

THE BOUNDED SLOPE CONDITION FOR FUNCTIONALS DEPENDING ON x , u AND ∇u

A. FIASCHI AND G. TREU

ABSTRACT. A global regularity result is proved for a class of minimizers of functionals of the form

$$\mathcal{I}(u) = \int_{\Omega} f(|\nabla u(x)|) + g(x, u(x)) dx \quad u \in \phi + W_0^{1,1}(\Omega)$$

where ϕ satisfies the Bounded Slope Condition.

1. INTRODUCTION

In this paper we address ourselves to the problem of the regularity of minimizers of scalar integral functionals of the form

$$\int_{\Omega} L(x, u(x), \nabla u(x)) dx \quad u \in \phi + W_0^{1,1}(\Omega, \mathbb{R})$$

where Ω is an open bounded subset of \mathbb{R}^n and ϕ is a given boundary datum. Stemming from the fundamental De Giorgi-Moser-Nash theorem a huge literature furnishes many results on the interior regularity. We cannot exhaustively review them here and we refer to [20] for a wide and quite up-to-date reference on the subject. The only fact that we want to underline is that, in all these results, the function $L(x, u, \cdot)$ is assumed to have controlled growth, both from below and from above, and to be uniformly convex too.

The situation is quite different when we deal with the global regularity of the minimizers. From one hand we cannot expect to obtain global regularity: in [9] an example can be found of a harmonic function on the unit ball B , coinciding with a Lipschitz function on ∂B , that is not Lipschitz on \bar{B} . This shows that, even in the case of the Dirichlet functional, depending just on ∇u , having good growth at infinity and satisfying the uniform convexity assumption, we cannot expect to improve the locally Lipschitz continuity of the minimizer. On the other hand, in some special cases, a different approach, that one can say it is inspired by works by Hilbert and Haar [15, 12], gives a different perspective of the problem. First of all we want to recall a very well known result originally due to Stampacchia [11]. Consider the functional

$$\int_{\Omega} L(\nabla u(x)) dx \tag{1.1}$$

where L is a strictly convex function. If ϕ satisfies the so called Bounded Slope Condition (BSC) of rank K (see Section 4 below), then there exists a Lipschitz function with the same rank of ϕ that is a minimizer in the class of Lipschitz functions coinciding with ϕ on $\partial\Omega$. Under the same assumptions, Cellina has recently proved in [4] that if (1.1) admits a minimum in $\phi + W_0^{1,1}(\Omega)$ then it is Lipschitz of rank K .

We want to sketch in few words the proof of this result. We recall that the boundary datum ϕ satisfies the (BSC) if for every point γ on the boundary of Ω there exist two affine functions coinciding in γ and such that $\phi(\gamma')$ is between them for every $\gamma' \in \partial\Omega$. One of the main tools for the proof is the Comparison Principle between minimizers satisfying different boundary conditions, i.e.: assume that v and w are minimizers of (1.1) respectively in $\phi + W_0^{1,1}(\Omega)$ and in $\psi + W_0^{1,1}(\Omega)$, with $\phi(x) \leq \psi(x)$ for every x in $\partial\Omega$, then $v \leq w$ a.e. in Ω . Since any affine function is a minimizer of (1.1) among the functions with the same affine boundary condition, whenever ϕ satisfies the (BSC) of rank K , the Comparison Principle allows us to box up the minimizer on $\phi + W_0^{1,1}(\Omega)$

between two Lipschitz functions, having again rank K and coinciding on $\partial\Omega$. A Haar Radò type theorem for Sobolev functions (see [16] for a precise statement) says that the slope of the minimizer is maximum at the boundary, closing the argument.

We underline the fact that in this approach no role is played by either growth properties or uniform convexity of L .

At this point the question whether we can apply the same method to more general functionals naturally arises. The first remark is that the Comparison Principle may fail if we drop the strict convexity assumption on L , see [5] for an example. Anyhow it has been proved in [5, 7, 6] that it holds, even in this case, for restricted classes of minimizers. This led to prove various results on this subject (see [1, 8, 18, 19, 17] for results concerning the functional (1.1) and [6] for a special case with u -dependence too). A second remark is that in the case of a function L explicitly depending also on x and/or u , the affine functions are no more minimizers of the functional, so that the (BSC) does not immediately give a ‘barrier’ for the minimizers. Anyhow a theorem by Miranda [21] ensures that the class of functions satisfying the (BSC) is quite large: if Ω is uniformly convex and $\partial\Omega$ is $C^{1,1}$, then any $C^{1,1}$ function satisfies it.

In this paper we consider the functional

$$\mathcal{I}(u) = \int_{\Omega} h(\nabla u(x)) + g(x, u(x)) dx \quad u \in \phi + W_0^{1,1}(\Omega) \quad (1.2)$$

where h is a convex function and g is Lipschitz of constant α w.r.t. the second variable. The first result we prove is a Comparison Principle between any minimizer of the functional \mathcal{I} and the functions

$$\omega_{\pm\alpha}(x) := \frac{n}{\pm\alpha} h^* \left(\pm \alpha \frac{x - x_0}{n} \right) + c \quad (1.3)$$

introduced by Cellina in [6]. Then we show that, under additional assumptions on h , the validity of the (BSC) implies that the functions (1.3) provide good barriers for the minimizers of \mathcal{I} . Suitable hypotheses on g (G2 in Section 4 below) are needed to apply the Haar-Radò theorem [16], which allows us to conclude that there exist minimizers inheriting the global Lipschitz regularity of the barriers.

We want to spend some words to explain the difficulties of the proof. To show that the functions (1.3) are suitable to construct barriers we have to proceed in the following way. For any fixed point γ on the boundary of Ω we consider the affine function involved in the (BSC) from below at the point γ . Assume that it is $a \cdot x + b$. We have to find $x_0 \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that the set $\Omega_{x_0,c} = \{x \in \mathbb{R}^n : \omega_{\alpha}(x) - a \cdot x - b < 0\}$ contains Ω and $\partial\Omega_{x_0,c} \cap \partial\Omega$ contains γ . This is essentially a geometric requirement on the two sets $\Omega_{x_0,c}$ and Ω : the normal cone to $\Omega_{x_0,c}$ at γ has to be contained in the normal cone to Ω at the same point and also suitable conditions on the principal curvatures of both sets are needed. For these reasons, essentially technique, we restrict our attention to the case of a uniformly convex set Ω and of a radially symmetric function $h(\xi) = f(|\xi|)$. To compute the curvatures of $\Omega_{x_0,c}$ and to guarantee that the estimates on them hold in a suitable neighborhood of γ , further assumptions on f , (F4) or (F5) of Section 4, are needed. Anyway these hypotheses do not imply the uniform convexity of h , which is instead needed in previous regularity results [3, 2, 22] obtained by barrier techniques for functionals of the type (1.2).

We underline that constant boundary data trivially satisfy the (BSC) and that, in this case, we construct in a easier way the barriers. We can drop assumptions (F4) and (F5) and we obtain the global Lipschitzianity of minimizers (see Theorem 4.5 in Section 4) whenever Ω is uniformly convex.

In the last section of the paper we present some examples and remarks with the aim of clarifying the role of the assumptions on the function f .

2. PRELIMINARY RESULTS AND A COMPARISON PRINCIPLE

We consider an open bounded domain $\Omega \subset \mathbb{R}^n$ and an integral functional on $W^{1,1}(\Omega)$ of the form

$$\mathcal{I}(u) := \int_{\Omega} L(x, u(x), \nabla u(x)) dx,$$

for a function $L: \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$, with $L(\cdot, u, \xi)$ measurable for every (u, ξ) and $L(x, \cdot, \cdot)$ continuous for a.e. $x \in \Omega$.

Definition 2.1. We say that a function $u \in W^{1,1}(\Omega)$ is a *minimizer of the functional \mathcal{I}* if $\mathcal{I}(u) \leq \mathcal{I}(v)$, for every $v \in u + W_0^{1,1}(\Omega)$.

We recall that, given a function $h: \mathbb{R}^n \rightarrow \mathbb{R}$, its polar function $h^*: \mathbb{R}^n \rightarrow [-\infty, +\infty]$ is defined by $h^*(\xi) := \sup_{x \in \mathbb{R}^n} \{x \cdot \xi - h(x)\}$, for every $\xi \in \mathbb{R}^n$ (see [10]).

We are interested in the particular case where the Lagrangian is in the form $L(x, u, \xi) := h(\xi) + g(x, u)$, for a lower bounded function $h: \mathbb{R}^n \rightarrow \mathbb{R}$ and a function $g: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying the following hypotheses:

- (C1) h is convex,
- (C2) h is superlinear, that is $\lim_{|x| \rightarrow \infty} \frac{h(x)}{|x|} = +\infty$;
- (C3) g is Lipschitz continuous in the second variable, with Lipschitz constant equal to α , i.e.: $|g(x, u_1) - g(x, u_2)| \leq \alpha|u_1 - u_2|$ for every $x \in \Omega$ and $u_1, u_2 \in \mathbb{R}$.

