QUASI-MONOTONICITY FORMULAS FOR CLASSICAL OBSTACLE PROBLEMS WITH SOBOLEV COEFFICIENTS AND APPLICATIONS

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Abstract. We establish Weiss' and Monneau's type quasi-monotonicity formulas for quadratic energies having matrix of coefficients in a Sobolev space $W^{1,p}$, $p > n$, and provide an application to the corresponding free boundary analysis for the related classical obstacle problems.

1. Introduction

The aim of this short note is to extend the range of validity of Weiss' and Monneau's type quasi-monotonicity formulas to classical obstacle problems involving quadratic forms having matrix of coefficients in a Sobolev space $W^{1,p}$, with $p > n$. Such results are instrumental to pursue the variational approach for the analysis of the corresponding free boundaries in classical obstacle problems. More precisely, we consider the functional $\mathscr{E} : W^{1,2}(\Omega) \to \mathbb{R}$ given by

$$\mathcal{E}(v) := \int_\Omega \big(\langle \mathbb{A}(x)\nabla v(x), \nabla v(x)\rangle + 2h(x)v(x)\big)\, dx, \tag{1.1}$$

and study regularity issues related to its unique minimizer w on the set

$$\mathcal{K}_{\psi,g} := \left\{ v \in W^{1,2}(\Omega) : \, v \geq \psi \;\; \mathcal{L}^n \text{ a.e. on } \Omega, \, \mathrm{Tr}(v) = g \text{ on } \partial\Omega \right\}.$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded open set, $n \geq 2$, $\psi \in C^{1,1}_{loc}(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$, are such that $\psi \leq g$ $\mathcal{H}^{n-1}$-a.e on $\partial\Omega$, $\mathbb{A} : \Omega \to \mathbb{R}^{n\times n}$ is a matrix-valued field and $f : \Omega \to \mathbb{R}$ is a function satisfying:

(H1) $\mathbb{A} \in W^{1,p}(\Omega; \mathbb{R}^{n\times n})$ with $p > n$;

(H2) $\mathbb{A}(x) = (a_{ij}(x))_{i,j=1,\dots,n}$ symmetric, continuous and coercive, that is $a_{ij} = a_{ji}$ in Ω for all i, $j \in \{1, \dots, n\}$, and for some $\Lambda \geq 1$

$$\Lambda^{-1}|\xi|^2 \leq \langle \mathbb{A}(x)\xi, \xi\rangle \leq \Lambda|\xi|^2 \tag{1.2}$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^n$;

(H3) $f := h - \mathrm{div}\,(\mathbb{A}\nabla\psi) > c_0 \; \mathcal{L}^n$ a.e. Ω, for some $c_0 > 0$, and f is Dini-continuous, namely

$$\int_0^1 \frac{\omega_f(t)}{t}\, dt < \infty, \tag{1.3}$$

where $\omega_f(t) := \sup_{x,y\in\Omega,\, |x-y|\leq t} |f(x) - f(y)|$.

In some instances in place of (H3) we will require the stronger condition

(H4) $f > c_0 \; \mathcal{L}^n$ a.e. Ω, for some $c_0 > 0$, and f is double-Dini continuous, that is

$$\int_0^1 \frac{\omega_f(r)}{r}\, |\log r|^a\, dr < \infty, \tag{1.4}$$

for some $a \geq 1$.

Note that for the zero obstacle problem, i.e. $\psi = 0$, assumptions (H3) and (H4) involve only the lower order term h in the integrand and not the matrix field $\mathbb{A}$.

Given the assumptions introduced above we provide a full free boundary stratification result.

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Theorem 1.1. *Assume (H1)-(H4) to hold, and let w be the (unique) minimizer of $\mathcal{E}$ in (1.1) on $\mathcal{K}_{\psi,g}$.*

Then, w is $W^{2,p}_{loc}\cap C^{1,1-n/p}_{loc}(\Omega)$, and the free boundary decomposes as $\partial\{w=\psi\}\cap\Omega=\mathrm{Reg}(w)\cup\mathrm{Sing}(w)$, where $\mathrm{Reg}(w)$ and $\mathrm{Sing}(w)$ are called its regular and singular part, respectively. Moreover, $\mathrm{Reg}(w)\cap\mathrm{Sing}(w)=\emptyset$ and

(i) *if $a>2$ in (H4), then $\mathrm{Reg}(w)$ is relatively open in $\partial\{w=\psi\}$ and, for every point $x_0\in\mathrm{Reg}(w)$, there exist $r=r(x_0)>0$ such that $\partial\{w=\psi\}\cap B_r(x)$ is a C^1 $(n-1)$-dimensional manifold with normal vector absolutely continuous.*

   In particular if f is Hölder continuous there exists $r>0$ such that $\partial\{w=\psi\}\cap B_r(x)$ is a $C^{1,\beta}$ $(n-1)$-dimensional manifold for some exponent $\beta\in(0,1)$.

