Brezis-Gallouet-Wainger type inequality with critical fractional Sobolev space and BMO

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Abstract. In this paper, we prove the Brezis-Gallouet-Wainger type inequality involving the BMO norm, the fractional Sobolev norm, and the logarithmic norm of $\dot{\mathcal{C}}^\eta$, for $\eta \in (0,1)$.

1 Introduction and main results

The main purpose of this paper is to established L^∞-bound by means of the BMO norm, or the critical fractional Sobolev norm with the logarithm of $\dot{\mathcal{C}}^\eta$ norm. Such a L^∞-estimate of this type is known as the Brezis-Gallouet-Waigner (BGW) type inequality. Let us remind that Brezis-Gallouet [3], and Brezis-Wainger [4] considered the relation between L^∞, $W^{k,r}$, and $W^{s,p}$, and proved that there holds

$$\|f\|_{L^\infty} \leq C \left(1 + \log^{\frac{r-1}{r}} \left(1 + \|f\|_{W^{s,p}}\right)\right), \quad sp > n \tag{1.1}$$

provided $\|f\|_{W^{k,r}} \leq 1$, for $kr = n$. Its application is to prove the existence of solutions of the nonlinear Schrödinger equations, see details in [3]. We also note that an alternative proof of (1.1) was given by H. Engler [5] for any bounded set in $\mathbb{R}^n$ with the cone condition. Similar embedding for vector functions u with $divu = 0$ was investigated by Beale-Kato-Majda:

$$\|\nabla u\|_{L^\infty} \leq C \left(1 + \|\mathrm{rot}u\|_{L^\infty} \left(1 + \log(1 + \|u\|_{W^{s+1,p}})\right) + \|\mathrm{rot}u\|_{L^2}\right), \tag{1.2}$$

for $sp > n$, see [1] (see also [10] for an improvement of (1.2) in a bounded domain). An application of (1.2) is to prove the breakdown of smooth solutions for the 3-D Euler equations. After that, estimate (1.2) was enhanced by Kozono and Taniuchi [6] in that $\|\mathrm{rot}u\|_{L^\infty}$ can be relaxed to $\|\mathrm{rot}u\|_{BMO}$:

$$\|\nabla u\|_{L^\infty} \leq C \left(1 + \|\mathrm{rot}u\|_{BMO} \left(1 + \log(1 + \|u\|_{W^{s+1,p}})\right)\right). \tag{1.3}$$

To obtain (1.3), Kozono-Taniuchi [6] proved a logarithmic Sobolev inequality in terms of BMO norm and Sobolev norm that for any $1 < p < \infty$, and for $s > n/p$, then there is a constant $C = C(n,p,s)$ such that the estimate

$$\|f\|_{L^\infty} \leq C \left(1 + \|f\|_{BMO} \left(1 + \log^+ \left(\|f\|_{W^{s,p}}\right)\right)\right) \tag{1.4}$$

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holds for all $f \in W^{s,p}$. Obviously, (1.4) is a generalization of (1.1).

Besides, it is interesting to note that Gagliardo-Nirenberg type inequality with critical Sobolev space directly yields BGW type inequality. For example, H. Kozono, and H. Wadade [8] proved the Gagliardo-Nirenberg type inequalities for the critical case and the limiting case of Sobolev space as follows:

$$\|u\|_{L^q} \leq C_n r' q^{\frac{1}{r'}} \|u\|_{L^p}^{\frac{p}{q}} \|(-\Delta)^{\frac{n}{2r}} u\|_{L^r}^{1-\frac{p}{q}}, \tag{1.5}$$

holds for all $u \in L^p \cap \dot{H}^{\frac{n}{r},r}$ with $1 \leq p < \infty$, $1 < r < \infty$, and for all q with $p \leq q < \infty$ (see also Ozawa [11]).
And

$$\|u\|_{L^q} \leq C_n q \|u\|_{L^p}^{\frac{p}{q}} \|u\|_{BMO}^{1-\frac{p}{q}}, \tag{1.6}$$

holds for all $u \in L^p \cap BMO$ with $1 \leq p < \infty$, and for all q with $p \leq q < \infty$.
As a result, (1.5) implies

$$\|u\|_{L^\infty} \leq C \left(1 + (\|u\|_{L^p} + \|(-\Delta)^{\frac{n}{2r}} u\|_{L^r}) \left(\log(1 + \|(-\Delta)^{\frac{s}{2}} u\|_{L^q}) \right)^{\frac{1}{r'}} \right), \tag{1.7}$$