Remark 2.2. We observe that the hypotheses on h guarantee that the effective domain of its polar function h^* is \mathbb{R}^n . Indeed, let us assume by contradiction that $\text{dom } h^* \neq \mathbb{R}^n$. This implies the existence of $\xi \in \mathbb{R}^n$ with

$$\sup_{x \in \mathbb{R}^n} \{x \cdot \xi - h(x)\} = h^*(\xi) = +\infty.$$

Therefore, we can find a sequence $(x_k)_k \subset \mathbb{R}^n$, such that $h(x_k) + k < x_k \cdot \xi \leq |x_k| |\xi|$. Hence, $\lim_k |x_k| = +\infty$ and $\lim_k \frac{h(x_k)}{|x_k|} \leq |\xi|$, which contradict the superlinearity hypothesis (C2).

We define the functional I on $W^{1,1}(\Omega)$ by

$$I(u) := \int_{\Omega} [h(\nabla u(x)) + g(x, u(x))] dx.$$

A standard application of the Direct Method of the Calculus of Variations ensures the existence of a minimizer of I in $\phi + W_0^{1,1}(\Omega)$, for any $\phi \in W^{1,1}(\Omega)$. It has been shown in [18] that if h is superlinear the pointwise minimum and the pointwise maximum of the minimizers of I are in $\phi + W_0^{1,1}(\Omega)$ and are still minimizers of the same functional. We recall here a special case of a Haar-Radò type Theorem, which has been proven in its general form in [16, Theorem 4.2].

Theorem 2.3. Let h be convex and superlinear, g be measurable, convex in the second variable. Assume moreover that there exists a positive constant K such that

$$\forall x, y \in \mathbb{R}^n, \forall u, v \in \mathbb{R} \quad v \geq u + K|y - x| \Rightarrow g_v^+(y, v) \geq g_v^+(x, u),$$

where g_v^+ denotes the right derivative of g with respect to the second variable. If there exist two Lipschitz continuous functions $l^-, l^+ \in \phi + W_0^{1,1}(\Omega)$ of rank L on $\bar{\Omega}$ such that

$$l^-(x) \leq u(x) \leq l^+(x) \quad \text{a.e. in } \Omega$$

where $u \in \phi + W_0^{1,1}(\Omega)$ is the maximum or the minimum of the minimizers of I , then $|u(x) - u(y)| \leq L|x - y|$, for every Lebesgue points x and y .

We now define the integral functionals $I_{\pm\alpha}$ on $W^{1,1}(\Omega)$ by setting

$$I_{\pm\alpha}(u) := \int_{\Omega} [h(\nabla u(x)) \pm \alpha u(x)] dx,$$

where α is the positive constant appearing in (C3). A result by Cellina (see [6]) states that under our hypotheses on Ω and h , for every $x_0 \in \mathbb{R}^n$ and $c \in \mathbb{R}$ the functions $\omega_{\pm\alpha}(x): \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\omega_{\pm\alpha}(x) := \frac{n}{\pm\alpha} h^* \left(\pm \alpha \frac{x - x_0}{n} \right) + c, \quad (2.1)$$

are unique minimizers of $I_{\pm\alpha}$ in the sense that $I_{\pm\alpha}(\omega_{\pm\alpha}) < I_{\pm\alpha}(v)$ for every $v \in \omega_{\pm\alpha} + W_0^{1,1}(\Omega)$. We remark that the hypotheses on h guarantee that $\omega_{\pm\alpha} \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^n)$.

Given $u, v \in W^{1,1}(\Omega)$, we write $u \leq v$ on $\partial\Omega$ if $(u - v)^+ \in W_0^{1,1}(\Omega)$. Now we state a comparison result between minimizers of I and minimizers of $I_{\alpha}, I_{-\alpha}$.

Theorem 2.4. *Let u be a minimizer of I and $\omega_\alpha, \omega_{-\alpha}$ be as in (2.1) for some x_0 and c . Under hypotheses (C1)-(C3), if $u \geq \omega_\alpha$ on $\partial\Omega$, then $u \geq \omega_\alpha$ a.e. in Ω , and if $u \leq \omega_{-\alpha}$ on $\partial\Omega$, then $u \leq \omega_{-\alpha}$ a.e. in Ω .*

Proof. Let us define $E := \{x \in \Omega : u(x) < \omega_\alpha(x)\}$, $v := \min\{u, \omega_\alpha\}$, and $w := \max\{u, \omega_\alpha\}$. We argue by contradiction and assume that E has positive measure. By assumption $u \geq \omega_\alpha$ on $\partial\Omega$, therefore $v \in \omega_\alpha + W_0^{1,1}(\Omega)$ and $w \in u + W_0^{1,1}(\Omega)$. Since ω_α is the unique minimizer of I_α , we have

$$\begin{aligned} I_\alpha(\omega_\alpha) &= \int_{\Omega \setminus E} [h(\nabla\omega_\alpha) + \alpha\omega_\alpha] dx + \int_E [h(\nabla\omega_\alpha) + \alpha\omega_\alpha] dx \\ &< I_\alpha(v) = \int_{\Omega \setminus E} [h(\nabla\omega_\alpha) + \alpha\omega_\alpha] dx + \int_E [h(\nabla u) + \alpha u] dx; \end{aligned}$$

therefore, we have

$$\int_E [h(\nabla\omega_\alpha) + \alpha\omega_\alpha] dx < \int_E [h(\nabla u) + \alpha u] dx. \quad (2.2)$$

Analogously, since u is a minimizer of I , we get

$$\begin{aligned} I(u) &= \int_{\Omega \setminus E} [h(\nabla u) + g(x, u)] dx + \int_E [h(\nabla u) + g(x, u)] dx \\ &\leq I(w) = \int_{\Omega \setminus E} [h(\nabla u) + g(x, u)] dx + \int_E [h(\nabla\omega_\alpha) + g(x, \omega_\alpha)] dx; \end{aligned}$$

hence it follows

$$\int_E [h(\nabla u) + g(x, u)] dx \leq \int_E [h(\nabla\omega_\alpha) + g(x, \omega_\alpha)] dx. \quad (2.3)$$

Putting together (2.2) and (2.3), we obtain

$$\begin{aligned} \int_E [h(\nabla\omega_\alpha) + \alpha\omega_\alpha - h(\nabla u) - \alpha u \\ + h(\nabla u) + g(x, u) - h(\nabla\omega_\alpha) - g(x, \omega_\alpha)] dx < 0; \end{aligned}$$

i.e.,

$$\int_E [g(x, \omega_\alpha) - g(x, u) - \alpha(\omega_\alpha - u)] dx > 0,$$

and this is a contradiction with hypothesis (C3), which implies

$$-\alpha(\omega_\alpha - u) \leq g(x, \omega_\alpha) - g(x, u) \leq \alpha(\omega_\alpha - u),$$

for every $x \in E$. In the same way we can prove that $u \leq \omega_{-\alpha}$ on $\partial\Omega$ implies $u \leq \omega_{-\alpha}$ a.e. in Ω . \square

3. BASIC PROPERTIES OF A CLASS OF CONVEX FUNCTIONS

We will focus now on the particular case depending on the norm of the gradient, i.e.: $h(\xi) = f(|\xi|)$, where $f : \mathbb{R} \rightarrow \mathbb{R}$. Therefore, from now on we will posit the following hypotheses on the function $f : \mathbb{R} \rightarrow \mathbb{R}$.

- (F1) f is a convex, even function, increasing in \mathbb{R}^+ , such that $f(0) = 0$;
- (F2) the effective domain of f , $\text{dom} f$, is equal to \mathbb{R} ;
- (F3) f is superlinear, i.e.: $\lim_{|t| \rightarrow +\infty} \frac{f(t)}{|t|} = +\infty$.

We recall that assumption (F1) implies that $h^*(\xi) = f^*(|\xi|)$ and we state some basic and well-known facts on the function f that follow from the above assumptions. We will use them in the next section.

Lemma 3.1. *If (F1) and (F2) hold, then f^* is superlinear, i.e.: $\lim_{|\xi| \rightarrow +\infty} \frac{f^*(\xi)}{|\xi|} = +\infty$.*

Proof. Assumption (F1) implies that f^* is convex, even and $f^*(0) = 0$. Assumption (F2) means that, for any $t > 0$ there exists $\xi \geq 0$ such that $t \in \partial f^*(\xi)$. Then, either $f^*(\xi) = +\infty$ for any ξ sufficiently large (and then f^* is superlinear), or $\lim_{\xi \rightarrow +\infty} f^*(\xi) = +\infty$. The conclusion follows. \square

Remark 3.2. In Lemma 3.1 we used the simple fact that the superlinearity of an even convex function with effective domain coinciding with \mathbb{R} is equivalent to the fact that its derivative goes to $+\infty$ as the variable goes to $+\infty$. We will use again this property in the rest of the paper.

Remark 3.3. We also recall that, as in Remark 2.2, (F1) and (F3) imply that $\text{dom} f^* = \mathbb{R}$.