(ii) *if $a\geq 1$ in (H4), then $\mathrm{Sing}(w)=\cup_{k=0}^{n-1}S_k$, with S_k contained in the union of at most countably many submanifolds of dimension k and class C^1.*

Theorem 1.1 has been proved by Caffarelli for suitably regular matrix fields, and it is the resume of his long term program on the subject (cf. for instance [2, 3, 4, 5] and the books [15, 6, 19] for more details and references also on related problems). Let us also remark that very recently the fine structure of the set of singular points for the Dirichlet energy has been unveiled in the papers by Colombo, Spolaor and Velichkov [7] and Figalli and Serra [8] by means of a logarithmic epiperimetric inequality and new monotonicity formulas, respectively.

In the last years Theorem 1.1 has been extended to the case in which $\mathbb{A}$ either is Lipschitz continuous in [9] or belongs to a fractional Sobolev space $W^{1+s,p}$ in [11], with s, p and n suitably related, and also in some nonlinear cases [10]. The last papers follow the variational approach to free boundary analysis developed remarkably by Weiss [21] and by Monneau [18]. The extensions of Weiss' and Monneau's quasi-monotonicity formulas obtained in the papers [9, 11] hinge upon a generalization of the Rellich and Nečas' inequality due to Payne and Weinberger (cf. [16]). On a technical side they involve the derivation of the matrix field $\mathbb{A}$. The main difference contained in the present note with respect to the papers [9, 11] concerns the monotone quantity itself. Indeed, rather than considering the natural quadratic energy associated to the obstacle problem under study, we establish quasi-monotonicity for a related constant coefficient quadratic form. The latter result is obtained thanks to a freezing argument inspired by some computations of Monneau (cf. [18, Section 6]) in combination with the well-known quadratic lower bound on the growth of solutions from free boundary points (see Section 3 for more details). Such an insight, though elementary, has been overlooked in the literature and enables us to obtain Weiss' and Monneau's quasi-monotonicity formulas under the milder assumptions (H1) and (H3) (the latter having no role if $\psi=0$), since the matrix field $\mathbb{A}$ is not differentiated along the derivation process of the quasi-monotonicity formulas.

To conclude this introduction we briefly resume the structure of the paper: standard preliminaries for the classical obstacle problem are collected in Section 2. The mentioned generalizations of Weiss' and Monneau's quasi-monotonicity formulas are dealt with in Section 3, finally Section 4 contains the applications to the free boundary stratification for quadratic problems.

2. Preliminaries

Throughout the section we use the notation introduced in Section 1 and adopt Einstein' summation convention.

We first reduce ourselves to the zero obstacle problem. Let w be the unique minimizer of $\mathcal{E}$ over $\mathcal{K}_{\psi,g}$, and define $u:=w-\psi$. Then, u is the unique minimizer of

$$\mathscr{E}(v):=\int_\Omega\big(\langle\mathbb{A}(x)\nabla v(x),\nabla v(x)\rangle+2f(x)v(x)\big)\,dx, \tag{2.1}$$

over

$$\mathbb{K}_{\psi,g}:=\left\{v\in W^{1,2}(\Omega):\,v\geq 0\;\;\mathcal{L}^n\text{ a.e. on }\Omega,\,\mathrm{Tr}(v)=g-\psi\text{ on }\partial\Omega\right\},$$

where $f=h-\mathrm{div}\,(\mathbb{A}\nabla\psi)$. Clearly, $\partial\{w=\psi\}\cap\Omega=\partial\{u=0\}\cap\Omega$, therefore we shall establish all the results in Theorem 1.1 for u (notice that assumptions (H3) and (H4) are formulated exactly in terms of f).

Note that u satisfies a PDE both in the distributional sense and a.e. on Ω, elliptic regularity then applies to u itself to establish its smoothness. The next result had been established by Ural'tseva in [20] with a different proof.

Proposition 2.1. *Let u be the minimum of $\mathscr{E}$ on $\mathbb{K}_{\psi,g}$. Then*

$$\mathrm{div}(\mathbb{A}\nabla u) = f\chi_{\{u>0\}} \tag{2.2}$$

$\mathcal{L}^n$-a.e. on Ω and in $\mathcal{D}'(\Omega)$. Moreover, $u \in W^{2,p}_{loc} \cap C^{1,1-\frac{n}{p}}_{loc}(\Omega)$.

Proof. For the validity of (2.2) we refer to [10, Proposition 3.2] where the result is proven in the broader context of variational inequalities (see also [9, Proposition 2.2]).

From this, by taking into account that $\mathbb{A} \in C^{0,1-n/p}_{loc}(\Omega, \mathbb{R}^{n\times n})$ in view of Morrey embedding theorem, Schauder estimates then yields $u \in C^{1,1-n/p}_{loc}(\Omega)$ (cf. [14, Theorem 3.13]).

