Weak continuity and lower semicontinuity results for determinants

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Abstract

Weak continuity properties of minors and lower semicontinuity properties of functionals with polyconvex integrands are addressed in this paper. In particular, it is shown that if $\{u_n\}$ is bounded in $W^{1,N-1}\left(\Omega;\mathbb{R}^N\right)$, $\{\mathrm{adj}\nabla u_n\} \subset L^{\frac{N}{N-1}}\left(\Omega;\mathbb{R}^{N\times N}\right)$, and if $u \in BV\left(\Omega;\mathbb{R}^N\right)$ are such that $u_n \to u$ in $L^1\left(\Omega;\mathbb{R}^N\right)$ and

$$\det \nabla u_n \overset{*}{\rightharpoonup} \mu$$

in the sense of measures, then for $\mathcal{L}^N$ a.e. $x \in \Omega$

$$\det \nabla u\,(x) = \frac{d\mu}{d\mathcal{L}^N}\,(x)\,.$$

The result is sharp and counterexamples are provided in the cases where regularity of $\{u_n\}$ or the type of weak convergence are weakened.

1 Introduction

The search for the least integrability spaces that ensure weak continuity of null lagrangians, and in particular of geometrically relevant operators such as the adjoint of the strain and the Jacobian determinant, has been for decades in the agenda of experts in geometric measure theory, in the calculus of variations, in partial differential equations and in nonlinear elasticity. This is a subject that attracts the interest of experts from a wide palette of analytical expertise, and it requests tools from different areas of mathematics and continuum mechanics.

In this paper we focus our attention on the Jacobian determinant, and the question we address is: what are the sharpest hypotheses under which we may guarantee that if $\{u_n\} \subset W^{1,1}(\Omega;\mathbb{R}^N)$ converges to a function u then

$$\det\nabla u_n \to \det\nabla u?$$

Here, and in what follows, $\Omega \subset \mathbb{R}^N$ is an open bounded set, and the choice of topologies of convergence to be considered for the sequences $\{u_n\}$ and $\{\det\nabla u_n\}$ are at the core of the problem. This question surfaces naturally in the study of cavitation of rubber-like materials (see e.g. Ball [7], James and Spector [27], Müller and Spector [39], Sivaloganathan [42]), and recently there have been deep developments in this theory and in related generalizations motivated by the study of vorticity in the Ginzburg-Landau model (see e.g. Alberti, Baldo and Orlandi [3], [4], Bethuel, Brezis and Heléin [8], Jerrard and Soner [28], [29], [30]). A positive statement is of relevance but so is a negative one as it surely underlines the formation of defects.

It is well known that

$$u_n \rightharpoonup u \text{ in } W^{1,N}(\Omega;\mathbb{R}^N) \Rightarrow \det\nabla u_n \overset{*}{\rightharpoonup} \det\nabla u \text{ in the sense of measures.}$$

This, by now classical, result has been established by Morrey [34] and Reshethnyak [41], and its proof relies on a simple integration by parts and on the fact that in $W^{1,N}(\Omega;\mathbb{R}^N)$ the Jacobian determinant $\det\nabla u$ agrees with the distributional determinant

$$\operatorname{Det}\nabla u := \sum_{i=1}^{N} (-1)^{i+1} \frac{\partial}{\partial x_i}\left(u_1 \frac{\partial (u_2,\dots,u_N)}{\partial (x_1,\dots,x_{i-1},x_{i+1},\dots,x_N)}\right).$$

It is clear that $\operatorname{Det}\nabla u$ is well defined in the sense of distributions on a class wider than $W^{1,N}\left(\Omega;\mathbb{R}^N\right)$. Typically adopted subclasses are those of functions $u \in W^{1,N-1}(\Omega;\mathbb{R}^N)$ with u_1 bounded, and also $u \in W^{1,p}\left(\Omega;\mathbb{R}^N\right)$ for some $p \geq N^2/(N+1)$. In the latter case, by the Sobolev Imbedding Theorem we have $u \in L^{N^2}\left(\Omega;\mathbb{R}^N\right)$, and thus Hölder inequality ensures that the products involved in the definition of $\operatorname{Det}\nabla u$ are well defined in L^1. More generally, we may also consider the case where $u \in W^{1,p}\left(\Omega;\mathbb{R}^N\right)$, $\operatorname{adj}\nabla u \in L^q(\Omega;\mathbb{R}^{N\times N})$, with $1/p + 1/q \leq 1 + 1/N$. We recall that $\operatorname{adj} A$ is the transpose of the matrix of cofactors of A, and it is called the *adjugate* of A.

There is an extensive literature in the subject (see e.g. Ball [6], Brezis, Fusco and Sbordone [10], Brezis and Nirenberg [11], [12], Coifman, Lions , Meyer and Semmes [14], Dacorogna and Murat [15], Giaquinta, Modica and Souček [23], Hajłasz [24], Iwaniec and Sbordone [26], Morrey [34], Müller [35], [36], [38], [37], Müller, Tang and Yang [40], Reshetnyak [41]), and this introduction by no means intends to be a survey exposition.

Recently Iwaniec and Onninen [25] proved the following result:

Theorem 1.1 *Let $u_n,\ u \in W^{1,N-1}\left(\mathbb{R}^N;\mathbb{R}^N\right)$ be such that $\{\operatorname{adj}\nabla u_n\}$ is a sequence bounded in $L^{\frac{N}{N-1}}\left(\mathbb{R}^N;\mathbb{R}^{N\times N}\right)$ and $u_n \rightharpoonup u$ in $W^{1,N-1}\left(\mathbb{R}^N;\mathbb{R}^N\right)$. Then*

$$\lim_{n\to\infty}\langle\det\nabla u_n, v\rangle = \langle\det\nabla u, v\rangle$$

for every $v \in VMO\left(\mathbb{R}^N\right)$, and, in particular,

$$\lim_{k\to\infty}\int_{\mathbb{R}^N}\det\nabla u_n v\,dx = \int_{\mathbb{R}^N}\det\nabla u_n v\,dx$$

for every $v \in C_0\left(\mathbb{R}^N\right)$.

Since $|\det A| \leq |\operatorname{adj} A|^{N/(N-1)}$, the hypotheses of the previous theorem imply that $\{\det\nabla u_n\}$ is bounded in $L^1\left(\mathbb{R}^N\right)$. Therefore we are led naturally to study the convergence of $\{\det\nabla u_n\}$ in the case where the assumption

$$\{\operatorname{adj}\nabla u_n\}\ \text{ is bounded in } L^{\frac{N}{N-1}}\left(\mathbb{R}^N;\mathbb{R}^{N\times N}\right)$$

is weakened to read

$$\{\operatorname{adj}\nabla u_n\}\subset L^{\frac{N}{N-1}}\left(\mathbb{R}^N;\mathbb{R}^{N\times N}\right)\ \text{ and }\ \{\det\nabla u_n\}\ \text{ is bounded in } L^1\left(\mathbb{R}^N\right).$$

Clearly, in this this case we can only expect $\{\det\nabla u_n\}$, or a subsequence of it, to converge weak-* to a Radon measure μ, and thus the all we can expect (or wish for) is that

$$\frac{d\mu}{d\mathcal{L}^N} = \det\nabla u.$$

This statement is in the same spirit of earlier work by Marcellini [32], Giaquinta, Modica and Souček [23], Fonseca and Marcellini [20], Jerrard and Soner [28], and others.

More recently, Fonseca, Fusco and Marcellini in [17] introduced the notion of *total variation of the Jacobian determinant*, defined via relaxation for $u \in L^\infty_{\text{loc}}\left(\Omega;\mathbb{R}^N\right)\cap W^{1,p}\left(\Omega;\mathbb{R}^N\right)$ for some $p > N-1$, by

$$TV(u,\Omega) = \inf\left\{\liminf_{h\to+\infty}\int_\Omega|\det\nabla u_n(x)|\ dx : u_n \in W^{1,N}\left(\Omega;\mathbb{R}^N\right)\right.$$
$$\left. u_n \rightharpoonup u \text{ in } W^{1,p}\left(\Omega;\mathbb{R}^N\right)\right\},$$

This definition is conform with the approach commonly used for variational problems with non-standard growth and coercivity conditions (see e.g. Acerbi, Bouchitté and Fonseca [1], Acerbi and Dal Maso [2], Bouchitté, Fonseca and Malý [9], Celada and Dal Maso [13], Fonseca and Malý [19], Fonseca and Marcellini [20], Malý [31], Giaquinta, Modica and Souček [23], Marcellini [32], [33]). In [17] an explicit characterization of $TV(u,\Omega)$ is given for some classes of locally Lipschitz-continuous away from a given point. When $\mathrm{Det}Du$ is a Radon a measure, in general its total variation differs from the total variation of the Jacobian determinant, $TV(u,\Omega)$, although several examples suggest that, at least for some ranges of p, its absolutely continuous part with respect to the $N-$dimensional Lebesgue measure reduces to $\det\nabla u$. Indeed, this holds when $u \in W^{1,p}\left(\Omega;\mathbb{R}^N\right)$ and $\mathrm{adj}\nabla u \in L^q(\Omega;\mathbb{R}^{N\times N})$ with $1/p+1/q \leq 1+1/N$ (see Müller [36]). We also recall the typical example of radially symmetric maps of the type $u(x) := x/|x|$, $\det\nabla u = 0$ a.e. in B_1, where $\mathrm{Det}\nabla u = \omega_N\,\delta_0$ (see Fonseca and Marcellini [20], Giaquinta, Modica and Souček [23]).

The main result of the paper is the following theorem:

Theorem 1.2 *Let $\{u_n\}$ be a sequence bounded in $W^{1,N-1}\left(\Omega;\mathbb{R}^N\right)$, with*

$$\{\mathrm{adj}\,\nabla u_n\} \subset L^{\frac{N}{N-1}}\left(\Omega;\mathbb{R}^{N\times N}\right). \tag{1.1}$$

Assume that $u_n \to u$ in $L^1\left(\Omega;\mathbb{R}^N\right)$ for some $u \in BV\left(\Omega;\mathbb{R}^N\right)$ and

$$\det\nabla u_n \overset{*}{\rightharpoonup} \mu$$

in the sense of measures. Then for $\mathcal{L}^N$ a.e. $x\in\Omega$

$$\det\nabla u\,(x) = \frac{d\mu}{d\mathcal{L}^N}\,(x)\,. \tag{1.2}$$

Here for $u \in BV\left(\Omega;\mathbb{R}^N\right)$ we denote by ∇u the Radon–Nikodym derivative of the distributional derivative Du of u, with respect to the N–dimensional Lebesgue measure $\mathcal{L}^N$.

Remark 1.3 (i) The hypothesis that $u \in BV\left(\Omega;\mathbb{R}^N\right)$ is clearly redundant, since it is implied by the boundedness of $\{u_n\}$ in $W^{1,N-1}\left(\Omega;\mathbb{R}^N\right)$ (when $N\geq 3$ we actually get that $u \in W^{1,N-1}\left(\Omega;\mathbb{R}^N\right)$).
(ii) Condition (1.1) is used, throughout the paper, to guarantee that $\mathrm{Det}\,\nabla u_n = \det\nabla u_n$ (see [36]). Note that it is automatically satisfied if $\{u_n\}\subset W^{1,N}\left(\Omega;\mathbb{R}^N\right)$.
(iii) We are unable to establish an analog result for minors of lower order since the proof of Theorem 1.2 is strongly hinged on a version of the isoperimetric inequality for the determinant.

However, based on the geometric measure theory results of Giaquinta, Modica, Souček [22], Celada and Dal Maso [13] have resolved this question under the additional assumption that $\{u_n\}$ is bounded in L^∞ (see Remark 3.8).
(iv) We point out that if we had assumed that $\{\mathrm{adj}\,\nabla u_n\}$ was bounded in $L^{\frac{N}{N-1}}\left(\Omega;\mathbb{R}^{N\times N}\right)$, then we would be (at least for $N\geq 3$) in the framework of Theorem 1.1 of Iwaniec and Onninen. This would allow us to conclude (1.2) immediately, as in this case Theorem 1.1 would yield $\mu = \det\nabla u\mathcal{L}^N\lfloor\Omega$.

The result of Theorem 1.2 is sharp and counterexamples are provided in Theorems 1.4, 1.5, 1.6, and 1.7 to address the cases where regularity of $\{u_n\}$ or the type of weak convergence are weakened. These theorems are stated and will be proved for Ω a N-dimensional interval, i.e.

$$\Omega = (a_1, b_1) \times \cdots \times (a_N, b_N),$$

with $a_i < b_i$, $i = 1, \ldots, N$. Analog conclusions may be derived for any bounded domain in $\mathbb{R}^N$, although proofs then become more technical.

Theorems 1.4 and 1.5 illustrate that care must be taken when relaxing the regularity of the sequence $\{u_n\}$.

Theorem 1.4 *Let $1 \leq p < N$, let $u \in W^{1,N}(\Omega; \mathbb{R}^N)$, and let $\mu \in \mathfrak{M}(\Omega; \mathbb{R})$. There exists a sequence $\{u_n\} \subset W^{1,p}(\Omega; \mathbb{R}^N)$ such that*

$$u_n \to u \quad \text{in } W^{1,p}(\Omega; \mathbb{R}^N), \quad \det \nabla u_n \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathfrak{M}(\Omega; \mathbb{R}). \tag{1.3}$$

Theorem 1.5 *Let $1 \leq p < N$, $1 \leq q < \infty$, let $u \in W^{1,N}(\Omega; \mathbb{R}^N)$, and let $f \in L^q(\Omega)$. There exists a sequence $\{u_n\} \subset W^{1,p}(\Omega; \mathbb{R}^N)$ such that*

$$u_n \rightharpoonup u \quad \text{in } W^{1,p}(\Omega; \mathbb{R}^N) \quad \text{and} \quad \det \nabla u_n \to f \quad \text{in } L^q(\Omega).$$

The two results below show that the boundedness of $\{u_n\}$ in $W^{1,N-1}\left(\Omega; \mathbb{R}^N\right)$ cannot be weakened.

Theorem 1.6 *Let $1 \leq p < N-1$, let $u \in W^{1,N}(\Omega; \mathbb{R}^N)$, and let $\mu \in \mathfrak{M}(\Omega; \mathbb{R})$. There exists a sequence $\{u_n\} \subset C^1(\overline{\Omega}; \mathbb{R}^N)$ such that*

$$u_n \to u \quad \text{in } W^{1,p}(\Omega; \mathbb{R}^N), \quad \det \nabla u_n \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathfrak{M}(\Omega; \mathbb{R}).$$

Theorem 1.7 *Let $1 \leq p < N-1$, $1 \leq q < \infty$, let $u \in W^{1,N}(\Omega; \mathbb{R}^N)$, and let $f \in L^q(\Omega)$. There exists a sequence $\{u_n\} \subset C^1(\overline{\Omega}; \mathbb{R}^N)$ such that*

$$u_n \rightharpoonup u \quad \text{in } W^{1,p}(\Omega; \mathbb{R}^N), \quad \det \nabla u_n \to f \quad \text{in } L^q(\Omega).$$

We conclude the introduction by noting that Theorem 1.2 may be exploited to derive new proofs of lower semicontinuity results for polyconvex integrands which previously had been obtained using cartesian currents techniques (see Theorem 5.2).

2 Preliminaries

In this section we recall well-known results concerning area and change of variable formulas. We end with a new version of the isoperimetric inequality for the determinant (see Theorem 2.6 below).

In the sequel $\Omega \subset \mathbb{R}^N$ is an open set, and $\mathfrak{M}(\Omega)$ and $\mathfrak{M}(\Omega; \mathbb{R})$ stand for the space of positive and real-valued Radon measures defined on Ω, respectively.

Proposition 2.1 *Let* $\{\mu_n\} \subset \mathfrak{M}(\Omega;\mathbb{R})$ *be a sequence of Radon measures such that* $\mu_n \overset{*}{\rightharpoonup} \mu$ *in* $\mathfrak{M}(\Omega;\mathbb{R})$. *Assume that* $|\mu_n| \overset{*}{\rightharpoonup} \nu$ *in* $\mathfrak{M}(\Omega;\mathbb{R})$, *and let* $E \subset \Omega$ *be a bounded, measurable set, with* $\nu(\partial E) = 0$ *and* $\operatorname{dist}(E,\partial\Omega) > 0$. *If* $u \in C(\Omega) \cap L^\infty(\Omega)$ *then*

$$\int_E u\,d\mu_n \to \int_E u\,d\mu.$$

Proof. Step 1: We consider first the case where $u \equiv 1$. Fix $\varepsilon > 0$ and choose an open set $D \supset \overline{E}$ such $\nu\left(\overline{D\backslash E}\right) < \frac{\varepsilon}{2}$. Let $\varphi \in C_c(D;[0,1])$ be such that $\varphi \equiv 1$ on E. We have

$$|\mu_n(E) - \mu(E)| \le \left| \int_D \varphi\,d\mu_n - \int_D \varphi\,d\mu \right| + |\mu_n|(D\backslash E) + |\mu|(D\backslash E).$$

Hence

$$\limsup_{n\to\infty} |\mu_n(E) - \mu(E)| \le 2\nu\left(\overline{D\backslash E}\right) < \varepsilon.$$

Given the arbitrariness of $\varepsilon > 0$ we obtain the desired result.
Step 2: Set $\lambda_n := u\,\mu_n$. It is clear that $\lambda_n \overset{*}{\rightharpoonup} \lambda$ in $\mathfrak{M}(\Omega;\mathbb{R})$, where $\lambda := u\,\mu$, and, since $|\lambda_n| \le \|u\|_\infty |\mu_n|$, up to the extraction of a subsequence, and without loss of generality we may assume that $|\lambda_n| \overset{*}{\rightharpoonup} \gamma$ in $\mathfrak{M}(\Omega;\mathbb{R})$. Observe that $\gamma(\partial E) = 0$, because $\gamma \le \|u\|_\infty \nu$. By part (i) it follows that $\lambda_n(E) \to \lambda(E)$. ■

The following proposition is due to Celada and Dal Maso [13].

