A VARIATIONAL CHARACTERISATION OF THE SECOND EIGENVALUE OF THE p-LAPLACIAN ON QUASI OPEN SETS

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Abstract. In this article, we prove a minimax characterisation of the second eigenvalue of the p-Laplacian operator on p-quasi open sets, using a construction based on minimizing movements. This leads also to an existence theorem for spectral functionals depending on the first two eigenvalues of the p-Laplacian.

1. Introduction

The Dirichlet eigenvalues of the p-Laplacian operator are defined as the numbers $\lambda > 0$ for which the following Dirichlet problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega, \end{cases} \tag{1}$$

admits a non-zero weak solution $u \in W_0^{1,p}(\Omega)$. Here $\Omega \subset \mathbb{R}^n$ is an open set of finite measure and $1 < p < \infty$. In fact, the eigenvalues are the critical values of the Rayleigh quotient

$$\mathcal{R}_\Omega(w) = \frac{\displaystyle\int_\Omega |\nabla w|^p \, dx}{\displaystyle\int_\Omega |w|^p \, dx}$$

and the corresponding weak solutions of equation (1) are the critical points of $\mathcal{R}_\Omega(w)$ among all non-zero functions in $w \in W_0^{1,p}(\Omega)$. While the first eigenvalue $\lambda_1(\Omega)$ is defined as the minimum value of $\mathcal{R}_\Omega(w)$, not much is known on higher order eigenvalues when $p \neq 2$. One way to obtain them is by using the so called *Krasnoselskii's genus* $\gamma(\mathcal{M})$ of a set $\mathcal{M} \subset W_0^{1,p}(\Omega)$. In fact, it was shown in [20] that, denoting by Σ_k, $k = 1, 2, \ldots$, the collection of all symmetric subsets $\mathcal{M}$ contained in $W_0^{1,p}(\Omega)$ with $\gamma(\mathcal{M}) \geq k$, the numbers

$$\lambda_k(\Omega) = \inf_{\mathcal{M} \in \Sigma_k} \Big[\sup_{u \in \mathcal{M}} \mathcal{R}_\Omega(w) \Big] \tag{2}$$

form an increasing sequence of eigenvalues. It is not known whether all the eigenvalues of the p-Laplacian are of this form if $p \neq 2$. However, it was proved by Anane-Tsouli [5] that, given a bounded and connected open set Ω, if $\lambda_1(\Omega)$ and $\lambda_2(\Omega)$ are defined as in (2), then $\lambda_1(\Omega)$ is the smallest eigenvalue and there are no other eigenvalues in the interval $(\lambda_1(\Omega), \lambda_2(\Omega))$, see also [24].

Another variational characterisation of $\lambda_2(\Omega)$ was given by Cuesta-de Figueiredo-Gossez in [12], who proved that for a bounded open and connected set Ω we have

$$\lambda_2(\Omega) = \inf_{\gamma \in \Gamma(u_1, -u_1)} \Big[\max_{w \in \gamma([0,1])} \int_\Omega |\nabla w|^p \, dx \Big], \tag{3}$$

where u_1 is the first nonnegative eigenfunction with $\|u_1\|_{L^p(\Omega)} = 1$ and $\Gamma(u_1, -u_1)$ is the family of all continuous maps from $[0,1]$ to $\mathcal{M}_p(\Omega) = \{u \in W_0^{1,p}(\Omega) : \|u\|_{L^p(\Omega)} = 1\}$ with endpoints u_1

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and $-u_1$. Later on, it was shown by Brasco-Franzina [9] that (3) still holds if Ω is any open set of finite measure, not necessarily connected. Finally, as pointed to us by L. Brasco, a different variational characterization of $\lambda_2(\Omega)$ can be obtained by combining a result proved in [15] with the argument used by Brasco and Franzina in the proof of [10, Th. 4.2]:

$$\lambda_2(\Omega) = \inf_{f\in\mathcal{C}_{odd}(\mathbb{S}^1,\mathcal{M}_p(\Omega))} \left[\max_{u\in Im(f)} \int_\Omega |\nabla w|^p\,dx \right], \tag{4}$$

where $\mathcal{C}_{odd}(\mathbb{S}^1,\mathcal{M}_p(\Omega))$ is the set of continuous and odd maps from $\mathbb{S}^1$ to $\mathcal{M}_p(\Omega)$.

In this paper, we study the properties of the first two eigenvalues of the p-Laplacian in a p-quasi open set A. Beside being of its own interest, this study is motivated by the existence Theorem 1.2 below. Recall that $A\subset\mathbb{R}^n$ is p-quasi open if there exists a p-quasi continuous nonnegative function $u\in W^{1,p}(\mathbb{R}^n)$ such that $A=\{u>0\}$ (see Section 2 for more details). In order to deal with the p-Laplacian on p-quasi open sets we need to introduce the p-fine topology, which turns out to be an important tool for our study. In particular some basic properties of Sobolev functions on quasi open sets are given in Theorem 2.7 and Lemmas 2.9 and 2.10, while the strong minimum principle for the p-Laplacian on p-quasi open sets is stated in Theorem 3.3.

For bounded open sets, it is known (see [29]) that if the first eigenvalue is simple, then it is isolated. Here we prove that the same holds in the framework of p-quasi open sets, see Proposition 3.11. This fact turns out to be useful in the proof of our main result, which states that formula (3) is still true when Ω is replaced by a p-quasi open set A of finite measure.

Theorem 1.1. *Let $A\subset\mathbb{R}^n$ be a p-quasi open set of finite measure, let $u_1\in W_0^{1,p}(A)$ be a normalized eigenfunction of $\lambda_1(A)$ and let $\Gamma(u_1,-u_1)=\{\gamma\in C([0,1],\mathcal{M}_p(A)) : \gamma(0)=u_1,\ \gamma(1)=-u_1\}$. Then*

$$\lambda_2(A) = \min_{\gamma\in\Gamma(u_1,-u_1)} \left[\max_{w\in\gamma([0,1])} \int_A |\nabla w|^p\,dx \right]. \tag{5}$$

Note that, differently from what was known before in the case of open sets, see (3) and (4), we prove here that the infimum at the right hand side of (5) is indeed attained.

A few words on the proof of the minimax formula (5) for a p-quasi open set A are in order. Assuming that $\lambda_1(A)$ is simple, the idea is to show the existence of a curve $\gamma:[0,1]\to\mathcal{M}_p(A)$ connecting u_1 and $-u_1$ such that

$$\max_{w\in\gamma([0,1])} \int_A |\nabla w|^p\,dx = \lambda_2(A).$$

In the case of a bounded connected open set, the construction of this path γ in [12] involves a delicate use of Ekeland's variational principle which does not seem to work for quasi open sets.

Therefore we have chosen here a completely different approach, based on De Giorgi's *minimizing movements*. Indeed, we construct the desired path by joining three different curves. One of them, connecting the negative and positive parts of a second eigenfunction u_2 is easily constructed by hands. The construction of the two other curves is where we use the minimizing movements, see Lemma 3.16. Precisely, we consider the limit of a sequence of maps $v_h : A\times[0,\infty)\to\mathcal{M}_p(A)$, where v_h are the gradient flows of the p-energy functional

$$E(u) = \int_\Omega |\nabla u|^p\,dx$$

restricted to the manifold $\mathcal{M}_p(A)$, with respect to the $L^p(A)$-distance in $W_0^{1,p}(A)$. The maps v_h are time-discretized weak solutions of the following doubly nonlinear evolution equation

$$\begin{cases} |\partial_t u|^{p-2}\partial_t u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \sigma(t)|u|^{p-2}u & \text{in } A\times(0,\infty) \\ u\in W_0^{1,p}(A)\cap\mathcal{M}_p(A) & \text{for all } t\geq 0 \end{cases} \tag{6}$$

with $u(0) = v_0$, where $\lambda_1(A) < E(v_0) \leq \lambda_2(A)$ and v_0 is either u_2^+ or $-u_2^-$. It turns out that for every h, the energy functional $E(v_h(t))$ is strictly decreasing along the flow for $t > 0$ and we have $E(v_h(t)) \to \lambda_1(A)$ and $v_h(t) \to u_1$ (or $-u_1$) in $W_0^{1,p}(A)$ as $t \to +\infty$. Then we show that the flows v_h converge to a map $v : A \times [0, \infty) \to \mathcal{M}_p(A)$ weakly in $W^{1,p}((0,\infty), L^p(A))$ and strongly in $W_0^{1,p}(A)$ for almost every $t \geq 0$, as $h \to +\infty$. Although these convergences do not imply that v is also a weak solution of the equation (6), they are enough to conclude that

$$E(v(t)) < \lambda_2(A) \ \ \forall\, t > 0 \quad \text{and} \quad v(t) \to u_1 \ (\text{or} - u_1) \text{ as } t \to +\infty.$$

An immediate application of the variational characterisation (5) of the second eigenvalue of the p-Laplacian is the lower semicontinuity of $\lambda_2(A)$ with respect to a suitable convergence in the family of all p-quasi open subsets of a bounded open set Ω. In turn, this lower semicontinuity leads to the following existence theorem, where we denote by $\mathscr{A}_p(\Omega)$ the family of all p-quasi open sets contained in Ω.

Theorem 1.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a lower semicontinuous function, separately increasing in both variables, and let $\Omega \subset \mathbb{R}^n$ be a bounded open set. For every $0 < c \leq |\Omega|$, there exists a p-quasi open minimizer of the following problem*

$$\min\left\{f\left(\lambda_1(A), \lambda_2(A)\right) : A \in \mathscr{A}_p(\Omega),\ |A| = c\right\}.$$

Throughout the paper we shall always assume that $1 < p \leq n$, unless otherwise stated. If $p > n$ then p-quasi open sets reduce to open sets for which all the results contained in Sections 2 and 3 are well known. On the other hand, the results proved in Sections 4 and 5, which are new also in the context of open sets, are proved exactly in the same way regardless of the fact that p is smaller or greater than n, see Remark 5.8.

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2. Quasi open and finely open sets

In this section we shall review the notions of p-capacity and p-quasi open sets and prove some results that will be crucial for the rest of the paper. However, since quasi open sets do not form a topology, we need to introduce also the related notion of p-finely open sets, which do form a topology. For all the main properties and the basic results needed in the sequel and not proven here, we refer to [19, 18, 1, 22, 25] and the references therein.

Finally, we warn the reader that sometimes we shall drop the notation p whenever it is clear from the context that we refer to p-quasi or p-finely open sets.

Given a measurable set $E \subset \mathbb{R}^n$, we define its *p-capacity* by setting

$$\mathrm{Cap}_p(E) := \inf\left\{\int_{\mathbb{R}^n} (|u|^p + |\nabla u|^p)\, dx : u \in W^{1,p}(\mathbb{R}^n),\ u \geq 1 \text{ a.e. in an open set } U \supset E\right\}.$$

Note that this definition is equivalent to the Bessel capacity $C_{1,p}$ defined via the Bessel kernel G_1 (see [25, Rem. 1.13] or [32, Sect. 2.6] for more details).

If a property holds everywhere except possibly in a set of zero p-capacity, we say that it holds *p-quasi everywhere* (and we write *p-q.e.* or *q.e.* for short).

Definition 2.1 ((Quasi open sets))**.** A set $A \subset \mathbb{R}^n$ is said to be *p-quasi open* if for every $\varepsilon > 0$ there exists an open set U_ε such that $\mathrm{Cap}_p(U_\varepsilon \triangle A) < \varepsilon$; equivalently, if there exists an open set A_ε such that $A \cup A_\varepsilon$ is open and $\mathrm{Cap}_p(A_\varepsilon) < \varepsilon$.

A function $f : A \to \mathbb{R}$ defined on a quasi open set A is said to be *p-quasi continuous* if for every $\varepsilon > 0$, there exists an open set A_ε such that $\mathrm{Cap}_p(A_\varepsilon) < \varepsilon$ and the restriction of f to $A \setminus A_\varepsilon$ is continuous. More equivalent definitions are contained in Theorem 2.4 below.

It is well known that any function $u \in W^{1,p}(\mathbb{R}^n)$ has a p-quasi continuous representative v. In particular, $u = v$ a.e. in $\mathbb{R}^n$. Moreover v is unique in the sense that if w is another p-quasi continuous representative of u, then $v = w$ q.e., see [25, Th. 1.3]. Henceforth, when dealing with a function in $W^{1,p}(\mathbb{R}^n)$, we shall always assume that u is p-quasi continuous.

Definition 2.2. For an open set $\Omega \subset \mathbb{R}^n$ we denote the collection of all p-quasi open subsets of Ω by $\mathscr{A}_p(\Omega)$. If A is p-quasi open and $\mathrm{Cap}_p(A) > 0$ we say that $u \in W^{1,p}(\mathbb{R}^n)$ belongs to the space $W^{1,p}_0(A)$ if any p-quasi continuous representative of u vanishes p-q.e. in $\mathbb{R}^n \setminus A$. The space $W^{1,p}_0(A)$, equipped with the norm naturally induced by $W^{1,p}(\mathbb{R}^n)$, is a Banach space. Setting $p' = p/(p-1)$, we denote by $W^{-1,p'}(A)$ the dual space of $W^{1,p}_0(A)$.

Note that the space $W^{1,p}_0(A)$ can be equivalently defined by setting

$$W^{1,p}_0(A) = \bigcap \{W^{1,p}_0(U) : U \text{ open}, U \supset A\},$$

see [25, Th. 2.10]. However, we shall never use this characterisation in the sequel.

Since p-quasi open sets do not form a topology, we introduce the *p-fine topology* which is the coarsest topology on $\mathbb{R}^n$ making all (classical) p-superharmonic functions continuous. A more robust equivalent definition can be given using the Wiener criteria, as follows.

Definition 2.3 ((Finely open sets))**.** A set $U \subset \mathbb{R}^n$ is *p-finely open* if for every $x \in U$

$$\int_0^1 \left(\frac{\mathrm{Cap}_p(B_r(x) \setminus U)}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty.$$

The fine topology has been extensively studied in the context of nonlinear potential theory. For more details we refer the reader to [19], [18], [1], [25] and to the references therein. We recall here the following result, see [25, Th. 1.4 and 1.5], which deals with the compatibility of finely open sets with quasi-open sets.

Theorem 2.4. *Given a set $A \subset \mathbb{R}^n$, the following are equivalent.*

(i) A is p-quasi open.
(ii) $A = U \cup E$ where U is p-finely open and $Cap_p(E) = 0$.
(iii) There exists a p-quasi continuous function $u \in W^{1,p}(\mathbb{R}^n)$, $u \geq 0$, such that $A = \{u > 0\}$.

Furthermore, given a p-quasi open set A and a function $f : A \to \mathbb{R}$, the following are equivalent.

(i) f is p-quasi continuous in A.
(ii) The sets $\{f > c\}$ and $\{f < c\}$ are p-quasi open for all $c \in \mathbb{R}$.
(iii) f is p-finely continuous in A up to a set of zero p-capacity.

Remark 2.5. From Definition 2.1 it is immediate that a p-quasi open set A remains quasi open if we change it by a set of zero p-capacity. This makes the characterisation $A = U \cup E$ in Theorem 2.4 unique up to sets of zero p-capacity. On the other hand, if U is p-finely open and E has zero p-capacity, it is easy to check from Definition 2.3 that also $U \setminus E$ is p-finely open. Note also that if U is p-finely open and $\mathrm{Cap}_p(U) = 0$ then U is empty.

We need to introduce also the notion of quasi connectedness, as follows.

Definition 2.6 ((Quasi connected sets))**.** A p-quasi open set $A \subset \mathbb{R}^n$ is *p-quasi connected* if for any p-quasi open sets A_1, A_2 such that $A = A_1 \cup A_2$ and $\mathrm{Cap}_p(A_1 \cap A_2) = 0$, then either $\mathrm{Cap}_p(A_1) = 0$ or $\mathrm{Cap}_p(A_2) = 0$.

The notion of p-quasi connectedness is closely related to the topological notion of p-finely connected set. Indeed a much stronger result holds, due to A. Björn-J. Björn [7, Th. 1.1].

Theorem 2.7. *Let A be a p-quasi open set. Then the following are equivalent.*

(i) If $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ and $\nabla u = 0$ a.e. in A, then there exists a constant c such that $u = c$ a.e. in A.

(ii) A is p-quasi connected.

(iii) $A = U \cup E$, where U is p-finely connected and p-finely open and $Cap_p(E) = 0$.

The equivalence between (ii) and (iii) in the above theorem is a straightforward consequence of the definition and of Theorem 2.8 below, while the equivalence between these two conditions and (i) is the main result in [7].

