HOMOGENIZATION OF TWO–PHASE METRICS AND APPLICATIONS

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ABSTRACT. We consider two–phase metrics of the form $\varphi(x,\xi) := \alpha\chi_{B_\alpha}(x)\,|\xi| + \beta\chi_{B_\alpha}(x)\,|\xi|$, where α,β are fixed positive constants, and B_α, B_β are disjoint Borel sets whose union is $\mathbb{R}^N$, and we prove that they are dense in the class of symmetric Finsler metrics φ satisfying

$$\alpha|\xi| \le \varphi(x,\xi) \le \beta|\xi| \quad \text{on } \mathbb{R}^N \times \mathbb{R}^N.$$

Then we study the closure $Cl(\mathcal{M}_\theta^{\alpha,\beta})$ of the class $\mathcal{M}_\theta^{\alpha,\beta}$ of two–phase periodic metrics with prescribed volume fraction θ of the phase α. We have not a complete answer to this problem at the moment: we give upper and lower bounds for the class $Cl(\mathcal{M}_\theta^{\alpha,\beta})$, and we localize the problem, generalizing the bounds to the non–periodic setting. Finally, we apply our results to study the closure, in terms of Γ–convergence, of two–phase gradient-constraints in composites of the type $f(x,\nabla u) \le C(x)$, with $C(x) \in \{\alpha,\,\beta\}$ for almost every x.

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1. Introduction

A *symmetric Finsler metric* on $\mathbb{R}^N$ is a Borel map $\varphi : \mathbb{R}^N \times \mathbb{R}^N \to [0, +\infty)$ such that $\varphi(x, \cdot)$ is a norm for every $x \in \mathbb{R}^N$. A symmetric *Finsler distance* on $\mathbb{R}^N$ can be geodesically associated with it as follows:

$$d_\varphi(x, y) := \inf \left\{ \int_0^1 \varphi(\gamma, \dot{\gamma})\, \mathrm{d}t \ : \ \gamma \in W^{1,\infty}\left([0,1]; \mathbb{R}^N\right),\ \gamma(0) = x,\ \gamma(1) = y \right\}.$$

Clearly, the family of Riemannian metrics is a subset of Finsler ones, which is however not closed with respect to the Γ–convergence of the associated length functionals, or, equivalently, with respect to the local uniform convergence of the corresponding distances. This was first pointed out by Acerbi and Buttazzo in the case of periodic homogenization [1], where the following example in dimension $N = 2$ is provided: consider a sequence of periodic coefficients $(a_n)_{n\in\mathbb{N}}$ of the form $a_n(x) = a(nx)$, where the function a takes only two different values $\beta > \alpha > 0$ on the white and black squares of a chessboard, respectively. When n goes to infinity, the distances associated with the metrics $a_n(x)|\xi|$ converge to a norm $\phi(\cdot)$. Moreover, if the quotient β/α is sufficiently large, the unit ball $B_\phi := \{\xi \in \mathbb{R}^2 \ : \ \phi(\xi) \le 1\}$ is a regular octagon, so ϕ is non–Riemannian.

A natural question therefore arises: what kind of metrics can we obtain as limit of (continuous) Riemannian ones? The conjecture, as stated in [9], was that any symmetric Finsler metric φ on $\mathbb{R}^N$ satisfying

$$\alpha|\xi| \le \varphi(x, \xi) \le \beta|\xi| \quad \text{on } \mathbb{R}^N \times \mathbb{R}^N, \tag{1.1}$$

with α and β positive constants, can be approximated by smooth Riemannian metrics satisfying the same bounds; more precisely, there exists a sequence $(\varphi_n)_n$ of smooth Riemannian metrics satisfying (1.1) such that $\varphi_n \rightrightarrows \varphi$, i.e., such that the associated distances d_{φ_n} converge, locally uniformly in $\mathbb{R}^N \times \mathbb{R}^N$, to d_φ.

This issue was first considered by Braides, Buttazzo, Fragalà in [5] and partially solved by additionally assuming φ lower semicontinuous. A complete answer has been subsequently provided in [15] by one of the authors, where the same result is proved without assuming any continuity of the metric in x. The sought sequence is obtained by approximating φ on a dense subset of geodesics. These arguments have been generalized in [16] to obtain analogous density results for non–symmetric Finsler metrics.

In this paper we prove another density result. Instead of regular Riemannian metrics, we consider the class of *two–phase* metrics of the form

$$\varphi(x, \xi) := \begin{cases} \alpha|\xi| & \text{if } x \in B_\alpha; \\ \beta|\xi| & \text{if } x \in B_\beta, \end{cases} \tag{1.2}$$

where B_α and B_β are two disjoint Borel sets whose union is $\mathbb{R}^N$. This class of metrics comes out naturally in the study of composite materials, where the *homogenized composite* is often obtained by a fine mixture of a finite number of homogeneous and isotropic materials. Our main result is Theorem 3.3 which states that this class of metrics is dense in the class of Finsler metrics satisfying (1.1).

The first result we give in this direction is the following (see Theorem 3.1). Let ϕ be a norm on $\mathbb{R}^N$ satisfying (1.1). Then there exists a sequence $(\varphi_n)_n$ of 1–periodic two–phase metrics of the type (1.2) such that $\varphi_n \rightrightarrows \phi$. Moreover the metrics φ_n can be chosen upper semicontinuous with respect to x, and this will be crucial in our applications.

The main idea in the construction of the approximating sequence of two–phase metrics is the following. Fix a vector $\xi \in \mathbb{R}^N$, and let $\gamma : \mathbb{R} \to \mathbb{R}^N$ be a curve which satisfies the following conditions:

i) γ is T–periodic on the N–dimensional torus;

ii) the *mean direction* of γ is ξ; i.e., $\xi = \gamma(T) - \gamma(0)$;

iii) γ is a geodesic curve for the periodic metric φ defined by

$$\varphi(x,\xi) = \alpha\,|\xi| \quad \text{on } \gamma + \mathbb{Z}^N, \qquad \varphi(x,\xi) = \beta\,|\xi| \quad \text{otherwise.}$$

Then we have

$$d_\varphi(\gamma(0), \gamma(T)) = \alpha L,$$

where $L > 1$ is the length of γ on the torus. Therefore if $L = \phi(\xi)/\alpha$, by periodicity and by iii) we obtain that

$$d_\varphi(\gamma(0), \gamma(nT)) = d_\phi(\gamma(0), \gamma(nT)) \qquad \text{for every } n \in \mathbb{N}.$$

It follows that the *stable norm* φ^{hom} of φ obtained by homogenization (cf. formula (2.14)), satisfies $\varphi^{hom}(\xi) = \phi(\xi)$. It is now clear that to prove the theorem it will be enough to repeat the previous construction for an arbitrarily big number of vectors ξ_i: the proof reduces to the construction of an arbitrarily big number of curves, with prescribed length and direction, which are geodesic with respect to the two–phase metric taking the value α on their supports and β elsewhere. The main difficulty is to check that the addition of any new curve preserves the geodesic character of the previous ones. We will perform such construction using *zig–zag* curves contained in small neighborhood of straight–lines of direction ξ_i. Finally a perturbation argument will allow us to make these approximating metrics upper semicontinuous with respect to x.

Once approximated a norm ϕ on $\mathbb{R}^N$, we can approximate any Finsler metric φ using a localization argument as in [5], consisting in freezing the x dependence of φ. The conclusion is that the class of two–phase metrics is dense in the class of Fisler ones.

This result is based on the assumption that the two–phase metrics can have arbitrarily amount of the two different phases, while often in applications to composites the volume fraction θ of one of the two, let us say of the phase α, is prescribed. Then a natural question is what is the closure $Cl(\mathcal{M}_\theta^{\alpha,\beta})$ of the class $\mathcal{M}_\theta^{\alpha,\beta}$ of two–phase periodic metrics with prescribed volume fraction θ. We have not a complete answer to this problem at the moment: in Theorem 4.6 we give upper and lower bounds for the set $Cl(\mathcal{M}_\theta^{\alpha,\beta})$. In Theorem 4.8 and Theorem 4.11 we localize these results, generalizing them to the non–periodic setting.

