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Potential theory/Partial differential equations

Nonstationary Navier–Stokes equations with singular time-dependent external forces

Équations de Navier–Stokes non stationnaires avec forces externes dépendant du temps et singulières

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A R T I C L E I N F O

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A B S T R A C T

We establish a sufficient condition for the existence of solutions to the incompressible Navier–Stokes equations, with singular time-dependent external forces defined in terms of capacity $\mathrm{Cap}_{\mathcal{H}_{1},2}(E)$.

R É S U M É

Nous établissons une condition suffisante pour l'existence de solutions aux équations de Navier–Stokes incompressibles, avec force externe dépendant du temps et singulière, dans un espace défini en termes de la capacité $\mathrm{Cap}_{\mathcal{H}_{1},2}(E)$.

1. Introduction and main result

In this paper, we establish a sufficient condition for the existence of solutions to the incompressible Navier–Stokes equations (in short NSE):

$$\begin{cases} \partial_t u - \Delta u + \mathrm{div}(u \otimes u) + \nabla p = F & \text{in } \mathbb{R}^n \times (0,\infty), \\ \mathrm{div}\, u = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(0) = u_0 & \text{in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where u with value in $\in \mathbb{R}^n$ $(n \geq 2)$ is the velocity, and p with value $\in \mathbb{R}$ is the pressure.

It is not hard to see that, if the pair $(u(x,t), p(x,t))$ solves NSE (1.1), then $(u_\lambda(x,t), p_\lambda(x,t))$ with

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t),$$

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$$p_\lambda(x,t)=\lambda^2 p(\lambda x,\lambda^2 t)$$

is a solution to the system (1.1) with the initial and the force data

$$u_{0,\lambda}(x)=\lambda u_0(\lambda x),$$
$$F_\lambda(x,t)=\lambda^3 F(\lambda x,\lambda^2 t).$$

It is well known that the following continuous embeddings hold

$$L^n(\mathbb{R}^n)\subset\mathcal{M}^{q,q}(\mathbb{R}^n)\subset BMO^{-1}(\mathbb{R}^n)\subset B^{-1,\infty}_\infty(\mathbb{R}^n), \tag{1.2}$$

where $\mathcal{M}^{q,q}(\mathbb{R}^n)$ is the Morrey space with order (q,q), $q\in[1,n]$, i.e. the set of functions $f\in L^q(\mathbb{R}^n)$ such that

$$||f||_{\mathcal{M}^{q,q}(\mathbb{R}^n)}:=\sup_{B_r(x_0)\subset\mathbb{R}^n}\left\{r^{q-n}\int\limits_{B_r(x_0)}|f(x)|^q\mathrm{d}x\right\}^{\frac{1}{q}},$$

and the space $BMO^{-1}(\mathbb{R}^n)$ is the set of distributions f satisfying

$$||f||_{BMO^{-1}(\mathbb{R}^n)}:=\sup_{B_r(x_0)\subset\mathbb{R}^n}\left\{r^{-n}\int\limits_0^{r^2}\int\limits_{B_r(x_0)}|\mathrm{e}^{s\Delta}f(x)|^2\mathrm{d}x\,\mathrm{d}s\right\}^{\frac{1}{2}}$$

and the space $B^{-1,\infty}_\infty(\mathbb{R}^n)$ is the Besov space equipped with the norm

$$||f||_{B^{-1,\infty}_\infty(\mathbb{R}^n)}:=\sup_{t>0}t^{\frac{1}{2}}||\mathrm{e}^{t\Delta}f(.)||_{L^\infty(\mathbb{R}^n)}.$$

Those spaces are invariant under the scaling $f(.)\to\lambda f(\lambda.)$, in the sense that $\|f\|_E=\|\lambda f\|_E$.

T. Kato [3] initiated the study of (1.1) with $F\equiv 0$ and the initial data belonging to the space $L^n(\mathbb{R}^n)$. He obtained the global existence of solutions in a subspace of $C([0,\infty),L^n(\mathbb{R}^n))$ if the norm $||u_0||_{L^n(\mathbb{R}^n)}$ is small enough. The global existence result also holds for the small initial data in the homogeneous Morrey space $\mathcal{M}^{q,q}(\mathbb{R}^n)$, for $1\le q\le n$ (see [4], [5], [11]). Later in 2001, H. Kock and D. Tataru [6] showed that the global well-posedness of NSE holds with small initial data in the space BMO^{-1}. Otherwise, J. Bourgain and N. Pavlović [1] showed that (1.1) with initial data in $B^{-1,\infty}_\infty(\mathbb{R}^n)$ is ill-posed no matter how the initial data are.

