.12) \\
&\le \frac{c}{HM^{p^-}}\Big[|E(\varsigma\lambda, B_i)| \\
&\quad + \Big(\frac{c\,M}{\lambda}\Big)^{p-1}\Psi\Big(\frac{\lambda}{c\,M}\Big)|\mu|(B_i)\Big].
\end{aligned}
$$

To estimate the term II_i we use the higher integrability (3.5): for $\chi > 1$ given in Proposition 3.2, we have

$$
\begin{aligned}
II_i &\le \Big(\frac{1}{H\lambda}\Big)^{p\chi}|5B_i| \fint_{5B_i} (|Dv_i| + s)^{p\chi}\,dz \\
&\le \frac{c}{(H\lambda)^{p\chi}}|20B_i|\Big(\fint_{10B_i}(|Dv_i| + s)\,dz\Big)^{p\chi} \\
&\le \frac{c}{H^{p\chi}}\Big[|E(\varsigma\lambda, B_i)| + \Big(\frac{c\,M}{\lambda}\Big)^{p-1}\Psi\Big(\frac{\lambda}{c\,M}\Big)|\mu|(B_i)\Big]. \qquad (4.13)
\end{aligned}
$$

Connecting the two estimates (4.12) and (4.13) and plugging the result into (4.11), taking into account that $H \ge 1$, gives

$$
\begin{aligned}
\big|5B_i \cap E(2H\lambda, Q_{r_2})\big| &\le c\Big[\frac{1}{HM^{p^-}} + \frac{1}{H^{p\chi}}\Big]\Big[|E(\varsigma\lambda, B_i)| \\
&\quad + \Big(\frac{c\,M}{\lambda}\Big)^{p-1}\Psi\Big(\frac{\lambda}{c\,M}\Big)|\mu|(B_i)\Big].
\end{aligned}
$$

At this point first we choose $M \equiv M(n,p,\nu,L,H)$ such that $HM^{p^-} = H^{p\chi}$; then, using the fact that the $\{B_i\}$ are pairwise disjoint, see (4.10), we sum up and we have

$$\big|E(2H\lambda, B_{r_1})\big| \le \frac{c_*}{H^{p\chi}}|E(\varsigma\lambda, B_{r_2})| + c\,\frac{\Psi(\lambda)}{\lambda^{p-1}}|\mu|(B_{2R})$$

with c_* depending only on n,p,ν,L but not on H; c depends also on H. Notice that

$$\Psi\Big(\frac{\lambda}{c\,M}\Big) \le c(M,\phi(\cdot))\Psi(\lambda)$$

follows from (2.4). In turn, recalling that ς is a constant depending only on n and p, we can further simplify the previous estimate in the following way, renaming λ and H ($\lambda \leftrightarrow \varsigma\lambda$ and $H \leftrightarrow 2H/\varsigma$):

$$\big|E(H\lambda, B_{r_1})\big| \le \frac{c_*}{H^{p\chi}}|E(\lambda, B_{r_2})| + c\,\frac{\Psi(\lambda)}{\lambda^{p-1}}|\mu|(B_{2R})$$

and this estimate holds for every $\lambda \ge B\lambda_{\rm ref}$, with $\lambda_{\rm ref}$ defined in (4.2) and B as in (4.3), and for every H sufficiently large, in particular at this point larger than $2H_0/\varsigma$. Now we multiply both sides for the quantity $(H\lambda)^{p-1}/\Psi(H\lambda)$ and we get

$$\begin{aligned}\frac{(H\lambda)^{p-1}}{\Psi(H\lambda)}\big|E(H\lambda, B_{r_1})\big| &\le \frac{c_*}{H^{p\chi}}\frac{(H\lambda)^{p-1}}{\Psi(H\lambda)}|E(\lambda, B_{r_2})| + c\,\frac{\Psi(\lambda)}{\lambda^{p-1}}|\mu|(B_{2R})\\ &\le \frac{c_*}{H^{p(\chi-1)+1}h_\Psi(H)}\frac{\lambda^{p-1}}{\Psi(\lambda)}|E(\lambda, B_{r_2})| + c\,|\mu|(B_{2R}).\end{aligned}$$