Let us now discuss some application of our results to the study of composites. Let Ω be a bounded open subset of $\mathbb{R}^N$ with Lipschitz continuous boundary, and consider the class of functions $u \in W^{1,\infty}(\Omega)$ satisfying the following gradient constraint:

$$|\nabla u(x)| \le C(x) \qquad \text{for a.e. } x \in \Omega. \tag{1.3}$$

Pointwise gradient constraints of type (1.3) may occur in several different contexts, we refer for instance to [4], [12], [25]. Assume now that Ω represents the reference configuration of a non homogeneous body, and $C(x)$ depends on the physical properties of the material at the point x. The main example is given by a body made up by two different homogeneous isotropic materials. In this case the function $C(\cdot)$ takes two values, α and β. If the body consists in a fine mixture of the two materials, i.e., if the body is the result of a homogenization process (for instance in the sense of Γ-convergence), the effect of condition (1.3) for the homogenized material has to be clarified. This problem was studied in [25] in the case of periodic homogenization to model the dielectric breakdown. Here we consider the general (non–periodic) case and we introduce some new techniques which make more general the arguments used in [25]. Let us also mention that constraints functionals of the type (1.3) have been largely studied in the last years in connection with problems of relaxation of integral functionals. We refer the reader to [18],[19] and to the references therein.

The main ingredient in our approach is the connection between gradient constraints of the type (1.3) and geodesic distances, already observed in [9], [10], [26]. The main point is that (1.3) can be equivalently restated in the framework of supremal functionals as

$$F(u) := \sup_{\Omega} f(x, \nabla u(x)) \leq 1, \tag{1.4}$$

where

$$f(x,\xi) := \frac{1}{C(x)}|\xi|.$$

Let us consider the class of supremal functionals F of the form (1.4), with $f(x,\xi)$ Carathéodory function satisfying, for a.e $x \in \Omega$,

$$\alpha'|\xi| \leq f(x,\xi) \leq \beta'|\xi| \qquad \text{for every } \xi \in \mathbb{R}^N, \tag{1.5}$$

$$f(x,\lambda\,\xi) = |\lambda| f(x,\xi) \qquad \text{for every } \xi \in \mathbb{R}^N \text{ and } \lambda \in \mathbb{R}, \tag{1.6}$$

where $\alpha' := 1/\beta$, $\beta' := 1/\alpha$. In [26] the following fact is proved: F coincides with a *difference quotient* functional of the type

$$R^{d_F}(u) := \sup_{x,y\in\Omega\, x\neq y} \frac{u(x)-u(y)}{d_F(x,y)},$$

where d_F is a geodesic distance associated with F (see (5.5)). On the contrary, a difference quotient functional R^d can be written in a supremal form only if the distance d is geodesic, and satisfies the additional condition to be an *intrinsic distance* (see Definition 2.15). Moreover a sequence of difference quotients R^{d_n} Γ-converges to some R^d if and only if d_n converges to d. In particular, a sequence of supremal functionals Γ-converges up to a subsequence to some difference quotient functional R^d, which can be written in a supremal form if and only if d is intrinsic. The class of difference quotient functionals is then the closure under Γ–convergence of supremal functionals, and it strictly contains supremal functionals, as geodesic distances strictly contain intrinsic distances (cf. [16], [26]).

In our applications we specialize these results to the case of two–phase supremal functionals, or, equivalently, of two–phase constraints. The problem of the closure under Γ–convergence of two–phase constraints is then reduced to the problem of the closure of two–phase intrinsic distances. The property to be intrinsic of a distance d associated with a metric φ is guaranteed by the upper semicontinuity of φ; this is why in our main results we require the approximating sequence of metrics to be upper semicontinuous. In view of our density results on two–phase metrics, we deduce (see Theorem 5.4 and Theorem 5.5) that the closure of two–phase constraint of the type (1.3), with $C(x) \in \{\alpha, \beta\}$, is given by the class of the constraints of the type $R^d(u) \leq 1$, where d varies in the class of geodesic distances associated with the Finsler metrics satisfying (1.1). This class contains, in particular, any constraint of the type $\operatorname{ess\,sup}_{\Omega} f(x, \nabla u(x)) \leq 1$, where f varies on the class of Charatéodory functions satisfying (1.5) and (1.6).

Finally we consider the periodic case: if the volume fraction θ is not prescribed, then the closure with respect to periodic homogenization of two–phase constraints is given (see Remark

5.10) by the class of all constraints of the type $\sup_{\mathbb{R}^N} f(\nabla u(x)) \leq 1$, where f varies on the class of norms defined on $\mathbb{R}^N$ satisfying (1.5).

In the case of prescribed volume fraction θ, we get the additional condition $f^* \in \mathcal{M}_\theta^{\alpha,\beta}$, where the function f^* is defined by duality in (2.10). The upper and lower bounds for the class $Cl(\mathcal{M}_\theta^{\alpha,\beta})$ improve the ones obtained in [25], and make more general that approach.

Finally let us mention that our density results can be used to model line–energies for composites. A relevant case is that of Griffith's surface energy associated with a crack in a planar hyper–elastic body. Given a one dimensional crack in the body, the associated surface energy in the Griffith's theory is essentially proportional to the length of the crack. If the body is made by two different materials, then it is natural to consider a two–phase surface energy. Therefore our density results can be involved to approach the problem of the G–closure of two–phase Griffith's energies in composites. The analogous problem concerning the G–closure of two–phase bulk energies is by now classic, and was completely solved in the linear case by Lurie and Cherkaev in [27], and by Tartar in [28].

The paper is organized as follows. In Section 2 we introduce the notion of Finsler metric and we recall its main properties. In Section 3 we give our density results of two–phase metrics. In Section 4 we consider the problem of periodic Finsler metrics with prescribed volume fraction, then we localize the analysis to the non–periodic case. In Section 5 we apply our results to study the asymptotic behavior of two–phase constraint functionals.

2. Notation and Preliminaries on Finsler metrics

We write below a list of symbols used throughout the paper.

N	an integer number
$B_r(x)$	the open ball in $\mathbb{R}^N$ of radius r centered at x
B_r	the open ball in $\mathbb{R}^N$ of radius r centered at 0
$\mathbb{S}^{N-1}$	the $(N-1)$–dimensional unitary sphere of $\mathbb{R}^N$
Q	the unitary cube $[0,1]^N$ in $\mathbb{R}^N$
$\mathcal{H}^k$	k–dimensional Hausdorff measure
$\lvert u \rvert$	Euclidean norm of the vector $u \in \mathbb{R}^k$
χ_E	the characteristic function of the set E

Given a subset U of $\mathbb{R}^k$, we denote by $\overline{U}$ its closure. We furthermore say that U is *well contained* in a subset V of $\mathbb{R}^k$ if $\overline{U}$ is compact and contained in V. If E is a Lebesgue measurable subset of $\mathbb{R}^k$, we denote by $|E|$ its k–dimensional Lebesgue measure, and we say that E is *negligible* whenever $|E| = 0$. We say that a property holds *almost everywhere* (*a.e.* for short) on $\mathbb{R}^k$ if it holds up to a negligible subset of $\mathbb{R}^k$. Given an integrable function $f : \mathbb{R}^k \to \mathbb{R}$, the notation $⨍_E f \, dx$ stands for $\frac{1}{|E|}\int_E f \, dx$.

An open and connected subset of $\mathbb{R}^N$ will be referred as *domain* in the sequel. Throughout the paper, α and β will always denote two fixed positive constants with $\beta > \alpha$.