for every $1 \leq p < \infty$, $1 < r < \infty$, $1 < q < \infty$ and $n/q < s < \infty$.
While (1.6) yields

$$\|u\|_{L^\infty} \leq C \left(1 + (\|u\|_{L^p} + \|u\|_{BMO}) \log(1 + \|(-\Delta)^{\frac{s}{2}} u\|_{L^q}) \right), \tag{1.8}$$

for every $1 \leq p < \infty$, $1 < q < \infty$, and $n/q < s < \infty$.
Thus, (1.7) and (1.8) may be regarded as a generalization of BGW inequality. Note that in (1.7) and (1.8), the logarithm term only contains the semi-norm $\|u\|_{\dot{W}^{s,p}}$. Furthermore, Kozono, Ogawa, Taniuchi [7] proved the logarithmic Sobolev inequalities in Besov space, generalizing the BGW inequality and the Beale-Kato-Majda inequality.

Motivated by these above results, in this paper, we study BGW type inequality by means of the BMO norm, the fractional Sobolev norm and the $\dot{C}^\eta$ norm, for $\eta \in (0, 1)$. Then, our first result is as follows:

Theorem 1.1 *Let $\eta \in (0, 1)$, and $\alpha \in (0, n)$. Then, there exists a constant $C = C(n, \eta) > 0$ such that the estimate*

$$\|f\|_{L^\infty} \leq C + C\|f\|_{BMO} \left(1 + \log^+ \left[\sup_{z \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(y)|}{(|z-y|+1)^\alpha} dy + \|f\|_{\dot{C}^\eta} \right] \right) \tag{1.9}$$

holds for all $f \in \dot{C}^\eta \cap BMO$. We accept the notation $\log^+ s = \log s$ if $s \geq 1$, and $\log^+ s = 0$ if $s \in (0, 1)$.

Remark 1.2 *It is clear that $\left(\sup_{z \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(y)|}{(|z-y|+1)^\alpha} dy \right)$ is finite if $f \in L^1$. On the other hand, if $f \in L^r$, $r > 1$, then for any $\alpha \in (\frac{n}{r}, n)$, we have*

$$\sup_{z \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(y)|}{(|z-y|+1)^\alpha} dy \leq C\|f\|_{L^r},$$

where the constant C is independent of f.

Remark 1.3 *If* supp f $\subset$ B_R, *then* (1.9) *implies*

$$\|f\|_{L^\infty} \leq C + C\|f\|_{BMO}\left(1 + \log^+\left[R^{n-\alpha+\eta} + \|f\|_{\dot{\mathcal{C}}^\eta}\right]\right). \tag{1.10}$$

Remark 1.4 *Note that if* $f \in W^{s,p}$ *with* $sp > n$, *then* (1.9) *is stronger than* (1.4) *since* $W^{s,p} \subset \mathcal{C}^{0,\eta} \subset \dot{\mathcal{C}}^\eta$, *with* $\eta = \frac{sp-n}{p}$.

Concerning the BGW type inequality involving the fractional Sobolev space, we have the following result:

Theorem 1.5 *Let* $s > 0, p \geq 1$ *be such that* $sp = n$. *Let* $\alpha > 0$, $\eta \in (0,1)$. *Then, there exists a constant* $C = C(n,s,p,\eta,\alpha) > 0$ *such that the estimate*

$$\|f\|_{L^\infty} \leq C + C\|f\|_{\dot{W}^{s,p}}\left(1 + \left(\log^+\left(\sup_{z\in\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(y)|}{(|z-y|+1)^\alpha}dy + \|f\|_{\dot{\mathcal{C}}^\eta}\right)\right)^{\frac{p-1}{p}}\right) \tag{1.11}$$

holds for all $f \in \dot{\mathcal{C}}^\eta \cap \dot{W}^{s,p}$, *where* $\dot{W}^{s,p}$ *is the homogeneous fractional Sobolev space, see its definition below.*

Remark 1.6 *As Remark 1.4, we can see that* (1.11) *is stronger then* (1.1). *Furthermore, if* supp f $\subset$ B_R, *then* (1.9) *implies*

$$\|f\|_{L^\infty} \leq C + C\|f\|_{\dot{W}^{s,p}}\left(1 + \left(\log^+\left[R^{n-\alpha+\eta} + \|f\|_{\dot{\mathcal{C}}^\eta}\right]\right)^{\frac{p-1}{p}}\right). \tag{1.12}$$