Recently, T.V. Phan and N.C. Phuc [9] proved the existence of solutions to the stationary equation of (1.1) with data singular external force F in space $\mathcal{V}^{1,2}(\mathbb{R}^n)$. We refer the detail of this space to [9].

In this paper, we consider the global existence of solutions of problem (1.1) with initial data and forcing term. In order to state it, we recall that the $(\mathcal{H}_1,2)$-capacity of a Borel set $E\subset\mathbb{R}^{n+1}$ is defined by

$$\mathrm{Cap}_{\mathcal{H}_1,2}(E)=\inf\left\{\int\limits_{\mathbb{R}^{n+1}}|f|^2\mathrm{d}x\,\mathrm{d}t: f\in L^2_+(\mathbb{R}^{n+1}),\mathcal{H}_1*f\ge\chi_E\right\}$$

where $\mathcal{H}_1$ is the Heat kernel of the first order:

$$\mathcal{H}_1(x,t)=\left((4\pi)^{n/2}\Gamma(1/2)\right)^{-1}\frac{\chi_{(0,\infty)}(t)}{t^{(n+1)/2}}\exp\left(-\frac{|x|^2}{4t}\right)\ \text{for } (x,t) \text{ in } \mathbb{R}^{n+1}.$$

The Riesz parabolic kernel of order one $\mathbb{I}_1$ is defined by:

$$\mathbb{I}_1[\mu](x,t)=\int\limits_0^\infty\frac{\mu(\tilde{Q}_\rho(x,t))}{\rho^{n+1}}\frac{\mathrm{d}\rho}{\rho},$$

where $\tilde{Q}_\rho(x,t)=B_\rho(x)\times(t-\rho^2/2,t+\rho^2/2)\subset\mathbb{R}^{n+1}$ and μ is a nonnegative Radon measure on $\mathbb{R}^{n+1}$.

Let us define

$$Y=\left\{g\in L^2_{\mathrm{loc}}(\mathbb{R}^{n+1}):\|g\|_Y=\sup_{E\subset\mathbb{R}^{n+1}}\left\{\frac{\int_E|g(x,t)|^2\mathrm{d}x\,\mathrm{d}t}{\mathrm{Cap}_{\mathcal{H}_1,2}(E)}\right\}^{\frac{1}{2}}<+\infty\right\},$$

with the supremum being taken over all compact sets $E\subset\mathbb{R}^{n+1}$.

For any $2 < l \le n+2$, we have the following embeddings:

$$L^{n+2}(\mathbb{R}^{n+1}) \subset \mathcal{M}_*^{l,l}(\mathbb{R}^{n+1}) \subset Y \subset \mathcal{M}_*^{2,2}(\mathbb{R}^{n+1}),$$

where $\mathcal{M}_*^{l,l}(\mathbb{R}^{n+1}) = \sup_{\tilde{Q}_\rho(x_0,t_0)\subset\mathbb{R}^{n+1}} \left(\rho^{l-(n+2)} \int_{\tilde{Q}_\rho(x_0,t_0)} |f(x,t)|^l \mathrm{d}x\,\mathrm{d}t \right)$ is the Morrey space corresponding to the parabolic problem.

