FREE GRADIENT DISCONTINUITIES

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ABSTRACT

Various problems in continuum mechanics or artificial intelligence can be modelled by integral functionals depending on second order derivatives of competing functions and on closed sets where discontinuities for either such functions or their first derivatives are allowed. The existence of minimizers for such functionals is shown together with some information about their regularity.

1. Introduction

Integral functionals depending on free discontinuities and first order derivatives have been studied recently by many authors, either in connection with continuum mechanics and visual reconstruction [15,21], or from the more abstract viewpoint of calculus of variations, geometric measure theory [1,10,11,12,13] and optimal partitions [7,8,11]. The interest of these functionals lies in the fact that any network that makes decisions (i.e. that can develope or modify discontinuities) cannot be described by an energy that is at the same time smooth and convex [2].

In the same perspective we want to discuss some minimization problems related to functionals depending on free gradient discontinuities and possibly on second order derivatives; the interest of this kind of functionals is originated by some models of image segmentation [2] or in the study of deformations for elastic plastic plates [5], rigid-plastic concrete slabs [23], or plastic yield lines in limit analysis of masonry structures [18,24], and finally in the analysis of piecewise affine approximation.

The introduction of spaces of Special Bounded Variation [1,12] or Special Bounded Hessian [4,25] provided the correct functional framework for the analysis of this kind of problems.

More precisely here we discuss the minimization of functionals of the following kind

$$\begin{aligned} F(K_0,K_1,u) \ &= \\ \int_{\Omega\setminus(K_0\cup K_1)} Q(D^2u)dy &+ \int_{\Omega\setminus(K_0\cup K_1)} R(Du)dy + \int_{K_0\cap\Omega} \varphi(|u^+-u^-|)d\mathcal{H}^1 \\ &+ \int_{K_1\cap\Omega} \psi(|Du^+-Du^-|)d\mathcal{H}^1 + \int_\Omega f(y,u)dy \end{aligned} \tag{1}$$

where $\Omega \subset \mathbf{R}^2$ is an open set, φ, ψ, f are given functions, Q, R are positive definite quadratic forms with constant coefficients, $\mathcal{H}^1$ denotes the $1-$dimensional Hausdorff measure. The functional F will be studied either over the class of functions with discontinuities and creases

$$A_d = \left\{(K_0, K_1, u); K_0, K_1 \subset \mathbf{R}^2 \text{ closed sets}, u \in C^0(\Omega \setminus K_0) \cap C^2(\Omega \setminus (K_0 \cup K_1))\right\}$$

or over the class of functions with creases

$$A_c = \left\{(K, u); K \subset \mathbf{R}^2 \text{ closed set}, u \in C^0(\overline{\Omega}) \cap C^2(\Omega \setminus K)\right\}.$$

We get the existence of minimizers by mean of relaxed energy functionals and regularity properties via techniques similar to [5,6], under compatibility or growth assumptions on f, provided F is coercive with respect to the total variation of D^2u in Ω. We prove also some additional informations about partial regularity of an essential minimizer in A_c (see Corollary 3.5), by showing that its $1-$dimensional upper density is positive, hence isolated gradient singularities are excluded.

Finally the existence of minimizers for the weak formulation of the segmentation model by Blake & Zisserman [2] is given in Theorem 5.2.

2. Elastic perfectly plastic plate

The first example of functional depending on second derivatives and free gradient discontinuities that we consider is the functional (6), where the first three terms represent a model formulation of the stored deformation energy for an elastic perfectly plastic thin plate made of mild iron and the last term represents a potential energy.

Assume

$$\begin{aligned}&\Omega \subset \mathbf{R}^2 \text{ open bounded strongly Lipschitz:}\\ &\partial\Omega \text{ is the union of finitely many } C^2 \text{ curves,}\end{aligned} \tag{2}$$

$$Q : M^2 \to [0, +\infty) \ : \ \exists a > 0, \ \ \sum Q_{ijlm}\eta_{ij}\eta_{lm} \geq a|\eta|^2 \ \ \forall \eta \in M^2, \tag{3}$$

where M^2 denotes the symmetric 2×2 matrices, and, either

$$\begin{aligned}&f(y, s) = g(y)s, \ \ \forall s \in \mathbf{R}, \ a.e. \ y \in \Omega, \ \ \text{with } g \in L^q(\Omega), \ q > 2,\\ &\int_\Omega gv\, dy = 0 \qquad \text{for every affine displacement } v\\ &\|g\|_{L^1(\Omega)} < C(\Omega)(a \wedge \mu)\end{aligned} \tag{4}$$

where $C(\Omega)$ is defined in Theorems 2.7,2.12,3.1 of [5], or

$$\begin{aligned}&0 \leq C_1|s|^q \leq f(y, s) \leq C_2(1 + |s|^q), \ 0 < C_1 \leq C_2,\\ &|f(y, s) - f(y, t)| \leq (g(y) + |s| + |t|)\,|s - t|, \ \ g \in L^q(\Omega), \ q > 2,\\ &\qquad \forall s, t \in \mathbf{R}, \ a.e. \ y \in \Omega,\end{aligned} \tag{5}$$

Here we state the existence of a solution for the following problem: find a minimizing pair for the functional F_1 defined by

$$F_1(K,u) = \int_{\Omega \setminus K} Q(D^2u)\,dy + \lambda \mathcal{H}^1(K \cap \Omega) + \mu \int_{K \cap \Omega} |[Du]|\,d\mathcal{H}^1 + \int_{\Omega} f(y,u)\,dy \quad (6)$$

over $(K,u) \in A_c$, where $|[Du]|$ denotes the jump of Du across K.