Next consider the equation

$$a_{ij}\,\frac{\partial^2 v}{\partial x_i \partial x_j} = f\chi_{\{u>0\}} - \mathrm{div}\mathbb{A}^j\,\frac{\partial u}{\partial x_j} =: \varphi, \tag{2.3}$$

where $\mathbb{A}^j$ denotes the j-column of $\mathbb{A}$. Since $\nabla u \in L^\infty_{loc}(\Omega, \mathbb{R}^n)$ and $\mathrm{div}\,\mathbb{A}^j \in L^p(\Omega)$ for all $j \in \{1,\dots,n\}$, then $\varphi \in L^p_{loc}(\Omega)$. [13, Corollary 9.18] implies the uniqueness of a solution $v \in W^{2,p}_{loc}(\Omega)$ to (2.3). By taking into account the identity $\mathrm{Tr}(\mathbb{A}\nabla^2 v) = \mathrm{div}(\mathbb{A}\nabla v) - \mathrm{div}\mathbb{A}^j\,\frac{\partial v}{\partial x_j}$, (2.3) rewrites as

$$\mathrm{div}(\mathbb{A}\nabla v) - \mathrm{div}\mathbb{A}^j\,\frac{\partial v}{\partial x_j} = \varphi, \tag{2.4}$$

we have that u and v are two solutions. Then by [17, Theorem 1.I] we obtain $u = v$. □

We recall next the standard notations for the coincidence set and for the free boundary

$$\Lambda_u = \{x \in \Omega :\ u(x) = 0\}, \qquad \Gamma_u = \partial\Lambda_u \cap \Omega. \tag{2.5}$$

For any point $x_0 \in \Gamma_u$, we introduce the family of rescaled functions

$$u_{x_0,r}(x) := \frac{u(x_0 + rx)}{r^2} \tag{2.6}$$

for $x \in \frac{1}{r}(\Omega - \{x_0\})$. The existence of $C^{1,\alpha}$-limits as $r \downarrow 0$ of the latter family is standard by noting that the rescaled functions satisfy an appropriate PDE and then uniform $W^{2,p}$ estimates.

Proposition 2.2 ([11, Proposition 4.1])**.** *Let u be the unique minimizer of $\mathscr{E}$ over $\mathbb{K}_{\psi,g}$, and $K \subset \Omega$ a compact set. Then for every $x_0 \in K \cap \Gamma_u$, for every $R > 0$ there exists a constant $C = C(n, p, \Lambda, R, K, \|f\|_{L^\infty}, \|\mathbb{A}\|_{W^{1,p}}) > 0$ such that, for every $r \in (0, \frac{1}{4R}\mathrm{dist}(K, \partial\Omega))$*

$$\|u_{x_0,r}\|_{W^{2,p}(B_R)} \le C. \tag{2.7}$$

In particular, $(u_{x_0,r})_r$ is equibounded in $C^{1,\gamma}_{\mathrm{loc}}$ for $\gamma \in (0, 1 - n/p]$.

The functions arising in this limit process are called blow-up limits.

Corollary 2.3 (Existence of blow-ups)**.** *Let u be the unique minimizer of $\mathscr{E}$ over $\mathbb{K}_{\psi,g}$, and let $x_0 \in \Gamma_u$. Then, for every sequence $r_k \downarrow 0$ there exists a subsequence $(r_{k_j})_j \subset (r_k)_k$ such that the rescaled functions $(u_{x_0,r_{k_j}})_j$ converge in $C^{1,\gamma}_{\mathrm{loc}}$, $\gamma \in (0, 1 - n/p)$.*

Elementary growth conditions of the solution from free boundary points are easily deduced from Proposition 2.2 and the condition $p > n$. In turn, such properties will be crucial in the derivation of the quasi-monotonicity formulas.

Proposition 2.4. *Let u be the unique minimizer of $\mathscr{E}$ over $\mathbb{K}_{\psi,g}$. Then for all compact sets $K \subset \Omega$ there exists a constant $C = C(n, p, \Lambda, K, \|f\|_{L^\infty}, \|\mathbb{A}\|_{W^{1,p}}) > 0$ such that for all points $x_0 \in \Gamma_u \cap K$, and for all $r \in \big(0, \frac{1}{2}\mathrm{dist}(K, \partial\Omega)\big)$ it holds*

$$\|u\|_{L^\infty(B_r(x_0))} \le C\,r^2, \qquad \|\nabla u\|_{L^\infty(B_r(x_0),\mathbb{R}^n)} \le C\,r. \tag{2.8}$$

and

$$\|\nabla^2 u\|_{L^p(B_r(x_0),\mathbb{R}^{n\times n})} \le C\,r^{n/p}. \tag{2.9}$$

Finally, we recall the fundamental quadratic detachment property from free boundary points that entails non triviality of blow up limits. It has been established by Blank and Hao in [1, Theorem 3.9] under the sole VMO regularity of $\mathbb{A}$, an assumption weaker than (H1).