Proposition 2.2 *Let* $\Omega \subset \mathbb{R}^N$ *be an open set, and let* $\{\mu_n\} \subset \mathfrak{M}(\Omega;\mathbb{R})$ *be a sequence of Radon measures such that*

(i) $\mu_n \to T \in \mathcal{D}'(\Omega)$ *in the sense of distributions;*

(ii) $\mu_n^+ \overset{*}{\rightharpoonup} \nu$ *in* $\mathfrak{M}(\Omega)$.

Then T *can be identified with a measure* μ *and* $\mu_n \overset{*}{\rightharpoonup} \mu$ *in* $\mathfrak{M}(\Omega;\mathbb{R})$.

For each matrix $A \in \mathbb{R}^{d\times N}$ let $\mathcal{M}_k(A)$ be the collection of all minors of order k, with $1 \le k \le \min\{d,N\}$, and define

$$\mathcal{M}(A) := \left(A, \mathcal{M}_2(A), \cdots, \mathcal{M}_{\min\{d,N\}}(A)\right),$$

where

$$\tau = \tau(d,N) := \sum_{k=1}^{\min\{d,N\}} \sigma(k),$$

$$\sigma(k) := \binom{d}{k}\binom{N}{k} = \frac{d!N!}{(k!)^2(d-k)!(N-k)!}.$$

It can be shown that

$$|\mathcal{M}_k(A+B)| \leq C\left(1+|A|^k\right)(1+|\mathcal{M}_1(B)|+\cdots+|\mathcal{M}_k(B)|). \tag{2.1}$$

When $d=N$ then

$$|\mathcal{M}_k(AB)| \leq |\mathcal{M}_k(A)|\,|\mathcal{M}_k(B)|, \tag{2.2}$$

$\mathcal{M}_N(A) = \det A$, and $\mathcal{M}_{N-1}(A)$ is the list, up to sign, of the coefficients of the *adjugate* of A. We recall that the adjugate of A, denoted by $\operatorname{adj} A$, is the transpose of the matrix of cofactors of A, $\operatorname{adj} A = (\operatorname{cof} A)^T$. In particular,

$$A \operatorname{adj} A = \det A\, \mathbb{I}_N, \tag{2.3}$$

thus

$$\det(A \operatorname{adj} A) = (\det A)^N,$$

and so

$$|\det A|^N = |\det A|\,|\det \operatorname{adj} A| \leq |\det A|\,|\operatorname{adj} A|^N,$$

which, in turn, yields

$$|\det A| \leq |\operatorname{adj} A|^{N/(N-1)}. \tag{2.4}$$

In the sequel we will use repeatedly the fact that if a, $b \in \mathbb{R}^N$ then

$$\det(\mathbb{I}_N + a \otimes b) = 1 + a \cdot b. \tag{2.5}$$

If $u : \mathbb{R}^N \to \mathbb{R}^N$ is a Lipschitz function and if $E \subset \mathbb{R}^N$ is measurable, we define the *counting function* (or *Banach indicatrix*) of u at y *with respect to the set* E as

$$N(u,E,y) := \mathcal{H}_0(\{x \in E : u(x) = y\}),$$

and the *degree* of u at y *with respect to the set* E as

$$\deg(u,E,y) := N(u, E \cap \{\det \nabla u > 0\}, y) - N(u, E \cap \{\det \nabla u < 0\}, y).$$

When E is open and $|u(\partial E)| = 0$ then the function $\deg(u,E,\cdot)$ coincides $\mathcal{L}^N$ a.e. with the *topological degree.* We refer to [18] for more details. Next we state the well-know area and change of variable formulas.

Theorem 2.3 (Area Formula) *If $u : \mathbb{R}^N \to \mathbb{R}^N$ is a Lipschitz function and if $E \subset \mathbb{R}^N$ is a measurable set, then $N(u,E,\cdot)$ is measurable,*

$$\int_E |\det \nabla u(x)|\, dx = \int_{u(E)} N(u,E,y)\, dy,$$

and

$$\int_E \det \nabla u(x)\, dx = \int_{\mathbb{R}^N} \deg(u,E,y)\, dy.$$

Theorem 2.4 (Change of variable) *Let $f : \mathbb{R}^N \to [0, \infty]$ be a Lebesgue measurable function, and let $u : \mathbb{R}^N \to \mathbb{R}^N$ be a Lipschitz function. Then*

$$\int_E f(u(x)) |\det \nabla u(x)| \; dx = \int_{\mathbb{R}^d} f(y) \; N(u, E, y) \; dy.$$

Moreover, if $f \in L^1(\mathbb{R}^N)$ then

$$\int_E f(u(x)) \det \nabla u(x) \; dx = \int_{\mathbb{R}^N} f(y) \deg(u, E, y) \; dy.$$

The following proposition prepares the ground for a new variant of the isoperimetric formula (see Theorem 2.6).

Proposition 2.5 *Let Ω be a Lipschitz domain. Suppose that $u \in C^1(\overline{\Omega}; \mathbb{R}^N)$. Then*

(i) for all $w \in C^1(\mathbb{R}^N; \mathbb{R}^N)$

$$\int_\Omega \operatorname{div}_y w(u(x)) \det \nabla u(x) \; dx = \int_{\partial\Omega} \nu(x) \cdot \operatorname{adj} \nabla u(x) \, w(u(x)) \; d\mathcal{H}^{N-1}; \tag{2.6}$$

(ii) for all $w \in C^1(\mathbb{R}^N; \mathbb{R}^N)$ with $\|w\|_\infty \leq 1$

$$\left| \int_\Omega \operatorname{div}_y w(u(x)) \det \nabla u(x) \, dx \right| \leq C \int_{\partial\Omega \cap \{u_1 \neq 0\}} | \operatorname{cof} \nabla u(x) \, \nu(x) \, | \, d\mathcal{H}^{N-1}.$$

Proof. (i) It is well known that (cf. [34])

$$\sum_{j=1}^N \frac{\partial}{\partial x_j} (\operatorname{cof} \nabla u(x))_{ij} = 0, \qquad i = 1, \ldots, N, \tag{2.7}$$

in the sense of distributions. Hence, also by (2.3), we have

$$\begin{aligned}
&\sum_{j=1}^N \frac{\partial}{\partial x_j} \left(\sum_{i=1}^N (\operatorname{cof} \nabla u(x))_{ij} \; w_i(u(x)) \right) \\
&\quad = \sum_{j=1}^N \sum_{i=1}^N (\operatorname{cof} \nabla u(x))_{ij} \; \frac{\partial}{\partial x_j} (w_i(u(x)) \\
&\quad = \sum_{j=1}^N \sum_{i=1}^N \sum_{k=1}^N (\operatorname{cof} \nabla u(x))_{ij} \; \frac{\partial w_i}{\partial y_k} (u(x)) \; \frac{\partial u_k}{\partial x_j} (x) \\
&\quad = \sum_{i=1}^N \sum_{k=1}^N \frac{\partial w_i}{\partial y_k} (u(x)) \; \delta_{ik} \det \nabla u.
\end{aligned}$$

By the Gauss-Green formula for Lipschitz sets we obtain (2.6).
(ii) Denote
$$\mathrm{p}(y) := (0, y_2, \dots, y_N).$$
By (2.6) we have
$$\begin{aligned}\int_\Omega \operatorname{div}_y w(u(x)) \det \nabla u(x)\, dx &= \int_{\partial\Omega} \operatorname{cof} \nabla u(x)\ \nu(x) \cdot w(u(x))\, d\mathcal{H}^{N-1}(x)\\ &= \int_{\partial\Omega} \operatorname{cof} \nabla u(x)\ \nu(x) \cdot [w(u(x)) - w(\mathrm{p}(u(x)))]\, d\mathcal{H}^{N-1}(x)\\ &\quad + \int_{\partial\Omega} \operatorname{cof} \nabla u(x)\ \nu(x) \cdot [w(\mathrm{p}(u(x))) - \mathrm{p}(w(\mathrm{p}(u(x))))]\, d\mathcal{H}^{N-1}(x)\\ &\quad + \int_{\partial\Omega} \operatorname{cof} \nabla u(x)\ \nu(x) \cdot \mathrm{p}(w(\mathrm{p}(u(x))))\, d\mathcal{H}^{N-1}(x)\\ &=: I_1 + I_2 + I_3.\end{aligned}$$
Since $\mathrm{p}(u(x)) = u(x)$ on $\{u_1 = 0\}$, we have
$$|I_1| \le 2 \int_{\partial\Omega \cap \{u_1 \neq 0\}} |(\operatorname{cof} \nabla u\,(x)\, \nu\,(x)\,|\, d\mathcal{H}^{N-1}(x).$$
For the second term we have
$$\operatorname{div}(w \circ \mathrm{p} - \mathrm{p} \circ w \circ \mathrm{p}) = 0,$$
and thus
$$I_2 = \int_\Omega \operatorname{div}_y (w \circ \mathrm{p} - \mathrm{p} \circ w \circ \mathrm{p})\,(u(x))\ \det \nabla u(x)\, dx = 0.$$
Finally, using the fact that the rows of $\operatorname{cof} \nabla u$ except the first one are exterior multiples of ∇u_1 (and thus orthogonal to ∇u_1), we deduce that
$$\operatorname{cof} \nabla u(x)\ \nu(x) \cdot \mathrm{p}(w(\mathrm{p}(u(x)))) = \sum_{i=2}^N (\operatorname{cof} \nabla u(x)\ \nu(x))_i\, w_i(\mathrm{p}(u(x))) = 0$$
on $\{u_1 = 0\} \subset \{\nabla u_1 \perp T_x(\partial\Omega)\}$, where the symbol $T_x(\partial\Omega)$ denotes the tangent space of $\partial\Omega$ at x and $\nu(x)$ is parallel to ∇u_1. It follows
$$|I_3| \le \int_{\partial\Omega \cap \{u_1 \neq 0\}} |\operatorname{cof} \nabla u\,(x)\, \nu\,(x)\,|\, d\mathcal{H}^{N-1}(x),$$
and this concludes the proof. ■

Theorem 2.6 (Isoperimetric Inequality for the Determinant) *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary and let $u \in C^1\left(\overline{\Omega}; \mathbb{R}^N\right)$. Then*
$$\left| \int_\Omega \det \nabla u\, dx \right| \le C\,(N) \left(\int_{\partial\Omega \cap \{u_1 \neq 0\}} |\operatorname{adj} \nabla u|\ d\mathcal{H}^{N-1} \right)^{\frac{N}{N-1}}.$$

Proof. Let

$$f(y) := \deg(u, \Omega, y).$$

We claim that f belongs to $BV(\mathbb{R}^N)$ and that

$$\|Df\|(\mathbb{R}^N) \le C(N) \int_{\partial\Omega\cap\{u_1\neq 0\}} |\mathrm{adj}\, \nabla u| \, d\mathcal{H}^{N-1}. \tag{2.8}$$

Indeed, if $w \in C_c^1(\mathbb{R}^N; \mathbb{R}^N)$ and $\|w\|_\infty \le 1$ then

$$\begin{aligned}\int_{\mathbb{R}^N} f(y) \operatorname{div} w(y)\, dy &= \int_{\mathbb{R}^N} \deg(u, \Omega, y) \operatorname{div} w(y)\, dy \\ &= \int_\Omega \operatorname{div}_y w(u(x)) \det \nabla u \, dx \\ &\le C \int_{\partial\Omega\cap\{u_1\neq 0\}} |\operatorname{cof} \nabla u(x)| \, d\mathcal{H}^{N-1}(x),\end{aligned}$$

where we have used the Change of Variable Formula, Proposition 2.5, and the fact that $u \in C^1(\overline{\Omega}; \mathbb{R}^N)$.

In view of (2.8), by the Area Formula and Sobolev Embedding Theorem we have

$$\begin{aligned}\left|\int_A \det \nabla u \, dx\right| = \left|\int_{\mathbb{R}^N} f(y)\, dy\right| &\le \int_{\mathbb{R}^N} |f(y)|^{N/(N-1)}\, dy \\ &\le C(N) \left(\|Df\|(\mathbb{R}^N)\right)^{N/(N-1)} \\ &\le C(N) \left(\int_{\partial\Omega\cap\{u_1\neq 0\}} |\mathrm{adj}\, \nabla u| \, d\mathcal{H}^{N-1}\right)^{\frac{N}{N-1}},\end{aligned}$$

where we used the fact that $f(y) \in \mathbb{N}$, and thus $|f(y)| \le |f(y)|^{N/(N-1)}$. ■

3 Weak convergence of determinants

In this section we assume throughout that Ω is an open, bounded subset of $\mathbb{R}^N$.

The following result is well-known (see Ball [6], Morrey [34]), and it is the starting point for the study of weak convergence of minors.

Theorem 3.1 *(i) If* $u_n \rightharpoonup u$ *in* $W^{1,N}(\Omega; \mathbb{R}^N)$ *then* $\det \nabla u_n \overset{*}{\rightharpoonup} \det \nabla u$ *in the sense of measures.*
(ii) If $u_n \rightharpoonup u$ *in* $W^{1,N-1}(\Omega; \mathbb{R}^N)$ *then* $\mathrm{adj}\, \nabla u_n \overset{*}{\rightharpoonup} \mathrm{adj}\, \nabla u$ *in the sense of measures.*

Although the theorem below may also be considered to be a classical result, we present a proof here for the convenience of the reader.

Theorem 3.2 *Let*

$$\{u_n\} \subset W^{1,N-1}\left(\Omega;\mathbb{R}^N\right), \quad \{\operatorname{adj}\nabla u_n\} \subset L^{\frac{N}{N-1}}\left(\Omega;\mathbb{R}^{N\times N}\right)$$

be such that $\{\mathcal{M}(\nabla u_n)\}$ is bounded in $L^1(\Omega;\mathbb{R}^\tau)$. Assume that $u_n \to u$ in $L^\infty\left(\Omega;\mathbb{R}^N\right)$ for some $u \in W^{1,N-1}\left(\Omega;\mathbb{R}^N\right) \cap C\left(\overline{\Omega};\mathbb{R}^N\right)$ with

$$\operatorname{adj}\nabla u \in L^{\frac{N}{N-1}}\left(\Omega;\mathbb{R}^{N\times N}\right).$$

Then $\mathcal{M}(\nabla u_n) \overset{*}{\rightharpoonup} \mathcal{M}(\nabla u)$ in the sense of measures.

Proof. To simplify the notation, in the sequel we will denote u_n by $u^{(n)}$. Since $\left\{\mathcal{M}\left(\nabla u^{(n)}\right)\right\}$ is bounded in $L^1(\Omega)$, it suffices to establish weak convergence to $\mathcal{M}(\nabla u)$ in $\mathcal{D}'(\Omega)$.

The proof follows by induction on the order k of the minors of ∇u. We claim that if $1 \le k \le N$, then

$$\int_\Omega \mathcal{M}_k\left(\nabla u^{(n)}\right)\varphi(x)\,dx \to \int_\Omega \mathcal{M}_k(\nabla u)\,\varphi(x)\,dx \tag{3.1}$$

for every $\varphi \in C_c^\infty(\Omega)$. For $k=1$ this holds trivially since $\nabla u^{(n)} \overset{*}{\rightharpoonup} \nabla u$ in the sense of measures. Assume that (3.1) holds for $1 \le k \le N-1$ and we show that (3.1) is still valid for $k+1$. Fix a minor $\mathcal{M}_{k+1}(\nabla u)$, written as

$$\mathcal{M}_{k+1}(\nabla u) = \det\left((\nabla u)_{I_{k+1},J_{k+1}}\right)_{i\in I_{k+1},\, j\in J_{k+1}},$$

where

$$(\nabla u)_{I_{k+1},J_{k+1}} := \left(\frac{\partial u_i}{\partial x_j}\right)_{i\in I_{k+1},\, j\in J_{k+1}},$$

and I_{k+1}, J_{k+1} are subsets of $\{1,\cdots,N\}$ with cardinality $k+1$. Under the assumed integrability assumptions on u and on $\operatorname{adj}\nabla u$ it follows (see Remark 1.3 (ii)) that for a fixed $i \in I_{k+1}$

$$\mathcal{M}_{k+1}(\nabla u) = \sum_{j\in J_{k+1}} \frac{\partial}{\partial x_j}\left(u_i\left(\operatorname{cof}(\nabla u)_{I_{k+1},J_{k+1}}\right)_{ij}\right)$$

in the sense of distributions, and an analog formula holds also for each $u^{(n)}$. Let $\varphi \in C_c^\infty(\Omega)$. We have

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega \mathcal{M}_{k+1}\left(\nabla u^{(n)}\right)\varphi\,dx \\
&= -\sum_{j\in J_{k+1}} \lim_{n\to\infty}\int_\Omega u_i^{(n)}\left(\operatorname{cof}\left(\nabla u^{(n)}\right)_{I_{k+1},J_{k+1}}\right)_{ij}\frac{\partial\varphi}{\partial x_j}\,dx \\
&= -\sum_{j\in J_{k+1}} \lim_{n\to\infty}\int_\Omega \left(u_i^{(n)}-u_i\right)\left(\operatorname{cof}\left(\nabla u^{(n)}\right)_{I_{k+1},J_{k+1}}\right)_{ij}\frac{\partial\varphi}{\partial x_j}\,dx \\
&\quad -\sum_{j\in J_{k+1}} \lim_{n\to\infty}\int_\Omega u_i\left(\operatorname{cof}\left(\nabla u^{(n)}\right)_{I_{k+1},J_{k+1}}\right)_{ij}\frac{\partial\varphi}{\partial x_j}\,dx.
\end{aligned}$$

As $\left\{ \operatorname{cof}\left(\nabla u^{(n)}\right)_{I_{k+1},J_{k+1}} \right\}$ is bounded in L^1, and since $u_n \to u$ in $L^\infty\left(\Omega;\mathbb{R}^N\right)$, the first term in the last sum converges to zero. Also, by the induction hypothesis

$$\operatorname{cof}\left(\nabla u^{(n)}\right)_{I_{k+1},J_{k+1}} \overset{*}{\rightharpoonup} \operatorname{cof}\left(\nabla u\right)_{I_{k+1},J_{k+1}}$$

in the sense of measures, and as $u_i \frac{\partial \varphi}{\partial x_j} \in C_c$, we conclude that the second term converges to

$$-\sum_{j\in J_{k+1}} \int_\Omega u_i \left(\operatorname{cof}\left(\nabla u\right)_{I_{k+1},J_{k+1}}\right)_{ij} \frac{\partial \varphi}{\partial x_j}\, dx = \int_\Omega \mathcal{M}_{k+1}\left(\nabla u\right)\, \varphi\left(x\right)\, dx.$$

■

Proposition 3.3 *Let*

$$\{u_n\} \subset W^{1,N-1}\left(\Omega;\mathbb{R}^N\right), \quad \{\operatorname{adj}\nabla u_n\} \subset L^{\frac{N}{N-1}}\left(\Omega;\mathbb{R}^{N\times N}\right)$$

be such that $\{\mathcal{M}\left(\nabla u_n\right)\}$ *is bounded in* $L^1\left(\Omega;\mathbb{R}^\tau\right)$, *and* $\{\det\nabla u_n\}$ *is equi-integrable. Assume that* $u_n \to u$ *in* $L^1\left(\Omega;\mathbb{R}^N\right)$ *for some* $u \in BV\left(\Omega;\mathbb{R}^N\right)$. *Then* $\det\nabla u_n \rightharpoonup \det\nabla u$ *in* $L^1\left(\Omega\right)$.