Remark 1.7 *We consider* $f_\delta(x) = -\log(|x|+\delta)\psi(|x|)$, *where* $\psi \in \mathcal{C}_c^1([0,\infty))$, $0 \leq \psi \leq 1$, $\psi(|x|) = 1$ *if* $|x| \leq \frac{1}{4}$, *and* $\delta > 0$ *is small enough. It is not hard to see that for any* $\delta > 0$ *small enough*

$$||f_\delta||_{L^\infty(\mathbb{R}^d)} \sim |\log(\delta)|, \quad ||f_\delta||_{BMO(\mathbb{R}^d)} \sim 1, \quad \|f_\delta\|_{\dot{W}^{\frac{n}{p},p}} \sim |\log(\delta)|^{\frac{1}{p}},$$

and

$$\sup_{z\in\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f_\delta(y)|}{(|z-y|+1)^\alpha}dy \sim 1, \quad \|f_\delta\|_{\dot{\mathcal{C}}^\eta(\mathbb{R}^n)} \lesssim \delta^{-1}.$$

Therefore, the power 1 *and* $\frac{p-1}{p}$ *of the term* $\log_2^+\left(\sup_{z\in\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(y)|}{(|z-y|+1)^\alpha}dy + \|f\|_{\dot{\mathcal{C}}^\eta}\right)$ *in* (1.9) *and* (1.11) *respectively are sharp that there are no such estimates of the form:*

$$\|f_1\|_\infty \leq C + C\|f_1\|_{BMO}\left(1 + \left(\log_2^+\left(\sup_{z\in\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f_1(y)|}{(|z-y|+1)^\alpha}dy + \|f_1\|_{\dot{\mathcal{C}}^\eta}\right)\right)^\gamma\right),$$

and

$$\|f_2\|_{L^\infty} \leq C + C\|f_2\|_{\dot{W}^{\frac{n}{p},p}}\left(1 + \left(\log^+\left(\sup_{z\in\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f_2(y)|}{(|z-y|+1)^\alpha}dy + \|f_2\|_{\dot{\mathcal{C}}^\eta}\right)\right)^{\gamma\frac{p-1}{p}}\right),$$

hold for all $f_1 \in BMO \cap \dot{\mathcal{C}}^\eta$, $f_2 \in \dot{\mathcal{C}}^\eta \cap \dot{W}^{s,p}$, *for some* $\gamma \in (0,1)$.

Before closing this section, let us introduce some functional spaces that we use through this paper. First of all, we recall $\dot{\mathcal{C}}^\eta$, $\eta \in (0,1)$, as the homogeneous Holder continuous of order η, endowed with the semi-norm:

$$\|f\|_{\dot{\mathcal{C}}^\eta} = \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\eta}.$$

Next, if $s \in (0,1)$, then we recall $\dot{W}^{s,p}$ the homogeneous fractional Sobolev space, endowed with the semi-norm:

$$\|f\|_{\dot{W}^{s,p}} = \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} dxdy\right)^{\frac{1}{p}}.$$

When $s > 1$, and s is not an integer, we denote $\dot{W}^{s,p}$ as the homogeneous fractional Sobolev space endowed with the semi-norm:

$$\|f\|_{\dot{W}^{s,p}} = \sum_{|\sigma|=[s]} \|D^\sigma f\|_{\dot{W}^{s-[s],p}}.$$

If s is an integer, then

$$\|f\|_{\dot{W}^{s,p}} = \sum_{|\sigma|=[s]} \|D^\sigma f\|_{L^p}.$$

We refer to [9] for details on the fractional Sobolev space.
After that, we accept the notation $(f)_\Omega := \fint_\Omega f = \frac{1}{|\Omega|}\int_\Omega f(x)dx$ for any Borel set Ω. Finally, C is always denoted as a constant which can change from line to line. And $C(k,n,l)$ means that this constant merely depends on k, n, l.

2 Proof of the Theorems

We first prove Theorem 1.1.

Proof of Theorem 1.1 . It is enough to prove that

$$|f(0)| \le C + C\|f\|_{BMO}\left(1+\log_2^+\left(\int_{\mathbb{R}^n}\frac{|f(y)|}{(|y|+1)^\alpha}dy + \|f\|_{\dot{\mathcal{C}}^\eta}\right)\right), \tag{2.1}$$