Next result, due to Latvala [27, Th. 1.1] will be used later in this section.

Theorem 2.8. *Let U be open and connected in the p-fine topology and let E be a set of zero p-capacity. Then $U \setminus E$ is also open and connected in the p-fine topology.*

It is known that the p-fine topology has the *quasi Lindelöf property*, i.e. every family $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of p-finely open sets contains an at most countable subfamily $\{U_h\}$ such that $\cup_{\alpha\in\mathcal{A}} U_\alpha = \cup_h U_h$ up to a set of zero p-capacity (see [19, Sect. 12] for the case $p = 2$ and [22] for $p \neq 2$). Moreover, it is also known that the p-fine topology is locally connected, see [22, Th. 3.15]. Using these properties it is straightforward to check that a p-quasi open set A can be always decomposed as

$$A = \bigcup_{j\in\mathbb{N}} U_j \cup E, \tag{7}$$

where the U_j are p-finely open and p-finely connected sets, pairwise disjoint, and $\mathrm{Cap}_p(E) = 0$. We shall refer to the sets U_j as to the *p-quasi connected components* of A. Note that they are uniquely determined up to a set of zero p-capacity, as explained in Remark 2.5.

The next lemma deals with the restrictions of a function in $W^{1,p}_0(A)$ on the p-quasi connected components of a quasi open set A. To this aim, given a set $E \subset \mathbb{R}^n$, for every $x' \in \mathbb{R}^{n-1}$ we set $E_{x'} = \{t \in \mathbb{R} : (x', t) \in E\}$ and we shall denote by $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure.

Lemma 2.9. *Let A be a p-quasi open set of finite measure and let $V \subset A$ be a p-quasi connected component of A. For any $u \in W^{1,p}_0(A)$, we have $u_{|V} \in W^{1,p}_0(V)$.*

Proof. Thanks to (7), we may write $A = V \cup U \cup E$, where V, U, E are mutually disjoint, V is as in the statement, U is p-finely open and $\mathrm{Cap}_p(E) = 0$. Let $u \in W^{1,p}(\mathbb{R})$ be a p-quasi continuous function, $u = 0$ q.e. in $\mathbb{R}^n \setminus A$ and set

$$v(x) := \begin{cases} u(x) & \text{if } x \in V, \\ 0 & \text{elsewhere.} \end{cases}$$

Since u is quasi continuous in $\mathbb{R}^n$, by Theorem 2.4 it coincides with a finely continuous function up to a set of zero p-capacity. Then, it is easily checked that also v is finely continuous up to a set of zero p-capacity and thus quasi continuous in $\mathbb{R}^n$. We claim that for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$ the function $v(x', \cdot)$ is in $W^{1,p}(\mathbb{R})$.

To this end recall that $\mathrm{Cap}_p(E) = 0$, hence $\mathcal{H}^{n-1}(E) = 0$, see [16, Sect. 4.7.2]. Using this fact and Corollary 6.3, we get that there exists a set $Z_0 \subset \mathbb{R}^{n-1}$ with $\mathcal{L}^{n-1}(Z_0) = 0$, such that for $x' \notin Z_0$ the sections $V_{x'}$ and $U_{x'}$ are open and $E_{x'} = \emptyset$.

Since the *precise representative* u^* of u is p-quasi continuous, see [32, Th. 3.10.2], with no loss of generality we may take $u = u^*$. Thus we may assume that $u(x', \cdot) \in W^{1,p}(\mathbb{R}) \cap C(\mathbb{R})$, see [16, Sect. 4.9.1]. In addition, since $u(x) = 0$ q.e. in $\mathbb{R}^n \setminus A$, enlarging Z_0 if needed, we may also assume that $u(x', \cdot) \equiv 0$ on the closed set $\mathbb{R} \setminus A_{x'}$ for all $x' \notin Z_0$. Since the sections $V_{x'}, U_{x'}$ form a partition of $A_{x'}$ and $u(x', \cdot) \in W^{1,p}_0(A_{x'})$, we conclude that $u(x', \cdot) \in W^{1,p}_0(V_{x'})$. This proves that $v(x', \cdot) \in W^{1,p}(\mathbb{R})$, as claimed.

Note that

$$\int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \left| \frac{d}{dt} v(x', t) \right|^p dt \right) dx' \le \int_A |\nabla u|^p \, dx.$$

Repeating the above argument in all coordinate directions we conclude that $v \in W^{1,p}(\mathbb{R}^n)$. In fact, $v \in W_0^{1,p}(V)$ since it is p-quasi continuous and vanishes q.e. outside V. This completes the proof in this case. □

The next lemma will be used in the proof of Proposition 3.12.

Lemma 2.10. *Let A be a p-quasi open and p-quasi connected set. Let $u, v \in W_0^{1,p}(A)$ be two functions such that $u, v > 0$ q.e. in A and*

$$\frac{\nabla u}{u} = \frac{\nabla v}{v} \qquad a.e.\ in\ A.$$

Then, there exists a constant $\kappa > 0$ such that $u = \kappa v$ q.e. in A.

Proof. By subtracting from A a set of zero p-capacity and using Theorem 2.8, we may assume that A is p-finely open and p-finely connected, that u and v are p-finely continuous in A and that $u(x), v(x) > 0$ for all $x \in A$. We claim that for any $x \in A$ there exists a p-finely open neighborhood of x where u/v is constant. Then the result will follow immediately.

To prove this claim, fix $x \in A$. Recall that the p-fine topology is locally connected. Thus we may find a p-finely connected neighborhood V_x of x contained in the p-finely open set $A \cap \{u > \varepsilon\} \cap \{v > \varepsilon\}$, where $0 < \varepsilon < \min\{u(x), v(x)\}$. Since $w_\varepsilon = \log \max\{u, \varepsilon\} - \log \max\{v, \varepsilon\} \in W_{loc}^{1,p}(\mathbb{R}^n)$ and $\nabla w_\varepsilon = 0$ in V_x, Theorem 2.7 yields that u/v is constant on V_x. This proves the claim, thus concluding the proof of the lemma. □

3. Eigenvalues of the p-Laplacian in a quasi open set

In this section we study the main properties of the first and second eigenvalue of the p-Laplacian in a p-quasi open set and establish the variational characterisation of Theorem 1.1. This will be used in Section 4 to establish the lower semicontinuity of $\lambda_2(A)$.

3.0.1. *The p-Laplacian and the Resolvent.*
Given a quasi open set $A \in \mathscr{A}_p(\Omega)$ and $f \in W^{-1,p'}(A)$, the Dirichlet problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f & \text{in } A \\ u \in W_0^{1,p}(A) \end{cases} \tag{8}$$

is defined in the usual weak sense. Precisely, we say that $u \in W_0^{1,p}(A)$ is a *weak solution* of the equation (8) if for every $\phi \in W_0^{1,p}(A)$, we have

$$\int_A |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx = \langle f, \phi \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W^{-1,p'}(A)$ and $W_0^{1,p}(A)$.

Let us set $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u) \in W^{-1,p'}(A)$. Following [25], we say that a p-quasi continuous function $u \in W^{1,p}(A)$ is a *fine supersolution* of the equation $-\Delta_p u = 0$, if for every nonnegative function $\phi \in W_0^{1,p}(A)$, we have

$$\int_A |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx \ge 0.$$

The monotonicity of the p-Laplacian operator ensures, as in the standard case of an open set, the existence of a unique weak solution of (8). This enable us to define the resolvent map as usual.

Definition 3.1 ((Resolvent))**.** For a quasi open set $A \in \mathscr{A}_p(\Omega)$, the *resolvent map* for the p-Laplacian operator, is defined for any $f \in W^{-1,p'}(A)$ by setting $\mathscr{R}_{p,A}(f) := u$, where $u \in W_0^{1,p}(A)$ is the unique weak solution of (8).

We recall from Kilpeläinen-Malý [25] the following theorem related to fine supersolutions.

Theorem 3.2 (([25, Th. 4.3]))**.** *Let $A \subset \mathbb{R}^n$ be p-quasi open and let $u_j \in W^{1,p}(A)$ be an increasing sequence of fine supersolutions of converging q.e. to a function $u : A \to \mathbb{R}$. Then u is p-quasi continuous.*

An immediate consequence of the above theorem is the following minimum principle for fine supersolutions, see [27, Th. 4.1]. We give its simple proof for the reader's convenience.

Theorem 3.3 ((Minimum principle))**.** *Let $u \in W_0^{1,p}(A)$ be a fine supersolution on a p-quasi open and quasi connected set $A \subset \mathbb{R}^n$, $u \geq 0$ q.e.. Then either $u > 0$ or $u = 0$ q.e. in A.*

Proof. First, we recall that the minimum of two fine supersolutions is also a fine supersolution, see [25, Prop. 3.5]. Therefore, given $u \in W_0^{1,p}(A)$ as in the statement, the functions $v_j := \min\{ju, 1\}$, $j \in \mathbb{N}$, form an increasing sequence of fine supersolutions converging q.e. in A to the function v given by

$$v(x) = \begin{cases} 0 & \text{if} \quad x \in \{u = 0\}, \\ 1 & \text{if} \quad x \in \{u > 0\}. \end{cases}$$

By Theorem 3.2, v is p-quasi continuous in A, hence by Theorem 2.4 the sets $\{v < 1\} = \{u = 0\}$ and $\{v > 0\} = \{u > 0\}$ are both quasi open. Then the conclusion follows immediately from the assumption that A is p-quasi connected. □

3.1. **Variational eigenvalues.** We now recall for the reader's convenience a few well known facts from the classical variational theory of eigenvalues which we will need later. We refer the reader to [21] for more details.

3.1.1. *Minimax characterisation.*
A useful way to deal with nonlinear eigenvalues is to obtain them via the Euler-Lagrange equation of constrained minimization problems. Given a Banach space X and two functionals $E, G \in C^1(X)$, the minimizers u of the constrained problem $\min\{E(w) : G(w) = 1\}$ satisfy the equation

$$DE(u) = \lambda DG(u) \tag{9}$$

for some Lagrange multiplier $\lambda \in \mathbb{R}$, where D denotes the Fréchet derivative. There are several ways of generating such constrained critical values λ, if the functional E is invariant with respect to some compact group of symmetries acting on the manifold

$$\mathcal{M} := \{w \in X : G(w) = 1\}. \tag{10}$$

Here we use a variant of the *mountain pass lemma* by Ambrosetti-Rabinowitz [3]. Recall that the norm of the Fréchet derivative of the restriction $\widetilde{E}$ of E to $\mathcal{M}$ at a point $u \in \mathcal{M}$, is defined as

$$\|D\widetilde{E}(u)\|_* := \min\{\|DE(u) - tDG(u)\|_{X^*} : t \in \mathbb{R}\},$$

where $\|\cdot\|_{X^*}$ denotes the norm of the dual space X^*. It is said that the functional E satisfies the *Palais-Smale condition on* $\mathcal{M}$, if for any sequence $u_h \in \mathcal{M}$ such that $E(u_h)$ is bounded and $\|D\widetilde{E}(u_h)\|_* \to 0$, there exists a subsequence of u_h converging strongly in X. Then, the following result holds, see [12, Prop. 2.5], [21, Th. 3.2].

Theorem 3.4. *Let X be a Banach space and $E, G \in C^1(X)$. Let $\mathcal{M}$ be as in (10) and assume that $DG \neq 0$ on $\mathcal{M}$ and that E satisfies the Palais-Smale condition on $\mathcal{M}$.*

Let $u_0, u_1 \in \mathcal{M}$ and $\rho > 0$ be such that $\|u_1 - u_0\|_X > \rho$ and

$$\inf\{E(u) : u \in \mathcal{M},\ \|u - u_0\|_X = \rho\} > \max\{E(u_0), E(u_1)\}.$$

If the set $\Gamma(u_0, u_1) := \{\gamma \in C([0,1], \mathcal{M}) : \gamma(0) = u_0,\ \gamma(1) = u_1\}$ is not empty, then

$$\alpha = \inf_{\gamma \in \Gamma(u_0,u_1)} \Big[\max_{w \in \gamma([0,1])} E(w) \Big]$$

is a critical value for $\widetilde{E}$, i.e., there exists $u \in \mathcal{M}$ such that $E(u) = \alpha$ and $\|D\widetilde{E}(u)\|_* = 0$.

3.1.2. *Eigenvalues of the p-Laplacian operator.*
Here we provide the definitions of eigenvalues and eigenfunctions of the p-Laplacian, along with the notion of simplicity of eigenvalues. Although these definitions are quite similar to the ones for open sets, some subtle aspects will be investigated later in this section.

Definition 3.5. Let $A \subset \mathbb{R}^n$ be a p-quasi open set of finite measure and $\lambda \in \mathbb{R}$. If there exists a non-zero weak solution of the eigenvalue problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \ \lambda |u|^{p-2}u \ \text{ in } A \\ u \in W_0^{1,p}(A), \end{cases} \tag{11}$$

then λ is an *eigenvalue* of the p-Laplacian on A and u is a corresponding *eigenfunction*; if in addition $\|u\|_{L^p(A)} = 1$, then u is called a *normalized* eigenfunction. The subspace of $W_0^{1,p}(A)$ generated by all the eigenfunctions corresponding to λ is the *eigenspace* associated to λ. When this eigenspace is one-dimensional, i.e. $\{u, -u\}$ are the only normalized eigenfunctions, then λ is said to be *simple*.

Note that (11) is a special case of the Euler-Lagrange equation (9) if one takes $X = W_0^{1,p}(A)$,

$$E(w) = \int_A |\nabla w|^p \, dx \quad \text{and} \quad G(w) = \int_A |w|^p \, dx. \tag{12}$$

Moreover, taking u as test function in (11), one gets

$$\lambda \ = \ \frac{\displaystyle\int_A |\nabla u|^p \, dx}{\displaystyle\int_A |u|^p \, dx}. \tag{13}$$

Hence, just as for open sets, it is easy to show that

$$\lambda \ \geq \ c(n,p)\, |A|^{-p/n}, \tag{14}$$

using the Sobolev inequality. Thus all the eigenvalues are bounded away from zero.

Definition 3.6 ((First eigenvalue))**.** The first eigenvalue of the p-Laplacian in a p-quasi open set $A \subset \mathbb{R}^n$ of finite measure, is defined as

$$\lambda_1(A) := \min\{\lambda > 0 : \lambda \text{ is an eigenvalue of the } p\text{-Laplacian in } A\}.$$

Note that the above definition is well posed thanks to Proposition 3.11 (i) below and to the fact that by (14) the eigenvalues are greater than a strictly positive constant. Introducing the manifold

$$\mathcal{M}_p(A) := \{w \in W_0^{1,p}(A) : \|w\|_{L^p(A)} = 1\}, \tag{15}$$

we have that

$$\lambda_1(A) = \inf_{w \in \mathcal{M}_p(A)} \int_A |\nabla w|^p \, dx. \tag{16}$$

Indeed, the fact that $\lambda_1(A)$ is greater than or equal to the infimum at the right hand side is an immediate consequence of (13). Instead, the opposite inequality is obtained by observing that if A has finite measure then the functional E admits a minimizer u on $\mathcal{M}_p(A)$ and u is a weak solution of (11) for some $\lambda > 0$. Thus, $\lambda = \lambda_1(A)$ and we have

$$\lambda_1(A) = \int_A |\nabla u|^p \, dx = \min_{w \in \mathcal{M}_p(A)} \int_A |\nabla w|^p \, dx.$$

In particular u is an eigenfunction for $\lambda_1(A)$.

Remark 3.7. For a p-quasi open set A of finite measure, set $X = W_0^{1,p}(A)$ and E, G as in (12). Note that for every $t \in \mathbb{R}$, $u \in W_0^{1,p}(A)$, the derivative $DE(u) - tDG(u)$ is the element of $W^{-1,p'}(A)$ such that for all $\phi \in W_0^{1,p}(A)$

$$\langle DE(u) - tDG(u), \phi \rangle = p \int_A (|\nabla u|^{p-2} \nabla u \cdot \nabla \phi - t|u|^{p-2} u \phi) \, dx.$$

From this equality it follows immediately that if u is a critical point for $\widetilde{E}$ on $\mathcal{M}_p$, then u is an eigenfunction and $E(u)$ is an eigenvalue of the p-Laplacian.

The following simple lemma will allow us to use Theorem 3.4.

