Sharp regularity estimates for solutions of the continuity equation drifted by Sobolev vector fields

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Abstract

The aim of this note is to prove sharp regularity estimates for solutions of the continuity equation, associated to $W^{1,p}$ vector fields, for $p > 1$. The regularity is of "logarithmic order" and is measured by means of suitable seminorms.

Key words: Ordinary differential equations with non smooth vector fields; continuity equation; transport equation; regular Lagrangian flow; BV function; log-Sobolev space; Bressan's mixing conjecture.

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Contents

1 Introduction and main result

In this paper we study regularity properties of solutions to the continuity equation,

$$\begin{cases} \partial_t u + \operatorname{div}(bu) = 0, \\ u(0,x) = u_0(x), \end{cases} \qquad \text{in } [0,T] \times \mathbb{R}^d, \tag{CE}$$

where $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a time dependent vector field, $u_0 : \mathbb{R}^d \to \mathbb{R}$ is the initial data and $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ the unknown of the problem. We are mainly interested in the study of (CE) when b is Sobolev regular and divergence free, more precisely we are going to assume

$$\int_0^T \|b_s\|_{W^{1,p}(\mathbb{R}^d)}\, \mathrm{d}s < \infty \quad \text{for some } p > 1, \text{ and } \operatorname{div} b_t = 0 \quad \text{for a.e. } t \in [0,T].$$

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Distributional solutions in the class $L^\infty([0,T]\times\mathbb{R}^d)$ are considered, precisely we study weak-star continuous maps $t \to u_t \in L^\infty(\mathbb{R}^d)$ such that, for every $\varphi \in C_c^\infty(\mathbb{R}^d)$, the function $t \to \int_{\mathbb{R}^d} \varphi u_t \,\mathrm{d}x$ is absolutely continuous and fulfills

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \varphi u_t \,\mathrm{d}x = \int_{\mathbb{R}^d} b_t \cdot \nabla\varphi u_t \,\mathrm{d}x \qquad \text{for a.e. } t \in [0,T].$$

Let us also recall that, the Cauchy problem (CE) is strictly related to the system of ordinary differential equations

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} X(t,x) = b(t, X(t,x)), \\ X(0,x) = x, \quad (t,x) \in [0,T]\times\mathbb{R}^d. \end{cases} \tag{ODE}$$

Indeed, when the vector field is regular enough (for instance globally bounded and Lipschitz in the spatial variable, uniformly in time) the classical Cauchy-Lipschitz theory guarantees the existence of a unique flow map $X : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$, and solutions to (CE) are provided by [1]

$$(X_t)_\# \bar{u}\mathscr{L}^d = u_t \mathscr{L}^d, \tag{1.1}$$

A link between (CE) and (ODE) is still present in our setting, but it is very subtle. The main technical issue is the loss of point-wise uniqueness for solutions of (ODE). To overcome this difficult, Ambrosio in [A04] has introduced the notion of *regular Lagrangian flow* (see Definition 2.6) and has established a link between well-posedness of the Cauchy problem (CE) and existence and uniqueness for regular Lagrangian flows. He has also shown well-posedness in L^∞ for (CE) when the vector field has the BV spatial regularity and bounded divergence (it means $\operatorname{div} b_t \ll \mathscr{L}^d$, with density in L^∞); this result provides an important extension of the celebrated DiPerna-Lions theory [DPL89].

In the last years the quantitative study of solutions to (CE) has received a lot of attentions. Two important problems in this regard are the study of the *propagation of regularity* and the *rate of mixing*. The importance of such an investigation relies on possible applications to partial non-linear PDE coming from the the fluid mechanics and the kinetic theory. We refer to [ACM16, IKX14, HSSS18, S13] for an overview of the topic of mixing in the Sobolev setting, and we are going to focus mostly on the regularity side of the problem.

In the smooth setting the picture is quite clear: the flow map X_t and its inverse inherit the Lipschitz regularity of the vector field, precisely it holds

$$e^{-tL}|x-y| \le |X_t(x) - X_t(y)| \le e^{tL}|x-y| \qquad \text{for any } x,y\in\mathbb{R}^d,\ t\in[0,T],$$

where $L := \sup_{t\in[0,T]} \|\nabla b_t\|_{L^\infty}$. This estimate, along with (1.1) and the icompressibility condition $\operatorname{div} b = 0$, leads to

$$\|\nabla u_t\|_{L^\infty} \le e^{tL}\|\nabla u_0\|_{L^\infty} \qquad \text{for any } t\in[0,T], \tag{1.2}$$

where u_t is the unique solution of (CE). Moreover, it is a simple exercise to see that (1.2) is sharp. In other words the Lipschitz seminorm increases at most exponentially fast in time along solutions of the incompressible continuity equation, drifted by Lipschitz velocity fields, and this rate is sharp.

When considering $W^{1,p}$ vector fields the situation is much more complicated, and new wild phenomena come up. The Lipschitz and Sobolev regularity, even of fractional order, might be instantaneously lost during the time evolution [J16, ACM16, ACM14], but some very weak notion of regularity seems to be propagated also in this case [CDL08, BJ15, BJ15, LF16].

[1] This is an identity between measures, where the left hand side is defined by $(X_t)_\#\bar{u}\mathscr{L}^d(E) := (u\mathscr{L}^d)((X_t)^{-1}(E))$ for every Borel set $E\subset\mathbb{R}^d$. It is equivalent to

$$\int_{\mathbb{R}^d}\phi(x)u_t(x)dx = \int_{\mathbb{R}^d}\phi(X(t,x))u_0(x)dx \ \ \forall\, \phi\in C_b(\mathbb{R}^d).$$

The main result of the present paper is the sharp characterisation of the regularity for solutions to (CE) drifted by $W^{1,p}$ divergence free vector fields, for $p > 1$, in the scale of *log-Sobolev functionals*

$$\left(\int_{B_{1/3}} \int_{\mathbb{R}^d} \frac{|f(x+h) - f(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^{1-p}} \, \mathrm{d}x \, \mathrm{d}h \right)^{1/2} \qquad \text{for } p \geq 0. \tag{1.3}$$

The functional (1.3) is inspired by the well-known Gagliardo semi-norm

$$\left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x+h) - f(x)|^2}{|h|^{d+2s}} \, \mathrm{d}x \, \mathrm{d}h \right)^{1/3} \qquad s \in (0,1),$$

that aims at measuring the L^2 norm of the derivative of order s of f. In the cases $p = 1$ and $p = 2$ the functional (1.3) is equivalent to the one considered in [LF16], se also [BN18b, Theorem 1.4]. Let us now state our main result.

Theorem 1.1. *Let $p > 1$ be fixed, and let $b \in L^1([0,T]; W^{1,p}(\mathbb{R}^d;\mathbb{R}^d))$ a divergence-free vector field. Then, for every initial data $u_0 \in L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$ with $\|u_0\|_{L^\infty} \leq 1$ the solution $u \in L^\infty([0,T]\times\mathbb{R}^d)$ to* (CE) *satisfies*

$$\int_{B_{1/3}} \int_{\mathbb{R}^d} \frac{|u_t(x+h) - u_t(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^{1-p}} \, \mathrm{d}x \, \mathrm{d}h \lesssim_{p,d} \left(\int_0^t \|\nabla b_s\|_{L^p} \, \mathrm{d}s \right)^p + \|u_0\|_{BV}^p + \|u_0\|_{L^1} . \tag{1.4}$$

Moreover, there exist a divergence-free vector field $b \in L^\infty([0,+\infty); W^{1,p}(\mathbb{R}^d))$ and an initial data $u_0 \in L^\infty(\mathbb{R}^d) \cap W^{1,d}(\mathbb{R}^d)$, such that the solution $u \in L^\infty([0,+\infty) \times \mathbb{R}^d)$ of the Cauchy problem (CE) *satisfies*

$$\int_{B_{1/2}} \int_{\mathbb{R}^d} \frac{|u_t(x+h) - u_t(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^{\gamma}} \, \mathrm{d}x \, \mathrm{d}h = \infty, \qquad \textit{for any } t > 0, \tag{1.5}$$

for any $\gamma < 1 - p$.

If we further assume $b \in L^\infty([0,T]; W^{1,p}(\mathbb{R}^d;\mathbb{R}^d))$ with $p > 1$, then (1.4) ensures that the log-Sobolev functional of order p increases at most polynomially fast in time, with exponent p. Also this rate is sharp as it is shown in Theorem 3.1.

We refer to section 2 and section 3 for technical remarks on Theorem 1.1 concerning the boundedness assumption on the vector field, the regularity of the initial data and simple generalizations to the case of velocity fields with nonzero divergence. Let us now spend a few words explaining the strategy of the proof. For what concerns the first part of Theorem 1.1, our starting point has been the Lusin-Lipschitz estimate for Lagrangian flows obtained by Crippa and De Lellis in [CDL08]. We show, indeed, that a suitable version of this estimate (see Proposition 2.9) implies (1.4) by means of a general result (Proposition 2.16) that links two different notions of "having a derivative of logarithm order" in L^p.

The second part of Theorem 1.1 builds upon a variant of the construction proposed by Alberti Crippa and Mazzucato in [ACM16] (see also [ACM14] and [ACM18]). The main new technical tool we introduce is the interpolation inequality proved in Proposition 3.5 (see also Corollary 3.7) that provides a bound for the log-Sobolev functional (1.3) in terms of L^2 and $\dot{H}^{-1}$ norms. As a byproduct of Theorem 1.1 and the interpolation inequality we recover the sharp lower bound on "mixing rate" for vector fields with uniformly bounded $W^{1,p}$ norms, for $p > 1$ (see also [CDL08, Theorem 6.2] and [IKX14, HSSS18, S13, LF16]).

The paper is organised as follows. In section 2 we deal with the first part of Theorem 1.1, while section 3 is devoted to the proof of the second part of the Theorem 1.1 (see also Theorem 3.2) and of Theorem 3.1. In this section we also collect two mixing estimates (see Proposition 3.10) obtained as a byproduct of the previously developed theory.