Let $m_0 \in \mathbb{N}$, set $B_\rho := B_\rho(0)$, we have,

$$\begin{aligned}
|f(0)| &= \left| f(0) - \fint_{B_{2^{-m_0}}} f + \sum_{j=-m_0}^{m_0-1}\left(\fint_{B_{2^j}} f - \fint_{B_{2^{j+1}}} f\right) + \fint_{B_{2^{m_0}}} f\right| \\
&\le \fint_{B_{2^{-m_0}}} |f - f(0)| + \sum_{j=-m_0}^{m_0-1}\fint_{B_{2^j}} |f-(f)_{B_{2^{j+1}}}| + C2^{-m_0(n-\alpha)}\int_{B_{2^{m_0}}}\frac{|f(y)|}{(|y|+1)^\alpha}dy \\
&\le \fint_{B_{2^{-m_0}}} |y|^\eta \|f\|_{\dot{\mathcal{C}}^\eta}dy + 2m_0\|f\|_{BMO} + C2^{-m_0(n-\alpha)}\int_{B_{2^{m_0}}}\frac{|f(y)|}{(|y|+1)^\alpha}dy \\
&\le C2^{-m_0\min\{n-\alpha,\eta\}}\left(\int_{\mathbb{R}^n}\frac{|f(y)|}{(|y|+1)^\alpha}dy + \|f\|_{\dot{\mathcal{C}}^\eta}\right) + Cm_0\|f\|_{BMO}.
\end{aligned}$$

Choosing

$$m_0 = \left[\frac{\log_2^+ \left(\int_{\mathbb{R}^n} \frac{|f(y)|}{(|y|+1)^\alpha} dy + \|f\|_{\dot{\mathcal{C}}^\eta} \right)}{\min\{n-\alpha, \eta\}} \right] + 1,$$

we get (2.1). The proof is complete. ■

Next, we prove Theorem 1.5.

Proof of Theorem 1.5. To prove it, we need the following lemmas:

Lemma 2.1 *Let $a_0 = 1$, and let $(a_0, a_1, ..., a_{k+1}) \in \mathbb{R}^{k+2}$, for any $k \geq 1$, be a unique solution of the following system:*

$$\sum_{j=0}^{k+1} a_j 2^{jl} = 0, \quad \forall l = 0, ..., k. \tag{2.2}$$

Then, we have

$$a := \sum_{j=0}^{k} (k-j+1) a_j \neq 0. \tag{2.3}$$

Moreover, for any $m \geq 1$, and for $b, b_l \in \mathbb{R}$, $l = -m, ..., m$, we have

$$\sum_{l=-m}^{m-1} \left[\sum_{j=0}^{k+1} a_j b_{j+l} \right] = \sum_{l=m}^{k+m} \left[\sum_{j=l-m+1}^{k+1} a_j \right] b_l + \sum_{l=-m}^{k-m} \left[\sum_{j=0}^{l+m} a_j \right] (b_l - b) + ab. \tag{2.4}$$

As a result, we obtain

$$|b| \leq \frac{1}{|a|} \left[\sum_{j=0}^{k+1} |a_j| \right] \sum_{l=-m}^{k-m} |b_l - b| + \frac{1}{|a|} \sum_{l=-m}^{m-1} \left| \sum_{j=0}^{k+1} a_j b_{j+l} \right| + \frac{1}{|a|} \left[\sum_{j=0}^{k+1} |a_j| \right] \sum_{l=m}^{k+m} |b_l|. \tag{2.5}$$

Proof. First of all, we note that $a_j \neq 0$, for $j = 0, ..., k+1$. Set

$$Q(x) = \sum_{j=0}^{k+1} a_j x^j.$$

Then,

$$Q'(1) = \sum_{j=1}^{k+1} j a_j.$$

On the other hand, by (2.2), we have $Q(2^l) = 0$, for $l = 0, ..., k$. Thus,

$$Q(x) = a_{k+1} \prod_{l=0}^{k} (x - 2^l), \quad \text{and } Q'(1) = \prod_{l=1}^{k} (1 - 2^l).$$

This implies

$$\sum_{j=1}^{k+1} ja_j = \prod_{j=1}^{k}(1-2^l) \neq 0. \tag{2.6}$$

Next, we observe that

$$0 = (k+1)\sum_{j=0}^{k+1} a_j = a + \sum_{j=1}^{k+1} ja_j = 0.$$

The last equation and (2.6) yield $a = -\prod_{j=1}^{k}(1-2^l) \neq 0$.