The free gradient discontinuity set K describes the contribution of internal variables on the "a priori" unknown set where the plastic deformation occurs [5]. We notice that 1-dimensional plasticity set occurs when testing thin metallic plates, contrarily to the case of 3-dimensional deformable bodies where the plasticity set is allowed to have Hausdorff dimension bigger than two [17].

Theorem 2.1 - *Assume (2), (3) and either (4) or (5). Then there exists a minimizing pair (Z,z) for the functional F_1 defined by (6) for every $(K,u) \in A_c$. Moreover Z is $(\mathcal{H}^1, 1)$ rectifiable.*

The study of functional F_1 contains the case considered in [6], the difference is due to the quadratic form Q and possibly to the last term when assumption (5) holds. To find minimizers of F_1, following the argument of [4], we introduce the space of functions with Special Bounded Hessian ($SBH(\Omega)$). We recall first some definitions from [12].

Let $v : \Omega \to \mathbf{R}^k$ be a Borel function; for $x \in \Omega$ and $z \in \widetilde{\mathbf{R}}^k = \mathbf{R}^k \cup \{\infty\}$ (the one point compactification of $\mathbf{R}^k$) we say that z is the approximate limit of v at x, and we write $z = \operatorname{ap}\lim_{y \to x} v(y)$, if

$$g(z) = \lim_{\rho \to 0} \frac{\int_{B_\rho(x)} g(v(y))dy}{|B_\rho|}$$

for every $g \in C^0(\widetilde{\mathbf{R}}^k)$ (by $|E|$ we denote the Lebesgue measure of the set E). The set

$$S_v = \{x \in \Omega;\ \operatorname{ap}\lim_{y \to x} v(y) \quad \text{does not exist }\}$$

is a Borel set, of negligible Lebesgue measure; for brevity's sake we denote by $\tilde{v} : \Omega \setminus S_v \to \widetilde{\mathbf{R}}^k$ the function

$$\tilde{v}(x) = \operatorname{ap}\lim_{y \to x} v(y).$$

Let $x \in \Omega \setminus S_v$ be such that $\tilde{v}(x) \in \mathbf{R}^k$; we say that v is approximately differentiable at x if there exists a $k \times 2$ matrix $\nabla v(x)$ such that

$$\operatorname{ap}\lim_{y \to x} \frac{|v(y) - \tilde{v}(x) - \nabla v(x)(y-x)|}{|y-x|} = 0.$$

If v is a smooth function then ∇v is the jacobian matrix.

We recall the definition of the space of functions of bounded variation in Ω with values in $\mathbf{R}^k$:

$$BV(\Omega, \mathbf{R}^k) = \{v \in L^1(\Omega, \mathbf{R}^k);\ Dv \text{ bounded variation vector measure }\}$$

where Dv denotes the distributional derivatives of v.

In recent papers [11,12,4,25], for studying free discontinuity problems, some classes of functions with special bounded variation have been considered.

Definition 2.2 - *$SBV(\Omega, \mathbf{R}^k)$ denotes the class of all functions $v \in BV(\Omega, \mathbf{R}^k)$ such that*

$$|Dv|_T = \int_\Omega |\nabla v|\, dy + \int_{S_v} |v^+ - v^-|\, d\mathcal{H}^1,$$

where $|\cdot|_T$ denotes the total variation of a measure.

Definition 2.3 - *$SBH(\Omega)$ denotes the class of all functions $v \in L^1(\Omega)$ such that $Dv \in SBV(\Omega, \mathbf{R}^2)$.*

At the end of the present section we have collected some simple examples and remarks about functions either in $SBV(\Omega)$ or in $SBH(\Omega)$.

On $SBH(\Omega)$ we have the following weak formulation of (6):

$$\begin{aligned}\mathcal{F}_1(v) &= \int_\Omega Q(\nabla^2 v)\, dy + \lambda \mathcal{H}^1(S_{Dv}) + \mu \int_{S_{Dv}} |[Dv]|\, d\mathcal{H}^1 + \int_\Omega f(y,v)\, dy \\ &:= \mathcal{E}(v, \lambda, \mu, \Omega) + \int_\Omega f(y,v)\, dy, \qquad (7)\end{aligned}$$

where $\nabla^2 v$ is the absolutely continuous part of the distributional hessian of v with respect to the Lebesgue measure and S_{Dv} is the discontinuity set of Dv. Under the natural conditions (4) on f, we proved in [5] the existence of minimizers of $\mathcal{F}_1$ submitted to various boundary or unilateral conditions (see [5] Theorems 3.1,4.1,5.3). By the same arguments we can prove the following statement.

Theorem 2.4 - *Assume (2), (3) and either (4) or (5). Then there is $v_0 \in SBH(\Omega)$ with*

$$\mathcal{F}_1(v_0) = \min\{\mathcal{F}_1(v) :\ v \in SBH(\Omega)\}.$$

Moreover we have

$$\mathcal{F}_1(v_0) \le \inf\{F_1(K,u) :\ (K,u) \in A_c\ \}.$$

Proof of Theorem 2.1 - In case (4) with $Q(\eta) = |\eta|^2$ the statement has been proved in [6]. For the sake of completeness, here we describe the argument and sketch the main modifications for the general case.

The energy (6) is meaningful for admissible pairs, see [6] Lemmas 2.6,2.8. The existence of a minimizing pair (Z, z) for the functional (6) is obtained by arguing on a minimizer of (7). Namely, let v_0 be a minimizer for the functional (7) on $SBH(\Omega)$ and set

$$\Omega_0 = \left\{x \in \Omega;\ \lim_{\rho \to 0} \rho^{-1} \mathcal{E}(v_0, \lambda, \mu, \overline{B}_\rho(x)) = 0\right\}$$

where $\mathcal{E}$ is defined in (7) and $B_\rho(x) = \{y \in \mathbf{R}^2; |y - x| < \rho\}$. The main point is showing that Ω_0 is an open set and $S_{Dv_0} \cap \Omega_0 = \emptyset$; then we choose $Z = \overline{S_{Dv_0}}$, $z = v_0$ and we prove that $z \in C^0(\overline{\Omega}) \cap C^2(\Omega \setminus Z)$, $F_1(Z, z) = \mathcal{F}_1(v_0)$ so that, by Theorem 2.4, the pair (Z, z) is a minimizing pair for F_1.