Lemma 3.4. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a convex function. Assume that there exist $\epsilon > 0$ and $0 < \tau^1 < \tau^2$ such that*

$$f(t) \geq f(s) + \xi_s(t-s) + \frac{\epsilon}{2}(t-s)^2 \quad (3.1)$$

for every $t, s \in (\tau^1, \tau^2)$ and $\xi_s \in \partial f(s)$. Then f^* is $C^{1,1}([\xi^1, \xi^2])$, where $\xi^1 = \sup \partial f(\tau^1)$, $\xi^2 = \inf \partial f(\tau^2)$,

$$\xi^2 - \xi^1 \geq \epsilon(\tau^2 - \tau^1) \quad \text{and} \quad |f^{*''}(\xi)| \leq \frac{1}{\epsilon} \text{ a.e. in } [\xi^1, \xi^2] \quad (3.2)$$

Proof. First of all we observe that 3.1 implies the strict convexity of f in (τ^1, τ^2) and, then, that f^* is $C^1(\xi^1, \xi^2)$, where $\xi^i \in \partial f(\tau^i)$, $i = 1, 2$. Indeed let us suppose that f^* is not differentiable in $\xi \in (\xi^1, \xi^2)$, i.e., $\partial f^*(\xi)$ is not a singleton; this means that there exist $s \neq t$ in $\partial f^*(\xi)$. From the monotonicity of ∂f^* we have that $s, t \in (\tau^1, \tau^2)$. Moreover $\xi \in \partial f(s) \cap \partial f(t)$, i.e. f is affine in $[s, t]$ and hence f is not strictly convex. Assume now that $f^* \notin C^1(\mathbb{R})$, so that there exists $\xi \in (\xi^1, \xi^2)$ such that $f^{*'}$ is not continuous in ξ ; since $f^{*'}$ is a monotone function, the left and right limits of $f^{*'}$ in ξ exist and they do not coincide; therefore, $\partial f^*(\xi)$, which is the convex envelope of these limits, cannot be a singleton, which is in contradiction with the fact that f^* is differentiable in ξ . By assumption, for any $t, s \in (\tau^1, \tau^2)$, we have

$$\begin{aligned} f(t) - f(s) - \xi_s(t-s) &\geq \frac{\epsilon}{2}(t-s)^2 \\ f(s) - f(t) - \xi_t(s-t) &\geq \frac{\epsilon}{2}(t-s)^2. \end{aligned}$$

By adding term by term, we get

$$(\xi_t - \xi_s)(t-s) \geq \epsilon(t-s)^2.$$

Passing to the limit for $t \rightarrow \tau^2$ and $s \rightarrow \tau^1$, dividing by $\tau^2 - \tau^1$, we obtain the first inequality in (3.2). Recalling that $\xi \in \partial f(\tau)$ if and only if $\tau \in \partial f^*(\xi) = f^{*'}(\xi)$, we get

$$|f^{*'}(\xi_t) - f^{*'}(\xi_s)| \leq \frac{1}{\epsilon} |\xi_t - \xi_s|,$$

proving the claim. \square

Lemma 3.5. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a convex function. Assume that there exist $\tau > 0$ such that $\partial f(\tau) = [\xi^1, \xi^2]$, where $\xi^2 > \xi^1$. Then $f^{*''}(\xi) = 0$ a.e. in (ξ^1, ξ^2) .*

Proof. The assumption $\partial f(\tau) = [\xi^1, \xi^2]$ holds if and only of $\tau \in \partial f^*(\xi^1)$, $i = 1, 2$. It follows that f^* is affine in (ξ^1, ξ^2) , proving the claim. \square

4. A REGULARITY RESULT

In this section we will consider the functional

$$\int_{\Omega} f(|\nabla u|) + g(x, u) \, dx$$

defined in $\phi + W_0^{1,1}(\Omega)$. We will assume that the boundary datum is in a special class: the one of the functions satisfying the Bounded Slope Condition. We will show that if Ω is uniformly convex, if suitable assumptions hold for f and g then the results of Section 3 and the Comparison Principle of Section 2 imply that all the minimizers of the functional are bounded by two Lipschitz barriers that coincide with ϕ on the boundary of Ω . Then the Haar-Radò type theorem (Theorem 2.3) will guarantee that the maximum and the minimum of the minimizers are Lipschitz continuous.

We recall here the Bounded Slope Condition introduced by Hartman and Stampacchia in [14].

Definition 4.1 (BSC). The function ϕ satisfies the *Bounded Slope Condition* of rank $M \geq 0$ if for every $\gamma \in \partial\Omega$ there exist $z_\gamma^-, z_\gamma^+ \in \mathbb{R}^n$ and $M \in \mathbb{R}$ such that

$$\forall \gamma' \in \partial\Omega \quad \phi(\gamma) + z_\gamma^- \cdot (\gamma' - \gamma) \leq \phi(\gamma') \quad (4.1)$$

$$\forall \gamma' \in \partial\Omega \quad \phi(\gamma) + z_\gamma^+ \cdot (\gamma' - \gamma) \geq \phi(\gamma') \quad (4.2)$$

and $|z_\gamma^\pm| \leq M$ for every $\gamma \in \partial\Omega$.

Remark 4.2. The (BSC) implies that ϕ is Lipschitz of rank M . Moreover it forces Ω to be convex, unless ϕ is affine. Necessary and sufficient conditions to the (BSC) are studied, respectively in [13] and [21].

In this section we will use the following set of assumptions on f , g and Ω . We assume that f satisfies either

(F4) for every $k \in \mathbb{N}$ there exist $\epsilon_k > 0$ and $\tau_k^i > k$, $i = 1, 2$ such that

$$(i) \quad f(t) \geq f(s) + \xi_s(t-s) + \frac{\epsilon_k}{2}(t-s)^2 \text{ for every } t, s \in (\tau_k^1, \tau_k^2) \text{ and } \xi_s \in \partial f(s)$$

$$(ii) \quad \lim_k \epsilon_k(\tau_k^2 - \tau_k^1) = \lambda > 0$$

$$(iii) \quad \lim_{k \rightarrow +\infty} \frac{1}{\epsilon_k(\tau_k^1)^3} = 0$$

or

(F5) for every $k \in \mathbb{N}$ there exist $\tau_k > k$ such that

$$(i) \quad \partial f(\tau_k) = [\xi_k^1, \xi_k^2]$$

$$(ii) \quad \lim_{k \rightarrow +\infty} \xi_k^2 - \xi_k^1 = \lambda > 0.$$

The function g is assumed to satisfy

(G1) g is Lipschitz continuous in the second variable, with Lipschitz constant equal to α , i.e.: $|g(x, u_1) - g(x, u_2)| \leq \alpha|u_1 - u_2|$ for every $x \in \Omega$ and $u_1, u_2 \in \mathbb{R}$

(G2) $g : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable and convex in the second variable. Denoting by g_v^+ the right derivative of g with respect to the second variable, we assume that there exists a positive constant K such that

$$\forall x, y \in \mathbb{R}^n, \forall u, v \in \mathbb{R} \quad v \geq u + K|y - x| \Rightarrow g_v^+(y, v) \geq g_v^+(x, u).$$

We consider a R -uniformly convex open bounded subset Ω of \mathbb{R}^n , where R -uniformly convex means that for every $\gamma \in \partial\Omega$ there exists $b_\gamma \in \mathbb{R}^n$ with $|b_\gamma| = 1$ such that

$$R b_\gamma \cdot (\gamma' - \gamma) \geq \frac{1}{2} |\gamma' - \gamma|^2, \quad (4.3)$$

As we previously recalled, in our setting, the existence of a solution of the minimum problem

$$\min_{v \in \phi + W_0^{1,1}(\Omega)} \int_{\Omega} [f(|\nabla v|) + g(x, v)] dx,$$

follows by the Direct Method of the Calculus of Variations.

The following theorem states the existence of Lipschitz barriers that coincide with the boundary datum on the boundary of Ω .

Theorem 4.3. *Assume that f satisfies hypotheses (F1)-(F3) and either (F4) or (F5). Let g satisfy assumption (G1). Let u be a minimizer of the functional*

$$\int_{\Omega} [f(|\nabla v|) + g(x, v)] dx \quad v \in \phi + W_0^{1,1}(\Omega)$$

where Ω is an open bounded R -uniformly convex set and $\phi : \Omega \rightarrow \mathbb{R}$ satisfies the (BSC) with rank M . Then there exist $\ell^+, \ell^- : \bar{\Omega} \rightarrow \mathbb{R}$, both Lipschitz of rank $L = L(R, f, M)$, such that

$$\ell^-(\gamma) = \phi(\gamma) = \ell^+(\gamma) \quad \text{for every } \gamma \in \partial\Omega$$

and

$$\ell^-(x) \leq u(x) \leq \ell^+(x) \quad \text{for almost every } x \in \Omega.$$

Proof. It is sufficient to construct the function ℓ^- , the construction of ℓ^+ going on exactly in the same way.

We fix a point $\gamma \in \partial\Omega$ and we consider also the vector z_γ^- involved in the definition of the (BSC) at the point γ . The proof of the theorem will be achieved when we will have shown that there exist x_γ, c_γ , such that the set

$$\Omega_{x_\gamma, c_\gamma} := \left\{ x \in \mathbb{R}^n : \frac{n}{\alpha} f^* \left(\frac{\alpha}{n} |x - x_\gamma| \right) + c_\gamma - z_\gamma^- \cdot (x - \gamma) - \phi(\gamma) < 0 \right\}$$

contains Ω and $\gamma \in \partial\Omega_{x_\gamma, c_\gamma} \cap \partial\Omega$. In fact, by this last property, we have immediately that

$$l_{x_\gamma, c_\gamma}(x) := \frac{n}{\alpha} f^* \left(\frac{\alpha}{n} |x - x_\gamma| \right) + c_\gamma \leq z_\gamma^- \cdot (x - \gamma) + \phi(\gamma) \leq \phi(x)$$

for any $x \in \partial\Omega$. The Comparison Principle (2.4) then implies that

$$l_{x_\gamma, c_\gamma}(x) \leq z_\gamma^- \cdot (x - \gamma) + \phi(\gamma) \leq u(x) \quad \text{a.e. on } \Omega.$$

We get the result simply by defining

$$\ell^-(x) = \sup_{\gamma \in \partial\Omega} l_{x_\gamma, c_\gamma}(x).$$

We divide the proof in several steps.