Notation. We denote by $\mathbb{R}^d$ the Euclidean space of dimension d endowed with the Lebesgue measure $\mathscr{L}^d$ and the Euclidean norm $|\cdot|$. $B_r(x)$ denotes the ball of radius $r > 0$ centred at $x \in \mathbb{R}^d$ and we will often write B_r instead of $B_r(0)$. $L^p = L^p(\mathbb{R}^d)$ are standard Lebesgue spaces of p-integrable functions while $W^{1,p}(\mathbb{R}^d)$ stands for the Sobolev space of functions endowed with the norm $\|f\|^p_{W^{1,p}} = \|f\|^p_{L^p} + \|\nabla f\|^p_{L^p}$.

We set

$$\fint_E f \, \mathrm{d}x = \frac{1}{\mathscr{L}^d(E)} \int_E f \, \mathrm{d}x, \qquad \forall\ E \subset \mathbb{R}^d \text{ Borel set},$$

and

$$Mf(x) := \sup_{r>0} \fint_{B_r(x)} |f(y)| \, \mathrm{d}y, \qquad \forall\ x \in \mathbb{R}^d,$$

to denote the Hardy-Littlewood maximal function.

We often use the expression $a \lesssim_c b$ to mean that there exists a universal constant C depending only on c such that $a \le Cb$. The same convention is adopted for $\gtrsim_c$ and $\simeq_c$.

2 Regularity result

In this section we deal with the positive part of Theorem 1.1 that is restated below for the reader convenience.

Theorem 2.1. *Let $p > 1$ be fixed, and let $b \in L^1([0,T]; W^{1,p}(\mathbb{R}^d; \mathbb{R}^d))$ a divergence-free vector field. Then, for every initial data $u_0 \in L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$ with $\|u_0\|_{L^\infty} \le 1$ the solution $u \in L^\infty([0,T] \times \mathbb{R}^d)$ to* (CE) *satisfies*

$$\int_{B_{1/3}} \int_{\mathbb{R}^d} \frac{|u_t(x+h) - u_t(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^{1-p}} \, \mathrm{d}x \, \mathrm{d}h \lesssim_{p,d} \left(\int_0^t \|\nabla b_s\|_{L^p} \, \mathrm{d}s \right)^p + \|u_0\|^p_{BV} + \|u_0\|_{L^1}. \tag{2.1}$$

Let us begin with a few technical remarks.

Remark 2.2. Building upon Remark 2.19, one can prove a following variant of (2.1):

$$\sup_{h \in B_{1/3}} \log(1/|h|)^p \int_{\mathbb{R}^d} |u_t(x+h) - u_t(x)|^2 \, \mathrm{d}x \lesssim_{p,d} \left(\int_0^t \|\nabla b_s\|_{L^p} \, \mathrm{d}s \right)^p + \|u_0\|^p_{BV} + \|u_0\|_{L^1}.$$

This will play a role in the study of the geometric mixing norm of solutions to (CE), see Proposition 3.10.

Remark 2.3. The assumption $b \in L^\infty([0,T] \times \mathbb{R}^d)$ can be replaced by more general growth conditions, for instance one can ask

$$\frac{b(t,x)}{1+|x|} = b_1(t,x) + b_2(t,x) \quad \text{with } b_1 \in L^1([0,T]; L^1(\mathbb{R}^d)) \text{ and } b_2 \in L^1([0,T]; L^\infty(\mathbb{R}^d)), \tag{2.2}$$

compare with [CDL08, page 12].

Remark 2.4. The divergence free condition can be weakened by assuming

$$\exp\left\{ \int_0^T \|\operatorname{div} b_s\|_{L^\infty} \, \mathrm{d}s \right\} = L < \infty, \tag{2.3}$$

provided we consider the *transport equation*

$$\begin{cases} \partial_t u + b \cdot \nabla u = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{TrE}$$

instead of (CE). In this case, one has

$$\int_{B_{1/3}} \int_{\mathbb{R}^d} \frac{|u_t(x+h) - u_t(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^{1-p}} \, \mathrm{d}x \, \mathrm{d}h \lesssim_{p,d} L^p \left(\int_0^t \|\nabla b_s\|_{L^p} \, \mathrm{d}s \right)^p + \|u_0\|^p_{BV} + \|u_0\|_{L^1}.$$

Remark 2.5. We have chosen to assume $u_0 \in BV(\mathbb{R}^d)$ in Theorem 2.1 just for sake of simplicity and for consistency with the counterexample in Theorem 3.2 and the geometric mixing bound (3.15). It is simple to see that, the regularity assumption on the initial data can be weakened in several ways, for instance by assuming either $\bar{u} \in W^{s,1}(\mathbb{R}^d)$ for $0 < s \leq 1$, or that $\bar{u}$ satisfies some Lusin-Lipschitz inequality as (2.18).

We expect that the right assumption should be

$$\int_{B_{1/3}} \int_{\mathbb{R}^d} \frac{|u_0(x+h) - u_0(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^{1-p}} \, \mathrm{d}x \, \mathrm{d}h < \infty, \tag{2.4}$$

but, unfortunately, our technique does not allow to cover this case.

The proof of Theorem 2.1 builds upon a well-known ingredient: the quantitative Lusin-Lipschitz estimate for Lagrangian flows associated to Sobolev fields, first introduced in [ALM05] and [CDL08]. In subsection 2.1 we state and prove a suitable version of this result and, as a corollary we get a quantitative Lusin-Lipschitz estimate for solutions to the continuity equation (see Corollary 2.11 and compare with [CDL08, Theorem 5.3]). Eventually, in subsection 2.2 we establish a general result that links a suitable quantitative Lusin-Lipschitz property with an estimate of the log-Sobolev functional (1.3).

2.1 Regularity of Lagrangian flows

In this subsection we present a regularity estimate for Lagrangian flows associated to Sobolev vector fields with exponent $p > 1$.

Definition 2.6 (Regular Lagrangian flow)**.** We say that $X : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a regular Lagrangian flow associated to b_t (RLF for short) if the following conditions hold:

(i) there exists an $\mathscr{L}^d$-negligible set $N \subset \mathbb{R}^d$ such that

$$X_t(x) = x + \int_0^t b_s(X_s(x)) \, \mathrm{d}s \qquad \text{for any } t \in [0,T] \text{ and } x \in \mathbb{R}^d \setminus N;$$

(ii) there exists $L > 0$, called compressibility constant, such that

$$(X_t)_\sharp \mathscr{L}^d \leq L \mathscr{L}^d \qquad \text{for any } t \in [0,T].$$

Condition (i) ensures that $t \to X_t(x)$ solves (ODE) for $\mathscr{L}^d$-a.e. initial data. Condition (ii), instead, has the role to select "good" trajectories imposing that the flow cannot concentrate too much the reference measure $\mathscr{L}^d$. It also plays an important technical role by guaranteeing that the notion of RLF is stable under modifications of the vector field on a $\mathscr{L}^d$-negligible set.

Remark 2.7. It has been shown in [A04, Theorem 6.2, Theorem 6,4] that, under the BV assumption on the vector field (i.e. $\int_0^T \|b_s\|_{BV} \, \mathrm{d}s < \infty$), the uniform bound on the negative part of the divergence (i.e. $\operatorname{div} b_t \ll \mathscr{L}^d$ and $\int_0^T \|[\operatorname{div} b_s]^-\|_{L^\infty} \, \mathrm{d}s < \infty$) and some growth conditions (for instance $b \in L^\infty([0,T] \times \mathbb{R}^d)$, see also Remark 2.3) there exists a unique RLF associated to b. See also the recent [QN18] for a different proof.

Moreover, assuming a bound on the whole divergence

$$\exp\left\{ \int_0^T \|\operatorname{div} b_s\|_{L^\infty} \, \mathrm{d}s \right\} \leq L,$$

condition (ii) in Definition 2.6 can be improved as

$$1/L \mathscr{L}^d \leq (X_t)_\sharp \mathscr{L}^d \leq L \mathscr{L}^d \qquad \text{for every } t \in [0,T]. \tag{2.5}$$

In particular, if b is divergence free then X_t is measure preserving.

Remark 2.8. It is convenient to adopt the following convection: any vector field is *point-wise* defined by setting

$$b_t(x) := \begin{cases} \lim_{r\to 0} \fint_{B_r(x)} b_t(y)\,\mathrm{d}y & \text{when it exists} \\ 0 & \text{otherwise} \end{cases}$$

This convention allows for a point-wise *Lusin-Lipschitz maximal estimate*:

$$|b_t(x) - b_t(y)| \lesssim_d |x-y|(M|\nabla b_t|(x) + M|\nabla b_t|(y)) \quad \text{for any } x, y \in \mathbb{R}^d \text{ and } \mathscr{L}^1\text{-a.e. } t \in [0,T], \tag{2.6}$$

where M denotes the Hardy-Littlewood maximal function. See [ST] for a proof of this result in the scalar case.

We are ready to state and prove the Lusin-Lipschitz regularity result for Lagrangian flows associated to $W^{1,p}$ vector fields with $p > 1$, compare it with [CDL08, Poposition 2.3].

Proposition 2.9. *Let $b \in L^1([0,T]; W^{1,p}(\mathbb{R}^d;\mathbb{R}^d))$ for some $p > 1$, and let X be a regular Lagrangian flow associated to b with compressibility constant L. Then, there exists a measurable function $g_t(x) = g(t,x) : [0,T] \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ such that*

$$e^{-g_t(x)-g_t(y)} \le \frac{|X_t(x) - X_t(y)|}{|x-y|} \le e^{g_t(x)+g_t(y)} \qquad \text{for any } x, y \in \mathbb{R}^d \text{ and } t \in [0,T], \tag{2.7}$$

and

$$\|g_t\|_{L^p} \lesssim_{p,d} L \int_0^t \|\nabla b_s\|_{L^p}\,\mathrm{d}s \qquad \forall t \in [0,T]. \tag{2.8}$$

Moreover, if b is divergence free we can assume $L = 1$.

