# On the consistency of the Rational Large Eddy Simulation model

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**Abstract**

In this paper we consider the Rational Large Eddy Simulation model for turbulent flows (RLES in the sequel), introduced by Galdi and Layton [11]. We recall some analytical results regarding the RLES model and the main result we will prove is the convergence of the strong solutions to the RLES model to those of the Navier-Stokes (in some Sobolev spaces), as the averaging radius goes to zero. Estimates on the rates of convergence are also obtained. These results give more weight to the validity of the method in either computational or physical experiments.

We also consider the error arising from the derivation of the model in presence of boundaries. In particular the equations present an extra-term involving the boundary value of the stress tensor. By using some recent estimates on this "commutation error" we show that, with a Smagorinsky sub-grid scale term, the kinetic energy remains bounded. bf Key words Large Eddy Simulation, Rational Model, Strong Solutions, Consistency.

## 1 Introduction

We consider the Rational Large Eddy Simulation (RLES) model, introduced by Galdi and Layton [11]:

$$\begin{cases} \dfrac{\partial w_\delta}{\partial t} + \nabla q_\delta + \nabla \cdot (w_\delta \otimes w_\delta) - \dfrac{1}{Re}\Delta w_\delta + \\ +\nabla \cdot \left(\mathrm{I} - \dfrac{\delta^2}{24}\Delta\right)^{-1}\left[\dfrac{\delta^2}{12}\nabla w_\delta \nabla w_\delta^T\right] = \overline{f}, \\ \nabla \cdot w_\delta = 0, \\ w_\delta(x,0) = w_\delta^0(x). \end{cases} \tag{1}$$

Here, $Re > 0$ is the Reynolds number, while the vector field $w_\delta : \Omega \times [0,T] \to \mathbf{R}^3$ is an approximation, formally of order $O(\delta^4)$, of the filtered velocity $w$ that comes out by filtering the solution $u$ to the Navier-Stokes equations (2), while

$\Omega \subset \mathbf{R}^3$ is a smooth bounded domain; more precisely $w_\delta$ is an approximation of

$$w(x,t) = g_\delta(x) * u(x,t)$$

where $g_\delta(x)$ is a Gaussian kernel

$$g_\delta(x) = \left(\frac{6}{\pi}\right)^{3/2} \frac{1}{\delta^3} \mathrm{e}^{-\frac{6|x|^2}{\delta^2}},$$

and $*$ denotes the usual convolution. The un-filtered velocity $u$ is a solution to the Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \nabla p + \nabla \cdot (u \otimes u) - \dfrac{1}{Re} \Delta u = f, \\ \nabla \cdot u = 0, \\ u(x,0) = u^0(x), \end{cases} \tag{2}$$

that are the well-known equations describing the motion of viscous, incompressible fluids. (As usual the Navier-Stokes equations are studied in $\Omega$ with homogeneous Dirichlet boundary conditions).

We point out that the system appearing in (1) is not a differential system, due to the presence of the non-local term

$$\begin{aligned} \left(\mathrm{I} - \frac{\delta^2}{24}\Delta\right)^{-1} \left[\frac{\delta^2}{12} \nabla w_\delta \nabla w_\delta^T\right]_{ij} &:= \\ := \left(\mathrm{I} - \frac{\delta^2}{24}\Delta\right)^{-1} \frac{\delta^2}{12} \sum_{l=1}^{3} \frac{\partial w_\delta^i}{\partial x_l} \frac{\partial w_\delta^j}{\partial x_l}. \end{aligned} \tag{3}$$

For the reader's convenience, we recall that system (1) for the approximate mean velocity is derived in the following way:

1. extend all the variables appearing in the Navier-Stokes equations (2) to 0 outside the domain $\Omega$;
2. apply the filter, that acts as a convolution with the kernel $g_\delta$;
3. assume that convolution and the linear operators commute;
4. pass to the frequency space *via* the Fourier transform;
5. by using the so called **(0,1) subdiagonal Padé** rational approximation of the exponential function, write the Fourier transform of $\overline{u \otimes u}$ in terms only of the Fourier transform of $\overline{u}$;
6. apply the inverse Fourier Transform to get system (1).

This way of reasoning leads to the system (1) that does not contain high order terms, with respect to $\delta$. The terms that involve higher powers of $\delta$ are supposed to be small, compared to those that are retained. For further details and comprehensive references on classical methods of LES, see Aldama [3] and the recent book by Sagaut [19].

The subdiagonal approximation is used to approximate the Fourier transform of the Gaussian kernel in a satisfactory physical consistent way. The complete derivation of the RLES model (1), together with the physical motivation that inspired it can be found in Reference [11].

**Remark 1.1.** *We observe that the derivation is based on the application of the Fourier Transform and then in presence of boundaries, the functions are simply extended by zero outside their domain. This introduce an additional error, as analyzed by Dunca, John, and Layton [7]; see also Section 4.*

To compare the RLES with the most known LES methods, we recall that the classical "gradient method" (see Leonard [17]) has been derived in the same way, but in the above "point 5." it is used a Taylor series expansion (with respect of $\delta$) of the Gaussian kernel $g_\delta$. This classical method has the drawback of a possible increasing of the high wave-numbers. The Padé approximant are introduced since they are decreasing at infinity and the corresponding approximate equations (approximate to the order of $\delta^4$) may have a better behavior.

