A MINIMIZATION APPROACH TO THE WAVE EQUATION ON TIME-DEPENDENT DOMAINS

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Abstract. We prove the existence of weak solutions to the homogeneous wave equation on a suitable class of time-dependent domains. Using the approach suggested by De Giorgi and developed by Serra and Tilli, such solutions are approximated by minimizers of suitable functionals in space-time.

Keywords: wave equation, time-dependent domains, minimization

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Introduction

Several problems in dynamic fracture mechanics lead to the study of the wave equation in time-dependent domains (see [6, 7, 3]). The main difficulty is that at every time t the solution belongs to a different function space V_t. It is not restrictive to assume that all spaces V_t are embedded in a given Hilbert space H.

In the case of fracture mechanics, a common situation is $V_t = H^1(\Omega \setminus \Gamma_t)$ and $H = L^2(\Omega)$, where Ω is a domain in $\mathbb{R}^d$ and Γ_t is a closed $(d-1)$-dimensional subset of Ω, representing the crack at time t. A natural assumption on Γ_t is that it is monotonically increasing with respect to t, thus encoding the fact that, once created, a crack cannot disappear. As a consequence, the spaces V_t are increasing in time too.

To deal with possibly irregular cracks a more general increasing family of spaces has been considered in [2]: $V_t = GSBV_2^2(\Omega, \Gamma_t)$, defined as the space of functions $u \in GSBV(\Omega)$ such that $u \in L^2(\Omega)$, $\nabla u \in L^2(\Omega; \mathbb{R}^d)$, and $J_u \subset \Gamma_t$ (see [1] for the definition and properties of these spaces and for the definition of the approximate gradient ∇u and of the jump set J_u).

Given $u^0 \in V_0$ and $u^1 \in H$, the Cauchy problem we are interested in is formally written as

$$\begin{cases} u''(t) + Au(t) = 0 & \text{for a.e. } t > 0\,, \\ u(t) \in V_t & \text{for a.e. } t > 0\,, \\ u(0) = u^0\,,\ u'(0) = u^1\,, \end{cases} \tag{0.1}$$

where $'$ denotes the time derivative and A is a continuous and coercive linear operator ($A = -\Delta$ with homogeneous Neumann boundary conditions in the examples considered above).

The existence of a solution for (0.1) has already been proven in [2], through a time-discrete approach, by solving suitable incremental minimum problems and then passing to the limit as the time step tends to zero.

The purpose of this paper is to prove that a solution of (0.1) can be approximated by global minimizers of suitable energy functionals defined as integrals on $[0, \infty)$ with respect to time. On the one hand this shows a link between solutions of the hyperbolic problem (0.1) and solutions of minimum problems for integral functionals on the same time domain. On the other hand this result provides a new proof of the existence of a solution to (0.1).

The seminal idea of this approximation process goes back to a conjecture by De Giorgi [5] on the nonlinear wave equation. Such a conjecture has been proven by Serra and Tilli in [8] and, in a more general setting, in [9].

In our paper we extend their result to the case of time-dependent domains. To illustrate the global minimization approach in our setting, we focus on the model case $V_t = H^1(\Omega \setminus \Gamma_t)$ and $A = -\Delta$. The main idea is to associate to the Cauchy problem (0.1) a functional of the form

$$\mathcal{F}_\varepsilon(u) := \frac{1}{2}\int_0^\infty e^{-t/\varepsilon}\Big(\varepsilon^2\|u''(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^2_{L^2(\Omega;\mathbb{R}^d)}\Big)\,\mathrm{d}t\,. \tag{0.2}$$

This functional is to be minimized, for every fixed $\varepsilon > 0$, among all the functions $t \mapsto u(t)$ satisfying the initial conditions $u(0) = u^0$ and $u'(0) = u^1$ and the time-dependent constraint $u(t) \in V_t$ for a.e. $t > 0$. Once the existence of a minimizer u_ε is proven, the Euler-Lagrange equation of (0.2) formally reads as

$$\varepsilon^2 u''''_\varepsilon(t) - 2\varepsilon u'''_\varepsilon(t) + u''_\varepsilon(t) - \Delta u_\varepsilon(t) = 0 \qquad \text{in } \Omega \setminus \Gamma_t\,,$$

and hence, letting $\varepsilon \to 0$, one *formally* obtains a solution to the wave equation in (0.1).

As mentioned above, a quite general scheme to pass to the limit rigorously has been introduced by Serra and Tilli in [9] when time-dependent constraint $u(t) \in V_t$ is not present. The proof consists in finding suitable estimates on the minimizers u_ε of the functionals $\mathcal{F}_\varepsilon$ and to exploit these estimates in order to obtain, by compactness, the convergence of u_ε to a weak solution u to the wave equation.

In this paper we implement this scheme in the case of time-dependent domains. This requires some changes in the proof, since all competitors of the minimum problem for (0.2) must satisfy the constraint $u(t) \in V_t$ for a.e. $t > 0$.

The main change is in the proof of the key estimate for $u_\varepsilon(t)$, which is obtained in [9] by using an inner variation $u_\varepsilon(\varphi_\delta(t))$ for a suitable function $\varphi_\delta\colon [0,\infty) \to [0,\infty)$. Since in our case we have to require that $u_\varepsilon(\varphi_\delta(t)) \in V_t$ for a.e. $t > 0$, this variation is admissible only if $\varphi_\delta(t) \le t$ for a.e. $t > 0$. By the technical definition of φ_δ, this leads to the constraint $\delta > 0$. Therefore the standard comparison between the functional on $u_\varepsilon(\varphi_\delta(t))$ and on the minimizer $u_\varepsilon(t)$, in the limit as $\delta \to 0+$, gives only an inequality, instead of the equality proven in [9, formula (4.7)]. This inequality, however, turns out to be enough to obtain the other estimates of [9] with minor changes.

A further difficulty appears when proving that the limit u of u_ε is a weak solution of (0.1), since also the test functions η must satisfy the constraint $\eta(t) \in V_t$ for a.e. $t > 0$. Therefore, to adapt the proof of [9], we have to approximate an arbitrary test function η satisfying the constraint $\eta(t) \in V_t$ for a.e. $t > 0$ by sums of functions of the form $\varphi(t)v$ with $v \in V_s$ and $\varphi \in C^2(\mathbb{R})$ with $\mathrm{supp}(\varphi) \subset [s,\infty)$, which still satisfy the constraint.

1. Description of the problem

1.1. **Setting.** To study the wave equation in time-dependent domains we adopt the functional setting introduced in [4]. Let H be a separable Hilbert space and let $(V_t)_{t\in[0,\infty)}$ be a family of separable Hilbert spaces with the following properties

- (H1) for every $t \in [0,\infty)$ the space V_t is contained and dense in H with continuous embedding;
- (H2) for every $s, t \in [0,\infty)$, with $s < t$, V_s is a closed subspace of V_t with the induced scalar product.