To our purpose later, we define the space:

$$Z = \left\{ F \in \mathcal{D}'(\mathbb{R}^{n+1}) : \sup_{E\subset\mathbb{R}^{n+1}} \left(\int_E \frac{|\int_0^t \mathrm{e}^{(t-s)\Delta}\mathbb{P}F\,\mathrm{d}s|^2}{\mathrm{Cap}_{\mathcal{H}_1,2}(E)} \mathrm{d}x\,\mathrm{d}t \right)^{1/2} < +\infty \right\},$$

where the norm is defined by

$$\|F\|_Z = \sup_{E\subset\mathbb{R}^{n+1}} \left(\int_E \frac{|\int_0^t \mathrm{e}^{(t-s)\Delta}\mathbb{P}F\,\mathrm{d}s|^2}{\mathrm{Cap}_{\mathcal{H}_1,2}(E)} \mathrm{d}x\,\mathrm{d}t \right)^{1/2}.$$

Next, we define

$$X = \left\{ g \in \mathcal{D}'(\mathbb{R}^n) : \|\mathrm{e}^{t\Delta}g\|_Y < +\infty \right\},$$

where the norm on X is defined by $\|g\|_X = \|\mathrm{e}^{t\Delta}g\|_Y$.

Then, we observe that

$$\mathrm{Cap}_{\mathcal{H}_1,2}(\tilde{Q}_\rho(x,t)) = \rho^n \mathrm{Cap}_{\mathcal{H}_1,2}(\tilde{Q}_1(0)) \text{ for any } \rho > 0,$$

and

$$\mathrm{Cap}_{\mathcal{H}_1,2}(E) \ge C|E|^{1-\frac{2}{n+2}},$$

for any Borel set $E \subset \mathbb{R}^{n+1}$, see [8]. Thus, it is not difficult to show that, for $1 < q < n$,

$$\mathcal{M}^{q,q}(\mathbb{R}^n) \subset X \subset BMO^{-1}(\mathbb{R}^n) \subset B_\infty^{-1,\infty}(\mathbb{R}^n).$$

Put

$$A(x,t) := \begin{cases} (\mathrm{e}^{t\Delta}u_0)(x) + \int_0^t (\mathrm{e}^{(t-s)\Delta}\mathbb{P}F)(x)\,\mathrm{d}s & \text{if } (x,t) \in \mathbb{R}^n \times [0,+\infty), \\ 0 & \text{otherwise} \end{cases}$$

where $\mathbb{P} = id - \nabla\Delta^{-1}\nabla.$ is the Helmholtz–Leray projection onto the vector fields of zero divergence, i.e. for any $f \in \mathbb{R}^n$, $\mathbb{P}f = f - \nabla u$ and $\Delta u = \operatorname{div} f$.

Then, we have the following theorem.

Theorem 1.1. *There exists a constant $c_1 = c_1(n) > 0$ such that, if $\|u_0\|_X + \|F\|_Z < c_1$, then problem (1.1) admits a global solution satisfying*

$$|u(x,t)| \le |A(x,t)| + c\mathbb{I}_1[|A|^2](x,t), \quad \forall (x,t) \in \mathbb{R}^n \times (0,\infty), \tag{1.3}$$

for some constant $c = c(n) > 0$.

In the particular case when $F \equiv 0$, the assumption reads

$$\int_E |(\mathrm{e}^{t\Delta}u_0)(x)|^2 \mathrm{d}x\,\mathrm{d}t \le C\,\mathrm{Cap}_{\mathcal{H}_1,2}(E), \tag{1.4}$$

for any compact set $E \subset \mathbb{R}^{n+1}$, and the pointwise estimate (1.3) becomes

$$|u(x,t)| \le |\mathrm{e}^{t\Delta}u_0|(x,t) + \tilde{C}\mathbb{I}_1[|\mathrm{e}^{t\Delta}u_0|^2](x,t).$$

Remark 1.2. Note that we have the following embeddings:

$$L^{(n+2)/3}(\mathbb{R}^{n+1}) \subset Z_1 \subset Z_0 \subset Z,$$

where $Z_0 = \left\{F : F = \operatorname{div}(f),\ f \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1}) : \sup_{E\subset\mathbb{R}^{n+1}} \frac{\int_E |f| \mathrm{d}x\,\mathrm{d}t}{\mathrm{Cap}_{\mathcal{H}_1,2}(E)} < +\infty\right\}$, with the norm $\|F\|_{Z_0} = \inf_{f:\operatorname{div}(f)=F} \sup_{E\subset\mathbb{R}^{n+1}} \frac{\int_E |f| \mathrm{d}x\,\mathrm{d}t}{\mathrm{Cap}_{\mathcal{H}_1,2}(E)}$; and