We also observe that the RLES model is very similar to the *Lagrangian Averaged Navier-Stokes* (LANS) $\alpha-$model (that has been derived in a completely different way) introduced by Holm *et al* [9] and recently analyzed by Foiaş *et al* [8] and Marsden and Shkoller [18]. From the point of view of modeling we note that the RLES model discards all the subgrid scale terms

$$(u-\overline{u})\otimes(u-\overline{u}) \stackrel{def}{=} u'\otimes u',$$

since they turn out to be (formally) of higher order, in that development. For this reason, other high order rational LES models (HOSFS), that involve high order Padé approximants, are going to be investigated, see Berselli and Iliescu [5]. Other LES models have been recently proposed; see for instance Hughes *et al* [12] Katopodes *et al* [15] and the review in Sagaut [19].

Passing to some numerical results, we recall that the solutions to (1) show a better behavior, with respect to other models commonly used in LES. In this respect see the comparison of various model performed by Iliescu *et al* [14].

## 1.1 Analytical results

The analysis concerning existence of weak solution has been performed by Galdi, Iliescu, and Layton (see Iliescu [13]) by adding an extra dissipative term of the Smagorinsky type

$$\left(c_1 + c_2|\nabla\overline{u} + \nabla\overline{u}^{\mathrm{T}}|^{2\mu}\right)(\nabla\overline{u} + \nabla\overline{u}^{\mathrm{T}})$$

with $c_1, c_2$ positive constants and $\mu \geq 0.1$.

The existence and uniqueness of strong solutions, *i.e.*, solutions such that

$$w_\delta \in L^\infty(0,T;H^1)\cap L^2(0,T;H^2)$$

and

$$\frac{\partial w_\delta}{\partial t} \in L^2(0,T;L^2)$$

and without extra dissipative terms has been provided in Berselli *et al* [6]. In that reference it is proved that given $w_0 \in H^1$ and $\overline{f} \in L^2(0,T;L^2)$, there exists a strictly positive $T^* = T^*(w_0, Re, \overline{f}, \delta)$ such that there exists a unique strong solution of (1) in $[0, T^*)$.

Here and in the sequel $Q = ]0, \mathcal{L}[^3 \subset \mathbf{R}^3$, while $H^s$ will denote the Hilbert space of periodic vector valued functions $u$, belonging to the Sobolev space $[H^s(Q)]^3$ and such that

$$\int_{]0,\mathcal{L}[^3} u(x)\, dx = 0.$$

The functions in $H^s$ are written as

$$u = \sum_{k\in\mathbf{Z}^3} c_k \mathrm{e}^{\frac{2i\pi k\cdot x}{\mathcal{L}}}, \quad \overline{c}_k = c_{-k},\, c_0 = 0$$

and the norm in the latter space is defined as

$$\|u\|^2_{H^s} = \sum_{k\in\mathbf{Z}^3} |k|^{2s}|c_k|^2.$$

Note that the latter formula allows to consider real (and also negative) values for $s$. The spaces $L^p := L^p(Q)$, $1 \leq p \leq \infty$ are the customary Lebesgue spaces, which are supposed to be known. We use this particular simple setting to avoid the big technical difficulties due to the boundary conditions. In this way we can focus on fine properties of the solutions to (1).

# 2 Some preliminary results

We start by improving some results regarding the life span of smooth solutions for the RLES model. In reference [6] it has been proved that the life-span of strong solutions to (1) is $O(\delta^4)$. Here we start by proving that in fact it is independent of $\delta$.

We need some preliminary results and we start by proving the following lemma:

**Lemma 2.1.** *Let be given $f \in H^1$ then*

$$\left\| \left(I - \frac{\delta^2}{24}\Delta\right)^{-1} \left[\frac{\delta^2}{12}\nabla f \nabla f^T\right] \right\|_{L^2} \leq \frac{\delta^2}{12}\|\nabla f\|^2_{L^4}. \tag{4}$$

*Furthermore, we have*

$$\left\| \left(I - \frac{\delta^2}{24}\Delta\right)^{-1} \left[\frac{\delta^2}{12}\nabla f \nabla f^T\right] \right\|_{H^2} \leq \frac{\mathcal{L}^2}{2\pi^2}\|\nabla f\|^2_{L^4}. \tag{5}$$

*Proof.* We start by observing that if we set (for smooth $\phi$ and $\psi$)

$$\left(\mathrm{I} - \frac{\delta^2}{24}\Delta\right)^{-1} \frac{\delta^2}{12}\phi = \psi, \tag{6}$$

then we have

$$\frac{\delta^2}{12}\phi = \psi - \frac{\delta^2}{24}\Delta\psi;$$

multiplying the above equation by $\psi$ and integrating over $Q$ we obtain

$$\|\psi\|_{L^2}^2 + \frac{\delta^2}{24}\|\nabla\psi\|_{L^2}^2 = \frac{\delta^2}{12}(\phi,\psi) \le \frac{\delta^2}{12}\|\phi\|_{L^2}\|\psi\|_{L^2},$$

that implies

$$\|\psi\|_{L^2} \le \frac{\delta^2}{12}\|\phi\|_{L^2}. \tag{7}$$

By recalling that in (7) $\psi$ is the left-hand side of the inequality appearing in (4), while $\phi = \nabla f \nabla f$ we have finally proved (4).