Lemma 2.5. *There exists a constant $\vartheta = \vartheta(n, \Lambda, c_0, \|f\|_{L^\infty}) > 0$ such that, for every $x_0 \in \Gamma_u$ and $r \in (0, \frac{1}{2}\mathrm{dist}(x_0, \partial\Omega))$, it holds*

$$\sup_{x \in \partial B_r(x_0)} u(x) \geq \vartheta\, r^2.$$

3. Quasi-monotonicity formulas

In this section we establish Weiss' and Monneau's type quasi-monotonicity formulas for the quadratic problem. As pointed out in Section 1 the main difference with the existing literature concerns the monotone quantity itself. Indeed, rather than considering the natural quadratic energy $\mathscr{E}$ associated to the obstacle problem under study, we may consider the classical Dirichlet energy thanks to a normalization. In doing this we have been inspired by Monneau [18, Section 6]. The advantage of this formulation is that the matrix field $\mathbb{A}$ is not differentiated in deriving the quasi-monotonicity formulas contrary to [10, 11]. Our additional insight is elementary but crucial: we further exploit the quadratic growth of solutions from free boundary points in Proposition 2.4 to establish quasi-monotonicity.

Let $x_0 \in \Gamma_u$ be any point of the free boundary, then the affine change of variables

$$x \mapsto x_0 + f^{-1/2}(x_0)\mathbb{A}^{1/2}(x_0)x =: x_0 + \mathbb{L}(x_0)\, x$$

leads to

$$\mathscr{E}(u) = f^{1-\frac{n}{2}}(x_0)\det(\mathbb{A}^{1/2}(x_0))\, \mathscr{E}_{\mathbb{L}(x_0)}(u_{\mathbb{L}(x_0)}), \tag{3.1}$$

where $\Omega_{\mathbb{L}(x_0)} := \mathbb{L}^{-1}(x_0)\,(\Omega - x_0)$, and we have set

$$\mathscr{E}_{\mathbb{L}(x_0)}(v) := \int_{\Omega_{\mathbb{L}(x_0)}} \Big(\langle \mathbb{C}_{x_0}\nabla v, \nabla v\rangle + 2\frac{f_{\mathbb{L}(x_0)}}{f(x_0)}\, v\Big)\, dx, \tag{3.2}$$

with

$$\begin{gathered} u_{\mathbb{L}(x_0)}(x) := u\big(x_0 + \mathbb{L}(x_0)x\big), \\ f_{\mathbb{L}(x_0)}(x) := f\big(x_0 + \mathbb{L}(x_0)x\big), \\ \mathbb{C}_{x_0}(x) := \mathbb{A}^{-1/2}(x_0)\mathbb{A}(x_0 + \mathbb{L}(x_0)x)\mathbb{A}^{-1/2}(x_0). \end{gathered} \tag{3.3}$$

Note that $f_{\mathbb{L}(x_0)}(\underline{0}) = f(x_0)$ and $\mathbb{C}_{x_0}(\underline{0}) = \mathrm{Id}$. Moreover, the free boundary is transformed under this map into

$$\Gamma_{u_{\mathbb{L}(x_0)}} = \mathbb{L}^{-1}(x_0)(\Gamma_u - x_0),$$

and the energy $\mathscr{E}$ in (1.1) is minimized by u if and only if $\mathscr{E}_{\mathbb{L}(x_0)}$ in (3.2) is minimized by the function $u_{\mathbb{L}(x_0)}$ in (3.3).

In addition, rewriting the Euler-Lagrange equation for $u_{\mathbb{L}(x_0)}$ in non-divergence form we get on $\Omega_{\mathbb{L}(x_0)}$:

$$c_{ij}(x)\,\frac{\partial^2 u_{\mathbb{L}(x_0)}}{\partial x_i \partial x_j} + \mathrm{div}\mathbb{A}^i(x)\,\frac{\partial u_{\mathbb{L}(x_0)}}{\partial x_i} = \frac{f_{\mathbb{L}(x_0)}(x)}{f(x_0)}\chi_{\{u_{\mathbb{L}(x_0)}>0\}}\,.$$

(using again Einstein's convention) with $\mathbb{C}_{x_0} = (c_{ij})_{i,j=1,\dots,n}$. Moreover, we may further rewrite the latter equation on $\Omega_{\mathbb{L}(x_0)}$ as

$$\begin{aligned} \Delta u_{\mathbb{L}(x_0)} &= 1 + \Big(\frac{f_{\mathbb{L}(x_0)}(x)}{f(x_0)}\chi_{\{u_{\mathbb{L}(x_0)}>0\}} - 1 - \big(c_{ij}(x) - \delta_{ij}\big)\frac{\partial^2 u_{\mathbb{L}(x_0)}}{\partial x_i \partial x_j} - \mathrm{div}\,\mathbb{C}^i_{x_0}(x)\,\frac{\partial u_{\mathbb{L}(x_0)}}{\partial x_i}\Big) \\ &=: 1 + f_{x_0}(x)\,. \end{aligned} \tag{3.4}$$

We are now ready to establish Weiss' and Monneau's quasi-monotonicity formulas for u by using equality (3.4) and Proposition 2.4.

3.1. **Weiss' quasi-monotonicity formula.** In this section we consider the Weiss' energy

$$\Phi_u(x_0,r) := \frac{1}{r^{n+2}}\int_{B_r}\big(|\nabla u_{\mathbb{L}(x_0)}|^2 + 2\,u_{\mathbb{L}(x_0)}\big)\,dx - \frac{2}{r^{n+3}}\int_{\partial B_r} u^2_{\mathbb{L}(x_0)}\,d\mathcal{H}^{n-1}\,, \tag{3.5}$$

$x_0\in\Gamma_u$, and prove its quasi-monotonicity.