The local result in (2.10) now follows in a standard way, reabsorbing the appropriate quantities on the right-hand side. Indeed, first we take the supremum with respect to λ in $[B\lambda_{\rm ref}, +\infty)$ and then we add to both sides the sup of the same quantity over $(0, B\lambda_{\rm ref})$. This gives, after using the monotonicity of $s \mapsto s^{p-1}/\Psi(s)$ and changing variable in the supremum in the left-hand side

$$\begin{aligned}\sup_{\lambda>0}\frac{\lambda^{p-1}}{\Psi(\lambda)}\big|E(\lambda, B_{r_1})\big| &\le \frac{1}{2}\sup_{\lambda>0}\frac{\lambda^{p-1}}{\Psi(\lambda)}|E(\lambda, B_{r_2})|\\ &\qquad + c\frac{(B\lambda_{\rm ref})^{p-1}}{\Psi(B\lambda_{\rm ref})}|B_{2R}| + c\,|\mu|(B_{2R}),\end{aligned}$$

since by (2.4) with $\beta = p(\chi-1)+1 > 1$ we can take $H \equiv H(n,p,\nu,L,\phi(\cdot))$ so large that

$$\frac{c_*}{H^{p(\chi-1)+1}h_\Psi(H)} \le \frac{1}{2}. \tag{4.14}$$

Since a direct computation, taking into account (1.3), shows that

$$\begin{aligned}B\lambda_{\rm ref} &\le \frac{c}{(r_2-r_1)^n}\left[\int_{B_{2R}}(|Du|+s)\,dx + R^n\left[\frac{R}{\phi(R)}\right]^{\frac{1}{p-1}}\|\mu\|_{L^{1,\phi(\cdot)}}^{\frac{1}{p-1}} + \lambda_0\right]\\ &=: \frac{\lambda_1}{(r_2-r_1)^n},\end{aligned}$$

with $c \equiv c(n,p,\nu,L,\phi(\cdot))$; a standard iteration result (we use the version in Lemma 2.8) implies that we can reabsorb the supremum on the right-hand side and obtain

$$\sup_{\lambda>0}\frac{\lambda^{p-1}}{\Psi(\lambda)}\big|E(\lambda, B_R)\big| \le \frac{(\lambda_1/R^n)^{p-1}}{\Psi(\lambda_1/R^n)}|B_{2R}| + c\,|\mu|(B_{2R}).$$

Therefore, with

$$\bar\lambda \equiv \bar\lambda(u) := C\fint_{B_{2R}}(|Du|+s)\,dx + C\left[\frac{R}{\phi(R)}\right]^{\frac{1}{p-1}}\|\mu\|_{L^{1,\phi(\cdot)}}^{\frac{1}{p-1}} + C\lambda_0 \tag{4.15}$$

$C \equiv C(n, p, \nu, L, \phi(\cdot))$ sufficiently large, we have

$$\frac{1}{|B_R|}\sup_{\lambda>0}\frac{\lambda^{p-1}}{\Psi(\lambda)}\big|E(\lambda, B_R)\big| \le \frac{\bar\lambda^{p-1}}{\Psi(\bar\lambda)} \tag{4.16}$$

using again the monotonicity of $s \mapsto s^{p-1}/\Psi(s)$; we indeed can estimate

$$\frac{|\mu|(B_{2R})}{|B_{2R}|} \le c(n)\|\mu\|_{L^{1,\phi}(\Omega)}\frac{2R}{\phi(2R)} \le \bar\lambda^{p-1} \le \frac{\bar\lambda^{p-1}}{\Psi(\bar\lambda)}$$

and this finally proves (2.10). Now, in order to get the estimate in (2.11), we use a rescaling argument: once fixed $B_{2R}(x_0) \subset \Omega$, if we define

$$\tilde u : B_1 \to \mathbb{R}, \qquad \tilde u(x) = \frac{u(x_0 + 2Rx)}{2Rf(R)\bar\lambda}$$

then it is easy so see that $\tilde u$ solves

$$\operatorname{div} \tilde a(x, D\tilde u) = \tilde\mu \qquad \text{in } B_1$$

with

$$\tilde a(x,\xi) = \frac{a(x, \bar\lambda f(R)\xi)}{[\bar\lambda f(R)]^{p-1}}, \qquad \tilde s = \frac{s}{\bar\lambda f(R)}, \qquad \tilde\mu(x) = R\,\frac{\mu(x_0 + Rx)}{[\bar\lambda f(R)]^{p-1}},$$