Lemma 3.8. *Let A be a p-quasi open set of finite measure. Let $E, G : W_0^{1,p}(A) \to \mathbb{R}$ be as in* (12) *and $\mathcal{M}_p$ as in* (15)*. Then E satisfies the Palais-Smale condition on $\mathcal{M}_p(A)$.*

Proof. Let $u_h \in \mathcal{M}_p(A)$ be a sequence such that $E(u_h)$ is bounded and $\|D\widetilde{E}(u_h)\|_* \to 0$. Then there exists a sequence $t_h \in \mathbb{R}$ such that $\|\Delta_p u_h + t_h |u_h|^{p-2} u_h\|_{W^{-1,p'}(A)} \to 0$. Setting $f_h := \Delta_p u_h + t_h |u_h|^{p-2} u_h$, we observe that

$$-\int_A |\nabla u_h|^p \, dx + t_h = \langle f_h, u_h \rangle \to 0 \qquad \text{as } h \to \infty.$$

Therefore, up to a subsequence we may assume that $t_h \to t \in \mathbb{R}$. Moreover, the compact imbedding of $W^{1,p}(B_r)$ in $L^p(B_r)$ for all $r > 0$ and the Sobolev inequality imply that, up to a subsequence, the sequence u_h converges strongly in $L^p(A)$. In order to prove the lemma it is enough to show that u_h converges strongly in $W_0^{1,p}(A)$. This follows immediately by observing that the sequence $\Delta_p u_h$ converges strongly in $W^{-1,p'}(A)$. Therefore, we have

$$\begin{aligned} \int_A (|\nabla u_k|^{p-2} \nabla u_k - |\nabla u_h|^{p-2} \nabla u_h) \cdot (\nabla u_k - \nabla u_h) \, dx &= -\langle \Delta_p u_k - \Delta_p u_h, u_k - u_h \rangle \\ &\leq C \|\Delta_p u_k - \Delta_p u_h\|_{W^{-1,p'}(A)} \to 0 \quad \text{as } h, k \to \infty, \end{aligned}$$

and the conclusion follows from Lemma 6.4. ☐

When X is a Hilbert space (corresponding to $W_0^{1,2}(A)$ in this setting), then the discreteness of the spectrum is well known from the classical theory of linear operators. But for the case of $W_0^{1,p}(A)$ with $p \neq 2$, the existence of a spectral gap is not known in general, except for the first and second eigenvalues on open connected sets (see [28], [24]).

To make matters worse, we work in the framework of quasi open sets with a weaker notion of connectedness than the standard one. Therefore some delicate issues have to be handled in order to characterize the second eigenvalue.

3.2. **Properties of the first eigenvalue.** For open sets of finite measure it is well known that every eigenvalue is the first eigenvalue in its nodal domains, i.e., if λ is an eigenvalue with eigenfunction u, then $\lambda = \lambda_1(\{u > 0\})$. The proof of this result for the eigenvalues of the p-Laplacian in an open set is due to Brasco-Franzina [8, Th. 3.1]. The same proof carries on in the framework of quasi open sets, so we omit it.

Lemma 3.9. *Let A be a p-quasi open set of finite measure, λ an eigenvalue of the p-Laplacian in A and $u \in W_0^{1,p}(A)$ a corresponding eigenfunction. Then $\lambda = \lambda_1(\{u > 0\})$ and u is a first eigenfunction of $\{u > 0\}$. Moreover, if $\lambda_1(A)$ is simple and $u \in W_0^{1,p}(A)$ is an eigenfunction of $\lambda_1(A)$, then u does not change sign.*

From the above lemma and the minimum principle Theorem 3.3, we have the following result.

Corollary 3.10. *Let A be a p-quasi open set of finite measure such that $\lambda_1(A)$ is simple and let u be a nonnegative first eigenfunction. Then $\{u > 0\}$ is a p-quasi connected component of A and $\lambda_1(A) = \lambda_1(\{u > 0\})$.*

Proof. First, observe that by the minimum principle if u is not identically zero in a p-quasi connected component A' then it is strictly positive in A'.

We argue by contradiction assuming that there exist two different quasi connected components A_1 and A_2 of A where u is not identically zero and denote by $\tilde{u}_i$ the restriction of u to A_i, for $i = 1, 2$. By Lemma 2.9, we have $\tilde{u}_i \in W_0^{1,p}(A_i)$. Moreover,

$$-\Delta_p \tilde{u}_i = \lambda_1(A)|\tilde{u}_i|^{p-2}\tilde{u}_i, \qquad \text{on } A_i.$$

Hence, the functions $\tilde{u}_1 \pm \tilde{u}_2$ are two linearly independent eigenfunctions of $\lambda_1(A)$, which contradicts the assumption that $\lambda_1(A)$ is simple. □

Using Lemma 3.9 and Corollary 3.10, we have the following proposition, which was proved for open, connected sets in [28], [29].

Proposition 3.11. *Let $A \subset \mathbb{R}^n$ be a p-quasi open set of finite measure. We have the following.*

(i) If λ_k is a sequence of eigenvalues such that $\lambda_k \to \lambda$, then λ is also an eigenvalue.
(ii) If the first eigenvalue $\lambda_1(A)$ is simple then it is isolated.

Proof. Statement (i) can be proved with the same argument used in the proof of [29, Th. 3].

In order to prove (ii), we first consider the case when A is p-quasi connected. Under this assumption, we claim that the eigenfunctions corresponding to an eigenvalue $\lambda > \lambda_1(A)$ must change sign. To see this, assume that u is a nonnegative eigenfunction. By Theorem 3.3, $\{u > 0\}$ coincides with A up to a set of capacity zero. Then by Lemma 3.9 we conclude that u is a first eigenfunction and the claim follows.

To show that the first eigenvalue is isolated we argue by contradiction, as in [29, Th. 9]. Assume that there exists a sequence of eigenvalues $\lambda_k > \lambda_1(A)$ converging to $\lambda_1(A)$ and let u_k be the normalized eigenfunction corresponding to λ_k. We may assume that, up to a not relabelled subsequence, u_k converges weakly in $W_0^{1,p}(A)$, strongly in $L^p(A \cap B_r)$ for all $r > 0$ and a.e. to a function $u \in W_0^{1,p}(A)$. Moreover, since A has finite measure, a simple argument based on Sobolev inequality shows (also for $p = n$) that $\|u\|_{L^p(A)} = 1$. Hence, by lower semicontinuity, we get that

$$\int_A |\nabla u|^p \, dx \le \lim_{k\to\infty} \lambda_k = \lambda_1(A).$$

Thus u is a normalized first eigenfunction. By Theorem 3.3, we may also assume without loss of generality that $u > 0$ q.e. on A. Now, arguing as in the proof of (14), one has

$$\min\{\, |\{u_k > 0\}|\,,\, |\{u_k < 0\}|\,\} \ge c(n,p)\lambda_k^{-\frac{n}{p}}$$

for all $k \in \mathbb{N}$. Therefore, setting $A^+ := \limsup_{k\to\infty}\{u_k > 0\}$, $A^- := \limsup_{k\to\infty}\{u_k < 0\}$, from the previous inequality, recalling that A has finite measure, we have immediately that

$$\min\{|A^+|, |A^-|\} \ge c(n,p)\lambda_1(A)^{-\frac{n}{p}}.$$

On the other hand, since the sequence u_k is converging a.e. to u, we have also that $A^+ \subset \{u \geq 0\}$ and $A^- \subset \{u \leq 0\}$ up to a set of zero Lebesgue measure. But the latter inclusion is impossible since $|A^-| > 0$ and $u > 0$ q.e. in A. This contradiction proves the result in this case.

Let us now assume that A is not p-quasi connected and again let us argue by contradiction assuming that there exists a sequence of eigenvalues $\lambda_k > \lambda_1(A)$ converging to $\lambda_1(A)$. As before, we denote by u_k a normalized eigenfunction of λ_k. Again, we may assume that, up to a not relabelled subsequence, u_k converges strongly in $L^p_{\text{loc}}(A)$ and a.e. in A to a nonnegative normalized first eigenfunction u. By Corollary 3.10 the set $\{u > 0\}$ is a p-quasi connected component of A. Thus by Lemma 2.9 we have that $u_k \in W^{1,p}_0(\{u > 0\})$ for all k. Therefore each eigenfunction u_k has to change sign in $\{u > 0\}$, otherwise by Lemma 3.9 u_k is a first eigenfunction in $\{u > 0\}$ and $\lambda_k = \lambda_1(\{u > 0\}) = \lambda_1(A)$, which is impossible. Then the conclusion of the proof goes exactly as the preceeding case. □

The following proposition extends to p-quasi open and p-quasi connected sets a property that is well known in the case of open sets, see for instance [2] or [6].

Proposition 3.12. *Let A be a p-quasi open and p-quasi connected set of finite measure. Then $\lambda_1(A)$ is simple.*

Proof. First, observe that if u is a first eigenfunction, then also $|u|$ is a first eigenfunction. Thus, by Theorem 3.3 $u \neq 0$ q.e. in A. Therefore, since A is p-quasi connected, u does not change sign in A. Thus, in order to prove $\lambda_1(A)$ is simple, it is enough to show that if u, v are two nonnegative normalized first eigenfunctions, then $u = v$.

To this end, fix $\varepsilon > 0$ and recall the following extension of the classical Picone's identity, see [2]. For every two nonnegative functions $u, v \in W^{1,p}(\mathbb{R}^n)$ it holds true that

$$\begin{aligned} 0 &\leq |\nabla u|^p + (p-1)\frac{u^p}{(v+\varepsilon)^p}|\nabla v|^p - p\frac{u^{p-1}}{(v+\varepsilon)^{p-1}}|\nabla v|^{p-2}\nabla v \cdot \nabla u \\ &= |\nabla u|^p - |\nabla v|^{p-2}\nabla\Big(\frac{u^p}{(v+\varepsilon)^{p-1}}\Big) \cdot \nabla v. \end{aligned} \tag{17}$$

Integrating the right hand side of the previous inequality in A and using the fact that v is a first eigenfunction we get

$$\int_A |\nabla u|^p \, dx - \int_A |\nabla v|^{p-2}\nabla v \cdot \nabla\Big(\frac{u^p}{(v+\varepsilon)^{p-1}}\Big) dx = \int_A |\nabla u|^p \, dx - \lambda_1(A)\int_A \frac{u^p v^{p-1}}{(v+\varepsilon)^{p-1}} dx.$$

Therefore, recalling (17) and using Fatou's lemma we have, letting $\varepsilon \to 0$,

$$\int_A \Big(|\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}|\nabla v|^{p-2}\nabla v \cdot \nabla u\Big) dx = 0.$$

Recalling that by the minimum principle $v > 0$ q.e. in A, a simple argument shows that the equality above implies that for a.e. $x \in A$

$$\nabla u(x) = \frac{u(x)}{v(x)}\nabla v(x).$$

The conclusion then follows from Lemma 2.10, recalling that $\|u\|_{L^p} = \|v\|_{L^p}$. □

3.3. **Variational characterisation of the second eigenvalue.** The fact that if $\lambda_1(A)$ is simple then it is isolated shows that there is a spectral gap between $\lambda_1(A)$ and the next eigenvalue. This naturally leads to the following definition of second eigenvalue, which is well posed due to Proposition 3.11.

Definition 3.13 ((Second Eigenvalue))**.** Let $A \subset \mathbb{R}^n$ be a p-quasi open set of finite measure. The second eigenvalue of the p-Laplacian on A is defined as follows.

$$\lambda_2(A) := \begin{cases} \min\{\lambda > \lambda_1(A) : \lambda \text{ is an eigenvalue}\} & \text{if } \lambda_1(A) \text{ is simple} \\ \lambda_1(A) & \text{otherwise} \end{cases} \tag{18}$$

Equipped with (18), now we restate Theorem 1.1, which is a *mountain pass characterisation* of the second eigenvalue. This characterisation is the main result of this paper.

Theorem 3.14. *Let A be a p-quasi open set of finite measure, $u_1 \in W_0^{1,p}(A)$ be a normalized eigenfunction of $\lambda_1(A)$, and*

$$\Gamma(u_1, -u_1) = \{\gamma \in C([0,1], \mathcal{M}_p(A)) : \gamma(0) = u_1,\ \gamma(1) = -u_1\},$$

where $\mathcal{M}_p(A)$ is the manifold defined in (15)*. Then*

$$\lambda_2(A) = \min_{\gamma \in \Gamma(u_1,-u_1)} \left[\max_{w \in \gamma([0,1])} \int_A |\nabla w|^p \, dx \right]. \tag{19}$$

The result above was proved in the case of open sets in [9]. However, the techniques used therein cannot be adopted in the setting of quasi open sets. It is noteworthy that when $\lambda_1(A)$ is simple the characterisation in (19) coincides with the one given using the Krasnoselskii genus. This latter definition was used by Anane-Tsouli [5] to prove the existence of spectral gap between the first and the second eigenvalue. We refer the reader to [28], [31], [24] for further details.

3.3.1. *A path of decreasing p-energy.*
To prove Theorem 3.14, we construct an appropriate path, so that the p-Dirichlet energy decreases throughout the path and remains lower than $\lambda_2(A)$. The idea is similar to the one used in [12], but the construction of the path in our case is completely different. Indeed, our construction relies on the theory of minimizing movements, a technique introduced by De Giorgi and developed in the book by Ambrosio-Gigli-Savaré [4]. To this end, we need to introduce the basic notation and definitions. For further details we refer to the first two chapters of [4].

Let $(\mathcal{S}, d)$ be a complete metric space and $\Phi : \mathcal{S} \to (-\infty, +\infty]$. Denote by

$$\mathscr{D}(\Phi) := \{v \in \mathcal{S} : \Phi(v) < \infty\} \tag{20}$$

the *effective domain* of Φ, i.e., the set of points where Φ is finite. Given a point $v \in \mathscr{D}(\Phi)$, the *local slope of* Φ *at* v is defined by setting

$$|\partial\Phi|(v) := \limsup_{w \to v} \frac{(\Phi(v) - \Phi(w))^+}{d(v,w)} \ . \tag{21}$$

In order to apply the results of [4] we now specify our choice for $(\mathcal{S}, d)$ and Φ. More precisely, in this section we are going to take as a metric space the space $L^p(A)$ with the distance induced by the norm, where A is a quasi open set of finite measure. The functional Φ will be defined as follows

$$\Phi(v) := \begin{cases} E(v) & \text{if } v \in \mathcal{M}_p(A) \\ +\infty & \text{if } v \in L^p(A) \setminus \mathcal{M}_p(A), \end{cases} \tag{22}$$

where E is the functional defined in (12) and $\mathcal{M}_p(A)$ is given in (15). Next lemma is a key ingredient in the proof of the existence of a path having all the properties stated in Lemma 3.16.

Lemma 3.15. *Let $A \subset \mathbb{R}^n$ be a p-quasi open set of finite measure and let Φ be the functional defined in* (22)*. For all $v \in \mathcal{M}_p(A)$, we have*

$$|\partial\Phi|(v) \geq \frac{p}{2} \|\Delta_p v + E(v)|v|^{p-2}v\|_{W^{-1,p'}(A)}. \tag{23}$$

Moreover, if $v_0 \in \mathcal{M}_p(A)$ is not an eigenfunction for the p-Laplacian, there exist $c_0, \delta > 0$ such that

$$\|\Delta_p v + E(v)|v|^{p-2}v\|_{W^{-1,p'}(A)} \geq c_0 \quad \textit{for all } v \in \mathcal{M}_p(A) \quad \textit{satisfying} \quad \|v - v_0\|_{L^p(A)} < \delta. \tag{24}$$

Proof. Fix $v \in \mathcal{M}_p(A)$ and $\varphi \in W_0^{1,p}(A)$, not parallel to v. Setting $w_t = (v + t\varphi)/\|v + t\varphi\|_{L^p(A)}$, a simple calculation shows that

$$\lim_{t \to 0^+} \frac{\Phi(w_t) - \Phi(v)}{\|w_t - v\|_{L^p(A)}} = p \frac{\langle -\Delta_p v - E(v)|v|^{p-2}v\ ,\ \varphi \rangle}{\|\langle |v|^{p-2}v, \varphi\rangle v - \varphi\|_{L^p(A)}},$$

where, as usual, we denote

$$\langle -\Delta_p v, \varphi \rangle = \int_A |\nabla v|^{p-2} \nabla v \cdot \nabla \varphi \, dx \quad \text{and} \quad \langle |v|^{p-2}v, \varphi \rangle = \int_A |v|^{p-2} v \varphi \, dx.$$

Thus, observing that $\|\langle |v|^{p-2}v, \varphi\rangle v - \varphi\|_{L^p(A)} \leq 2\|\varphi\|_{L^p(A)}$, we have

$$\begin{aligned}
|\partial\Phi|(v) &\geq p \sup \left\{ \frac{|\langle \Delta_p v + E(v)|v|^{p-2}v, \varphi\rangle|}{\|\langle |v|^{p-2}v, \varphi\rangle v - \varphi\|_{L^p(A)}} : \varphi \in W_0^{1,p}(A),\ \varphi \neq tv \text{ for } t \in \mathbb{R} \right\} \\
&\geq p \sup \left\{ \frac{|\langle \Delta_p v + E(v)|v|^{p-2}v, \varphi\rangle|}{2\|\varphi\|_{W_0^{1,p}(A)}} : \varphi \in W_0^{1,p}(A) \right\} \\
&= \frac{p}{2} \|\Delta_p v + E(v)|v|^{p-2}v\|_{W^{-1,p'}(A)}.
\end{aligned}$$

This proves (23).