The set Ω_0 is open since we have that $\rho^{-1}\mathcal{E}(v_0, \lambda, \mu, \overline{B}_\rho(x))$ decays like a positive power of ρ as $\rho \to 0$ near $x \in \Omega_0$.

The proof is obtained arguing by contradiction and involves the study of a sequence (u_h) of local minimizers of $\mathcal{F}_1$ for which $\rho^{-1}\mathcal{E}(u_h, \lambda, \mu, \overline{B}_\rho(x))$ do not decay fast enough and the consideration of their rescaled versions v_h on a unit ball. The presence in $\mathcal{E}$ of three terms with different kind of homogeneity under rescaling, causes a loss of compactness in the L^2 space both for the functions v_h and for their gradients. This is amended by the use of a Poincaré–Wirtinger inequality in *SBV* (Theorem 3.1 of [6]): we obtain that a subsequence of the blown–up sequence (v_h) converges a.e. to a function w with convergence a.e. of the gradients to Dw. Next we prove that w is solution of a strongly elliptic p.d.e. of fourth order with constant coefficients (see (8) below), hence $\rho^{-1}\mathcal{E}(w, \lambda, \mu, \overline{B}_\rho(x))$ decays like ρ as $\rho \to 0$, and we transfer the decay estimates back on u_h, to obtain a contradiction. This is obtained by comparison of v_h with a solution of a suitable Dirichlet problem for the elliptic operator associated to Q. Here is used a trace theorem for $SBH(\Omega)$ (see the Polar Slicing Theorem 4.4 of [6]).
The main differences with respect to the argument of [6] are the following two lemmas.

Lemma 2.5 - *Let $B_r \subset \mathbf{R}^m$ and let $b \in H^{\frac{3}{2}}(\partial B_r)$, $l \in H^{\frac{1}{2}}(\partial B_r)$ and let $\mathcal{L}$ be the fourth order differential operator associated to the quadratic form Q : $\mathcal{L} = \sum_{ijhk} D_{ij} Q_{ijhk} D_{hk}$.*

Then there is a unique $v \in H^2(B_r)$ solution of the Dirichlet problem

$$\begin{cases} \mathcal{L}v = 0 & \text{in } B_r, \\ v = b & \text{on } \partial B_r, \\ \dfrac{\partial v}{\partial \nu} = l & \text{on } \partial B_r; \end{cases} \tag{8}$$

moreover the following inequality holds:

$$\int_{B_\rho} |D^2 v|^2 dy \le c_m \left(\frac{\rho}{r}\right)^m \int_{B_r} |D^2 v|^2 dy \qquad \forall \rho < r,$$

where c_m is an absolute constant depending only on the dimension and the coefficients of Q. The Green operator $E_r : H^{\frac{3}{2}}(\partial B_r) \times H^{\frac{1}{2}}(\partial B_r) \to H^2(B_r)$ defined by $E_r(b,l) = v$ is continuous[16,20].

We set the following definitions. For $\lambda, \mu > 0$, for every relatively closed set $C \subseteq \Omega$ and $v \in SBH(\Omega)$ we define

$$\mathcal{E}(v,\lambda,\mu,C) = \int_C Q(\nabla^2 v) dy + \lambda \mathcal{H}^1(S_{Dv} \cap C) + \mu \int_{S_{Dv}\cap C} |[Dv]| d\mathcal{H}^1,$$
$$\Psi(v,\lambda,\mu,C) = \mathcal{E}(v,\lambda,\mu,C) - \inf\{\mathcal{E}(u,\lambda,\mu,C);\ u \in SBH(\Omega),\ u = v \text{ in } \Omega \setminus C\}.$$

The functional Ψ is called the *excess* of v in C.

Lemma 2.6 - (Excess estimate) *Under the assumptions of Theorem 2.4, fix a minimizer v_0 of $\mathcal{F}_1$ in $SBH(\Omega)$. There exists a constant $c = c(v_0,\lambda,\mu,g,\Omega)$ such that, for every $\overline{B}_\rho(x) \subset \Omega$ the following estimate holds (uniformly in $x \in \Omega$)*

$$\Psi(v_0,\lambda,\mu,\overline{B}_\rho(x)) \le c\rho^{2-\frac{2}{q}}.$$

Proof - Assume (4) and $x = 0$, $\overline{B}_\rho = \overline{B}_\rho(0) \subset \Omega$ and let $v \in SBH(\Omega)$ such that $v = v_0$ in $\Omega \setminus \overline{B}_\rho$. Taking into account Theorem 2.6 of [5] we have

$$\begin{aligned}
\mathcal{E}(v_0,\lambda,\mu,\overline{B}_\rho) &= \mathcal{F}_1(v_0,\overline{B}_\rho) + \int_{B_\rho} g v_0\, dy \\
&\le \mathcal{F}_1(v,\overline{B}_\rho) + \int_{B_\rho} g v_0\, dy = \mathcal{E}(v,\lambda,\mu,\overline{B}_\rho) - \int_{B_\rho} g(v - v_0)\, dy \\
&\le \mathcal{E}(v,\lambda,\mu,\overline{B}_\rho) + \|g\|_{L^q}\|v - v_0\|_{L^\infty}|B_\rho|^{1-\frac{1}{q}} \\
&\le \mathcal{E}(v,\lambda,\mu,\overline{B}_\rho) + c'|D^2(v - v_0)|_T \rho^{2-\frac{2}{q}} \\
&\le \mathcal{E}(v,\lambda,\mu,\overline{B}_\rho) + c'(|D^2 v|_T + |D^2 v_0|_T)\rho^{2-\frac{2}{q}},
\end{aligned}$$

where $\mathcal{F}_1(\cdot,\overline{B}_\rho)$ is the functional $\mathcal{F}_1(\cdot)$ localized on $\overline{B}_\rho$. By choosing v such that $\Psi(v,\lambda,\mu,\overline{B}_\rho) = 0$, by the estimate in [6] Remark 4.2 and by the minimality of v we achieve the thesis.