Proof. Step 1: Assume first that $u \in C^1\left(\overline{\Omega};\mathbb{R}^N\right)$. Consider a sequence

$$\{u_n\} \subset W^{1,N-1}\left(\Omega;\mathbb{R}^N\right), \quad \{\operatorname{adj}\nabla u_n\} \subset L^{\frac{N}{N-1}}\left(\Omega;\mathbb{R}^{N\times N}\right)$$

such that $u_n \to u$ in $L^1\left(\Omega;\mathbb{R}^N\right)$, $\{\mathcal{M}\left(\nabla u_n\right)\}$ is bounded in $L^1\left(\Omega;\mathbb{R}^\tau\right)$, and $\{\det\nabla u_n\}$ is equi-integrable. Extract a subsequence of $\{u_n\}$ (not relabelled). The idea of the proof is to obtain a further subsequence (still not relabelled), and truncate each u_n to obtain a function w_n such that $w_n \to u$ in $L^\infty\left(\Omega;\mathbb{R}^N\right)$, $\{\mathcal{M}\left(\nabla w_n\right)\}$ is bounded in $L^1\left(\Omega;\mathbb{R}^\tau\right)$, and $\det\nabla u_n - \det\nabla w_n \to 0$ in $L^1\left(\Omega\right)$. The result then follows immediately from Theorem 3.2. The remaining of the proof of this step is devoted to the construction of w_n.

Since $u_n \to u$ in $L^1\left(\Omega;\mathbb{R}^N\right)$, and extracting a subsequence, if necessary, we may assume that

$$4^n \left\|u_n - u\right\|_{L^1(\Omega;\mathbb{R}^N)} \to 0$$

and so

$$\left|\left\{x \in \Omega : \left|u_n - u\right| \geq \frac{1}{4^n}\right\}\right| \to 0. \tag{3.2}$$

Let

$$C := \sup_n \int_\Omega \left|\mathcal{M}\left(\nabla u_n\right)\right|\, dx. \tag{3.3}$$

Find $k_n \in \{n+1, \cdots, 2n\}$ such that

$$\int_{\left\{\frac{1}{2^{k_n}} < |u_n - u| < \frac{1}{2^{k_n - 1}}\right\}} \left|\mathcal{M}\left(\nabla u_n\right)\right|\, dx \leq \frac{C}{n}, \tag{3.4}$$

and set

$$\varphi(t) := \begin{cases} 1 & \text{if } t \le 1, \\ \frac{2-t}{t} & \text{if } 1 < t < 2, \\ 0 & \text{if } t \ge 2. \end{cases}$$

Define

$$w_n(x) := u(x) + \varphi\left(2^{k_n} |u_n - u|(x)\right)(u_n - u)(x).$$

Observe that

$$\{w_n\} \subset W^{1,N-1}\left(\Omega; \mathbb{R}^N\right), \quad \{\operatorname{adj} \nabla w_n\} \subset L^{\frac{N}{N-1}}\left(\Omega; \mathbb{R}^{N\times N}\right)$$

and

$$w_n = u_n \text{ on } \left\{|u_n - u| \le \frac{1}{2^{k_n}}\right\}, \quad w_n = u \text{ on } \left\{|u_n - u| > \frac{1}{2^{k_n-1}}\right\}. \tag{3.5}$$

Moreover,

$$\begin{aligned} \|w_n - u\|_{L^\infty(\Omega;\mathbb{R}^N)} &= \left\|\varphi\left(2^{k_n}|u_n-u|\right)(u_n-u)\right\|_{L^\infty(\Omega;\mathbb{R}^N)} \\ &\le 2^{-k_n+1} \le 2^{-n} \to 0 \end{aligned}$$

as $n \to \infty$.

On $\left\{\frac{1}{2^{k_n}} < |u_n - u| < \frac{1}{2^{k_n-1}}\right\}$ we have

$$\nabla w_n = (1-\varphi)\nabla u + \varphi \nabla u_n + \varphi' R_n (\nabla u_n - \nabla u)$$

where it is understood that φ and φ' are evaluated at $2^{k_n}|u_n - u|$, and

$$R_n := 2^{k_n}(u_n - u) \otimes \frac{(u_n - u)}{|u_n - u|} \text{ satisfy } |R_n| \le 2.$$

By (2.1) and (2.2) we have for all $1 \le k \le N$

$$\begin{aligned} |\mathcal{M}_k(\nabla w_n)| &\le C\left(1 + |\nabla u|^k\right)(1 + |\mathcal{M}(\nabla u_n)|) \\ &\le C(|\mathcal{M}(\nabla u_n)| + 1), \end{aligned} \tag{3.6}$$

where we have used the fact that ∇u is bounded.

It remains to show that $\det \nabla u_n - \det \nabla w_n \to 0$ in $L^1(\Omega)$. We have

$$\begin{aligned} \int_\Omega |\det \nabla u_n - \det \nabla w_n|\, dx &= \int_{\left\{\frac{1}{2^{k_n}} < |u_n - u| < \frac{1}{2^{k_n-1}}\right\}} |\det \nabla u_n - \det \nabla w_n|\, dx \\ &+ \int_{\left\{|u_n-u| > \frac{1}{2^{k_n-1}}\right\}} |\det \nabla u_n - \det \nabla u|\, dx \end{aligned}$$

by (3.5). The second term in the above equality converges to zero due to the equi-integrability of $\{\det \nabla u_n\}$, by (3.2) and the fact that $k_n \le 2n$. The first term may be bounded as

$$C \int_{\left\{ \frac{1}{2^{k_n}} < |u_n - u| < \frac{1}{2^{k_n - 1}} \right\}} \left(|\mathcal{M}(\nabla u_n)| + 1 \right) dx$$
$$\leq \frac{C}{n} + C \left| \left\{ \frac{1}{2^{k_n}} < |u_n - u| < \frac{1}{2^{k_n - 1}} \right\} \right| \to 0$$

by (3.2) and (3.4).

Step 2: Here we remove the extra regularity condition imposed on u. Since $\{\det \nabla u_n\}$ is equi-integrable, by De La Vallé-Poussin criterion there exists a (non negative) Young function Φ such that

$$\lim_{t \to \infty} \frac{\Phi(t)}{t} = \infty \tag{3.7}$$

and

$$\{\Phi(\det \nabla u_n)\} \text{ is bounded in } L^1.$$

Without loss of generality, and up to the extraction of a subsequence, we may assume that there exist two finite positive Radon measures μ and ν and a function $f \in L^1(\Omega)$ such that

$$|\mathcal{M}(\nabla u_n)| \, \mathcal{L}^N \lfloor \Omega \overset{*}{\rightharpoonup} \mu, \quad \Phi(\det \nabla u_n) \mathcal{L}^N \lfloor \Omega \overset{*}{\rightharpoonup} \nu,$$

and $\det \nabla u_n \rightharpoonup f$ in $L^1(\Omega)$.

We seek to prove that $f(x) = \det \nabla u(x)$ for $\mathcal{L}^N$ a.e. $x \in \Omega$. Let $x_0 \in \Omega$ be a Lebesgue point for f. Suppose, in addition, that

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^{N+1}} \int_{B(x_0, \varepsilon)} |u(x) - T(x; x_0)| \, dx = 0, \tag{3.8}$$

where

$$T(x; x_0) := u(x_0) - \nabla u(x_0)(x - x_0),$$

and that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \text{ and } \frac{d\nu}{d\mathcal{L}^N}(x_0) \text{ exist and are finite.} \tag{3.9}$$

Let $\varepsilon \to 0^+$ with $\mu(\partial B(x_0, \varepsilon)) = \nu(\partial B(x_0, \varepsilon)) = 0$, and define

$$u_{n,\varepsilon}(y) := \frac{u_n(x_0 + \varepsilon y) - u(x_0)}{\varepsilon} \quad \text{for } y \in B := B(0, 1).$$

By (3.8) we have

$$\lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \|u_{n,\varepsilon} - u_0\|_{L^1(B)} = 0,$$

where $u_0(y) := \nabla u(x_0) y$, and by (3.9) we obtain

$$\infty > \frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \int_B |\mathcal{M}(u_{n,\varepsilon})| \, dy,$$
$$\infty > \frac{d\nu}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \int_B \Phi(\det \nabla u_{n,\varepsilon}) \, dy.$$

Choose a countable, dense family $\{\theta_k\} \subset C_c(B)$. Since $\det \nabla u_n \rightharpoonup f$ in $L^1(\Omega)$ and $x_0 \in \Omega$ is a Lebesgue point for f, we deduce that

$$\begin{aligned}
&\lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \int_B \det \nabla u_{n,\varepsilon}(y)\, \theta_k(y)\, dy \\
&\qquad = \lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \frac{1}{|B(x_0,\varepsilon)|} \int_{B(x_0,\varepsilon)} \det \nabla u_n(x)\, \theta_k\left(\frac{x - x_0}{\varepsilon}\right) dx \qquad (3.10) \\
&\qquad = \lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \frac{1}{|B(x_0,\varepsilon)|} \int_{B(x_0,\varepsilon)} f(x)\, \theta_k\left(\frac{x - x_0}{\varepsilon}\right) dx \\
&\qquad = f(x_0) \int_B \theta_k\, dy
\end{aligned}$$

for every $k \in \mathbb{N}$. We may now diagonalize the sequence $\{u_{n,\varepsilon}\}$ to obtain a sequence $\{v_n\}$ such that $v_n \to u_0$ in L^1, $\{\mathcal{M}(v_n)\}$ and $\{\Phi(\det \nabla v_n)\}$ are bounded in L^1, and

$$\lim_{n \to \infty} \int_B \det \nabla v_n\, \theta_k\, dy = f(x_0) \int_B \theta_k\, dy \qquad (3.11)$$

for every $k \in \mathbb{N}$. Since $\{\Phi(\det \nabla v_n)\}$ is bounded in L^1, it follows from (3.7) that $\{\det \nabla v_n\}$ is equi-integrable.

We are now in position to apply Step 1 to the sequence $\{v_n\}$ to deduce that $\det \nabla v_n \rightharpoonup \det \nabla u(x_0)$ in $L^1(B)$. On the other hand, by (3.11) $\det \nabla v_n \overset{*}{\rightharpoonup} f(x_0)$ in the sense of measures and so we conclude that

$$f(x_0) = \det \nabla u(x_0).$$

■

Remark 3.4 (i) Note that, in general, we can only expect u to belong to $BV(\Omega; \mathbb{R}^N)$. Therefore, there is no guarantee apriori that $\mathrm{Det}\, \nabla u$ is well defined as a distribution, nor that $\det \nabla u \in L^1(\Omega)$.
(ii) A similar proof to that of Proposition 3.3 asserts that if $1 \le k \le \min\{N, d\}$,

$$\{u_n\} \subset W^{1,k-1}(\Omega; \mathbb{R}^d), \quad \{\mathcal{M}_{k-1}(\nabla u_n)\} \subset L^{\frac{k}{k-1}}\left(\Omega; \mathbb{R}^{\sigma(k-1)}\right)$$

is such that $\{\mathcal{M}_j(\nabla u_n)\}$ are bounded in $L^1(\Omega)$ for $1 \le j \le k-1$, where $1 \le k \le \min\{N, d\}$, $\{\mathcal{M}_k(\nabla u_n)\}$ is equi-integrable, and $u_n \to u$ in $L^1(\Omega; \mathbb{R}^N)$ for some $u \in BV(\Omega; \mathbb{R}^N)$, then

$$\mathcal{M}_k(\nabla u_n) \rightharpoonup \mathcal{M}_k(\nabla u) \ \textit{ in } L^1(\Omega).$$

The argument used in Proposition 3.3 may be used to obtain uniformly converging sequences from initially L^1 converging sequences without energetically increasing the sequence of minors. Precisely:

Proposition 3.5 *Let*

$$\{u_n\} \subset W^{1,N-1}(\Omega; \mathbb{R}^N), \quad \{\mathrm{adj}\, \nabla u_n\} \subset L^{\frac{N}{N-1}}(\Omega; \mathbb{R}^{N \times N})$$

be such that $\{\mathcal{M}(\nabla u_n)\}$ *is bounded in* $L^1(\Omega;\mathbb{R}^\tau)$, *and* $\{\det \nabla u_n\}$ *is equi-integrable. Assume that* $u_n \to u$ *in* $L^1(\Omega;\mathbb{R}^N)$ *for some* $u \in C^1(\overline{\Omega};\mathbb{R}^N)$. *Then there exists a sequence*

$$\{w_n\} \subset W^{1,N-1}\left(\Omega;\mathbb{R}^N\right) \cap L^\infty\left(\Omega;\mathbb{R}^N\right), \quad \{\operatorname{adj} \nabla w_n\} \subset L^{\frac{N}{N-1}}\left(\Omega;\mathbb{R}^{N\times N}\right)$$

such that $w_n \to u$ *in* $L^\infty(\Omega;\mathbb{R}^N)$, $\mathcal{M}(\nabla w_n) \overset{*}{\rightharpoonup} \mathcal{M}(\nabla u)$ *in the sense of measures. Moreover*

$$|\{x \in \Omega : \ w_n(x) \neq u_n(x)\}| \to 0 \tag{3.12}$$

and

$$\int_{\{w_n \neq u_n\}} |\mathcal{M}(\nabla w_n)| \, dx \to 0. \tag{3.13}$$

Proof. The construction of the sequence $\{w_n\}$ is entirely identical to that of Step 1 of the previous theorem.

We notice (3.5) and observe that the uniform convergence of $\{w_n\}$ to u follows from (2.1), (3.12) is a consequence of (3.2), and (3.13) is easily obtained from (3.6) and (3.4). ■

Remark 3.6 (i) If $u_n \in W^{1,N}(\Omega;\mathbb{R}^N)$ then the truncated function w_n also belongs to $W^{1,N}(\Omega;\mathbb{R}^N)$.
(ii) Proposition 3.5 admits a generalization for minors of lower order similar to that of Proposition 3.5 as described in Remark 3.4 (ii).

Proof of Theorem 1.2. Step 1: Assume first that $\Omega = B := B(0,1)$, $u(x) = Ax$ for some $A \in \mathbb{R}^{N\times N}$, $\{u_n\} \subset C^1(\overline{B};\mathbb{R}^N)$, $u_n \to u$ in $L^1(B;\mathbb{R}^N)$,

$$|\nabla u_n|^{N-1} \mathcal{L}^N \lfloor B \overset{*}{\rightharpoonup} \alpha \mathcal{L}^N \lfloor B, \tag{3.14}$$

$$|\det \nabla u_n| \mathcal{L}^N \lfloor B \overset{*}{\rightharpoonup} \beta \mathcal{L}^N \lfloor B, \tag{3.15}$$

$$|\operatorname{adj} \nabla u_n| \mathcal{L}^N \lfloor B \overset{*}{\rightharpoonup} \gamma \mathcal{L}^N \lfloor B, \tag{3.16}$$

for some α, β, $\gamma \geq 0$. We claim that

$$\frac{1}{|B|} \int_B \det \nabla u_n \, dx \to \det A. \tag{3.17}$$

To prove the claim we consider separately the cases $N \geq 3$ and $N = 2$.
Step 2: We prove (3.17) for $N \geq 3$. In what follows we denote u_n by $u^{(n)}$. We have

$$\left| \int_B \left(\det \nabla u^{(n)} - \det \nabla u \right) dx \right| \leq \left| \int_B \left(\det \nabla u^{(n)} - \frac{\partial \left(u_1, u_2^{(n)}, \cdots, u_N^{(n)} \right)}{\partial (x_1, \cdots, x_N)} \right) dx \right|$$

$$+ \left| \int_B \sum_{i=1}^N \frac{\partial u_1}{\partial x_i}(x) \left[\left(\operatorname{cof} \nabla u^{(n)}(x) \right)_{1i} - (\operatorname{cof} A)_{1i} \right] dx \right|.$$

By Theorem 3.1, Proposition 2.1, and (3.16) we obtain

$$\lim_{n\to\infty}\int_B \sum_{i=1}^N \frac{\partial u_1}{\partial x_i}(x)\left[\left(\operatorname{cof}\nabla u^{(n)}(x)\right)_{1i} - (\operatorname{cof} A)_{1i}\right] dx = 0.$$

In the remaining of the proof we show that

$$\lim_{n\to\infty}\int_B \left(\det\nabla u^{(n)} - \frac{\partial\left(u_1, u_2^{(n)}, \cdots, u_N^{(n)}\right)}{\partial\left(x_1, \cdots, x_N\right)}\right) dx = 0. \tag{3.18}$$