The scalar product in H is denoted by $(\cdot,\cdot)$ and the corresponding norm by $\|\cdot\|$. The norm in V_t is denoted by $\|\cdot\|_t$. By (H2) for every $0\le s<t$ we have $\|v\|_s=\|v\|_t$ for every $v\in V_s$.

The dual of H is identified with H, while for every $t\in[0,T]$ the dual of V_t is denoted by V_t^*. Note that the adjoint of the continuous embedding of V_t into H provides a continuous embedding of H into V_t^* and that H is dense in V_t^*. Let $\langle\cdot,\cdot\rangle_t$ be the duality product between V_t^* and V_t and let $\|\cdot\|_t^*$ be the corresponding dual norm. Note that $\langle\cdot,\cdot\rangle_t$ is the unique continuous bilinear map on $V_t^*\times V_t$ satisfying

$$\langle h,v\rangle_t=(h,v) \qquad \text{for every } h\in H \text{ and } v\in V_t\,. \tag{1.1}$$

Let $V_\infty:=\bigcup_{t\ge0}V_t$ and let $a\colon V_\infty\times V_\infty\to\mathbb{R}$ be a bilinear symmetric form satisfying the following conditions:

(H3) continuity: there exists $M_0>0$ such that

$$|a(u,v)|\le M_0\|u\|_t\|v\|_t \quad \text{for every } t\ge0 \text{ and every } u,v\in V_t\,; \tag{1.2}$$

(H4) coercivity: there exist $\lambda_0\ge0$ and $\nu_0>0$ such that

$$a(u,u)+\lambda_0\|u\|^2\ge\nu_0\|u\|_t^2 \quad \text{for every } t\ge0 \text{ and every } u\in V_t\,; \tag{1.3}$$

(H5) positive semidefiniteness:

$$a(u,u)\ge0 \quad \text{for every } u\in V_\infty\,. \tag{1.4}$$

For every $\tau,t\in[0,\infty)$ let $A_\tau^t:V_t\to V_\tau^*$ be the continuous linear operator defined by

$$\langle A_\tau^t u,v\rangle_\tau:=a(u,v) \quad \text{for every } u\in V_t \text{ and } v\in V_\tau\,. \tag{1.5}$$

Note that

$$\|A_\tau^t u\|_\tau^*\le M_0\|u\|_t \quad \text{for every } u\in V_t\,. \tag{1.6}$$

Finally, we set $Q(u):=a(u,u)$ for every $u\in V_\infty$.

Definition 1.1. Given $T>0$, we define $\mathcal{W}_T^{0,1}:=L^2((0,T);V_T)\cap H^1((0,T);H)$, with the Hilbert space structure induced by the scalar product

$$(u,v)_{\mathcal{W}_T^{0,1}}=(u,v)_{L^2((0,T);V_T)}+(u',v')_{L^2((0,T);H)}\,,$$

where u' and v' denote the distributional derivatives. The norm induced by the scalar product $(\cdot,\cdot)_{\mathcal{W}_T^{0,1}}$ is denoted by $\|\cdot\|_{\mathcal{W}_T^{0,1}}$. Moreover, we define

$$\mathcal{V}_T^{0,1}:=\{u\in\mathcal{W}_T^{0,1}\,:\,u(t)\in V_t \text{ for a.e. } t\in(0,T)\}\,,$$

and note that it is a closed subspace of $\mathcal{W}_T^{0,1}$.

Analogously, we define $\mathcal{W}_T^{0,2}:=L^2((0,T);V_T)\cap H^2((0,T);H)$, with the Hilbert space structure induced by the scalar product

$$(u,v)_{\mathcal{W}_T^{0,2}}=(u,v)_{L^2((0,T);V_T)}+(u',v')_{L^2((0,T);H)}+(u'',v'')_{L^2((0,T);H)}\,,$$

and the space

$$\mathcal{V}_T^{0,2}:=\{u\in\mathcal{W}_T^{0,2}\,:\,u(t)\in V_t \text{ for a.e. } t\in(0,T)\}\,,$$

which is a closed subspace of $\mathcal{W}_T^{0,2}$.

Finally, $\mathcal{V}^{0,1}$ (resp. $\mathcal{V}^{0,2}$) is defined as the space of functions $u\colon(0,+\infty)\to H$ whose restrictions to $(0,T)$ belong to $\mathcal{V}_T^{0,1}$ (resp. $\mathcal{V}_T^{0,2}$) for every $T>0$.

Remark 1.2. It is well known that every function $u \in H^1((0,T);H)$ (resp. $u \in H^2((0,T);H)$) admits a representative, still denoted by u, which belongs to the space $C^0([0,T];H)$ (resp. $C^1([0,T];H)$). With this convention we have $\mathcal{V}_T^{0,1} \subset C^0([0,T];H)$ (resp. $\mathcal{V}_T^{0,2} \subset C^1([0,T];H)$) for every $T > 0$.

Definition 1.3. We say that u is a weak solution of the equation

$$u''(t) + A_t^t u(t) = 0, \quad u(t) \in V_t \qquad \text{for } t \in [0,\infty) \tag{1.7}$$

if $u \in \mathcal{V}^{0,1}$ and for every $T > 0$

$$\int_0^T (u'(t), \psi'(t))\,\mathrm{d}t = \int_0^T a(u(t), \psi(t))\,\mathrm{d}t \tag{1.8}$$

for every $\psi \in \mathcal{V}_T^{0,1}$ with $\psi(0) = \psi(T) = 0$.

For every Banach space X let $C_w([0,T];X)$ be the space of functions $u\colon [0,T] \to X$ that are continuous for the weak topology of X.

Remark 1.4. If u is a weak soltution of (1.7) with $u \in L^\infty((0,T);V_T)$ and $u' \in L^\infty((0,T);H)$ for every $T > 0$, then [4, Theorem 2.17 and Proposition 2.18] imply that, after a modification on a set of measure zero, $u \in C_w([0,T];V_T)$ and $u' \in C_w([0,T];H)$ for every $T > 0$.

1.2. **Main results.** Throughout the paper we fix $u^0 \in V_0$, $u^1 \in H$, and a sequence $\{u^1_\varepsilon\} \subset V_0$ such that

$$\|u^1_\varepsilon - u^1\|_H \to 0 \text{ as } \varepsilon \to 0+ \quad \text{and} \quad \varepsilon \|u^1_\varepsilon\|_0 \le C_1\,, \tag{1.9}$$

for some constant $C_1 < \infty$. For every $\varepsilon > 0$ we consider the functional

$$\mathcal{F}_\varepsilon(u) := \frac{1}{2}\int_0^\infty e^{-t/\varepsilon}\Big(\varepsilon^2 \|u''(t)\|^2 + Q(u(t))\Big)\,\mathrm{d}t\,, \tag{1.10}$$

defined on the set

$$\mathcal{V}^{0,2}(u^0, u^1_\varepsilon) := \{u \in \mathcal{V}^{0,2} \ : u(0) = u^0,\ u'(0) = u^1_\varepsilon\}\,, \tag{1.11}$$

which is well-defined in view of Remark 1.2.