Assuming (5), the argument is the same, taking into account the estimate

$$\int_{B_\rho} |f(y,v)-f(y,v_0)|\,dy \le \left(\|g\|_{L^q(\Omega)} + \|v\|_{L^q(\Omega)} + \|v_0\|_{L^q(\Omega)}\right) \|v-v_0\|_{L^\infty} |B_\rho|^{1-\frac{1}{q}}. \blacksquare$$

Remark 2.7 - We notice that there are functions $v \in BV(\Omega)$ such that the singular set S_v does not coincide with $\mathrm{spt}(Dv)^s$ (the support of the singular part of Dv); moreover there are $w \in BH(\Omega)$ such that $S_{Dw} \neq \mathrm{spt}(D^2w)^s$. Actually either of the inclusions may fail, as shown by the examples below, where $\Omega = (-1,1)^2 \subset \mathbf{R}^2$. When derivatives are "special measures" we have additional informations:

$$\mathrm{spt}(Dv)^s \subset \overline{S_v} \qquad \forall v \in SBV(\Omega),$$

$$\mathrm{spt}(D^2w)^s \subset \overline{S_{Dw}} \qquad \forall w \in SBH(\Omega),$$

but the inclusions can still be strict.

Ex.1 : $v(x_1,x_2) = c(x_1)$ where c is the odd extension of the Cantor function. We have $v \in BV(\Omega)$ but $v \notin SBV(\Omega)$. Since v is continuous, then $S_v = \emptyset$, $\mathrm{spt}(Dv) = \mathrm{spt}(Dv)^s$.

Ex.2 : $w(x_1,x_2) = \int_0^{x_1} v(t,x_2)dt$ is a function with bounded hessian, say $w \in BH(\Omega)$, $S_{Dw} = \emptyset$, $\mathrm{spt}(D^2w)^s \neq \emptyset$.

Ex.3 : $z(x) = x/|x|$ is a function in $SBV(\Omega,\mathbf{R}^2)$, $S_z = \{0\}$, $\mathrm{spt}(Dz)^s = \emptyset$ since $Dz \in L^1$.

Ex.4 : $\eta(x) = |x|$ is a function in $SBH(\Omega)$, $D\eta = z$; we notice that since $\nabla^2\eta = D^2\eta \notin L^2(\Omega)$ then η does not belong to $\mathrm{dom}\,\mathcal{F}_1$.

Ex.5 : $\varphi(x_1,x_2) = \sqrt{x_1^4 + x_2^2}$ is a function in $SBH(\Omega)$, $S_{D\varphi} = \{0\}$, $\mathrm{spt}(D^2\varphi)^s = \emptyset$.

Remark 2.8 - The notion of singular set is unstable with respect to the trace operators as shown by the following examples. More precisely (see [1,6]) there exist bounded linear maps

$$\gamma^t : BV(\Omega) \to L^1(-1,1)$$
$$\gamma_1^t : BH(\Omega) \to L^1((-1,1),\mathbf{R}^2)$$

such that

$$\gamma^t(v)(s) = v(t,s) \qquad \forall v \in C^1(\overline{\Omega})$$

$$\gamma_1^t(v)(s) = \nabla v(t,s) \qquad \forall v \in C^2(\overline{\Omega}),$$

moreover, by the slicing theorems in [1,6], for a.e. t we have only $\widetilde{v}(t,s) = \widetilde{\gamma^t(v)}(s)$, $\widetilde{\left(\frac{\partial v}{\partial s}\right)}(t,s) = \widetilde{\gamma_1^t(v)}(s) \cdot e_2$ for a.e. s, where $\widetilde{\ }$ denotes the approximate limit.

By Theorem 3.1 in [1] for every $v \in BV(\Omega)$ we have

$$\gamma^t(v) \in BV(-1,1), \quad S_{\gamma^t(v)} = S_v \cap \{x_1 = t\} \ \text{ for a.e. } t \in (-1,1).$$

The property given above for a.e. t is sharp, since the two singular sets may be different for some values of t.

Ex.6 : Let H be the Heaviside function and set $\psi(x_1,x_2) = H(x_1 - x_2^2)$. Then $S_{\gamma^0(\psi)} = \emptyset$ but $S_\psi \cap \{x_1 = 0\} = \{0\}$.

Ex.7 : Set $\xi(x_1,x_2) = H(x_2 - \sqrt{|x_1|})$. Then $S_{\gamma(\xi)} = \{0\}$ but $S_\xi \cap \{x_1 = 0\} = \emptyset$. Notice that $S_\xi \subsetneq \mathrm{spt}(D\xi)^s$.

3. Additional information on regularity

In this section we deduce some further information about the regularity of the singular set Z of an essential minimizing pair for the functional (6) which follows by the arguments in [6]. For brevity's sake in this section we consider only the case $Q(\eta) = |\eta|^2$. The main result is contained in Theorem 3.4.