We recall the notion of Γ–convergence (we refer the reader to [14] for an exhaustive treatment of this topic). Given a metric space X, we will say that a sequence of functionals $F_n : X \to \mathbb{R} \cup \{\pm\infty\}$ Γ*–converge* to a functional F on X if

$$F(x) = \Gamma - \liminf_n F_n(x) = \Gamma - \limsup_n F_n(x) \qquad \text{for every } x \in X,$$

where

$$\begin{aligned}\Gamma-\liminf_n F_n(x) &= \inf\{\liminf_n F_n(x_n)\ :\ x_n\to x\}\\ \Gamma-\limsup_n F_n(x) &= \inf\{\limsup_n F_n(x_n)\ :\ x_n\to x\}.\end{aligned}$$

If F_n Γ–converge to F on X, then F is lower semicontinuous, and is coercive too provided the functionals F_n are equi–coercive. In this case we have the crucial property of the Γ–convergence, that is, the sequence $\inf_X F_n$ converges to the minimum of F on X.

2.1. **Finsler metrics: definition and main properties.** We collect in this section the main definitions and properties of Finsler metrics we shall need in the sequel. For a more detailed presentation of this material, we refer to [17].

Let Ω be a domain in $\mathbb{R}^N$ with Lipschitz boundary and denote by M either Ω or its closure $\overline{\Omega}$. We will denote by $\Gamma(M)$ the family of all Lipschitz curves from $[0,1]$ to M, and we will always assume that such curves are parametrized with constant velocity. The space $\Gamma(M)$ is equipped with the metric given by the uniform convergence, namely we say that the sequence $(\gamma_n)_n$ converges to γ to mean that $\sup_{t\in I}|\gamma_n(t)-\gamma(t)|$ tends to zero as n goes to infinity. We will denote by $\Gamma_{x,y}(M)$ the subset of curves in $\Gamma(M)$ joining x to y; i.e., such that $\gamma(0)=x$ and $\gamma(1)=y$. Throughout the paper, we will adopt the concise notation $\mathcal{H}^1(\gamma)$ to denote the Euclidean length $\mathcal{H}^1\big(\gamma([0,1])\big)$ of a curve $\gamma\in\Gamma(M)$.

Let us set

$$|x-y|_M := \inf\left\{\mathcal{H}^1(\gamma)\ :\ \gamma\in\Gamma_{x,y}(M)\right\}$$

for every $x,y\in M$. Since $\partial\Omega$ is Lipschitz, there exists a constant $C>0$ such that

$$|x-y|\le|x-y|_M\le C|x-y| \qquad \text{for all } x,\ y\in M.$$

Let d be a distance on M. The d–length of a curve $\gamma\in\Gamma(M)$ is defined as the supremum of the d–lengths of inscribed polygonal curves, namely as

$$\mathrm{L}_d(\gamma) := \sup\left\{\sum_{i=0}^{m-1} d\big(\gamma(t_i),\gamma(t_{i+1})\big)\ :\ 0=t_0<t_1<..<t_m=1,\ m\in\mathbb{N}\right\}. \tag{2.1}$$

We will say that d is a *geodesic distance* on M if

$$d(x,y)=\inf\{\mathrm{L}_d(\gamma)\ :\ \gamma\in\Gamma_{x,y}(M)\} \qquad \text{for every } x,y\in M.$$

Let us now fix two positive constants α and β. We denote by $\mathcal{D}(M)$ the family of geodesic distances on M satisfying

$$\alpha|x-y|_M\le d(x,y)\le\beta|x-y|_M \qquad \text{for all } x,\ y\in M. \tag{2.2}$$

We endow $\mathcal{D}(M)$ with the metric given by the uniform convergence on compact subsets of $M\times M$. The convergence of a sequence $(d_n)_n$ to d in $\mathcal{D}(M)$ will be hereafter denoted by $d_n\rightrightarrows d$ (in $M\times M$). We have (cf. [9, Theorem 3.1]):

Theorem 2.1. *When M is closed, $\mathcal{D}(M)$ is a compact metric space.*

Any distance in $\mathcal{D}(M)$ induces on M a topology which is equivalent to the Euclidean one. Therefore the following proposition holds (cf. [3, Busemann–Theorem 4.3.1]).

Theorem 2.2. *Let $d \in \mathcal{D}(M)$. The length functional L_d is lower semicontinuous on $\Gamma(M)$ with respect to the uniform convergence of paths; i.e., if $(\gamma_n)_n$ converges to γ, then*

$$\mathrm{L}_d(\gamma) \leq \liminf_n \mathrm{L}_d(\gamma_n).$$

When M is closed, we have in particular that, for every pair of points x, y in M,

$$d(x,y) = \mathrm{L}_d(\gamma) \qquad \text{for some } \gamma \in \Gamma_{x,y}(M).$$

Any path of minimal d–length joying two points x,$y \in M$ will be referred as *geodesic*.
Let us now introduce the notion of *symmetric Finsler metric*.

Definition 2.3. A *symmetric Finsler metric* on M is a Borel–measurable function $\varphi : M \times \mathbb{R}^N \to [0, +\infty)$ such that

i) $\varphi(x, \lambda\,\xi) = \lambda\,\varphi(x,\xi)$ for every $(x,\xi) \in M \times \mathbb{R}^N$, and for every $\lambda \geq 0$;
ii) $\varphi(x, \cdot)$ is convex on $\mathbb{R}^N$, for a.e. $x \in M$;
iii) for every curve $\gamma \in \Gamma(M)$

$$\varphi(\gamma(t), \dot\gamma(t)) = \varphi(\gamma(t), -\dot\gamma(t)) \qquad \text{for a.e. } t \in [0,1].$$

We set

$$\mathcal{M}(M) := \left\{\varphi \;\text{ symmetric Finsler metrics on } M\text{: } \alpha|\xi| \leq \varphi(x,\xi) \leq \beta|\xi| \text{ on } M \times \mathbb{R}^N\right\}.$$

To any $\varphi \in \mathcal{M}(M)$, we can associate a distance $d_\varphi \in \mathcal{D}(M)$ through the formula

$$d_\varphi\,(x,y) := \inf\left\{\mathbb{L}_\varphi\,(\gamma) :\; \gamma \in \Gamma_{x,y}(M)\,\right\}, \tag{2.3}$$

where the length functional $\mathbb{L}_\varphi$ is defined as

$$\mathbb{L}_\varphi(\gamma) := \int_0^1 \varphi(\gamma(t), \dot\gamma(t))\,\mathrm{d}t, \qquad \gamma \in \Gamma(M). \tag{2.4}$$

When $\varphi(x,\xi) := a(x)|\xi|$ for some Borel coefficient $a : M \to [\alpha, \beta]$, the related distance d_φ will be denoted by d_a.

The following holds (cf. [20, Theorem 4.3]):

Proposition 2.4. *Let $\varphi \in \mathcal{M}(M)$. Then L_{d_φ} is the relaxed functional of $\mathbb{L}_\varphi$ on $\Gamma(M)$; i.e.,*

$$\mathrm{L}_{d_\varphi}(\gamma) = \inf\left\{\liminf_{n\to+\infty} \mathbb{L}_\varphi(\gamma_n) \;:\; (\gamma_n)_n \text{ converges to } \gamma \text{ in } \Gamma(M)\;\right\}$$

for any $\gamma \in \Gamma(M)$.

Remark 2.5. The functional L_d agrees with $\mathbb{L}_\varphi$ whenever the latter is lower semicontinuous on $\Gamma(M)$. This happens, for instance, when φ is lower semicontinuous on $M \times \mathbb{R}^N$ and $\varphi(x,\cdot)$ is convex on $\mathbb{R}^N$ for every $x \in M$ (cf. [11, Theorem 4.1.1]).