and for $B_r(y_0) \subset B_2$, $y_0 \in B_1$, $r \le 1$,

$$|\tilde\mu|(B_r(y_0)) = \frac{R^{1-n}}{[\bar\lambda f(R)]^{p-1}}|\mu|(B_{rR}(x_0 + y_0)),$$

so that

$$\tilde\mu \in L^{1,\tilde\phi(\cdot)}(B_1) \quad \text{for} \quad \tilde\phi(r) = \frac{\phi(rR)}{\phi(R)} \quad \text{and} \quad \|\tilde\mu\|_{L^{1,\tilde\phi(\cdot)}(B_1)} \le \frac{\|\mu\|_{L^{1,\phi(\cdot)}}}{[\bar\lambda(u)f(R)]^{p-1}}\frac{R}{\phi(R)}.$$

Simple computations show that

$$\tilde f(r) = \left[\frac{r}{\tilde\phi(r)}\right]^{\frac{1}{p-1}} = \left[\frac{\phi(R)r}{\phi(rR)}\right]^{\frac{1}{p-1}},$$

$$\widetilde\Psi(\lambda) = \tilde f^{-1}(\lambda) = \frac{1}{R}\Psi\left(\left[\frac{R}{\phi(R)}\right]^{\frac{1}{p-1}}\lambda\right) = \frac{1}{R}\Psi\big(f(R)\lambda\big);$$

we observe that (2.4) holds for $\widetilde\Psi$ with λ_0 replaced by $\lambda_0/f(R)$ but for the same function h_Ψ; finally, we also have $\tilde\phi(1) = 1$ and $\widetilde\Psi(1) = 1$. Therefore, if we want to apply the estimate in (4.16) to $\tilde u$ between $B_{1/2}$ and B_1, we compute

$$\begin{aligned}\bar\lambda(\tilde u) &= C\fint_{B_1}(|D\tilde u| + \tilde s)\,dx + C\left[\frac{1}{\tilde\phi(1)}\right]^{\frac{1}{p-1}}\|\tilde\mu\|^{\frac{1}{p-1}}_{L^{1,\tilde\phi(\cdot)}(B_1)} + C\frac{\lambda_0}{f(R)}\\ &\le \frac{1}{f(R)}\left[\frac{C}{\bar\lambda(u)}\fint_{B_{2R}(x_0)}(|Du| + s)\,dx + C\left[\frac{R}{\phi(R)}\right]^{\frac{1}{p-1}}\frac{\|\mu\|^{\frac{1}{p-1}}_{L^{1,\phi(\cdot)}}}{\bar\lambda(u)} + C\lambda_0\right]\\ &\le \frac{2C\lambda_0}{f(R)}\end{aligned}$$

so we get

$$\sup_{\lambda>0}\frac{\lambda^{p-1}}{\widetilde\Psi(\lambda)}\big|\{y \in B_{1/2} : |D\tilde u(y)| > \lambda\}\big| \le \frac{[\bar\lambda(\tilde u)]^{p-1}}{\widetilde\Psi(\bar\lambda(\tilde u))} \le c\frac{\lambda_0^{p-1}}{\Psi(\lambda_0)}\frac{\phi(R)}{R}\cdot R = c\phi(R).$$

We stress that

$$c \ge \sup_{\lambda>0}\frac{\lambda^{p-1}}{\widetilde\Psi(\lambda)\phi(R)}\big|\{y \in B_{1/2} : |D\tilde u(y)| > \lambda\}\big|$$

$$= \sup_{\lambda>0} \frac{\lambda^{p-1}}{\widetilde{\Psi}(\lambda)\phi(R)} \frac{|\{x \in B_R(x_0) : |Du(x)| > \bar{\lambda}(u)f(R)\lambda\}|}{|B_R(x_0)|}$$
$$= \sup_{\lambda>0} \frac{[\lambda/f(R)]^{p-1}}{\widetilde{\Psi}(\lambda/f(R))\phi(R)} \frac{|\{x \in B_R(x_0) : |Du(x)|/\bar{\lambda}(u) > \lambda\}|}{|B_R(x_0)|}$$

and then, in view of (2.9) and the explicit expression of the function on the left-hand side

$$\frac{[\lambda/f(R)]^{p-1}}{\widetilde{\Psi}(\lambda/f(R))\phi(R)} = \frac{\lambda^{p-1}}{\Psi(\lambda)} \cdot \frac{1}{[f(R)]^{p-1}} \cdot \frac{R}{\phi(R)} = \frac{\lambda^{p-1}}{\Psi(\lambda)},$$

this gives

$$\|Du/\bar{\lambda}(u)\|_{\mathcal{M}^{\tilde{\Phi}(\cdot)}(B_R(x_0))} = \frac{\|Du\|_{\mathcal{M}^{\tilde{\Phi}(\cdot)}(B_R(x_0))}}{\bar{\lambda}(u)} \le 1$$

that is (2.11).