Theorem 3.1 (Weiss' quasi-monotonicity formula)**.** *Under assumptions (H1)-(H3), if $K\subset\Omega$ is a compact set, there is a constant $C = C(n,p,\Lambda,c_0,K,\|f\|_{L^\infty},\|\mathbb{A}\|_{W^{1,p}}) > 0$ such that for all $x_0\in K\cap\Gamma_u$*

$$\frac{d}{dr}\Big(\Phi_u(x_0,r) + C\int_0^r \frac{\omega(t)}{t}\,dt\Big) \geq \frac{2}{r^{n+4}}\int_{\partial B_r}(\langle\nabla u_{\mathbb{L}(x_0)},x\rangle - 2u_{\mathbb{L}(x_0)})^2 d\mathcal{H}^{n-1}, \tag{3.6}$$

for $\mathcal{L}^1$ a.e. $r\in(0,\frac{1}{2}\mathrm{dist}(K,\partial\Omega))$, where $\omega(r) := \omega_f(r) + r^{1-\frac{n}{p}}$.

In particular, $\Phi_u(x_0,\cdot)$ has finite right limit $\Phi_u(x_0,0^+)$ in zero, and for all $r\in(0,\frac{1}{2}\mathrm{dist}(K,\partial\Omega))$,

$$\Phi_u(x_0,r) - \Phi_u(x_0,0^+) \geq -C\int_0^r\frac{\omega(t)}{t}\,dt. \tag{3.7}$$

Proof of Theorem 3.1. We analyse separately the volume and the boundary terms appearing in the definition of the Weiss energy in (3.5). For the sake of notational simplicity we write u_{x_0} in place of $u_{\mathbb{L}(x_0)}$. In what follows $C = C(n,p,\Lambda,c_0,K,\|f\|_{L^\infty},\|\mathbb{A}\|_{W^{1,p}}) > 0$ denotes a constant that may vary from line to line.

We start off with the bulk term. The Coarea Formula implies for $\mathcal{L}^1$-a.e. $r\in(0,\mathrm{dist}(K,\partial\Omega))$

$$\begin{aligned}\frac{d}{dr}\Big(\frac{1}{r^{n+2}}&\int_{B_r}\big(|\nabla u_{x_0}|^2 + 2\,u_{x_0}\big)\,dx\Big) = \\ &-\frac{n+2}{r^{n+3}}\int_{B_r}\big(|\nabla u_{x_0}|^2 + 2\,u_{x_0}\big)\,dx + \frac{1}{r^{n+2}}\int_{\partial B_r}\big(|\nabla u_{x_0}|^2 + 2\,u_{x_0}\big)\,dx.\end{aligned} \tag{3.8}$$

We use the Divergence Theorem together with the following identities

$$|\nabla u_{x_0}|^2 = \frac{1}{2}\mathrm{div}\,(\nabla(u^2_{x_0})) - u_{x_0}\,\Delta u_{x_0}\,,$$

$$\mathrm{div}\,\Big(|\nabla u_{x_0}|^2\,\frac{x}{r}\Big) = \frac{n-2}{r}|\nabla u_{x_0}|^2 - 2\Delta u_{x_0}\langle\nabla u_{x_0},\frac{x}{r}\rangle + 2\,\mathrm{div}\,\Big(\langle\nabla u_{x_0},\frac{x}{r}\rangle\nabla u_{x_0}\Big)\,,$$

$$\mathrm{div}\,\Big(u_{x_0}\,\frac{x}{r}\Big) = u_{x_0}\,\frac{n}{r} + \langle\nabla u_{x_0},\frac{x}{r}\rangle,$$

to deal with the first, third and fourth addend in (3.8), respectively. Hence, we can rewrite the right hand side of equality (3.8) as follows

$$\begin{aligned}\frac{d}{dr}\Big(\frac{1}{r^{n+2}}\int_{B_r}\big(|\nabla u_{x_0}|^2 + 2\,u_{x_0}\big)\,dx\Big) &= \frac{2}{r^{n+2}}\int_{B_r}(\Delta u_{x_0}-1)\Big(2\,\frac{u_{x_0}}{r} - \langle\nabla u_{x_0},\frac{x}{r}\rangle\Big)\,dx \\ + \frac{2}{r^{n+2}}\int_{\partial B_r}\langle\nabla u_{x_0},\frac{x}{r}\rangle^2 d\mathcal{H}^{n-1} &- \frac{4}{r^{n+2}}\int_{\partial B_r}\frac{u_{x_0}}{r}\,\langle\nabla u_{x_0},\frac{x}{r}\rangle d\mathcal{H}^{n-1}\,.\end{aligned} \tag{3.9}$$