$$Z_1 = \left\{ F : F = \operatorname{div}(f),\ \text{with} \sup_{\tilde{Q}_\rho(x_0,t_0)\subset\mathbb{R}^{n+1}} \left(\rho^{2p-(n+2)} \int_{\tilde{Q}_\rho(x_0,t_0)} |f(x,t)|^p \mathrm{d}x\,\mathrm{d}t \right)^{1/p} < \infty \right\},$$

for $1 < p < (n+2)/2$, with the norm

$$\|F\|_{Z_1} = \inf_{f:\operatorname{div}(f)=F} \sup_{\tilde{Q}_\rho(x_0,t_0)\subset\mathbb{R}^{n+1}} \left(\rho^{2p-(n+2)} \int_{\tilde{Q}_\rho(x_0,t_0)} |f(x,t)|^p \mathrm{d}x\,\mathrm{d}t \right)^{1/p}.$$

As a consequence of above Theorem and Remark 1.2, we have the following result.

Corollary 1.3. *If $u_0 \in X$, and $F \in Z_1$, such that $\|u_0\|_X + \|F\|_{Z_1}$ is small enough then equation (1.1) admits a global solution.*

2. Proof of Theorem 1.1

Let u be a mild solution of (1.1), i.e. $u \in L^2_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$\begin{cases} \partial_t u - \Delta u + \mathbb{P}\operatorname{div}(u \otimes u) = \mathbb{P}F & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(0) = u_0 & \text{in } \mathbb{R}^n. \end{cases} \tag{2.1}$$

By Duhamel's principle (see [10]), we get

$$u(t) = \mathrm{e}^{t\Delta} u_0 + \int_0^t \mathrm{e}^{(t-s)\Delta}\mathbb{P}F\,\mathrm{d}s - \int_0^t \mathrm{e}^{(t-s)\Delta}\mathbb{P}\operatorname{div}(u\otimes u)\,\mathrm{d}s. \tag{2.2}$$

For any distribution $G : \mathbb{R}^N \times (0,\infty) \to \mathbb{R}^n$, we can write

$$\int_0^t \mathrm{e}^{(t-s)\Delta}(\mathbb{P}G)^i \mathrm{d}s = \int_0^t \int_{\mathbb{R}^n} k_{i,j}(x-y, t-s) G^j(y,s)\,\mathrm{d}y\,\mathrm{d}s,$$

where $(k_{i,j})$ is the Oseen kernel; it is well known that this kernel satisfies the following estimates:

$$|k_{i,j}(x,t)| \le c_1 \frac{1}{(\max\{|x|, \sqrt{|t|}\})^N},$$

$$\left|\frac{\partial^{l_1+l_2} k_{i,j}}{\partial x^{l_1}\partial t^{l_2}}(x,t)\right| \le c_2 \frac{1}{(\max\{|x|, \sqrt{|t|}\})^{N+l_1+2l_2}}, \quad \text{for } l_1, l_2 \in \mathbb{N},$$

for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$, where c_1, c_2 are positive constants depending only on n, i, j, l_1, l_2 (see Lerner [7], and Lemarié-Rieusset [2]). Therefore, we get for any $G \in (L^1_{\mathrm{loc}}(\mathbb{R}^n))^n$

$$\left| \int_0^t \left(\mathrm{e}^{(t-s)\Delta}\mathbb{P}\operatorname{div}(G)\right)(x)\,\mathrm{d}s \right| \le c_3 \mathbb{I}_1[|G|](x,t), \quad \forall (x,t) \in \mathbb{R}^{n+1}. \tag{2.3}$$

From (2.2) and (2.3), we obtain

$$|u(x,t)| \le |A(x,t)| + cI_1[|u|^2](x,t), \quad \forall (x,t) \in \mathbb{R}^{n+1}. \tag{2.4}$$

Now, consider the sequence $\{u_k\}_{k\geq 1} \subset L^2_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ of functions defined by $u_1 = 0$ and

$$u_{k+1}(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} \mathbb{P} F \, ds - \int_0^t e^{(t-s)\Delta} \mathbb{P} \operatorname{div}(u_k \otimes u_k) \, ds, \quad \forall k \geq 1.$$

Hence, from (2.3) we have

$$|u_{k+1}(x,t)| \leq |A(x,t)| + c_4 \mathbb{I}_1[|u_k|^2](x,t), \tag{2.5}$$

$$|u_{k+1}(x,t) - u_k(x,t)| \leq c_5 \mathbb{I}_1[|u_k - u_{k-1}|(|u_k| + |u_{k-1}|)](x,t), \tag{2.6}$$

for some positive constants c_4, c_5.