Conversely, to any geodesic distance $d \in \mathcal{D}(M)$, we can associate a symmetric Finsler metric $\varphi_d \in \mathcal{M}(M)$ in the following way: for every $(x,\xi) \in M \times \mathbb{R}^N$, set

$$\varphi_d\,(x,\xi) := \limsup_{h\to 0^+} \frac{d\,(\gamma(0), \gamma(h))}{h}$$

if there exists a Lipschitz curve $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $(\gamma(0), \dot{\gamma}(0)) = (x, \xi)$, and

$$\varphi_d\,(x,\xi) := \beta\,|\xi| \qquad \text{otherwise.}$$

Since d is locally equivalent to the Euclidean distance, this definition makes sense, that is, $\varphi_d(x, \xi)$ does not depend on the choice of the curve γ.

When $M = \Omega$, the above definition agrees with the following (see [23, Section 2]):

$$\varphi_d\,(x,\xi) := \limsup_{h\to 0^+} \frac{d\,(x, x+h\xi)}{h} \qquad \text{for every } (x,\xi) \in \Omega \times \mathbb{R}^N. \tag{2.5}$$

The following integral representation result holds (cf. [23, Theorem 2.5]):

Proposition 2.6. *Let $d \in \mathcal{D}(M)$. Then*

$$\mathrm{L}_d(\gamma) = \int_0^1 \varphi_d(\gamma(t), \dot{\gamma}(t))\,\mathrm{d}t \qquad \textit{for every } \gamma \in \Gamma(M).$$

In particular, $d = d_{\varphi_d}$.

Remark 2.7. Proposition 2.6 implies that the map $\mathcal{M}(M) \ni \varphi \mapsto d_\varphi \in \mathcal{D}(M)$ is surjective. It is not injective, however. In fact, the inequality $\mathrm{L}_{d_\varphi}(\gamma) \le \mathbb{L}_\varphi(\gamma)$, which holds true for any $\gamma \in \Gamma(M)$, yields

$$\varphi_{d_\varphi}(x,\xi) \le \varphi(x,\xi) \qquad \text{for every } (x,\xi) \in M \times \mathbb{R}^N,$$

and this inequality can be strict when φ is not continuous, as easy examples show.

In the sequel, we will write $\varphi_n \rightrightarrows \varphi$ in $\mathcal{M}(M)$ to mean that $d_{\varphi_n} \rightrightarrows d_\varphi$ in $\mathcal{D}(M)$. Next proposition gives sufficient conditions for the convergence of metrics in $\mathcal{M}(M)$.

Proposition 2.8. *Assume M closed, and let φ, $\varphi_n \in \mathcal{M}(M)$. Then $\varphi_n \rightrightarrows \varphi$ in the following cases:*

- *i) (φ_n) converges uniformly to φ on compact subsets of $M \times M$;*
- *ii) the metrics φ_n are lower semicontinuous and converge increasingly to φ pointwise on $M \times \mathbb{R}^N$;*
- *iii) (φ_n) converges decreasingly to φ pointwise on $M \times \mathbb{R}^N$.*

The following proposition establishes the equivalence between the convergence of distances in $\mathcal{D}(M)$, and the Γ-convergence of the associated length functionals.

Proposition 2.9. *Assume M closed. Let φ_n and φ be in $\mathcal{M}(M)$, and denote by d_n and d the associated distances in $\mathcal{D}(M)$. The following facts are equivalent.*

- *i) $d_n \rightrightarrows d$;*
- *ii) $d_n \to d$ pointwise in $M \times M$;*
- *iii) The functionals $\mathbb{L}_{\varphi_n}$ Γ-converge to the relaxation L_d of $\mathbb{L}_\varphi$, with respect to the uniform convergence in $\Gamma(M)$.*

The following density result has been been established in [15]

Theorem 2.10. *Assume M closed. Then continuous metrics are dense in $\mathcal{M}(M)$. More precisely, for every $\varphi \in \mathcal{M}(M)$ there exists a sequence of continuous functions $a_n : M \to [\alpha, \beta]$ such that*

$$d_{a_n} \rightrightarrows d_\varphi \quad \text{in } \mathcal{D}(M).$$

We now study the relation between $\mathcal{D}(\Omega)$ and $\mathcal{D}(\overline{\Omega})$. Any distance belonging to $\mathcal{D}(\Omega)$ is Lipschitz continuous, so it can be uniquely extended by continuity to $\overline{\Omega} \times \overline{\Omega}$. The following result holds.

Proposition 2.11. *Given $\varphi \in \mathcal{M}(\Omega)$, denote by $\overline{\varphi}$ the extension of φ to $\overline{\Omega} \times \mathbb{R}^N$ obtained by setting $\overline{\varphi}(x, \cdot) := \beta \,|\cdot|$ on $\partial\Omega$. Then*

$$d_\varphi(x, y) := \inf_{\gamma \in \Gamma_{x,y}(\Omega)} \int_0^1 \varphi(\gamma(t), \dot\gamma(t))\,\mathrm{d}t = \inf_{\gamma \in \Gamma_{x,y}(\overline{\Omega})} \int_0^1 \overline{\varphi}(\gamma(t), \dot\gamma(t))\,\mathrm{d}t \tag{2.6}$$

for every $x, y \in \Omega$. In particular, the continuous extension of d_φ to $\overline{\Omega} \times \overline{\Omega}$ is the distance $d_{\overline{\varphi}}$ defined as

$$d_{\overline{\varphi}}(x, y) = \inf\left\{ \int_0^1 \overline{\varphi}(\gamma(t), \dot\gamma(t))\,\mathrm{d}t : \ \gamma \in \Gamma_{x,y}(\overline{\Omega}) \right\} \quad \textit{for every } x, y \in \overline{\Omega}. \tag{2.7}$$

Proof. For every $n \in \mathbb{N}$, let

$$\overline{\varphi}_n(x, \xi) := \begin{cases} \beta|\xi| & \text{if } x \in A_n \\ \varphi(x, \xi) & \text{elsewere,} \end{cases}$$

where $A_n := \{\, x \in \overline{\Omega} \,:\, \mathrm{dist}(x, \mathbb{R}^N \setminus \Omega) < 1/n \,\}$, and let $d_{\overline{\varphi}_n}$ be the distance on $\overline{\Omega} \times \overline{\Omega}$ defined according to (2.7). Since the $\overline{\varphi}_n$ are continuous on $A_n \times \mathbb{R}^N$, it is easy to see that

$$d_{\overline{\varphi}_n}(x, y) = \inf\left\{ \int_0^1 \overline{\varphi}_n(\gamma(t), \dot\gamma(t))\,\mathrm{d}t : \ \gamma \in \Gamma_{x,y}(\Omega) \right\} \quad \text{for every } x, y \in \Omega.$$

As $\overline{\varphi} = \varphi \le \overline{\varphi}_n$ on $\Omega \times \mathbb{R}^N$, we deduce that

$$d_{\overline{\varphi}}(x, y) \le d_\varphi(x, y) \le d_{\overline{\varphi}_n}(x, y) \quad \text{for every } x, y \in \Omega.$$

Since $\overline{\varphi}_n(x, \xi)$ converges decreasingly to $\overline{\varphi}(x, \xi)$ on $\overline{\Omega} \times \overline{\Omega}$, the assertion follows by Proposition 2.8. □

Remark 2.12. We can define a continuous immersion $i : \mathcal{D}(\Omega) \hookrightarrow \mathcal{D}(\overline{\Omega})$ by identifying any element of $\mathcal{D}(\Omega)$ with its unique continuous extension to $\overline{\Omega} \times \overline{\Omega}$. By Proposition 2.11 we deduce

$$\left\{ d_\varphi \,:\, \varphi \in \mathcal{M}(\overline{\Omega}),\ \varphi(x, \cdot)_{|\partial\Omega} = \beta|\cdot| \,\right\} \subseteq i\left(\mathcal{D}(\Omega)\right). \tag{2.8}$$

Moreover $i\left(\mathcal{D}(\Omega)\right)$ is dense in $\mathcal{D}(\overline{\Omega})$, since the set

$$\left\{ d_\varphi \,:\, \varphi \in \mathcal{M}(\overline{\Omega}),\ \varphi \text{ continuous on } \overline{\Omega} \times \mathbb{R}^N \,\right\} \tag{2.9}$$

is dense in $\mathcal{D}(\overline{\Omega})$ by Theorem 2.10, and is clearly contained in $i\left(\mathcal{D}(\Omega)\right)$ (cf. Proposition 2.18). In general $i\left(\mathcal{D}(\Omega)\right)$ is a strict subset of $\mathcal{D}(\overline{\Omega})$, in particular $\mathcal{D}(\Omega)$ is not closed. The example is easy: take $\Omega := (0, 1)^N$ and consider the distance $d \in \mathcal{D}(\overline{\Omega})$ associated through (2.3) to the isotropic metric $a(\cdot)$ identically equal to β on Ω and to α on $\partial\Omega$. Since $\varphi_d(x, \xi) = \beta|\xi|$ for every $x \in \Omega$, it is clear that d cannot belong to $i\,(\mathcal{D}(\Omega))$.