STEP 1. In this step we state some properties of the auxiliary domain defined as follows: fix $a \in \mathbb{R}^n$, $b > 0$ and let

$$\Omega_b := \{x \in \mathbb{R}^n : \frac{n}{\alpha} f^* \left(\frac{\alpha}{n} |x| \right) - a \cdot x - b < 0\}$$

Whenever we assume that (F1) and (F2) hold, Lemma 3.1 implies that Ω_b is bounded for every b and that, for $b > 0$, 0 is contained in its interior. Moreover, by the continuity of f^* there exist $x_{1,b}, x_{2,b}$ in the set $\partial\Omega_b = \{x \in \mathbb{R}^n : \frac{n}{\alpha} f^* \left(\frac{\alpha}{n} |x| \right) - a \cdot x - b = 0\}$ such that $\frac{n}{\alpha} f^* \left(\frac{\alpha}{n} |x_{1,b}| \right) \leq \frac{n}{\alpha} f^* \left(\frac{\alpha}{n} |x| \right) \leq \frac{n}{\alpha} f^* \left(\frac{\alpha}{n} |x_{2,b}| \right)$ for every $x \in \partial\Omega_b$ and such that $\overline{B}(0, |x_{1,b}|) \subset \Omega_b \subset B(0, |x_{2,b}|)$. It is immediate to see that

$$\lim_{b \rightarrow +\infty} |x_{1,b}| = +\infty \tag{4.4}$$

Indeed, as in Remark 2.2, the superlinearity of f implies that $\text{dom } f^* = \mathbb{R}$; hence $\frac{n}{\alpha} f^* \left(\frac{\alpha}{n} |x| \right) - a \cdot x$ is bounded on every bounded set. This implies that if $b \rightarrow +\infty$, then $|x| \rightarrow +\infty$ for $x \in \partial\Omega_b$; in particular (4.4) holds true.

STEP 2. In this step we fix $\gamma \in \partial\Omega$ and η a unit vector in the normal cone to Ω in γ and we aim to select a special domain Ω_{b_η} , for $a = z_\gamma^-$.

We first assume that (F4) holds. Let $k \in \mathbb{N}$ be such that $k > M \geq |z_\gamma^-|$ be fixed. Thanks to (4.4), we can choose $\bar{b} > 0$ such that $|x_{1,b}| > k$, for every $b > \bar{b}$. According to assumption (i) of (F4), let $\epsilon_k > 0$ and $\tau_k^i > k$, $i = 1, 2$ be such that $f(t) \geq f(s) + \xi_s(t - s) + \frac{\epsilon_k}{2}(t - s)^2$ for every $t, s \in (\tau_k^1, \tau_k^2)$ and $\xi_s \in \partial f(s)$. We notice that, thanks to assumption (F.3), we can also assume that $\xi_s > k$ and $\xi_s > |x_{2,\bar{b}}|$ for any $\xi_s \in \partial f(s)$, $s \in [\tau_k^1, \tau_k^2]$. By Lemma 3.4, the function f^* is $C^{1,1}(\xi_k^1, \xi_k^2)$, where $\xi_k^i \in \partial f(\tau_k^i)$, $i = 1, 2$, and $|f^{*''}(\xi)| \leq \frac{1}{\epsilon_k}$ for a.e. $\xi \in (\xi_k^1, \xi_k^2)$. In the case where (F5) holds, we immediately obtain that f^* is in $C^2(\xi_k^1, \xi_k^2)$ and $f^{*''} = 0$ in (ξ_k^1, ξ_k^2) . In both cases we choose now $\frac{\alpha}{n} |\xi_k| = \frac{\xi_k^1 + \xi_k^2}{2}$ and we observe that

$$f^{*'} \left(\frac{\alpha}{n} |\xi_k| \right) =: \tau_k \geq \tau_k^1 > k > M > |z_\gamma^-|, \tag{4.5}$$

$$\frac{\alpha}{n} |\xi_k| > |x_{2,\bar{b}}|. \tag{4.6}$$

We prove now that there exist $x_\eta \in \mathbb{R}^n$ and $b_\eta \in \mathbb{R}$ such that $|x_\eta| = |\xi_k|$,

$$\frac{n}{\alpha} f^* \left(\frac{\alpha}{n} |\xi_k| \right) - z_\gamma^- \cdot x_\eta - b_\eta = 0, \quad \text{i.e. } x_\eta \in \partial\Omega_{b_\eta}, \tag{4.7}$$

and $f^{*'} \left(\frac{\alpha}{n} |x_\eta| \right) \frac{x_\eta}{|x_\eta|} - z_\gamma^- = \lambda \eta$ for some $\lambda \neq 0$; i.e., the outward normal to $\partial\Omega_{b_\eta}$ in x_η is parallel to η .

First of all we observe that the estimate (4.5) guarantees the existence of a one-to-one correspondence between S^{n-1} and $f^{*'} \left(\frac{\alpha}{n} |\xi_k| \right) S^{n-1} - z_\gamma^-$. Therefore, for every $\eta \in S^{n-1}$,

there exists a unique $\nu_\eta \in S^{n-1}$ such that $f^{*\prime}(\frac{\alpha}{n}|\xi_k|)\nu_\eta - z_\gamma^- = \lambda\eta$, for a suitable $\lambda > 0$. Hence, let us define b_η by

$$b_\eta := \frac{n}{\alpha} f^*\left(\frac{\alpha}{n}|\xi_k|\right) - z_\gamma^- \cdot \nu_\eta |\xi_k|.$$

We consider now $\Omega_{b_\eta} := \{x \in \mathbb{R}^n : \frac{n}{\alpha} f^*(\frac{\alpha}{n}|x|) - z_\gamma^- \cdot x - b_\eta < 0\}$; we have that $\nu_\eta |\xi_k| \in \partial\Omega_{b_\eta}$ by definition of b_η and $\partial\Omega_{b_\eta}$ is \mathcal{C}^1 in a neighborhood of $\nu_\eta |\xi_k|$. Then the outward normal to $\partial\Omega_{b_\eta}$ in $\nu_\eta |\xi_k|$ is parallel to

$$f^{*\prime}\left(\frac{\alpha}{n}|\nu_\eta|\xi_k|\right) \frac{\nu_\eta|\xi_k|}{|\xi_k|} - z_\gamma^- = f^{*\prime}\left(\frac{\alpha}{n}|\xi_k|\right) \nu_\eta - z_\gamma^- = \lambda\eta.$$

This proves that $\nu_\eta |\xi_k|$ is the point x_η we were looking for.

By the implicit function theorem we also infer that there exists a neighborhood of the point x_η in which $\partial\Omega_{b_\eta} = \{x \in \mathbb{R}^n : f^*(\frac{\alpha}{n}|x|) - z_\gamma^- \cdot x - b_\eta = 0\}$ is at least $\mathcal{C}^{1,1}$.

We notice that $|x_{1,b_\eta}| > k$, because $b_\eta > \bar{b}$ by (4.6).

STEP 3. We are interested in proving that, for k sufficiently large, we can find a ball of radius R contained in Ω_{b_η} that touches $\partial\Omega_{b_\eta}$ in x_η . To reach this aim in this step we compute the principal curvatures of $\partial\Omega_{b_\eta}$ in the neighborhood of x_η .

We can estimate the principal curvatures of $\partial\Omega_{b_\eta}$ in almost every point $x \in \partial\Omega_{b_\eta}$ such that $\xi_k^1 \leq \frac{\alpha}{n}|x| \leq \xi_k^2$. Each principal curvature is less or equal to the greater eigenvalue of the Hessian matrix of the implicit function ψ defined by (4.7) and it can be estimated by the norm of the matrix itself. This means to estimate

$$\frac{\partial_{ij}\psi(\hat{x})}{(1 + |\nabla\psi(\hat{x})|^2)^{3/2}},$$

for $i, j = 1, \dots, n-1$, where we have assumed, without loss of generality, that the n -component x_n of the vector x is implicitly defined with respect to the first $n-1$ components $\hat{x} := (x_1, x_2, \dots, x_{n-1})$.