Since

$$\begin{aligned}&\int_B \left(\det\nabla u^{(n)} - \frac{\partial\left(u_1, u_2^{(n)}, \cdots, u_N^{(n)}\right)}{\partial\left(x_1, \cdots, x_N\right)}\right) dx\\ &= \int_{\partial B}\sum_{i=1}^N \left(u_1^{(n)} - u_1\right)\left(\operatorname{cof}\nabla u^{(n)}(x)\right)_{1i}\nu_i\, d\mathcal{H}^{N-1},\end{aligned}$$

if we knew that $\left\|u_1^{(n)} - u_1\right\|_{L^\infty(\partial B)} \to 0$ and if the sequence $\{u_n\}$ was bounded in $W^{1,N-1}\left(\partial B;\mathbb{R}^N\right)$ then (3.18) would follow immediately. It is clear that apriori this may not be true. To overcome this difficulty we must construct a surface ∂U "close" to ∂B, and a new sequence $\{w_n\}$ obtained by an appropriate truncation of $\left\{u_1^{(n)}\right\}$, for which this argument would hold and such that

$$\int_{\partial B}\sum_{i=1}^N \left(u_1^{(n)} - w_n\right)\left(\operatorname{cof}\nabla u^{(n)}(x)\right)_{1i}\nu_i\, d\mathcal{H}^{N-1} \to 0.$$

Fix $r > 0$. We cover the boundary of the ball B with finitely many balls $\{B(x_i, r)\}$, $i = 1, , \cdots, M = M(r)$, with $M = O\left(r^{1-N}\right)$. We now construct another covering of the boundary of the form $\{B(x_i, r_i)\}$ for certain $r \le r_i \le 2r$. The choice of r_i will be done recursively. The selection of r_1 is done as follows. Set $\alpha_0 := \alpha\left(2^N - 1\right)|B|$, and define

$$E_1 := \left\{s \in (r, 2r) : \liminf_{n\to\infty}\int_{\partial B(x_1,s)\cap B}\left|\nabla u^{(n)}\right|^{N-1} d\mathcal{H}^{N-1} \le \alpha_0\, r^{N-1}\right\}. \tag{3.19}$$

By Proposition 2.1 and Fatou's Lemma we have

$$\begin{aligned}
\alpha_0 r^N &= \alpha \left|B\left(x_1, 2r\right) \backslash B\left(x_1, r\right)\right| \geq \alpha \left|\left(B\left(x_1, 2r\right) \backslash B\left(x_1, r\right)\right) \cap B\right| \\
&= \lim_{n\to\infty} \int_{(B(x_1,2r)\backslash B(x_1,r))\cap B} \left|\nabla u^{(n)}\right|^{N-1} dx \\
&\geq \liminf_{n\to\infty} \int_{(r,2r)\backslash E_1} \int_{\partial B(x_1,s)\cap B} \left|\nabla u^{(n)}\right|^{N-1} d\mathcal{H}^{N-1} ds \\
&\geq \int_{(r,2r)\backslash E_1} \liminf_{n\to\infty} \int_{\partial B(x_1,s)\cap B} \left|\nabla u^{(n)}\right|^{N-1} d\mathcal{H}^{N-1} ds \\
&> \mathcal{L}^1\left(\left(r, 2r\right) \backslash E_1\right) \alpha_0 r^{N-1},
\end{aligned}$$

unless $\mathcal{L}^1\left(\left(r, 2r\right) \backslash E_1\right) = 0$. In both cases we conclude that

$$r - \mathcal{L}^1\left(E_1\right) < r$$

and thus E_1 is non empty. Since $\mathcal{L}^1\left(E_1\right) > 0$ and $u^{(n)} \to u$ in $L^1\left(B; \mathbb{R}^N\right)$ we may select $r_1 \in E_1$ and extract a subsequence of $\left\{u^{(n)}\right\}$ (not relabelled) such that

$$\lim_{n\to\infty} \int_{\partial B(x_1,r_1)\cap B} \left|\nabla u^{(n)}\right|^{N-1} d\mathcal{H}^{N-1} \leq \alpha_0 r^{N-1}, \tag{3.20}$$

$$u^{(n)} \to u \quad \text{in} \quad L^1\left(\partial B\left(x_1, r_1\right) \cap B; \mathbb{R}^N\right). \tag{3.21}$$

Assume now that $r_1, \cdots, r_{i-1}$ have been selected, an appropriate subsequence of $\left\{u^{(n)}\right\}$ (not relabelled) has been extracted, and define

$$\begin{aligned}
E_i := \Bigg\{ s \in (r, 2r) : \liminf_{n\to\infty} \Bigg(&\int_{\partial B(x_i,s)\cap B} \left|\nabla u^{(n)}\right|^{N-1} d\mathcal{H}^{N-1} \\
&+ \sum_{k=1}^{i-1} \frac{r}{2^{2^{k-1}}} \int_{\partial B(x_i,s)\cap\partial B(x_k,r_k)\cap B} \left|\nabla u^{(n)}\right|^{N-1} d\mathcal{H}^{N-2} \Bigg) \\
&\leq \alpha_0 r^{N-1} 2 \left(1 - \frac{1}{2^{2^{i-1}}}\right) \Bigg\}.
\end{aligned}$$

We show that E_i is non empty. By Proposition 2.1 (ii) and using the fact that

$r_k \in E_k$ for $1 \leq k \leq i-1$, we have

$$\begin{aligned}
&\liminf_{n\to\infty} \int_{(r,2r)\setminus E_i} \left(\int_{\partial B(x_i,s)\cap B} \left|\nabla u^{(n)}\right|^{N-1} d\mathcal{H}^{N-1} \right. \\
&\qquad \left. + \sum_{k=1}^{i-1} \frac{r}{2^{2^{k-1}}} \int_{\partial B(x_i,s)\cap\partial B(x_k,r_k)\cap B} \left|\nabla u^{(n)}\right|^{N-1} d\mathcal{H}^{N-2} \right) ds \\
&\quad \leq \limsup_{n\to\infty} \int_{(B(x_i,2r)\setminus B(x_i,r))\cap B} \left|\nabla u^{(n)}\right|^{N-1} dx \\
&\qquad + \limsup_{n\to\infty} \sum_{k=1}^{i-1} \frac{r}{2^{2^{k-1}}} \int_{\partial B(x_k,r_k)\cap B} \left|\nabla u^{(n)}\right|^{N-1} d\mathcal{H}^{N-1} \\
&\quad \leq \alpha \left|B\left(x_i,2r\right)\setminus B\left(x_i,r\right)\right| + \sum_{k=1}^{i-1} \frac{1}{2^{2^{k-1}}} \alpha_0 r^N 2 \left(1 - \frac{1}{2^{2^{k-1}}}\right) \\
&\quad = \alpha_0 r^N 2 \left(1 - \frac{1}{2^{2^{i-1}}}\right).
\end{aligned}$$

On the other hand, by Fatou's Lemma and the definition of E_i

$$\begin{aligned}
&\liminf_{n\to\infty} \int_{(r,2r)\setminus E_i} \left(\int_{\partial B(x_i,s)\cap B} \left|\nabla u^{(n)}\right|^{N-1} d\mathcal{H}^{N-1} \right. \\
&\qquad \left. + \sum_{k=1}^{i-1} \frac{r}{2^{2^{k-1}}} \int_{\partial B(x_i,s)\cap\partial B(x_k,r_k)\cap B} \left|\nabla u^{(n)}\right|^{N-1} d\mathcal{H}^{N-2} \right) ds \\
&\quad \geq \int_{(r,2r)\setminus E_i} \liminf_{n\to\infty} \left(\int_{\partial B(x_i,s)\cap B} \left|\nabla u^{(n)}\right|^{N-1} d\mathcal{H}^{N-1} \right. \\
&\qquad \left. + \sum_{k=1}^{i-1} \frac{r}{2^{2^{k-1}}} \int_{\partial B(x_i,s)\cap\partial B(x_k,r_k)\cap B} \left|\nabla u^{(n)}\right|^{N-1} d\mathcal{H}^{N-2} \right) ds \\
&\quad > \mathcal{L}^1\left(\left(r,2r\right)\setminus E_i\right) \alpha_0 r^{N-1} 2 \left(1 - \frac{1}{2^{2^{i-1}}}\right),
\end{aligned}$$

unless $\mathcal{L}^1\left(\left(r,2r\right)\setminus E_i\right) = 0$. We conclude that $\mathcal{L}^1\left(E_i\right) > 0$, and thus we may select $r_i \in E_i$ and extract a subsequence of $\left\{u^{(n)}\right\}$ (not relabelled) such that

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\partial B(x_i,r_i)\cap B} \left|\nabla u^{(n)}\right|^{N-1} d\mathcal{H}^{N-1} \\
&\qquad + \sum_{k=1}^{i-1} \frac{r}{2^{2^{k-1}}} \int_{\partial B(x_i,r_i)\cap\partial B(x_k,r_k)\cap B} \left|\nabla u^{(n)}\right|^{N-1} d\mathcal{H}^{N-2} \\
&\quad \leq \alpha_0 r^{N-1} 2 \left(1 - \frac{1}{2^{2^{i-1}}}\right)
\end{aligned} \tag{3.22}$$

and $u^{(n)} \to u$ in $L^1\left(\partial B\left(x_i, r_i\right) \cap B; \mathbb{R}^N\right)$. Set

$$U := B \setminus \bigcup_{i=1}^{M} B\left(x_i, r_i\right).$$

Observe that ∂U has the geometry of a "lunar surface", and the ridges of the craters, C_i, $j = 1, \cdots, M$, are exactly sets of the form $\partial B\left(x_i, r_i\right) \cap \partial B\left(x_k, r_k\right) \cap B$. By choice of the radii, the sequence $\left\{u^{(n)}\right\}$ is bounded on

$$W^{1,N-1}\left(\partial B\left(x_i, r_i\right) \cap \partial B\left(x_k, r_k\right) \cap B; \mathbb{R}^N\right),$$

which is compactly embedded in $C\left(\partial B\left(x_i, r_i\right) \cap \partial B\left(x_k, r_k\right) \cap B; \mathbb{R}^N\right)$. Therefore we may find $n_0\left(r\right)$ large enough such that

$$\left\|u_1^{(n)} - u_1\right\|_{L^\infty\left(\partial B\left(x_i, r_i\right) \cap \partial B\left(x_k, r_k\right) \cap B\right)} \leq r \tag{3.23}$$

for all $n \geq n_0\left(r\right)$ and $i, k = 1, \cdots, M$, $i \neq k$.

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a non decreasing smooth function such that $|\varphi(s)| \leq |s|$ for all $s \in \mathbb{R}$,

$$\varphi\left(s\right) = \begin{cases} s & \text{if } |s| \geq 2r, \\ 0 & \text{if } -r < s < r, \end{cases}$$

with $\varphi\left(s\right) \neq 0$ outside $\left(-r, r\right)$, and consider the truncations of $u_1^{(n)}$ on ∂U defined by

$$w_n\left(x\right) := u_1^{(n)}\left(x\right) + \varphi\left(u_1\left(x\right) - u_1^{(n)}\left(x\right)\right).$$

Note that $|\varphi(t) - t| \leq 2t$ for all t, and thus

$$\left\|w_n - u_1\right\|_{L^\infty(\partial U)} \leq 2r. \tag{3.24}$$

Consider a partition of U into cones K_i with vertices at the origin and whose intersection with ∂U are exactly the craters $C_i = B\left(x_i, r_i\right) \cap \partial U$, $i = 1, \cdots, M$. In view of (3.23), $w_n = u_1^{(n)}$ on the relative boundaries of C_i with respect to ∂U, and thus we extend w_n as equal to $u_1^{(n)}$ on $\partial K_i \cap U$. This now defines w_n as a function in $C^1\left(\bigcup_{i=1}^M \partial K_i\right)$. By Whitney's Extension Theorem, we extend

w_n to $\overline{U}$ as C^1 functions. Then

$$\begin{aligned}
&\limsup_{n\to\infty}\left|\int_B\left(\det\nabla u^{(n)}-\frac{\partial\left(u_1,u_2^{(n)},\cdots,u_N^{(n)}\right)}{\partial\left(x_1,\cdots,x_N\right)}\right)dx\right| \\
&\quad\leq\limsup_{n\to\infty}\left|\int_U\left(\det\nabla u^{(n)}-\frac{\partial\left(w_n,u_2^{(n)},\cdots,u_N^{(n)}\right)}{\partial\left(x_1,\cdots,x_N\right)}\right)dx\right| \qquad (3.25)\\
&\qquad+\limsup_{n\to\infty}\left|\int_U\left(\frac{\partial\left(w_n,u_2^{(n)},\cdots,u_N^{(n)}\right)}{\partial\left(x_1,\cdots,x_N\right)}-\frac{\partial\left(u_1,u_2^{(n)},\cdots,u_N^{(n)}\right)}{\partial\left(x_1,\cdots,x_N\right)}\right)dx\right| \\
&\qquad+C\limsup_{n\to\infty}\int_{B\setminus U}\left(\left|\det\nabla u^{(n)}\right|+\left|\nabla u^{(n)}\right|^{N-1}\right)dx.
\end{aligned}$$

By Proposition 2.1 (i) and (3.14), (3.15) we have

$$C\limsup_{n\to\infty}\int_{B\setminus U}\left(\left|\det\nabla u^{(n)}\right|+\left|\nabla u_n\right|^{N-1}\right)dx\leq C\left(\lambda+\gamma\right)\left|B\setminus U\right|=O\left(r\right),$$

while

$$\begin{aligned}
&\limsup_{n\to\infty}\left|\int_U\left(\frac{\partial\left(w_n,u_2^{(n)},\cdots,u_N^{(n)}\right)}{\partial\left(x_1,\cdots,x_N\right)}-\frac{\partial\left(u_1,u_2^{(n)},\cdots,u_N^{(n)}\right)}{\partial\left(x_1,\cdots,x_N\right)}\right)dx\right| \\
&=\limsup_{n\to\infty}\left|\int_{\partial U}\left(w_n-u_1\right)\left(\operatorname{cof}\nabla u^{(n)}\left(x\right)\right)_{1i}\nu_i\,d\mathcal{H}^{N-1}\right| \\
&\leq Cr\sum_{i=1}^M\int_{\partial B(x_i,r_i)\cap B}\left|\nabla u_n\right|^{N-1}d\mathcal{H}^{N-1}\leq 2C\alpha_0 r^N M\leq Cr,
\end{aligned}$$

by (3.22), (3.23) and (3.24), and where we have used the fact that $M=\mathrm{O}\left(r^{1-N}\right)$. It remains to estimate

$$\limsup_{n\to\infty}\left|\int_U\left(\det\nabla u^{(n)}-\frac{\partial\left(w_n,u_2^{(n)},\cdots,u_N^{(n)}\right)}{\partial\left(x_1,\cdots,x_N\right)}\right)dx\right|.$$

We have

$$\begin{aligned}
&\limsup_{n\to\infty}\left|\int_U\left(\det\nabla u^{(n)}-\frac{\partial\left(w_n,u_2^{(n)},\cdots,u_N^{(n)}\right)}{\partial\left(x_1,\cdots,x_N\right)}\right)dx\right| \\
&\leq\limsup_{n\to\infty}\sum_{i=1}^M\left|\int_{K_i}\det\nabla z^{(n)}\,dx\right| \\
&\leq C\limsup_{n\to\infty}\sum_{i=1}^M\left(\int_{\partial K_i\cap\left\{z_1^{(n)}\neq 0\right\}}\left|\operatorname{adj}\nabla z^{(n)}\right|\,d\mathcal{H}^{N-1}\right)^{N/(N-1)}
\end{aligned}$$

where we have used Theorem 2.6 and where

$$z^{(n)} := \left(u_1^{(n)} - w_n, u_2^{(n)}, \cdots, u_N^{(n)}\right).$$

Since $w_n = u_1^{(n)}$ on the lateral boundaries of the cone K_i, the right hand side of the previous inequality equals to

$$\begin{aligned}
&C \limsup_{n\to\infty} \sum_{i=1}^{M} \left(\int_{\partial K_i \cap \left\{ z_1^{(n)} \neq 0 \right\} \cap \partial U} \left| \operatorname{adj} \nabla z^{(n)} \right| \, d\mathcal{H}^{N-1} \right)^{N/(N-1)} \\
&\leq C \limsup_{n\to\infty} \sum_{i=1}^{M} \left(\int_{\partial B(x_i, r_i) \cap B} \left(1 + \left| \nabla u^{(n)} \right|^{N-1} \right) d\mathcal{H}^{N-1} \right)^{N/(N-1)} \\
&\leq C r^N M \leq C r,
\end{aligned}$$

by (3.22) and where we have used the fact that

$$\left| \nabla z^{(n)} \right| \leq C \left(1 + \|\varphi'\|_\infty\right) \left(1 + \left| \nabla u^{(n)} \right|\right).$$

Here C depends on $||\nabla u||_\infty$ and $\|\varphi'\|_\infty$ may be bounded from above by a constant independent of r. The conclusion follows by letting $r \to 0^+$.

Step 3: The proof of (3.17) for $N = 2$ is very similar to that of Step 2, although somewhat simpler. We only indicate the main changes. We proceed as in the previous step until (3.20). Assume now that $r_1, \cdots, r_{i-1}$ have been selected, an appropriate subsequence of $\{u_n\}$ (not relabelled) has been extracted, and define

$$E_i := \left\{ s \in (r, 2r) : \liminf_{n\to\infty} \int_{\partial B(x_i, s) \cap B} |\nabla u_n| \, d\mathcal{H}^1 \leq \alpha_0 r \right\}.$$

As before we conclude that $\mathcal{L}^1(E_i) > 0$. Hence we may select $r_i \in E_i$ and extract a subsequence of $\{u_n\}$ (not relabelled) such that

$$\lim_{n\to\infty} \int_{\partial B(x_i, r_i) \cap B} |\nabla u_n| \, d\mathcal{H}^1 \leq \alpha_0 r,$$

and $u_n(x) \to u(x)$ for all $x \in \partial B(x_i, r_i) \cap \partial B(x_k, r_k) \cap B$, $k = 1, \cdots, i-1$.

Condition (3.23) should now be replaced by

$$\left| \left(u_1^{(n)} - u_1 \right)(x) \right| \leq r \tag{3.26}$$

for all $n \geq n_0(r)$ and for all $x \in \partial B(x_i, r_i) \cap \partial B(x_k, r_k) \cap B$, $i \neq k$.

The remaining of the proof is similar.