We conclude this paragraph by proving an auxiliary lemma that will be required in Section 5.

Lemma 2.13. *Let Ω be a convex subset of $\mathbb{R}^N$, and $\varphi \in \mathcal{M}(\Omega)$. Extend φ to a metric $\overline{\varphi} \in \mathcal{M}(\mathbb{R}^N)$ by setting $\overline{\varphi}(x, \cdot) := \beta\,|\cdot|$ outside Ω, and denote by $d_\varphi \in \mathcal{D}(\Omega)$ and $d_{\overline{\varphi}} \in \mathcal{D}(\mathbb{R}^N)$ the distances associated through* (2.3) *to φ and $\overline{\varphi}$, respectively. Then*

$$d_\varphi(x, y) = d_{\overline{\varphi}}(x, y) \qquad \text{for every } x, y \in \Omega.$$

Remark 2.14. In the definition of $d_\varphi(x, y)$ the minimization is made among curves constraint to stay in Ω. Hence $d_\varphi(x, y) \geq d_{\overline{\varphi}}(x, y)$ for $x, y \in \Omega$, in general, and the inequality can be strict for non–convex domains, as easy examples show.

Proof. It is enough to prove that

$$d_{\overline{\varphi}}(x, y) = \inf \left\{ \int_0^1 \overline{\varphi}(\gamma(t), \dot{\gamma}(t)) \mathrm{d}t : \ \gamma \in \Gamma_{x,y}(\overline{\Omega}) \right\} \quad \text{for every } x, y \in \Omega,$$

for the right–hand side term coincides with $d_\varphi(x, y)$ by Proposition 2.11. Let γ be any admissible curve in $\Gamma_{x,y}(\mathbb{R}^N)$, and let $\xi(t) := \pi(\gamma(t))$ for every $t \in [0, 1]$, where $\pi : \mathbb{R}^N \to \overline{\Omega}$ denotes the projection on the convex set $\overline{\Omega}$. Since $\xi \in \Gamma_{x,y}(\overline{\Omega})$, it suffices to prove that

$$\int_0^1 \overline{\varphi}(\xi(t), \dot{\xi}(t))\, \mathrm{d}t \leq \int_0^1 \overline{\varphi}(\gamma(t), \dot{\gamma}(t))\, \mathrm{d}t.$$

But this easily follows from the fact that the inequality $|\dot{\xi}(t)| \leq |\dot{\gamma}(t)|$ for a.e. $t \in [0, 1]$, which holds true for π is 1–Lipschitz continuous, implies that $\overline{\varphi}(\xi(t), \dot{\xi}(t)) \leq \overline{\varphi}(\gamma(t), \dot{\gamma}(t))$ for a.e. $t \in [0, 1]$. □

2.2. **Intrinsic distances.** We recall here the definition of intrinsic distance introduced by De Cecco and Palmieri and its main properties (for details see [20], [21], [22], [23]).

Definition 2.15. We say that a distance $d \in \mathcal{D}(M)$ is *intrinsic* if

$$d(x, y) = \sup_{|E|=0} \left\{ \inf \left\{ \int_0^1 \varphi_d(\gamma, \dot{\gamma})\, dt \ : \ \gamma \in \Gamma^E_{x,y}(M) \right\} \right\},$$

where $\Gamma^E_{x,y}(M)$ denotes the set of all Lipschitz curves in $\Gamma_{x,y}(M)$ which are transversal to E; i.e., such that

$$|\{t \in [0, 1] \ : \ \gamma(t) \in E\}| = 0.$$

To any Finsler metric $\varphi \in \mathcal{M}(M)$ we associate an intrinsic distance d^φ on M in the following way. We define the *dual metric* of φ on $M \times \mathbb{R}^N$ as

$$\varphi^*(x, \eta) := \sup_{|\xi|=1} \left\{ \frac{\xi \cdot \eta}{\varphi(x, \xi)} \right\}. \tag{2.10}$$

Clearly, φ^* enjoys

$$\frac{1}{\beta}\,|\eta| \leq \varphi^*(x, \eta) \leq \frac{1}{\alpha}\,|\eta| \qquad \text{for all } (x, \eta) \in M \times \mathbb{R}^N.$$

The intrinsic distance d^φ is defined by

$$d^\varphi(x,y) := \sup \Big\{ u(y) - u(x) \,:\, \operatorname*{ess\,sup}_M \varphi^*(x, \nabla u(x)) \le 1 \Big\} \tag{2.11}$$

for every x, $y \in M$. It is known that the above definition is equivalent to the following:

$$d^\varphi(x,y) = \sup_{|E|=0} \left\{ \inf \left\{ \int_0^1 \varphi(\gamma, \dot\gamma)\, dt \,:\, \gamma \in \Gamma^E_{x,y}(M) \right\} \right\}.$$

Moreover we have (see [22, Theorem 2.10] and [8, Theorem 3.1]):

Theorem 2.16. *Let $\varphi \in \mathcal{M}(M)$ and d^φ as above. Then there exists a metric $\tilde\varphi \in \mathcal{M}(M)$ with $\tilde\varphi(x,\cdot) = \varphi(x,\cdot)$ for almost every $x \in M$ such that $d^\varphi = d_{\tilde\varphi}$. In particular, d^φ belongs to $\mathcal{D}(M)$.*

Remark 2.17. In fact, $\tilde\varphi$ can be taken of the form $\varphi(x,\xi)\chi_{M\setminus F}(x) + \beta|\xi|\chi_F(x)$ for a suitable negligible Borel–measurable subset F of M.

Given $\varphi \in \mathcal{M}(M)$, the intrinsic distance d^φ is in general different from d_φ. The following result however holds (see Theorem 4.5 in [23], or Proposition 3.6 in [8]).

Proposition 2.18. *Let $\varphi \in \mathcal{M}(M)$ be upper semicontinuous on $M \times \mathbb{R}^N$. Then $d_\varphi = d^\varphi$.*

We now state a proposition that will be required later. It could be deduced by the results in [13]. A more direct proof is provided below for the reader's convenience.

Proposition 2.19. *Let $\varphi \in \mathcal{M}(\mathbb{R}^N)$ be such that $d_\varphi = d^\varphi$. For each $n \in \mathbb{N}$, let φ_n be the metric in $\mathcal{M}(\mathbb{R}^N)$ defined as*

$$\varphi_n(x,\xi) := (\rho_n * \varphi)(x,\xi) = \int_{\mathbb{R}^N} \rho_n(x-y)\, \varphi(y,\xi)\, \mathrm{d}y \quad \text{for every } (x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where $(\rho_n)_n$ is a sequence of standard mollifiers. Then $\varphi_n \rightrightarrows \varphi$ in $\mathcal{M}(\mathbb{R}^N)$.