From now on, to simplify the notations, we drop the indices η and γ and we denote by F the function $F(x) = \frac{n}{\alpha} f^*(\frac{\alpha}{n}|x|) - z \cdot x - b$. It follows, again by the implicit function theorem, that, for \mathcal{H}^{n-1} -a.e. $\hat{x} \in \mathbb{R}^{n-1}$

$$\begin{aligned} & \frac{\partial_{ij}\psi}{(1 + |\nabla\psi|^2)^{3/2}} \\ &= \frac{-\partial_{ij}F(\partial_n F)^2 + \partial_{in}F\partial_j F\partial_n F + \partial_{jn}F\partial_i F\partial_n F - \partial_{nn}F\partial_i F\partial_j F}{|\nabla F|^3} \end{aligned} \quad (4.8)$$

with F and its derivative computed in the point $x = (\hat{x}, \psi(\hat{x}))$. Hence, we obtain that (4.8) is equal to

$$\begin{aligned} &= \left((f^{*\prime})^2 \left(\frac{\alpha}{n}|x| \right) - 2f^{*\prime} \left(\frac{\alpha}{n}|x| \right) \frac{zx}{|x|} + |z|^2 \right)^{-3/2} \\ & \cdot \left\{ \left[-\frac{\alpha}{n} f^{*\prime\prime} \left(\frac{\alpha}{n}|x| \right) \frac{x_i x_j}{|x|^2} + f^{*\prime} \left(\frac{\alpha}{n}|x| \right) \frac{x_i x_j}{|x|^3} \right] \cdot \left[f^{*\prime} \left(\frac{\alpha}{n}|x| \right) \frac{x_n}{|x|} - z_n \right]^2 \right. \\ & + \left[\frac{\alpha}{n} f^{*\prime\prime} \left(\frac{\alpha}{n}|x| \right) \frac{x_i x_n}{|x|^2} - f^{*\prime} \left(\frac{\alpha}{n}|x| \right) \frac{x_i x_n}{|x|^3} \right] \cdot \left[f^{*\prime} \left(\frac{\alpha}{n}|x| \right) \frac{x_j}{|x|} - z_j \right] \left[f^{*\prime} \left(\frac{\alpha}{n}|x| \right) \frac{x_n}{|x|} - z_n \right] \\ & + \left[\frac{\alpha}{n} f^{*\prime\prime} \left(\frac{\alpha}{n}|x| \right) \frac{x_j x_n}{|x|^2} - f^{*\prime} \left(\frac{\alpha}{n}|x| \right) \frac{x_j x_n}{|x|^3} \right] \cdot \left[f^{*\prime} \left(\frac{\alpha}{n}|x| \right) \frac{x_i}{|x|} - z_i \right] \left[f^{*\prime} \left(\frac{\alpha}{n}|x| \right) \frac{x_n}{|x|} - z_n \right] \\ & + \left[-\frac{\alpha}{n} f^{*\prime\prime} \left(\frac{\alpha}{n}|x| \right) \frac{x_n^2}{|x|^2} - f^{*\prime} \left(\frac{\alpha}{n}|x| \right) \frac{|x|^2 - x_n^2}{|x|^3} \right] \cdot \left[f^{*\prime} \left(\frac{\alpha}{n}|x| \right) \frac{x_i}{|x|} - z_i \right] \\ & \left. \cdot \left[f^{*\prime} \left(\frac{\alpha}{n}|x| \right) \frac{x_j}{|x|} - z_j \right] \right\}, \end{aligned}$$

with $(x_1, \dots, x_{n-1}) = \hat{x}$ and $x_n = \psi(\hat{x})$. By computation, it follows that

$$\begin{aligned} & \left| \frac{\partial_{ij}\psi(\hat{x})}{(1 + |\nabla\psi(\hat{x})|^2)^{3/2}} \right| \leq \\ & \leq \left[\alpha f^{*''}\left(\frac{\alpha}{n}|x|\right)|z|^2 + \frac{(f^{*'})^3\left(\frac{\alpha}{n}|x|\right)n}{|x|} + \frac{(f^{*'})^2\left(\frac{\alpha}{n}|x|\right)n|z|}{|x|} + \frac{f^{*'}\left(\frac{\alpha}{n}|x|\right)n|z|^2}{|x|} \right] \\ & \quad \cdot \left| f^{*'}\left(\frac{\alpha}{n}|x|\right) \frac{x}{|x|} - z \right|^{-3} \end{aligned}$$

In the case where (F4) holds we use Lemma 3.4 to estimate

$$\left[\alpha f^{*''}\left(\frac{\alpha}{n}|x|\right)|z|^2 \right] \left| f^{*'}\left(\frac{\alpha}{n}|x|\right) \frac{x}{|x|} - z \right|^{-3} \leq C \frac{1}{\epsilon_k (\tau_k^1)^3}$$

for a suitable constant C . In the other case

$$\left[\alpha f^{*''}\left(\frac{\alpha}{n}|x|\right)|z|^2 \right] \left| f^{*'}\left(\frac{\alpha}{n}|x|\right) \frac{x}{|x|} - z \right|^{-3} = 0$$

Lemma 3.1 implies also that

$$\begin{aligned} & \lim_{|x| \rightarrow +\infty} \frac{1}{|x|} \left[(f^{*'})^3\left(\frac{\alpha}{n}|x|\right)n + (f^{*'})^2\left(\frac{\alpha}{n}|x|\right)n|z| + f^{*'}\left(\frac{\alpha}{n}|x|\right)n|z|^2 \right] \\ & \quad \cdot \left| f^{*'}\left(\frac{\alpha}{n}|x|\right) \frac{x}{|x|} - z \right|^{-3} = 0. \end{aligned}$$

Therefore, under either assumption (iii) of (F4) or (i) of (F5), we can choose k such that all the principal curvatures of $\partial\Omega_{b_\eta} \cap \{x \in \mathbb{R}^n : \xi_k^1 \leq \frac{\alpha}{n}|x| \leq \xi_k^2\}$ are bounded by a constant $\frac{1}{R_k}$ less than $\frac{1}{R}$. This implies that, for k sufficiently large, the ball $B(x_\eta - R\eta, R)$ contains x_η in its boundary and is “locally” included in Ω_{b_η} : this means that $U_{x_\eta} \cap B(x_\eta - R\eta, R) \subseteq U_{x_\eta} \cap \Omega_{b_\eta}$, for a suitable neighborhood U_{x_η} of x_η .

STEP 4. Now, in the next three steps we want to show that, for k sufficiently large, the ball $B(x_\eta - R\eta, R)$ is entirely contained in Ω_{b_η} . In the present step and in the next one, we suppose that $\partial\Omega_{b_\eta}$ is globally $\mathcal{C}^{1,1}$, and we want to compute the norm of the points of $\partial\Omega_{b_\eta}$ in a suitable neighborhood of x_η . First of all, we recall that, given a function $\psi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$, such that $\psi(0) = 0$, $\nabla\psi(0) = 0$, and $\psi \in \mathcal{C}^{1,1}(B(0, \delta))$, with the absolute value of the curvature almost everywhere bounded by $1/R_k > 0$, an easy computation shows that

$$|\psi(\zeta)| \leq R_k - \sqrt{R_k^2 - \delta^2}, \quad (4.9)$$

for $|\zeta| \leq \delta$. We can assume without loss of generality that the system of coordinates x in \mathbb{R}^n is such that the tangent plane ρ to $B(0, |x_\eta|)$ in x_η is $\{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n = x_{\eta_n}\}$, so that $|x_\eta| = |x_{\eta_n}|$. Now we fix the tangent plane π to $\partial\Omega_{b_\eta}$ in x_η . We consider a new system of coordinates (ζ, t) , $\zeta \in \mathbb{R}^{n-1}$, $t \in \mathbb{R}$, such that the plane π is the set $\{(\zeta, t) : t = 0\}$, x_η corresponds to $(\zeta, t) = 0$, and $\partial\Omega_{b_\eta}$ is the graph of a function $\psi : A \subseteq \mathbb{R}^{n-1} \rightarrow \mathbb{R}$, for a suitable open set A . We denote by $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ the change of variables that brings back (ζ, t) into x . We have

$$|T(\zeta, \psi(\zeta)) - T(\zeta, 0)| \leq R_k - \sqrt{R_k^2 - \delta^2} \quad \text{for every } |\zeta| \leq \delta, \quad (4.10)$$

by (4.9), using the curvature estimate of the previous step. Then, we have

$$\begin{aligned} & \left| (T(\zeta, 0)_1, \dots, T(\zeta, 0)_{n-1}, x_{\eta_n}) \right. \\ & \quad \left. - (T(\zeta, 0)_1, \dots, T(\zeta, 0)_{n-1}, \text{sign } x_{\eta_n} \sqrt{|x_\eta|^2 - T(\zeta, 0)_1^2 + \dots - T(\zeta, 0)_{n-1}^2}) \right| \\ & \leq |x_\eta| - \sqrt{|x_\eta|^2 - \delta^2} \quad \text{for } |\zeta| \leq \delta. \end{aligned} \quad (4.11)$$

We also have

$$|T(\zeta, 0) - (T(\zeta, 0)_1, \dots, T(\zeta, 0)_{n-1}, x_{\eta_n})| \leq \delta \sin \theta_\eta, \quad \text{for } |\zeta| \leq \delta, \quad (4.12)$$

where θ_η is the angle between the planes π and ρ . We observe that θ_η is also the angle between the normal directions to π and ρ , η and $x_\eta/|x_\eta|$ respectively. We recall that $|x_{1,b_\eta}|$ and $|x_{2,b_\eta}|$ are the points of $\partial\Omega_{b_\eta}$ such that $\frac{n}{\alpha}f^*(\frac{\alpha}{n}|x_{1,b_\eta}|) \leq \frac{n}{\alpha}f^*(\frac{\alpha}{n}|x|) \leq \frac{n}{\alpha}f^*(\frac{\alpha}{n}|x_{2,b_\eta}|)$ for every $x \in \partial\Omega_{b_\eta}$. The set $\overline{\Omega}_{b_\eta}$ contains $\overline{c\partial}(x_\eta, B(0, |x_{1,b_\eta}|))$ so that the normal cone to $\overline{c\partial}(x_\eta, B(0, |x_{1,b_\eta}|))$ contains the normal to Ω_{b_η} in x_η , i.e. the angle θ_η between η and $\frac{x_\eta}{|x_\eta|}$ satisfies

$$\theta_\eta = \arccos \eta \cdot \frac{x_\eta}{|x_\eta|} \leq \arcsin \left(\sup_{|x_{1,b_\eta}| < |x| < |x_{2,b_\eta}|} \inf_{y \in B(0, |x_{1,b_\eta}|)} \frac{x-y}{|x-y|} \cdot \frac{x}{|x|} \right). \quad (4.13)$$