Step 4: We now prove the theorem in the general case. Fix $\Omega' \subset\subset \Omega$ and let $\{u_n\}$ be as in the statement of theorem. Consider a standard sequence of mollifiers $\{\varphi_j\}$ and set

$$u_{n,j} := \varphi_j * u_n.$$

Clearly, for every $n \in \mathbb{N}$ we have $u_{n,j} \to u_n$ in $W^{1,N-1}\left(\Omega'; \mathbb{R}^N\right)$ as $j \to \infty$, and also $\operatorname{adj} \nabla u_{n,j} \to \operatorname{adj} \nabla u_n$ in $L^{\frac{N}{N-1}}\left(\Omega'; \mathbb{R}^{N \times N}\right)$ as $j \to \infty$. In addition, since

$$|\det \nabla u_n| \leq |\operatorname{adj} \nabla u_n|^{\frac{N}{N-1}},$$

in view of (1.1) it follows that $\det \nabla u_{n,j} \to \det \nabla u_n$ in $L^1(\Omega')$. We diagonalize the sequence $\{u_{n,j}\}$ to obtain a sequence $\{w_n\} \subset C^1\left(\Omega'; \mathbb{R}^N\right)$ with $w_n := u_{n,j(n)}$ such that

$$w_n \to u \text{ in } L^1(\Omega'), \quad \det \nabla w_n \overset{*}{\rightharpoonup} \mu,$$

and

$$|\nabla w_n|^{N-1} \mathcal{L}^N \lfloor \Omega \overset{*}{\rightharpoonup} \nu, \quad |\det \nabla w_n| \mathcal{L}^N \lfloor \Omega \overset{*}{\rightharpoonup} \lambda, \quad |\operatorname{adj} \nabla w_n| \mathcal{L}^N \lfloor \Omega \overset{*}{\rightharpoonup} \sigma,$$

for some non negative Radon measures ν, λ, σ.

Let $x_0 \in \Omega'$ be such that

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^{N+1}} \int_{B(x_0,\varepsilon)} |u(x) - T(x;x_0)| \, dx = 0, \tag{3.27}$$

where

$$T(x;x_0) := u(x_0) - \nabla u(x_0)(x - x_0),$$

$$\frac{d\mu}{d\mathcal{L}^N}(x_0), \quad \frac{d\nu}{d\mathcal{L}^N}(x_0), \quad \frac{d\lambda}{d\mathcal{L}^N}(x_0), \quad \frac{d\sigma}{d\mathcal{L}^N}(x_0) \tag{3.28}$$

exist and are finite. Let $\varepsilon \to 0^+$ be such that

$$|\mu|(\partial B(x_0,\varepsilon)) = \nu(\partial B(x_0,\varepsilon)) = \lambda(\partial B(x_0,\varepsilon)) = \sigma(\partial B(x_0,\varepsilon)) = 0,$$

and define

$$w_{n,\varepsilon}(y) := \frac{w_n(x_0 + \varepsilon y) - u(x_0)}{\varepsilon} \quad \text{for } y \in B.$$

By (3.8) we have

$$\lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \|w_{n,\varepsilon} - u_0\|_{L^1(B)} = 0,$$

where $u_0(y) := \nabla u(x_0) y$. Choose a countable, dense family $\{\theta_k\} \subset C_c(B)$. For every $k \in \mathbb{N}$

$$\lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \int_B \theta_k(y) |\nabla w_{n,\varepsilon}|^{N-1} dy = \left\langle \frac{d\nu}{d\mathcal{L}^N}(x_0) \mathcal{L}^N \lfloor B, \theta_k \right\rangle,$$

$$\lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \int_B \theta_k(y) |\det \nabla w_{n,\varepsilon}| \, dy = \left\langle \frac{d\lambda}{d\mathcal{L}^N}(x_0) \mathcal{L}^N \lfloor B, \theta_k \right\rangle,$$

$$\lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \int_B \theta_k(y) |\operatorname{adj} \nabla w_{n,\varepsilon}| \, dy = \left\langle \frac{d\gamma}{d\mathcal{L}^N}(x_0) \mathcal{L}^N \lfloor B, \theta_k \right\rangle,$$

and, in addition,

$$\lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \frac{1}{|B|} \int_B \det \nabla w_{n,\varepsilon} \, dy = \frac{d\mu}{d\mathcal{L}^N}(x_0).$$

We may now diagonalize the sequence $\{w_{n,\varepsilon}\}$ to obtain a sequence $\{v_n\} \subset C^1\left(\overline{B};\mathbb{R}^N\right)$ such that $v_n \to u_0$ in L^1, and

$$\begin{aligned} &|\nabla v_n|^{N-1}\mathcal{L}^N\lfloor B \overset{*}{\rightharpoonup} \frac{d\nu}{d\mathcal{L}^N}(x_0)\,\mathcal{L}^N\lfloor B,\\ &|\det\nabla v_n|\,\mathcal{L}^N\lfloor B \overset{*}{\rightharpoonup} \frac{d\lambda}{d\mathcal{L}^N}(x_0)\,\mathcal{L}^N\lfloor B,\\ &|\operatorname{adj}\nabla v_n|\,\mathcal{L}^N\lfloor B \overset{*}{\rightharpoonup} \frac{d\gamma}{d\mathcal{L}^N}(x_0)\,\mathcal{L}^N\lfloor B, \end{aligned} \tag{3.29}$$

with

$$\lim_{n\to\infty}\frac{1}{|B|}\int_B \det\nabla v_n\,dy = \frac{d\mu}{d\mathcal{L}^N}(x_0)\,. \tag{3.30}$$

We may now apply (3.17) to the sequence $\{v_n\}$ to conclude that

$$\lim_{n\to\infty}\frac{1}{|B|}\int_B \det\nabla v_n\,dy = \det\nabla u\,(x_0)\,,$$

and this, in view of (3.30), yields

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \det\nabla u\,(x_0)\,.$$

The arbitrariness of Ω' concludes the proof. ■

We use Theorem 1.2 to obtain a new proof of the result below, which, in turn, will be used to prove the lower semicontinuity result in Theorem 5.2.

Corollary 3.7 *Let $\{u_n\} \subset W^{1,N}\left(\Omega;\mathbb{R}^N\right)$ be a sequence bounded in $L^\infty\left(\Omega;\mathbb{R}^N\right)\cap W^{1,N-1}\left(\Omega;\mathbb{R}^N\right)$ and such that*

$$\sup_n \int_\Omega (\det\nabla u_n)^+\,dx < \infty.$$

Assume that $u_n \to u$ in $L^1\left(\Omega;\mathbb{R}^N\right)$ for some $u \in BV\left(\Omega;\mathbb{R}^N\right)$. Then there exists a subsequence (not relabelled) and a Radon measure μ such that

$$\det\nabla u_n \overset{*}{\rightharpoonup} \mu$$

in the sense of measures, with

$$\det\nabla u\,(x) = \frac{d\mu}{d\mathcal{L}^N}(x)$$

for $\mathcal{L}^N$ a.e. $x \in \Omega$.

Proof. For convenience of notation, in the proof we will denote u_n by $u^{(n)}$. For every $i = 1,\cdots,N$, the sequence $\left\{u_1^{(n)}\left(\operatorname{cof}\nabla u^{(n)}(x)\right)_{1i}\right\}$ is bounded in

$L^1(\Omega)$, thus, up to a subsequence (not relabelled), there exist Radon measures ν, $\mu_1, \cdots, \mu_N$, such that

$$u_1^{(n)}\left(\operatorname{cof}\nabla u^{(n)}(x)\right)_{1i} \overset{*}{\rightharpoonup} \mu_i, \quad \left(\det\nabla u^{(n)}\right)^+ \overset{*}{\rightharpoonup} \nu$$

in the sense of measures. Since

$$\det\nabla u^{(n)} = \sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(u_1^{(n)}(x)\left(\operatorname{cof}\nabla u^{(n)}(x)\right)_{1i}\right),$$

we deduce that

$$\det\nabla u^{(n)} \to \sum_{i=1}^{N}\frac{\partial\mu_i}{\partial x_i} \text{ in } \mathcal{D}'(\Omega).$$

By Proposition 2.2 we conclude that $\mu := \sum_{i=1}^{N}\frac{\partial\mu_i}{\partial x_i}$ is a Radon measure and that $\det\nabla u^{(n)} \overset{*}{\rightharpoonup} \mu$ in the sense of measures. The result now follows from the previous theorem. ■

Remark 3.8 Using cartesian currents, Celada and Dal Maso [13] proved the previous corollary, and, more generally, that if

$$\mathcal{M}_k(\nabla u_n) \overset{*}{\rightharpoonup} \mu_k$$

in the sense of measures, where $\mathcal{M}_k(A)$ is a generic minor of order k, with $1 \le k \le N$, then

$$\mathcal{M}_k(\nabla u) = \frac{d\mu_k}{d\mathcal{L}^N},$$

(see Corollary 3.3 in [13]).

4 Counterexamples

In what follows, Ω is an N-dimensional interval, i.e.

$$\Omega = (a_1, b_1) \times \cdots \times (a_N, b_N),$$

with $a_i < b_i$, $i = 1, \ldots, N$.

The proofs of Theorems 1.4 1.5, 1.6, and 1.7 are hinged on a series of lemma that we present in the sequel.

Lemma 4.1 *Suppose that $u \in C^2(\overline{\Omega};\mathbb{R}^N)$, $g \in C^1(\overline{\Omega})$, and $\varepsilon > 0$. Then there exist a function $v \in W^{1,\infty}(\Omega;\mathbb{R}^N)$ and a constant $C = C(N) > 0$ such that*

$$|\nabla v| \le C(|\nabla u| + |g| + 1) \quad \mathcal{L}^N \text{ a.e. in } \Omega, \tag{4.1}$$

$$\|v\|_{L^\infty(\Omega;\mathbb{R}^N)} \le \varepsilon \tag{4.2}$$

and

$$|\det(\nabla u + \nabla v)| \ge |g|^N \quad \mathcal{L}^N \text{ a.e. in } \Omega. \tag{4.3}$$

Proof. We set

$$h := N2^{N+2}\left(1+|\nabla u|^2+|g|^2\right)^{1/2}, \nu := \sqrt{N}\left(1+\|\nabla h\|_{L^\infty(\Omega;\mathbb{R}^N)}+\frac{1}{\varepsilon}\|h\|_{L^\infty(\Omega)}\right).$$

Consider a 2-periodic saw-tooth function $S:\mathbb{R}\to\mathbb{R}$ such that

$$S(t) := |t|, \quad t\in[-1,1],$$

and define

$$v(x) := h(x)\,\psi(x)$$

where

$$\psi(x) := \frac{1}{\nu}\sum_{i=1}^{N} S(\nu x_i)\mathbf{e}_i.$$

Note that

$$|\psi(x)| \le \min\left\{\frac{\varepsilon}{\|h\|_{L^\infty(\Omega)}}, \frac{1}{1+\|\nabla h\|_{L^\infty(\Omega;\mathbb{R}^N)}}\right\} \quad \text{in } \mathbb{R}^N, \tag{4.4}$$

$$|\nabla\psi(x)| = 1 \quad \mathcal{L}^N \text{ a.e. in } \mathbb{R}^N, \tag{4.5}$$

$$|\det\nabla\psi(x)| = 1 \quad \mathcal{L}^N \text{ a.e. in } \mathbb{R}^N. \tag{4.6}$$

From the definition of ν and (4.4)–(4.5) we immediately obtain (4.1) and (4.2). To prove (4.3), we first recall the elementary inequality

$$|\det B-\det A| \le N|A-B|\,(|A|^{N-1}+|B|^{N-1}) \tag{4.7}$$

which holds for any matrices A, $B\in\mathbb{R}^{N\times N}$, where the matrix norm $|A|$ is defined by

$$|A| := \sup\{|A\xi| : \ \xi\in\mathbb{R}^N,\ |\xi|=1\}.$$

Observe that

$$|\nabla u+\psi\otimes\nabla h| \le |\nabla u|+1 \le N^{-1}2^{-N-1}|h| \tag{4.8}$$

$\mathcal{L}^N$ a.e. in Ω. By (4.7) and (4.8) we have

$$\begin{aligned}
&|\det(\nabla u+\nabla v)-\det(h\nabla\psi)|\\
&\quad\le N|\nabla u+\psi\otimes\nabla h|\,(|\nabla u+\psi\otimes\nabla h+h\nabla\psi|^{N-1}+|h\nabla\psi|^{N-1})\\
&\quad\le 2^{-N-1}|h|\,(|2h|^{N-1}+|h|^{N-1}) \le \frac{1}{2}|h|^N
\end{aligned}$$

$\mathcal{L}^N$ a.e. in Ω, therefore by (4.6)

$$|\det(\nabla u+\nabla v| \ge |\det(h\nabla\psi)|-\frac{1}{2}\,|h|^N = \frac{1}{2}|h|^N \ge |g|^N$$

$\mathcal{L}^N$ a.e. in Ω, and this proves (4.3). ∎

Lemma 4.2 *Suppose that $N > 1$. Let Q_0 be the cube $(-1,1)^N$ and let $-1 \le \ell \le 1$. Then there exist a function $\omega \in W^{1,1}(Q_0, \mathbb{R}^N)$ and a constant $C = C(N) > 0$ such that*

$$\omega - \mathrm{id} \in \bigcap_{1 \le p < N} W_0^{1,p}(Q_0; \mathbb{R}^N), \tag{4.9}$$

$$|\nabla \omega(x)| \le \frac{C}{|x|} \quad \textit{for } \mathcal{L}^N \textit{ a.e. } x \in Q_0, \tag{4.10}$$

$$\omega(Q_0) \subset (-2,2)^N, \tag{4.11}$$

and

$$\det \nabla \omega(x) = \ell \quad \textit{for } \mathcal{L}^N \textit{ a.e. } x \in Q_0. \tag{4.12}$$

Proof. Define

$$\omega_i(x) := x_i \frac{\varphi(|x|_\infty)}{|x|_\infty}, \qquad i = 1, \dots, N,$$

where for $r \in [0,1]$

$$\varphi(r) := \left(\ell r^N - \ell + 1\right)^{1/N} \quad \text{and} \quad |x|_\infty := \max\{|x_1|, \dots, |x_N|\}.$$

For $\mathcal{L}^N$ a.e. $x \in Q_0$ we have

$$\frac{\partial \omega_i}{\partial x_j} = \delta_{ij} \frac{\varphi(|x|_\infty)}{|x|_\infty} + \left(\varphi'(|x|_\infty) - \frac{\varphi(|x|_\infty)}{|x|_\infty}\right) \frac{x_i}{|x|_\infty} \chi_{E_j}$$

with

$$\frac{\partial |x|_\infty}{\partial x_i} = \chi_{E_i}, \quad E_i := \{x : |x_i| = \max_j |x_j|\}.$$

Hence by (2.5)

$$\begin{aligned} \det \nabla \omega = & \frac{1}{|x|_\infty^N} \varphi^N(|x|_\infty) \\ & + |x|_\infty^{1-N} \sum_{i=1}^N \varphi^{N-1}(|x|_\infty) \left(\varphi'(|x|_\infty) - \frac{\varphi(|x|_\infty)}{|x|_\infty}\right) \frac{x_i}{|x|_\infty} \chi_{E_i} \\ = & |x|_\infty^{1-N} \varphi^{N-1}(|x|_\infty) \varphi'(|x|_\infty), \end{aligned}$$

and we obtain (4.12), i.e.

$$\det \nabla \omega = \ell.$$

Now, standard arguments show that the function ω is weakly differentiable, (4.10) and (4.11) can be obtained by a straightforward computation, the L^p-integrability of gradient follows from (4.11) and the boundary condition is preserved because $\varphi(1) = 1$. ∎

Lemma 4.3 *Suppose that $u \in C^2(\overline{\Omega};\mathbb{R}^N)$, $f \in C^1(\overline{\Omega})$, $1 \le p < N$, $1 \le q < \infty$, and $\varepsilon > 0$. Then there exist a function $w \in W^{1,p}(\Omega;\mathbb{R}^N)$ and a constant $C = C(N,p) > 0$ such that*

$$\int_\Omega |\nabla w|^p \, dx \le C \int_\Omega (|\nabla u|^p + |f|^{p/N} + 1) \, dx, \tag{4.13}$$

$$\|w - u\|_{L^\infty(\Omega;\mathbb{R}^N)} \le \varepsilon, \tag{4.14}$$

$$w = u \text{ on } \partial\Omega \tag{4.15}$$

and

$$\int_\Omega |\det \nabla w - f|^q \, dx \le \varepsilon. \tag{4.16}$$

Proof. Without loss of generality we assume that $\Omega = (0,1)^N$. By mollification of $\left(2|f| + \frac{7}{2}\right)^{1/N}$ we obtain a function $g \in C^1(\overline{\Omega})$ such that

$$2|f| + 3 \le g^N \le 2|f| + 4. \tag{4.17}$$

Choose $\delta \in (0,\varepsilon) \cap \mathbb{Q}$, and use Lemma 4.1 to construct a function $v \in W^{1,\infty}(\Omega;\mathbb{R}^N)$ satisfying (4.1)–(4.3) with $\frac{\delta}{2}$ in place of ε. Mollifying v we obtain a smooth function $\tilde{v}$ with support in Ω' such that

$$\|\tilde{v}\|_{L^\infty(\Omega;\mathbb{R}^N)} \le \delta, \quad \|\tilde{v}\|_{W^{1,\infty}(\Omega;\mathbb{R}^N)} \le C(\|\nabla u\|_{L^\infty(\Omega;\mathbb{R}^{N\times N})} + \|f\|_{L^\infty(\Omega)} + 1), \tag{4.18}$$

$$\|\tilde{v} - v\|_{W^{1,p}(\Omega;\mathbb{R}^N)} < \delta,$$

for a suitable constant $C > 1$ depending only on N. Set

$$z := u + \tilde{v},$$

and

$$c_0 := C(\|\nabla u\|_{L^\infty(\Omega;\mathbb{R}^{N\times N})} + \|f\|_{L^\infty(\Omega)} + 1), \tag{4.19}$$

$$c_1 := 1 + \|u\|_{W^{2,\infty}(\Omega;\mathbb{R}^N)} + \|z\|_{W^{2,\infty}(\Omega;\mathbb{R}^N)} + \|f\|_{W^{1,\infty}(\Omega)}.$$