We consider next the boundary term in the expression of Φ_u. By scaling and a direct calculation we get

$$\begin{aligned}\frac{d}{dr}\Big(\frac{2}{r^{n+3}}\int_{\partial B_r}u^2_{x_0}\,d\mathcal{H}^{n-1}\Big) &\overset{x=ry}{=} 2\int_{\partial B_1}\frac{d}{dr}\left(\frac{u_{x_0}(ry)}{r^2}\right)^2 d\mathcal{H}^{n-1} \\ &= 4\int_{\partial B_1}\frac{u_{x_0}(ry)}{r^4}\left(\langle\nabla u_{x_0}(ry),y\rangle - 2\frac{u_{x_0}(ry)}{r}\right)d\mathcal{H}^{n-1} \\ &\overset{x=ry}{=} \frac{4}{r^{n+2}}\int_{\partial B_r}\frac{u_{x_0}}{r}\,\langle\nabla u_{x_0},\frac{x}{r}\rangle\,d\mathcal{H}^{n-1} - \frac{8}{r^{n+2}}\int_{\partial B_r}\frac{u^2_{x_0}}{r^2}\,d\mathcal{H}^{n-1}\,.\end{aligned} \tag{3.10}$$

Then by combining together the equations (3.9) and (3.10) and recalling equation (3.4) we obtain

$$\begin{aligned}\Phi_u'(x_0,r) &= \frac{2}{r^{n+2}}\int_{B_r} f_{x_0}\,\Big(2\,\frac{u_{x_0}}{r}-\langle\nabla u_{x_0},\frac{x}{r}\rangle\Big)\,dx+\frac{2}{r^{n+2}}\int_{\partial B_r}\Big(\langle\nabla u_{x_0},\frac{x}{r}\rangle-2\,\frac{u_{x_0}}{r}\Big)^2\,d\mathcal{H}^{n-1}\\ &= \frac{2}{r^{n+2}}\int_{B_r\setminus\Lambda_{u_{x_0}}} f_{x_0}\,\Big(2\,\frac{u_{x_0}}{r}-\langle\nabla u_{x_0},\frac{x}{r}\rangle\Big)\,dx+\frac{2}{r^{n+2}}\int_{\partial B_r}\Big(\langle\nabla u_{x_0},\frac{x}{r}\rangle-2\,\frac{u_{x_0}}{r}\Big)^2\,d\mathcal{H}^{n-1},\end{aligned}$$

where in the last equality we used the unilateral obstacle condition to deduce that $\Lambda_{u_{x_0}}\subseteq\{\nabla u_{x_0}=\underline{0}\}$. Therefore, by the growth of u and ∇u from x_0 in (2.8) we obtain

$$\Phi_u'(x_0,r)\geq-\frac{C}{r^{n+1}}\int_{B_r\setminus\Lambda_{u_{x_0}}}|f_{x_0}|\,dx+\frac{2}{r^{n+2}}\int_{\partial B_r}\Big(\langle\nabla u_{x_0},\frac{x}{r}\rangle-2\,\frac{u_{x_0}}{r}\Big)^2\,d\mathcal{H}^{n-1}\,. \tag{3.11}$$

Next note that by (H1), (H3), and by the very definition of f_{x_0} in (3.4) it follows that

$$\frac{1}{r^{n+1}}\int_{B_r\setminus\Lambda_{u_{x_0}}}|f_{x_0}|\,dx\leq\frac{\omega_f(r)}{c_0\,r}+\frac{C}{r^{n(1+\frac{1}{p})}}\int_{B_r}|\nabla^2u_{x_0}|\,dx+\frac{C}{r^n}\int_{B_r}|\mathrm{div}\,\mathbb{C}_{x_0}|\,dx\,. \tag{3.12}$$

By (2.9) we estimate the second addend on the right hand side of the last inequality as follows

$$\frac{1}{r^{n(1+\frac{1}{p})}}\int_{B_r}|\nabla^2u_{x_0}|\,dx\leq\frac{C}{r^{n(1+\frac{1}{p})}}\|\nabla^2u_{x_0}\|_{L^p(B_r,\mathbb{R}^{n\times n})}(\omega_nr^n)^{1-\frac{1}{p}}\leq C\,r^{-\frac{n}{p}}\,, \tag{3.13}$$

by Hölder inequality we get for the third addend

$$\frac{1}{r^n}\int_{B_r}|\mathrm{div}\,\mathbb{C}_{x_0}|\,dx\leq\frac{1}{r^n}\|\mathrm{div}\,\mathbb{C}_{x_0}\|_{L^p(B_r,\mathbb{R}^n)}\,(\omega_nr^n)^{1-\frac{1}{p}}\leq C\,r^{-\frac{n}{p}}\,. \tag{3.14}$$

Therefore, we conclude from (3.11)-(3.14)

$$\Phi_u'(x_0,r)\geq-C\,\frac{\omega(r)}{r}+\frac{2}{r^{n+2}}\int_{\partial B_r}\Big(\langle\nabla u_{x_0},\frac{x}{r}\rangle-2\,\frac{u_{x_0}}{r}\Big)^2\,d\mathcal{H}^{n-1}\,,$$

where $\omega(r):=\omega_f(r)+r^{1-\frac{n}{p}}$. □

Remark 3.2. Recalling that f is Dini-continuous by (H3), the modulus of continuity ω provided by Theorem 3.1 is in turn Dini-continuous.