In order to prove (24) we argue by contradiction assuming that (24) does not hold. If so, there exists a sequence $v_h \in \mathcal{M}_p(A)$ converging to v_0 in $L^p(A)$ and such that

$$\Delta_p v_h + E(v_h)|v_h|^{p-2}v_h \to 0 \ \text{ in } W^{-1,p'}(A).$$

Observe that the sequence v_h is bounded in $W_0^{1,p}(A)$. In fact, if for a not relabelled subsequence $\|\nabla v_h\|_{L^p(A)} \to +\infty$, then we have

$$\int_A |v_h|^{p-2} v_h v_0 \, dx = \frac{1}{E(v_h)} \int_A |\nabla v_h|^{p-2} \nabla v_h \cdot \nabla v_0 \, dx + \frac{1}{E(v_h)} \langle \Delta_p v_h + E(v_h)|v_h|^{p-2}v_h, v_0 \rangle.$$

Thus, by Hölder inequality, we would have

$$\int_A |v_h|^{p-2} v_h v_0 \, dx \leq \frac{\|\nabla v_0\|_{L^p(A)}}{\|\nabla v_h\|_{L^p(A)}} + \varepsilon_h \|v_0\|_{W_0^{1,p}(A)},$$

where $\varepsilon_h \to 0$ as $h \to +\infty$, from which we would conclude that $v_0 = 0$, which is impossible since $v_0 \in \mathcal{M}_p(A)$. Thus the sequence $E(v_h)$ is bounded and, up to a subsequence, we may assume that it converges to some number $E_0 \geq E(v_0) > 0$. Moreover, since $\Delta_p v_h + E(v_h)|v_h|^{p-2}v_h \to 0$ in $W^{-1,p'}(A)$ and $v_h \to v_0$ in $L^p(A)$, also $\Delta_p v_h$ converges strongly in $W^{-1,p'}(A)$. Thus, arguing exactly as in the final part of the proof of Lemma 3.8, we conclude that $v_h \to v_0$ in $W_0^{1,p}(A)$. Note that from the strong convergence of v_h to v_0 in $W_0^{1,p}(A)$ and of $\Delta_p v_h$ in $W^{-1,p'}(A)$, we have that indeed $\Delta_p v_h \to \Delta_p v_0$ in $W^{-1,p'}(A)$. Thus we get that

$$\Delta_p v_0 + E_0 |v_0|^{p-2} v_0 = 0,$$

which is impossible, since v_0 is not an eigenfunction. This contradiction concludes the proof of (24) and hence the proof of the lemma. ☐

We are now ready to give the proof of the next crucial lemma, which provides the construction of a low energy path connecting the first eigenfunction of the p-Laplacian to a function which is not an eigenfunction. With this lemma in hand, the proof of Theorem 3.14 will follow quickly.

Lemma 3.16. *Let $A \subset \mathbb{R}^n$ be a p-quasi open set of finite measure. Suppose that $\lambda_1(A)$ is simple and let u_1 be the first nonnegative normalized eigenfunction. If $v_0 \in \mathcal{M}_p(A)$ is not an eigenfunction and $\lambda_1(A) < E(v_0) \le \lambda_2(A)$, then there exists a curve $v \in C^{0,(p-1)/p}([0,\infty), \mathcal{M}_p(A)) \cap W^{1,p}([0,\infty), L^p(A))$ with $v(0) = v_0$, such that the following hold:*

$$(i) \quad E(v(t)) < \lambda_2(A) \quad \forall\, t > 0, \quad \text{and} \quad \int_0^\infty \|v'(t)\|^p_{L^p(A)}\, dt \le E(v_0); \tag{25}$$

$$(ii) \quad \lim_{t\to\infty} E(v(t)) = \lambda_1(A); \tag{26}$$

$$(iii) \quad \lim_{t\to\infty} v(t) = u_1 \quad \text{or} \quad \lim_{t\to\infty} v(t) = -u_1 \quad \text{in} \quad W_0^{1,p}(A). \tag{27}$$

Proof. Step 1 (The discrete scheme). Fix $\tau > 0$ and set $v_0^\tau := v_0$. Then, for all $k \ge 1$ we define recursively the function v_k^τ by selecting a minimizer of the following problem:

$$\min_{w\in\mathcal{M}_p(A)} \Big[\, \frac{1}{\tau^{p-1}} \int_A |w - v_{k-1}^\tau|^p\, dx + \int_A |\nabla w|^p\, dx \,\Big]. \tag{28}$$

The existence of a minimizer follows from coercivity and weak lower semicontinuity. Moreover, there exists a Lagrange multiplier $\sigma_k^\tau \in \mathbb{R}$ such that for every $\phi \in W_0^{1,p}(A)$ we have

$$\frac{1}{\tau^{p-1}} \int_A |v_k^\tau - v_{k-1}^\tau|^{p-2}(v_k^\tau - v_{k-1}^\tau)\phi\, dx + \int_A |\nabla v_k^\tau|^{p-2}\nabla v_k^\tau \cdot \nabla\phi\, dx = \sigma_k^\tau \int_A |v_k^\tau|^{p-2} v_k^\tau\, \phi\, dx. \tag{29}$$

Since $v_k^\tau \in \mathcal{M}_p(A)$, choosing $\phi = v_k^\tau$, we have

$$\sigma_k^\tau = \int_A |\nabla v_k^\tau|^p\, dx + \frac{1}{\tau^{p-1}} \int_A |v_k^\tau - v_{k-1}^\tau|^{p-2}(v_k^\tau - v_{k-1}^\tau) v_k^\tau\, dx \tag{30}$$

Then, comparing the values of the functional in (28) at v_k^τ and v_{k-1}^τ we get for all $k \ge 1$

$$\frac{1}{\tau^{p-1}} \|v_k^\tau - v_{k-1}^\tau\|^p_{L^p(A)} \le E(v_{k-1}^\tau) - E(v_k^\tau). \tag{31}$$

Now we choose an uniform partition of $[0,\infty)$ with $\{0, \tau, 2\tau, \ldots\}$ and define the piecewise constant flow $v^\tau : [0,\infty) \to \mathcal{M}_p(A)$ by setting $v^\tau(t)(x) = v^\tau_{[t/\tau]}(x)$ for all $t > 0$, $x \in \mathbb{R}^n$, where $[\cdot]$ denotes the integer part function. Similarly, we denote by σ^τ the piecewise constant function from $[0,\infty)$ to $\mathbb{R}$ defined by setting $\sigma^\tau(t) = \sigma^\tau_{[t/\tau]}$. Using (31), we have that for all $t > s \ge 0$ with $[t/\tau] > [s/\tau]$

$$\begin{aligned}
\|v^\tau(t) - v^\tau(s)\|_{L^p(A)} &\le \sum_{k=[s/\tau]+1}^{[t/\tau]} \|v_k^\tau - v_{k-1}^\tau\|_{L^p(A)} \\
&\le \Big([t/\tau] - [s/\tau]\Big)^{\frac{p-1}{p}} \Big(\sum_{k=[s/\tau]+1}^{[t/\tau]} \|v_k^\tau - v_{k-1}^\tau\|^p_{L^p(A)} \Big)^{\frac{1}{p}} \\
&\le \Big([t/\tau] - [s/\tau]\Big)^{\frac{p-1}{p}} \tau^{\frac{p-1}{p}} \Big[E(v^\tau_{[s/\tau]}) - E(v^\tau_{[t/\tau]})\Big]^{\frac{1}{p}}.
\end{aligned}$$

Thus, we find that

$$\|v^\tau(t) - v^\tau(s)\|_{L^p(A)} \le (E(v_0))^{\frac{1}{p}} (t - s + \tau)^{\frac{p-1}{p}}.$$

From this inequality, recalling that the functions $\{v^\tau\}_{0<\tau<1}$ are bounded in $W_0^{1,p}(A)$ uniformly with respect to t, we deduce, thanks to a refined version of Arzelà-Ascoli theorem (see [4, Prop. 3.3.1]), that there exists a sequence $\tau_i \to 0$ such that for all $t > 0$ the curves $v^{\tau_i}(t)$ converge in $L^p(A)$, uniformly with respect to $t \in [0,T]$, to a curve $v \in C^{0,(p-1)/p}([0,\infty), L^p(A))$. Moreover, since for every $t > 0$ the sequence $v^{\tau_i}(t)$ is bounded in $W_0^{1,p}(A)$, a simple compactness argument shows that it converges weakly in $W_0^{1,p}(A)$ to $v(t)$. Thus $v(t) \in \mathcal{M}_p(A)$ for all $t \geq 0$.

Step 2 (Convergence of the discrete scheme). We set

$$\hat{v}_i(t) := \frac{(t-(k-1)\tau_i)v_k^{\tau_i} + (k\tau_i - t)v_{k-1}^{\tau_i}}{\tau_i} \qquad \text{for } t \in [(k-1)\tau_i, k\tau_i]$$

and observe that, up to another not relabelled subsequence, the functions $\hat{v}_i'$ converge weakly in $L^p([0,\infty), L^p(A))$ to v'. Indeed, this follows immediately from (31) since for every $T > 0$

$$\int_0^T \|\hat{v}_i'(t)\|_{L^p(A)}^p \, dt \leq \sum_{k=1}^{[T/\tau_i]+1} (E(v_{k-1}^{\tau_i}) - E(v_k^{\tau_i})) \leq E(v_0). \tag{32}$$

Note that from this inequality we get in particular the second estimate in (25). Next, observe that, again up to a not relabelled subsequence, we may assume that the functions σ^{τ_i} converge weakly in $L^{p'}(0,T)$ for all $T > 0$. In fact, from (30) we have, using Hölder inequality and recalling (32),

$$\begin{aligned}\int_0^T |\sigma^{\tau_i}(t)|^{p'} \, dt &\leq C\,T E(v_0)^{p'} + C\int_0^T \left(\int_A |\hat{v}_i'(t)|^{p-1}|v^{\tau_i}(t)| \, dx\right)^{\frac{p}{p-1}} dt \\ &\leq C\,T E(v_0)^{p'} + C\int_0^T \|\hat{v}_i'(t)\|_{L^p(A)}^p \, dt \leq C(TE(v_0)^{p'} + E(v_0)),\end{aligned}$$

for a suitable constant C depending only on p. Finally, we claim that $v^{\tau_i}(t) \to v(t)$ strongly in $W_0^{1,p}(A)$ for a.e. $t > 0$. To prove this last claim we are going to use (29) and the convexity of the functional $E(v)$. Precisely, we have that for a.e. $t > 0$,

$$\begin{aligned}\int_A |\nabla v(t)|^p \, dx &\geq \int_A |\nabla v^{\tau_i}(t)|^p \, dx + p\int_A |\nabla v^{\tau_i}(t)|^{p-2}\nabla v^{\tau_i}(t) \cdot (\nabla v(t) - \nabla v^{\tau_i}(t)) \, dx \\ &= \int_A |\nabla v^{\tau_i}(t)|^p \, dx + p\sigma^{\tau_i}(t)\int_A |v^{\tau_i}(t)|^{p-2}v^{\tau_i}(t)(v(t) - v^{\tau_i}(t)) \, dx \\ &\quad - p\int_A |\hat{v}_i'(t)|^{p-2}\hat{v}_i'(t)(v(t) - v^{\tau_i}(t)) \, dx.\end{aligned}$$

Integrating the above inequality with respect to time, with some easy calculations we get that for every $T > 0$

$$\begin{aligned}\int_0^T\int_A |\nabla v(t)|^p \, dxdt &\geq \int_0^T\int_A |\nabla v^{\tau_i}(t)|^p \, dxdt - p\int_0^T |\sigma^{\tau_i}(t)|\left(\int_A |v(t) - v^{\tau_i}(t)|^p \, dx\right)^{\frac{1}{p}} dt \\ &\quad - p\int_0^T\int_A |\hat{v}_i'(t)|^{p-2}\hat{v}_i'(t)(v(t) - v^{\tau_i}(t)) \, dxdt.\end{aligned}$$

Therefore, recalling that the σ^{τ_i} are bounded in $L^{p'}(0,T)$, that v^{τ_i} converge to v in L^p locally uniformly with respect to t and that the $\hat{v}_i'$ are bounded in $L^p([0,\infty), L^p(A))$, passing to the limit we immediately get

$$\begin{aligned}\int_0^T\int_A |\nabla v(t)|^p \, dxdt &\geq \liminf_{i\to\infty}\int_0^T\int_A |\nabla v^{\tau_i}(t)|^p \, dxdt \\ &\geq \int_0^T \left(\liminf_{i\to\infty}\int_A |\nabla v^{\tau_i}(t)|^p \, dx\right)dt \geq \int_0^T\int_A |\nabla v(t)|^p \, dxdt,\end{aligned}$$

where we used Fatou lemma and the lower semicontinuity of the energy E with respect to the weak convergence in $W_0^{1,p}$. Thus we have proved that for a.e. $t > 0$

$$\int_A |\nabla v(t)|^p \, dx = \liminf_{i\to\infty}\int_A |\nabla v^{\tau_i}(t)|^p \, dx. \tag{33}$$

Note that by (31) for every i the fuctions $t \mapsto E(v^{\tau_i}(t))$ are decreasing. Therefore, by Helly's lemma (see [4, Lemma 3.3.3]), there exists a not relabelled subsequence such that for every $t > 0$ there exists the limit of $E(v^{\tau_i}(t))$. This shows that, up to a subsequence, the lim inf in (33) is indeed a limit, hence $\|\nabla v^{\tau_i}(t)\|_{L^p(A)} \to \|\nabla v(t)\|_{L^p(A)}$. This, together with the weak convergence in $W_0^{1,p}(A)$ of $v^{\tau_i}(t)$ to $v(t)$ proved in Step 1, implies that, up to a not relabelled subsequence, $v^{\tau_i}(t)$ converges to $v(t)$ in $W_0^{1,p}(A)$ for a.e. $t > 0$.

Step 3 (An energy inequality). We claim that there exists $c(p) > 0$ such that for a.e. $t > 0$

$$\int_0^t \|v'(s)\|^p_{L^p(A)}\, ds + c(p) \int_0^t \|\Delta_p v(s) + E(v(s))|v(s)|^{p-2} v(s)\|^{p'}_{W^{-1,p'}(A)}\, ds \le E(v_0) - E(v(t)). \quad (34)$$

To this end, we introduce a third kind of interpolation due to De Giorgi. For every $t \in (\tau_i(k-1), \tau_i k]$, $k \ge 1$, we denote by $\tilde{v}_i(t)$ a minimizer of

$$\min_{w \in \mathcal{M}_p(A)} \Big[\frac{1}{(t - \tau_i(k-1))^{p-1}} \int_A |w - v^{\tau_i}_{k-1}|^p\, dx + \int_A |\nabla w|^p\, dx \Big].$$

Just as in (31), here we have that for every $t \in (\tau_i(k-1), \tau_i k]$

$$\frac{1}{(t - \tau_i(k-1))^{p-1}} \|\tilde{v}_i(t) - v^{\tau_i}_{k-1}\|^p_{L^p(A)} \le E(v^{\tau_i}_{k-1}) - E(\tilde{v}_i(t)). \quad (35)$$

Hence, we have that for every $t > 0$

$$\|\tilde{v}_i(t) - v^{\tau_i}(t)\|_{L^p(A)} \le E(v_0)^{\frac{1}{p}} \tau_i^{\frac{p-1}{p}}.$$

and from this inequality we conclude at once that for all $T > 0$ also the curves $\tilde{v}_i(t)$ converge strongly in $L^p(A)$ to $v(t)$ uniformly with respect to $t \in [0, T]$. Note also that from (35), for every $t > 0$ we have $E(\tilde{v}_i(t)) \le E(v^{\tau_i}(t))$. Thus, for a.e. $t > 0$

$$E(v(t)) \le \liminf_{i\to\infty} E(\tilde{v}_i(t)) \le \limsup_{i\to\infty} E(\tilde{v}_i(t)) \le \lim_{i\to\infty} E(v^{\tau_i}_k(t)) = E(v(t)).$$

Therefore we may conclude that also the functions $\tilde{v}_i(t)$ converge strongly in $W_0^{1,p}(A)$ to $v(t)$ for a.e. $t > 0$. Now a very general argument which uses only the definition and no special properties of the local slope defined in (21) shows that for the interpolation defined above one has for every i and every $k \ge 1$

$$\int_0^{k\tau_i} \|\hat{v}_i'(s)\|^p_{L^p(A)}\, ds + \frac{(p-1)}{p^{p'}} \int_0^{k\tau_i} (|\partial\Phi|(\tilde{v}_i(s)))^{p'}\, ds \le E(v_0) - E(v^{\tau_i}_k),$$

see the inequalities (3.2.16) and (3.2.17) in [4], where Φ is defined as in (22). Thus, recalling (23) we deduce that for all i and for all $t > 0$, setting $c(p) := (p-1)/2^{p'}$, we have

$$\int_0^t \|\hat{v}_i'(s)\|^p_{L^p(A)} ds + c(p) \int_0^t \|\Delta_p \tilde{v}_i(s) + E(\tilde{v}_i(s))|\tilde{v}_i(s)|^{p-2} \tilde{v}_i(s)\|^{p'}_{W^{-1,p'}(A)} ds \le E(v_0) - E(v^{\tau_i}(t)).$$

Recalling that $\hat{v}_i'$ converges weakly in $L^p([0,\infty), L^p(A))$ to v' and that $\tilde{v}_i(t)$ and $v^{\tau_i}(t)$ converge in $W_0^{1,p}(A)$ to $v(t)$ for a.e. $t > 0$, (34) follows letting $i \to \infty$.