We now state our main results, which are proven in Sections 2, 3, and 4.

Theorem 1.5. *For every $\varepsilon \in (0,1)$ the functional $\mathcal{F}_\varepsilon$ admits a unique global minimizer u_ε in the set $\mathcal{V}^{0,2}(u^0, u^1_\varepsilon)$. Moreover,*

$$\mathcal{F}_\varepsilon(u_\varepsilon) \le \bar{C}\varepsilon, \tag{1.12}$$

for some constant $\bar{C} < \infty$ depending only on $\|u^0\|_0$ and C_1.

In particular, if $\varepsilon\|u^1_\varepsilon\|_0 \to 0$ as $\varepsilon \to 0+$, then

$$\mathcal{F}_\varepsilon(u_\varepsilon) \le \varepsilon\Big(\frac{1}{2}Q(u^0) + r_\varepsilon\Big)\,, \tag{1.13}$$

where $r_\varepsilon \to 0$ as $\varepsilon \to 0+$.

Theorem 1.6. *There exists a constant $C < \infty$ such that for every $\varepsilon \in (0,1)$ the minimizer u_ε of $\mathcal{F}_\varepsilon$ in $\mathcal{V}^{0,2}(u^0, u^1_\varepsilon)$ satisfies the following estimates:*

$$\int_t^{t+\tau} Q(u_\varepsilon(s))\,\mathrm{d}s \le C\,\tau \qquad \text{for every } t \ge 0\,,\ \tau \ge \varepsilon\,, \tag{1.14}$$

$$\|u_\varepsilon(t)\|^2 \le C(1 + t^2) \qquad \text{for every } t \ge 0\,, \tag{1.15}$$

$$\|u'_\varepsilon(t)\| \le C \qquad \text{for every } t \ge 0\,. \tag{1.16}$$

Theorem 1.7. *For every $\varepsilon \in (0,1)$ let u_ε be the minimizer of $\mathcal{F}_\varepsilon$ in $\mathcal{V}^{0,2}(u^0,u^1_\varepsilon)$. Then for every sequence $\{\varepsilon_n\} \subset (0,1)$, with $\varepsilon_n \to 0$ as $n\to\infty$, there exist a subsequence, not relabeled, and a weak solution u of (1.7) such that $u_{\varepsilon_n} \rightharpoonup u$ weakly in $\mathcal{W}^{0,1}_T$ for every $T>0$. Moreover the following properties hold:*

(a) *weak continuity: $u\in C_w([0,T];V_T)$ and $u'\in C_w([0,T];H)$ for every $T>0$;*
(b) *initial conditions: $u(0)=u^0$ and $u'(0)=u^1$.*

If, in addition, $\varepsilon\|u^1_\varepsilon\|_0 \to 0$ as $\varepsilon\to 0+$, then the following energy inequality holds:

$$\|u'(t)\|^2 + Q(u(t)) \le \|u^1\|^2 + Q(u^0) \quad \text{for every } t>0\,. \tag{1.17}$$

2. Proof of Theorem 1.5

Before proving our results we introduce a change of variables that will be useful throughout the paper.

Remark 2.1. For every $\varepsilon>0$ and every $T>0$ we set

$$\mathcal{W}^{0,2}_{\varepsilon,T} := L^2((0,T);V_{\varepsilon T}) \cap H^2((0,T);H)\,,$$
$$\mathcal{V}^{0,2}_{\varepsilon,T} := \{v\in \mathcal{W}^{0,2}_{\varepsilon,T} \,:\, v(t)\in V_{\varepsilon t} \text{ for a.e. } t\in(0,T)\}\,.$$

Note that $\mathcal{W}^{0,2}_{\varepsilon,T}$ is a Hilbert space with the scalar product

$$(u,v)_{\mathcal{W}^{0,2}_{\varepsilon,T}} = (u,v)_{L^2((0,T);V_{\varepsilon T})} + (u',v')_{L^2((0,T);H)} + (u'',v'')_{L^2((0,T);H)}\,,$$

and $\mathcal{V}^{0,2}_{\varepsilon,T}$ is a closed subspace of $\mathcal{W}^{0,2}_{\varepsilon,T}$. Furthermore, $\mathcal{V}^{0,2}_{\varepsilon}$ denotes the space of functions $u\colon[0,\infty)\to H$ whose restrictions to the interval $(0,T)$ belong to $\mathcal{V}^{0,2}_{\varepsilon,T}$ for every $T>0$. By Remark 1.2 every $u\in\mathcal{W}^{0,2}_{\varepsilon,T}$ admits a representative, still denoted by u, which belongs to $C^1([0,T];H)$. With this convention we have $\mathcal{V}^{0,2}_{\varepsilon,T}\subset C^1([0,T];H)$ for every $T>0$. Finally, we define

$$\mathcal{V}^{0,2}_{\varepsilon}(u^0,\varepsilon u^1_\varepsilon) := \{v\in\mathcal{V}^{0,2}_{\varepsilon} \,:\, v(0)=0, v'(0)=\varepsilon u^1_\varepsilon\}\,.$$

It is easy to see that if $u\in\mathcal{V}^{0,2}(u^0,u^1_\varepsilon)$, then the function v defined by

$$v(t) := u(\varepsilon t) \tag{2.1}$$

belongs to $\mathcal{V}^{0,2}_{\varepsilon}(u^0,\varepsilon u^1_\varepsilon)$ and

$$\mathcal{F}_\varepsilon(u) = \varepsilon\mathcal{G}_\varepsilon(v)\,, \tag{2.2}$$

where

$$\mathcal{G}_\varepsilon(v) := \frac{1}{2}\int_0^\infty e^{-t}\Big(\frac{\|v''(t)\|^2}{\varepsilon^2} + Q(v(t))\Big)\,\mathrm{d}t\,.$$

In view of Remark 2.1, Theorem 1.5 is a consequence of the following result for the functional $\mathcal{G}_\varepsilon$.

Theorem 2.2. *For every $\varepsilon\in(0,1)$ the functional $\mathcal{G}_\varepsilon$ admits a unique global minimizer v_ε in $\mathcal{V}^{0,2}_{\varepsilon}(u^0,\varepsilon u^1_\varepsilon)$. Moreover,*

$$\mathcal{G}_\varepsilon(v_\varepsilon) \le \bar{C}, \tag{2.3}$$

for some constant $\bar{C}<\infty$ depending only on $\|u^0\|_0$ and C_1 .

Furthermore $u_\varepsilon(t) := v_\varepsilon(\frac{t}{\varepsilon})$ is the unique global minimizer of $\mathcal{F}_\varepsilon$ in $\mathcal{V}^{0,2}(u^0,u^1_\varepsilon)$ and satisfies (1.12) .