The second part is easier. In fact, we can rewrite (6) as

$$2\left(\frac{24}{\delta^2}I - \Delta\right)^{-1}\phi = \psi, \tag{8}$$

and since we can write $\phi$ and $\psi$ as

$$\phi(x) = \sum_{k\in\mathbf{Z}^3}\phi_k \mathrm{e}^{\frac{2i\pi k\cdot x}{\mathcal{L}}}, \quad \psi(x) = \sum_{k\in\mathbf{Z}^3}\psi_k \mathrm{e}^{\frac{2i\pi k\cdot x}{\mathcal{L}}},$$

Eq. (8) implies immediately that

$$\psi_k = 2\frac{1}{\dfrac{24}{\delta^2} + \dfrac{4\pi^2}{\mathcal{L}^2}|k|^2}\,\phi_k.$$

By calculating the $H^2$-norm of $\psi$ we obtain

$$\|\psi\|_{H^2}^2 = \sum_{k\in\mathbf{Z}^3}\frac{4|k|^4}{\left[\dfrac{24}{\delta^2} + \dfrac{4\pi^2}{\mathcal{L}^2}|k|^2\right]^2}\,\phi_k^2 \le \frac{\mathcal{L}^4}{4\pi^4}\sum_{k\in\mathbf{Z}^3}\phi_k^2.$$

The last estimate proves finally that

$$\|\psi\|_{H^2} \le \frac{\mathcal{L}^2}{2\pi^2}\|\phi\|_{L^2}$$

from which it follows (5). □

With the above lemma we can prove the following result regarding life-span. The following results is useful for the consistency of the method, since it proves that for a given smooth (periodic and divergence-free) initial datum, we can construct the solution of both the Navier-Stokes and RLES equations in a common non-empty time interval; in the same interval we can then compare those solutions.

**Theorem 2.1.** *The life span of a strong solution to the RLES model depends on $\|\nabla w_\delta^0\|_{L^2}$, Re, and $\nu$, but it is independent of $\delta$.*

*Proof.* First we recall that in reference [6] it has been proved that the boundedness of the $L^\infty(0,T;L^2)$ norm of the gradient of a strong solution is enough to employ a standard continuation argument. Then, if we are able to prove a uniform bound for

$$\sup_{0<t<T}\|\nabla w_\delta(t)\|_{L^2},$$

then we can infer that the life span is bounded from below by $T$.

For simplicity, from now on, we may suppose that $\overline{f} \equiv 0$. We multiply (1) by $-\Delta w_\delta$ and we integrate over $Q$ to get

$$\frac{1}{2}\frac{d}{dt}\|\nabla w_\delta\|_{L^2}^2 + \frac{1}{Re}\|\Delta w_\delta\|_{L^2}^2 \leq \left|\int_Q (w_\delta \cdot \nabla) w_\delta \, \Delta w_\delta\right|$$

$$+\left|\int_Q \left(\mathrm{I} - \frac{\delta^2}{24}\Delta\right)^{-1} \left[\frac{\delta^2}{12}\nabla w_\delta \nabla w_\delta^T\right] \, \nabla\Delta w_\delta \, dx\right|.$$

The first term can be estimated in a standard way as follows (see for instance Temam [20])

$$|((w_\delta \cdot \nabla)\, w_\delta, \Delta w_\delta)| \leq \frac{1}{4Re}\|\Delta w_\delta\|_{L^2}^2 + c\, Re^3 \|\nabla w_\delta\|_{L^2}^6, \tag{9}$$

where $c$ is a positive constant, depending only on $Q$.

To increase the second one we need Lemma 2.1. In fact, we obtain

$$\left|\int_Q \left(\mathrm{I} - \frac{\delta^2}{24}\Delta\right)^{-1} \left[\frac{\delta^2}{12}\nabla w_\delta \nabla w_\delta^T\right] \, \nabla\Delta w_\delta \, dx\right| \leq$$

$$\leq \left\|\left(\mathrm{I} - \frac{\delta^2}{24}\Delta\right)^{-1} \left[\frac{\delta^2}{12}\nabla w_\delta \nabla w_\delta^T\right]\right\|_{H^2} \|\nabla\Delta w_\delta \, dx\|_{H^{-2}}.$$

Now by using the definition of negative norm of a Sobolev space, Lemma 2.1, and the Young inequality we obtain

$$\left|\int_Q \left(\mathrm{I} - \frac{\delta^2}{24}\Delta\right)^{-1} \left[\frac{\delta^2}{12}\nabla w_\delta \nabla w_\delta^T\right] \, \nabla\Delta w_\delta \, dx\right| \leq$$

$$\leq \frac{1}{4Re}\|\Delta w_\delta\|_{L^2}^2 + \frac{1}{2^{10} Re^3}\|\nabla w_\delta\|_{L^2}^6.$$