We set the following definitions. Let B be an open disk in $\mathbf{R}^2$; for every measurable function $v : B \to \mathbf{R}$ we define the least median of v in B as

$$\mathrm{med}(v, B) = \inf\{\ t \in \mathbf{R};\ |\{u < t\} \cap B| \geq \frac{1}{2}|B|\}.$$

In case v is vector-valued, the med operator acts componentwise. If $B = B_r(x)$ and $u \in \mathit{SBH}(B)$, we set $(M_x u)(y) = \mathrm{med}(Du, B) \cdot (y - x)$ and we define the affine function

$$(\mathcal{P}u)(y) = (M_x u)(y) + \mathrm{med}(u - M_x u, B) \qquad \forall y \in \mathbf{R}^2.$$

Lemma 3.1 - (Blow-up equation) *Let $\overline{B}_r(x) \subset \Omega \subset \mathbf{R}^2$, $(v_h) \subset \mathit{SBH}(\Omega)$, (λ_h) and (μ_h) two sequences of positive numbers and let $w \in H^2(B_r(x))$. Assume*

(i) $\lim_h \mathcal{H}^1(S_{Dv_h}) = 0,$

(ii) $\lim_h \mathcal{E}(v_h, \lambda_h, \mu_h, \overline{B}_\rho(x)) = \alpha(\rho) < +\infty$ *for almost all* $\rho < r$,

(iii) $\lim_h \Psi(v_h, \lambda_h, \mu_h, \overline{B}_\rho(x)) = 0$ *for every* $\rho < r$,

(iv) $\lim_h \big(v_h - \mathcal{P}v_h\big) = w$ a.e. *on* $B_r(x)$.

Then w is biharmonic in $B_r(x)$ and

$$\alpha(\rho) = \int_{B_\rho(x)} |D^2 w|^2 dy \qquad \textit{for almost all } \rho < r.$$

Proof - Let $c_h = \lambda_h \vee \big(\mathcal{H}^1(S_{Dv_h}) + \frac{1}{h}\big)^{-\frac{1}{2}}$. Then $\lim_h \mathcal{E}(v_h, c_h, \mu_h, \overline{B}_\rho(x)) = \alpha(\rho) < +\infty$ for almost all $\rho < r$, and $\lim_h \Psi(v_h, c_h, \mu_h, \overline{B}_\rho(x)) = 0$ for every $\rho < r$. By [6] Theorem 4.7 the thesis follows. ∎

Lemma 3.2 - (Decay) *For every $\alpha, \beta \in]0,1[$, such that $\alpha^\beta < c_2^{-1}$ (c_2 is given by Lemma 2.5), and for every $\lambda, \mu > 0$, there exist two positive constants ϵ and ϑ, depending only on λ, μ, α and β, such that if $\Omega \subset \mathbf{R}^2$ is open, $\rho > 0$, $\overline{B}_\rho(x) \subset \Omega$ and if $u \in SBH(\Omega)$ with*

$$\mathcal{H}^1(S_{Du} \cap \overline{B}_\rho(x)) \le \epsilon\rho$$

and

$$\Psi(u, \lambda, \mu, \overline{B}_\rho(x)) \le \vartheta \mathcal{E}(u, \lambda, \mu, \overline{B}_\rho(x)),$$

then

$$\mathcal{E}(u, \lambda, \mu, \overline{B}_{\alpha\rho}(x)) \le \alpha^{2-\beta} \mathcal{E}(u, \lambda, \mu, \overline{B}_\rho(x)).$$

Proof - Suppose the lemma is not true. Then there exist $\alpha \in]0,1[$, $\beta \in]0,1[$, such that $\alpha^\beta < c_2^{-1}$, $\lambda, \mu > 0$, two sequences (ϵ_h), (ϑ_h) such that $\lim_h \epsilon_h = \lim_h \vartheta_h = 0$, a sequence (u_h) in $SBH(\Omega)$ and a sequence of disks $\overline{B}_{\rho_h}(x_h) \subset \Omega$, such that

$$\mathcal{H}^1(S_{Du_h} \cap B_{\rho_h}(x_h)) = \epsilon_h \rho_h$$

$$\Psi(u_h, \lambda, \mu, \overline{B}_{\rho_h}(x_h)) \le \vartheta_h \mathcal{E}(u_h, \lambda, \mu, \overline{B}_{\rho_h}(x_h))$$

and

$$\mathcal{E}(u_h, \lambda, \mu, \overline{B}_{\alpha\rho_h}(x_h)) > \alpha^{2-\beta} \mathcal{E}(u_h, \lambda, \mu, \overline{B}_{\rho_h}(x_h)).$$

For each h, setting $\sigma_h = \rho_h^{-1} \mathcal{E}(u_h, \lambda, \mu, B_{\rho_h}(x_h))$, we define

$$v_h(x) = \frac{1}{\sqrt{\sigma_h \rho_h^3}} u_h(x_h + \rho_h x) \qquad x \in B_1,$$

and setting $\lambda_h = \frac{\lambda}{\sigma_h}$ and $\mu_h = \mu\sqrt{\frac{\rho_h}{\sigma_h}}$, we have

$$\mathcal{E}(v_h, \lambda_h, \mu_h, \overline{B}_1) = 1,$$

$$\Psi(v_h, \lambda_h, \mu_h, \overline{B}_1) \le \vartheta_h$$

and

$$\mathcal{E}(v_h, \lambda_h, \mu_h, \overline{B}_\alpha) > \alpha^{2-\beta}.$$

By Theorem 3.2 in [6], there exist a subsequence of (v_h), still denoted by (v_h), and a function $w \in H^2(B_1)$ such that (assuming, without loss of generality, $\mathcal{P}v_h = 0$) $\lim_h v_h = w$ a.e. in B_1. By Lemma 3.1, we argue that the function w is biharmonic in B_1 and that

$$\limsup_h \mathcal{E}(v_h, \lambda_h, \mu_h, \overline{B}_\alpha) = \int_{B_\alpha} |D^2 w|^2 \, dy \ .$$