Step 4 (Conclusion of the proof). By (31), for every i, the function $t \mapsto E(v^{\tau_i}(t))$ is decreasing. Therefore, denoting by $Z_0 \subset (0, \infty)$ a set of zero $\mathcal{L}^1$ measure such that $v^{\tau_i}(t)$ converges to $v(t)$ in $W_0^{1,p}(A)$ for all $t \in (0,\infty) \setminus Z_0$, we have

$$E(v(t)) \le E(v(s)) \quad \text{for all } 0 < s < t \text{ with } s, t \notin Z_0. \quad (36)$$

Since $v \in C^{0,(p-1)/p}([0,\infty), L^p(A))$, if $s \to t$ then $\|v(s) - v(t)\|_{L^p(A)} \to 0$, hence by lower semicontinuity we have also

$$E(v(t)) \le \liminf_{s\to t} E(v(s)) \quad \text{for all } t > 0. \tag{37}$$

Moreover, since by assumption v_0 is not an eigenfunction, from Lemma 3.15 and from the fact that $v \in C^{0,(p-1)/p}([0,\infty), L^p(A))$, it follows that there exist $t_0, c_0 > 0$ such that

$$\|\Delta_p v(t) + E(v(t))|v(t)|^{p-2}v(t)\|_{W^{-1,p'}(A)} \ge c_0 \;\text{ for all } t \in [0, t_0].$$

Hence, (34), (36) and (37) yield that $E(v(t)) < E(v_0) \le \lambda_2(A)$ for all $t > 0$. This proves the first inequality in (25). Note that the assumption that v_0 is not an eigenfunction is crucial for the validity of such estimate. Indeed, if v_0 were an eigenfunction then the above construction would produce the limit flow $v(t) \equiv v_0$ for all $t > 0$.

Now, let us set

$$\alpha := \lim_{\substack{t\to+\infty\\ t\notin Z_0}} E(v(t)) < \lambda_2(A). \tag{38}$$

This limit exists and it is strictly smaller than $\lambda_2(A)$ since the function $E(v(t))$ is decreasing for $t \notin Z_0$ and $E(v(t)) < \lambda_2(A)$ for all $t > 0$.

Note that (32) yields

$$\int_0^\infty \||\hat{v}_i'(t)|^{p-2}\hat{v}_i'(t)\|_{L^{p'}(A)}^{p'}\, dt \le E(v_0),$$

for every i. Therefore, we may assume that, up to a not relabelled subsequence, $|\hat{v}_i'|^{p-2}\hat{v}_i'$ converges weakly in $L^{p'}([0,\infty), L^{p'}(A))$ to a curve q such that

$$\int_0^\infty \|q(t)\|_{L^{p'}(A)}^{p'}\, dt \le E(v_0). \tag{39}$$

Now, let us integrate (29) in $(0,t)$ and let us pass to the limit as $i \to \infty$. From the weak convergence of $|\hat{v}_i'|^{p-2}\hat{v}_i'$ in $L^{p'}([0,\infty), L^{p'}(A))$ and all the convergences proved in Step 2 we have, that for all $t > 0$

$$\int_0^t \int_A q(t)\phi \; dxdt + \int_0^t \int_A |\nabla v(t)|^{p-2}\nabla v(t) \cdot \nabla\phi \; dxdt = \int_0^t \sigma(t)\, dt \int_A |v(t)|^{p-2}v(t)\phi \; dx.$$

for all $\phi \in \mathcal{D}$, where $\mathcal{D}$ is a dense sequence in $W_0^{1,p}(A)$. Differentiating this equality with respect to t yields that for a.e. $t > 0$ and for all $\phi \in \mathcal{D}$

$$\int_A q(t)\phi \; dx + \int_A |\nabla v(t)|^{p-2}\nabla v(t) \cdot \nabla\phi \; dx = \sigma(t) \int_A |v(t)|^{p-2}v(t)\phi \; dx. \tag{40}$$

By density, this equation holds for a.e. $t > 0$ and for every $\phi \in W_0^{1,p}(A)$. Now, let us choose a sequence $t_h \in (0,\infty) \setminus Z_0$ such that (40) holds, $t_h \to +\infty$, $\|q(t_h)\|_{L^{p'}(A)}^{p'} \to 0$ as $h \to \infty$. Note that this is possible thanks to (39). Observe that since the sequence $v(t_h)$ is bounded in $W_0^{1,p}(A)$ up to a subsequence it converges strongly in $L^p(A)$ and weakly in $W_0^{1,p}(A)$ to a function $w \in \mathcal{M}_p(A)$. Testing the equation satisfied by $v(t_h)$ with $v(t_h)$, we have that

$$\lim_{h\to\infty} \sigma(t_h) = \lim_{h\to\infty} \left[E(v(t_h)) + \int_A q(t_h)v(t_h)\, dx\right] = \alpha. \tag{41}$$

Let us now fix $k > h \geq 1$ and let us test with $v(t_k) - v(t_h)$ the equation (40) satisfied by $v(t_k)$ and the equation satisfied by $v(t_h)$. Subtracting the two resulting equations we have

$$\int_A (|\nabla v(t_k)|^{p-2}\nabla v(t_k) - |\nabla v(t_h)|^{p-2}\nabla v(t_h)) \cdot (\nabla v(t_k) - \nabla v(t_h))\, dx$$
$$= \int_A (\sigma(t_k)|v(t_k)|^{p-2}v(t_k) - \sigma(t_h)|v(t_h)|^{p-2}v(t_h))(v(t_k) - v(t_h))\, dx$$
$$- \int_A (q(t_k) - q(t_h))(v(t_k) - v(t_h))\, dx.$$

From this equation, recalling Lemma 6.4 and (41), and using the convergence of $v(t_h)$ to w in L^p and the convergence of $q(t_h)$ to 0 in $L^{p'}(A)$ we get that the sequence $\nabla v(t_h)$ converges to ∇w in $L^p(A)$. Now, considering (40) at the time t_h and letting $h \to \infty$, from (41) we finally get that for all $\phi \in W_0^{1,p}(A)$

$$\int_A |\nabla w|^{p-2}\nabla w \cdot \nabla \phi\; dx = \alpha \int_A |w|^{p-2} w\phi\; dx.$$

Therefore w is an eigenfunction. Then, from (38) we deduce that $\alpha = \lambda_1(A)$ and that w is either equal to u_1 or to $-u_1$.

Finally, observe that $E(v(t)) \geq \lambda_1(A)$ for all $t > 0$. From this inequality, (37) and the fact that $E(v(t)) \to \lambda_1(A)$ as $t \to +\infty$, $t \notin Z_0$, we conclude that

$$\lim_{t\to+\infty} E(v(t)) = \lambda_1(A). \tag{42}$$

This establishes (26).

To conclude the proof we need to show that $v(t) \to w$ in $L^p(A)$ as $t \to +\infty$. To this end we argue by contradiction assuming that there exists a sequence s_h, with $s_h \to +\infty$, such that $\|v(s_h) - w\|_{L^p(A)} \geq c > 0$ for all h. Then, since the sequence $v(s_h)$ is bounded in $W_0^{1,p}(A)$, we may assume that, up to a not relabelled subsequence, it converges strongly in $L^p(A)$ and weakly in $W_0^{1,p}(A)$ to some function $z \in \mathcal{M}_p(A), z \neq w$ and that $0 < s_h < t_h$ for all h. Then, from (42) it follows that $E(z) = \lambda_1(A)$. This means that z is a normalized eigenvalue and thus, since $\lambda_1(A)$ is simple, $z = -w$. In particular, for h sufficiently large we have

$$\|v(t_h) - w\|_{L^p(A)} < \frac{1}{2}, \qquad \|v(s_h) + w\|_{L^p(A)} < \frac{1}{2}.$$

Note that, since $v \in C([0,\infty), L^p(A))$, the function $f(t) = \|v(t) - w\|_{L^p(A)} - \|v(t) + w\|_{L^p(A)}$ is continuous. From the two inequalities above it follows that

$$f(s_h) \geq 2\|w\|_{L^p(A)} - 2\|v(s_h) + w\|_{L^p(A)} > 1.$$

Similarly, we have that $f(t_h) < -1$. Therefore, there exist $r_h \in (s_h, t_h)$, $r_h \to +\infty$, such that $f(r_h) = 0$. However, arguing as above, we have that up to a not relabelled subsequence, $v(r_h)$ converges in $L^p(A)$ either to w or to $-w$, that is $f(r_h)$ converges either to -2 or to 2, which is impossible. This contradiction proves (27). □

Remark 3.17. The fact that the limit for the path of Lemma 3.16 exists as $t \to \infty$ allows us to reparametrize the path to finite time, preserving continuity. Hence, for our purposes we will assume that $v \in C([0,1], \mathcal{M}_p(A))$ with $v(1) = u_1$ or $-u_1$.

of Theorem 3.14. Recalling that $\Gamma(u_1, -u_1)$ is the set of all continuous paths with values in $\mathcal{M}_p(A)$ joining u_1 to $-u_1$, let us define

$$\lambda := \inf_{\gamma\in\Gamma(u_1,-u_1)} \Big[\max_{w\in\gamma([0,1])} \int_A |\nabla w|^p\, dx \Big].$$

Clearly, $\lambda_1(A) \leq \lambda$. Observe that to prove the result it is enough to show that there exists an admissible curve $\gamma \in \Gamma(u_1, -u_1)$ such that

$$\max_{t\in[0,1]} \int_A |\nabla\gamma(t)|^p \, dx = \lambda_2(A). \tag{43}$$

Indeed if $\lambda_1(A)$ is not simple from the previous equality we trivially have $\lambda = \lambda_2(A) = \lambda_1(A)$. On the other hand, if $\lambda_1(A)$ is simple, then by Theorem 3.4 and Lemma 3.8 we have that λ is an eigenvalue; since by Definition 3.13 there is no other eigenvalue between $\lambda_1(A)$ and $\lambda_2(A)$, from (43) we get $\lambda = \lambda_2(A)$, thus concluding the proof of the theorem.

Case 1 : $\lambda_1(A)$ is simple. In this case, thanks to Lemma 3.9 we may assume with no loss of generality that $u_1 \geq 0$. Then we set $U := \{u_1 > 0\}$ and recall that by Corollary 3.10, U is a p-quasi connected component of A. Denote by u_2 a normalized second eigenfunction.

Assume first that u_2 changes sign in U, hence u_2^+ cannot be an eigenfunction, otherwise by the minimum principle either $u_2^+ > 0$ or $u_2^+ = 0$ q.e. in U. Similarly u_2^- is not an eigenfunction. In this case the goal is to construct a continuous curve on $\mathcal{M}_p(A)$ from u_1 to $-u_1$, such that the energy E reaches the maximum value $\lambda_2(A)$ at a point and and stays below this value elsewhere. Wee denote, for $t \in [0,1]$,

$$w(t) := \frac{(1-t)u_2^+}{\|u_2^+\|_{L^p(A)}} - \frac{(1-(1-t)^p)^{1/p}u_2^-}{\|u_2^-\|_{L^p(A)}}. \tag{44}$$

Note that w is a curve with values in $\mathcal{M}_p(A)$ connecting $u_2^+/\|u_2^+\|_{L^p(A)}$ to $-u_2^-/\|u_2^-\|_{L^p(A)}$ and that $E(w(t)) = \lambda_2(A)$ for all t. Since u_2^+ is not an eigenfunction, using Lemma 3.16 and Remark 3.17, we may construct two curves $v_i \in C\big([0,1], \mathcal{M}_p(A)\big)$, $i = 1, 2$, with the property that $E(v_i(t)) \leq \lambda_2(A)$ for all $t \in [0,1]$ and such that v_1 connects $u_2^+/\|u_2^+\|_{L^p(A)}$ to u_1 and v_2 connects $-u_2^-/\|u_2^-\|_{L^p(A)}$ to $-u_1$. Then, denoting by ${v_1}^{-1}$ the path v_1 covered in the opposite direction, we set

$$\gamma := {v_1}^{-1} * w * v_2,$$

where $*$ denotes the concatenation of two curves. By this construction, it is evident that γ is an admissible curve satisfying (43).

Assume now that u_2 does not change sign in U, say $u_2 \geq 0$ in U. By the minimum principle, either $u_2 = 0$ or $u_2 > 0$ q.e. in U. In the latter case from Lemma 3.9, we have that $\lambda_2(A) = \lambda_1(U) = \lambda_1(A)$ which is impossible since $\lambda_1(A)$ is simple. Hence, $u_2 = 0$ q.e. in U. Following Brasco-Franzina [9], we define a curve $\gamma \in \Gamma(u_1, -u_1)$ by setting

$$\gamma(t) := \frac{\cos(\pi t)u_1 \, + \, t(1-t)u_2}{\big(|\cos(\pi t)|^p \, + \, t^p(1-t)^p\big)^{1/p}} \tag{45}$$

for all $t \in [0,1]$. As before, γ is an admissible curve satisfying (43).

Case 2 : $\lambda_1(A)$ is not simple. Assume first that u_1 is supported in a p-quasi connected component U of A. Then there exists another first eigenfunction v which is different from both u_1 and $-u_1$. If v is supported in U, then by Proposition 3.12 v must coincide in U either with u_1 or $-u_1$. Therefore there exists a p-quasi connected component U' of A, with $\mathrm{Cap}_p(U \cap U') = 0$ where v is not identically zero. Denote by u_2 the restriction of v to U', normalized so to have L^p norm equal to 1. Using Lemma 2.9 we have that u_2 is still a first eigenfunction. Moreover (45) provides again a curve satisfying (43).

Finally if there exist two or more connected components where u_1 is not identically zero, let us denote by U one of these connected components and let us set

$$\gamma(t) := \frac{\cos(\pi t)u_1\chi_U \, + \, a(t)u_1\chi_{A\setminus U}}{\big(|\cos(\pi t)|^p\|u_1\|^p_{L^p(U)} \, + \, |a(t)|^p\|u_1\|^p_{L^p(A\setminus U)}\big)^{1/p}},$$

where $a : [0,1] \to [-1,1]$ is a strictly decreasing smooth function such that $a(0) = 1$, $a(1/2) > 0$, $a(1) = -1$. Then it is easily checked that γ is again an admissible curve satisfying (43). $\square$

We conclude this section with the following simple consequence of Theorem 3.14.

Corollary 3.18. *Let $A \subset B$ be two p-quasi open sets of finite measure. Then $\lambda_i(B) \le \lambda_i(A)$ for $i = 1, 2$.*

Proof. The inequality $\lambda_1(B) \le \lambda_1(A)$ is an immediate consequence of (16).