This finally implies the following estimate:

$$\frac{1}{2}\frac{d}{dt}\|\nabla w_\delta\|_{L^2}^2 + \frac{1}{2Re}\|\Delta w_\delta\|_{L^2}^2 \leq c\|\nabla w_\delta\|_{L^2}^6, \tag{10}$$

where $c$ is a positive constant independent of $\delta$. Then the basic theory of differential inequalities, applied to (10) implies that $\|\nabla w_\delta\|$ is bounded in some interval $[0, T^*)$, where $T^*$ is independent of $\delta$. If $\overline{f} \not\equiv 0$, but

$$\overline{f} \in L^2(0, T^*; L^2),$$

then an analogous result may be achieved by following the same lines of the previous one. $\square$

We now quote the following result that has been proved in [4]:

**Theorem 2.2.** *Let $w_\delta$ be a strong solution to* (1) *in* $[0, T^*[$. *If* $\overline{f} \in C^\infty$, *then*

$$w_\delta \in C^\infty(]0, T^*[\times]0, \mathcal{L}[^3).$$

The proof of this result is based on a boot strap argument, together with classical regularity results for the Stokes operator. Full details can be found in [4].

The full regularity is not needed for our result. As we will see, what is really needed is just the first step of regularity, *i.e.*, we need the result that if $w_\delta$ is a strong solution in $[0, T^*)$, then

$$w_\delta \in L^\infty(0, T^*; H^2) \cap L^2(0, T^*; H^3). \tag{11}$$

The results of Theorem 2.2 will be used to prove convergence of the solution $w_\delta$ to the solution of the Navier-Stokes equations, as the averaging parameter $\delta \to 0$. It seems that it is not possible to prove the consistency result (just in the $L^2$-norm) if the additional regularity results on the solution were not been previously proved. The results of Theorem 2.2 is then important in this respect, but it is also interesting itself, because the philosophy of the Large Eddy Simulation supposes the underlying "mean velocity" to be a smooth function.

# 3 Proof of the consistency result

In this section we prove the main result of this paper, namely that $w_\delta$ converges, as $\delta$ goes to zero, to a strong solution to the Navier-Stokes equations. We recall that in general such kind of results have been proved only for very few models. In particular, similar results have been proved for the $\alpha$-model studied in reference [8]. We recall that the $\alpha$-model is obtained through the so-called Kelvin filtering, *i.e.*, integrating the Navier-Stokes equations around a loop that moves with a spatially filtered Eulerian flow.

Regarding other LES models (that are more similar to the RLES) the only one for which the consistency is known is the so called "LES scale similarity model" studied in Layton [16]. This model consists in finding $(w, q)$ that satisfy $\nabla \cdot w = 0$ and

$$\begin{aligned} &\frac{\partial w}{\partial t} + \nabla \cdot (\overline{w w}) + \nabla \cdot (\overline{w}(w - \overline{w}) + (w - \overline{w})\overline{w}) + \\ &- \nabla \cdot (\nu_T(\delta, w)\mathbb{D}(w)) + \nabla q - \frac{1}{Re}\Delta w - A(\delta) w = f. \end{aligned} \tag{12}$$

The operators appearing in (12) are defined as follows: $\mathbb{D}(f) = (\nabla f + \nabla f^T)$, for each vector field $f$; furthermore,

$$(A(\delta)w, v) = -(\nu_F(\delta)\mathbb{D}(w - \overline{w}), \mathbb{D}(v - \overline{v})),$$

where $\nu_F(\delta) \to 0$ as $\delta \to 0$. The "turbulent viscosity" must satisfy $\nu_T(\delta, w) = \nu_T(\delta) \to 0$ as $\delta \to 0$. The hypotheses cover also the case in which $\nu_T(\delta, w)$ is the usual Smagorinsky dissipative term. For the model (12) it has been proved a result of $L^\infty(0, T; L^2)$ and $L^2(0, T; H^1)$ convergence, provided that both solutions are strong. (These condition may slightly be relaxed, see [16]).

In this paper we prove a stronger result for the RLES model, since we show the convergence in $L^\infty(0, T; H^1)$. We can now state the main result of this paper:

**Theorem 3.1.** *Let $w_\delta$ be a strong solution to* (1)*, while let $w$ be a solution to the Navier-Stokes equations, in the common time interval $[0, T]$. Let us suppose that both the initial data are smooth (say $w_\delta(x, 0)$ and $w(x, 0)$ belong also to $H^2$) and that*

$$\exists\, c_1 > 0 : \qquad \|w_\delta(x, 0) - w(x, 0)\|_{L^2} \leq c_1 \delta^2.$$

*Then we have, for some $c_2 > 0$,*

$$\sup_{t\in[0,T]} \|w_\delta(x, t) - w(x, t)\|_{L^2} \leq c_2\, \delta^2.$$

*If, in addition,*

$$\exists\, c_3 > 0 : \qquad \|w_\delta(x, 0) - w(x, 0)\|_{H^1} \leq c_3\, \delta, \tag{13}$$

*then we have, for some $c_4 > 0$,*

$$\sup_{t\in[0,T]} \|w_\delta(x, t) - w(x, t)\|_{H^1} \leq c_4 \delta.$$

To prove the above theorem we need the first part of estimate (11). In the following lemma we propose, for the reader convenience, a sketch of the proof. The lemma below is then a subcase of Theorem 2.2.