5. Proofs of Corollaries

In order to prove quickly the corollaries, we first give recall some calculus results for a class of functions having "almost monomial" behavior. We suppose $\varphi : [0,\infty) \to [0,\infty)$ to be $C^1(0,\infty)$ and to satisfy the condition

$$1 < g_0 \le \frac{\varphi'(s)s}{\varphi(s)} \le g_1 < \infty \qquad \text{for all } s > 0. \tag{5.1}$$

We introduce the notation $A \approx B$ to denote in short that there exists a constant $c_\varphi \equiv c(g_0, g_1) \ge 1$ such that $B/c_\varphi \le A \le c_\varphi B$. The first property we need, that can be checked by a simple computation, is that if φ satisfies (5.1), then all the functions

$$\varphi^{-1}(s), \qquad \frac{1}{\varphi(1/s)} \quad \text{and} \quad [\varphi(s)]^\ell \quad \text{for any } \ell > 0$$

satisfy (5.1) for a constant depending on g_0, g_1 and possibly on ℓ (notice that (5.1) implies that φ is increasing thus invertible). Moreover,

$$\frac{d}{ds}\frac{s}{\varphi(s)} = \frac{\varphi(s) - s\varphi'(s)}{[\varphi(s)]^2} \approx -\frac{1}{\varphi(s)}, \qquad \frac{d}{ds}\varphi^{-1}(s) \approx \frac{\varphi^{-1}(s)}{s}. \tag{5.2}$$

We then note that for $q > 1, \gamma, \varsigma \in \mathbb{R}$ the functions

$$t \mapsto t^q \log^\gamma(e/t), \qquad t \mapsto t^q \log^\gamma(1/t), \log^\varsigma \log(1/t) \tag{5.3}$$

satisfy (5.1) for $t \in (0, \bar{R})$, $\bar{R} \equiv \bar{R}(\gamma, \varsigma) \le R_0$ and for constants depending only on q.

5.1. **A scaling procedure.** We start with a rescaling similar to that used to prove Remark 2.6. We set $\tilde{u}(x) = u(x_0 + x)/\lambda$ where λ is the quantity

$$c \fint_{B_{2R}(x_0)} (|Du| + s + 1)\,dx + c\,\|\mu\|_{L^{1,\phi(\cdot)}(\Omega)}^{\frac{1}{p-1}} \left[\frac{R}{\phi(R)}\right]^{\frac{1}{p-1}},$$

with c the constant appearing in (2.10); $\tilde{u}$ solves

$$\operatorname{div} a(x, D\tilde{u}) = \tilde{\mu} \quad \text{in } B_{2R}(0), \qquad \tilde{a}(x,\xi) = \frac{a(x,\lambda\xi)}{\lambda^{p-1}}, \qquad \tilde{\mu} = \frac{\mu}{\lambda^{p-1}},$$

$\tilde{s} = s/\lambda$, and is such that

$$c \fint_{B_{2R}(0)} (|D\tilde{u}| + \tilde{s} + 1)\,dx + c\,\|\tilde{\mu}\|_{L^{1,\phi(\cdot)}(\Omega)}^{\frac{1}{p-1}} \left[\frac{R}{\phi(R)}\right]^{\frac{1}{p-1}}$$
$$= \frac{1}{\lambda}\left[c \fint_{B_{2R}(x_0)} (|Du| + s)\,dx + c\,\|\mu\|_{L^{1,\phi(\cdot)}(\Omega)}^{\frac{1}{p-1}} \left[\frac{R}{\phi(R)}\right]^{\frac{1}{p-1}}\right] + c \le c.$$

Thus applying (2.10) to $\tilde{u}$ gives

$$\sup_{\lambda>0} \Phi(\lambda) \frac{|\{x \in B_R : |D\tilde{u}(x)| > \lambda\}|}{|B_R|} \le c, \tag{5.4}$$

where c is a universal constant. Once we will get (5.6) and the analogue for Corollary 1.2, it will be easy to scale back to u, obtaining respectively (1.7) and (1.11).