Note that c_0 does not depend on δ. Write $1 - \delta = \frac{r}{s}$ with r, $s \in \mathbb{N}$, and choose $m = ks$ with $k \in \mathbb{N}$ large enough so that

$$\frac{c_1\left(2\sqrt{N} + 2Nc_0^{N-1}\right)}{m} \le \delta. \tag{4.20}$$

We consider a decomposition of $\Omega_\delta := \left(\frac{\delta}{2}, 1 - \frac{\delta}{2}\right)^N$ into $(rk)^N$ cubes

$$Q^\alpha, \qquad \alpha \in \{1, \dots, rk\}^N,$$

where

$$Q^\alpha := x_\alpha + \left(\frac{-1}{2m}, \frac{1}{2m}\right)^N.$$

We define

$$\mathcal{A}_1 := \left\{ \alpha \in \{1, \ldots, rk\}^N : \ |\det \nabla z(x^\alpha)| < 2|f(x^\alpha)| + 1 \right\},$$
$$\mathcal{A}_2 := \{1, \ldots, rk\}^N \setminus \mathcal{A}_1.$$

If $\alpha \in \mathcal{A}_2$, then set

$$\ell^\alpha := \frac{f(x^\alpha)}{\det \nabla z(x^\alpha)}.$$

Note that $\ell^\alpha \in [-1, 1]$. Define

$$h(x) := \begin{cases} x^\alpha + \dfrac{\varphi^\alpha(|x - x^\alpha|_\infty)}{|x - x^\alpha|_\infty} (x - x^\alpha), & \alpha \in \mathcal{A}_2, \ x \in \overline{Q}^\alpha \setminus \{x^\alpha\}, \\ x & \text{otherwise}, \end{cases}$$

where

$$\varphi^\alpha(r) := \left(\ell^\alpha r^N + (1 - \ell^\alpha)(2m)^{-N} \right)^{1/N}.$$

Observe that

$$\varphi^\alpha(r) = \frac{1}{2m} \left(\ell^\alpha \, (2rm)^N - \ell^\alpha + 1 \right)^{1/N},$$

and thus by a rescaled and translated version of Lemma 4.2 we deduce that $h \in W^{1,p}(\Omega; \mathbb{R}^N)$, and if $\alpha \in \mathcal{A}_2$ then

$$\int_{Q^\alpha} |\nabla h|^p \leq \frac{C}{m^N}, \tag{4.21}$$

$$(h - \mathrm{id})\lfloor Q^\alpha \in W_0^{1,p}(Q^\alpha; \mathbb{R}^N), \tag{4.22}$$

$$h(Q^\alpha) \subset x_\alpha + \left(-\frac{1}{m}, \frac{1}{m} \right)^N \tag{4.23}$$

and

$$\det \nabla h = \ell^\alpha \text{ on } Q^\alpha. \tag{4.24}$$

Consider a cut-off function $\eta \in C_c^2(\Omega)$ such that

$$0 \leq \eta \leq 1, \quad \eta = 1 \text{ on } \Omega_\delta, \quad |\nabla \eta| \leq \frac{C}{\delta}, \tag{4.25}$$

and set

$$w(x) := u(x) \, (1 - \eta(x)) + z(h(x)) \, \eta(x), \quad x \in \Omega.$$

Then w is differentiable for all $x \neq x_\alpha$, and (4.15) is trivially verified.

By (4.18) we have

$$\begin{aligned} |w(x) - u(x)| &\leq |\tilde{v}(h(x))| + |u(h(x)) - u(x)| \\ &\leq \delta + \|\nabla u\|_{L^\infty(\Omega; \mathbb{R}^{N \times N})} |h(x) - x| \\ &\leq \delta + \frac{c_0}{m} \leq 2\delta, \end{aligned} \tag{4.26}$$

where in the last inequality we have used (4.20) and (4.23). Hence (4.14) is satisfied.

Note that $w = z \circ h$ in Ω_δ, and thus

$$\int_{Q^\alpha} |\nabla w|^p \, dx \le C \sup_{Q^\alpha} |\nabla z(h)|^p \int_{Q^\alpha} |\nabla h|^p \, dx \le \frac{C}{m^N} \sup_{Q^\alpha} |\nabla z(h)|^p \tag{4.27}$$

by (4.21). Since $\nabla z(h(\cdot))$ and $\nabla z\,(\cdot)$ are continuous functions, we may find x_1, $x_2 \in Q^\alpha$ such that

$$\sup_{x \in Q^\alpha} |\nabla z(h(x))| = |\nabla z(h(x_1))|, \quad \inf_{x \in Q^\alpha} |\nabla z(x)| = |\nabla z(x_2)|,$$

and thus by (4.19), (4.23) and (4.20)

$$\begin{aligned} \sup_{x \in Q^\alpha} |\nabla z(h(x))|^p &\le \left(\inf_{x \in Q^\alpha} |\nabla z(x)| + \|z\|_{W^{2,\infty}(\Omega;\mathbb{R}^N)} |h(x_1) - x_2| \right)^p \\ &\le \left(\inf_{x \in Q^\alpha} |\nabla z(x)| + \frac{2c_1\sqrt{N}}{m} \right)^p \le \left(\inf_{x \in Q^\alpha} |\nabla z(x)| + \delta \right)^p . \end{aligned}$$

Hence by (4.17) and (4.18)

$$\begin{aligned} \int_{Q^\alpha} |\nabla w|^p \, dx &\le \frac{C}{m^N} \left(\inf_{x \in Q^\alpha} |\nabla z(x)| + \delta \right)^p \le C \int_{Q^\alpha} (1 + |\nabla z|^p) \, dx \\ &\le C \int_{Q^\alpha} (1 + |\nabla u|^p + |\nabla v|^p + \delta^p) \, dx \\ &\le C \int_{Q^\alpha} (1 + |\nabla u|^p + |g|^p) \, dx \\ &\le C \int_{Q^\alpha} \left(1 + |\nabla u|^p + |f|^{p/N} \right) \, dx. \end{aligned} \tag{4.28}$$

Since $h\,(x) = x$ in $L_\delta := \Omega \setminus \Omega_\delta$ we have

$$\begin{aligned} \nabla w &= \nabla u\,(1 - \eta) + \eta \nabla z + (z - u) \otimes \nabla \eta \\ &= \nabla u + \eta \nabla \tilde{v} + \tilde{v} \otimes \nabla \eta \\ &= \nabla u + \eta \nabla v + \eta \nabla\,(\tilde{v} - v) + \tilde{v} \otimes \nabla \eta \end{aligned} \tag{4.29}$$

and so

$$\begin{aligned} \int_{L_\delta} |\nabla w|^p \, dx &\le C \left(\int_{L_\delta} (|\nabla u|^p + |\nabla v|^p + |\nabla\,(\tilde{v} - v)\,|^p) \, dx + \frac{1}{\delta^p} \|\tilde{v}\|^p_{L^\infty(\Omega;\mathbb{R}^N)} |L_\delta| \right) \\ &\le C \left(\int_{L_\delta} \left(1 + |\nabla u|^p + |f|^{p/N} \right) dx \right) \end{aligned} \tag{4.30}$$

by (4.17), (4.18) and (4.25). Combining this inequality with (4.28) yields (4.13).

To prove (4.16) we first estimate

$$\int_{L_\delta} |\det \nabla w - f|^q \, dx.$$

By $(4.29)_2$, (4.18) and (4.25), we have

$$|\nabla w| \le |\nabla u| + |\nabla \tilde{v}| + |\tilde{v} \otimes \nabla \eta| \le 2c_0 + C,$$

hence

$$\int_{L_\delta} |\det \nabla w - f|^q \, dx \le C \, |L_\delta| \left(\|\nabla w\|^{Nq}_{L^\infty(\Omega;\mathbb{R}^{N\times N})} + \|f\|^q_{L^\infty(\Omega)} \right) \le C\delta c_0^{Nq}. \tag{4.31}$$

In Ω_δ we consider separately the cubes Q^α with $\alpha \in \mathcal{A}_1$, and Q^α with $\alpha \in \mathcal{A}_2$. If $\alpha \in \mathcal{A}_1$ then $w = z$ in Q^α and

$$|\det \nabla z(x^\alpha)| < 2|f(x^\alpha)| + 1.$$

Using (4.20), (4.7) and (4.19) we infer that

$$\begin{aligned}
|\det \nabla z\,(x)\,| \le& |\det \nabla z\,(x) - \det \nabla z(x^\alpha)| + 2|f\,(x) - f(x^\alpha)| + 1 + 2\,|f\,(x)| \\
\le& N \,|\nabla z\,(x) - \nabla z(x^\alpha)| \left(|\nabla z\,(x)\,|^{N-1} + |\nabla z(x^\alpha)|^{N-1}\right) \\
&+ 2c_1|x - x^\alpha| + 1 + 2\,|f\,(x)| \\
\le& 2c_1 \left(1 + Nc_0^{N-1}\right) |x - x^\alpha| + 1 + 2\,|f\,(x)| \\
\le& \frac{2c_1 \left(1 + Nc_0^{N-1}\right)}{m} + 1 + 2\,|f\,(x)| \le 2|f\,(x)\,| + 2
\end{aligned} \tag{4.32}$$

for $\mathcal{L}^N$ a.e. $x \in Q^\alpha$. From (4.3) and (4.17) we obtain

$$|\det \nabla z| + 1 < 2|f| + 3 \le g^N \le |\det(\nabla u + \nabla v)| \quad \mathcal{L}^N \text{ a.e. in } Q^\alpha,$$

and thus, using (4.7),

$$\begin{aligned}
1 &\le |\det \nabla z - \det(\nabla u + \nabla v)| \le N\,|\nabla v - \nabla \tilde{v}| \, \left(|\nabla z|^{N-1} + |\nabla u + \nabla v|^{N-1}\right) \\
&\le Cc_0^{N-1}|\nabla v - \nabla \tilde{v}|.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_{Q^\alpha} |\det \nabla w - f|^q \, dx &= \int_{Q^\alpha} |\det \nabla z - f|^q \, dx \\
&\le Cc_0^{Nq}\,|Q^\alpha| \le Cc_0^{Nq+N-1} \int_{Q^\alpha} |\nabla v - \nabla \tilde{v}| \, dx.
\end{aligned} \tag{4.33}$$

If $\alpha \in \mathcal{A}_2$ then

$$f(x^\alpha) = \ell^\alpha \det \nabla z(x^\alpha).$$

Recall that $\ell^\alpha \in [-1,1]$. Then by (4.20), (4.23), (4.24) we obtain

$$\begin{aligned}|\det \nabla w(x) - f(x)| &\le |\ell^\alpha \det \nabla z(h(x)) - \ell^\alpha \det \nabla z(x^\alpha)| + |f(x^\alpha) - f(x)| \\ &\le \frac{c_1\left(1 + 2Nc_0^{N-1}\right)}{m} \le \delta\end{aligned}$$

for each $x \in Q^\alpha$, and thus

$$\int_{Q^\alpha} |\det \nabla w - f|^q \, dx \le \frac{C\delta^q}{m^N}. \tag{4.34}$$

Putting together (4.31), (4.33) and (4.34) by Hölder inequality we obtain

$$\begin{aligned}\int_\Omega |\det \nabla w - f|^q \, dx &\le C\delta c_0^{Nq} + Cc_0^{Nq+N-1} \int_\Omega |\nabla v - \nabla \tilde{v}| \, dx + C\delta^q \, |\Omega_\delta| \\ &\le C(\delta c_0^{Nq} + c_0^{Nq+N-1}\delta + \delta^q),\end{aligned}$$

and this, with a suitable choice of δ, concludes the proof. ■

As an immediate consequence of Lemma 4.3 we obtain:

Proof of Theorem 1.5. We may suppose that $\Omega = (0,1)^N$. It is clear that it suffices to treat the case where $u \in C^2(\overline{\Omega}; \mathbb{R}^N)$ and $f \in C^1(\overline{\Omega})$. The general case follows using a standard mollification and diagonalization argument. With each $n = 1, 2, \ldots$ we associate a partition of Ω into cubes

$$Q^\alpha, \qquad \alpha \in \{1, \ldots, n\}^N,$$

where

$$Q^\alpha := \left(\frac{\alpha_1 - 1}{n}, \frac{\alpha_N}{n}\right) \times \cdots \times \left(\frac{\alpha_N - 1}{n}, \frac{\alpha_N}{n}\right).$$

We apply Lemma 4.3 to Q^α, u, f, n^{-N-1} in place of Ω, u, f, ε and find a function $w_n^\alpha \in W^{1,p}(Q^\alpha)$ such that

$$\int_{Q^\alpha} |\nabla w_n^\alpha|^p \, dx \le C \int_{Q^\alpha} (|\nabla u|^p + |f|^{p/N} + 1) \, dx, \tag{4.35}$$

$$\|w_n^\alpha - u\|_{L^\infty(Q^\alpha; \mathbb{R}^N)} \le \frac{1}{n^{N+1}}, \tag{4.36}$$

$$w_n^\alpha = u \text{ on } \partial Q^\alpha,$$

and

$$\int_{Q^\alpha} |\det \nabla w_n^\alpha - f|^q \, dx \le \frac{1}{n^{N+1}}. \tag{4.37}$$

It suffices to set

$$u_n(x) := w_n^\alpha(x) \text{ if } x \in Q^\alpha$$

Then, by (4.35) and (4.36), $u_n \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^N)$, while by (4.37)

$$\int_\Omega |\det \nabla u_n - f|^q \, dx = \sum_\alpha \int_{Q^\alpha} |\det \nabla u_n - f|^q \, dx \le \sum_\alpha \frac{1}{n^{N+1}} = \frac{1}{n},$$

and this concludes the proof. ■

Proof of Theorem 1.4. Assume that $\Omega = (0,1)^N$, $u \in C^2(\overline{\Omega};\mathbb{R}^N)$ and $\mu = f\mathcal{L}^N$ with $f \in C^1(\overline{\Omega})$. Given $n = 2^k$, $k = 1, 2, \ldots$, we partition Ω into cubes

$$Q^\alpha, \qquad \alpha \in \{1, \ldots, n\}^N,$$

where

$$Q^\alpha := \left(\frac{\alpha_1 - 1}{n}, \frac{\alpha_N}{n}\right) \times \cdots \times \left(\frac{\alpha_N - 1}{n}, \frac{\alpha_N}{n}\right). \tag{4.38}$$

We denote by x^α the center of Q^α and write

$$\tilde{Q}^\alpha := \left(x_1^\alpha - \tfrac{1}{2n^2}, x_1^\alpha + \tfrac{1}{2n^2}\right) \times \cdots \times \left(x_N^\alpha - \tfrac{1}{2n^2}, x_N^\alpha + \tfrac{1}{2n^2}\right),$$

$$\Omega_n := \bigcup_\alpha \tilde{Q}^\alpha.$$

We note that

$$\left|\tilde{Q}^\alpha\right| = \frac{|Q^\alpha|}{n^N}, \quad |\Omega_n| = \frac{|\Omega|}{n^N}.$$

Set $\tilde{f} := f - \det \nabla u$ and

$$f_n(x) := n^N \tilde{f}(x^\alpha + n(x - x^\alpha)), \qquad x \in \tilde{Q}^\alpha.$$

We apply Lemma 4.3 to $\tilde{Q}^\alpha$, u, f_n, n^{-N-1} in place of Ω, u, f, ε to find a function $w_n^\alpha \in W^{1,p}(\tilde{Q}^\alpha)$ such that

$$\int_{\tilde{Q}^\alpha} |\nabla w_n^\alpha|^p \, dx \leq C \int_{\tilde{Q}^\alpha} (|\nabla u|^p + |f_n|^{p/N} + 1) \, dx, \tag{4.39}$$

$$\|w_n^\alpha - u\|_{L^\infty(\tilde{Q}^\alpha;\mathbb{R}^N)} \leq \frac{1}{n}, \tag{4.40}$$

$$w_n^\alpha = u \text{ on } \partial\tilde{Q}^\alpha, \tag{4.41}$$

and

$$\int_{\tilde{Q}^\alpha} |\det \nabla w_n^\alpha - f_n| \, dx \leq \frac{1}{n^{N+1}}. \tag{4.42}$$

Set

$$u_n := \begin{cases} w_n^\alpha & \text{on each } \tilde{Q}^\alpha, \\ u & \text{on } \Omega \setminus \Omega_n \,. \end{cases}$$

Then, by (4.39)–(4.42), u_n obviously satisfies

$$\int_\Omega |u_n - u|^p \, dx \leq \frac{1}{n^p}, \tag{4.43}$$

and

$$\begin{aligned}\int_\Omega |\nabla u_n - \nabla u|^p\,dx &= \int_{\Omega_n} |\nabla w_n^\alpha - \nabla u|^p\,dx \\ &\le C\int_{\Omega_n} (|\nabla u|^p + |f_n|^{p/N} + 1)\,dx \\ &= C\left(\int_{\Omega_n} (|\nabla u|^p + 1)\,dx + \frac{1}{n^{N-p}}\int_\Omega |\tilde f|^{p/N}\,dx\right) \\ &\le C\left(\int_{\Omega_n} (|\nabla u|^p + 1)\,dx + \frac{1}{n^{N-p}}\int_\Omega |\det\nabla u - f|^{p/N}\,dx\right)\end{aligned} \tag{4.44}$$

where we have used the fact that a simple change of variables yields

$$\int_{\Omega_n} |f_n|^{p/N}\,dx = \frac{1}{n^{N-p}}\int_\Omega |\tilde f|^{p/N}\,dx.$$

Moreover

$$\begin{aligned}\int_\Omega |\det\nabla u_n|\,dx &\le \int_\Omega |\det\nabla u|\,dx + \int_{\Omega_n} |f_n|\,dx + \int_{\Omega_n} |\det\nabla w_n^\alpha - f_n|\,dx \\ &\le \int_\Omega |\det\nabla u|\,dx + \int_\Omega |\tilde f|\,dx + \frac{1}{n} \\ &\le C(1 + \|\det\nabla u\|_{L^1(\Omega)} + \|f\|_{L^1(\Omega)}).\end{aligned} \tag{4.45}$$