**Lemma 3.1.** *Let $w_\delta(x, 0) \in H^2$ and $\nabla \cdot w_\delta(x, 0) = 0$. Then a strong solution in the time interval $[0, T]$ satisfies also*

$$w_\delta \in L^\infty(0, T; H^2),$$

*and the bound is independent of $\delta$.*

*Proof.* To prove the lemma we use standard techniques. As usual $A$ denotes the Stokes operator associated to the periodic boundary conditions. Recall that on $D(A^\alpha)$ the norm $\|A^\alpha u\|_{L^2}$ is equivalent to $\|u\|_{H^{2\alpha}}$, for details see Temam [21]. We multiply Eq. (1) by $A^2 w_\delta$ and perform suitable integration by parts. Some calculations are formal, but can be completely justified through a Galerkin approximation; see for instance Temam [20].

Let us start by estimating the additional nonlinear term appearing in the RLES model, all the other are estimated in the classical way of the theory of the Navier-Stokes equations. We have

$$\left|\left(\left(\mathrm{I} - \frac{\delta^2}{24}\Delta\right)^{-1} \left[\frac{\delta^2}{12}\nabla w_\delta \nabla w_\delta^T\right], \nabla A^2 w_\delta\right)\right| \leq$$

$$\leq \left\|\left(\mathrm{I} - \frac{\delta^2}{24}\Delta\right)^{-1} \left[\frac{\delta^2}{12}\nabla w_\delta \nabla w_\delta^T\right]\right\|_{H^2} \|\nabla A^2 w_\delta\|_{H^{-2}}$$

and, by using Lemma 2.1, we obtain the following (note that the constant $c$

does not depend on $\delta$)

$$\left|\left(\left(\mathrm{I}-\frac{\delta^2}{24}\Delta\right)^{-1}\left[\frac{\delta^2}{12}\nabla w_\delta \nabla w_\delta^T\right], \nabla A^2 w_\delta\right)\right| \leq$$

$$\leq c\|\nabla w_\delta\|_{L^4}^2\,\|w_\delta\|_{H^3}$$

and, by inequality (15),

$$\leq c_1\|\nabla w_\delta\|_{L^2}^{1/2}\|w_\delta\|_{H^2}^{3/2}\|w_\delta\|_{H^3}.$$

With the standard estimates and by using that $w_\delta \in L^\infty(0, T^*; H^1)$ we finally obtain

$$\frac{d}{dt}\|Aw_\delta\|_{L^2}^2 + \frac{1}{Re}\|w_\delta\|_{H^3}^2 \leq c(\|Aw_\delta\|^3 + \|Aw_\delta\|^4).$$

By using the Gronwall inequality (recall that since $w_\delta$ is a strong solution in $[0, T^*[$ we have $w_\delta \in L^2(0, T^*; H^2)$) we finally obtain that

$$w_\delta \in L^\infty(0, T^*; H^2).$$

□

We can now prove the main result.

*of Theorem 3.1.* To prove the theorem we start by considering $w_\delta$ a strong solution to (1), while $w$ is a strong solution to (2). We subtract the equation satisfied by $w$ to that satisfied by $w_\delta$ and we multiply the difference by $U = w_\delta - w$. We integrate over $Q$ and we use suitable integration by parts to obtain

$$\frac{1}{2}\frac{d}{dt}\|U\|_{L^2}^2 + \frac{1}{Re}\|\nabla U\|_{L^2}^2 \leq |((U\cdot\nabla)w, U)| + + \left|\left(\left(I-\frac{\delta^2}{24}\Delta\right)^{-1}\left[\frac{\delta^2}{12}\nabla w_\delta \nabla w_\delta^T\right], \nabla U\right)\right|. \tag{14}$$

We also used the standard fact that

$$\int_Q (u\cdot\nabla)v\,v\,dx = 0,$$

if $\nabla\cdot u = 0$ and $v$ is periodic (or $v = 0$ on $\partial Q$).

By using the first part of Lemma 2.1 we obtain that the last term on the right hand side of (14) can be bounded by

$$\frac{\delta^2}{12}\|\nabla w_\delta\|_{L^4}^2\|\nabla U\|_{L^2}.$$

The other one can be bounded with the usual Hölder inequality

$$|((U\cdot\nabla)w, U)| \leq \|U\|_{L^4}^2\|\nabla w\|_{L^2}.$$

We now use the interpolation inequality

$$\|f\|_{L^4} \leq c\|f\|^{1/4}\|\nabla f\|^{3/4}, \quad \forall\, f \in H^1, \tag{15}$$

together with the Young's inequality to obtain

$$\frac{d}{dt}\|U\|^2 + \frac{1}{Re}\|\nabla U\|^2 \leq$$

$$\leq c(\delta^4\|\nabla w_\delta\|\|\Delta w_\delta\|^3 + \|U\|^2\|\nabla w\|^2).$$