Proof. Let $N \subset \mathbb{R}^d$ be as in (i)Definition 2.6, $\varepsilon > 0$, $x, y \in \mathbb{R}^d \setminus N$ and $t \in [0,T]$. We have

$$\left|\log\left(\frac{\varepsilon + |X_t(x) - X_t(y)|}{\varepsilon + |x-y|}\right)\right| = \left|\int_0^t \frac{\mathrm{d}}{\mathrm{d}s} \log\left(\varepsilon + |X_s(x) - X_s(y)|\right)\mathrm{d}s\right| \le \int_0^t \frac{|b_s(X_s(x)) - b_s(X_s(y))|}{|X_s(x) - X_s(y)|}\,\mathrm{d}s.$$

Using (2.6) an letting $\varepsilon \to 0$ we get

$$\left|\log\left(\frac{|X_t(x) - X_t(y)|}{|x-y|}\right)\right| \le C_d \int_0^t M|\nabla b_s|(X_s(x))\,\mathrm{d}s + C_d \int_0^t M|\nabla b_s|(X_s(y))\,\mathrm{d}s.$$

Set $g_t(x) := C_d \int_0^t M|\nabla b_s|(X_s(x))\,\mathrm{d}s$ when $x \in \mathbb{R}^d \setminus N$ and $g_t(x) := +\infty$ otherwise. The condition (ii)Definition 2.6, the boundness of the maximal function between L^p spaces when $p > 1$, together with Minkowski's inequality (see [ST, Appendix]), yields (2.8). The proof is complete. □

Remark 2.10. It is clear from the proof of Proposition 2.9 that g_t in (2.7) can be taken independent of $t \in [0,T]$, simply by considering $g_T := C_d \int_0^T M|\nabla b_s|(X_s(x))\,\mathrm{d}s$.

As a simple consequence of Proposition 2.9 we get a Lusin-Lipschitz estimate for solutions to the continuity equation (CE), compare with [CDL08, Theorem 5.3].

Corollary 2.11. *Let $b \in L^1([0,T]; W^{1,p}(\mathbb{R}^d;\mathbb{R}^d))$ be bounded and divergence free, with $p > 1$. Then, there exists a measurable function $\tilde{g}_t(x) = \tilde{g}(t,x) : [0,T] \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, such that, for every $u_0 \in BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, the solution $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ in $L^\infty([0,T] \times \mathbb{R}^d)$ to* (CE) *satisfies*

$$|u_t(x) - u_t(y)| \le |x-y| \exp\{\tilde{g}_t(x) + \tilde{g}_t(y)\}, \tag{2.9}$$

for $\mathscr{L}^{2d}$-a.e $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ and for any $t \in [0,T]$. Moreover

$$\|\tilde{g}_t\|_{L^p} \lesssim_{p,d} \int_0^t \|\nabla b_s\|_{L^p}\,\mathrm{d}s + \|u_0\|_{BV} \quad \text{for any } t \in [0,T]. \tag{2.10}$$

Proof. Let X be a unique RLF associated to b. For any $t \in [0,T]$, the map $x \to X_t(x)$ is essentially invertible, i.e. there exists $Y : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ such that

$$X(t, Y(t,x)) = Y(t, X(t,x)) = x \qquad \text{for } \mathscr{L}^d\text{-a.e. } x \in \mathbb{R}^d, \tag{2.11}$$

see [A04, Theorem 6.2]. It is immediate to check that Y_t satisfies (2.7) provided we replace $g_t(x)$ with $\bar{g}_t(x) := g_t(Y_t(x))$. By using that Y_t preserves $\mathscr{L}^d$ (compare with Remark 2.7) we deduce

$$\|\bar{g}_t\|_{L^p} \lesssim_{p,d} \int_0^t \|\nabla b_s\|_{L^p}\, \mathrm{d}s \qquad \forall t \in [0,T]. \tag{2.12}$$

Thus, we get

$$\frac{|u_t(x) - u_t(y)|}{|x-y|} \leq C_d(M|\nabla u_0|(Y_t(x)) + M|\nabla u_0|(Y_t(y))) \exp\{\bar{g}_t(x) + \bar{g}_t(y)\}$$

for $\mathscr{L}^{2d}$-a.e $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ and for any $t \in [0,T]$, where we used the $\mathscr{L}^d$-a.e. identity $u_t = \bar{u}(Y_t)$ (it can be checked by observing that $u_t(X_t(x)) = u_0(x)$ for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$) and (2.6) for $u_0 \in BV(\mathbb{R}^d)$.

Finally observe that, for $x, y \in \mathbb{R}^d$ and $t \in [0,T]$, one has

$$C_d(M|\nabla \bar{u}|(Y_t(x)) + M|\nabla \bar{u}|(Y_t(y))) \exp\{\bar{g}_t(x) + \bar{g}_t(y)\} \leq \exp\{\tilde{g}_t(x) + \tilde{g}_t(x)\},$$

where $\tilde{g}_t(x) = \bar{2}g_t(x) + 2\log(\max\{C_d M \mid \nabla u_0|(Y_t(x)); 1\})$. This implies (2.9). Thanks to the L^p estimate for the maximal function (see [ST, Theorem 1]), the fact that Y_t is measure preserving and (2.12) we obtain (2.10). The proof is complete. ☐

We conclude this subsection by proving a converse of Proposition 2.9. Roughly speaking we show that, for a given vector field $b \in L^1([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$, the existence of a RLF that satisfies the Lusin-Lipschitz estimate (2.7) with exponent $p > 1$ implies that b is of class $W^{1,p}$. We are grateful to Luigi Ambrosio for having pointed this out to us.

Let us recall that a vector field $b \in L^1(\mathbb{R}^d)$ belongs to $\mathrm{BD}(\mathbb{R}^d)$ if its distributional *symmetric derivative* Eb is a finite Radon measure, we refer to [ACDM97] for more explanations.

Theorem 2.12. *Let $b \in L^1([0,T] \times \mathbb{R}^d; \mathbb{R}^d)$ be fixed. If there exists a Regular Lagrangian flow X_t associated to b that satisfies*

$$\begin{aligned} &\exp\left\{-\int_s^t h_r(X_r(x))\,\mathrm{d}r - \int_s^t h_r(X_r(y))\,\mathrm{d}r\right\} \\ &\quad \leq \frac{|X_t(x) - X_t(y)|}{|X_s(x) - X_s(y)|} \\ &\quad \leq \exp\left\{\int_s^t h_r(X_r(x))\,\mathrm{d}r + \int_s^t h_r(X_r(y))\,\mathrm{d}r\right\} \end{aligned} \tag{2.13}$$

for any $x, y \in \mathbb{R}^d$ and $0 \leq s < t \leq T$, where $h : [0,T] \times \mathbb{R}^d \to [0,+\infty]$ fulfills

$$\int_0^T \|h_t\|_{L^p}\,\mathrm{d}t < \infty, \qquad \text{for some } p \geq 1, \tag{2.14}$$

then

(i) when $p = 1$, $b \in L^1([0,T]; BD(\mathbb{R}^d))$, and $\int_0^T \|b_t\|_{BD} \leq \int_0^T \|b_t\|_{L^1}\,\mathrm{d}t + 2\int_0^T \|h_t\|_{L^1}\,\mathrm{d}t$;

(ii) when $p > 1$, b admits a distributional derivative in L^p for $\mathscr{L}^d$-a.e. $t \in [0,T]$ and

$$\int_0^T \|\nabla b_t\|_{L^p}\,\mathrm{d}t \lesssim_{d,p} \int_0^T \|h_t\|_{L^p}\,\mathrm{d}t.$$

Remark 2.13. It is clear from the proof of Proposition 2.9 that, if $b \in L^1([0,T]; W^{1,p}(\mathbb{R}^d;\mathbb{R}^d))$, for some $p > 1$ and $\operatorname{div} b \in L^\infty([0,T] \times \mathbb{R}^d)$, then (2.13) holds true with $h_r = M|\nabla b_r| \in L^p(\mathbb{R}^d)$.

Proof of Theorem 2.12. Let $N \subset \mathbb{R}^d$ as in Definition 2.6, we have

$$\int_s^t h_r(X_r(x))\,\mathrm{d}r + \int_s^t h_r(X_r(y))\,\mathrm{d}r \geq \left|\log\left(\frac{|X_t(x)-X_t(y)|}{|X_s(x)-X_s(y)|}\right)\right| = \left|\int_s^t \frac{\mathrm{d}}{\mathrm{d}r}|X_r(x)-X_r(y)|\,\mathrm{d}r\right|$$
$$= \left|\int_s^t \frac{\langle b_r(X_r(x)) - b_r(X_r(y));\ X_r(x)-X_r(y)\rangle}{|X_r(x)-X_r(y)|^2}\,\mathrm{d}r\right|$$

for any $x, y \in \mathbb{R}^d \setminus N$. Therefore, for any $x, y \in \mathbb{R}^d \setminus N$, it holds

$$\left|\frac{\langle b_r(X_r(x)) - b_r(X_r(y));\ X_r(x)-X_r(y)\rangle}{|X_r(x)-X_r(y)|^2}\right| \leq h_r(X_r(x)) + h_r(X_r(y)) \quad \text{for } \mathscr{L}^1\text{-a.e. } r\in[0,T],$$

thus

$$\left|\frac{\langle b_r(x) - b_r(y);\ x-y\rangle}{|x-y|^2}\right| \leq h_r(x) + h_r(y) \quad \text{for } \mathscr{L}^1\text{-a.e. } r\in[0,T],\ \forall x,y\in\mathbb{R}^d\setminus N_r, \tag{2.15}$$

where $\mathscr{L}^d(N_r) = 0$. Our conclusion follows from Lemma 2.14 below. □

Lemma 2.14. *Let $b \in L^1(\mathbb{R}^d;\mathbb{R}^d)$ and $h \in L^p(\mathbb{R}^d)$ for $p \in [0,1]$. If*

$$\left|\frac{\langle b(x) - b(y);\ x-y\rangle}{|x-y|^2}\right| \leq h(x) + h(y) \quad \textit{for any } x,y\in\mathbb{R}^d\setminus N, \textit{ where } \mathscr{L}^d(N)=0, \tag{2.16}$$

then

(i) when $p = 1$, $b \in BD(\mathbb{R}^d)$ and $\|b\|_{BD} \leq \|b\|_{L^1} + 2\|h\|_{L^1}$;

(ii) when $p > 1$, b admits a distributional derivative in L^p and $\|\nabla b\|_{L^p} \lesssim_{d,p} \|h\|_{L^p}$.