So that we obtain

$$\begin{aligned} & |T(\zeta, 0) - (T(\zeta, 0)_1, \dots, T(\zeta, 0)_{n-1}, x_{\eta_n})| \\ & \leq \delta \left(\sup_{|x_{1,b_\eta}| < |x| < |x_{2,b_\eta}|} \inf_{y \in B(0, |x_{1,b_\eta}|)} \frac{x-y}{|x-y|} \cdot \frac{x}{|x|} \right). \end{aligned} \quad (4.14)$$

Hence, we can conclude that

$$\begin{aligned} & \left| |T(\zeta, \psi(\zeta))| - |x_\eta| \right| \\ & \leq \left| T(\zeta, \psi(\zeta)) - \left(T(\zeta, 0)_1, \dots, T(\zeta, 0)_{n-1}, \text{sign } x_{\eta_n} \sqrt{|x_\eta|^2 - T(\zeta, 0)_1^2 + \dots - T(\zeta, 0)_{n-1}^2} \right) \right| \\ & \leq R_k - \sqrt{R_k^2 - \delta^2} + |x_\eta| - \sqrt{|x_\eta|^2 - \delta^2} \\ & \quad + \delta \left(\sup_{|x_{1,b_\eta}| < |x| < |x_{2,b_\eta}|} \inf_{y \in B(0, |x_{1,b_\eta}|)} \frac{x-y}{|x-y|} \cdot \frac{x}{|x|} \right). \end{aligned} \quad (4.15)$$

STEP 5. In this step we show that fixed δ , by choosing k large enough (in particular $k > R$) we can make the quantity in (4.15) as small as we want. In particular, for $\delta > 4R$ (where R is the constant appearing in (4.3)), we can choose k such that the norm of the points of $\partial\Omega_{b_\eta}$, $T(\zeta, \psi(\zeta))$, with $|\zeta| \leq \delta$ is between $|x_\eta| - \frac{\lambda}{4}$ and $|x_\eta| + \frac{\lambda}{4}$ (where λ is the constant appearing in assumption (ii) of (F.4) and (F.5)).

We denote by Λ_k the minimum between R_k and $|x_{1,b_\eta}|$. We recall that both are greater than R . We fix $\delta > 4R$. We want now to prove that we can choose k such that

$$\delta \left(\sup_{|x_{1,b_\eta}| < |x| < |x_{2,b_\eta}|} \inf_{y \in B(0, |x_{1,b_\eta}|)} \frac{x-y}{|x-y|} \cdot \frac{x}{|x|} \right) \leq \frac{\lambda}{8} \quad (4.16)$$

and

$$2|\Lambda_k - \sqrt{\Lambda_k^2 - \delta^2}| \leq \frac{\lambda}{8} \quad (4.17)$$

By the estimates in Step 3, we have $\lim_{k \rightarrow \infty} R_k = +\infty$. Hence,

$$\lim_k 2|\Lambda_k - \sqrt{\Lambda_k^2 - \delta^2}| = 0.$$

On the other hand, we remark that

$$\inf_{y \in B(0, |x_{1,b}|)} \frac{x-y}{|x-y|} \cdot \frac{x}{|x|} = \sqrt{1 - \frac{|x_{1,b}|^2}{|x|^2}}$$

and $\sqrt{1 - \frac{|x_{1,b}|^2}{|x|^2}} \leq \sqrt{1 - \frac{|x_{1,b}|^2}{|x_{2,b}|^2}}$. Therefore, if we prove that

$$\lim_{b \rightarrow +\infty} \frac{|x_{2,b}| - |x_{1,b}|}{|x_{1,b}|} = 0 \quad (4.18)$$

we are done.

For every $t > 0$ we identify $\frac{n}{\alpha}f^*(\frac{\alpha}{n}t) =: \tilde{f}(t)$. Then \tilde{f} is convex and satisfies assumption (F1) and (F2). By Lemma 3.1, recalling Remark 3.2, $\lim_{t \rightarrow +\infty} \tilde{f}'(t) = +\infty$. We can consider the inequality $|x_{2,b}| - |x_{1,b}| \leq |y_{2,b}| - |y_{1,b}|$ where

$$\begin{aligned} |z_\gamma^-| |y_{2,b}| + b &= \tilde{f}'(|x_{1,b}|)(|y_{2,b}| - \tilde{f}^{-1}(b)) + b \\ -|z_\gamma^-| |y_{1,b}| + b &= \tilde{f}'(|x_{1,b}|)(|y_{1,b}| - \tilde{f}^{-1}(b)) + b \end{aligned}$$

so that $|y_{2,b}| = \frac{\tilde{f}'(|x_{1,b}|)\tilde{f}^{-1}(b)}{\tilde{f}'(|x_{1,b}|)-|z_\gamma^-|}$ and $|y_{1,b}| = \frac{\tilde{f}'(|x_{1,b}|)\tilde{f}^{-1}(b)}{\tilde{f}'(|x_{1,b}|)+|z_\gamma^-|}$. We have $|x_{1,b}| \geq |y_{1,b}|$ and then $\frac{\tilde{f}^{-1}(b)}{|x_{1,b}|} \leq \frac{\tilde{f}'(|x_{1,b}|)+|z_\gamma^-|}{\tilde{f}'(|x_{1,b}|)}$ so that $\lim_{b \rightarrow +\infty} \frac{\tilde{f}^{-1}(b)}{|x_{1,b}|}$ is finite; then we obtain

$$\lim_{b \rightarrow +\infty} \frac{|x_{2,b}| - |x_{1,b}|}{|x_{1,b}|} \leq \lim_{b \rightarrow +\infty} \frac{2|z_\gamma^-|\tilde{f}'(|x_{1,b}|)\tilde{f}^{-1}(b)}{|x_{1,b}|[(\tilde{f}'(|x_{1,b}|))^2 - |z_\gamma^-|^2]} = 0.$$

Therefore, we can conclude by putting together estimates (4.15), (4.16), and (4.17), that

$$\left|T(\zeta, \psi(\zeta)) - |x_\eta|\right| = \left|T(\zeta, \psi(\zeta)) - \frac{n}{2\alpha}(\xi_k^1 + \xi_k^2)\right| \leq \lambda/4, \quad \text{for } |\zeta| \leq \delta. \quad (4.19)$$

STEP 6.

In this step we conclude the proof of the existence of the ball of radius R contained in Ω_{b_η} and touching $\partial\Omega_{b_\eta}$ in x_η .

We recall that in Steps 4 and 5, we assumed $\partial\Omega_{b_\eta}$ to be globally $C^{1,1}$. We already know that $\partial\Omega_{b_\eta}$ is $C^{1,1}$ near x_η by hypothesis (F.4) (i)/ (F.5)(i). Now, by choosing k sufficiently large, we can make $\partial\Omega_{b_\eta}$ actually $C^{1,1}$ at least in the points $T(\zeta, \psi(\zeta))$, for $|\zeta| \leq \delta$. Indeed, hypothesis (F.4)(ii) (or (F.5)(ii)) implies $\xi_k^2 - \xi_k^1 > \lambda/2$ for k large enough (see (3.2)), and f^* is $C^{1,1}(\xi_k^1, \xi_k^2)$. Hence, estimate (4.19) and a comparison argument show that $\partial\Omega_{b_\eta}$ is $C^{1,1}$ in its points $T(\zeta, \psi(\zeta))$, for $|\zeta| \leq \delta$, since they lie in the set $\{x \in \mathbb{R}^n : \xi_k^1 \leq \frac{\alpha}{n}|x| \leq \xi_k^2\}$. In particular, in the same points, $\partial\Omega_{b_\eta}$ has curvature less than $1/R_k \leq 1/R$.

Now we consider the distances of the points $(T(\zeta, \psi(\zeta))_1, \dots, T(\zeta, \psi(\zeta))_{n-1}, x_{\eta_n})$ from x_η , for $|\zeta| = \delta$. We can choose k sufficiently large such that the minimum of these distances, p_δ , can be estimated from below by $\delta/2 > 2R$. Indeed, we have

$$\begin{aligned} p_\delta &= \left[|T(\zeta, 0) - T(0, 0)| - \frac{|T(\zeta, \psi(\zeta)) - T(\zeta, 0)|}{\cos \theta_\eta} \sin \theta_\eta \right] \cos \theta_\eta \\ &= \left[|\zeta| - \frac{|\psi(\zeta)|}{\cos \theta_\eta} \sin \theta_\eta \right] \cos \theta_\eta \\ &\geq \left[\delta - \frac{R_k - \sqrt{R_k^2 - \delta^2}}{\cos \theta_\eta} \sin \theta_\eta \right] \cos \theta_\eta \\ &= \delta \cos \theta_\eta - \left(R_k - \sqrt{R_k^2 - \delta^2} \right) \sin \theta_\eta, \end{aligned} \quad (4.20)$$

where we recall that θ_η is the angle between the vector η and the direction $\frac{x_\eta}{|x_\eta|}$. Using estimate (4.13), we get

$$p_\delta \geq \delta \frac{|x_{1,b_\eta}|}{|x_{2,b_\eta}|} - \left(R - \sqrt{R^2 - \delta^2} \right) \sqrt{1 - \frac{|x_{1,b_\eta}|^2}{|x_{2,b_\eta}|^2}} > \frac{\delta}{2} > 2R, \quad (4.21)$$

for k sufficiently large, thanks to (4.18).