Now, we prove (2.4). We denote LHS (resp. RHS) is the left hand side (resp. the right hand side) side of (2.4). It is not difficult to verify that

$$\sum_{l=-m}^{k-m}\left[\sum_{j=0}^{l+m} a_j\right] b = ab.$$

Then, a direct computation shows

$$\begin{aligned}
RHS &= a_0b_{-m} + (a_0+a_1)b_{1-m} + ... + (a_0+...+a_k)b_{k-m} \\
&\quad + (a_1+...+a_{k+1})b_m + (a_2+...+a_{k+1})b_{m+1} + ... + a_{k+1}b_{k+m} = a_0\sum_{l=-m}^{k-m} b_l \\
&\quad + a_1\left(\sum_{l=1-m}^{k-m} b_l + \sum_{l=m}^{m} b_l\right) + ... + a_k\left(\sum_{l=k-m}^{k-m} b_{k-m} + \sum_{l=m}^{m+k-1} b_l\right) + a_{k+1}\left(\sum_{l=m}^{m+k} b_l\right).
\end{aligned}$$

Note that $\left(\sum_{j=0}^{k+1} a_j\right)\sum_{l=k+1-m}^{m-1} b_l = 0$. Thus,

$$\begin{aligned}
RHS &= RHS + \left(\sum_{j=0}^{k+1} a_j\right)\sum_{l=k+1-m}^{m-1} b_l \\
&= \sum_{j=0}^{k+1} a_j\left(\sum_{l=j-m}^{j+m-1} b_l\right) \\
&= \sum_{l=m}^{k+m}\left(\sum_{j=l-m+1}^{k+1} a_j\right) b_l + \sum_{l=k+1-m}^{m-1}\left(\sum_{j=0}^{k+1} a_j\right) b_l + \sum_{l=-m}^{k-m}\left(\sum_{j=0}^{l+m} a_j\right) b_l \\
&= \sum_{l=m}^{k+m}\left(\sum_{j=l-m+1}^{k+1} a_j\right) b_l + \sum_{l=-m}^{k-m}\left(\sum_{j=0}^{l+m} a_j\right) b_l \\
&= LHS.
\end{aligned}$$

Or, we get (2.4).
Finally, (2.5) follows from (2.4) by using the triangle inequality. In other words, we

get Lemma 2.1. $\blacksquare$

Next, we have

Lemma 2.2 *Assume $a_0, a_1, ..., a_{k+1}$ as in Lemma 2.1. Let $\Omega_j = B_{2^{j+1}} \backslash B_{2^j}$, where $B_\rho := B_\rho(0)$ for any $\rho > 0$. Then, there holds*

$$\left| \sum_{j=0}^{k+1} a_j \fint_{\Omega_j} f \right| \leq C \fint_{B_{2^{k+3}} \backslash B_{2^{-1}}} \left| D^k f(y) - \left(D^k f \right)_{B_{2^{k+3}} \backslash B_{2^{-1}}} \right| dy. \tag{2.7}$$

For any $l \in \mathbb{R}$, we set $E_l = B_{2^{k+l+3}} \backslash B_{2^{l-1}}$. As a consequence of (2.7)*, we obtain*

$$\left| \sum_{j=0}^{k+1} a_j \fint_{\Omega_{j+l}} f \right| \leq C 2^{kl} \fint_{E_l} \fint_{E_l} \left| D^k f(y) - D^k f(y') \right| dy dy'. \tag{2.8}$$

Moreover, by the triangle inequality we get from (2.8)

$$\left| \sum_{j=0}^{k+1} a_j \fint_{\Omega_{j+l}} f \right| \leq C \ 2^{kl} \fint_{E_l} \left| D^k f(y) \right| dy. \tag{2.9}$$

Proof. Assume a contradiction that (2.7) is not true. There exists then a sequence $(f_m)_{m \geq 1} \subset W^{k,1}(B_{2^{k+3}} \backslash B_{2^{-1}})$ such that

$$\int_{B_{2^{k+3}} \backslash B_{2^{-1}}} |D^k f_m(y) - \left(D^k f_m \right)_{B_{2^{k+3}} \backslash B_{2^{-1}}} | dy \leq \frac{1}{m}, \tag{2.10}$$

and

$$|\sum_{j=0}^{k+1} a_j \fint_{\Omega_j} f_m| = 1, \quad \forall m \geq 1.$$

Let us put

$$\tilde{f}_m(x) = f_m(x) - P_{k,m}(x), \quad \text{with } P_{k,m}(x) = \sum_{l=0}^{k} \sum_{\alpha_1 + ... + \alpha_n = l} c_{l,k,m}(\alpha_1, ..., \alpha_n) x_1^{\alpha_1} x_2^{\alpha_2} ... x_n^{\alpha_n},$$

and $c_{l,k,m}(\alpha_1, ..., \alpha_n)$ is a constant such that

$$\left(D^l \tilde{f}_m \right)_{B_{2^{k+3}} \backslash B_{2^{-1}}} = 0, \ \forall l = 0, ..., k. \tag{2.11}$$