Next, we need the following result, which is proved in Theorem 4.36, [8].

Proposition 2.1. *Let μ be a nonnegative Radon measure on $\mathbb{R}^{n+1}$. Then the following statements are equivalent.*

1. *For every compact set $E \subset \mathbb{R}^{n+1}$,*

$$\mu(E) \leq c_6 \text{Cap}_{\mathcal{H}_1,2}(E), \tag{2.7}$$

for some positive constant c_6.

2. $\mathbb{I}_1[\mu] < \infty$ *a.e., and*

$$\mathbb{I}_1[(\mathbb{I}_1[\mu])^2] \leq c_7 \mathbb{I}_1[\mu] \ \text{a.e. in } \mathbb{R}^{n+1}, \tag{2.8}$$

for some positive constant c_7.

3. *For every compact set $E \subset \mathbb{R}^{n+1}$,*

$$\int_E (\mathbb{I}_1[\mu])^2 dx\, dt \leq c_8 \text{Cap}_{\mathcal{H}_1,2}(E), \tag{2.9}$$

for some positive constant c_8.

Applying Proposition 2.1 to $d\mu = |A(x,t)|^2 dx\, dt$, we obtain if, for some $\lambda > 0$ and for every compact set $E \subset \mathbb{R}^{n+1}$ such that

$$\int_E |A(x,t)|^2 dx\, dt \leq \lambda\, \text{Cap}_{\mathcal{H}_1,2}(E), \tag{2.10}$$

the following inequalities

$$\mathbb{I}_1[|A|^2] < \infty, \quad \text{a.e. in } \mathbb{R}^{n+1},$$

and

$$\mathbb{I}_1[(\mathbb{I}_1[|A|^2])^2] \leq c_7 c_6^{-1} \lambda \mathbb{I}_1[|A|^2], \quad \text{a.e. in } \mathbb{R}^{n+1}. \tag{2.11}$$

a. Suppose

$$\mathbb{I}_1[(\mathbb{I}_1[|A|^2])^2] \leq \frac{1}{16 c_4^2} \mathbb{I}_1[|A|^2] < \infty, \quad \text{a.e. in } \mathbb{R}^{n+1}, \tag{2.12}$$

we claim that

$$|u_k(x,t)| \leq |A(x,t)| + 4c_4 \mathbb{I}_1[|A|^2](x,t), \quad \text{for } k \geq 1. \tag{2.13}$$

Clearly, (2.13) is true for $k = 1$. Now assume that (2.13) holds for $k = m$:

$$|u_m(x,t)| \leq |A(x,t)| + 4c_4 \mathbb{I}_1[|A|^2](x,t), \quad \forall (x,t) \in \mathbb{R}^{n+1}.$$

From (2.5), we obtain

$$\begin{aligned} |u_{m+1}(x,t)| &\leq |A(x,t)| + c_4 \mathbb{I}_1[|u_m|^2](x,t) \\ &\leq |A(x,t)| + 2c_4 \mathbb{I}_1[|A|^2](x,t) + 32 c_4^2 \mathbb{I}_1[\left(\mathbb{I}_1[|A|^2]\right)^2](x,t) \\ &\leq |A(x,t)| + 4c_4 \mathbb{I}_1[|A|^2](x,t). \end{aligned}$$

Note that we use (2.12) in the last inequality. Then, (2.13) is true with $k = m+1$. In other words, we get the claim above.