To show that $\lambda_2(B) \le \lambda_2(A)$, let us denote by $u_{1,A}$, $u_{1,B}$ two nonnegative normalized first eigenfunctions of A, and B respectively. Setting, for $t \in [0,1]$

$$v_1(t) = \big(t u_{1,A}^p + (1-t) u_{1,B}^p\big)^{1/p}, \qquad v_2 = -v_1(t),$$

we have, see [8, Lemma 2.1],

$$\int_\Omega |\nabla v_i(t)|^p \, dx \le t \int_\Omega |\nabla u_{1,A}(t)|^p \, dx + (1-t) \int_\Omega |\nabla u_{1,B}(t)|^p \, dx \le \lambda_1(A).$$

On the other hand, thanks to Theorem 3.14 there exists a map $\gamma \in C([0,1], \mathcal{M}_p(A))$ such that $\gamma(0) = u_{1,A}$, $\gamma(1) = -u_{1,A}$ and (43) holds. Therefore, setting $w = v_1 * \gamma * {v_2}^{-1}$, we have $w \in C([0,1], \mathcal{M}_p(B))$ and by construction

$$\max_{t \in [0,1]} \int_B |\nabla w(t)|^p \, dx = \lambda_2(A).$$

From this equality and Theorem 3.14 applied to B we then get $\lambda_2(B) \le \lambda_2(A)$. $\square$

4. γ_p-lower semicontinuity of eigenvalues

In this section we fix a bounded open set $\Omega \subset \mathbb{R}^n$. Henceforth, given a p-quasi open set $A \in \mathscr{A}_p(\Omega)$ and a function $u \in W_0^{1,p}(A)$ we shall still denote by u its zero extension in $\Omega \setminus A$, which is a function in $W_0^{1,p}(\Omega)$.

4.1. γ_p-convergence and properties.

We now introduce the γ_p-convergence of p-quasi open sets. Differently from the case $p = 2$ considered in [11], in the following definition we require the *weak* convergence in $W^{1,p}$ of the resolvents and not the strong one. Indeed, in view of the nonlinearity of the p-Laplacian, requiring the strong convergence of the resolvent operators would end up in a too strong topology in $\mathscr{A}_p(\Omega)$ with very few compact sets. Instead, the definition below provides plenty of compact families in $\mathscr{A}_p(\Omega)$. However, the drawback is that now the proof of the lower semicontinuity of the eigenvalues requires a more delicate argument.

Definition 4.1. Let A_m, A be p-quasi open sets in $\mathscr{A}_p(\Omega)$ for every $m \in \mathbb{N}$. We say that the sequence A_m *γ_p-converges* to A as $m \to \infty$ and we write $A_m \xrightarrow{\gamma_p} A$, if $\mathscr{R}_{p,A_m}(f) \rightharpoonup \mathscr{R}_{p,A}(f)$ weakly in $W_0^{1,p}(\Omega)$ for every $f \in W^{-1,p'}(\Omega)$, where the operators $\mathscr{R}_{p,A_m}$ are defined as in Definition 3.1.

The above definition of γ_p-convergence of p-quasi open sets is strongly related to a convergence in the space $\mathcal{M}_0^p(\Omega)$ of Borel measures with values in $[0,\infty]$ vanishing on sets of zero p-capacity introduced by Dal Maso-Murat in [14]. They say that a sequence $\mu_m \in \mathcal{M}_0^p(\Omega)$ *γ-converges* to a measure $\mu \in \mathcal{M}_0^p(\Omega)$ if for any $f \in W^{-1,p'}(\Omega)$ the solutions $u_m \in W_0^{1,p}(\Omega)$ of the equations

$$\int_\Omega |\nabla u_m|^{p-2} \nabla u_m \cdot \nabla \varphi \, dx + \int_\Omega |u_m|^{p-2} u_m \varphi \, d\mu_m = \langle f, \varphi \rangle \qquad \text{for all } \varphi \in W_0^{1,p}(\Omega)$$

converge weakly in $W^{1,p}(\Omega)$ to the solution of the corresponding equation with μ_m replaced by μ. It is evident that the Definition 4.1 is equivalent to the γ-convergence of the measures ∞_{A_m} to ∞_A in the sense of Dal Maso-Murat [14], where by ∞_A we denote the measure in $\mathcal{M}_0^p(\Omega)$ defined by

$$\infty_A(B) := \begin{cases} 0 & \text{if } \mathrm{Cap}_p(B \cap A) = 0, \\ +\infty & \text{if } \mathrm{Cap}_p(B \setminus A) > 0 \end{cases}$$

for all Borel sets $B \subset \Omega$. With this observation in mind, the next theorem follows immediately from a general result of Dal Maso and Murat, see [14, Th. 6.3].

Theorem 4.2. *$A_m \xrightarrow{\gamma_p} A$ in $\mathscr{A}_p(\Omega)$ if and only if $\mathscr{R}_{p,A_m}(1) \rightharpoonup \mathscr{R}_{p,A}(1)$ weakly in $W_0^{1,p}(\Omega)$.*

The following theorem is also contained in the above mentioned paper, see [14, Th. 6.8].

Theorem 4.3. *Let $A_m, A \in \mathscr{A}_p(\Omega)$ be such that $A_m \xrightarrow{\gamma_p} A$. Then for every $f \in W^{-1,p'}(\Omega)$ we have that $\mathscr{R}_{p,A_m}(f) \to \mathscr{R}_{p,A}(f)$ strongly in $W_0^{1,r}(\Omega)$ for all $1 \le r < p$.*

Now we show that, if the underlying quasi open sets γ_p-converge, then the limit of the sequence of eigenvalues is still an eigenvalue and the corresponding eigenfunctions converge strongly in $W^{1,r}(\Omega)$ for all $1 \le r < p$.

Proposition 4.4. *Let $A_m \in \mathscr{A}_p(\Omega)$ be a sequence of p-quasi open sets and let λ_m be, for every $m \in \mathbb{N}$, an eigenvalue of the p-Laplacian in A_m with a normalized eigenfunction $u_m \in W_0^{1,p}(A_m)$. If there exist $A \in \mathscr{A}_p(\Omega)$ and $\lambda \in \mathbb{R}$ such that $A_m \xrightarrow{\gamma_p} A$ and $\lambda_m \to \lambda$ as $m \to \infty$, then λ is an eigenvalue of the p-Laplacian in A (hence $\lambda > 0$) and the eigenfunctions u_m converge in $W^{1,r}(\Omega)$, up to a subsequence, to an eigenfunction u of λ, for all $1 \le r < p$.*

Proof. Since u_m is a normalized eigenfunction of λ_m and $\lambda_m \to \lambda$, the sequence $\{u_m\}$ is bounded in $W_0^{1,p}(\Omega)$. Therefore there exists a function $u \in W_0^{1,p}(\Omega)$ such that, up to a not relabelled subsequence, $u_m \rightharpoonup u$ in $W^{1,p}(\Omega)$, $u_m \to u$ in $L^p(\Omega)$ and a.e. in Ω. Hence $\|u\|_{L^p(\Omega)} = 1$. Let us now set

$$v_m := \mathscr{R}_{p,A_m}(\lambda |u|^{p-2} u).$$

We claim that

$$\lim_{m\to\infty} \|u_m - v_m\|_{W_0^{1,p}(\Omega)} = 0. \tag{46}$$

Since $A_m \xrightarrow{\gamma_p} A$, by Theorem 4.3 we have that $v_m \to \mathscr{R}_{p,A}(\lambda|u|^{p-2}u)$ in $W^{1,r}(\Omega)$ for all $1 \le r < p$. Then the claim (46) yields that $u_m \to \mathscr{R}_{p,A}(\lambda|u|^{p-2}u)$ in $W^{1,r}(\Omega)$ for all $1 \le r < p$. But since $u_m \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$, we conclude that $u = \mathscr{R}_{p,A}(\lambda|u|^{p-2}u)$. Thus $u \in W_0^{1,p}(A)$, λ is an eigenvalue of A and u is a corresponding eigenfunction. This concludes the proof, provided we show that the claim (46) holds.

To this end, note that u_m and v_m satisfy the following equations in A_m.

$$\begin{aligned} -\operatorname{div}(|\nabla u_m|^{p-2}\nabla u_m) &= \lambda_m |u_m|^{p-2} u_m; \\ -\operatorname{div}(|\nabla v_m|^{p-2}\nabla v_m) &= \lambda\, |u|^{p-2} u. \end{aligned}$$

Testing both equations by the function $u_m - v_m$ and subtracting the resulting equalities, we obtain

$$\begin{aligned} &\int_{A_m} \Big(|\nabla u_m|^{p-2}\nabla u_m - |\nabla v_m|^{p-2}\nabla v_m \Big) \cdot (\nabla u_m - \nabla v_m)\, dx \\ &\qquad = \int_{A_m} \Big[\lambda_m |u_m|^{p-2} u_m - \lambda |u|^{p-2} u \Big] (u_m - v_m)\, dx. \end{aligned}$$

By the a.e. convergence of u_m to u, using a well known variant of the Lebesgue dominated convergence theorem, we get that the sequence $\lambda_m |u_m|^{p-2} u_m$ converges in $L^{p'}(\Omega)$ to $\lambda |u|^{p-2} u$. Since the sequence $u_m - v_m$ is bounded in $L^p(\Omega)$, we get that the right hand side of the above equality converges to zero. Then, from Lemma 6.4 we get immediately that

$$\lim_{m\to\infty} \int_\Omega |\nabla u_m - \nabla v_m|^p \, dx = 0.$$

This proves the claim (46), thereby completing the proof of the lemma. ☐

4.2. **γ_p-lower semicontinuity of eigenvalues.** Now we investigate the behavior of the p-Laplacian eigenvalues with respect to the γ_p-convergence of quasi open sets. The case of the first eigenvalue is easy to deal with.

Corollary 4.5 ((Lower semicontinuity of λ_1))**.** *Let $A_m, A \in \mathcal{A}_p(\Omega)$ be such that $A_m \xrightarrow{\gamma_p} A$. Then*

$$\lambda_1(A) \le \liminf_{m\to\infty} \lambda_1(A_m). \tag{47}$$

Proof. Without loss of generality we may assume that the above lim inf is indeed a limit, say λ, and that λ is finite. From Proposition 4.4, we know that λ is an eigenvalue of A. As $\lambda_1(A)$ is the minimum of all eigenvalues, $\lambda_1(A) \le \lambda$ and (47) follows. ☐

The proof of lower semicontinuity for the second eigenvalue, is more involved. To this end we have to use both the result stated in Theorem 3.14 and a construction based on Lemma 3.16.

Proposition 4.6 ((Lower semicontinuity of λ_2))**.** *Let $A_m, A \in \mathcal{A}_p(\Omega)$ be such that $A_m \xrightarrow{\gamma_p} A$. Then*

$$\lambda_2(A) \le \liminf_{m\to\infty} \lambda_2(A_m). \tag{48}$$

Proof. If $\lambda_1(A)$ is not simple then (48) follows at once from Definition 3.13 and Corollary 4.5. Hence in the rest of the proof we assume that $\lambda_1(A)$ is simple.

Let $u_1$be the first nonnegative normalized eigenfunction of $\lambda_1(A)$. Without loss of generality we may assume that the lim inf in (48) is a limit and that it is finite. Then

$$\lambda_2(A_m) \to \lambda \quad \text{as } m \to \infty \tag{49}$$

and by Proposition 4.4, λ is an eigenvalue. For every $m \in \mathbb{N}$, let $u_{1,m}$ be a normalized nonnegative eigenfunction of $\lambda_1(A_m)$ supported in a p-quasi connected component U_m of A_m. Note that such eigenfunction always exists thanks to Lemma 2.9. Moreover, by Proposition 4.4, we may assume that the sequence $u_{1,m}$ converges weakly in $W^{1,p}(\Omega)$ to an eigenfunction w.

For every m let us denote by $u_{2,m}$ a normalized eigenfunction for $\lambda_{2,m}(A_m)$. We now construct a suitable sequence of curves $\gamma_m \in W^{1,p}([0,1], \mathcal{M}_p(A_m))$ with endpoints $\pm u_{1,m}$, by considering the following cases.

Case 1 : $u_{2,m}$ changes sign in U_m.
We first construct a continuous curve w_m on $\mathcal{M}_p$ from $u^+_{2,m}/\|u^+_{2,m}\|_{L^p(\Omega)}$ to $-u^-_{2,m}/\|u^-_{2,m}\|_{L^p(\Omega)}$, in a different way from what we did in (44). We set

$$w_m(t) := \frac{\frac{(1-t)u^+_{2,m}}{\|u^+_{2,m}\|_{L^p(\Omega)}} - \frac{tu^-_{2,m}}{\|u^-_{2,m}\|_{L^p(\Omega)}}}{((1-t)^p + t^p)^{1/p}}.$$

Note that the above construction yields that there exists a constant C independent of m such that for all $m \in \mathbb{N}$

$$\int_0^1 \|w'_m(t)\|^p_{L^p(\Omega)} \, dt \le C \tag{50}$$

and furthermore that $E(w_m(t)) = \lambda_2(A_m)$ for all $t \in [0,1]$. Now, since $u^+_{2,m}$ and $u^-_{2,m}$ are not eigenfunctions, using Lemma 3.16 and Remark 3.17, we can find two curves $v_{i,m} \in W^{1,p}([0,1], \mathcal{M}_p(A_m))$ with $i = 1, 2$, which have the property that $E(v_{i,m}(t)) \le \lambda_2(A_m)$ for all $t \in [0,1]$ and such that

$v_{1,m}$ connects $u^+_{2,m}/\|u^+_{2,m}\|_{L^p(A_m)}$ to $u_{1,m}$ and $v_{2,m}$ connects $-u^-_{2,m}/\|u^-_{2,m}\|_{L^p(A_m)}$ to $-u_{1,m}$. Then, denoting by ${v_{1,m}}^{-1}$ the path $v_{1,m}$ covered in the opposite direction, we set

$$\gamma_m := {v_{1,m}}^{-1} * w_m * v_{2,m}.$$

Note that from (50) and the second inequality in (25), we have that

$$\int_0^1 \|\gamma_m'(t)\|^p_{L^p(\Omega)}\,dt \le C.$$

Case 2 : $u_{2,m}$ has constant sign in U_m.
In this case, arguing as in the proof of Theorem 3.14, we may always find another second eigenfunction, still denoted by $u_{2,m}$, whose support is disjoint from U_m up to a set of zero p-capacity. Thus we define the curves $\gamma_m \in \Gamma(u_{1,m}, -u_{1,m})$ by setting

$$\gamma_m(t) = \frac{\cos(\pi t)u_{1,m} + t(1-t)u_{2,m}}{\left(|\cos(\pi t)|^p + t^p(1-t)^p\right)^{1/p}} \qquad t\in[0,1].$$

Then it is easily checked that $E(\gamma_m(t)) \le \lambda_2(A_m)$ for all m and t and that also in this case there exists a constant C such that for all m

$$\int_0^1 \|\gamma_m'(t)\|^p_{L^p(\Omega)}\,dt \le C.$$

Combining the two cases, we conclude that there exists a sequence $\gamma_m \in W^{1,p}([0,1],\mathcal{M}_p(A_m))$ of curves with endpoints $\pm u_{1,m}$, such that for all $m\in\mathbb{N}$, we have

$$\int_0^1 \|\gamma_m'(t)\|^p_{L^p(\Omega)}\,dt \le C \quad \text{and} \quad E(\gamma_m(t)) \le \lambda_2(A_m), \qquad \text{for all } t>0.$$

Now we prove that $\lambda \ge \lambda_2(A)$, where λ is the limit in (49).

To this end we argue by contradiction, assuming that $\lambda < \lambda_2(A)$. Then we have that $\lambda = \lambda_1(A)$ and that $u_{1,m}$ converges to u_1. Therefore, using Arzelà-Ascoli theorem as in Step 1 of the proof of Theorem 3.14, we conclude that there exists $\gamma \in W^{1,p}([0,1], L^p(\Omega))$ such that, up to a subsequence, $\gamma_m(t) \to \gamma(t)$ in $L^p(\Omega)$ and weakly in $W^{1,p}_0(\Omega)$ for all $t\in[0,1]$.

Note also that $\gamma(t) \in W^{1,p}_0(A)$ for all $t\in[0,1]$, hence $\gamma \in W^{1,p}([0,1],\mathcal{M}_p(A))$. Indeed, the functions $w_{A_m} = \mathscr{R}_{p,A_m}(1) \rightharpoonup \mathscr{R}_{p,A}(1)$ weakly in $W^{1,p}_0(\Omega)$. Thus, Lemma 5.6 below yields that $\gamma(t) = 0$ q.e. in $\Omega\setminus A$.

Since the endpoints of γ are $\pm u_1$, from Theorem 3.14 we conclude that

$$\lambda_2(A) \le \max_{t\in[0,1]} \int_A |\nabla\gamma(t)|^p\,dx \le \liminf_{m\to\infty}\left[\max_{t\in[0,1]}\int_A |\nabla\gamma_m(t)|^p\,dx\right] \le \lim_{m\to\infty}\lambda_2(A_m) = \lambda_1(A),$$

which is impossible since $\lambda_1(A)$ is simple. This contradiction concludes the proof. □

5. A shape optimization problem

In this section we prove the following theorem, which is the p-Laplacian counterpart of the existence theorem of Buttazzo-Dal Maso [11]. With this theorem in hand, Theorem 1.2 follows at once, thanks to Corollaries 3.18 and 4.5 and to Proposition 4.6.