Finally, if $\varepsilon\|u_1^\varepsilon\|_0 \to 0$ *as* $\varepsilon \to 0+$ *, then*

$$\mathcal{G}_\varepsilon(v_\varepsilon) \le \frac{1}{2}Q(u^0) + r_\varepsilon\,, \tag{2.4}$$

where $r_\varepsilon \to 0$ *as* $\varepsilon \to 0$ *and* u_ε *satisfies* (1.13) *.*

Proof. Fix $\varepsilon > 0$ and set $v(t) := u^0 + \varepsilon t u^1_\varepsilon$ for every $t \ge 0$. Note that $v \in \mathcal{V}^{0,2}_\varepsilon(u^0, \varepsilon u^1_\varepsilon)$, since $u^0,\, u^1_\varepsilon \in V_0 \subset V_t$ for every $t \ge 0$. By (H3) and by (1.9), we have

$$\mathcal{G}_\varepsilon(v) = \frac{1}{2}\int_0^\infty e^{-t}Q(v(t))\,\mathrm{d}t \le \frac{1}{2}Q(u^0) + M_0\,\varepsilon\|u^1_\varepsilon\|_0(\varepsilon\|u^1_\varepsilon\|_0 + \|u^0\|_0) \le \bar{C}\,, \tag{2.5}$$

where $\bar{C}$ is a constant depending only on C_1 and $\|u_0\|_0$. Note that, if $\varepsilon\|u^1_\varepsilon\|_0 \to 0$ as $\varepsilon \to 0+$, then by (2.3) it follows that

$$\mathcal{G}_\varepsilon(v) \le \frac{1}{2}Q(u^0) + r_\varepsilon\,, \tag{2.6}$$

where $r_\varepsilon \to 0$ as $\varepsilon \to 0$.

In particular, $\mathcal{G}_\varepsilon$ has a finite infimum and (2.3) (as well as (2.4)) follows as soon as $\mathcal{G}_\varepsilon$ has an absolute minimizer v_ε. To show this, consider a minimizing sequence $\{v_{\varepsilon,n}\} \subset \mathcal{V}^{0,2}_\varepsilon(u^0, \varepsilon u^1_\varepsilon)$ and fix $T > 0$. By the very definition of $\mathcal{G}_\varepsilon$ and by (2.5),

$$\int_0^T \|v''_{\varepsilon,n}(t)\|^2\,\mathrm{d}t \le e^T \int_0^T e^{-t}\|v''_{\varepsilon,n}(t)\|^2\,\mathrm{d}t \le 2\varepsilon^2 e^T \mathcal{G}_\varepsilon(v_{\varepsilon,n}) \le \varepsilon^2 C_T\,, \tag{2.7}$$

for some constant $C_T < \infty$. The bound (2.7), together with the boundary conditions

$$v_{\varepsilon,n}(0) = u^0 \quad \text{and} \quad v'_{\varepsilon,n}(0) = \varepsilon u^1_\varepsilon\,, \tag{2.8}$$

implies

$$\|v_{\varepsilon,n}\|_{H^2((0,T);H)} \le C_{T,\varepsilon} \tag{2.9}$$

for some constant $C_{T,\varepsilon} < \infty$ independent of n. Moreover, by (H2) and (H4), for $t \in [0,T]$ we have

$$\nu_0\|v_{\varepsilon,n}(t)\|^2_T = \nu_0\|v_{\varepsilon,n}(t)\|^2_t \le \lambda_0\|v_{\varepsilon,n}(t)\|^2 + Q(v_{\varepsilon,n}(t))$$

from which, using (2.5) and (2.9), we get

$$\nu_0\|v_{\varepsilon,n}\|^2_{L^2((0,T);V_T)} \le \lambda_0\|v_{\varepsilon,n}\|^2_{L^2((0,T);H)} + \int_0^T Q(v_{\varepsilon,n}(t))\,\mathrm{d}t \le \widehat{C}_{T,\varepsilon}$$

for some constant $\widehat{C}_{T,\varepsilon} < \infty$ independent of n. It follows that $\|v_{\varepsilon,n}\|_{\mathcal{W}^{0,2}_{\varepsilon,T}}$ is uniformly bounded and hence, up to a subsequence, $v_{\varepsilon,n} \rightharpoonup v_\varepsilon$ in $\mathcal{W}^{0,2}_{\varepsilon,T}$ as $n \to \infty$, for some $v_\varepsilon \in \mathcal{W}^{0,2}_{\varepsilon,T}$. Moreover, since $\mathcal{V}^{0,2}_{\varepsilon,T}$ is closed, $v_\varepsilon \in \mathcal{V}^{0,2}_{\varepsilon,T}$. By the arbitrariness of T we have $v_\varepsilon \in \mathcal{V}^{0,2}_\varepsilon$ and by (2.8) we get $v_\varepsilon \in \mathcal{V}^{0,2}_\varepsilon(u^0, \varepsilon u^1_\varepsilon)$. Finally, since $\mathcal{G}_\varepsilon$ is lower semi-continuous and strictly convex by (H5), v_ε is the unique minimizer of $\mathcal{G}_\varepsilon$ in $\mathcal{V}^{0,2}_\varepsilon(u^0, \varepsilon u^1_\varepsilon)$. The statements about $u_\varepsilon(t)$ follow from Remark 2.1. □

3. Proof of Theorem 1.6

We first introduce some notations. Let v_ε be the minimizer of $\mathcal{G}_\varepsilon$ in $\mathcal{V}^{0,2}_\varepsilon(u^0,\varepsilon u^1_\varepsilon)$ and let L_ε be the corresponding Lagrangian defined as

$$L_\varepsilon(t) := D_\varepsilon(t) + Q_\varepsilon(t)\,, \tag{3.1}$$

where

$$D_\varepsilon(t) := \frac{\|v''_\varepsilon(t)\|^2}{2\varepsilon^2} \quad \text{and} \quad Q_\varepsilon(t) := \frac{Q(v_\varepsilon(t))}{2}\,. \tag{3.2}$$

Moreover, we define the kinetic energy function K_ε as

$$K_\varepsilon(t) := \frac{\|v'_\varepsilon(t)\|^2}{2\varepsilon^2}\,. \tag{3.3}$$

We shall use the following result, which can be proven as in [9, Lemma 3.4].

Lemma 3.1. *There exists a constant $C<\infty$ (depending only on $\|u^0\|_0$, $\|u^1\|$, and C_1 in (1.9)) such that for every $\varepsilon\in(0,1)$ the minimizer v_ε of $\mathcal{G}_\varepsilon$ in $\mathcal{V}^{0,2}_\varepsilon(u^0,\varepsilon u^1_\varepsilon)$ satisfies*

$$\int_0^\infty e^{-t} D_\varepsilon(t)\,\mathrm{d}t = \int_0^\infty e^{-t}\frac{\|v''_\varepsilon(t)\|^2}{2\varepsilon^2}\,\mathrm{d}t \le C\,, \tag{3.4}$$

$$\int_0^\infty e^{-t} K_\varepsilon(t)\,\mathrm{d}t = \int_0^\infty e^{-t}\frac{\|v'_\varepsilon(t)\|^2}{2\varepsilon^2}\,\mathrm{d}t \le C\,. \tag{3.5}$$

In particular, in view of Lemma 3.1, we have $K_\varepsilon \in W^{1,1}(0,T)$ for all $T>0$ and

$$K'_\varepsilon(t) = \frac{1}{\varepsilon^2}(v'_\varepsilon(t), v''_\varepsilon(t)) \qquad \text{for a.e. } t>0\,. \tag{3.6}$$

Following the approach in [9], we introduce the *average operator* $\mathcal{A}$, defined by

$$(\mathcal{A}f)(s) := \int_s^\infty e^{-(t-s)} f(t)\,\mathrm{d}t\,, \qquad s\ge 0\,.$$

for every measurable function $f\colon [0,\infty)\to[0,\infty]$.