5.2. **Proof of Corollary 1.1.** We take ϕ satisfying (5.1) and such that f, as defined in (2.3), is decreasing and smooth. We write, using Fubini's theorem,

$$\begin{aligned} \fint_{B_R} |D\tilde{u}|^{\frac{\vartheta(p-1)}{\vartheta-1}}\, dx &= c(n,p) \int_0^\infty \lambda^{\frac{\vartheta(p-1)}{\vartheta-1}} \frac{|\{x \in B_R : |D\tilde{u}(x)| > \lambda\}|}{|B_R|} \frac{d\lambda}{\lambda} \\ &\le c + c \int_{f(\bar{R})}^\infty \lambda^{\frac{\vartheta(p-1)}{\vartheta-1}} \frac{|\{x \in B_R : |D\tilde{u}(x)| > \lambda\}|}{|B_R|} \frac{d\lambda}{\lambda} \\ &=: c\,[1+I]; \end{aligned}$$

the constant c depends on n, p, R_0, α, or n, p, R_0, ς, depending whether we are considering (1.4) or (1.5). To estimate the last term, in both cases we use (5.4), recalling that $\Psi = f^{-1}$:

$$\begin{aligned} I \le c \int_{f(\bar{R})}^\infty \lambda^{\frac{\vartheta(p-1)}{\vartheta-1}} \frac{f^{-1}(\lambda)}{\lambda^{p-1}} \frac{d\lambda}{\lambda} &= c \int_0^{\bar{R}} [f(\rho)]^{\frac{\vartheta(p-1)}{\vartheta-1}} \frac{\rho}{[f(\rho)]^{p-1}} \frac{d\rho}{\rho} \\ &= c \int_0^{\bar{R}} [f(\rho)]^{\frac{p-1}{\vartheta-1}}\, d\rho \\ &= c \int_0^{\bar{R}} \left[\frac{\rho}{\phi(\rho)}\right]^{\frac{1}{\vartheta-1}} d\rho, \end{aligned} \tag{5.5}$$

c depending only on n, p and R_0. In the first equality we changed variable $\rho = f^{-1}(\lambda)$ and we used the following facts:

$$\frac{d}{d\rho}\frac{\rho}{\phi(\rho)} \approx -\frac{1}{\phi(\rho)}, \qquad f'(\rho) \approx -f(\rho)\frac{\phi(\rho)}{\rho}\frac{1}{\phi(\rho)} = -\frac{f(\rho)}{\rho}, \qquad \frac{d\lambda}{\lambda} \approx -\frac{d\rho}{\rho},$$

due to (5.2). Now it is immediate to see that if ϕ is as in (1.4) or as in (1.5) in $(0, \bar{R})$ the integral on the right-hand side in (5.5) is finite; moreover the functions satisfy (5.1) (see (5.3)). The condition in (2.4) is satisfied due to Remark 2.1 in the case $\vartheta > p$. If $\vartheta = p$, since $\alpha > 0$, we cannot anymore take $\tilde{L} \ge 2^p$. On the other hand, a careful examination of the proof (see (4.14)) shows that it is sufficient that (2.4) holds for a α fixed, precisely with $\bar{\alpha} = p(\chi-1)+1$. This allows to estimate

$$\phi(2R) \ge 2^p \left(\frac{\log(1/[2R])}{\log(1/R)}\right)^\alpha \phi(R) \ge 2^{p-p(\chi-1)/2}\phi(R)$$

for R sufficiently small, since $\frac{\log(1/[2R])}{\log(1/R)} \to 1$ as $R \to 0$. Hence the choice

$$h_\Psi(H) = H^{-\frac{\log_2(\tilde{L}/2)}{p-1}} = H^{-\frac{\log_2(2^{p-p(\chi-1)/2})}{p-1}} = H^{-(1+p(\chi-1)/2)}$$

is such that

$$h_\Psi(H) H^{\bar{\alpha}} = H^{p(\chi-1)/2} \to +\infty$$

and the proof above still works. We hence got

$$\fint_{B_R} |D\tilde{u}|^{\frac{\vartheta(p-1)}{\vartheta-1}}\, dx \le c. \tag{5.6}$$