Proof. We want to show that every convergent subsequence of $(d_{\varphi_n})_n$ has d_φ as limit, which is enough to prove the statement by the compactness of $\mathcal{D}(\mathbb{R}^N)$.

Let us then consider such a subsequence (not relabelled to ease notations) and call δ its limit distance. We start by showing that

$$\delta(x,y) \le d_\varphi(x,y) \qquad \text{for every } x, y \in \mathbb{R}^N. \tag{2.12}$$

We already know that

$$\varphi(x,\xi) = \lim_{n\to+\infty} \varphi_n(x,\xi) \qquad \text{for every } (x,\xi) \in \big(\mathbb{R}^N \setminus E\big) \times \mathbb{R}^N$$

for some negligible subset E of $\mathbb{R}^N$. The equality $d_\varphi = d^\varphi$ implies that d_φ is not affected by modification of the metric on negligible subsets of $\mathbb{R}^N$ with respect to x, hence, up to setting $\varphi(x,\xi) := \beta|\xi|$ on $E \times \mathbb{R}^N$, we can assume as well that

$$\varphi(x,\xi) \ge \limsup_{n\to+\infty} \varphi_n(x,\xi) \qquad \text{for every } (x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N.$$

Now, by Fatou's Lemma, for every curve $\gamma \in \Gamma(\mathbb{R}^N)$ we get

$$\int_0^1 \varphi(\gamma,\dot\gamma)\, \mathrm{d}t \ge \limsup_{n\to+\infty} \int_0^1 \varphi_n(\gamma,\dot\gamma)\, \mathrm{d}t \ge \limsup_{n\to+\infty} d_{\varphi_n}(\gamma(0),\gamma(1)),$$

so (2.12) follows by letting γ vary in $\Gamma_{x,y}(\mathbb{R}^N)$, for any fixed $x, y \in \mathbb{R}^N$.

Let us pass to the proof of the converse inequality

$$\delta(x,y) \geq d_\varphi(x,y) \qquad \text{for every } x,y \in \mathbb{R}^N. \tag{2.13}$$

Let u be a Lipschitz function on $\mathbb{R}^N$ such that $\varphi^*(x,\nabla u(x)) \leq 1$ for almost every $x \in \mathbb{R}^N$. For each $n \in \mathbb{N}$, set $u_n := \rho_n * u$. We claim that $\varphi_n^*(x,\nabla u_n(x)) \leq 1$ for every $x \in \mathbb{R}^N$; i.e.

$$\nabla u_n(x) \cdot \eta \leq \varphi_n(x,\eta) \qquad \text{for every } x,\, \eta \in \mathbb{R}^N.$$

By the fact that $\nabla u_n = \rho_n * \nabla u$, we indeed have:

$$\begin{aligned} \nabla u_n(x)\cdot\eta &= \langle \int_{\mathbb{R}^N} \rho_n(x-y)\,\nabla u(y)\,\mathrm{d}y, \eta\rangle = \int_{\mathbb{R}^N} \rho_n(x-y)\,\langle \nabla u(y), \eta\rangle\,\mathrm{d}y \\ &\leq \int_{\mathbb{R}^N} \rho_n(x-y)\,\varphi(y,\eta)\,\mathrm{d}y = \varphi_n(x,\eta) \end{aligned}$$

for every x, $\eta \in \mathbb{R}^N$, as claimed. Now each φ_n is continuous, hence, by Proposition 2.18, we deduce that

$$d_{\varphi_n}(x,y) = d^{\varphi_n}(x,y) \geq u_n(y) - u_n(x)$$

for every $x, y \in \mathbb{R}^N$, hence, letting $n \to +\infty$, we get

$$\delta(x,y) \geq u(y) - u(x).$$

Since $d_\varphi = d^\varphi$, (2.13) follows taking the supremum of the right–hand side term in the previous inequality over all functions u such that $\varphi^*(x,\nabla u(x)) \leq 1$ almost everywhere on $\mathbb{R}^N$. □

2.3. **Homogenization of periodic Finsler metrics.** A Finsler metric φ on $\mathbb{R}^N$ will be called *1–periodic* if $\varphi(x,\cdot) = \varphi(x+z,\cdot)$ for every $x \in \mathbb{R}^N$ and $z \in \mathbb{Z}^N$. We will denote by $\mathcal{M}_p$ the family of 1–periodic Finsler metrics $\varphi \in \mathcal{M}(\mathbb{R}^N)$, and by $\mathcal{N}$ the space of norms on $\mathbb{R}^N$ belonging to $\mathcal{M}(\mathbb{R}^N)$. For any $\phi \in \mathcal{N}$ we set

$$\|\phi\| := \max_{|\xi|=1} |\phi(\xi)|.$$

Given $\varphi \in \mathcal{M}_p$, let us set $\varphi_\varepsilon(x,\xi) := \varphi(x/\varepsilon,\xi)$ on $\mathbb{R}^N \times \mathbb{R}^N$ for every $\varepsilon > 0$. Next proposition establishes a Γ–convergence result for the functionals $\mathbb{L}_{\varphi_\varepsilon}$ defined by (2.4) (see [2]).

Proposition 2.20. *Let φ be in $\mathcal{M}_p$. The length functionals $\mathbb{L}_{\varphi_\varepsilon}$ Γ–converge to $\mathbb{L}_{\varphi^{hom}}$ in $\Gamma(\mathbb{R}^N)$ as $\varepsilon \to 0$, where $\varphi^{hom} \in \mathcal{N}$ is defined by*

$$\varphi^{hom}(\xi) = \lim_{\varepsilon\to 0^+} d_{\varphi_\varepsilon}(0,\xi) = \lim_{t\to+\infty} \frac{1}{t}\; d_\varphi(0,t\xi). \tag{2.14}$$

By Proposition 2.9 this implies $d_{\varphi_\varepsilon} \rightrightarrows d_{\varphi^{hom}}$ in $\mathcal{D}(\mathbb{R}^N)$, or, equivalently, $\varphi_\varepsilon \rightrightarrows \varphi_{hom}$ in $\mathcal{M}(\mathbb{R}^N)$. The norm φ^{hom} is called the *stable* or *asymptotic norm* of the metric φ.

We conclude this section by proving a lemma that will be crucial in the sequel. Let us denote by $Y := [-1/2, 1/2]^N$ the closed unit cube centered at 0.

Lemma 2.21. *Let $\varphi_1, \varphi_2 \in \mathcal{M}_p$. Then there exists a positive constant $C > 0$, depending on α and β only, such that, for every x, $y \in \mathbb{R}^N$,*

$$|d_{\varphi_1}(x,y) - d_{\varphi_1}(x,y)| \leq C(1+|x-y|)\,\|d_{\varphi_1} - d_{\varphi_2}\|_{L^\infty(2Y\times 2Y)}. \tag{2.15}$$

In particular if $\varphi_n \rightrightarrows \varphi$ in $\mathcal{M}_p$, then $\varphi_n^{hom} \rightrightarrows \varphi^{hom}$ in $\mathcal{N}$.

Proof. Let x, y in $\mathbb{R}^N$ and let γ be a geodesic curve for x, y with respect to the distance d_{φ_1}. Let $m := \left[2\mathcal{H}^1(\gamma)\right] + 1$ and set $t_i := i/m$ for each $i \in \{0, \dots, m\}$. Then $0 = t_0 < t_1 < \cdots < t_m = 1$ is a partition of $[0, 1]$ such that

$$\gamma([t_i, t_{i+1}]) \subset \gamma(t_i) + Y \qquad \text{for each } i = 0, \dots, m-1.$$

By 1-periodicity of φ_i, together with the fact that $m \leq C(1 + |x - y|)$ for a positive constant C depending on α and β only, we deduce

$$\begin{aligned} d_{\varphi_2}(x, y) &\leq \sum_i d_{\varphi_2}(\gamma(t_i), \gamma(t_{i+1})) \\ &\leq \sum_i d_{\varphi_1}(\gamma(t_i), \gamma(t_{i+1})) + m \, \|d_{\varphi_1} - d_{\varphi_2}\|_{L^\infty(2Y \times 2Y)} \\ &\leq d_{\varphi_1}(x, y) + C(1 + |x - y|) \, \|d_{\varphi_1} - d_{\varphi_2}\|_{L^\infty(2Y \times 2Y)}, \end{aligned}$$

and (2.15) follows by interchanging the roles of φ_1 and φ_2.