We note that $\mathcal{A}f$ is well defined (possibly ∞) since $f\ge 0$. Moreover, the equality

$$\mathcal{A}f(0) = \int_0^\infty e^{-t} f(t)\,\mathrm{d}t\,, \tag{3.7}$$

implies that, if $\mathcal{A}f(0)<\infty$, then $\mathcal{A}f$ is absolutely continuous on all intervals $[0,T]$ and

$$(\mathcal{A}f)' = \mathcal{A}f - f \quad \text{a.e. in } [0,\infty)\,. \tag{3.8}$$

In any case, since $\mathcal{A}f\ge 0$, starting from $f\ge 0$ one can iterate $\mathcal{A}$, and a simple computation gives

$$(\mathcal{A}^2 f)(s) = \int_s^\infty e^{-(t-s)}(t-s) f(t)\,\mathrm{d}t\,, \tag{3.9}$$

thus in particular

$$(\mathcal{A}^2 f)(0) = \int_0^\infty e^{-t} t f(t)\,\mathrm{d}t\,. \tag{3.10}$$

Finally, we define the approximate energy

$$E_\varepsilon(t) := K_\varepsilon(t) + (\mathcal{A}^2 Q_\varepsilon)(t)\,. \tag{3.11}$$

The key ingredient in order to prove Theorem 1.6 is given by the following proposition.

Proposition 3.2. *The function E_ε is uniformly bounded and monotonically nonincreasing. More precisely, there exists $C_1' < \infty$, depending only on $\|u^0\|_0$, $\|u^1\|$, and C_1 in* (1.9)*, such that*

$$E_\varepsilon(t) \leq C_1' \quad \text{for every } t \geq 0 . \tag{3.12}$$

Moreover, if $\varepsilon\|u^1_\varepsilon\|_0 \to 0$ as $\varepsilon \to 0+$, then

$$E_\varepsilon(t) \leq \frac{1}{2}\|u^1_\varepsilon\|^2 + \frac{1}{2}Q(u^0) + \widetilde{r}_\varepsilon , \tag{3.13}$$

where $\widetilde{r}_\varepsilon \to 0$ as $\varepsilon \to 0+$.

Proof. The proof of Proposition 3.2 closely follows the strategy adopted in [9] to prove [9, Theorem 4.8]. We briefly sketch the main steps, underlining the main differences with respect to the case treated in [9]. The proof is divided into four steps.

Step 1. For every $g \in C^{1,1}(\mathbb{R};[0,\infty))$, with $g(0) = 0$ and $g(t)$ affine for t sufficiently large, there exists a constant $C_1(g) < \infty$, depending on g, $\|u^0\|_0$, and C_1 in (1.9)*, such that*

$$\int_0^\infty e^{-s}(g'(s) - g(s))L_\varepsilon(s)\,\mathrm{d}s - \int_0^\infty e^{-s}(4D_\varepsilon(s)g'(s) + K_\varepsilon'(s)g''(s))\,\mathrm{d}s + R_\varepsilon \geq 0 , \tag{3.14}$$

where

$$R_\varepsilon := \varepsilon g'(0)\int_0^\infty e^{-s}s\,a(v_\varepsilon(s),u^1_\varepsilon)\,\mathrm{d}s$$

satisfies

$$|R_\varepsilon| < C_1(g) . \tag{3.15}$$

In particular, if $\varepsilon\|u^1_\varepsilon\|_0 \to 0$ as $\varepsilon \to 0+$, then

$$|R_\varepsilon| \to 0 \qquad \text{as } \varepsilon \to 0+ . \tag{3.16}$$

Using the approximation argument in [9, Corollary 4.5], it is enough to prove (3.14) for $g \in C^2(\mathbb{R};[0,\infty))$ with $g(0) = 0$ and $g(t)$ constant for t large enough.

For $\delta \geq 0$ small enough, the function $\varphi_\delta(t) := t - \delta g(t)$ is a C^2-diffeomorphism of $[0,\infty)$ into itself. We consider the function $v_{\varepsilon,\delta}(t) := v_\varepsilon(\varphi_\delta(t)) + t\delta\varepsilon g'(0)u^1_\varepsilon$. By construction $\varphi_\delta(t) \leq t$ so that, in view of (H2), $v_{\varepsilon,\delta} \in \mathcal{V}^{0,2}_\varepsilon$. Note that in the proof of this property the condition $\delta \geq 0$ is crucial. Moreover, $v_{\varepsilon,\delta}(0) = v_\varepsilon(0) = u^0$ and

$$v'_{\varepsilon,\delta}(t)_{|t=0} = v'_\varepsilon(0)(1 - \delta g'(0)) + \delta\varepsilon g'(0)u^1_\varepsilon = \varepsilon u^1_\varepsilon ,$$

whence $v_{\varepsilon,\delta} \in \mathcal{V}^{0,2}_\varepsilon(u^0, \varepsilon u^1_\varepsilon)$.

Set $\psi_\delta(s) := \varphi_\delta^{-1}(s)$ for every $s \geq 0$. By the change of variables $t = \psi_\delta(s)$, it is straightforward to check that

$$\begin{aligned}\mathcal{G}_\varepsilon(v_{\varepsilon,\delta}) =& \frac{1}{2\varepsilon^2}\int_0^\infty \psi_\delta'(s)e^{-\psi_\delta(s)}\|v''_\varepsilon(s)|\varphi'_\delta(\psi_\delta(s))|^2 + v'_\varepsilon(s)\varphi''_\delta(\psi_\delta(s))\|^2\,\mathrm{d}s\\ &+ \frac{1}{2}\int_0^\infty \psi_\delta'(s)e^{-\psi_\delta(s)}Q(v_\varepsilon(s) + \delta\varepsilon g'(0)\psi_\delta(s)u^1_\varepsilon)\,\mathrm{d}s .\end{aligned} \tag{3.17}$$