Remark 3.3. More generally, the argument in Theorem 3.1 works for solutions to second order elliptic PDEs in nondivergence form of the type

$$a_{ij}(x)\,u_{ij}+b_i(x)\,u_i+c(x)\,u=f(x)\chi_{\{u>0\}}\,,$$

the only difference with the statement of Theorem 3.1 being that in this framework $\omega(r):=\omega_f(r)+r^{1-\frac{n}{p}}+r^2\sup_{B_r}c$.

3.2. **Monneau's quasi-monotonicity formula.** Let v be a positive 2-homogeneus polynomial solution of

$$\Delta v=1\quad\text{on }\mathbb{R}^n. \tag{3.15}$$

Then by 2-homogeneity, elementary calculations lead to

$$\Phi_v(\underline{0},r)=\Phi_v(\underline{0},1)=\int_{B_1}v\,dy, \tag{3.16}$$

for all $r>0$. We prove next a quasi-monotonicity formula for solutions of the obstacle problem in case $x_0\in\Gamma_u$ is a singular point of the free boundary, namely it is such that

$$\Phi_u(x_0,0^+)=\Phi_v(\underline{0},1) \tag{3.17}$$

for some v 2-homogeneous solution of (3.15).

Theorem 3.4 (Monneau's quasi-monotonicity formula)**.** *Under hypotheses (H1), (H2), (H4) with* $a = 1$*, if* $K \subset \Omega$ *is a compact set and* (3.16) *holds for* $x_0 \in K \cap \Gamma_u$ *then there exists a constant* $C = C(n, p, \Lambda, c_0, K, \|f\|_{L^\infty}, \|\mathbb{A}\|_{W^{1,p}}) > 0$ *such that the function*

$$\big(0, \frac{1}{2}\mathrm{dist}(K, \partial\Omega)\big) \ni r \longmapsto \frac{1}{r^{n+3}} \int_{\partial B_r} (u_{\mathbb{L}(x_0)} - v)^2 \, dx + C \int_0^r \frac{dt}{t} \int_0^t \frac{\omega(s)}{s} \, ds \tag{3.18}$$

is nondecreasing, where v *is any* 2*-homogeneous polynomial solution of* (3.15)*, and* ω *is the modulus of continuity provided by Theorem 3.1.*

Proof of Theorem 3.4. As in the proof of Theorem 3.1 for the sake of notational simplicity we write u_{x_0} rather than $u_{\mathbb{L}(x_0)}$.

Set $w := u_{x_0} - v$, then arguing as in (3.10) and by applying the Divergence Theorem we get

$$\begin{aligned} \frac{d}{dr}\left(\frac{1}{r^{n+3}} \int_{\partial B_r} w^2 \, d\mathcal{H}^{n-1}\right) &= \frac{2}{r^{n+3}} \int_{\partial B_r} w \left(\langle \nabla w, \frac{x}{r} \rangle - 2\frac{w}{r}\right) d\mathcal{H}^{n-1} \\ &= \frac{2}{r^{n+3}} \int_{B_r} \mathrm{div}\,(w \nabla w) \, dx - \frac{4}{r^{n+4}} \int_{\partial B_r} w^2 \, d\mathcal{H}^{n-1} \\ &= \frac{2}{r^{n+3}} \int_{B_r} w \Delta w \, dx + \frac{2}{r^{n+3}} \int_{B_r} |\nabla w|^2 \, dx - \frac{4}{r^{n+4}} \int_{\partial B_r} w^2 \, d\mathcal{H}^{n-1} \end{aligned} \tag{3.19}$$

For what concerns the first term on the right hand side of (3.19) recall that $u \in W^{2,p}_{loc}(\Omega)$, thus by locality of the weak derivatives $\mathcal{L}^n\big(\{\nabla u_{x_0} = \underline{0}\} \setminus \{\nabla^2 u_{x_0} = \underline{0}\}\big) = 0$. Being $\Lambda_{u_{x_0}} \subseteq \{\nabla u_{x_0} = \underline{0}\}$, we conclude that $\Delta u_{x_0} = 0$ $\mathcal{L}^n$-a.e. in $\Lambda_{u_{x_0}}$, and therefore in view of (3.4) we infer

$$w \Delta w = (u_{x_0} - v)(\Delta u_{x_0} - 1) = \begin{cases} (u_{x_0} - v)\, f_{x_0} & \mathcal{L}^n\text{-a.e. } \Omega \setminus \Lambda_{u_{x_0}} \\ v & \mathcal{L}^n\text{-a.e. } \Lambda_{u_{x_0}}. \end{cases}$$