By Lemmas 3.1, 2.5 and by the assumption on α, β, we have

$$\int_{B_\alpha} |D^2 w|^2 dy \le c_2\alpha^2 \int_{B_1} |D^2 w|^2 dy = c_2\alpha^2 < \alpha^{2-\beta},$$

whereas by the assumptions on (u_h) we have

$$\int_{B_\alpha} |D^2 w|^2 dy \ge \alpha^{2-\beta}. \qquad \blacksquare$$

Lemma 3.3 - *Let $\alpha, \beta, \epsilon, \vartheta$ be as in Lemma 3.2. If v_0 is a minimizer of the functional $\mathcal{F}_1$ in SBH(Ω) under the assumptions (2)-(4) (or (2), (3), (5)), then for every $x \in \overline{S}_{Dv_0} \cap \Omega$*

$$\mathcal{E}(v_0, \lambda, \mu, \overline{B}_\rho(x)) > \epsilon\rho$$

for every $\rho < \operatorname{dist}(x, \partial\Omega) \wedge \left(\vartheta\epsilon\alpha c^{-1}\right)^{\frac{q}{q-2}}$, where c is given by Lemma 2.6.

Proof - Assume that $\mathcal{E}(v_0, \lambda, \mu, \overline{B}_\rho(x)) \le \epsilon\rho$ for some $\rho < \operatorname{dist}(x, \partial\Omega) \wedge \left(\vartheta\epsilon\alpha c^{-1}\right)^{\frac{q}{q-2}}$. Then, by Lemma 2.6, we have

$$\Psi(v_0, \lambda, \mu, \overline{B}_\rho(x)) \le c\rho^{2-\frac{2}{q}} \le \epsilon\vartheta\alpha\rho,$$

hence, by Lemma 4.12 in [6], we get $x \in \overline{S}_{Dv_0} \cap \Omega_0$ and this contradicts the property $\overline{S}_{Dv_0} \cap \Omega_0 = \emptyset$ (see [6], Lemma 4.11 and Theorem 4.13). $\blacksquare$

Theorem 3.4 - *If v_0 is a minimizer of the functional*

$$\mathcal{F}_1(v) = \int_\Omega |\nabla^2 v|^2 \, dy + \lambda\mathcal{H}^1(S_{Dv}) + \mu \int_{S_{Dv}} |[Dv]| \, d\mathcal{H}^1 + \int_\Omega f(y, v) \, dy,$$

in SBH(Ω), then for every $x \in \overline{S}_{Dv_0}$

$$\limsup_{\rho \to 0} \frac{\mathcal{H}^1(S_{Dv_0} \cap \overline{B}_\rho(x))}{\rho} \ge \epsilon.$$

Proof - Let $\alpha, \beta, \epsilon, \vartheta$ be as in Lemma 3.2. Assume by contradiction that there is $x \in \overline{S}_{Dv_0}$ and $r > 0$ such that $\rho^{-1}\mathcal{H}^1(S_{Dv_0} \cap \overline{B}_\rho(x)) < \epsilon$ for every $\rho \le r$ and also that $cr^{1-\frac{2}{q}} \le \vartheta\epsilon$, where c is given by Lemma 2.5. By Lemma 3.2, we obtain for every $h \in \mathbf{N}$

$$\mathcal{E}(u, \lambda, \mu, \overline{B}_{\alpha^h r}(x)) \le \alpha^{(2-\beta)h}\mathcal{E}(u, \lambda, \mu, \overline{B}_r(x)),$$

and, for h large enough, this contradicts Lemma 3.3. ■

Corollary 3.5 - *If $(Z,z) \in A_c$ is a minimizing pair for the functional F_1, then there exists a minimizing pair $(Z',z') \in A_c$ such that $Z' \subset Z$ and for every $x \in Z'$ we have*

$$\limsup_{\rho \to 0} \frac{\mathcal{H}^1(Z' \cap \overline{B}_\rho(x))}{\rho} \geq \epsilon_1.$$

Proof - If $(Z,z) \in A_c$ is a minimizing pair for the functional F_1, then z is a minimizer of the functional $\mathcal{F}_1$ in $SBH(\Omega)$. Setting $Z' = \overline{S_{Dz}}$ and $z' = z$, by Theorem 3.4 the thesis follows immediately. ■

A pair (Z',z') like the one defined in the previous proof is called *essential minimizing* pair. A consequence of Corollary 3.5 is the following statement.

Corollary 3.6 - *Any essential minimizing pair of F_1 in A_c cannot have isolated gradient discontinuities.*

4. Rigid perfectly plastic slab

Concrete slabs undergoing small deformations under transverse loads, are sometimes described as thin plates. Actually the elastic deformation turns out to be irrelevant if compared to the plastic flow occurring along "a priori" unknown plastic yield lines, so that it is natural to couple rigid deformations with plastic hinges along an unknown pattern of lines; on these lines the deformation is still continuous but the gradients may undergo jump discontinuities of rank 1.

Actually finding the yield lines patterns associated to prescribed clamped edges or supporting columns by energy minimization, does not correspond to finding quasi-static equilibria [23,24], but can provide a way of computing safety criteria for admissible transversal loads by limit analysis.

For the sake of simplicity here we assume isotropy, which actually fails in "real life" slabs due to the presence of steel reinforcements.