Notice that

$$s = \varphi_\delta(\psi_\delta(s)) = \psi_\delta(s) - \delta g(\psi_\delta(s)) \tag{3.18}$$

so that, in view of the assumptions on g, we have $e^{-\psi_\delta(s)} \leq e^{\delta\|g\|_{L^\infty}} e^{-s}$. Moreover, since

$$\psi'_\delta(s) = 1 + \delta g'(\psi_\delta(s))\psi'_\delta(s) \quad \text{ and } \quad \psi''_\delta(s) = \delta(g''(\psi_\delta(s))(\psi'_\delta(s))^2 + g'(\psi_\delta(s))\psi''_\delta(s)),$$

for δ sufficiently small both $\psi'_\delta(s)$ and $\psi''_\delta(s)$ are bounded uniformly with respect to s. This fact, together with Lemma 3.1, implies that the first integral in (3.17) is finite. As for the second integral we have

$$\frac{1}{2}\int_0^\infty \psi'_\delta(s) e^{-\psi_\delta(s)} Q(v_\varepsilon(s) + \delta\varepsilon g'(0)\psi_\delta(s)u^1_\varepsilon)\,\mathrm{d}s \leq \frac{1}{2}\|\psi'_\delta\|_{L^\infty} e^{\delta\|g\|_{L^\infty}}(A_1 + A_2 + A_3)\,, \tag{3.19}$$

where

$$\begin{aligned} A_1 &:= \int_0^\infty e^{-s} Q(v_\varepsilon(s))\,\mathrm{d}s\,,\\ A_2 &:= \delta^2 (g'(0))^2 \varepsilon^2 Q(u^1_\varepsilon)\int_0^\infty e^{-s}(\psi_\delta(s))^2\,\mathrm{d}s\,,\\ A_3 &:= 2\delta\varepsilon g'(0)\int_0^\infty e^{-s}\psi_\delta(s) a(v_\varepsilon(s), u^1_\varepsilon)\,\mathrm{d}s\,. \end{aligned}$$

Now, $A_1 < \infty$ by (2.3) and $A_2 < +\infty$ in view of (3.18). Finally, by (H5) and the Cauchy inequality, we have $A_3 \leq A_1 + A_2 < \infty$. It follows $\mathcal{G}_\varepsilon(v_{\varepsilon,\delta}) < \infty$ for δ sufficiently small. Analogously, one can show that differentiation under the integral sign in (3.17) is possible.

Since $v_{\varepsilon,0} = v_\varepsilon$ and $v_{\varepsilon,\delta} \in \mathcal{V}^{0,2}_\varepsilon(u^0, \varepsilon u^1_\varepsilon)$ only for $\delta \geq 0$, the minimality of v_ε implies

$$\frac{\mathrm{d}}{\mathrm{d}\delta}\mathcal{G}_\varepsilon(v_{\varepsilon,\delta})\Big|_{\delta=0} \geq 0\,,$$

while in [9] the equality holds. One can compute this derivative as in [9, pages 2031-2032] and one can check that it coincides with the left-hand side of (3.14).

As for R_ε, by assumptions (H3) and (H5) and by (1.9) and (2.2), we have

$$\begin{aligned} |R_\varepsilon| &= \varepsilon|g'(0)|\int_0^\infty e^{-s}s\,|a(v_\varepsilon(s), u^1_\varepsilon)|\,\mathrm{d}s\\ &\leq \varepsilon|g'(0)|\Big(\int_0^\infty e^{-s}Q(v_\varepsilon(s))\,\mathrm{d}s + M_0\|u^1_\varepsilon\|_0\int_0^\infty e^{-s}s^2\,\mathrm{d}s\Big)\\ &\leq |g'(0)|(2\varepsilon\mathcal{G}_\varepsilon(v_\varepsilon) + 2M_0\varepsilon\|u^1_\varepsilon\|_0) \leq 2g'(0)(\varepsilon\bar{C} + C_1) =: C_1(g)\,, \end{aligned} \tag{3.20}$$

thus proving (3.15). By the last but one inequality in (3.20) and by (2.2), it follows that, if $\varepsilon\|u^1_\varepsilon\|_0 \to 0$ as $\varepsilon \to 0+$, then $R_\varepsilon \to 0$ as $\varepsilon \to 0+$.

Step 2. We have $(\mathcal{A}^2 L_\varepsilon)(0) \leq (\mathcal{A}L_\varepsilon)(0) - 4(\mathcal{A}D_\varepsilon)(0) + R_\varepsilon$.

The claim follows by applying (3.14) with $g(t) = t$.

Step 3. We have $K'_\varepsilon(t) \leq (\mathcal{A}L_\varepsilon)(t) - (\mathcal{A}^2 L_\varepsilon)(t) - 4(\mathcal{A}D_\varepsilon)(t)$ *for almost every* $t > 0$.

The proof closely resembles the one of [9, Corollary 4.7]. Fix $t > 0$ and for every $\delta > 0$ let $g_{t,\delta}$ be defined by

$$g_{t,\delta}(s) := \begin{cases} 0 & \text{if } s \leq t\,,\\ \frac{(s-t)^2}{2\delta} & \text{if } s \in [t, t+\delta]\,,\\ s - t - \frac{\delta}{2} & \text{if } s \geq t+\delta\,. \end{cases} \tag{3.21}$$

The claim follows by considering $g = g_{t,\delta}$ in (3.14) and sending $\delta \to 0$.

Step 4. Inequality (3.12) *holds true.*

In view of Step 2 and (3.6), $\mathcal{A}^2 Q_\varepsilon$ and K_ε are absolutely continuous on the intervals $[0,T]$ for every $T>0$. Therefore, we can differentiate E_ε and, using Step 3, (3.8), and the very definition of L_ε in (3.1), we get

$$\begin{aligned} E'_\varepsilon &= K'_\varepsilon + (\mathcal{A}^2 Q_\varepsilon)' = K'_\varepsilon + \mathcal{A}^2 Q_\varepsilon - \mathcal{A} Q_\varepsilon \\ &\le \mathcal{A} L_\varepsilon - \mathcal{A}^2 L_\varepsilon - 4\mathcal{A} D_\varepsilon + \mathcal{A}^2 Q_\varepsilon - \mathcal{A} Q_\varepsilon = -\mathcal{A}^2 D_\varepsilon - 3\mathcal{A} D_\varepsilon \le 0\,, \end{aligned}$$

and hence $E_\varepsilon(t) \le E_\varepsilon(0)$ for a.e. $t \ge 0$. Moreover, by the very definition of E_ε and L_ε, together with (2.3), Step 2, and (3.15), it follows that

$$\begin{aligned} E_\varepsilon(0) &= K_\varepsilon(0) + (\mathcal{A}^2 Q_\varepsilon)(0) = \frac{1}{2}\|u^1_\varepsilon\|^2 + (\mathcal{A}^2 Q_\varepsilon)(0) \\ &\le \frac{1}{2}\|u^1_\varepsilon\|^2 + (\mathcal{A}^2 L_\varepsilon)(0) \le \frac{1}{2}\|u^1_\varepsilon\|^2 + (\mathcal{A} L_\varepsilon)(0) + R_\varepsilon \\ &= \frac{1}{2}\|u^1_\varepsilon\|^2 + \mathcal{G}_\varepsilon(v_\varepsilon) + R_\varepsilon < C'_1\,, \end{aligned} \tag{3.22}$$

where C'_1 depends on $\|u^0\|_0$, $\|u^1\|$, and C_1 in (1.9). This concludes the proof of (3.12). Finally, by using (3.16) and (2.4) in the last line in (3.22), we obtain that, if $\varepsilon\|u^1_\varepsilon\|_0 \to 0$ as $\varepsilon \to 0+$, then

$$E_\varepsilon(0) \le \frac{1}{2}\|u^1_\varepsilon\|^2 + \frac{1}{2}Q(u^0) + r_\varepsilon + R_\varepsilon \le \frac{1}{2}\|u^1_\varepsilon\|^2 + \frac{1}{2}Q(u^0) + \widetilde{r}_\varepsilon\,,$$

where $\widetilde{r}_\varepsilon \to 0$ as $\varepsilon \to 0+$. Therefore also (3.13) holds true. □