The last assertion is an easy consequence of (2.15) and of the homogenization formula (2.14). ∎

3. Homogenization of two–phase metrics

In this section we consider the problem of finding the closure (i.e. all possible limits) of two–phase metrics. We call *two–phase* metric every metric $\varphi \in \mathcal{M}(\mathbb{R}^N)$ of the form

$$\varphi(x, \xi) := \begin{cases} \alpha|\xi| & \text{if } x \in B_\alpha; \\ \beta|\xi| & \text{if } x \in B_\beta, \end{cases} \tag{3.1}$$

where B_α and B_β are two disjoint Borel sets whose union is $\mathbb{R}^N$. We denote by $\mathcal{M}^{\alpha,\beta}$ the subset of $\mathcal{M}(\mathbb{R}^N)$ given by two–phase metrics, and by $\mathcal{M}_p^{\alpha,\beta}$ the class of 1-periodic metrics of type (3.1); namely, with B_α (and B_β) 1-periodic.

3.1. The closure of $\mathcal{M}_p^{\alpha,\beta}$ is $\mathcal{N}$.

In this paragraph we will prove (see Theorem 3.1) that every norm $\phi \in \mathcal{N}$ can be approximated by a sequence φ_n in $\mathcal{M}_p^{\alpha,\beta}$. Moreover (see Theorem 3.3) ϕ is approximated also by the stable norms φ_n^{hom} obtained through (2.14).

Theorem 3.1. *Let $\phi \in \mathcal{N}$. There exists a sequence $\varphi_n \in \mathcal{M}_p^{\alpha,\beta}$ such that $\varphi_n \rightrightarrows \phi$.*

Proof. Let $(\xi)_{i\in\mathbb{N}}$ be a sequence of unit vectors with rational directions, and dense in $\mathbb{S}^{N-1}$. For each i let us denote by R_i the straight line of direction ξ_i passing through the origin. We now replicate R_i by 1–periodicity: for each $z \in \mathbb{Z}^N$, set $R_i^z := R_i + z$. For every positive integer M let us consider the Finsler metric $\phi_M \in \mathcal{M}(\mathbb{R}^N)$ defined by

$$\phi_M(x, \xi) := \begin{cases} \phi(\xi_i) \, |\xi| & \text{if } x \in R_i^z \text{ for some } i \leq M,\ z \in \mathbb{Z}^N; \\ \beta|\xi| & \text{otherwise.} \end{cases} \tag{3.2}$$

By construction, for every $i \leq M$ we have

$$d_{\phi_M}(0, t\xi_i) = |t|\phi(\xi_i) \qquad \text{for every } t \in \mathbb{R}.$$

By the homogenization formula (2.14) we deduce $\phi_M^{hom}(\xi_i) = \phi(\xi_i)$ for every $i \leq M$, which yields $\phi_M^{hom} \rightrightarrows \phi$ in $\mathcal{N}$ for $M \to +\infty$. Using a diagonal argument, the proof of the theorem reduces to approximate every ϕ_M with metrics in $\mathcal{M}_p^{\alpha,\beta}$. Namely, we have to construct a sequence of periodic Borel functions $a_n : \mathbb{R}^N \to \{\alpha, \beta\}$ such that $d_{a_n} \rightrightarrows d_{\phi_M}$.

Given $\delta > 0$, we denote by $R_i^z(\delta)$ the closed tubular neighborhood of R_i^z of width δ; i.e.,

$$R_i^z(\delta) := \{x \in \mathbb{R}^N \,:\, \mathrm{dist}(x, R_i^z) \le \delta\,\}$$

for each $1 \le i \le M$ and $z \in \mathbb{Z}^N$, and we define a zig–zag polygonal curve $Z_i^z(\delta)$ contained in $R_i^z(\delta)$ as follows: each $Z_i^z(\delta)$ lives in a 2–dimensional plane containing the straight line R_i^z (whose choice plays no role in our construction). The angle θ_i that the segment lines of $Z_i^z(\delta)$ form with the straight line R_i^z is chosen in such a way that

$$\alpha = \phi(\xi_i)\cos\theta_i. \tag{3.3}$$

Finally we can assume that every $Z_i^z(\delta)$ is 1-periodic. Let

$$a_\delta(x) := \begin{cases} \alpha & \text{if } x \in Z_i^z(\delta), \text{ for some } 1 \le i \le M \text{ and } z \in \mathbb{Z}^N; \\ \beta & \text{elsewhere.} \end{cases} \tag{3.4}$$

The map a_δ is 1–periodic and lower semicontinuous on $\mathbb{R}^N$. We want to show that every convergent subsequence $(d_{a_{\delta_n}})_n$ with $\delta_n \downarrow 0$ has d_{ϕ_M} as limit, which implies that the whole sequence converges to d_{ϕ_M}, by compactness of $\mathcal{D}(\mathbb{R}^N)$.

Let us then fix such a sequence, denoted by $(d_n)_n$ to ease notations, and let d be its limit. In order to prove that $d = d_{\phi_M}$, we will actually show that

$$\int_0^1 \varphi_d(\gamma, \dot\gamma)\,\mathrm{d}t = \int_0^1 \phi_M(\gamma, \dot\gamma)\,\mathrm{d}t \qquad \text{for every curve } \gamma \in \Gamma(\mathbb{R}^N). \tag{3.5}$$

Let us set $L := \cup_{i=1}^M \cup_{z\in\mathbb{Z}^N} R_i^z$. To prove (3.5) it will be enough to show that

$$\varphi_d(x,\xi) = \beta|\xi| = \phi_M(x,\xi) \qquad \text{on } \big(\mathbb{R}^N \setminus L\big) \times \mathbb{R}^N, \tag{3.6}$$

$$\varphi_d(x,\xi_i) = \phi_M(x,\xi_i) \qquad \text{for } \mathcal{H}^1\text{–a.e. } x \in \bigcup_{z\in\mathbb{Z}^N} R_i^z,\ 1 \le i \le M. \tag{3.7}$$

To prove (3.6), note that if $x \in \mathbb{R}^N \setminus L$ then there exist $n_x \in \mathbb{N}$ and $h_x > 0$ such that

$$\frac{d_n(x, x+h\xi)}{h} = \beta|\xi| \qquad \text{for all } n \ge n_x,\ \xi \in \mathbb{S}^{N-1} \text{ and } h \in (0, h_x).$$

Let us pass to the proof of (3.7). Let Σ be the $\mathcal{H}^1$–negligible subset of L containing all points belonging to (at least) two distinct straight lines. Pick up a point $x \in L \setminus \Sigma$. Then $x \in R_i^z$ for some $1 \le i \le M$ and $z \in \mathbb{Z}^N$. We want to show that there exists $h_x > 0$ such that

$$\lim_{n\to\infty} \frac{d_n(x, x+h\xi_i)}{h} = \phi(\xi_i) \qquad \text{for all } h \in (0, h_x). \tag{3.8}$$

This immediately implies (3.7) by letting $h \to 0$.