We now choose k such that also $|x_{1,b_\eta}| > 4R$ holds true. We recall that the curvature of $\partial\Omega_{b_\eta}$ in its points $T(\zeta, \psi(\zeta))$, for $|\zeta| \leq \delta$, is less than $1/R$. This property together with estimate (4.21) implies that the convex envelope between $B(0, |x_{1,b_\eta}|)$ and the points $T(\zeta, \psi(\zeta))$, for $|\zeta| \leq \delta$, contains in its interior the ball $B(x_\eta - R\eta, R)$. Since $\bar{\Omega}_{b_\eta}$ contains this convex envelope, we conclude that $B(x_\eta - R\eta, R)$ is contained in Ω_{b_η} and touches $\partial\Omega_{b_\eta}$ in x_η .

STEP 7. In this step we conclude the proof. It is enough to define $x_\gamma := \gamma - x_\eta$ and $c_\gamma := \phi(\gamma) - \frac{n}{\alpha}f^*\left(\frac{\alpha}{n}|x_\eta|\right)$ and to consider the sets $\Omega_{x_\gamma, c_\gamma}$ and the function l_{x_γ, c_γ} defined at the beginning of the proof. In this way, using the fact that Ω is R -uniformly convex, we have

$$\Omega \subseteq B(\gamma - R\eta, R) = \gamma - x_\eta + B(x_\eta - R\eta, R) \subseteq \gamma - x_\eta + \Omega_{b_\eta} = \Omega_{x_\gamma, c_\gamma}.$$

Indeed,

$$\begin{aligned}
& \gamma - x_\eta + \Omega_{b_\eta} \\
&= \left\{ x \in \mathbb{R}^n : \frac{n}{\alpha} f^* \left(\frac{\alpha}{n} |x - \gamma + x_\eta| \right) - z_\gamma^- \cdot (x - \gamma + x_\eta) - b_\eta < 0 \right\} \\
&= \left\{ x \in \mathbb{R}^n : \frac{n}{\alpha} f^* \left(\frac{\alpha}{n} |x - x_\gamma| \right) + \phi(\gamma) - z_\gamma^- \cdot x_\eta - b_\eta - z_\gamma^- \cdot (x - \gamma) - \phi(\gamma) < 0 \right\} \\
&= \left\{ x \in \mathbb{R}^n : \frac{n}{\alpha} f^* \left(\frac{\alpha}{n} |x - x_\gamma| \right) + c_\gamma - z_\gamma^- \cdot (x - \gamma) - \phi(\gamma) < 0 \right\},
\end{aligned}$$

since

$$z_\gamma^- \cdot x_\eta + b_\eta = \frac{n}{\alpha} f^* \left(\frac{\alpha}{n} |x_\eta| \right).$$

It is immediate to see that $\gamma \in \partial\Omega_{x_\gamma, c_\gamma} \cap \partial\Omega$. \square

We are now ready to state the main result.

Theorem 4.4. *Under the same assumption of Theorem 4.3, and the additional assumption (G2), the maximum and the minimum of the minimizers of the functional*

$$\int_{\Omega} [f(|\nabla v|) + g(x, v)] dx \quad v \in \phi + W_0^{1,1}(\Omega)$$

are Lipschitz continuous of rank $L = L(R, f, M)$.

Proof. The result is immediate by applying Theorem 2.3 with the barriers constructed in Theorem 4.3. \square

The following theorem is a very special case of the previous one and we state it because we think that it has an interest in itself. Thanks to the fact that here we assume ϕ to be constant we can drop all the ‘technical’ assumptions on f . Above all we underline that assumptions (F4) and (F5) are no more needed, but not even (F2) has to be assumed.

Theorem 4.5. *Assume that f satisfies hypotheses (F1) and (F3). Let g satisfy assumptions (G1) and (G2). Let u be the maximum or the minimum of the minimizers of the functional*

$$\int_{\Omega} [f(|\nabla v|) + g(x, v)] dx \quad v \in \phi + W_0^{1,1}(\Omega)$$

where Ω is an open bonded R -uniformly convex set and the boundary datum ϕ is a constant function. Then u is Lipschitz continuous of rank $L = L(R, f, M)$

Proof. Since ϕ is constant, we can choose in (4.1) and (4.2) $z_\gamma^- = z_\gamma^+ = 0$ for every $\gamma \in \partial\Omega$. In particular, the radial symmetry of f^* implies that Ω_{b_η} is a ball. The conclusion immediately follows. \square

The next corollary is in the same flavor of some results in [18, 19], where an extra geometrical assumption on the faces of the epigraph of f implies regularity of all minimizers.

Corollary 4.6. *Let us assume that f and g satisfy assumptions (F1), (F3), (G1), and (G2). Assume moreover that the faces of the epigraph of f are bounded by a positive constant K . Then, if either ϕ is constant, or the assumptions (F2) and (F.4) or (F.5) hold true, every minimizer of the functional*

$$\int_{\Omega} [f(|\nabla v|) + g(x, v)] dx \quad v \in \phi + W_0^{1,1}(\Omega) \quad (4.22)$$

is Lipschitz continuous of rank $L + K$.

Proof. We follow closely the argument in [18]. Let u and v be two minimizers, and let A be the subset of Ω where $(\nabla u(x), f(|\nabla u(x)|))$ and $(\nabla v(x), f(|\nabla v(x)|))$ belong to the

projection of the same face of $\text{epi}(f)$. If $|\Omega \setminus A| > 0$, the convexity of f and g implies

$$\begin{aligned}
& \int_{\Omega} \left[f\left(\left|\frac{1}{2}\nabla u(x) + \frac{1}{2}\nabla v(x)\right|\right) + g\left(x, \frac{1}{2}u(x) + \frac{1}{2}v(x)\right) \right] dx \\
& < \int_{\Omega \setminus A} \left[\frac{1}{2}f(|\nabla u(x)|) + \frac{1}{2}f(|\nabla v(x)|) + \frac{1}{2}g(x, u(x)) + \frac{1}{2}g(x, v(x)) \right] dx \\
& \quad + \int_A \left[\frac{1}{2}f(|\nabla u(x)|) + \frac{1}{2}f(|\nabla v(x)|) + \frac{1}{2}g(x, u(x)) + \frac{1}{2}g(x, v(x)) \right] dx \\
& = \frac{1}{2} \int_{\Omega} [f(|\nabla u(x)|) + g(x, u(x))] dx + \frac{1}{2} \int_{\Omega} [f(|\nabla v(x)|) + g(x, v(x))] dx \\
& = \frac{1}{2} \int_{\Omega} [f(|\nabla u(x)|) + g(x, u(x))] dx,
\end{aligned}$$

which is a contradiction with the fact that u is a minimizer. Therefore, $(\nabla u(x), f(|\nabla u(x)|))$ and $(\nabla v(x), f(|\nabla v(x)|))$ belong to the projection of the same face of the epigraph of f , for a.e. $x \in \Omega$. This implies that, if there exists a Lipschitz continuous minimizer u of rank L , $u - v$ is a function in $W_0^{1,1}(\Omega)$, with $\nabla(u - v)$ bounded in L^∞ by $L + K$. Hence, v is also Lipschitz continuous of rank $L + K$. Since, by Theorem 4.4 and Theorem 4.5, our assumptions guarantee the L -Lipschitz continuity of the maximum and the minimum of minimizers, we are done. \square

5. EXAMPLES

The aim of this section is to clarify the role of assumptions (F4) and (F5) by means of some examples. The first one and the subsequent remark underline the fact that our assumptions allow us to consider non uniformly convex functions. The next function, in particular, is piecewise affine and satisfies (F5).

Example 5.1. Given $\lambda > 0$, let us consider the function $f: [0, +\infty) \rightarrow \mathbb{R}$ defined by

$$\begin{aligned}
f(0) &:= 0 \\
f(x) &:= f(k) + \lambda(k+1)(x-k) \quad \text{whenever } x \in [k, k+1], k \in \mathbb{N}.
\end{aligned}$$

We extend f to an even function on \mathbb{R} . It is easy to see that f has superlinear growth and is convex. Moreover, $\partial f(k) = [k\lambda, (k+1)\lambda]$. Therefore, it is enough to choose $\tau_k := k$ to obtain (F.5).

Remark 5.2. Let us observe that a convex superlinear function f which is ε -uniformly convex, with $\varepsilon > 0$, on a countable sequence of open intervals with fixed length in $[0, +\infty)$ and affine on each connected component of the complement of the union of those intervals satisfies (F.4).

We want now to focus on the fact that some of the properties required in assumption (F4) are very close to those satisfied by a quite large class of function. The next remark is devoted to this aim.