Theorem 5.1. *Let $\Omega\subset\mathbb{R}^n$ be a bounded open set and $F : \mathscr{A}_p(\Omega)\to\mathbb{R}$ be a decreasing function, lower semicontinuous with respect to γ_p-convergence. Then the minimization problem*

$$\min\left\{F(A) : A\in\mathscr{A}_p(\Omega),\ |A| = c\right\}, \tag{51}$$

where $0 < c \le |\Omega|$, always has a solution.

For every $A \in \mathscr{A}_p(\Omega)$, we set $w_A := \mathscr{R}_{p,A}(1)$. We claim that w_A is a subsolution of the equation $-\Delta_p u = 1$ in Ω. This is the content of the following lemma. The proof is similar to the one given in [11, p. 190], for the case $p = 2$. For the reader's convenience, we provide the proof in the Appendix.

Lemma 5.2 ((Comparison principle))**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $A \in \mathscr{A}_p(\Omega)$. Then $w_A = \mathscr{R}_{p,A}(1)$ is a subsolution of the equation $-\Delta_p u = 1$ in Ω, i.e.*

$$\int_\Omega |\nabla w_A|^{p-2} \nabla w_A \cdot \nabla \varphi \, dx \le \int_\Omega \varphi \, dx \tag{52}$$

for all nonnegative functions $\varphi \in W_0^{1,p}(\Omega)$. Moreover, if $w \in W_0^{1,p}(\Omega)$ is another subsolution satisfying (52) *and such that $w \le 0$ q.e. on $\Omega \setminus A$, then $w_A \ge w$ q.e. in Ω.*

5.1. **The main construction.** Following [11], we now fix a closed convex subset $K \subset W_0^{1,p}(\Omega)$ defined by imposing an obstacle condition. Precisely, we set

$$K := \{ w \in W_0^{1,p}(\Omega) : w \ge 0, \ -\Delta_p w - 1 \le 0 \}.$$

From Lemma 5.2 we have that $w_A \in K$ for every $A \in \mathscr{A}_p(\Omega)$. Moreover, if $w \in K$, multiplying the inequality $-\Delta_p w \le 1$ by w, we get

$$\int_\Omega |\nabla w|^p \, dx \le \int_\Omega w \, dx. \tag{53}$$

Thus, by the Poincaré inequality we conclude that K is bounded in $W_0^{1,p}(\Omega)$ and compact in $L^p(\Omega)$. At this point, still following [11], we define an auxiliary functional $G : K \to \mathbb{R}$ and reduce the proof of the existence of a minimizer of the problem (51) to showing the existence of a minimizer of the problem $\min \{ G(w) : w \in K, \, |\{w > 0\}| \le c \}$. Before defining G, we list the properties that we require from this functional.

(i) G is decreasing, i.e. for every $u, v \in K$ with $u \le v$ q.e. in Ω, then $G(u) \ge G(v)$;
(ii) G is lower semicontinuous on K with respect to the strong topology of $L^p(\Omega)$;
(iii) $G(w_A) = F(A)$ for every $A \in \mathscr{A}_p(\Omega)$, where $w_A = \mathscr{R}_{p,A}(1)$.

Definition 5.3. For every $w \in K$ we set $J(w) = \inf \{ F(A) : A \in \mathscr{A}_p(\Omega), \, w_A \le w \}$ and define G as the $L^p(\Omega)$-lower semicontinuous envelope of J. In other words $G : K \to \mathbb{R}$ is defined by setting

$$G(w) = \inf \{ \liminf_{h \to \infty} J(w_h) : w_h \in K, \ w_h \to w \text{ in } L^p(\Omega) \}.$$

We now show that the functional G satisfies properties i, ii and iii. The verification of the first two is relatively easy, as shown in the next lemma.

Lemma 5.4. *The functional G satisfies i and ii.*

Proof. Property ii is an immediate consequence of the definition of G.

In order to show i, first note that J is decreasing. Fix $u, v \in K$ with $u \le v$ q.e. in Ω. Then by the definition of $G(u)$, there exists a sequence of $\{u_h\} \in K$ such that u_h converge strongly in $L^p(\Omega)$ and pointwise a.e. to u and

$$G(u) = \lim_{h \to \infty} J(u_h).$$

Set $v_h = \max\{v, u_h\}$. Then $v_h \in K$ since the maximum of two subsolutions of the equation $-\Delta_p w = 1$ is still a subsolution. Moreover $v_h \to v$ in in $L^p(\Omega)$ by the dominate convergence theorem and the assumption that $u \le v$ q.e. in Ω. Hence

$$G(v) \le \liminf_{h \to \infty} J(v_h) \le \lim_{h \to \infty} J(u_h) = G(u),$$

which proves i. □

Property iii will follow from Lemma 5.7 below. First, we recall the definition of Γ-convergence and prove the auxiliary Lemma 5.6. This lemma is the p-Laplacian counterpart of Lemma 3.2 in [11] and the main result of this section.

Definition 5.5 ((Γ-convergence))**.** Let $\Phi_h, \Phi : L^p(\Omega) \to \mathbb{R} \cup \{+\infty\}$. We say that the functionals Φ_h Γ-*converge in* $L^p(\Omega)$ *to* Φ if the following two conditions are satisfied:

(1) for every $u_h \in L^p(\Omega)$ such that $u_h \to u \in L^p(\Omega)$, then

$$\Phi(u) \leq \liminf_{h\to\infty} \Phi_h(u_h);$$

(2) for every $u \in L^p(\Omega)$ there exists a sequence $u_h \to u$ such that

$$\Phi(u) \geq \limsup_{h\to\infty} \Phi_h(u_h).$$

This convergence shall be denoted by $\Phi_h \xrightarrow{\Gamma_p} \Phi$.

With this definition in hand we are ready to prove the key lemma of this section.

Lemma 5.6. *Let* $\{A_h\}$ *be a sequence in* $\mathscr{A}_p(\Omega)$ *such that the functions* w_{A_h} *converge weakly to* w *in* $W_0^{1,p}(\Omega)$ *and let* u_h *be a sequence in* $W_0^{1,p}(\Omega)$ *such that* $u_h = 0$ *q.e. in* $\Omega \setminus A_h$. *If* $u_h \rightharpoonup u$ *in* $W_0^{1,p}(\Omega)$, *then* $u = 0$ *q.e. in* $\{w = 0\}$.

Proof. Following [11], we define the functionals $\Phi_h : W^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by setting

$$\Phi_h(v) = \begin{cases} \dfrac{1}{p}\displaystyle\int_{A_h} |\nabla v|^p \, dx & \text{for } \; v \in W_0^{1,p}(A_h), \\ +\infty & \text{otherwise.} \end{cases}$$

By a general compactness result, see [13, Th. 4.18 and Prop. 4.11], there exists a functional $\Phi : W^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ such that, up to a not relabelled subsequence, $\Phi_h \xrightarrow{\Gamma_p} \Phi$. Let $\mathscr{D}(\Phi_h)$ and $\mathscr{D}(\Phi)$ be the effective domains of Φ_h and Φ, respectively, see (20).

Observe that if $u_h \in W_0^{1,p}(A_h)$ and $u_h \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ then by condition (i) in Definition 5.5, we get that

$$\Phi(u) \leq \liminf_{h\to\infty} \Phi_h(u_h) = \liminf_{h\to\infty} \frac{1}{p}\int_\Omega |\nabla u_h|^p \, dx < \infty,$$

hence $u \in \mathscr{D}(\Phi)$. Conversely, for every $v \in \mathscr{D}(\Phi)$, from Definition 5.5 it follows that there exists a sequence $v_h \in W^{1,p}(\Omega)$ converging strongly in $L^p(\Omega)$ to v such that $\Phi_h(v_h) \to \Phi(v)$. Thus the sequence v_h actually converges weakly to v in $W^{1,p}(\Omega)$.

Note that Φ is convex, actually strictly convex on $\mathscr{D}(\Phi)$. The latter property is proved by observing that if $\Phi((z+z')/2) = 1/2(\Phi(z)+\Phi(z')) < \infty$, then denoting by z_h and z'_h two sequences, converging in $L^p(\Omega)$ to z and z' respectively, such that $\Phi(z) = \lim_{h\to\infty} \Phi(z_h)$ and $\Phi(z') = \lim_{h\to\infty} \Phi(z'_h)$, then we have also that $\Phi((z+z')/2) = \lim_{h\to\infty} \Phi_h((z_h + z'_h)/2)$. Therefore we have in particular that

$$\lim_{h\to\infty} \left[\frac{1}{2}\int_\Omega (|\nabla z_h|^p + |\nabla z'_h|^p) \, dx - \int_\Omega \Big| \frac{\nabla z_h + \nabla z'_h}{2} \Big|^p \, dx \right] = 0,$$

from which we easily conclude, using the Clarkson's inequality, that $\nabla z_h - \nabla z'_h \to 0$ in $L^p(\Omega)$ and thus $z = z'$.

In view of the above remarks, the proof of the lemma will be achieved if we prove the following **Claim :** For every $v \in \mathscr{D}(\Phi)$, one has $v = 0$ q.e. on $\{w = 0\}$.

To prove the claim, it is enough to assume that $v \in Int(\mathscr{D}(\Phi))$, since the general case follows by approximation. Since $v \in Int(\mathscr{D}(\Phi))$, the subdifferential $\partial\Phi(v)$ is not empty, hence we may fix $f \in \partial\Phi(v)$. Denoting by $\langle\, .\, , .\,\rangle$ the duality action between $W^{-1,p'}(\Omega)$ and $W_0^{1,p}(\Omega)$, we have

$$v = \mathrm{argmin}\{\Phi(z) - \langle\, f, z\,\rangle : z \in W_0^{1,p}(\Omega)\}.$$

Let us show that v is the weak limit in $W_0^{1,p}(\Omega)$ of a sequence of minimizers of the functional Φ_h, i.e., if

$$v_h = \mathrm{argmin}\{\Phi_h(z) - \langle\, f, z\,\rangle : z \in W_0^{1,p}(\Omega)\},$$

then $v_h \rightharpoonup v$ in $W_0^{1,p}(\Omega)$. To this end, observe that up to a subsequence $v_h \rightharpoonup \tilde{v}$ weakly in $W_0^{1,p}(\Omega)$ and that $\tilde{v} \in \mathscr{D}(\Phi)$. On the other hand from (2) of Definition 5.5 there exists $w_h \in W_0^{1,p}(A_h)$ such that w_h converge in $L^p(\Omega)$ to v and $\Phi_h(w_h) \to \Phi(v)$. Moreover, passing possibly to a not relabelled subsequence, we may assume that $w_h \rightharpoonup v$ weakly in $W_0^{1,p}(\Omega)$. Thus, by lower semicontinuity, using the minimality of v_h, we get

$$\Phi(\tilde{v}) - \langle\, f, \tilde{v}\,\rangle \le \liminf_{h\to\infty} \Phi_h(v_h) - \langle\, f, v_h\,\rangle \le \lim_{h\to\infty} \Phi_h(w_h) - \langle\, f, w_h\,\rangle \le \Phi(v) - \langle\, f, v\,\rangle.$$

Therefore by uniqueness we have $\tilde{v} = v$ and that, up to another not relabelled subsequence, $v_h \rightharpoonup v$ in $W^{1,p}(\Omega)$. At this point, a standard compactness argument shows the convergence of the whole sequence v_h.

We now fix $\varepsilon \in (0,1)$ and a function $f^\varepsilon \in L^\infty(\Omega)$ such that $\|f - f^\varepsilon\|_{W^{-1,p'}(\Omega)} \le \varepsilon$. Then, for every h we denote by v_h^ε the function $v_h^\varepsilon := \mathscr{R}_{p,A_h}(f^\varepsilon)$. Testing the equations satisfied by v_h^ε and v_h with the function $v_h^\varepsilon - v_h$ and subtracting the two resulting equalities, we have

$$\int_\Omega (|\nabla v_h^\varepsilon|^{p-2}\nabla v_h^\varepsilon - |\nabla v_h|^{p-2}\nabla v_h)\cdot(\nabla v_h^\varepsilon - \nabla v_h)\,dx = \langle f^\varepsilon - f\,,\ v_h^\varepsilon - v_h\rangle. \tag{54}$$

If $p \ge 2$, we recall (61) which combined with (54), followed by a standard use of Young's inequality and Poincaré inequality, yields that for some $c(p) > 0$

$$\int_\Omega |\nabla v_h^\varepsilon - \nabla v_h|^p\,dx \le c(p)\|f - f^\varepsilon\|_{W^{-1,p'}(\Omega)}^{\frac{p}{p-1}} \le c(p)\,\varepsilon^{\frac{p}{p-1}}. \tag{55}$$

If $1 < p < 2$, we recall (62), which combined with (54), yields the following.

$$\begin{aligned}\int_\Omega |\nabla v_h^\varepsilon - \nabla v_h|^p\,dx \ &\le |\langle f^\varepsilon - f\,,\ v_h^\varepsilon - v_h\rangle|^{\frac{p}{2}}\Big(\int_\Omega(|\nabla v_h^\varepsilon|^p + |\nabla v_h|^p)\,dx\Big)^{1-\frac{p}{2}}\\ &= \ |\langle f^\varepsilon - f\,,\ v_h^\varepsilon - v_h\rangle|^{\frac{p}{2}}\Big(\langle\, f^\varepsilon, v_h^\varepsilon\,\rangle + \langle\, f, v_h\,\rangle\Big)^{1-\frac{p}{2}}.\end{aligned}$$

Since the sequence v_h is bounded in $W^{1,p}(\Omega)$, we easily get

$$\begin{aligned}\int_\Omega |\nabla v_h^\varepsilon - \nabla v_h|^p\,dx \ &\le \|f - f^\varepsilon\|_{W^{-1,p'}(\Omega)}^{\frac{p}{2}}\|v_h^\varepsilon - v_h\|_{W_0^{1,p}(\Omega)}^{\frac{p}{2}}\Big(\langle\, f^\varepsilon, v_h^\varepsilon - v_h\,\rangle + \langle\, f + f_\varepsilon, v_h\,\rangle\Big)^{1-\frac{p}{2}}\\ &\le C\,\varepsilon^{\frac{p}{2}}\Big(\,\|v_h^\varepsilon - v_h\|_{W_0^{1,p}(\Omega)} \ + \ \|v_h^\varepsilon - v_h\|_{W_0^{1,p}(\Omega)}^{\frac{p}{2}}\,\Big),\end{aligned}$$

for some constant C depending on $\|f\|_{W^{-1,p'}(\Omega)}$ and p, but independent of h and ε. Thus, from Young's inequality and Poincaré inequality, we have

$$\begin{aligned}\int_\Omega |\nabla v_h^\varepsilon - \nabla v_h|^p\,dx &\le \frac{1}{2}\int_\Omega |\nabla v_h^\varepsilon - \nabla v_h|^p\,dx + C\big(\,\varepsilon^{\frac{p^2}{2(p-1)}} \ + \ \varepsilon^p\,\big)\\ &\le \frac{1}{2}\int_\Omega |\nabla v_h^\varepsilon - \nabla v_h|^p\,dx + C\varepsilon^p,\end{aligned} \tag{56}$$

for $0<\varepsilon<1$. Thus, from (55) and (56) we may conclude that there exists a positive constant C depending only on Ω, $\|f\|_{W^{-1,p'}(\Omega)}$ and p but independent of h and ε, such that for every $1<p<\infty$,

$$\|v_h - v_h^\varepsilon\|_{W_0^{1,p}(\Omega)} \le C\varepsilon^{\frac{1}{p}}.$$

Note that for every $\varepsilon\in(0,1)$ there exists a not relabelled subsequence v_h^ε converging weakly and a.e. to a function $v^\varepsilon \in W_0^{1,p}(\Omega)$. Then, from the previous inequality, we have by lower semicontinuity

$$\|v - v^\varepsilon\|_{W_0^{1,p}(\Omega)} \le C\varepsilon^{\frac{1}{p}}.$$

Note that by the comparison principle (Lemma 5.2) we have $|v_h^\varepsilon| \le \|f^\varepsilon\|_{L^\infty(\Omega)}^{1/(p-1)}\, w_{A_h}$ q.e. in Ω, hence $|v^\varepsilon| \le \|f^\varepsilon\|_{L^\infty(\Omega)}^{1/(p-1)}\, w$ a.e. in Ω. In particular, we have that the precise representative of v^ε is 0 q.e. in the set where the precise representative of w vanishes. Then, the claim follows from the strong convergence in $W_0^{1,p}(\Omega)$ of v^ε to v. Hence the proof of the lemma is completed. □

5.2. **Proof of Theorem 5.1.** Now we have all the ingredients to prove the existence theorem. The proof follows the one in [11] with some extra difficulties due to the nonlinearity of the p-Laplacian.