We confine our attention to the following "rigid-plastic" energy

$$F_2(K,u) = \lambda \mathcal{H}^1(K \cap \Omega) + \mu \int_{K\cap\Omega} |[Du]|\, d\mathcal{H}^1 + \int_\Omega gu\, dy \tag{9}$$

over the class of piecewise affine functions with creases

$$\begin{aligned} A = \{(K,u);\ & K \subset \mathbf{R}^2 \text{ closed set, } u \in C^0(\overline{\Omega}), \\ & u \text{ affine on the connected components of } \Omega \setminus K\}\,. \end{aligned}$$

Theorem 4.1 - *Assume (2) and $g \in L^q(\Omega)$ with $q > 2$, with*

$$\int_\Omega gv\, dy = 0 \qquad \text{for every affine displacement } v,$$

$$\|g\|_{L^1(\Omega)} < C(\Omega)\mu,$$

where $C(\Omega)$ is defined in[5] *Theorems 2.7, 2.12, 3.1. Then there is a minimizing pair of (9) over A.*

The proof of Theorem 4.1 can be obtained suitably adapting the proof of Theorem 2.1. We prove here the main steps. The weak formulation of (9) is

$$\begin{aligned}\mathcal{F}_2(v) =& \lambda\mathcal{H}^1(S_{Dv}) + \mu \int_{S_{Dv}} |[Dv]|\, d\mathcal{H}^1 + \int_\Omega gv\, dy \\ :=& \,\mathcal{U}(v, \lambda, \mu, \Omega) + \int_\Omega gv\, dy, \end{aligned} \tag{10}$$

over the set of almost everywhere affine functions

$$\mathcal{A} = \left\{ v \in SBH(\Omega) : \ \nabla^2 v \equiv 0 \text{ in } \Omega \right\}$$

We notice that $Dv = \nabla v$ in $SBH(\Omega)$, hence $\mathcal{F}_2$ is coercive on $\mathcal{A}$ since it provides a bound on $|D^2 v|_T$.

Theorem 4.2 - *If $\{v_h\}_{h\in\mathbf{N}}$ is a minimizing sequence for $\mathcal{F}_2$, there exists $v \in SBH(\Omega)$ such that, possibly extracting a subsequence,*

$$v_h \to v \ \ \text{strongly } W^{1,1}(\Omega),$$

$$\nabla^2 v_h \rightharpoonup \nabla^2 v \ \ \text{weakly in } L^2(\Omega),$$

$$D^2 v_h - \nabla^2 v_h dy = Dv_h \otimes \nu\, d\mathcal{H}^1 \llcorner S_{Dv_h} \ \ \text{converges weakly}^* \text{ in the sense of}$$

$$\text{matrix valued measures to} \quad D^2 v - \nabla^2 v dy = Dv \otimes \nu\, d\mathcal{H}^1 \llcorner S_{Dv} \ ,$$

$$\nabla^2 v = 0 \qquad \text{a.e. in } \Omega \text{ and } v \in \mathcal{A}.$$

Proof - By Theorem 2.17 of[5] we get the semicontinuity of $\mathcal{F}_2$ in $\mathcal{A}$ with respect to the $W^{1,1}$ strong convergence. Hence $\nabla^2 v = 0$ a.e. in Ω and $v \in \mathcal{A}$. ∎

Remark 4.3 - Existence theorems analogous to 4.2 may be proved for concentrated loads, clamped plate or obstacle constraints, (see Remark 3.2 and Theorems 3.4, 3.5, 4.1, 5.3 of[5]).

A useful consequence of Poincaré-Wirtinger inequality in [6] is the following statement.

Theorem 4.4 - *(Compactness) Let $B \subset \mathbf{R}^2$ be an open ball centered at $x \in \mathbf{R}^2$, $(u_h) \subset SBH(B)$, and let*

$$\nabla^2 u_h = 0 \quad \text{a.e. in } \Omega,$$
$$\lim_h \mathcal{H}^1(S_{Du_h}) = 0.$$

Then a subsequence (u_{h_i}) and an affine function w exist such that

$$\lim_i \left(u_{h_i} - \mathcal{P}u_{h_i}\right) = w, \qquad \lim_i \left(Du_{h_i} - \text{med}(Du_{h_i}, B)\right) = Dw \qquad \text{a.e. on } B.$$

Proof - As like as in the first part of the proof of Th 3.2 of [6] one gets the existence of $w \in H^2(B)$ and the convergence properties. The semicontinuity inequality for the functional $\mathcal{F}_2$ gives

$$\int_B |D^2 w|^2 dy \leq \liminf_i \int_B |\nabla^2 u_{h_i}|^2 dy$$

hence $w \in \mathcal{A}$ and w is affine. ■

Theorem 4.5 - *(Decay) For every $\alpha \in]0,1[$, and for every $\lambda, \mu > 0$, there exist two positive constants ϵ and ϑ, depending only on λ, μ, and α, such that if $\Omega \subset \mathbf{R}^2$ is open, $\rho > 0$, $\overline{B}_\rho(x) \subset \Omega$ and if $u \in \mathcal{A}$ with*

$$\mathcal{U}(u, \lambda, \mu, \overline{B}_\rho(x)) \leq \epsilon\rho$$

and

$$\Psi(u, \lambda, \mu, \overline{B}_\rho(x)) \leq \vartheta\mathcal{U}(u, \lambda, \mu, \overline{B}_\rho(x)),$$

then

$$\mathcal{U}(u, \lambda, \mu, \overline{B}_{\alpha\rho}(x)) \leq \alpha^2\mathcal{U}(u, \lambda, \mu, \overline{B}_\rho(x)).$$