Instead, estimating the second and third terms on the right hand side of (3.19) thanks to (3.15) yields

$$\begin{aligned} \frac{1}{r^{n+3}} \int_{B_r} |\nabla w|^2 \, dx - \frac{2}{r^{n+4}} \int_{\partial B_r} w^2 \, d\mathcal{H}^{n-1} &= \frac{1}{r^{n+3}} \int_{B_r} \left(|\nabla u_{x_0}|^2 + |\nabla v|^2\right) dx \\ - \frac{2}{r^{n+3}} \int_{B_r} \mathrm{div}\,(u_{x_0} \nabla v) \, dx &+ \frac{2}{r^{n+3}} \int_{B_r} u_{x_0} \, dx - \frac{2}{r^{n+4}} \int_{\partial B_r} w^2 \, d\mathcal{H}^{n-1} \\ &\overset{(3.16)}{=} \frac{1}{r}\big(\Phi_{u_{x_0}}(x_0, r) - \Phi_v(x_0, r)\big) - \frac{2}{r^{n+4}} \int_{\partial B_r} u_{x_0} \left(\langle \nabla v, \frac{x}{r} \rangle - 2v\right) dx \\ &\overset{(3.17)}{=} \frac{1}{r}\left(\Phi_{u_{x_0}}(x_0, r) - \Phi_{u_{x_0}}(x_0, 0^+)\right). \end{aligned}$$

Then, (3.19) rewrites as

$$\begin{aligned} \frac{d}{dr}\left(\frac{1}{r^{n+3}} \int_{\partial B_r} w^2 \, d\mathcal{H}^{n-1}\right) &= \frac{2}{r}\left(\Phi_u(x_0, r) - \Phi_u(x_0, 0^+)\right) \\ &+ \frac{2}{r^{n+3}} \int_{B_r \setminus \Lambda_{u_{x_0}}} (u_{x_0} - v) f_{x_0} \, dx + \frac{2}{r^{n+3}} \int_{B_r \cap \Lambda_{u_{x_0}}} v \, dx. \end{aligned}$$

Inequality (3.7) in Theorem 3.1, the growth of the solution u from free boundary points in (2.8), the 2-homogeneity and positivity of v yield the conclusion (cf. (3.12)-(3.14)):

$$\frac{d}{dr}\left(\frac{1}{r^{n+3}} \int_{\partial B_r} w^2 \, d\mathcal{H}^{n-1}\right) \geq -\frac{C}{r} \int_0^r \frac{\omega(t)}{t} \, dt - \frac{C}{r^{n+1}} \int_{B_r \setminus \Lambda_{u_{x_0}}} |f_{x_0}| \, dx = -\frac{C}{r} \int_0^r \frac{\omega(t)}{t} \, dt$$

for some $C = C(n, p, \Lambda, c_0, K, \|f\|_{L^\infty}, \|\mathbb{A}\|_{W^{1,p}}) > 0$. □

4. Free boundary analysis

Weiss' and Monneau's quasi-monotonicity formulas proved in the Section 3 are important tools to deduce regularity of free boundaries for classical obstacle problems for variational energies both in the quadratic and in the nonlinear setting (see [9, 11, 10, 18, 19, 21]). In this section we improve [9, Theorems 4.12 and 4.14] in the quadratic case weakening the regularity of the coefficients of the relevant energies. This is possible thanks to the above mentioned new quasi-monotonicity formulas.

In the ensuing proof we will highlight only the substantial changes since the arguments are essentially those given in [9, 11]. In particular, we remark again that in the quadratic case the main differences concern the quasi-monotonicity formulas established for the quantity Φ_u rather than for the natural candidate related to $\mathscr{E}$.

We follow the approach by Weiss [21] and Monneau [18] for the free boundary analysis in Theorem 1.1.

Proof of Theorem 1.1. First recall that we may establish the conclusions for the function $u = w-\psi$ introduced in Section 2. Given this, the only minor change to be done to the arguments in [9, Section 4] is related to the freezing of the energy where the regularity of the coefficients plays a substantial role. More precisely, in the current framework for all $v \in W^{1,2}(B_1)$ we have

$$\left|\int_{B_1} \big(\langle\mathbb{A}(rx)\nabla v, \nabla v\rangle + 2f(rx)v\big)\, dx - \int_{B_1} \big(|\nabla v|^2 + 2v\big)dx\right| \leq (r^{1-\frac{n}{p}} + \omega_f(r)) \int_{B_1} \big(|\nabla v|^2 + 2v\big)dx.$$

We then describe shortly the route to the conclusion. To begin with recall that the quasi-monotonicity formulas established in [9, Section 3] are to be substituted by those in Section 3. Then the 2-homogeneity of blow up limits in [9, Proposition 4.2] now follows from Theorem 3.1. The quadratic growth of solutions from free boundary points contained in [9, Lemma 4.3], that implies non degeneracy of blow up limits, is contained in Lemma 2.5. The classification of blow up limits is performed exactly as in [9, Proposition 4.5]. The conclusions of [9, Lemma 4.8], a result instrumental for the uniqueness of blow up limits at regular points, can be obtained with essentially no difference. The proofs of [9, Propositions 4.10, 4.11, Theorems 4.12, 4.14] remain unchanged. The theses then follow at once. □

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