Proof. The proof is based on a simple approximation argument. Let $\rho_\varepsilon(x) := \varepsilon^{-d}\rho(x/\varepsilon)$ where $\rho \in C_c^\infty(\mathbb{R}^d)$ is nonnegative, supported in B_1 with $\int \rho = 1$. Set $b_\varepsilon := \rho_\varepsilon * b$ and $h_\varepsilon := \rho_\varepsilon * h$. It is easily seen that

$$\left|\frac{\langle b_\varepsilon(x) - b_\varepsilon(y);\ x-y\rangle}{|x-y|^2}\right| \leq h_\varepsilon(x) + h_\varepsilon(y), \quad \forall x,y\in\mathbb{R}^d. \tag{2.17}$$

Since b_ε and h_ε are smooth functions, using the Taylor expansion of b^ε around $x \in \mathbb{R}^d$, from (2.17), we deduce $|\nabla_{\mathrm{sym}} b_\varepsilon(x)| \leq 2h_\varepsilon(x)$ where we have denoted by ∇_{sym} the symmetric part of the gradient. It gives

$$\sup_{\varepsilon\in(0,1)} \|\nabla_{\mathrm{sym}} b_\varepsilon\|_{L^p} \leq 2\|h\|_{L^p},$$

that implies (i). When $p > 1$ Korn's inequality (see [ACDM97, Remark 7.11]) implies

$$\sup_{\varepsilon\in(0,1)} \|\nabla b_\varepsilon\|_{L^p} \lesssim_{p,d} \sup_{\varepsilon\in(0,1)} \|\nabla_{\mathrm{sym}} b_\varepsilon\|_{L^p} \leq 2\|h\|_{L^p}$$

and thus (ii) follows. □

Remark 2.15. When $p > 1$, the condition (2.16), along with $b \in L^p(\mathbb{R}^d;\mathbb{R}^d)$, characterises $W^{1,p}(\mathbb{R};\mathbb{R}^d)$; see for instance (2.6). When $p = 1$, the situation is different, indeed, $BD(\mathbb{R}^d)$ is bigger than the class of vector fields $b \in L^1(\mathbb{R}^d;\mathbb{R}^d)$ satisfying (2.16), even if we further require $\operatorname{div} b = 0$.

For instance, we can consider

$$b(x_1, x_2) = (\mathbf{1}_{x_2>0}, 0) \quad \text{for } x = (x_1,x_2)\in\mathbb{R}^2,$$

that belongs to $BV(\mathbb{R}^2;\mathbb{R}^2) \subset BD(\mathbb{R}^2)$. Assume now the existence of a nonnegative function g, such that

$$|\langle b(x) - b(y), x-y\rangle| \leq |x-y|^2\,(g(x)+g(y)) \quad \text{for any } x,y\in\mathbb{R}^2,$$

then, for any $x_2 > 0$ and $y_2 < 0$ we have $|x_1 - y_1| \leq |x-y|^2 (g(x) + g(y))$. Thus, choosing $y_2 = -x_2 = -|x_1 - y_1|$, one has

$$\frac{1}{|x_1 - y_1|} \leq g(x_1, |x_1 - y_1|) + g(y_1, -|x_1 - y_1|),$$

Setting $s = x_1 - y_1 \in \mathbb{R}$ and integrating we get

$$\begin{aligned} 8|\log(\varepsilon)| = \int_\varepsilon^1 \int_{-4}^4 \frac{1}{|s|} \, \mathrm{d}y_1 \, \mathrm{d}s &\leq \int_0^1 \left(\int_{-4+s}^{4+s} g(y_1, |s|) \, \mathrm{d}y_1 + \int_{-4}^4 g(y_1, -|s|) \, \mathrm{d}y_1 \right) \mathrm{d}s \\ &\leq \int_0^1 \left(\int_{-5}^5 (g(y_1, |s|) + g(y_1, -|s|)) \, \mathrm{d}y_1 \right) \mathrm{d}s \\ &= \int_{-1}^1 \int_{-5}^5 g(y_1, s) \, \mathrm{d}y_1 \, \mathrm{d}s < \infty, \end{aligned}$$

for any $\varepsilon \in (0, 1/10)$. This implies that g does not belong to $L^1(\mathbb{R}^2)$.

2.2 The key lemma

This section is devoted to the proof of the following.

Proposition 2.16. *Let $p \geq 1$ be fixed. For any $f \in L^1(\mathbb{R}^d)$ satisfying the exponential Lusin-Lipschitz estimate*

$$|f(x) - f(y)| \leq |x-y| \exp\{g(x) + g(y)\} \quad \text{for } \mathscr{L}^{2d}\text{-a.e. } x, y \in \mathbb{R}^d \times \mathbb{R}^d, \tag{2.18}$$

for some $g \in L^p(\mathbb{R}^d)$, it holds

$$\int_{B_{1/3}} \int_{\mathbb{R}^d} \frac{1 \wedge |f(x+h) - f(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^{1-p}} \, \mathrm{d}x \, \mathrm{d}h \lesssim_{p,d} \|g\|_{L^p}^p + \|f\|_{L^1}. \tag{2.19}$$

Roughly speaking, this result establishes an implication between two different notions of "having a derivative of logarithmic order". Note that these notions are not equivalent, indeed every Hölder function f satisfies $\int_{B_{1/3}} \int_{\mathbb{R}^d} \frac{1 \wedge |f(x+h)-f(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^{1-p}} \, \mathrm{d}x \, \mathrm{d}h < \infty$, but there are Hölder functions fulfilling $\liminf_{y \to x} \frac{|f(x)-f(y)|}{|x-y|} = \infty$ for any $x \in \mathbb{R}^d$.

Remark 2.17. The result in Proposition 2.16 is written in a form useful for our purposes and can be generalised in many different ways. For instance, one can assume that $f \in L^1(\mathbb{R}^d)$ satisfies a Hölder-Lipschitz inequality

$$|f(x) - f(y)| \leq |x-y|^\alpha \exp\{g(x) + g(y)\} \qquad \text{for } \mathscr{L}^{2d}\text{-a.e. } x, y \in \mathbb{R}^d \times \mathbb{R}^d,$$

for some $\alpha \in (0, 1]$ and some $g \in L^p(\mathbb{R}^d)$ and prove

$$\int_{B_{1/3}} \int_{\mathbb{R}^d} \frac{1 \wedge |f(x+h) - f(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^{1-p}} \, \mathrm{d}x \, \mathrm{d}h \lesssim_{p,\alpha,d} \|g\|_{L^p}^p + \|f\|_{L^1}.$$

The proof follows verbatim the one in Proposition 2.16.

Let us begin by proving a technical lemma.

Lemma 2.18. *Let $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$ be as in Proposition 2.16. Then it holds*

$$\int_{\mathbb{R}^d} 1 \wedge |f(x+h) - f(x)|^2 \, \mathrm{d}x \lesssim_d |h|^2 \int_1^{\log(1/|h|)} e^{2\lambda} \mathscr{L}^d(\{ 2g > \lambda \}) \, \mathrm{d}\lambda + |h| \, \|f\|_{L^1}, \tag{2.20}$$

for any $h \in \mathbb{R}^d$ with $|h| \leq 1/e$.

Proof. By using (2.18) and Cavalieri's formula we get

$$
\begin{aligned}
\int_{\mathbb{R}^d} 1 \wedge |f(x+h) - f(x)|^2 \, \mathrm{d}x \\
&= 2\int_0^{e|h|} t\mathscr{L}^d \left(\{ x : \ | f(x+h) - f(x)| > t \} \right) \mathrm{d}t \\
&\quad + 2\int_{e|h|}^1 t\mathscr{L}^d \left(\{ x : \ | f(x+h) - f(x)| > t \} \right) \mathrm{d}t \\
&\lesssim |h| \, \|f\|_{L^1} + \int_{e|h|}^1 t\mathscr{L}^d \left(\{ x : \ | f(x+h) - f(x)| > t \} \right) \mathrm{d}t \\
&\lesssim |h| \, \|f\|_{L^1} + \int_{e|h|}^1 t\mathscr{L}^d (\{ x : \ g(x) + g(x+h) > \log(t/ \mid h|) \}) \, \mathrm{d}t,
\end{aligned}
$$

for $\mathscr{L}^d$-a.e. $h \in \mathbb{R}^d$ with $|h| \le 1/e$. Estimating $\int_{e|h|}^1 t\mathscr{L}^d(\{ x : \ g(x) + g(x+h) > \log(t/ \mid h|) \}) \, \mathrm{d}t$ with

$$
2\int_{e|h|}^1 t\mathscr{L}^d(\{ 2g > \log(t/ \mid h|) \}) \, \mathrm{d}t, \tag{2.21}
$$

setting $\lambda = \log(t/|h|)$ and changing variables in (2.21) we get (2.20) for $\mathscr{L}^d$-a.e $h \in \mathbb{R}^d$ with $|h| \le 1/e$. The general statement follows by exploiting the continuity in h of the terms appearing in (2.20). □

Remark 2.19. Observe that Lemma 2.18 immediately gives a weak version of Proposition 2.16:

$$
\sup_{h \in B_{1/3}} \log(1/|h|)^p \int_{\mathbb{R}^d} 1 \wedge |f(x+h) - f(x)|^2 \, \mathrm{d}x \lesssim_{p,d} \|g\|_{L^{p,\infty}}^p + \|f\|_{L^1} \,,
$$

where $L^{p,\infty}$ is the weak L^p space. Indeed we have

$$
|h|^2 \int_1^{\log(1/|h|)} e^{2\lambda} \mathscr{L}^d(\{ 2g > \lambda \}) \, \mathrm{d}\lambda \lesssim |h|^2 \, \|g\|_{L^{p,\infty}}^p \int_1^{\log(1/|h|)} e^{2\lambda} \lambda^p \, \mathrm{d}\lambda \lesssim_p \log(1/|h|)^p \, \|g\|_{L^{p,\infty}}^p \,.
$$