Consider a fixed cube Q^β where $\beta \in \{1,\dots,m\}^N$ for some $m = 2^j$, $j \in \mathbb{N}$. If $n = 2^k$ with $k \ge j$ then

$$\begin{aligned}\left|\int_{Q^\beta} \det\nabla u_n\,dx - \int_{Q^\beta} f\,dx\right| &= \left|\int_{Q^\beta} (\det\nabla u_n - \det\nabla u - \tilde f)\,dx\right| \\ &= \left|\int_{Q^\beta} (\det\nabla u_n - \det\nabla u)\,dx - \int_{\Omega_n\cap Q^\beta} f_n\,dx\right| \\ &= \left|\int_{\Omega_n\cap Q^\beta} (\det\nabla u_n - f_n - \det\nabla u)\,dx\right| \\ &\le \frac{|Q^\beta|}{n} + \int_{\Omega_n\cap Q^\beta} |\det\nabla u|\,dx.\end{aligned} \tag{4.46}$$

Using the fact that $|\Omega_n| \to 0$ we conclude that $u_n \to u$ in $W^{1,p}(\Omega;\mathbb{R}^N)$, and

$$\int_{Q^\beta} \det\nabla u_n\,dx \to \int_{Q^\beta} f\,dx \tag{4.47}$$

as $n \to \infty$ for every Q^β as above. The weak-* convergence of $\{\det\nabla u_n\}$ to f in the sense of measures now follows from (4.47), the uniform boundedness of

$\{\det \nabla u_n\}$ in $L^1(\Omega)$ as implied by (4.45), and the fact that the function $\varphi \in C_c(\Omega)$ maybe approximated uniformly by linear combinations of characteristic functions of cubes Q^β as above. ∎

The approximating sequence obtained in Theorem 1.5 may be required to be in $C^1(\overline{\Omega};\mathbb{R}^N)$ if $p < N-1$. This improvement in the regularity of u_n asks for a more elaborated version of Lemma 4.3, precisely:

Lemma 4.4 *Let R' be a bounded $N-1$-dimensional interval and $\Omega = R' \times (a,b)$. Suppose that $u \in C^2\left(\overline{\Omega};\mathbb{R}^N\right)$, $f \in C^1\left(\overline{\Omega}\right)$, $1 \le p < N-1$, $1 \le q < \infty$, and $\varepsilon > 0$. Then there exist $w \in C^1\left(\overline{\Omega};\mathbb{R}^N\right)$ and a constant $C = C(N,p) > 0$ such that*

$$\int_\Omega |\nabla w|^p \, dx \le C \int_\Omega (|\nabla u|^p + |f|^{p/N} + 1)\, dx, \tag{4.48}$$

$$\|w-u\|_{L^p(\Omega;\mathbb{R}^N)} \le \varepsilon, \tag{4.49}$$

$$w = u \text{ in a neighborhood of } \partial R' \times [a,b] \tag{4.50}$$

and

$$\int_\Omega |\det \nabla w - f|^q \, dx \le \varepsilon. \tag{4.51}$$

Proof. Clearly it suffices to construct a Lipschitz functions satisfying (4.48)–(4.51), as a standard smoothing argument will yield from it a C^1 function with analog properties. The proof is similar to that of Lemma 4.3. One of main differences is that some constructions are carried out in dimension $N-1$. We may assume that $R' = (0,1)^{N-1}$ and that $(a,b) = (0,1)$. We proceed exactly as in the proof of Lemma 4.3 up to (4.19). We decompose $R'_\delta := \left(\frac{\delta}{2}, 1-\frac{\delta}{2}\right)^{N-1}$ into $(rk)^{N-1}$ cubes

$$P^\alpha, \qquad \alpha \in \{1,\dots,rk\}^{N-1},$$

where

$$P^\alpha = x'_\alpha + \left(\frac{-1}{2m}, \frac{1}{2m}\right)^{N-1}.$$

Now we introduce the function L^α on $\mathbb{R}$ in the following way: on the set

$$E^\alpha := \{x_N \in [0,1] : mx_N \text{ is integer}, \ |\det \nabla z(x'_\alpha, x_N)| \ge 2|f(x'_\alpha, x_N)| + 1\}$$

we define

$$L^\alpha(x_N) := \frac{f(x'_\alpha, x_N)}{\det \nabla z(x'_\alpha, x_N)}$$

and then we extend it to be bounded and continuous on $\mathbb{R}$ and linear on each interval complementary to E^α (if $E^\alpha = \emptyset$ then $L^\alpha :\equiv 0$). We have

$$\|L^\alpha\|_{L^\infty([0,1])} \le \frac{1}{2}. \tag{4.52}$$

We refer to the $(N-1)$-dimensional version of Lemma 4.2 and set

$$h(x) := \begin{cases} \left(x'_\alpha + \dfrac{\varphi^{\alpha,x_N}(|x'-x'_\alpha|_\infty)}{|x'-x'_\alpha|_\infty}(x'-x'_\alpha), x_N \right) & \text{if } x' \in \overline{P}^\alpha \setminus \{x'_\alpha\}, \\ (x', x_N) & \text{otherwise,} \end{cases}$$

where

$$\varphi^{\alpha,x_N}(r) := \left(L^\alpha(x_N) r^{N-1} + (1 - L^\alpha(x_N))(2m)^{1-N} \right)^{1/(N-1)}.$$

We have $h \in W^{1,p}(\Omega; \mathbb{R}^{N-1})$,

$$|\nabla_{x'} h(x', x_N)| \leq \frac{C}{m|x'-x'_\alpha|_\infty}, \quad \left| \frac{\partial h}{\partial x_N}(x', x_N) \right| \leq C, \tag{4.53}$$

$$h(x', x_N) = (x', x_N) \text{ on } \partial P^\alpha \times [0,1], \tag{4.54}$$

$$h(P^\alpha \times \{x_N\}) \subset \left\{ x' \in R^{N-1} : \operatorname{dist}(x', P^\alpha) \leq \frac{1}{m} \right\} \times \{x_N\}, \tag{4.55}$$

and

$$\det \nabla h(x', x_N) = L^\alpha(x_N) \text{ on } P^\alpha \times [0,1]. \tag{4.56}$$

Fix $\tau \in \left(0, \frac{1}{4}\right)$ and define the functions $\gamma : \overline{R'_\delta} \setminus \left\{ x'_\alpha : \alpha \in \{1, \ldots, rk\}^{N-1} \right\} \to \mathbb{R}^{N-1}$ and $H : \overline{\Omega} \to \mathbb{R}^N$ by

$$\gamma(x') := x'_\alpha + \frac{2\tau}{m|x'-x'_\alpha|_\infty}(x'-x'_\alpha) \quad \text{if } x' \in P^\alpha$$

and

$$H(x) := \begin{cases} h(x', x_N) & \text{if } |x'-x'_\alpha|_\infty \geq \dfrac{2\tau}{m}, x' \in P^\alpha \\ h\left(\gamma(x'), \left(\dfrac{m}{\tau}|x'-x'_\alpha|_\infty - 1 \right) x_N \right) & \text{if } \dfrac{\tau}{m} < |x'-x'_\alpha|_\infty < \dfrac{2\tau}{m}, \\ (x'_\alpha, 0) + \dfrac{m}{\tau}|x'-x'_\alpha|_\infty \left(h\left(\gamma(x'), 0\right) - (x'_\alpha, 0) \right) & \text{if } |x'-x'_\alpha|_\infty \leq \dfrac{\tau}{m}, \\ (x', x_N) & \text{otherwise.} \end{cases}$$

Note that by writing

$$\mathbb{R}^{N-1} = \bigcup_{i=1}^{N-1} A_i, \quad A_i := \left\{ x' \in \mathbb{R}^{N-1} : |x'-x'_\alpha|_\infty = |x_i - (x_\alpha)_i| \right\}$$

it is clear that the i^{th} component of γ is constant on A_i and so

$$\det \nabla_{x'} \gamma = 0 \text{ for } \mathcal{L}^{N-1} \text{ a.e. } x' \in \mathbb{R}^{N-1}. \tag{4.57}$$

Consider a cut-off function $\eta \in C_c^2(R')$ such that

$$0 \leq \eta \leq 1, \quad \eta = 1 \text{ on } R'_\delta, \quad |\nabla \eta| \leq \frac{C}{\delta},$$

and set

$$w(x', x_N) := u(x', x_N)\,(1 - \eta(x')) + z(H(x', x_N))\,\eta(x'), \quad x \in \Omega.$$

The function w is Lipschitz continuous and (4.50) is trivially verified. Using (4.53) we have

$$|\nabla H(x', x_N)| \le \begin{cases} \dfrac{C}{m|x' - x'_\alpha|_\infty} & \text{if } |x' - x'_\alpha|_\infty \ge \dfrac{2\tau}{m}, \quad x' \in P^\alpha, \\ \dfrac{Cm}{\tau} & \text{if } |x' - x'_\alpha|_\infty < \dfrac{2\tau}{m}. \end{cases} \tag{4.58}$$

Let $\alpha \in \{1, \dots, rk\}^{N-1}$ and denote

$$\hat{P}^\alpha := \left\{ x' \in P^\alpha : |x' - x'_\alpha|_\infty < \frac{2\tau}{m} \right\}.$$

As in (4.26), by (4.18), (4.20) and (4.23) on $(P^\alpha \setminus \hat{P}^\alpha) \times [0,1]$ we have

$$|w(x) - u(x)| \le |\tilde{v}(H(x))| + |u(H(x)) - u(x)| \le 2\delta.$$

Hence

$$\begin{aligned} \int_\Omega |w-u|^p\,dx &= \sum_\alpha \int_{\hat{P}^\alpha \times [0,1]} |w-u|^p\,dx + \sum_\alpha \int_{(P^\alpha \setminus \hat{P}^\alpha)\times[0,1]} |w-u|^p\,dx \\ &\le \sum_\alpha \mathcal{L}^{N-1}(\hat{P}^\alpha)\,\|w-u\|^p_{L^\infty(\Omega;R^N)} + (2\delta)^p \\ &\le C\tau^{N-1}\,\|w-u\|^p_{L^\infty(\Omega;R^N)} + (2\delta)^p, \end{aligned}$$

which, by choosing τ sufficiently small, asserts (4.49).

Now we want to estimate ∇w. Suppose first that $\alpha \in \{1, \dots, rk\}^{N-1}$. Using (4.20) and (4.55) we obtain

$$\begin{aligned} &\int_{(P^\alpha \setminus \hat{P}^\alpha)\times[0,1]} |\nabla w(x)|^p\,dx \qquad (4.59) \\ &\quad \le C \sup_{x \in (P^\alpha \setminus \hat{P}^\alpha)\times[0,1]} |\nabla z(h(x))|^p \int_{(P^\alpha \setminus \hat{P}^\alpha)\times[0,1]} |\nabla h(x)|^p\,dx \\ &\quad \le \frac{C}{m^{N-1}} \sup_{x \in P^\alpha \times [0,1]} |\nabla z(h(x))|^p, \qquad (4.60) \end{aligned}$$

where we have used the fact that

$$\begin{aligned} \int_{(P^\alpha \setminus \hat{P}^\alpha)\times[0,1]} |\nabla h|^p\,dx &\le \frac{C}{m^p} \int_{P^\alpha \setminus \hat{P}^\alpha} \frac{1}{|x' - x'_\alpha|^p_\infty}\,dx' \\ &\le \frac{C}{m^p} \int_{\frac{\tau}{2m}}^{\frac{1}{2m}} r^{N-2-p}dr \le \frac{C}{m^{N-1}}. \end{aligned}$$

As in (4.27) we conclude that

$$\int_{\left(P^\alpha\setminus\hat{P}^\alpha\right)\times[0,1]} |\nabla w|^p\,dx \le C\int_{P^\alpha\times[0,1]}\left(1+|\nabla u|^p+|f|^{p/N}\right)dx. \tag{4.61}$$

On the other hand, by (4.58)

$$\begin{aligned}\int_{\hat{P}^\alpha\times[0,1]} |\nabla w|^p\,dx &\le C\sup_{x\in\hat{P}^\alpha\times[0,1]}|\nabla z(H(x))|^p\int_{\hat{P}^\alpha\times[0,1]}|\nabla H(x)|^p\,dx \\ &\le Cc^p\left(\frac{\tau}{m}\right)^{N-1}\left(\frac{m}{\tau}\right)^p.\end{aligned} \tag{4.62}$$

Since $H(x) = x$ in $(R'\setminus R'_\delta)\times[0,1]$ we have

$$\begin{aligned}\nabla w &= \nabla u\,(1-\eta)+\eta\nabla z+(z-u)\otimes\nabla\eta \\ &= \nabla u+\eta\nabla\tilde{v}+\tilde{v}\otimes\nabla\eta \\ &= \nabla u+\eta\nabla v+\eta\nabla(\tilde{v}-v)+\tilde{v}\otimes\nabla\eta\end{aligned}$$

and so, as in (4.30),

$$\int_{\left(R'\setminus R'_\delta\right)\times[0,1]} |\nabla w|^p\,dx \le C\int_{\left(R'\setminus R'_\delta\right)\times[0,1]}\left(1+|\nabla u|^p+|f|^{p/N}\right)dx \tag{4.63}$$

Hence by (4.1), (4.17), (4.18), (4.60) and (4.62)

$$\begin{aligned}\int_\Omega |\nabla w|^p\,dx &\le C\int_\Omega\left(1+|\nabla u|^p+|f|^{p/N}\right)dx \\ &\quad + Cc^p\left(\frac{\tau}{m}\right)^{N-1}\left(\frac{m}{\tau}\right)^p m^{N-1} \\ &\le C\int_\Omega(1+|\nabla u|^p+|f|^{p/N})\,dx + Cc^p m^p\tau^{N-1-p}.\end{aligned}$$

Since $p<N-1$ by choosing τ sufficiently small we may ensure that $m^p\tau^{N-1-p}\le 1$, which gives (4.48).

Finally we estimate $\det\nabla w - f$. Denote

$$\begin{aligned}Q^{\alpha,j} &:= P^\alpha\times\left(\frac{j-1}{m},\frac{j}{m}\right),\quad \alpha\in\{1,\dots,rk\}^{N-1},\ j\in\{1,\dots,m\}, \\ \hat{Q}^{\alpha,j} &:= \left\{(x',x_N)\in Q^{\alpha,j}:\ |x'-x'_\alpha|_\infty<\frac{2\tau}{m}\right\}.\end{aligned}$$

Since $w = z\circ H$ on $Q^{\alpha,j}$ we have that

$$\det\nabla w(x',x_N) = \det\nabla z(H(x',x_N))\det\nabla H(x',x_N). \tag{4.64}$$

We claim that $\det\nabla H = 0$ on each $\hat{Q}^{\alpha,j}$. Indeed if $|x'-x'_\alpha|_\infty\le\frac{\tau}{m}$ then this follows from the fact that H does not depend on the variable x_N. If $\frac{\tau}{m}<$

$|x'-x'_\alpha|_\infty < \dfrac{2\tau}{m}$ then

$$\det \nabla H(x', x_N) = \left(\det \nabla h \left(\gamma(x'), \left(\frac{m}{\tau}|x'-x'_\alpha|_\infty - 1\right) x_N\right)\right) \times$$
$$(\det \nabla_{x'}\gamma(x')) \left(\frac{m}{\tau}|x'-x'_\alpha|_\infty - 1\right) = 0$$

by (4.57). Hence if $x \in Q^{\alpha,j}$ then

$$\det \nabla w(x', x_N) = \begin{cases} L^\alpha(x_N) \det \nabla z(h(x', x_N)) & (x', x_N) \in Q^{\alpha,j} \setminus \hat{Q}^{\alpha,j}, \\ 0 & (x', x_N) \in \hat{Q}^{\alpha,j}. \end{cases}$$

We consider $\alpha \in \{1, \ldots, rk\}^{N-1}$, $j \in \{1, \ldots, m\}$ and distinguish two cases. If at least one of the points $\frac{j-1}{m}$, $\frac{j}{m}$ does not belong to E^α, say $\frac{j}{m} \notin E^\alpha$ then an argument entirely similar to that of (4.31) and (4.32) with $\left(x'_\alpha, \frac{j}{m}\right)$ in place of x_α gives

$$\int_{Q^{\alpha,j}\setminus\hat{Q}^{\alpha,j}} |\det \nabla w - f|^q \, dx \le C c^{Nq+N-1} \int_{Q^{\alpha,j}\setminus\hat{Q}^{\alpha,j}} |\nabla v - \nabla \tilde{v}| \, dx. \tag{4.65}$$

If $\frac{j-1}{m}$, $\frac{j}{m} \in E^\alpha$ then

$$f\left(x'_\alpha, \tfrac{j-1}{m}\right) = L^\alpha\left(\tfrac{j-1}{m}\right) \det \nabla z \left(x'_\alpha, \tfrac{j-1}{m}\right),$$
$$f\left(x'_\alpha, \tfrac{j}{m}\right) = L^\alpha\left(\tfrac{j}{m}\right) \det \nabla z \left(x'_\alpha, \tfrac{j}{m}\right),$$

and

$$L^\alpha(x_N) \text{ is a convex combination of } \frac{f\left(x'_\alpha, \frac{j-1}{m}\right)}{\det \nabla z\left(x'_\alpha, \frac{j-1}{m}\right)} \text{ and } \frac{f\left(x'_\alpha, \frac{j}{m}\right)}{\det \nabla z\left(x'_\alpha, \frac{j}{m}\right)}$$

whenever $\frac{j-1}{m} \le x_N \le \frac{j}{m}$. Hence

$$\left|L^\alpha(x_N) - L^\alpha\left(\tfrac{j}{m}\right)\right| \le \left| \frac{f\left(x'_\alpha, \frac{j-1}{m}\right)}{\det \nabla z\left(x'_\alpha, \frac{j-1}{m}\right)} - \frac{f\left(x'_\alpha, \frac{j}{m}\right)}{\det \nabla z\left(x'_\alpha, \frac{j}{m}\right)} \right|$$
$$\le \frac{C c_1^N}{m},$$

where we have used the fact that by definition of E^α we have

$$\min\left\{\left|\det \nabla z\left(x'_\alpha, \tfrac{j-1}{m}\right)\right|, \left|\det \nabla z\left(x'_\alpha, \tfrac{j}{m}\right)\right|\right\} \ge 1.$$

This, together with (4.52) yields

$$|\det \nabla w(x) - f(x)| \le \left|L^\alpha(x_N) \det \nabla z(h(x)) - L^\alpha(x_N) \det \nabla z\left(x'_\alpha, \tfrac{j}{m}\right)\right|$$
$$+ \left|\det \nabla z\left(x'_\alpha, \tfrac{j}{m}\right)\right| \, \left|L^\alpha(x_N) - L^\alpha\left(\tfrac{j}{m}\right)\right|$$
$$+ \left|f\left(x'_\alpha, \tfrac{j}{m}\right) - f(x)\right| \le C\delta$$

for each $x \in Q^{\alpha,j} \setminus \hat{Q}^{\alpha,j}$, and thus

$$\int_{Q^{\alpha,j}\setminus\hat{Q}^{\alpha,j}} |\det \nabla w - f|^q \, dx \leq \frac{C\delta^q}{m^N}. \tag{4.66}$$

From (4.31), (4.65) and (4.66) we obtain

$$\int_\Omega |\det \nabla w - f|^q \, dx \leq Cc^{Nq+N-1} \int_\Omega |\nabla v - \nabla \tilde{v}| \, dx + C\delta^q + \int_{\bigcup_{\alpha,j} \hat{Q}_j} |f|^q \, dx$$
$$+ C\delta c_0^{Nq} \leq C(c^{Nq+N-1}\delta + \delta^q) + \int_{\bigcup_{\alpha,j} \hat{Q}_j} |f|^q \, dx + C\delta c_0^{Nq}.$$

With an appropriately small choice of τ and δ, using absolute continuity of the Lebesgue integral of $|f|^q$ we obtain (4.51). This concludes the proof. ■

Proof of Theorem 1.6. The reasoning used follows that of Theorem 1.4, where now we invoke Lemma 4.4 in place of Lemma 4.3, and we partition Ω into thin cylinders instead of small cubes. ■

Proof of Theorem 1.7. The proof is similar to that of Theorem 1.5, where now Lemma 4.4 is used in place of Lemma 4.3, and we partition Ω into thin cylinders instead of small cubes. ■

5 Lower semicontinuity

In this section we use Theorems 3.3 and 1.2 to derive new proofs of some well-known lower semicontinuity results for polyconvex integrands. The following proposition was proved by Dal Maso and Sbordone [16] using cartesian currents and by Fusco and Hutchinson [21] with a more direct proof.