5.3. **Proof of Corollary 1.2.** We set, for $t \geq 0$, $h(t) = t^{\frac{(p-1)\vartheta}{\vartheta-1}} \log^\gamma(e+t)$ and we compute

$$\frac{1}{c(p,\vartheta,\gamma)} \leq \frac{h'(t)t}{h(t)} \leq c(p,\vartheta,\gamma).$$

We have, using Fubini's theorem

$$\begin{aligned} \fint_{B_R} |D\tilde{u}|^{\frac{(p-1)\vartheta}{\vartheta-1}} \log^\gamma\left(e + |D\tilde{u}|^{p-1}\right) dx &\leq c \fint_{B_R} h(|D\tilde{u}|)\, dx \\ &= c\int_0^\infty h'(\lambda) \frac{|\{x \in B_R : |D\tilde{u}(x)| > \lambda\}|}{|B_R|}\, d\lambda \\ &\leq c\int_0^\infty h(\lambda) \frac{|\{x \in B_R : |D\tilde{u}(x)| > \lambda\}|}{|B_R|}\, \frac{d\lambda}{\lambda}; \end{aligned}$$

as before, it is sufficient to show that the integral of the above integrand over $(f(\bar{R}), \infty)$ is bounded by a constant. We have

$$\begin{aligned} \int_{f(\bar{R})}^\infty h(\lambda) \frac{|\{x \in B_R : |D\tilde{u}(x)| > \lambda\}|}{|B_R|}\, \frac{d\lambda}{\lambda} &\leq c\int_{f(\bar{R})}^\infty h(\lambda) \frac{f^{-1}(\lambda)}{\lambda^{p-1}}\, \frac{d\lambda}{\lambda} \\ &= c\int_0^{\bar{R}} h(f(\rho)) \frac{\rho}{[f(\rho)]^{p-1}}\, \frac{d\rho}{\rho} \\ &= c\int_0^{\bar{R}} [f(\rho)]^{\frac{p-1}{\vartheta-1}} \log^\gamma(e + f(\rho))\, d\rho \\ &\leq c\int_0^{\bar{R}} \left[\frac{\rho}{\phi(\rho)}\right]^{\frac{p-1}{\vartheta-1}} \log^\gamma\left(e + \frac{\rho}{\phi(\rho)}\right) d\rho. \end{aligned}$$

Now we use the definition of ϕ in (1.8) to get

$$\begin{aligned} \int_0^{\bar{R}} \left[\frac{\rho}{\phi(\rho)}\right]^{\frac{1}{\vartheta-1}} \log^\gamma\left(e + \frac{\rho}{\phi(\rho)}\right) d\rho &= \int_0^{\bar{R}} \frac{\log^\gamma(e + \rho^{1-\vartheta}\log^{-\alpha}(1/\rho))}{\log^{\frac{\alpha}{\vartheta-1}}(1/\rho)}\, \frac{d\rho}{\rho} \\ &\leq c\int_0^{\bar{R}} \log^{\gamma - \frac{\alpha}{\vartheta-1}}(1/\rho)\, \frac{d\rho}{\rho}, \end{aligned}$$

while for (1.9) the computation is similar: we have

$$\begin{aligned} &\int_0^{\bar{R}} \left[\frac{\rho}{\phi(\rho)}\right]^{\frac{1}{\vartheta-1}} \log^\gamma\left(e + \frac{\rho}{\phi(\rho)}\right) d\rho \\ &\qquad = \int_0^{\bar{R}} \frac{\log^\gamma[e + \rho^{1-\vartheta}\log^{-(\gamma+1)(\vartheta-1)}(1/\rho)\log^{-\varsigma}(\log(1/R))]}{\log^{\gamma+1}(1/\rho)\log^{\frac{\varsigma}{\vartheta-1}}(\log(1/R))}\, \frac{d\rho}{\rho} \\ &\qquad \leq c\int_0^{\bar{R}} \frac{1}{\log(1/\rho)\log^{\frac{\varsigma}{\vartheta-1}}(\log(1/\rho))}\, \frac{d\rho}{\rho}. \end{aligned}$$

In both cases our ranges of exponents guarantee that the integrals are finite.

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