Proof of Proposition 2.16. In order to keep notation short we set $\mu(\lambda) := \mathscr{L}^d(\{ 2g > \lambda \}) \, \mathrm{d}\lambda$. By using the result in Lemma 2.18 we get

$$
\begin{aligned}
\int_{B_{1/e}} \int_{\mathbb{R}^d} \frac{1 \wedge |f(x+h) - f(x)|^2}{|h|^d \log(1/|h|)^{1-p}} \, \mathrm{d}x \, \mathrm{d}h \\
&\lesssim \int_{B_{1/e}} \frac{\log(1/|h|)^{p-1}}{|h|^d} \left(|h|^2 \int_1^{\log(1/|h|)} e^{2\lambda} \, \mathrm{d}\mu(\lambda) + |h| \, \|f\|_{L^1} \right) \mathrm{d}h \\
&\lesssim_{p,d} \int_0^1 \log(1/r)^{p-1} r \int_1^{\log(1/r)} e^{2\lambda} \, \mathrm{d}\mu(\lambda) \, \mathrm{d}r + \|f\|_{L^1} \,.
\end{aligned}
$$

We change variables according to $\log(1/r) = t$, and we apply Fubini theorem

$$
\begin{aligned}
\int_0^1 \log(1/r)^{p-1} r \int_1^{\log(1/r)} e^{2\lambda} \, \mathrm{d}\mu(\lambda) \, \mathrm{d}r \\
&= \int_0^\infty e^{-2t} t^{p-1} \int_1^t e^{2\lambda} \, \mathrm{d}\mu(\lambda) \, \mathrm{d}t \\
&= \int_1^\infty e^{2\lambda} \int_\lambda^\infty e^{-2t} t^{p-1} \, \mathrm{d}t \, \mathrm{d}\mu(\lambda).
\end{aligned}
$$

Observe now that, by using the integration by part formula and the inequality $\lambda \ge 1$ one has

$$
e^{2\lambda} \int_\lambda^\infty e^{-2t} t^{p-1} \, \mathrm{d}t \lesssim_p \lambda^{p-1}, \tag{2.22}
$$

that along with the definition of $\mu(\lambda)$ implies

$$\int_1^\infty e^{2\lambda} \int_\lambda^\infty e^{-2t} t^{p-1} \, \mathrm{d}t \, \mathrm{d}\mu(\lambda) \lesssim_p \int_0^\infty \lambda^{p-1} \, \mathrm{d}\mu(\lambda) \lesssim_p \|g\|_{L^p}^p .$$

Putting all together we get

$$\int_{B_{1/e}} \int_{\mathbb{R}^d} \frac{1 \wedge |f(x+h) - f(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^{1-p}} \, \mathrm{d}x \, \mathrm{d}h \lesssim_{p,d} \|g\|_{L^p}^p + \|f\|_{L^1} , \tag{2.23}$$

that clearly implies our conclusion □

Proof of Theorem 2.1. The sought conclusion follows by applying Proposition 2.16 to $f = u_t$, recalling Corollary 2.11 and Remark 2.20 below. □

Remark 2.20. If $b \in L^1([0,T]; BV(\mathbb{R}^d;\mathbb{R}^d))$ has bounded divergence then, any solution $u_t \in L^\infty([0,T] \times \mathbb{R}^d)$ to (CE) has the *renormalization property* (see [A04]) i.e., for any $\beta \in C_c^1(\mathbb{R})$, the function $\beta(u(t,x))$ is a solution to (CE) as well. It implies that, for every $1 \le p < \infty$, the L^p norm of u_t is preserved in time.

3 Counterexamples and mixing estimates

The aim of this section is to show the sharpness of Theorem 2.1 in the scale of log-Sobolev functionals, by means of two examples. The first one ensures that the polynomial growth of order p proved in (2.1) is sharp. In the case $p = 1$ the result was already proven by Léger in [LF16].

Theorem 3.1. *Let $p \ge 1$. There exist a divergence free vector field $b \in L^\infty([0,+\infty); W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d))$ supported in $B_1 \times [0,+\infty)$, and a smooth initial data u_0 supported in B_1, such that the unique solution $u \in L^\infty([0,+\infty) \times \mathbb{R}^d)$ to* (CE) *satisfies*

$$\int_{B_{1/3}} \int_{\mathbb{R}^d} \frac{|u_t(x+h) - u_t(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^{1-p}} \, \mathrm{d}x \, \mathrm{d}h \gtrsim t^p \quad \textit{for any } t \ge 0.$$

The second example, that has been already presented in the introduction (part two of Theorem 1.1), is restated below for the reader convenience.

Theorem 3.2. *Let $p \ge 1$. There exist a divergence free vector field $b \in L^\infty([0,+\infty); W^{1,p}(\mathbb{R}^d))$ supported in $B_1 \times [0,+\infty)$, and $u_0 \in L^\infty(\mathbb{R}^d) \cap W^{1,d}(\mathbb{R}^d)$ also supported in B_1, such that the solution $u \in L^\infty([0,+\infty) \times \mathbb{R}^d)$ to* (CE) *satisfies*

$$\int_{B_{1/3}} \int_{\mathbb{R}^d} \frac{|u_t(x+h) - u_t(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^{\gamma}} \, \mathrm{d}x \, \mathrm{d}h = \infty, \qquad \textit{for any } t > 0, \tag{3.1}$$

for any $\gamma < 1 - p$.

As a consequence of Proposition 2.16 (see also Remark 2.17) and Proposition 2.9 the result in Theorem 3.2 immediately implies the following.

Proposition 3.3. *Let $p \ge 1$ fixed. For every $q > p$ there exists a compact supported divergence free vector field $b \in L^\infty([0,+\infty); W^{1,p}(\mathbb{R}^d))$ whose regular Lagrangian flow X satisfies the following property: for every $t > 0$, $g \in L^q(\mathbb{R}^d)$ and $\alpha \in (0,1]$ there exists a set $E \subset \mathbb{R}^d$ of positive $\mathscr{L}^d$ measure such that*

$$|X_t(x) - X_t(y)| > |x-y|^\alpha \exp\{g(x) + g(y)\} \qquad \textit{for any } x, y \in E. \tag{3.2}$$

In other words, the exponential Lusin-Lipschitz regularity of order p (see Proposition 2.9), for Lagrangian flows associated to $W^{1,p}$ vector fields, cannot be improved. Even an exponential Lusin-Hölder regularity of order $q > p$ cannot be proved. Compare this with Theorem 2.12.

The main idea behind our constructions comes from the work [ACM16] by Alberti, Crippa and Mazzucato. In this paper the authors have built a solution to (CE), drifted by a divergence free

Sobolev vector field, that is smooth at time zero but it does not belong to any Sobolev space for positive times. The construction of the vector field b and the solution u_t is achieved by patching together a countable number of pairs (v_n, ρ_n) of velocity fields and solutions to (CE) with disjoint supports. They are obtained by rescaling in space, time and size the building blocks provided by the following:

Proposition 3.4. *Assume $d \geq 2$ and let Q be the open cube with unit side centered at the origin of $\mathbb{R}^d$. There exist a velocity field $v \in C^\infty([0,+\infty) \times \mathbb{R}^d)$ and a solution $\rho \in L^\infty([0,+\infty) \times \mathbb{R}^d)$ to* (CE) *such that*

- *(i) v_t is bounded, divergence free and compactly supported in Q for any $t \geq 0$;*
- *(ii) ρ_t has zero average and it is bounded and compactly supported in Q for any $t \geq 0$;*
- *(iii) $\sup_{t\geq 0} \|v_t\|_{W^{1,p}(\mathbb{R}^d)} < \infty$ for any $t \geq 0$, for any $1 \leq p \leq \infty$;*
- *(iv) there exists a constant $c > 0$ such that*

$$\|\rho_t\|_{\dot{H}^{-1}(\mathbb{R}^d)} \lesssim e^{-ct}, \qquad \forall t \geq 0, \tag{3.3}$$

  where $\|\cdot\|_{\dot{H}^{-1}}$ denotes the homogeneous Sobolev norm of order -1.

This result is taken from [ACM18, Theorem 8], and provides a solution of (CE) whose $\dot{H}^{-1}$ norm decays exponentially fast in time. This rate cannot be improved, indeed, for $W^{1,p}$ velocity fields, with $p > 1$, one has $\|u_t\|_{H^{-1}} \geq Ce^{-ct}$ (see Proposition 3.10 and the discussion above for more details).

Observe that, by means of (3.3), Remark 2.20 and the interpolation inequality

$$\|\rho_t\|_{L^2}^2 \leq \|\rho_t\|_{\dot{H}^{-1}} \|\rho_t\|_{\dot{H}^1},$$

one can show the exponential loss of Sobolev regularity for solutions to (CE):

$$\|\rho_t\|_{\dot{H}^1} \gtrsim \|\rho_0\|_{L^1} e^{ct} \qquad \text{for any } t \geq 0.$$

This scheme suggests to introduce an interpolation inequality involving log-Sobolev functionals, negative Sobolev norms and L^p norms. This is the content of the next subsection.

3.1 Interpolation inequality and proof of Theorem 3.1

The main result of this subsection is inspired by [DDN18, Proof of Theorem 2.4] and reads as follows.

Proposition 3.5. *Let us fix the parameters $\gamma \in (-\infty, 1)$, $\lambda \in (0, 1/100)$ and $\delta \in (0,1]$. For any $f \in L^2(\mathbb{R}^d)$ it holds*

$$\|f\|_{L^2}^2 \lesssim_{d,\gamma} \frac{1}{|\log(\delta\lambda)|^{1-\gamma}} \int_{B_{\frac{1}{5\delta}}} \int_{\mathbb{R}^d} \frac{|f(x+h)-f(x)|^2}{|h|^d |\log(\delta|h|)|^\gamma} \, \mathrm{d}x \, \mathrm{d}h + |\log(\lambda)| \frac{\|f\|_{L^2}^2}{\log\left(2 + \frac{\|f\|_{L^2}}{\|f\|_{\dot{H}^{-1}}}\right)}. \tag{3.4}$$

Let us now describe two important corollaries of Proposition 3.5, that among other things, imply that v and ρ in Proposition 3.4 fulfill the assumptions of Theorem 3.1.