Proof of Theorem 1.6. By using Proposition 3.2, Theorem 1.6 can be proven as in [9, Section 5]. □

4. Proof of Theorem 1.7

Before proving Theorem 1.7, we introduce a suitable subset of $\mathcal{V}^{0,2}_{\varepsilon,T}$, which is dense in $\{\eta \in C^2_c((0,T);V_T) : \eta(t) \in V_t \text{ for every } t \in (0,T)\}$. For every $\varepsilon > 0$ and $T > 0$, we define $\mathcal{D}_T$ as the set of all functions $\eta \in C^2_c((0,T);V_T)$ of the form

$$\eta(t) = \sum_{i=2}^{N-2}\sum_{j=0}^{2} \varphi_{i,j}(t) h_{i,j}$$

for some $N \in \mathbb{N}$, $0 = t_0 < t_1 < \ldots < t_N = T$, $\varphi_{i,j} \in C^2(\mathbb{R})$ with supp $\varphi_{i,j} \subset [t_{i-1}, t_{i+1}]$, and $h_{i,j} \in V_{t_{i-1}}$ for $i = 2, \ldots, N-2$ and $j = 0,1,2$. By (H2) the last two conditions imply that $\eta(t) \in V_t$ for every $t \in [0,T]$. We are now in a position to state and prove our density result.

Lemma 4.1. *Let $T > 0$. For every $\eta \in C^2_c((0,T);V_T)$, with $\eta(t) \in V_t$ for every $t \in (0,T)$, there exists a sequence $\{\eta_N\} \subset \mathcal{D}_T$ such that*

$$\|\eta - \eta_N\|_{C^2([0,T];V_T)} \to 0 \qquad \text{as } N \to \infty\,. \tag{4.1}$$

Proof. Let $\eta \in C^2_c((0,T);V_T)$, with $\eta(t) \in V_t$ for every $t \in (0,T)$. In order to construct the approximating sequence $\{\eta_N\} \subset \mathcal{D}_T$ we make use of quintic Hermite interpolants, that we construct here through the Bernstein polynomials. Let $N \in \mathbb{N}$ and set $t_i = i\frac{T}{N}$ for

$i = 0, 1, \dots, N$. Fix $i = 0, \dots, N$. For $n \in \mathbb{N}$, we define the Bernstein polynomials in the interval $[t_i, t_{i+1}]$ as

$$B^i_{k,n}(t) := \begin{cases} \binom{n}{k}(t-t_i)^k(t_{i+1}-t)^{n-k} & \text{for } k = 0, \dots, n\,, \\ 0 & \text{for } k < 0 \text{ or } k > n\,, \end{cases}$$

and we define the polynomials of the spline basis as follows

$$\psi_{i,0,+}(t) := \frac{N^5}{T^5}(B^i_{0,5}(t) + B^i_{1,5}(t) + B^i_{2,5}(t))\,, \qquad \psi_{i,0,-}(t) := \frac{N^5}{T^5}(B^i_{3,5}(t) + B^i_{4,5}(t) + B^i_{5,5}(t))\,,$$

$$\psi_{i,1,+}(t) := \frac{N^4}{5T^4}(B^i_{1,5}(t) + 2B^i_{2,5}(t))\,, \qquad \psi_{i,1,-}(t) := -\frac{N^4}{5T^4}(2B^i_{3,5}(t) + B^i_{4,5}(t))\,,$$

$$\psi_{i,2,+}(t) := \frac{N^3}{20T^3}B^i_{2,5}(t)\,, \qquad \psi_{i,2,-}(t) := \frac{N^3}{20T^3}B^i_{3,5}(t)\,.$$

By construction, it is easy to see that

$$\psi_{i,0,+}(t) + \psi_{i,0,-}(t) = 1 \qquad \text{for } t \in [t_i, t_{i+1}]\,. \tag{4.2}$$

Moreover, by using that

$$\frac{\mathrm{d}}{\mathrm{d}t}B^i_{k,n}(t) = n(B^i_{k-1,n-1}(t) - B^i_{k,n-1}(t))\,,$$

one can easily show that

$$-\frac{T}{N}\psi'_{i,0,+}(t) + \psi'_{i,1,+}(t) + \psi'_{i,1,-}(t) = 1\,, \tag{4.3}$$

$$-\frac{T^2}{2N^2}\psi''_{i,0,+}(t) + \frac{T}{N}\psi''_{i,1,-}(t) + \psi''_{i,2,+}(t) + \psi''_{i,2,-}(t) = 1\,. \tag{4.4}$$

For every $i = 1, \dots, N-1$ and $j = 0, 1, 2$ we set

$$\varphi_{i,j}(t) := \begin{cases} \psi_{i-1,j,-}(t) & \text{if } t \in [t_{i-1}, t_i]\,, \\ \psi_{i,j,+}(t) & \text{if } t \in [t_i, t_{i+1}]\,, \\ 0 & \text{elsewhere}\,. \end{cases}$$

Finally, we define the function

$$\eta_N(t) := \sum_{i=2}^{N-2} \big(\varphi_{i,0}(t)\eta(t_{i-1}) + \varphi_{i,1}(t)\eta'(t_{i-1}) + \varphi_{i,2}(t)\eta''(t_{i-1})\big)\,.$$

By (H2) we have $\eta(t_{i-1})$, $\eta'(t_{i-1})$, $\eta''(t_{i-1}) \in V_{t_{i-1}}$, hence $\eta_N \in \mathcal{D}_T$ for every $N \in \mathbb{N}$.