By a sake of brief, we denote $c_{l,m} = c_{l,k,m}(\alpha_1, ..., \alpha_n)$. Since $P_{k,m}$ is a polynomial of at most k-degree, then $D^k P_{k,m} = const$. This fact, (2.10), and (2.11) imply

$$\int_{B_{2^{k+3}} \backslash B_{2^{-1}}} |D^k \tilde{f}_m(y)| dy = \int_{B_{2^{k+3}} \backslash B_{2^{-1}}} |D^k f_m(y) - \left(D^k f_m \right)_{B_{2^{k+3}} \backslash B_{2^{-1}}} | dy \leq \frac{1}{m}.$$

It follows form the compact embeddings that there exists a subsequence of $(\tilde{f}_m)_{m\geq 1}$ (still denoted as $(\tilde{f}_m)_{m\geq 1}$) such that that $\tilde{f}_m \to \tilde{f}$ strongly in $L^1(B_{2^{k+3}}\backslash B_{2^{-1}})$, and

$$D^k\tilde{f} = 0, \quad \text{in } \; B_{2^{k+3}}\backslash B_{2^{-1}}.$$

This implies that $\tilde{f}$ is a polynomial of at most $(k-1)$-degree, i.e:

$$\tilde{f}(x) = \sum_{l=0}^{k-1} \sum_{\alpha_1+...+\alpha_n=l} c'_{l,k}(\alpha_1,...,\alpha_n)x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}, \quad \forall x \in B_{2^{k+3}}\backslash B_{2^{-1}}.$$

On the other hand, we observe that for any $l = 0, ..., k$

$$\begin{aligned}
&\sum_{j=0}^{k+1} a_j \fint_{\Omega_j} \sum_{\alpha_1+...+\alpha_n=l} c(\alpha_1,...,\alpha_n)x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}dx_1dx_2...dx_n \\
&= \sum_{j=0}^{k+1} a_j \fint_{\Omega_1} \sum_{\alpha_1+...+\alpha_n=l} c(\alpha_1,...,\alpha_n)(2^jx_1)^{\alpha_1}(2^jx_2)^{\alpha_2}...(2^jx_n)^{\alpha_n}dx_1dx_2...dx_n \\
&= \fint_{\Omega_1} \sum_{\alpha_1+...+\alpha_n=l} c(\alpha_1,...,\alpha_n)\left(\sum_{j=0}^{k+1} a_j 2^{jl}\right) x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}\; dx_1dx_2...dx_n = 0,
\end{aligned}$$

by (2.2). This implies

$$\sum_{j=0}^{k+1} a_j \fint_{\Omega_j} \tilde{f} = 0, \tag{2.12}$$

and

$$\left|\sum_{j=0}^{k+1} a_j \fint_{\Omega_j} \tilde{f}_m\right| = \left|\sum_{j=0}^{k+1} a_j \fint_{\Omega_j} f_m\right| = 1.$$

Remind that $\tilde{f}_m \to \tilde{f}$ strongly in $L^1(B_{2^{k+3}}\backslash B_{2^{-1}})$, then we have

$$\left|\sum_{j=0}^{k+1} a_j \fint_{\Omega_j} \tilde{f}\right| = 1.$$

Or, we complete the proof of (2.7).
The proof of (2.8) (resp. (2.9)) is trivial then we leave it to the reader. This puts an end to the proof of Lemma 2.2. ■

Now, we are ready to prove Theorem 1.5.
It is enough to show that

$$|f(0)| \leq C + C\|f\|_{\dot{W}^{s,p}}\left(1 + \log_2^+\left(\int_{\mathbb{R}^n} \frac{|f(y)|}{(|y|+1)^\alpha}dy + \|f\|_{\dot{C}^\eta}\right)\right)^{\frac{p-1}{p}}. \tag{2.13}$$

Set $s_1 = s - k$, $s_1 \in [0, 1)$. Then, we divide our study into the two cases:

i) Case: $s_1 \in (0,1)$**:**
We apply Lemma 2.1 with $b = f(0)$, $b_j = \fint_{\Omega_j} f$. Then, for any $m_0 \geq 1$, there is a constant $C = C(k) > 0$ such that

$$|f(0)| \leq C \left(\sum_{l=-m_0}^{k-m_0} \left| \fint_{\Omega_l} f - f(0) \right| + \sum_{l=-m_0}^{m_0-1} \left| \sum_{j=0}^{k+1} a_j \fint_{\Omega_{j+l}} f \right| + \sum_{l=m_0}^{k+m_0} \left| \fint_{\Omega_l} f \right| \right). \quad (2.14)$$