Proposition 5.1 *Let $h : \mathbb{R}^{\tau(d,N)} \to [0, \infty)$ be a convex function such that*

$$h(v) \to \infty \qquad \text{as } |v| \to \infty. \tag{5.1}$$

Let $u \in W^{1,p}(\Omega; \mathbb{R}^d)$, and let $\{u_n\}$ be a sequence of functions in $W^{1,p}(\Omega; \mathbb{R}^d)$ which converges to u in $L^1(\Omega; \mathbb{R}^d)$, where $p = \min\{d, N\}$. Then

$$\int_\Omega h(\mathcal{M}(\nabla u)) \, dx \leq \liminf_{n\to\infty} \int_\Omega h(\mathcal{M}(\nabla u_n)) \, dx.$$

Proof. A blow-up argument allows us to reduce to the case where $\Omega = B$ and u is an affine function (see e.g. Step 2 in the proof of Theorem 1.2). In view of (5.1) we may assume that $\{\mathcal{M}(\nabla u_n)\}$ is bounded in $L^1(B; \mathbb{R}^\tau)$, and thus we apply Proposition 3.5 (see also Remark 3.6) to construct a sequence $\{w_n\} \subset W^{1,p}\left(B; \mathbb{R}^d\right) \cap L^\infty\left(B; \mathbb{R}^d\right)$ such that $w_n \to u$ in $L^\infty\left(B; \mathbb{R}^d\right)$, $M\left(\nabla w_n\right) \overset{*}{\rightharpoonup} M\left(\nabla u\right)$ in the sense of measures, and

$$|\{x \in B : \, w_n(x) \neq u_n(x)\}| \to 0 \tag{5.2}$$

and

$$\int_{\{w_n \neq u_n\}} |\mathcal{M}(\nabla w_n)| \, dx \to 0. \tag{5.3}$$

Let $\varphi \in C_c(B;[0,1])$ and let $v \mapsto a + b \cdot v$ be any affine function such that $h(v) \geq a + b \cdot v$ for all $v \in \mathbb{R}^\tau$. Then

$$\begin{aligned}
\int_B h(\mathcal{M}(\nabla u_n))\, dx &\geq \int_B \varphi(x)\, h(\mathcal{M}(\nabla u_n))\, dx \\
&\geq \int_{\{w_n = u_n\}} \varphi(x)\, h(\mathcal{M}(\nabla w_n))\, dx \\
&\geq \int_{\{w_n = u_n\}} \varphi(x)\, (a + b \cdot \mathcal{M}(\nabla w_n))\, dx.
\end{aligned}$$

By (5.3) we have

$$\begin{aligned}
\liminf_{n\to\infty} \int_B h(\mathcal{M}(\nabla u_n))\, dx &\geq \lim_{n\to\infty} \int_B \varphi(x)\, (a + b \cdot \mathcal{M}(\nabla w_n))\, dx \\
&= \int_B \varphi(x)\, (a + b \cdot \mathcal{M}(\nabla u))\, dx \\
&= (a + b \cdot \mathcal{M}(\nabla u)) \int_B \varphi(x)\, dx,
\end{aligned}$$

where we have used the fact that ∇u is constant. Letting $\varphi \nearrow 1$ and taking the supremum over all affine functions below h we conclude the proof. ■

The previous result may be improved when $d = N$ and when the integrand depends only on the determinant. Precisely:

Theorem 5.2 *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set, and let $f : \mathbb{R} \to [0, \infty]$ be a lower semicontinuous convex function. Let $\{u_n\} \subset W^{1,N}(\Omega; \mathbb{R}^N)$ be a sequence bounded in $W^{1,N-1}(\Omega; \mathbb{R}^N)$. Assume that $u_n \to u$ in $L^1(\Omega; \mathbb{R}^N)$, for some $u \in BV(\Omega; \mathbb{R}^N)$. Then*

$$\liminf_{n\to\infty} \int_\Omega f(\det \nabla u_n)\, dx \geq \int_\Omega f(\det \nabla u)\, dx.$$

Theorem 5.2 was proved by Celada and Dal Maso [13] using cartesian currents. Below we present an alternative argument that bypasses this technique.

Proof. Without loss of generality, we may assume that

$$\liminf_{n\to\infty} \int_\Omega f(\det \nabla u_n)\, dx = \lim_{n\to\infty} \int_\Omega f(\det \nabla u_n)\, dx < \infty. \tag{5.4}$$

Following [13] let $\psi \in C^1([0,\infty))$ be such that $\psi(t) = t$ for $t \in [0,1]$, ψ is constant for $t \geq 2$, and $0 \leq \psi'(t) \leq 1$ for all $t \geq 0$. Define $\Psi \in C^1(\mathbb{R}^N; \mathbb{R}^N)$ by

$$\Psi(y) := \begin{cases} \psi(|y|) \frac{y}{|y|} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0, \end{cases}$$

and for $k \in \mathbb{N}$ set

$$\Psi_k(y) := k\Psi\left(\frac{y}{k}\right).$$

It is easy to see that $\Psi_k(y) = y$ in $B(0,k)$, and as

$$\nabla\Psi_k(y) = \left(\mathbb{I}_N - \frac{y}{|y|}\otimes\frac{y}{|y|}\right)\frac{k}{|y|}\psi\left(\frac{|y|}{k}\right) + \frac{y}{|y|}\otimes\frac{y}{|y|}\psi'\left(\frac{|y|}{k}\right),$$

we have

$$\det\nabla\Psi_k(y) = \begin{cases} 1 & \text{if } |y| < k, \\ \left(\frac{k}{|y|}\psi\left(\frac{|y|}{k}\right)\right)^{N-1}\psi'\left(\frac{|y|}{k}\right) & \text{if } k \le |y| \le 2k \\ 0 & \text{if } |y| > 2k. \end{cases}$$

Therefore as $0 \le \psi'(t) \le 1$ and $\psi(t) \le t$ for all $t \ge 0$, we observe that

$$0 \le \det\nabla\Psi_k(y) \le 1 \text{ for all } y \in \mathbb{R}^N. \tag{5.5}$$

Let

$$u_{n,k}(x) := (\Psi_k \circ u_n)(x), \quad v_k(x) := (\Psi_k \circ u)(x).$$

Note that since $\|\nabla\Psi_k\|_\infty \le C(N)$, the sequence $\{u_{n,k}\}$ remains bounded in $W^{1,N-1}(\Omega;\mathbb{R}^N)$, uniformly with respect to n and k. Let g be any nonnegative, convex, piecewise affine function such that $f(v) \ge g(v)$ for all $v \in \mathbb{R}$. By the convexity of g and (5.5) we have

$$\begin{aligned} g(\det\nabla u_{n,k}) &= g\left(\det\left(D\Psi\left(\frac{u_n}{k}\right)\right)\det\nabla u_n\right) \\ &\le \det\left(D\Psi\left(\frac{u_n}{k}\right)\right)g(\det\nabla u_n) + \left(1 - \det\left(D\Psi\left(\frac{u_n}{k}\right)\right)\right)g(0) \\ &\le g(\det\nabla u_n) + \chi_{\{|u_n|>k\}}g(0). \end{aligned}$$

Hence

$$\begin{aligned} \infty > \int_\Omega f(\det\nabla u_n)\,dx &\ge \int_\Omega g(\det\nabla u_n)\,dx \\ &\ge \int_\Omega g(\det\nabla u_{n,k})\,dx - g(0)\left|\{x\in\Omega:\ |u_n|>k\}\right|. \end{aligned}$$

For a fixed $k \in \mathbb{N}$,

$$\lim_{n\to\infty}\left|\{x\in\Omega:\ |u_n|>k\}\right| = 0, \tag{5.6}$$

because $u_n \to u$ in $L^1(\Omega,\mathbb{R}^N)$. We claim that

$$\liminf_{n\to\infty}\int_\Omega g(\det\nabla u_{n,k})\,dx \ge \int_\Omega g(\det\nabla v_k)\,dx. \tag{5.7}$$

If the claim holds, then by (5.6)

$$\begin{aligned}\int_\Omega g(\det \nabla v_k)\, dx &\le \liminf_{n\to\infty} \int_\Omega g(\det \nabla u_{n,k})\, dx \\ &\le \liminf_{n\to\infty} \int_\Omega f(\det \nabla u_n)\, dx + g(0)\, |\{x\in\Omega :\ |u_n| > k\}| \\ &= \liminf_{n\to\infty} \int_\Omega f(\det \nabla u_n)\, dx.\end{aligned}$$

Using the Lebesgue Monotone Convergence Theorem, letting $k\to\infty$ we deduce that

$$\begin{aligned}\liminf_{n\to\infty} \int_\Omega f(\det \nabla u_n)\, dx &\ge \limsup_{k\to\infty} \int_\Omega g(\det \nabla v_k)\, dx \\ &\ge \lim_{k\to\infty} \int_{|\{x\in\Omega :\, |u|\le k\}|} g(\det \nabla u)\, dx = \int_\Omega g(\det \nabla u)\, dx.\end{aligned}$$

Here we have use the Chain Rule for the composition of C^1 with a BV function [5] to assert that

$$\nabla v_k = \nabla\left(\Psi_k \circ u\right) = \nabla \Psi_k \nabla u = \nabla u$$

when $|u| \le k$. Taking the supremum over all such piecewise affine functions g below f, and using again Lebesgue Monotone Convergence Theorem, we obtain

$$\lim_{n\to\infty} \int_\Omega f(\det \nabla u_n)\, dx \ge \int_\Omega f(\det \nabla u)\, dx.$$

The remaining of the proof is dedicated to asserting (5.7).

Either g is constant, in which case there is nothing to prove, or there exists $C>0$ such that

$$g\left(t\right) \ge Ct^+ - \frac{1}{C} \tag{5.8}$$

for every $t\in\mathbb{R}$, or

$$g\left(t\right) \ge Ct^- - \frac{1}{C}$$

for every $t\in\mathbb{R}$. Without loss of generality, we assume that (5.8) holds and that

$$\infty > \liminf_{n\to\infty} \int_\Omega g(\det \nabla u_{n,k})\, dx = \lim_{n\to\infty} \int_\Omega g(\det \nabla u_{n,k})\, dx. \tag{5.9}$$

Fix $k\in\mathbb{N}$ and, for simplicity of notation, denote $u_{n,k}$ and u_k simply by w_n and w, respectively. In view of (5.8), together with (5.9), we have

$$\limsup_{n\to\infty} \int_\Omega \left(\det \nabla w_n\right)^+\, dx < \infty. \tag{5.10}$$

Since $\{w_n\}$ is bounded in $L^\infty\left(\Omega;\mathbb{R}^N\right)$ and in $W^{1,N-1}\left(\Omega;\mathbb{R}^N\right)$, by Corollary 3.7 there exist Radon measures λ, ν and μ such that, up to a subsequence (not relabelled),

$$\det \nabla w_n \mathcal{L}^N \lfloor \Omega \overset{*}{\rightharpoonup} \lambda, \quad \left|\nabla w_n\right|^{N-1} \mathcal{L}^N \Big\lfloor \Omega \overset{*}{\rightharpoonup} \nu, \quad g(\det \nabla w_n)\, \mathcal{L}^N \lfloor \Omega \overset{*}{\rightharpoonup} \mu$$

in the sense of measures, with

$$\det \nabla w\,(x) = \frac{d\lambda}{d\mathcal{L}^N}\,(x)$$

for $\mathcal{L}^N$ a.e. $x \in \Omega$.

We claim that

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon \to 0^+} \frac{\mu(B(x_0,\varepsilon))}{|B(x_0,\varepsilon)|} \geq g(\det \nabla w(x_0)) \qquad \text{for } \mathcal{L}^N \text{ a.e. } x_0 \in \Omega. \tag{5.11}$$

If (5.11) holds, then the conclusion of the theorem follows immediately. Indeed, let $\varphi \in C_c(\Omega;\mathbb{R})$, $0 \leq \varphi \leq 1$. Since

$$\mu = \frac{d\mu}{d\mathcal{L}^N}\,\mathcal{L}^N + \mu_s,$$

where $\mu_s \geq 0$, we have

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega g(\det \nabla w_n)\,dx &\geq \liminf_{n\to\infty}\int_\Omega \varphi\, g(\det\nabla w_n)\,dx = \int_\Omega \varphi\,d\mu \\ &\geq \int_\Omega \varphi\,\frac{d\mu}{d\mathcal{L}^N}\,dx \geq \int_\Omega \varphi\, g(\det \nabla w)\,dx.\end{aligned}$$

By letting $\varphi \to 1$, and using Lebesgue Dominated Convergence Theorem, we obtain the desired result. Thus, to conclude the proof of the theorem, it suffices to show (5.11).

Let $x_0 \in \Omega$ be such that

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon^{N+1}}\int_{B(x_0,\varepsilon)} |w\,(x) - T\,(x;x_0)|\;dx = 0,$$

where

$$T\,(x;x_0) := w\,(x_0) - \nabla w\,(x_0)\,(x - x_0)\,,$$

and

$$\frac{d\mu}{d\mathcal{L}^N}\,(x_0)\,,\ \frac{d\lambda}{d\mathcal{L}^N}\,(x_0)\,,\ \frac{d\nu}{d\mathcal{L}^N}\,(x_0)\ \text{ exist and are finite.}$$

Choose $\varepsilon \to 0^+$ such that

$$\mu\,(\partial B\,(x_0,\varepsilon)) = \lambda\,(\partial B\,(x_0,\varepsilon)) = \nu\,(\partial B\,(x_0,\varepsilon)) = 0,$$

and define

$$w_{n,\varepsilon}\,(y) := \frac{w_n\,(x_0+\varepsilon y) - w\,(x_0)}{\varepsilon} \quad \text{for } y \in B.$$

As in Theorem 1.2, we may now diagonalize the sequence $\{w_{n,\varepsilon}\}$ to obtain a sequence $\{v_n\}$ bounded in $W^{1,N-1}\left(B;\mathbb{R}^N\right)$, such that $v_n \to v_0$ in L^1, where $v_0\,(y) := \nabla w\,(x_0)\,y$, and

$$\det \nabla v_n \mathcal{L}^N \lfloor B \overset{*}{\rightharpoonup} \det \nabla w\,(x_0)\,\mathcal{L}^N \lfloor B, \tag{5.12}$$

$$\frac{d\mu}{d\mathcal{L}^N}\,(x_0) = \lim_{n\to\infty}\frac{1}{|B|}\int_B g\,(\det\nabla v_n)\,.$$

Let $\varphi \in C_c(B;[0,1])$ and let $v \mapsto a+bv$ be any affine function such that $g(v) \geq a+bv$ for all $v \in \mathbb{R}$. Then

$$\begin{aligned}\frac{d\mu}{d\mathcal{L}^N}(x_0) &= \lim_{n\to\infty} \frac{1}{|B|}\int_B g(\det \nabla v_n)\ dy \\ &\geq \liminf_{n\to\infty} \frac{1}{|B|}\int_B \varphi(y)\ g(\det \nabla v_n)\ dy \\ &\geq \liminf_{n\to\infty} \frac{1}{|B|}\int_B \varphi(y)\,(a\ +\ b\det \nabla v_n(y))\ dy \\ &= \frac{1}{|B|}\int_B \varphi(y)\,(a\ +\ b\det \nabla w(x_0))\ dy \\ &= (a\ +\ b\det \nabla w(x_0))\,\frac{1}{|B|}\int_B \varphi(y)\ dy,\end{aligned}$$

where we have used (5.12). By letting first $\varphi \nearrow 1$ we obtain

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq a\ +\ b\det \nabla w(x_0)$$

and taking the supremum over all affine functions below g, we conclude

$$\frac{d\mu}{d\mathcal{L}^N}(x_0) \geq g(\det \nabla w(x_0)),$$

that (5.11) is established. ∎

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