The first one provides an estimate for a scaled version of the log-Sobolev functional (1.3) applied to ρ_t (the solution obtained in Proposition 3.4) that will play a role in the proof of Theorem 3.2. The proof follows by using (3.4), (iv)Proposition 3.4 and Remark 2.20.

Corollary 3.6. *Let ρ be as in Proposition 3.4, and let $t > 0$ be fixed. For every $\gamma \in (-\infty, 1)$, $\lambda \in (0, 1/100)$ and $\gamma \in (0,1]$, it holds*

$$\int_{B_{\frac{1}{5\delta}}} \int_{\mathbb{R}^d} \frac{|\rho_t(x+h)-\rho_t(x)|^2}{|h|^d |\log(\delta|h|)|^\gamma} \, \mathrm{d}x \, \mathrm{d}h \gtrsim_{d,\gamma} \|\rho_0\|_{L^2}^2 |\log(\delta\lambda)|^{1-\gamma} \left(C_\gamma - |\log(\lambda)| \frac{C(\|\rho_0\|_{L^2})}{1+ct} \right),$$

where c is the constant in Proposition 3.4, $C_\gamma > 0$ depends only on γ and $C(\|\rho_0\|_{L^2}) > 0$ depends only on $\|\rho_0\|_{L^2}$.

The second corollary of Proposition 3.5 follows from (3.4) setting $\delta = 1$ and

$$|\log(\lambda)| = \left[\frac{\log\left(2 + \frac{\|f\|_{L^2}}{\|f\|_{\dot H^{-1}}}\right)}{\|f\|_{L^2}^2} \int_{B_{1/5}} \int_{\mathbb{R}^d} \frac{|f(x+h) - f(x)|^2}{|h|^d \log(1/|h|)^\gamma} \, \mathrm{d}x \, \mathrm{d}h \right]^{\frac{1}{2-\gamma}}.$$

Corollary 3.7. *For every parameter $\gamma \in (-\infty, 1)$ it holds*

$$\log\left(2 + \frac{\|f\|_{L^2}}{\|f\|_{\dot H^{-1}}}\right)^{1-\gamma} \|f\|_{L^2} \lesssim_{d,\gamma} \int_{B_{1/5}} \int_{\mathbb{R}^d} \frac{|f(x+h) - f(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^\gamma} \, \mathrm{d}x \, \mathrm{d}h, \tag{3.5}$$

for any $f \in L^2(\mathbb{R}^d)$.

Observe that Corollary 3.7 immediately implies Theorem 3.1. Indeed it suffices to apply (3.5) with $\gamma = 1 - p$ to ρ_t (the solution built in Proposition 3.4) and use (iv)Proposition 3.4 and Remark 2.20.

The remaining part of this section is devoted to the proof of Proposition 3.5.

Proof of Proposition 3.5. Fix $\varepsilon > 0$. Let us fix $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $\varphi = 1$ in $B_1 \setminus B_{1/2}$, $\varphi = 0$ in $\left(B_{5/4} \setminus B_{1/4}\right)^c$ and $\int_{\mathbb{R}^d} \varphi = 1$. Set $\varphi_\varepsilon(x) := \varepsilon^{-d} \varphi(x/\varepsilon)$. Thus we have

$$\begin{aligned} \|f * \varphi_\varepsilon\|_{L^2}^2 = \left\|\hat f \hat\varphi_\varepsilon\right\|_{L^2}^2 &\le \left\|\log(2 + |\cdot|)|\hat\varphi_\varepsilon(\cdot)|^2\right\|_{L^\infty} \int_{\mathbb{R}^d} \frac{|\hat f(\xi)|^2}{\log(2 + |\xi|)} \, \mathrm{d}\xi \\ &= \left\|\log(2 + \varepsilon^{-1}|\cdot|)|\hat\varphi(\cdot)|^2\right\|_{L^\infty} \int_{\mathbb{R}^d} \frac{|\hat f(\xi)|^2}{\log(2 + |\xi|)} \, \mathrm{d}\xi \\ &\lesssim_d \left|\log\left(\varepsilon \wedge \frac{1}{2}\right)\right| \int_{\mathbb{R}^d} \frac{|\hat f(\xi)|^2}{\log(2 + |\xi|)} \, \mathrm{d}\xi, \end{aligned}$$

and

$$\|f - f * \varphi_\varepsilon\|_{L^2}^2 \lesssim \int_{\varepsilon \le 4|h| \le 5\varepsilon} \int_{\mathbb{R}^d} \frac{|f(x+h) - f(x)|^2}{|h|^d} \, \mathrm{d}x \, \mathrm{d}h.$$

Therefore

$$\|f\|_{L^2}^2 \lesssim_d \int_{\varepsilon \le 4|h| \le 5\varepsilon} \int_{\mathbb{R}^d} \frac{|f(x+h) - f(x)|^2}{|h|^d} \, \mathrm{d}x \, \mathrm{d}h + \left|\log\left(\varepsilon \wedge \frac{1}{2}\right)\right| \int_{\mathbb{R}^d} \frac{|\hat f(\xi)|^2}{\log(2 + |\xi|)} \, \mathrm{d}\xi. \tag{3.6}$$

Now we integrate (3.6) with respect to the variable ε against a suitable kernel obtaining

$$\begin{aligned} \int_\lambda^{\frac{1}{10\delta}} \frac{1}{|\log(\delta\varepsilon)|^\gamma} \frac{\mathrm{d}\varepsilon}{\varepsilon} \|f\|_{L^2(\mathbb{R}^d)}^2 \lesssim_d \int_\lambda^{\frac{1}{10\delta}} \int_{\varepsilon \le 4|h| \le 5\varepsilon} \int_{\mathbb{R}^d} \frac{|f(x+h) - f(x)|^2}{|h|^d} \, \mathrm{d}x \, \mathrm{d}h \frac{1}{|\log(\delta\varepsilon)|^\gamma} \frac{\mathrm{d}\varepsilon}{\varepsilon} \\ + \int_\lambda^{\frac{1}{10\delta}} \frac{\left|\log\left(\varepsilon \wedge \frac{1}{2}\right)\right|}{|\log(\delta\varepsilon)|^\gamma} \frac{\mathrm{d}\varepsilon}{\varepsilon} \int_{\mathbb{R}^d} \frac{|\hat f(\xi)|^2}{\log(2 + |\xi|)} \, \mathrm{d}\xi. \end{aligned} \tag{3.7}$$

Starting from the elementary inequalities

$$-\frac{1}{1-\gamma} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} |\log(\delta\varepsilon)|^{1-\gamma} = \frac{1}{|\log(\delta\varepsilon)|^\gamma} \frac{1}{\varepsilon} \qquad \text{for } \varepsilon < \frac{1}{\delta},$$

and

$$-\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \frac{|\log(\varepsilon)|^2}{|\log(\delta\varepsilon)|^\gamma} = \frac{|\log(\varepsilon)|}{|\log(\delta\varepsilon)|^\gamma} \frac{1}{\varepsilon} \left(2 - \gamma \frac{|\log(\varepsilon)|}{|\log(\delta\varepsilon)|}\right) \ge \frac{|\log(\varepsilon)|}{|\log(\delta\varepsilon)|^\gamma} \frac{1}{\varepsilon} \qquad \text{for } \varepsilon < 1,$$

we deduce

$$\int_\lambda^{\frac{1}{10\delta}} \frac{1}{|\log(\delta\varepsilon)|^\gamma} \frac{\mathrm{d}\varepsilon}{\varepsilon} \simeq_\gamma |\log(\delta\lambda)|^{1-\gamma}, \tag{3.8}$$

when $\delta\lambda$ is small enough (for instance we can ask $\delta\lambda < 1/100$, that is verified under our assumptions), and

$$\int_{\lambda}^{\frac{1}{10\delta}} \frac{\left|\log\left(\varepsilon \wedge \frac{1}{2}\right)\right|}{|\log(\delta\varepsilon)|^{\gamma}} \frac{\mathrm{d}\varepsilon}{\varepsilon} \lesssim_{\gamma} |\log(\delta\lambda)|^{1-\gamma} \left(\frac{|\log(\lambda)^2}{\log(\delta\lambda)} + \left(\frac{|\log(\delta)|}{|\log(\delta\lambda)|} \right)^{1-\gamma} \right) \lesssim_{\gamma} |\log(\delta\lambda)|^{1-\gamma} |\log(\lambda)|, \tag{3.9}$$

for every $\delta > 0$. Putting (3.7), (3.8) and (3.9) together we get

$$\|f\|_{L^2}^2 \lesssim_{d,\gamma} \frac{1}{|\log(\delta\lambda)|^{1-\gamma}} \int_{B_{\frac{1}{5\delta}}} \int_{\mathbb{R}^d} \frac{|f(x+h) - f(x)|^2}{|h|^d |\log(\delta|h|)|^{\gamma}} \,\mathrm{d}x\,\mathrm{d}h + |\log(\lambda)| \int_{\mathbb{R}^d} \frac{|\hat{f}(\xi)|^2}{\log(2+|\xi|)} \,\mathrm{d}\xi.$$

In order to conclude the proof it remains to show

$$\int_{\mathbb{R}^d} \frac{|\hat{f}(\xi)|^2}{\log(2+|\xi|)} \,\mathrm{d}\xi \le \frac{2\,\|f\|_{L^2}^2}{\log(2 + \frac{\|f\|_{L^2}}{\|f\|_{\dot{H}^{-1}}})}. \tag{3.10}$$

To this end we fix a parameter $\nu > 0$, and we estimate

$$\begin{aligned}
\int_{\mathbb{R}^d} \frac{|\hat{f}(\xi)|^2}{\log(2+|\xi|)} \,\mathrm{d}\xi &= \int_{|\xi|\le\nu} \frac{|\xi|^2}{\log(2+|\xi|)} |\xi|^{-2} |\hat{f}(\xi)|^2 \,\mathrm{d}\xi + \int_{|\xi|\ge\nu} \frac{1}{\log(2+|\xi|)} |\hat{f}(\xi)|^2 \,\mathrm{d}\xi \\
&\le \frac{\nu^2}{\log(2+\nu)} \int_{|\xi|\le\nu} |\xi|^{-2} |\hat{f}(\xi)|^2 \,\mathrm{d}\xi + \frac{1}{\log(2+\nu)} \int_{|\xi|\ge\nu} |\hat{f}(\xi)|^2 \,\mathrm{d}\xi \\
&\le \frac{\nu^2}{\log(2+\nu)} \|f\|_{\dot{H}^{-1}}^2 + \frac{1}{\log(2+\nu)} \|f\|_{L^2}^2 .
\end{aligned}$$

Choosing $\nu = \frac{\|f\|_{L^2}}{\|f\|_{\dot{H}^{-1}}}$, one gets (3.10). □

3.2 Proof of Theorem 3.2

Before going into details with the proof of Theorem 3.2 we present the last technical ingredient: an approximate orthogonality property for log-Sobolev functionals (1.3) of functions with disjoint supports.