Due to the regularity results of Theorem 2.2 (more specifically the bound of $\Delta w_\delta$ in $L^\infty(0,T;L^2)$ of Lemma 3.1) and the classical ones for the Navier-Stokes equations, we obtain

$$\frac{d}{dt}\|U\|^2 + \frac{1}{Re}\|\nabla U\|^2 \leq c(\delta^4 + \|U\|^2), \quad \forall\, t \in [0,T]. \tag{16}$$

A simple integration with respect to $t$, together with the hypothesis on the initial data, implies that

$$\|U(t)\| \leq C\delta^2, \qquad \forall\, t \in [0,T].$$

Now, we look for high order estimates. By following the same guide-lines of the previous result, we subtract the two equations (satisfied by $w_\delta$ and $w$, respectively), we multiply by $AU = A(w_\delta - w)$, and integrate over $Q$. We obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla U\|^2 + \frac{1}{Re}\|AU\|^2 \leq$$

$$\leq \left|\left(\left(I - \frac{\delta^2}{24}\Delta\right)^{-1}\left[\frac{\delta^2}{12}\nabla w_\delta \nabla w_\delta^T\right], \nabla AU\right)\right| + \tag{17}$$

$$+ \left|(U\cdot\nabla)\, w_\delta, AU)\right| + \left|(w\cdot\nabla)\, U, AU)\right|.$$

We recall that we can write $H^1$ as an interpolation space (namely $H^1 = [L^2, H^2]_{1/2}$) and by using Lemma 2.1 we obtain:

$$\left\|\left(\mathrm{I} - \frac{\delta^2}{24}\Delta\right)^{-1}\left[\frac{\delta^2}{12}\nabla f \nabla f\right]\right\|_{H^1} \leq c\delta\|\nabla f\|^2_{L^4}.$$

With the last inequality we get

$$\left|\left(\left(I - \frac{\delta^2}{24}\Delta\right)^{-1}\left[\frac{\delta^2}{12}\nabla w_\delta \nabla w_\delta^T\right], \nabla AU\right)\right| \leq$$

$$\leq \left\|\left(I - \frac{\delta^2}{24}\Delta\right)^{-1}\left[\frac{\delta^2}{12}\nabla w_\delta \nabla w_\delta^T\right]\right\|_{H^1} \|\nabla AU\|_{H^{-1}}$$

$$\leq c\delta\|\nabla w_\delta\|^2_{L^4}\|AU\| \leq \frac{1}{4Re}\|AU\|^2 + c\delta^2\|\nabla w_\delta\|^4_{L^4}.$$

To bound the usual nonlinear terms (appearing also in the Navier-Stokes) we use the classical inequality

$$|((u \cdot \nabla)\, v, w)| \leq c\|\nabla u\| \|\nabla v\|^{1/2} \|Av\|^{1/2} \|w\|,$$

that holds for $\forall\, u \in V,\ \forall\, v \in \mathcal{D}(A),\ \forall\, w \in H$ (see for instance Temam [20]) and we easily obtain

$$|(U \cdot \nabla)\, w_\delta, AU)| + |(w \cdot \nabla)\, U, AU)| \leq$$

$$\leq \frac{1}{4Re}\|AU\|^2 + c(\|\nabla w_\delta\|^2 + \|\nabla w\|^2)\|\nabla U\|^2.$$

Collecting all the inequalities we finally get

$$\begin{aligned} \frac{d}{dt}\|\nabla U\|^2 + \frac{1}{Re}\|AU\|^2 \leq & c(\|\nabla w_\delta\|^2 + \|\nabla w\|^2)\|\nabla U\|^2 \\ & + c\delta^2 \|\nabla w_\delta\|^4_{L^4}. \end{aligned} \tag{18}$$

The differential inequality (18), together with the Gronwall lemma, implies that there exists a positive constant $C$, independent of $\delta$, such that

$$\|\nabla U(t)\| \leq C\delta, \qquad \forall\, t \in [0, T].$$

$\square$

All the consistency results are going to be validated also from the numerical point of view, see [4].

# 4 On error bounds

As remarked in the introduction, the presence of the boundaries introduces some terms that are generally disregarded, even if they are not zero. The effect of boundary terms has been recently pointed out in Reference [7] together with the way to handle the additional terms in a reasonable functional setting.

In this section we consider the space averaged equations in a bounded smooth domain $\Omega$. We extend the velocity, the pressure, and the external force as zero outside $\Omega$ (recall "point 1." in the derivation of the model)

$$u = 0,\ u_0 = 0,\ p = 0,\ f = 0 \quad \text{if } x \notin \overline{\Omega}.$$

In this way the extended velocity belongs to $H^1_0(\mathbf{R}^3)$, but $u$ does not necessarily belong to $H^2(\mathbf{R}^3)$, even if it belongs to $H^2(\Omega)$. Convolving the first equation in (2) (with a filter function $g_\delta$) we obtain the following *momentum averaged equation* for $\overline{u} = g_\delta * u$ :

$$\begin{aligned} & \frac{\partial \overline{u}}{\partial t} - 2\frac{1}{Re}\nabla \cdot \mathbb{D}(\overline{u}) + \nabla \cdot (\overline{u \otimes u}) + \nabla \overline{p} = \overline{f} + \\ & + \int_{\partial\Omega} g_\delta(x - s)\, [2\nu \mathbb{D}(u)(s) n(s) - p(s) n(s)]\ ds, \end{aligned} \tag{19}$$

where $n(s)$ is the exterior normal vector. We observe that, in addition to the usual terms appearing in the classical LES models, in the right-hand side there is an additional commutation error.