Concerning the first term on the right hand side of (2.14), we have

$$\sum_{l=-m_0}^{k-m_0} \left| \fint_{\Omega_l} f - f(0) \right| \leq \sum_{l=-m}^{k-m} \fint_{\Omega_l} |f - f(0)| \leq \sum_{l=-m}^{k-m} \fint_{\Omega_l} |x|^\eta \|f\|_{\dot{C}^\eta} dx.$$

Thus,

$$\sum_{l=-m}^{k-m} \left| \fint_{\Omega_l} f - f(0) \right| \leq \sum_{l=-m}^{k-m} 2^{(l+1)\eta} \|f\|_{\dot{C}^\eta} \leq C(\eta, k) 2^{-m\eta} \|f\|_{\dot{C}^\eta}. \quad (2.15)$$

Next, we use (2.8) in Lemma 2.2 to obtain

$$\sum_{l=-m}^{m-1} \left| \sum_{j=0}^{k+1} a_j \fint_{\Omega_{j+l}} f \right| \leq C \sum_{l=-m}^{m-1} 2^{kl} \fint_{E_l} \fint_{E_l} \left| D^k f(y) - D^k f(z) \right| dydz, \quad (2.16)$$

where $E_l = B_{2^{k+l+3}} \backslash B_{2^{l-1}}$. It follows from Hölder's inequality

$$\sum_{l=-m_0}^{m_0-1} 2^{kl} \fint_{E_l} \fint_{E_l} |D^k f(y) - D^k f(z)| dydz \leq$$
$$\sum_{l=-m_0}^{m_0-1} 2^{kl} |E_l|^{-2} \left(\int_{E_l} \int_{E_l} \frac{|D^k f(y) - D^k f(z)|^p}{|y-z|^{n+s_1 p}} dydz \right)^{\frac{1}{p}} \left(\int_{E_l} \int_{E_l} |y-z|^{\frac{n+s_1 p}{p-1}} dydz \right)^{\frac{p-1}{p}}.$$

Since $y, z \in E_l$, we have $|y-z| \leq |y| + |z| \leq 2^{k+l+4}$. Thus, the right hand side of the indicated inequality is less than

$$C(n,p,k) 2^{kl + \frac{l(n+s_1 p)}{p}} |E_l|^{\frac{-2}{p}} \sum_{l=-m_0}^{m_0-1} \left(\int_{E_l} \int_{E_l} \frac{|D^k f(y) - D^k f(z)|^p}{|y-z|^{n+s_1 p}} dydz \right)^{\frac{1}{p}}.$$

Note that $n = sp = (k+s_1)p$, and $|E_l|^{\frac{-2}{p}} \leq C(n,p,k) 2^{-2l\frac{n}{p}}$.
Then, there is a constant $C = C(k,s,n) > 0$ such that

$$\sum_{l=-m_0}^{m_0-1} 2^{kl} \fint_{E_l} \fint_{E_l} |D^k f(y) - D^k f(z)| dydz \leq C \sum_{l=-m_0}^{m_0-1} \left(\int_{E_l} \int_{E_l} \frac{|D^k f(y) - D^k f(z)|^p}{|y-z|^{n+s_1 p}} dydz \right)^{\frac{1}{p}}. \quad (2.17)$$

Thanks to the inequality

$$\sum_{j=-m_0}^{m_0-1} c_j^{\frac{1}{p}} \leq (2m_0)^{\frac{p-1}{p}} \left(\sum_{j=-m_0}^{m_0-1} c_j \right)^{\frac{1}{p}}, \quad (2.18)$$

we have

$$\sum_{l=-m_0}^{m_0-1}\left(\int_{E_l}\int_{E_l}\frac{|D^kf(y)-D^kf(z)|^p}{|y-z|^{n+s_1p}}dydz\right)^{\frac{1}{p}}$$

$$\leq (2m_0)^{\frac{p-1}{p}}\left(\sum_{l=-m_0}^{m_0-1}\int_{E_l}\int_{E_l}\frac{|D^kf(y)-D^kf(z)|^p}{|y-z|^{n+s_1p}}dydz\right)^{\frac{1}{p}}. \tag{2.19}$$

Moreover, we observe that $\sum_{l=-\infty}^{+\infty}\chi_{E_l\times E_l}(y_1,y_2)\leq k+4,$ for all $(y_1,y_2)\in\mathbb{R}^n\times\mathbb{R}^n$. Thus,

$$\sum_{l=-m_0}^{m_0-1}\int_{E_l}\int_{E_l}\frac{|D^kf(y)-D^kf(z)|^p}{|y-z|^{n+s_1p}}dydz\leq (k+4)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|D^kf(y)-D^kf(z)|^p}{|y-z|^{n+s_1p}}dydz. \tag{2.20}$$