Proof - Suppose the lemma is not true. Then there exist $\alpha \in]0,1[$, $\lambda, \mu > 0$, two sequences (ϵ_h), (ϑ_h) such that $\lim_h \epsilon_h = \lim_h \vartheta_h = 0$, a sequence (u_h) in $SBH(\Omega)$ and a sequence of disks $\overline{B}_{\rho_h}(x_h) \subset \Omega$, such that

$$\mathcal{U}(u_h, \lambda, \mu, \overline{B}_{\rho_h}(x_h)) = \epsilon_h\rho_h,$$

$$\Psi(u_h, \lambda, \mu, \overline{B}_{\rho_h}(x_h)) \leq \vartheta_h\mathcal{U}(u_h, \lambda, \mu, \overline{B}_{\rho_h}(x_h))$$

and

$$\mathcal{U}(u_h, \lambda, \mu, \overline{B}_{\alpha\rho_h}(x_h)) > \alpha^2\mathcal{U}(u_h, \lambda, \mu, \overline{B}_{\rho_h}(x_h)).$$

For each h, translating x_h into the origin and blowing up, i.e. setting

$$v_h(x) = \frac{u_h(x_h + \rho_h x)}{\sqrt{\epsilon_h \rho_h^3}} \qquad x \in B_1,$$

and setting $\lambda_h = \frac{\lambda}{\epsilon_h}$ and $\mu_h = \mu\sqrt{\frac{\rho_h}{\epsilon_h}}$, we have, by using Lemma 4.5 in [6],

$$\mathcal{U}(v_h, \lambda_h, \mu_h, \overline{B}_1) = 1,$$

$$\Psi(v_h, \lambda_h, \mu_h, \overline{B}_1) \leq \vartheta_h$$

and

$$\mathcal{U}(v_h, \lambda_h, \mu_h, \overline{B}_\alpha) > \alpha^2. \tag{11}$$

By Theorem 3.2 of [6], there exist a subsequence of (v_h), still denoted by (v_h), and a function $w \in H^2(B_1)$ such that (assuming, without loss of generality, $\mathcal{P}v_h = 0$) $\lim_h v_h = w$ a. e. in B_1. Then we deduce that the function w is affine in B_1 and, arguing as in Theorem 4.7 of [6], we get

$$\limsup_h \ \mathcal{U}(v_h, \lambda_h, \mu_h, \overline{B}_\alpha) \ = \ \int_{B_\alpha} |D^2 w|^2 \ dy \ , \tag{12}$$

whereas by (11) and (12) we would have

$$0 < \alpha^2 \leq \int_\Omega |D^2 w|^2 dy = 0. \qquad \blacksquare$$

Finally we may use the previous Decay Theorem to achieve the proof of Theorem 4.1 as like as in the proof of Theorem 2.1.

Remark 4.6 - We remark that the previous theorem holds with α^2 replaced by α^n for any $n \in \mathbf{N}$, $n > 2$, but in such case ϵ and ϑ depend also on n.

5. Blake-Zisserman model of image segmentation.

In this section we state two theorems about some functionals defined over spaces of functions allowing both discontinuities and creases. Since we want to consider functionals which do not give a bound on the first order derivatives of an admissible function, we recall the following definition from [12].

Definition 5.1 - *We define for* $\Omega \subset \mathbf{R}^m$

$$GSBV(\Omega, \mathbf{R}^k) = \{v : \Omega \to \mathbf{R}^k \text{ Borel function};$$
$$\varphi \circ v \in SBV_{\mathrm{loc}}(\Omega) \ \ \forall \varphi \in C^1(\mathbf{R}^k) \text{ with } \mathrm{spt}(D\varphi) \text{ compact}\}.$$

Then we may define the following function space:

$$X(\Omega) = \{v : \Omega \to \mathbf{R} : \ v \in L^2(\Omega), \ v \in GSBV(\Omega, \mathbf{R}), \ \nabla v \in GSBV(\Omega, \mathbf{R}^2)\}$$

We stress the fact that $Dv \neq \nabla v$ since admissible functions may be discontinuous, and in addition $X(\Omega) \not\subset BV(\Omega)$.

Given $g \in L^2(\Omega)$, set

$$\mathcal{F}_3 : X(\Omega) \to [0, +\infty]$$

$$\mathcal{F}_3(v) \ = \int_\Omega |\nabla^2 v|^2 \, dy + \ \alpha \mathcal{H}^1(S_v) + \beta \mathcal{H}^1(S_{\nabla v} \setminus S_v) + \int_\Omega |v - g|^2 dy \qquad (13)$$

Theorem 5.2 - *Assume $g \in L^2(\Omega)$ and $0 < \beta \leq \alpha \leq 2\beta$. Then there is $v_0 \in X(\Omega)$ such that*

$$\mathcal{F}_3(v_0) \leq \mathcal{F}_3(v) \qquad \forall v \in X(\Omega).$$

$\mathcal{F}_3$ is a weak form of a functional proposed by Blake & Zisserman [2] in computer vision theory with the aim of detecting both discontinuities and creases of a given digital image g. For the proof of Theorem 5.2 we refer to a paper by the authors in preparation.

Another functional considered in [2] is the following

$$F_4(K_0, K_1, u) = \int_{\Omega \setminus (K_0 \cup K_1)} (|\nabla^2 u|^2 + |\nabla u|^2) dy +$$
$$\alpha \mathcal{H}^1(K_0) + \beta \mathcal{H}^1(K_1 \setminus K_0) + \int_\Omega |u - g|^2 dy \qquad (14)$$

defined for every $(K_0, K_1, u) \in A_d$. In the following theorem we give the existence of a minimizer for a weak formulation of F_4.

Theorem 5.3 - *Given $g \in L^2(\Omega)$ and $0 < \beta \leq \alpha$, the following functional*

$$\mathcal{F}_4(v) = \int_\Omega (|\nabla^2 v|^2 + |\nabla v|^2) dy + \alpha \mathcal{H}^1(S_v) + \beta \mathcal{H}^1(S_{\nabla v} \setminus S_v) + \int_\Omega |v - g|^2 dy$$

achieves a minimum over $X(\Omega)$.

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