**Definition 4.1.** *The commutation error $A_\delta(\mathbb{S}(u,p))$ in the space averaged Navier-Stokes equations is defined to be*

$$A_\delta(\mathbb{S}(u,p)) \stackrel{def}{=} \int_{\partial\Omega} g_\delta(x-y)\,\mathbb{S}(u,p)(y)\,dy,$$

*where $\mathbb{S}$ is the stress tensor, given by*

$$\mathbb{S}(u,p) \stackrel{def}{=} \frac{1}{Re}\left(\nabla u + \nabla u^{\mathrm{T}}\right) - p\mathbb{I}$$

*being $\mathbb{I}$ the identity tensor.*

**Remark 4.1.** *The commutation error depends on the normal stress on $\partial\Omega$ of the un-filtered variables $(u,p)$ and it does not depend on the filtered variables $(\overline{u},\overline{p})$.*

Together with the usual "closure"approximation, needed to model $\overline{u\otimes u}$, it is necessary to take into account the commutation error appearing on the right-hand side.

## 4.1 Estimates on the commutation error

To handle the commutation error in Reference [7] some estimates are obtained. The following propositions can be proved with the usual techniques of Sobolev spaces together with some some technical lemmas of geometric measure theory. The proof of the Propositions below can be found in Sections 4–6 of [7].

**Proposition 4.1.** *Let $\psi \in L^p(\partial\Omega)$, $1 \le p \le \infty$. Then*

$$\lim_{\delta\to 0}\left\| \int_{\partial\Omega} g_\delta(x-y)\psi(y)\,dy \right\|_{L^p(\mathbf{R}^3)} = 0$$

*if and only if $\psi(y)$ vanishes almost everywhere on $\partial\Omega$.*

**Proposition 4.2.** *Let $\psi(s) \in L^2(\partial\Omega)$, then there exists a positive constant $C = C(\Omega)$ such that, $\forall\,\delta > 0$*

$$\left\| \int_{\partial\Omega} g_\delta(x-s)\psi(s)\,ds \right\|_{H^{-1}(\Omega)} \le C\delta^{1/2}\|\psi\|_{L^2(\partial\Omega)}.$$

Finally, we recall a result that derives from the previous ones and that will be used in the sequel.

**Proposition 4.3.** *Let $v \in H^1(\mathbf{R}^3)$ such that $v_{|\Omega}$, its restriction to $\Omega$, belongs to $H^1_0(\Omega)\cap H^2(\Omega)$ and $v$ vanishes identically outside $\overline{\Omega}$. Furthermore, let $\psi$ belong to $L^p(\partial\Omega)$, for some $p \in [1,\infty]$. Then, if $\overline{v} = g_\delta * v$ we have:*

$$\lim_{\delta\to 0^+}\int_{\mathbf{R}^3}\overline{v}(x)\left(\int_{\partial\Omega} g_\delta(x-s)\psi(s)\,ds\right)dx = 0.$$

**Remark 4.2.** *We are assuming that $\psi$ belongs to $L^p$ since for strong solution to the Navier-Stokes equations it is possible to prove (see for instance Galdi [10]) that*

$$\psi = \mathbb{S}(u,p) \quad \textit{belongs to } L^p(\partial\Omega), \quad 1 \leq p \leq 4.$$

The first question to be answered regarding the influence of such commutation terms in the study of LES models regards the effect of the commutation error on the kinetic energy. By using the above proposition in Reference [7] it is shown that this term does not affects the usual energy estimates for several LES models, namely:

- the classical Smagorinsky model;
- the gradient (or Taylor) model plus a Smagorinsky dissipative term;
- a variant of the Rational LES model (the so-called "RLES model without additional problem") plus a Smagorinsky dissipative term.

**Remark 4.3.** *The RLES model without additional problem is a slightly different model, in which the additional (turbulent) stress tensor* (3) *is replaced by the simpler*

$$g_\delta * \left[\frac{\delta^2}{12}\nabla w_\delta \nabla w_\delta^T\right]_{ij} := g_\delta * \left(\frac{\delta^2}{12}\sum_{l=1}^{3}\frac{\partial w_\delta^i}{\partial x_l}\frac{\partial w_\delta^j}{\partial x_l}\right).$$

By using essentially the classical techniques and the above propositions 1-3 of [7], we prove in a straightforward manner the following bound for the kinetic energy for the RLES model (1).