Combining (2.17), (2.19) and (2.20) yields

$$\sum_{l=-m_0}^{m_0-1}2^{kl}\fint_{E_l}\fint_{E_l}|D^kf(y)-D^kf(z)|dydz\leq C(k,s,n)m_0^{\frac{p-1}{p}}\|f\|_{\dot{W}^{s,p}}. \tag{2.21}$$

It remains to treat the last term. Then, it is not difficult to see that for any $\alpha>0$

$$\sum_{l=m_0}^{k+m_0}\left|\fint_{\Omega_l}f\right|\leq C(k,n)2^{-m_0n}\int_{B_{2^{k+m_0}}}|f|$$

$$\leq C(k,n,\alpha)2^{-m_0(n-\alpha)}\int_{B_{2^{k+m_0}}}\frac{|f(x)|dx}{(|x|+1)^\alpha}. \tag{2.22}$$

Inserting (2.15), (2.21), and (2.22) into (2.14) yields

$$|f(0)|\leq C2^{-m_0\min\{n-\alpha,\eta\}}\left(\int_{\mathbb{R}^n}\frac{|f(y)|}{(|y|+1)^\alpha}dy+\|f\|_{\dot{C}^\eta}\right)+Cm_0^{\frac{p-1}{p}}\|f\|_{\dot{W}^{s,p}}. \tag{2.23}$$

By choosing

$$m_0=\left[\frac{\log_2^+\left(\int_{\mathbb{R}^n}\frac{|f(y)|}{(|y|+1)^\alpha}dy+\|f\|_{\dot{C}^\eta}\right)}{\min\{n-\alpha,\eta\}}\right]+1,$$

we obtain (2.13).

ii) Case: $s_1=0$ $(s=k)$**:**
The proof is similar to the one of the case $s_1\in(0,1)$. There is just a difference of estimating the second term on the right hand side of (2.14) as follows:

Use (2.9), we get

$$\sum_{l=-m_0}^{m_0-1}\left|\sum_{j=0}^{k+1}a_j\fint_{\Omega_{j+l}}f\right|\leq C\sum_{l=-m_0}^{m_0-1}2^{kl}\fint_{E_l}|D^kf|. \tag{2.24}$$

Applying Hölder's inequality, we have

$$\begin{aligned}\sum_{l=-m_0}^{m_0-1} 2^{kl} \fint_{E_l} |D^k f| &\leq \sum_{l=-m_0}^{m_0-1} 2^{kl} |E_l|^{-1/p} \left(\int_{E_l} |D^k f|^p \right)^{1/p} \\ &\leq C(n,k) \sum_{l=-m_0}^{m_0-1} \left(\int_{E_l} |D^k f|^p \right)^{1/p} \\ &\leq C m_0^{\frac{p-1}{p}} \left(\sum_{l=-m_0}^{m_0-1} \int_{E_l} |D^k f|^p \right)^{1/p}. \end{aligned} \tag{2.25}$$

We utilize the fact $\sum_{l=-\infty}^{\infty} \chi_{E_l}(y) \leq k+4, \ \forall y \in \mathbb{R}^n$ again in order to get

$$\left(\sum_{l=-m_0}^{m_0-1} \int_{E_l} |D^k f|^p \right)^{1/p} \leq (k+4) \left(\int_{\mathbb{R}^n} |D^k f|^p \right)^{1/p}. \tag{2.26}$$

From (2.26), (2.25), and (2.24), we get

$$\sum_{l=-m_0}^{m_0-1} \left| \sum_{j=0}^{k+1} a_j \fint_{\Omega_{j+l}} f \right| \leq C(k,n) \|f\|_{\dot{W}^{s,p}}. \tag{2.27}$$

Thus, we obtain another version of (2.23) as follows:

$$|f(0)| \leq C 2^{-m_0 \min\{n-\alpha, \eta\}} \left(\int_{\mathbb{R}^n} \frac{|f(y)|}{(|y|+1)^\alpha} dy + \|f\|_{\dot{C}^\eta} \right) + C m_0^{\frac{p-1}{p}} \|f\|_{\dot{W}^{s,p}}. \tag{2.28}$$

By the same argument as above (after (2.23)), we get the proof of the case $s_1 = 0$. This completes the proof of Theorem 1.5. ■

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