Hence, from (2.6) and Holder inequality, we have

$$\begin{aligned}|u_{k+1}-u_k| &\le 2c_5\mathbb{I}_1[|u_k-u_{k-1}||A|]+8c_5c_4\mathbb{I}_1\left[|u_k-u_{k-1}|\mathbb{I}_1[|A|^2]\right]\\ &\le 2c_5\left(\mathbb{I}_1[|u_k-u_{k-1}|^2]\mathbb{I}_1[|A|^2]\right)^{1/2}+8c_5c_4\left(\mathbb{I}_1\left[|u_k-u_{k-1}|^2\right]\mathbb{I}_1\left[\left(\mathbb{I}_1[|A|^2]\right)^2\right]\right)^{1/2}.\end{aligned}$$

b. Now, we suppose

$$\mathbb{I}_1[(\mathbb{I}_1[|A|^2])^2]\le M\mathbb{I}_1[|A|^2]<\infty,\quad \text{a.e. in } \mathbb{R}^{n+1}. \tag{2.14}$$

Then, we have

$$|u_{k+1}-u_k|\le 2c_5(1+4c_4M^{1/2})\left(\mathbb{I}_1[|u_k-u_{k-1}|^2]\mathbb{I}_1[|A|^2]\right)^{1/2}. \tag{2.15}$$

We need to prove that

$$|u_{k+1}-u_k|\le c_5b^{k-2}\mathbb{I}_1[|A|^2],\quad \forall k\ge 1, \tag{2.16}$$

where $b=2c_5(1+4c_4M^{1/2})M^{1/2}$.

In fact, (2.16) is true for $k=1$. Next, we assume that (2.16) holds with $k=m$. Then, from (2.15) and (2.14), we have

$$\begin{aligned}|u_{m+2}-u_{m+1}| &\le 2c_5(1+4c_4M^{1/2})c_5b^{m-2}\left(\mathbb{I}_1[\left(\mathbb{I}_1[|A|^2]\right)^2]\mathbb{I}_1[|A|^2]\right)^{1/2}\\ &\le 2c_5(1+4c_4M^{1/2})c_5b^{m-2}M^{1/2}\mathbb{I}_1[|A|^2]\\ &= c_5b^{m-1}\mathbb{I}_1[|A|^2].\end{aligned}$$

Thus, (2.16) is also true with $k=m+1$. Or, (2.16) holds for all $k\ge 1$.

Hence, if $b<1$ then u_k converges to $u=u_1+\sum_{j=1}^{\infty}(u_{j+1}-u_j)$ in $L^2_{\text{loc}}(\mathbb{R}^n\times(0,\infty),\mathbb{R}^n)$, and $\mathbb{I}_1[|u_k-u|^2]\to 0$ a.e. in $\mathbb{R}^n$. Moreover, we have

$$|u|\le|A|+4c_4\mathbb{I}_1[|A|^2].$$

Note that $b<1$ is equivalent to

$$M<\frac{1}{4c_4}\left(\sqrt{\frac{1}{4c_4}+\frac{1}{2c_5}}-\frac{1}{4c_4}\right)^2.$$

Combining this with (2.12) and (2.10)–(2.11), we conclude that the problem (1.1) admits a solution u satisfying (1.3) with

$$C(N)=\frac{c_6}{c_7}\min\left\{\frac{1}{16c_4^2},\frac{1}{8c_4}\left(\sqrt{\frac{1}{4c_4}+\frac{1}{2c_5}}-\frac{1}{4c_4}\right)^2\right\}.$$

Thus, the proof of Theorem 1.1 is complete.

Remark 2.2. We can show that

$$\sup_{\text{compact } E\subset\mathbb{R}^{n+1}}\left\{\frac{\int_E|u_k-u|^2\mathrm{d}x\,\mathrm{d}t}{\mathrm{Cap}_{\mathcal{H}_1,2}(E)}\right\}^{\frac{1}{2}}\to 0\quad\text{as } k\to\infty.$$

Remark 2.3. By (2.4), if we consider the equation

$$U=c\mathbb{I}_1[U^2]+\varepsilon f, \tag{2.17}$$

for some $\varepsilon>0$, with $U\in L^2_{\text{loc}}(\mathbb{R}^{n+1})$ then the following two statements are equivalent.

a. For every compact set $E\subset\mathbb{R}^{n+1}$, $\int_E f^2\mathrm{d}x\,\mathrm{d}t\le C\,\mathrm{Cap}_{\mathcal{H}_1,2}(E)$ for some constant $C>0$.
b. There exists a solution $U\in L^2_{\text{loc}}(\mathbb{R}^{n+1})$ of equation (2.17). In particular, we can apply $f=A(x,t)$.

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