Lemma 3.8. *Let $\gamma \in (-\infty, 1)$ be fixed. For every $n \in \mathbb{N}$ consider an open set Ω_n, a function $f_n \in L^2(\mathbb{R}^d)$ and a parameter $0 < \lambda_n < 1/4$. Assume that the family $\{\,\Omega_n\,\}_{n\in\mathbb{N}}$ is disjoint and that the distance between $\operatorname{supp} f_n$ and $\mathbb{R}^d \setminus \Omega_n$ is bigger than λ_n.*

Then it holds

$$\begin{aligned}
&\int_{B_{1/3}} \int_{\mathbb{R}^d} \frac{|\sum_n f_n(x+h) - \sum_n f_n(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^{\gamma}} \,\mathrm{d}x\,\mathrm{d}h \\
&\qquad \ge \limsup_{N\to\infty} \sum_{n=1}^{N} \left(\int_{B_{1/3}} \int_{\mathbb{R}^d} \frac{|f_n(x+h) - f_n(x)|^2}{|h|^d \log(1/|h|)^{\gamma}} \,\mathrm{d}x\,\mathrm{d}h - \frac{4\,\|f_n\|_{L^2}^2}{1-\gamma} |\log(\lambda_n)|^{1-\gamma} \right).
\end{aligned} \tag{3.11}$$

Proof. Let us call $\bar{\Omega}_n \subset \Omega_n$ the set of $x \in \mathbb{R}^d$ whose distance from $\operatorname{supp} f_n$ is smaller than $\lambda_n/3$. Observe that

$$\begin{aligned}
&\int_{B_{1/3}} \int_{\mathbb{R}^d} \frac{|\sum_n f_n(x+h) - \sum_n f_n(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^{\gamma}} \,\mathrm{d}x\,\mathrm{d}h \\
&\qquad \ge \limsup_{N\to\infty} \sum_{n=1}^{N} \int_{B_{\lambda_n/3}} \int_{\bar{\Omega}_n} \frac{|f_n(x+h) - f_n(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^{\gamma}} \,\mathrm{d}x\,\mathrm{d}h \\
&\qquad = \limsup_{N\to\infty} \sum_{n=1}^{N} \Big(\int_{B_{1/3}} \int_{\mathbb{R}^d} \frac{|f_n(x+h) - f_n(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^{\gamma}} \,\mathrm{d}x\,\mathrm{d}h \\
&\qquad\qquad - \int_{B_{1/3}\setminus B_{\lambda_n/3}} \int_{\mathbb{R}^d} \frac{|f_n(x+h) - f_n(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^{\gamma}} \,\mathrm{d}x\,\mathrm{d}h \Big).
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&\int_{B_{1/3}\setminus B_{\lambda_n/3}} \int_{\mathbb{R}^d} \frac{|f_n(x+h)-f_n(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^\gamma} \,\mathrm{d}x\,\mathrm{d}h \\
&\le 2 \int_{B_{1/3}\setminus B_{\lambda_n/3}} \int_{\mathbb{R}^d} \frac{|f_n(x)|^2}{|h|^d \log(1/|h|)^\gamma} \,\mathrm{d}x\,\mathrm{d}h + 2 \int_{B_{1/3}\setminus B_{\lambda_n/3}} \int_{\mathbb{R}^d} \frac{|f_n(x+h)|^2}{|h|^d \log(1/|h|)^\gamma} \,\mathrm{d}x\,\mathrm{d}h \\
&\le 4 \|f_n\|_{L^2}^2 \int_{B_{1/3}\setminus B_{\lambda_n/3}} \frac{1}{|h|^d} \frac{1}{\log(1/|h|)^\gamma} \,\mathrm{d}h \le \frac{4\|f_n\|_{L^2}^2}{1-\gamma} |\log(\lambda_n)|^{1-\gamma}.
\end{aligned}
$$

This concludes the proof. □

Proof of Theorem 3.2. Let $p \ge 1$ be fixed. We consider v and ρ as in Proposition 3.4, and a family of disjoint open cubes $\{ Q_n \}_{n\in\mathbb{N}}$ contained in B_1. Let us define

$$
\lambda_n := e^{-n}, \qquad \gamma_n := \frac{1}{n^2}, \qquad \tau_n := \left(n^2 e^{-dn}\right)^{1/p} \qquad \text{for } n \in \mathbb{N}. \tag{3.12}
$$

Assuming that Q_n has side of length $3\lambda_n$ and center at $x_n \in B_1$, we set

$$
v_n(t,x) := \frac{\lambda_n}{\tau_n} v\left(\frac{t}{\tau_n}, \frac{x-x_n}{\lambda_n}\right), \quad \rho_n(t,x) := \gamma_n \rho\left(\frac{t}{\tau_n}, \frac{x-x_n}{\lambda_n}\right), \quad \text{for } x \in \mathbb{R}^d,\ t \ge 0,\ n \in \mathbb{N};
$$

observe that u_n is supported in Q_n and $\operatorname{dist}(\operatorname{supp} u_n, \mathbb{R}^d \setminus Q_n) \ge \lambda_n$ for every $n \in \mathbb{N}$.

Setting

$$
b(t,x) := \sum_n v_n(t,x), \qquad u(t,x) := \sum_n \rho_n(t,x) \qquad \forall x \in \mathbb{R}^d, \quad \forall t > 0,
$$

the following facts hold true:

(i) b is divergence-free, supported in $B_1 \times [0,+\infty)$ and belongs to $L^\infty([0,+\infty); W^{1,p})$;

(ii) u is bounded and supported in $B_1 \times [0,+\infty)$;

(iii) the initial data $\bar{u}$ belongs to $W^{1,d}$;

(iv) u is a solution of (CE) with vector field b.

Let us prove (i), (ii), (iii) and (iv). For any $t > 0$ we have

$$
\begin{aligned}
\|b_t\|_{W^{1,p}}^p &\le \sum_n \left(\|v_n(t,\cdot)\|_{L^p}^p + \|\nabla v_n(t,\cdot)\|_{L^p}^p\right) \\
&= \sum_n \left(\frac{\lambda_n}{\tau_n}\right)^p \left(\lambda_n^d \left\|v_{t/\tau_n}\right\|_{L^p}^p + \lambda_n^{d-p} \left\|\nabla v_{t/\tau_n}\right\|_{L^p}^p\right) \\
&\le \sup_{s\ge 0} \|v_s\|_{W^{1,p}}^p \sum_n \frac{\lambda_n^d}{\tau_n^p} = \sup_{s \ge 0} \|v_s\|_{W^{1,p}}^p \sum_n \frac{1}{n^2} < \infty,
\end{aligned}
$$

this gives the non trivial part of (i). Point (ii) follows observing that $\sup_n \gamma_n < \infty$. In order to prove (iii) we estimate

$$
\begin{aligned}
\|u_0\|_{W^{1,d}}^d &\le \sum_n \left(\|\rho_n(0,\cdot)\|_{L^d}^d + \|\nabla \rho_n(0,\cdot)\|_{L^d}^d\right) \\
&= \sum_n \gamma_n^d \left(\lambda_n^d \|\rho_0\|_{L^d}^d + \|\nabla \rho_0\|_{L^d}^d\right) \\
&\le \|\rho_0\|_{W^{1,d}}^d \sum_n \gamma_n^d < \infty.
\end{aligned}
$$

Point (iv) is an immediate consequence of the construction.

We are now ready to show (3.1). Fix $t > 0$ and $\gamma \in (-\infty, 1)$. Thanks to Lemma 3.8 and Remark 2.20 we have

$$\begin{aligned}
&\int_{B_{1/3}} \int_{\mathbb{R}^d} \frac{|u(t,x+h) - u(t,x)|^2}{|h|^d \log(1/|h|)^\gamma} \, \mathrm{d}x \, \mathrm{d}h \\
&\geq \limsup_{N\to\infty} \sum_{n=1}^N \left(\int_{B_{1/3}} \int_{\mathbb{R}^d} \frac{|\rho_n(t,x+h) - \rho_n(t,x)|^2}{|h|^d \log(1/|h|)^\gamma} \, \mathrm{d}x \, \mathrm{d}h - \frac{4\,\|\rho_n(t,\cdot)\|_{L^2}^2}{1-\gamma} |\log(\lambda_n)|^{1-\gamma} \right) \\
&= \limsup_{N\to\infty} \sum_{n=1}^N \gamma_n^2 \left(\int_{B_{1/3}} \int_{\mathbb{R}^d} \frac{|\rho\left(\frac{t}{\tau_n}, \frac{x+h-x_n}{\lambda_n}\right) - \rho\left(\frac{t}{\tau_n}, \frac{x-x_n}{\lambda_n}\right)|^2}{|h|^d \log(1/|h|)^\gamma} \, \mathrm{d}x \, \mathrm{d}h - \frac{4\left\|\rho\left(\frac{t}{\tau_n}, \frac{\cdot}{\lambda_n}\right)\right\|_{L^2}^2}{1-\gamma} |\log(\lambda_n)|^{1-\gamma} \right) \\
&= \limsup_{N\to\infty} \sum_{n=1}^N \gamma_n^2 \lambda_n^d \left(\int_{B_{\frac{1}{3\lambda_n}}} \int_{\mathbb{R}^d} \frac{|\rho\left(\frac{t}{\tau_n}, x+h\right) - \rho\left(\frac{t}{\tau_n}, x\right)|^2}{|h|^d |\log(|\lambda_n h|)|^\gamma} \, \mathrm{d}x \, \mathrm{d}h - \frac{4\,\|\rho_0\|_{L^2}^2}{1-\gamma} |\log(\lambda_n)|^{1-\gamma} \right).
\end{aligned}$$