**Theorem 4.1.** *Let $(u,p)$ be a strong solution to* (2) *in $[0,T]$. Let $w_\delta$ be a solution, in $\mathbf{R}^3$, to the problem:*

$$\begin{cases} \dfrac{\partial w_\delta}{\partial t} + \nabla q_\delta + \nabla\cdot(w_\delta \otimes w_\delta) - \nabla\cdot(c_s|\nabla w_\delta|\nabla w_\delta) + \\ -\dfrac{1}{Re}\nabla\cdot\mathbb{D}(w_\delta) + \nabla\cdot\left(I - \dfrac{\delta^2}{24}\Delta\right)^{-1}\left(\dfrac{\delta^2}{12}\nabla w_\delta\nabla w_\delta^T\right) = \\ = \overline{f} + \displaystyle\int_{\partial\Omega} g_\delta(x-s)\mathbb{S}(u,p)\,n(s)\,ds \quad \textit{in } \boldsymbol{R}^3\times(0,T) \\ \nabla\cdot w_\delta = 0 \quad \textit{in } \boldsymbol{R}^3\times(0,T) \\ w_\delta(x,0) = w_\delta^0(x) \quad \textit{in } \boldsymbol{R}^3, \end{cases} \tag{20}$$

*where $|\nabla w_\delta| = \left(\sum_{ij}(\partial_i w_\delta^j)^2\right)^{1/2}$ and $g_\delta$ is the Gaussian kernel. Then, if $c_s$ is large enough, it results the following bound for the kinetic energy:*

$$\|w_\delta(t)\|_{L^2}^2 \leq e^t\left(\|w_\delta^0\|_{L^2}^2 + \int_0^t e^{-\tau}\Big[\|\overline{f}(\tau)\|_{L^2}^2 + \epsilon_\delta(\tau)\Big]d\tau\right)$$

*where $\lim\limits_{\delta\to 0^+}\epsilon_\delta(\tau) = 0$ for every $\tau \geq 0$.*

*Proof.* The proof is based on classical Sobolev techniques. Let us multiply equation (20) by $w_\delta$ and integrate over the whole space $\mathbf{R}^3$. Using standard arguments we obtain that

$$\int_{\mathbf{R}^3} \nabla q_\delta w_\delta \, dx = 0$$

and

$$\int_{\mathbf{R}^3} \nabla \cdot (w_\delta \otimes w_\delta) \cdot w_\delta \, dx = 0$$

Moreover,

$$-\frac{1}{Re} \int_{\mathbf{R}^3} \nabla \cdot \mathbb{D}(w_\delta) \cdot w_\delta \, dx = \frac{1}{Re} \|\nabla w_\delta\|_{L^2}^2 \geq 0.$$

Let us now estimate the following term:

$$\left| \int_{\mathbf{R}^3} \nabla \left( I - \frac{\delta^2}{24} \Delta \right)^{-1} \left( \frac{\delta^2}{12} \nabla w_\delta \nabla w_\delta^T \right) \cdot w_\delta \, dx \right| \leq$$

$$\leq \left\| \left( I - \frac{\delta^2}{24} \Delta \right)^{-1} \left( \frac{\delta^2}{12} \nabla w_\delta \nabla w_\delta^T \right) \right\|_{H^1} \|\nabla w_\delta\|_{H^{-1}} \leq$$

$$\leq c \left\| \left( I - \frac{\delta^2}{24} \Delta \right)^{-1} \left( \frac{\delta^2}{12} \nabla w_\delta \nabla w_\delta^T \right) \right\|_{W^{2,3/2}} \|\nabla w_\delta\|_2 \leq$$

$$\leq c \|\nabla w_\delta \nabla w_\delta^T\|_{3/2} \|\nabla w_\delta\|_3 \leq c \|\nabla w_\delta\|_3^3$$

(we used the regularity theory for elliptic PDEs, see Agmon, Douglis, and Nirenberg [2] and the Sobolev embedding $W^{2,3/2}(\mathbf{R}^3) \hookrightarrow W^{1,2}(\mathbf{R}^3)$; see [1]). In the end we obtain, using Schwartz and Minkowski inequalities to estimate $\int \overline{f} w_\delta \, dx$,

$$\frac{1}{2} \frac{d}{dt} \|w_\delta\|_{L^2}^2 \leq \frac{\|w_\delta\|_{L^2}^2}{2} + \frac{\|\overline{f}\|_{L^2}^2}{2} + (c - c_s) \|\nabla w_\delta\|_{L^3}^3 +$$

$$+ \int_{\mathbf{R}^3} w_\delta \left( \int_{\partial\Omega} g_\delta(x-s) \mathbb{S}(u,p) \, n(s) \, ds \right) dx.$$

Using Gronwall lemma, if $c_s > c$, we obtain:

$$\|w_\delta(t)\|_{L^2}^2 \leq e^t \Bigg( \|w_\delta^0\|_{L^2}^2 + \int_0^t e^{-\tau} \big( \|\overline{f}(\tau)\|_{L^2}^2 +$$

$$+2 \int_{\mathbf{R}^3} w_\delta \int_{\partial\Omega} g_\delta(x-s) \mathbb{S}(u,p) n(s) \, ds dx \big) d\tau \Bigg)$$

hence, by setting

$$\epsilon_\delta(\tau) = 2 \int_{\mathbf{R}^3} w_\delta \int_{\partial\Omega} g_\delta(x-s) \mathbb{S}(u,p) \, n(s) \, ds dx$$

Proposition 3 gives the requested bound for the kinetic energy. □

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