It remains to prove (4.1). Let $t \in \operatorname{supp} \eta$. For $N \in \mathbb{N}$ large enough there exists $i = 2, \dots, N-3$ such that $t \in [t_i, t_{i+1})$, so that by (4.2) and by the very definition of η_N, $\psi_{i,1,\pm}$, and $\psi_{i,2,\pm}$, we have

$$\begin{aligned} \|\eta_N(t) - \eta(t)\|_T &\le \|\psi_{i,0,+}(t)\eta(t_{i-1}) + \psi_{i,0,-}(t)\eta(t_i) - \eta(t)\|_T + \mathrm{O}(1/N) \\ &\le \|\eta(t_{i-1}) - \eta(t)\|_T + \|\eta(t_i) - \eta(t)\|_T + \mathrm{O}(1/N)\,, \end{aligned}$$

and hence η_N converges to η in V_T uniformly in $[0,T]$. Analogously, by (4.3), we obtain

$$\begin{aligned} \|\eta'_N(t) - \eta'(t)\|_T \le{}& \Big\|\psi'_{i,0,+}(t)\eta(t_{i-1}) + \psi'_{i,0,-}(t)\eta(t_i) + \frac{T}{N}\psi'_{i,0,+}(t)\eta'(t)\Big\|_T \\ &+ \|\psi'_{i,1,+}\|_{L^\infty}\|\eta'(t_{i-1}) - \eta'(t)\|_T + \|\psi'_{i,1,-}\|_{L^\infty}\|\eta'(t_i) - \eta'(t)\|_T + \mathrm{O}(1/N)\,, \end{aligned}$$

which, using that (by (4.2)) the first term on the right-hand side is bounded by

$$\frac{T}{N}\|\psi'_{i,0,+}(t)\|_{L^\infty}\Big\| - \frac{\eta(t_i)-\eta(t_{i-1})}{T/N} + \eta'(t)\Big\|_T\,,$$

implies that η'_N converges to η' in V_T uniformly in $[0,T]$. Analogously, using (4.2), (4.3), and (4.4), one can show that η''_N converges uniformly to η'' in $[0,T]$. □

Lemma 4.2. *Let $\varepsilon > 0$ and $T > 0$. For every $\eta \in C^2_c((0,T);V_T)$, with $\eta(t) \in V_t$ for every $t \in (0,T)$, we have*

$$\int_0^T e^{-s/\varepsilon}\Big(\varepsilon^2\left(u''_\varepsilon(s),\eta''(s)\right) + a(u_\varepsilon(s),\eta(s))\Big)\,\mathrm{d}s = 0\,. \tag{4.5}$$

Proof. In view of Lemma 4.1, it is sufficient to prove (4.5) for $\eta \in \mathcal{D}_T$. The proof is analogous to the one of [9, Lemma 5.1]. Let $\delta \in [-1,1]$ and set $u_{\varepsilon,\delta} := u_\varepsilon + \delta\eta$. By construction, $u_{\varepsilon,\delta} \in \mathcal{V}^{0,2}_T$ and, since η has compact support, also the initial conditions are satisfied. Therefore $u_{\varepsilon,\delta} \in \mathcal{V}^{0,2}(u^0,u^1_\varepsilon)$, and, again by construction, $\mathcal{F}_\varepsilon(u_{\varepsilon,\delta})$ is finite. Then the Euler-Lagrange equation (4.5) easily follows by differentiating $\mathcal{F}_\varepsilon(u_{\varepsilon,\delta})$ with respect to δ at $\delta = 0$. □

We are now in a position to prove Theorem 1.7.

Proof of Theorem 1.7. Let us fix a sequence $\{\varepsilon_n\} \subset (0,1)$, with $\varepsilon_n \to 0$ as $n \to \infty$. We divide the proof into five steps.

Step 1: There exist a subsequence, not relabeled, and a function $u \in \mathcal{V}^{0,1}$ such that

$$u_{\varepsilon_n} \rightharpoonup u \quad \text{in } \mathcal{W}^{0,1}_T \qquad \text{for every } T > 0\,. \tag{4.6}$$

Moreover, $u' \in L^\infty((0,\infty);H)$ and $u \in L^\infty((0,T);V_T)$ for every $T > 0$.

Let $T > 0$. By (1.15) and (1.16),

$$\sup_{n\in\mathbb{N}} \|u_{\varepsilon_n}\|_{H^1((0,T);H)} < \infty\,.$$

This inequality, together with (H4) and (1.14), implies that there exists $C_T < \infty$ such that

$$\nu_0\|u_{\varepsilon_n}\|^2_{L^2((0,T);V_T)} \le \int_0^T Q(u_{\varepsilon_n}(t))\,\mathrm{d}t + \lambda_0\|u_{\varepsilon_n}\|^2_{L^2((0,T);H)} \le C_T\,.$$

As a result $\{u_{\varepsilon_n}\}$ is equibounded in $\mathcal{W}^{0,1}_T$ and hence there exist a subsequence, not relabeled, and a function $u \in \mathcal{W}^{0,1}_T$ such that $u_{\varepsilon_n} \rightharpoonup u$ weakly in $\mathcal{W}^{0,1}_T$. Moreover, since $\{u_{\varepsilon_n}\} \subset \mathcal{V}^{0,2}_T \subset \mathcal{V}^{0,1}_T$ and $\mathcal{V}^{0,1}_T$ is a closed subspace of $\mathcal{W}^{0,1}_T$, we have that $u \in \mathcal{V}^{0,1}_T$. By the arbitrariness of T, the function u belongs to $\mathcal{V}^{0,1}$ and (4.6) holds true. Furthermore, in view of (4.6), inequality (1.16) implies $u' \in L^\infty((0,\infty);H)$ and (1.15) gives $u \in L^\infty((0,T);V_T)$ for every $T > 0$.

Step 2: Let $T > 0$. For every $\psi \in C^\infty_c((0,T);V_T)$, with $\psi(t) \in V_t$ for every $t \in (0,T)$, we have

$$\int_0^T (u'_{\varepsilon_n}(t), \varepsilon_n^2\psi'''(t) + 2\varepsilon_n\psi''(t) + \psi'(t))\,\mathrm{d}t = \int_0^T a(u_{\varepsilon_n}(t),\psi(t))\,\mathrm{d}t\,. \tag{4.7}$$

The claim follows by considering $\eta(t) = e^{t/\varepsilon_n}\psi(t)$ in (4.5) and integrating by parts.

Step 3: The function u is a weak solution of (1.7).

By [4, Lemma 2.8], it is enough to prove the claim for $\psi \in C^\infty_c((0,T);V_T)$ with $\psi(t) \in V_t$ for every $t \in (0,T)$. In view of (4.6), one can pass to the limit as $n \to \infty$ in (4.7), thus obtaining (1.8).

Step 4: u satisfies (a) and (b).

Since $u' \in L^\infty((0,\infty);H)$ and $u \in L^\infty((0,T);V_T)$ for every $T>0$ by Step 1, property (a) follows from Step 3, thanks to Remark 1.4. Claim (b) is obtained by combining (a), (1.9), and (4.6), together with the fact that $u_{\varepsilon_n} \in \mathcal{V}^{0,1}(u^0, u^1_{\varepsilon_n})$.

Step 5: The function u satisfies the energy inequality (1.17).

By using [9, Lemma 6.1] and (3.13), one can argue as in [9, Section 6] to obtain that the energy inequality (1.17) is satisfied for almost every $t>0$. Actually, in view of (a), this inequality is satisfied for every $t>0$. □

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