Let us fix $n \in \mathbb{N}$ and $\lambda \in (0, 1/100)$. Applying Corollary 3.6 with parameters γ, λ, and $\delta = \lambda_n$ (we need to consider n bigger than a suitable integer n_γ depending only on γ) we get

$$\begin{aligned}
&\int_{B_{\frac{1}{3\lambda_n}}} \int_{\mathbb{R}^d} \frac{|\rho\left(\frac{t}{\tau_n}, x+h\right) - \rho\left(\frac{t}{\tau_n}, x\right)|^2}{|h|^d |\log(|\lambda_n h|)|^\gamma} \, \mathrm{d}x \, \mathrm{d}h \\
&\qquad \gtrsim_\gamma \|\rho_0\|_{L^2}^2 \, |\log(\lambda_n \lambda)|^{1-\gamma} \left(C_\gamma - |\log(\lambda)| \frac{\tau_n C(\|\rho_0\|_{L^2})}{\tau_n + ct} \right).
\end{aligned}$$

For $n \in \mathbb{N}$ big enough we take λ such that,

$$a_t \tau_n^{-1} = |\log(\lambda)| \quad \text{where} \quad a_t := \frac{C_\gamma}{2C(\|\rho_0\|_{L^2})} ct,$$

obtaining

$$|\log(\lambda_n \lambda)|^{1-\gamma} \left(C_\gamma - |\log(\lambda)| \frac{\tau_n C(\|\rho_0\|_{L^2})}{\tau_n + ct} \right) \geq (|\log(\lambda_n)| + a_t(\tau_n)^{-1})^{1-\gamma} \frac{C_\gamma}{2} \geq \bar{C} t^{1-\gamma} \tau_n^{\gamma-1},$$

for every $n \geq n_\gamma$, where $\bar{C}$ is a positive constant depending only on, c (see Proposition 3.4), $\|\rho_0\|_{L^2}$ and γ.

Putting all together and recalling (3.12) we get

$$\begin{aligned}
&\int_{B_{1/3}} \int_{\mathbb{R}^d} \frac{|u(t,x+h) - u(t,x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^\gamma} \, \mathrm{d}x \, \mathrm{d}h \\
&\geq \bar{C} t^{1-\gamma} \sum_{n=n_\gamma}^\infty \gamma_n^2 \lambda_n^d \tau_n^{\gamma-1} - \frac{4\,\|\rho_0\|_{L^2}}{1-\gamma} \sum_{n=1}^\infty \gamma_n^2 \lambda_n^d |\log(\lambda_n)|^{1-\gamma} \\
&= \bar{C} t^{1-\gamma} \sum_{n=n_\gamma}^\infty n^{\frac{2(\gamma-1)}{p} - 4} e^{-dn \frac{\gamma+p-1}{p}} - \frac{4\,\|\rho_0\|_{L^2}}{1-\gamma} \sum_{n=1}^\infty n^{-\gamma-3} e^{-dn},
\end{aligned}$$

that is equal to $+\infty$ when $\gamma < 1-p$ and $t > 0$. □

3.3 Mixing estimates

As a simple byproduct of Theorem 2.1 and Corollary 3.7 we get two mixing estimates for solutions to (CE) drifted by divergence free vector fields bounded in $W^{1,p}$, uniformly in time, for $p > 1$.

These results are already present in the literature (see [CDL08, Theorem 6.2], [IKX14], [S13], [LF16]), and it is worth recalling that their extension to the case $p = 1$ is an important open problem related to Bressan's mixing conjecture (see [B03]).

Lemma 3.9. *Let us fix $p > 0$ and $f \in L^\infty(\mathbb{R}^d)$ satisfying $\|f\|_{L^\infty} = 1$ and $\mathscr{L}^d(\{|f| = 1\}) \geq c_0 > 0$. Then, for any $\kappa \in (0,1)$ and $\varepsilon \in (0, 1/3)$, it holds*

$$\sup_{x\in\mathbb{R}^d} \left| \fint_{B_\varepsilon} f(x+y)\,\mathrm{d}y \right| < \kappa \implies \varepsilon \geq \exp\left\{ -C \left(\sup_{h\in B_{1/3}} \log(1/|h|)^p \int_{\mathbb{R}^d} |f(x+h) - f(x)|\,\mathrm{d}x \right)^{1/p} \right\}, \tag{3.13}$$

where $C = (c_0(1-\kappa))^{-1/p}$.

Proof. For any $x \in \{|f| = 1\}$, we have $1 - \kappa < \left| \fint_{B_\varepsilon} (f(x) - f(x+h))\,\mathrm{d}h \right|$. Therefore

$$\begin{aligned} c_0(1-\kappa) &< \fint_{B_\varepsilon} \int_{\mathbb{R}^d} |f(x) - f(x+h)|\,\mathrm{d}x\,\mathrm{d}h \\ &\leq \left(\sup_{h\in B_{1/3}} \log(1/|h|)^p \int_{\mathbb{R}^d} |f(x) - f(x+h)|dx \right) \fint_{B_\varepsilon} \log(1/|h|)^{-p}\,\mathrm{d}h \\ &\leq \left(\sup_{h\in B_{1/3}} \log(1/|h|)^p \int_{\mathbb{R}^d} |f(x) - f(x+h)|\,\mathrm{d}x \right) |\log(\varepsilon)|^{-p}, \end{aligned}$$

this implies (3.13). The proof is complete. □

We are now ready to state and prove the aforementioned mixing estimates.

Proposition 3.10. *Let $p > 1$ be fixed. Let us consider a bounded divergence-free vector field b such that*

$$\|\nabla b_t\|_{L^p} \leq B < \infty \qquad \text{for a.e. } t \geq 0.$$

Then for any $u_0 \in BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ the solution $u \in L^\infty([0,+\infty) \times \mathbb{R}^d)$ to (CE) *satisfies*

$$\|u_t\|_{\dot{H}^{-1}} \geq C \exp(-cBt), \qquad \text{for any } t \geq 0, \tag{3.14}$$

where $C > 0$ and $c > 0$ depend only on $\|u_0\|_{L^2}$, $\|u_0\|_{BV}$, p and d.

Furthermore, if we assume $u_0(x) \in \{1, -1, 0\}$ for every $x \in \mathbb{R}^d$ and $\int_{\mathbb{R}^d} |u_0|\,\mathrm{d}x \geq c_0 > 0$, then for any $\kappa \in (0,1)$, and $\varepsilon \in (0, 1/3)$ it holds

$$\sup_{x\in\mathbb{R}^d} \left| \fint_{B_\varepsilon(x)} u_t(y)\,\mathrm{d}y \right| < \kappa \Longrightarrow \varepsilon \geq \exp(-CBt), \tag{3.15}$$

where $C > 0$ depends only on p, d, κ, c_0 and $\|u_0\|_{BV}$.

Proof. The first part of the statement follows by applying Corollary 3.7 with $\gamma = 1 - p$ and Theorem 2.1. The second part is a consequence of Lemma 3.9, Remark 2.2 and the following elementary observation. If f is a measurable function that takes only the values 1, 0 and -1 then

$$\int_{\mathbb{R}^d} |f(x+h) - f(x)|\,\mathrm{d}x \leq \int_{\mathbb{R}^d} |f(x+h) - f(x)|^2\,\mathrm{d}x \qquad \text{for every } h \in \mathbb{R}^d.$$

□

Two remarks are in order.

Remark 3.11. The mixing estimate (3.14) holds true also using the seminorm $\dot{H}^{-s}$ with $s > 0$ and reads

$$\|u_t\|_{\dot{H}^{-s}} \geq C \exp(-cBst) \qquad \text{for any } t \geq 0.$$

It can be proved by modifying the interpolation inequality (3.4).

Remark 3.12. Let us assume b_t to be smooth and compactly supported in $Q = [0,1]^d \subset \mathbb{R}^d$ and let us call X_t its flow. Consider the initial data $u_0 := \mathbf{1}_A - \mathbf{1}_{Q\setminus A} \in BV(\mathbb{R}^d)$ with $A \subset [0,1]^d$ satisfying $\mathscr{L}^d(A) = \frac{1}{2}$. It is immediate to see that $u_t := \mathbf{1}_{A_t} - \mathbf{1}_{Q\setminus A_t}$ where $A_t = X_t(A)$. Setting $\kappa = 1/2$, fixing $p > 1$ and using (3.15) we get

$$\frac{1}{4} \leq \frac{\mathscr{L}^d(B_\varepsilon(x) \cap A_t)}{\omega_d \varepsilon^d} \leq \frac{3}{4} \implies \int_0^t \|\nabla b_s\|_{L^p} \, \mathrm{d}s \geq C|\log(\varepsilon)|, \tag{3.16}$$

where C depends on p, d and $\|u_0\|_{BV}$. The implication (3.16) is the statement of Bressan's mixing conjecture for $p > 1$ (see [B03]) that has been proved for a first time in [CDL08].

Let us conclude the paper with an open question.

Open Question 3.13. Let $b \in L^\infty([0,+\infty); W^{1,1}(\mathbb{R}^d, \mathbb{R}^d))$ be a divergence free vector field with compact support. Fix an initial data $u_0 \in C_c^\infty(\mathbb{R}^d)$ and consider te solution $u \in L^\infty([0,+\infty) \times \mathbb{R}^d)$ to (CE). Is there an increasing function $\psi : (0,+\infty) \to (0,+\infty)$ with $\lim_{s\to\infty} \psi(s) = \infty$ and $\psi^{-1}(2s) \leq C\psi^{-1}(s)$ such that

$$\sup_{h \in B_{1/3}} \psi\left(\log(1/|h|)\right) \int_{\mathbb{R}^d} |u_t(x+h) - u_t(x)|^2 \, \mathrm{d}x \leq C\psi\left(t\right),$$

for every $t \geq 0$?

A positive answer to Open Question 3.13, along with the proof of Lemma 3.9, would give an exponential bound on mixing in the case $p = 1$, providing a positive answer to Bressan's mixing conjecture [B03].

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