Remark 5.3. Let us consider the set \mathcal{F} consisting of all the convex even functions $f: \mathbb{R} \rightarrow \mathbb{R}$ with $f(0) = 0$, and for which there exist $p > 1$, $a > 0$, $b \in \mathbb{R}$ such that:

- (1) $f(t) \geq a|t|^p + b$, for every $t \in \mathbb{R}$,
- (2) for every $\epsilon > 0$ there exist $a_\epsilon > 0$, $b_\epsilon \in \mathbb{R}$ with $f(t) < a_\epsilon|t|^{p+\epsilon} + b_\epsilon$, for all $t \in \mathbb{R}$.

It follows by simple computations that, being q the conjugate exponent of p , for every $f \in \mathcal{F}$ there exist $\alpha > 0$, $\beta \in \mathbb{R}$ such that $f^*(\xi) \leq \alpha|\xi|^q + \beta$, for all $\xi \in \mathbb{R}$, and for every $\epsilon > 0$ there exist $\alpha_\epsilon > 0$, $\beta_\epsilon \in \mathbb{R}$ for which $f^*(\xi) > \alpha_\epsilon|\xi|^{q-\epsilon} + \beta_\epsilon$, for every $\xi \in \mathbb{R}$.

We then obtain the following properties for the first and the second derivatives of f^* .

- i) for any $\epsilon > 0$ there exists $\alpha'_\epsilon > 0$ such that $f^{*\prime}(\xi) > \alpha'_\epsilon|\xi|^{q-1-\epsilon}$
- ii) there exists $c > 0$, such that for any $k \in \mathbb{N}$ we can find a non negligible measurable set A_k , with $A_k \subset (k, +\infty)$ and $f^{*\prime\prime}(\xi) \leq c|\xi|^{q-2}$ for any $\xi \in A_k$.

Let ξ_k a point of A_k of density equal to 1, so that we can find $\delta_k > 0$ and $B_k \subset A_k \cap (\xi_k - \delta_k, \xi_k + \delta_k)$ such that $|B_k| > 2(\delta_k - \frac{1}{k})$. Then we have

- a) $f^{*\prime\prime}(\xi) \leq \frac{1}{k}|\xi|^{q-2} \leq \max\{\frac{1}{k}|\xi_k - \delta_k|^{q-2}, \frac{1}{k}|\xi_k + \delta_k|^{q-2}\} = \gamma_k$ for every $\xi \in B_k$

b) $f^{*'}(\xi) > \alpha'_\epsilon |\xi_k - \delta_k|^{q-1-\epsilon}$ for every $\xi \in B_k$.

We underline also that

$$\lim_{k \rightarrow +\infty} \frac{\gamma_k}{\alpha'_\epsilon |\xi_k - \delta_k|^{q-1-\epsilon}} = 0 \quad (5.1)$$

Now we can compare assumption (F.4) with the results contained in this remark.

Assumption (F.4) i) together with Lemma 3.4 imply the existence of the uniform bound $f^{*''}(\xi) \leq \frac{1}{\epsilon_k}$ in the interval (ξ_k^1, ξ_k^2) . Here we have obtained for every function $f \in \mathcal{F}$ a bound on $f^{*''}$ that holds in a subset of an interval of measure close to the measure of the interval itself.

Assumption (F.4) ii) and iii) involve the value τ_k^1 that is, by polarity, a lower bound for $f^{*'}$ in the interval (ξ_k^1, ξ_k^2) . Here we have obtained that $\alpha'_\epsilon |\xi_k - \delta_k|^{q-1-\epsilon}$ is a lower bound for $f^{*'}$ in $(\xi_k - \delta_k, \xi_k + \delta_k)$.

Hence, we can say that (5.1) is exactly the analogue of (F.4) iii).

We can conclude by saying that assumption (F.4) adds to the properties satisfied by the functions in \mathcal{F} the fact that $f^* \in \mathcal{C}^{1,1}$ in a family of intervals on which the bounds on $f^{*''}$ hold for almost every point; moreover assumption (F.4) ii) has no analogue in the properties of the class \mathcal{F} . On the other hand, functions satisfying assumption (F.4) may of course have different kinds of growth properties from those considered in \mathcal{F} .

The next example emphasizes the existence of convex functions satisfying neither (F4) nor (F5).

Example 5.4. Let us consider the Cantor-Vitali function $g: [0, 1] \rightarrow \mathbb{R}$, and let us fix $p > 1$. We define $h: \mathbb{R} \rightarrow \mathbb{R}$ by setting

$$h(x) := \begin{cases} g(x) & \text{if } x \in [0, 1] \\ [(n+1)^p - n^p]g(x-n) + h(n) & \text{if } x \in (n, n+1] \end{cases}$$

for every $x \in \mathbb{R}$. Then we set $f(y) := \int_0^y h(x) dx$, for every $y \in [0, +\infty)$ and we extend f to an even function on \mathbb{R} . The convexity of f is straightforward from the continuity and the monotonicity of its derivative. It is also immediate to see that f has at least p -growth at infinity, in fact, for any $x \in (n, n+1]$, $f(x) > f(n) = \frac{1}{2}n^p$. Anyhow $f'' = h' = 0$ almost everywhere in \mathbb{R} . Hence, f is nowhere uniformly convex and does not satisfy (F.5) because f' is continuous.

ACKNOWLEDGMENTS

A.F. was partially supported by the *FP7-IDEAS-ERC-StG Grant 200497 BioSMA: Mathematics for Shape Memory Technologies in Biomechanics*.

REFERENCES

- [1] Pierre Bousquet, *Boundary continuity of solutions to a basic problem in the calculus of variations*, Adv. Calc. Var. **3** (2010), no. 1, 1–27.
- [2] ———, *Continuity of solutions of a problem in the calculus of variations*, (to appear).
- [3] Pierre Bousquet and Francis Clarke, *Local Lipschitz continuity of solutions to a problem in the calculus of variations*, J. Differential Equations **243** (2007), no. 2, 489–503.
- [4] Arrigo Cellina, *On the bounded slope condition and the validity of the Euler Lagrange equation*, SIAM J. Control Optim. **40** (2001/02), no. 4, 1270–1279 (electronic).
- [5] ———, *Comparison results and estimates on the gradient without strict convexity*, SIAM J. Control Optim. **46** (2007), no. 2, 738–749 (electronic).
- [6] ———, *Uniqueness and comparison results for functionals depending on ∇u and on u* , SIAM J. Optim. **18** (2007), no. 3, 711–716 (electronic).
- [7] Arrigo Cellina, Carlo Mariconda, and Giulia Treu, *Comparison results without strict convexity*, Discrete Contin. Dyn. Syst. Ser. B **11** (2009), no. 1, 57–65.
- [8] Arrigo Cellina and Mihai Vornicescu, *On the existence of solutions to a special variational problem*, Calc. Var. Partial Differential Equations **35** (2009), no. 2, 263–270.
- [9] Francis Clarke, *Continuity of solutions to a basic problem in the calculus of variations*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **4** (2005), no. 3, 511–530.
- [10] Ivar Ekeland and Roger Témam, *Convex analysis and variational problems*, english ed., Classics in Applied Mathematics, vol. 28, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999, Translated from the French.

- [11] Enrico Giusti, *Direct methods in the calculus of variations*, World Scientific Publishing Co. Inc., River Edge, NJ, 2003.
- [12] Alfred Haar, *Über das Plateausche Problem*, Math. Ann. **97** (1927), no. 1, 124–158.
- [13] Philip Hartman, *On the bounded slope condition*, Pacific J. Math. **18** (1966), 495–511.
- [14] Philip Hartman and Guido Stampacchia, *On some non-linear elliptic differential-functional equations*, Acta Math. **115** (1966), 271–310.
- [15] David Hilbert, *Über das Dirichletsche Prinzip*, Math. Ann. **59** (1904), no. 1-2, 161–186.
- [16] Carlo Mariconda and Giulia Treu, *A Haar-Rado type theorem for minimizers in Sobolev spaces*, ESAIM Control. Optim. Calc. Var., Published online, DOI: 10.1051/cocv/2010038.
- [17] ———, *Lipschitz regularity for minima without strict convexity of the Lagrangian*, J. Differential Equations **243** (2007), no. 2, 388–413.
- [18] ———, *Local Lipschitz regularity of minima for a scalar problem of the calculus of variations*, Commun. Contemp. Math. **10** (2008), no. 6, 1129–1149.
- [19] ———, *Hölder regularity for a classical problem of the calculus of variations*, Adv. Calc. Var. **2** (2009), no. 4, 311–320.
- [20] Giuseppe Mingione, *Regularity of minima: an invitation to the dark side of the calculus of variations*, Appl. Math. **51** (2006), no. 4, 355–426.
- [21] Mario Miranda, *Un teorema di esistenza e unicità per il problema dell'area minima in n variabili*, Ann. Scuola Norm. Sup. Pisa (3) **19** (1965), 233–249.
- [22] Guido Stampacchia, *On some regular multiple integral problems in the calculus of variations*, Comm. Pure Appl. Math. **16** (1963), 383–421.

IMATI-CNR, VIA FERRATA 1 I-27100, PAVIA
E-mail address: alice.fiaschi@imati.cnr.it

DIPARTIMENTO DI MATEMATICA PURA E APPLICATA, VIA TRIESTE 63 I-35121, PADOVA
E-mail address: treu@math.unipd.it