Lemma 5.7. *Let w_h be a sequence in K converging in $L^p(\Omega)$ to w_A for some $A \in \mathscr{A}_p(\Omega)$. Then*

$$F(A) \le \liminf_{h\to\infty} J(w_h).$$

Proof. Without loss of generality we may assume that the above lim inf is indeed a finite limit. From Definition 5.3 we have there exists $A_h \in \mathscr{A}_p(\Omega)$ such that $w_{A_h} \le w_h$ and

$$F(A_h) \le J(w_h) + 1/h.$$

Thanks to (53), the sequence $\{w_{A_h}\}$ is bounded in $W_0^{1,p}(\Omega)$, and hence there exists $w \in W_0^{1,p}(\Omega)$ such that, up a not relabelled subsequence, $w_{A_h} \rightharpoonup w$ weakly in $W_0^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and a.e. in Ω. Therefore since $w_{A_h} \le w_h$ a.e., we have that also $w \le w_A$ a.e. in Ω.

Let us fix $\varepsilon>0$ and set $A^\varepsilon := \{w_A > \varepsilon\}$. Clearly $(w_A-\varepsilon)^+ \in W_0^{1,p}(A^\varepsilon)$. Passing possibly to another not relabelled subsequence we may assume that the functions $w_{A_h\cup A^\varepsilon}$ converge weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ to a function w^ε. Define now $v^\varepsilon := 1-\min\{w_A,\,\varepsilon\}/\varepsilon$. Then $v^\varepsilon \in W^{1,p}(\Omega)$ and $0\le v^\varepsilon\le 1$ q.e. in Ω, $v^\varepsilon = 0$ q.e. in A^ε, $v^\varepsilon=1$ q.e. in $\Omega\setminus A$. Now set

$$u_h = \min\{v^\varepsilon,\, w_{A_h\cup A^\varepsilon}\}.$$

Then $u_h = 0$ q.e. on $\Omega\setminus A_h$, and, up to another not relabelled subsequence, u_h converge weakly in $W_0^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ to $\min\{v^\varepsilon,\, w^\varepsilon\}$. By Lemma 5.6 we conclude that $\min\{v^\varepsilon,\, w^\varepsilon\}=0$ q.e. in $\{w=0\}$ and in particular that $\min\{v^\varepsilon,\, w^\varepsilon\}=0$ q.e. in $\Omega\setminus A$. In turn, recalling that $v^\varepsilon=1$ in $\Omega\setminus A$ we have that $w^\varepsilon=0$ q.e. in $\Omega\setminus A$, hence $w^\varepsilon\in W_0^{1,p}(A)$.

Now, let us take a sequence ε_i converging to zero and such that w^{ε_i} converge strongly in $L^p(\Omega)$ and weakly in $W_0^{1,p}(\Omega)$ to some function $\widetilde{w}\in W_0^{1,p}(A)$. By a standard diagonal argument we may find a subsequence $w_{A_{h_i}\cup A_{\varepsilon_i}}$ also converging to $\widetilde{w}$ in $L^p(\Omega)$ and weakly in $W_0^{1,p}(\Omega)$. By the minimality of $w_{A_{h_i}\cup A^{\varepsilon_i}}$, since $(w_A-\varepsilon_i)^+ \in W_0^{1,p}(A_{h_i}\cup A^{\varepsilon_i})$ we have

$$\frac{1}{p}\int_\Omega |\nabla w_{A_{h_i}\cup A^{\varepsilon_i}}|^p\,dx - \int_\Omega w_{A_{h_i}\cup A^{\varepsilon_i}}\,dx \le \frac{1}{p}\int_\Omega |\nabla (w_A-\varepsilon_i)^+|^p\,dx - \int_\Omega (w_A-\varepsilon_i)^+\,dx. \tag{57}$$

Passing to the limit, by lower semicontinuity we get that

$$\frac{1}{p}\int_\Omega |\nabla \widetilde{w}|^p\,dx - \int_\Omega \widetilde{w}\,dx \le \frac{1}{p}\int_\Omega |\nabla w_A|^p\,dx - \int_\Omega w_A\,dx. \tag{58}$$

Thus, since $\widetilde{w} \in W_0^{1,p}(A)$, by uniqueness we conclude that $\widetilde{w} = w_A$. Moreover, combining (57) and (58) we have also that $\|\nabla w_{A_{h_i} \cup A^{\varepsilon_i}}\|_{L^p(\Omega)} \to \|\nabla w_A\|_{L^p(\Omega)}$, hence $w_{A_{h_i} \cup A^{\varepsilon_i}}$ converges strongly in $W_0^{1,p}(\Omega)$ to w_A. In turn, Theorem 4.2 implies that $A_{h_i} \cup A^{\varepsilon_i}$ γ_p-converges to A. Thus, using the lower semicontinuity of F we conclude that

$$F(A) \leq \liminf_{i\to\infty} F(A_{h_i} \cup A^{\varepsilon_i}) \leq \liminf_{i\to\infty} F(A_{h_i}) \leq \liminf_{h\to\infty} J(w_h),$$

thereby finishing the proof. □

Combining the definition of G with Lemma 5.7, we conclude that G satisfies iii, and hence it is the desired functional.

of Theorem 5.1. First, we observe that if $0 < c \leq |\Omega|$ the following problem has a solution

$$\inf\{G(w) \colon w \in K, \, |\{w > 0\}| \leq c\}.$$

Indeed, if w_h is a minimizing sequence, from (53) it follows that w_h is bounded in $W_0^{1,p}(\Omega)$. Therefore, up to a not relabelled subsequence, we may assume that w_h converges strongly in $L^p(\Omega)$ and weakly in $W_0^{1,p}(\Omega)$ to a function $w_0 \in W_0^{1,p}(\Omega)$. Since K is closed and convex, by Hahn-Banach theorem it is also weakly closed, hence $w_0 \in K$. Moreover, $|\{w_0 > 0\}| \leq c$. Thus, by the lower semicontinuity property ii, we conclude that w_0 is a minimizer of the above problem.

Denote now by A_0 a p-quasi open set such that $\{w_0 > 0\} \subset A_0 \subset \Omega$, with $|A_0| = c$. We claim that A_0 is a solution to $\min\{F(A) : A \in \mathscr{A}_p(\Omega), \ |A| \leq c\}$, hence a solution to the problem (51). Indeed, by Lemma 5.2, $w_0 \leq w_{A_0}$ q.e. in Ω, and then the properties i and iii of G imply

$$F(A_0) = G(w_{A_0}) \leq G(w_0).$$

For any $A \in \mathscr{A}_p(\Omega)$ and $|A| \leq c$, we have $w_A \in K$ and $|\{w_A > 0\}| \leq c$. Hence the minimality of w_0 yields

$$G(w_0) \leq G(w_A) = F(A),$$

which implies $F(A_0) \leq F(A)$. By the arbitrariness of A the claim follows and the proof is complete. □

Remark 5.8. All the statements in this section and in the previous one have been given in the context of p-quasi open sets, assuming that $1 < p \leq n$. However, all the arguments and tools used in the proofs, including the characterization of the second eigenvalue given by Theorem 3.4, do apply without changes also when $p > n$ and $\mathcal{A}_p(\Omega)$ reduces to the family of open sets contained in Ω. Therefore both the lower semicontinuity results of Section 4 and Theorem 5.1 still hold in this case.

6. Appendix

First, for the reader's convenience, we provide some details on the behavior of quasi open sets, when restricted to lines parallel to axis. These are useful for the purpose of Lemma 2.9. Let Υ be the set of all compact rectifiable curves $\gamma : [0,1] \to \mathbb{R}^n$. Given any family of curves $\Gamma \subset \Upsilon$, the *p-modulus* of the family Γ is defined as

$$M_p(\Gamma) := \inf\left\{\int_{\mathbb{R}^n} \rho(x)^p \, dx \mid \rho : \mathbb{R}^n \to [0,\infty] \text{ is Borel}, \ \int_\gamma \rho \, ds \geq 1 \ \ \forall \gamma \in \Gamma\right\} \tag{59}$$

for every $1 \leq p < \infty$. The notion of p-modulus appeared first in [17] and later on was extended in the framework of general metric spaces in [23]. It is easy to see that M_p is an outer measure on Υ.

Lemma 6.1. *Let $A \subset \mathbb{R}^n$ be a p-quasi open set. Then the set $\gamma^{-1}(A)$ is open in $[0,1]$ for M_p-a.e. rectifiable curve $\gamma \in \Upsilon$.*

For a proof of the above lemma, we refer to [30, Remark 3.5].

Lemma 6.2. *Let $E \subset \mathbb{R}^{n-1}$ be a Borel set and*

$$\Gamma_E := \{\gamma_{x'} : [0,1] \to E \times [0,1] \mid x' \in E,\ \gamma_{x'}(t) = (x', t)\}.$$

If $p > 1$, then we have $M_p(\Gamma_E) = 0$ if and only if $\mathcal{L}^{n-1}(E) = 0$.

Proof. Let $M_p(\Gamma_E) = 0$. From the definition (59) for every $m \in \mathbb{N}$ there exists $\rho_m : \mathbb{R}^n \to [0, \infty]$ such that $\|\rho_m\|^p_{L^p(\mathbb{R}^n)} \le 1/m$ and $\int_\gamma \rho_m\, ds \ge 1$ for every $\gamma \in \Gamma_E$. Hence, if K is a compact subset of E, we have

$$\mathcal{L}^{n-1}(K) \le \int_K \int_0^1 \rho_m(x', t)\, dt\, dx' \le \mathcal{L}^{n-1}(K)^{1-\frac{1}{p}} \Big(\int_{\mathbb{R}^n} \rho_m(x)^p\, dx \Big)^{\frac{1}{p}}$$

which implies $\mathcal{L}^{n-1}(K) \le 1/m$ for all $m \in \mathbb{N}$, hence $\mathcal{L}^{n-1}(K) = 0$. This proves that $\mathcal{L}^{n-1}(E) = 0$.

The converse follows by observing that if $\mathcal{L}^{n-1}(E) = 0$, then the function $\rho_E = \mathbb{1}_{E \times [0,1]}$ satisfies $\int_\gamma \rho_E\, ds = 1$ for all $\gamma \in \Gamma_E$ and $\|\rho_E\|_{L^p(\mathbb{R}^n)} = 0$. □

Corollary 6.3. *Let $A \subset \mathbb{R}^n$ be a p-quasi open set. Then $A_{x'} := \{t \in \mathbb{R} : (x', t) \in A\}$ is an open set for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$.*

Proof. Let A be a quasi open set $A \subset \mathbb{R}^n$. Fix $h \in \mathbb{N}$ and set

$$E_h = \Big\{ x' \in \mathbb{R}^{n-1} \mid A \cap (\{x'\} \times [-h, h]) \text{ is not open in } \{x'\} \times [-h, h] \Big\}.$$

From Lemma 6.1, we have $M_p(\Gamma_{E_h}) = 0$, where $\Gamma_{E_h} = \{(x', t) : t \in [-h, h]\}$. Thus from Lemma 6.2 we have $\mathcal{L}^{n-1}(E_h) = 0$. Hence, the result follows. □

Here, we provide the proof of Lemma 5.2.

of Lemma 5.2. Given a closed, convex subset $K \subset W^{1,p}_0(\Omega)$, we denote by $u_K \in K$ a solution of the following variational inequality

$$\int_\Omega |\nabla u_K|^{p-2} \nabla u_K \cdot \nabla(v - u_K)\, dx \ge \int_\Omega (v - u_K)\, dx \tag{60}$$

for all $v \in K$. Let us now define the operator $\mathcal{L} : W^{1,p}_0(\Omega) \to W^{-1,p'}(\Omega)$ setting $\mathcal{L}u := -\Delta_p u$ for all $u \in W^{1,p}_0(\Omega)$. Using Lemma 6.4, it is immediate to check that $\mathcal{L}$ satisfies all the monotonicity and coercivity assumptions that guarantee the existence of a solution of the variational inequality (60), see Corollary 1.8 in Ch. III of [26].

Given $A \in \mathcal{A}_p(\Omega)$, let us now choose

$$K := \{v \in W^{1,p}_0(\Omega) : v \le 0 \ \text{ q.e. in } \Omega \setminus A\}.$$

Let $w \in W^{1,p}_0(\Omega)$ be any subsolution of the equation $-\Delta_p w = 1$ in Ω such that $w \le 0$ q.e. in $\Omega \setminus A$. Setting $\varphi := \min\{u_K - w, 0\} \in W^{1,p}_0(A)$, using the fact the w is a subsolution, that $\varphi \le 0$ and that u_K satisfies the variational inequality (60), we get

$$\begin{aligned}\int_\Omega |\nabla w|^{p-2} \nabla w \cdot \nabla \varphi\, dx &\ge \int_\Omega \varphi\, dx = \int_\Omega (u_K - \max\{u_K, w\})\, dx \\ &\ge \int_\Omega |\nabla u_K|^{p-2} \nabla u_K \cdot \nabla(u_K - \max\{u_K, w\})\, dx = \int_\Omega |\nabla u_K|^{p-2} \nabla u_K \cdot \nabla \varphi\, dx.\end{aligned}$$

In turn, this equivalently can be written as

$$\int_{\{u_K < w\}} (|\nabla w|^{p-2} \nabla w - |\nabla u_K|^{p-2} \nabla u_K) \cdot (\nabla w - \nabla u_K)\, dx \le 0.$$

Thus, from Lemma 6.4 we conclude that

$$\int_{\{u_K < w\}} |\nabla u_K - \nabla w|^p\, dx = 0,$$

hence $|\{u_K < w\}| = 0$ and thus $u_K \geq w$ q.e. in Ω.

In particular, taking $w = 0$, we have that $u_K \geq 0$ q.e. in Ω. On the other hand $u_K \in K$, hence $u_K \leq 0$ q.e. in $\Omega \setminus A$. Thus, recalling Definition 2.2 we have that $u_K \in W_0^{1,p}(A)$. At this point, choosing as a test function in (60) $v = u_K + \psi$ for any $\psi \in W_0^{1,p}(A)$, we get that u_K is a weak solution of $-\Delta_p u = 1$ in A. By uniqueness,

$$u_K = w_A.$$

In a similar way, choosing $v = w_A - \psi$ in (60) for any nonnegative $\psi \in W_0^{1,p}(\Omega)$ yields that w_A is a subsolution in Ω of the equation $-\Delta_p u = 1$. Hence, the proof is complete. □

We conclude this section by recalling some well known inequalities in the following technical lemma. These have been applied in various places in the preceeding sections. For the proof we refer the interested reader to the paper [28].

Lemma 6.4. *Let $1 < p < \infty$. There exists $c(p) > 0$ such that*

$$(i) \quad |\xi|^p - |\eta|^p - p\,|\eta|^{p-2}\eta \cdot (\xi - \eta) \geq c(p) \begin{cases} |\xi - \eta|^2 (|\xi| + |\eta|)^{p-2} & \text{if } 1 < p < 2 \\ |\xi - \eta|^p & \text{if } p \geq 2; \end{cases}$$

$$(ii) \quad \Big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\Big) \cdot (\xi - \eta) \geq c(p) \begin{cases} |\xi - \eta|^2 (|\xi| + |\eta|)^{p-2} & \text{if } 1 < p < 2 \\ |\xi - \eta|^p & \text{if } p \geq 2. \end{cases}$$

The second inequality of the above implies, in particular, that there exists $c = c(p) > 0$ such that for every $u, v \in W^{1,p}(\mathbb{R}^n)$ and a measurable set $E \subset \mathbb{R}^n$, if $p \geq 2$ then we have

$$\int_E |\nabla u - \nabla v|^p \, dx \leq c \int_E (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot (\nabla u - \nabla v) \, dx, \tag{61}$$

while if $1 < p < 2$, we have

$$\begin{aligned} &\int_E |\nabla u - \nabla v|^p \, dx \\ &\quad \leq c\Big(\int_E (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot (\nabla u - \nabla v) \, dx \Big)^{\frac{p}{2}} \Big(\int_E (|\nabla u| + |\nabla v|)^p \, dx \Big)^{1-\frac{p}{2}}. \end{aligned} \tag{62}$$

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