To this end, let us choose h_x suitably small in such a way that $\overline{B}(x, \frac{\beta}{\alpha}\,h_x)$ does not intersect any other straight line $R_j^{z'}$ different from R_i^z. Then there exists a suitably large integer n_x such that for every $n \ge n_x$ we have

$$\overline{B}(x, \frac{\beta}{\alpha}\,h_x) \cap Z_j^{z'}(\delta_n) = \emptyset \qquad \text{if either } j \ne i \text{ or } z' \ne z.$$

Fix $n \ge n_x$ and $h \in (0, h_x)$. Let γ_n be an optimal curve for $d_n(x, x+h\xi_i)$ (which does exist as a_{δ_n} is lower semicontinuous, cf. Theorem 2.2 and Remark 2.5). As

$$\alpha\,\mathcal{H}^1(\gamma_n) \le d_n(x, x+h\xi_i) \le \beta\,h_x,$$

the curve γ_n is contained in $\overline{B}(x, \frac{\beta}{\alpha}\, h_x)$, hence $Z_i^z(\delta_n)$ is the only zig–zag polygonal curve it can intersect. Let us set

$$I_1 := \{t \in [0,1] \,:\, \gamma_n(t) \in Z_i^z(\delta_n)\}, \qquad I_2 := [0,1] \setminus I_1.$$

Finally let us set

$$v_1 := \int_{I_1} \dot\gamma_n \, \mathrm{d}t, \qquad v_2 := \int_{I_2} \dot\gamma_n \, \mathrm{d}t.$$

By (3.3), we have

$$\int_{I_1} a_{\delta_n}(\gamma_n)|\dot\gamma_n| \, \mathrm{d}t = \phi(\xi_i) \int_{I_1} \cos(\theta_i)|\dot\gamma_n| \, \mathrm{d}t \geq \phi(\xi_i) \int_{I_1} \dot\gamma_n \cdot \xi_i \, \mathrm{d}t = \phi(\xi_i)\, v_1 \cdot \xi_i.$$

Moreover, by construction,

$$\int_{I_2} a_{\delta_n}(\gamma_n)|\dot\gamma_n| \, \mathrm{d}t \geq \beta |v_2| \geq \phi(\xi_i)\, v_2 \cdot \xi_i.$$

We deduce that

$$d_n(x, x + h\xi_i) = \int_0^1 a_{\delta_n}(\gamma_n)|\dot\gamma_n| \, \mathrm{d}t \geq (v_1 + v_2) \cdot \xi_i \, \phi(\xi_i) = h\phi(\xi_i),$$

and that proves that (3.8) holds with an inequality. The inverse inequality is easier, and follows directly by setting

$$\underline{h} := \min\{t \in [0,h] \,:\, x + t\xi_i \in Z_i^z(\delta_n)\}, \qquad \overline{h} := \max\{t \in [0,h] \,:\, x + t\xi_i \in Z_i^z(\delta_n)\},$$

and choosing as competitor curve in (2.4) the curve γ_n joining x to $x + h\xi_i$, obtained by gluing the polygonal curve which moves along $Z_i^z(\delta_n)$ from $x + \underline{h}\xi_i$ to $x + \overline{h}\xi_i$ with the two segments having $x, x + \underline{h}\xi_i$ and $x + \overline{h}\xi_i, x + h\xi_i$ as end–points, respectively. □

Remark 3.2. In Theorem 3.1 we can also require the sequence φ_n to be upper semicontinuous with respect to x. This can be done by a suitable modification of the metric a_δ defined in (3.4). More precisely, for every $\rho > 0$ we consider the open ρ-neighborhood $(Z_i^z(\delta))^\rho$ of $Z_i^z(\delta)$, and define $a_{\delta,\rho}$ by

$$a_{\delta,\rho}(x) := \begin{cases} \alpha & \text{if } x \in (Z_i^z(\delta))^\rho, \text{ for some } 1 \leq i \leq M \text{ and } z \in \mathbb{Z}^N \\ \beta & \text{elsewhere.} \end{cases}$$

It is easy to see that $a_{\delta,\rho}$ and its lower semicontinuous envelope $\underline{a}_{\delta,\rho}$ induce the same distance. Since $a_\delta = \sup_{\rho>0} \underline{a}_{\delta,\rho}$, we get $a_{\delta,\rho} \rightrightarrows a_\delta$ as $\rho \to 0$ in view of Proposition 2.8. The conclusion follows by a diagonal argument.

Thanks to Lemma 2.21 and Theorem 3.1 we immediately deduce the following result.

Theorem 3.3. *Let ϕ in $\mathcal{N}$. Then there exists a sequence $\varphi_n \in \mathcal{M}_p^{\alpha,\beta}$ such that $\varphi_n^{hom} \rightrightarrows \phi$.*

3.2. **Localization: the closure of $\mathcal{M}^{\alpha,\beta}$ is $\mathcal{M}(\mathbb{R}^N)$.** Here we show that Theorem 3.1 can be localized, leading to a density result of (non periodic) two–phase Finsler metrics in $\mathcal{M}(\mathbb{R}^N)$. As in [5], the strategy is to freeze the dependence on the x variable.

Theorem 3.4. *Let $\varphi \in \mathcal{M}(\mathbb{R}^N)$. Then there exists a sequence of upper semicontinuous metrics $\varphi_n \in \mathcal{M}^{\alpha,\beta}$ such that $\varphi_n \rightrightarrows \varphi$.*

Proof. Thanks to Theorem 2.10, we already know that continuous metrics are dense in $\mathcal{M}(\mathbb{R}^N)$. It is then enough to prove the statement when φ is continuous. The assertion in the general case follows via a diagonal argument.

Given $k \in \mathbb{N}$ and $\lambda \in (0,1]$, we define an upper semicontinuous metric $\varphi_k^\lambda \in \mathcal{M}(\mathbb{R}^N)$ by setting

$$\varphi_k^\lambda(x,\xi) := \begin{cases} \varphi(x_i^k,\xi) & \text{if } x \in x_i^k + (-\frac{\lambda}{2k}, \frac{\lambda}{2k})^N \quad \text{for } x_i^k \in \mathbb{Z}^N/k,\, i \in \mathbb{N} \\ \beta|\xi| & \text{elsewhere.} \end{cases} \tag{3.9}$$

The metrics φ_k^1 converge to φ locally uniformly on $\big(\mathbb{R}^N \setminus E\big) \times \mathbb{R}^N$, where E is a negligible subset of $\mathbb{R}^N$. That easily implies $d^{\varphi_k^1} \rightrightarrows d^\varphi$ in $\mathbb{R}^N \times \mathbb{R}^N$, hence $\varphi_k^1 \rightrightarrows \varphi$ by Proposition 2.18. Moreover, for each $k \in \mathbb{N}$,

$$\varphi_k^1(x,\xi) = \inf_{\lambda\in(0,1)} \varphi_k^\lambda(x,\xi) \qquad \text{for every } (x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N,$$

so Proposition 2.8 yields $\varphi_k^\lambda \rightrightarrows \varphi_k^1$ as $\lambda \to 1^-$. Therefore, using a diagonal argument, the proof reduces to approximate each φ_k^λ using metrics in $\mathcal{M}^{\alpha,\beta}$, for every $k \in \mathbb{N}$ and $\lambda \in (0,1)$.

So, let us fix $k \in \mathbb{N}$ and $\lambda \in (0,1)$. To ease notations, we will write Q_i in place of $x_i^k + \big(-\lambda/(2k), (\lambda/(2k)\big)^N$. By Theorem 3.1, for every i there exists a sequence $\phi_n^i \in \mathcal{M}^{\alpha,\beta}$ such that

$$\phi_n^i \rightrightarrows \varphi(x_i^k,\cdot) \quad \text{in } \mathcal{M}(\mathbb{R}^N) \qquad \text{for } n \to +\infty. \tag{3.10}$$

Let us define the metric φ_n in $\mathcal{M}^{\alpha,\beta}$ as follows:

$$\varphi_n(x,\xi) := \begin{cases} \phi_n^i(x,\xi) & \text{if } x \in Q_i \quad \text{for some } i \in \mathbb{N} \\ \beta|\xi| & \text{elsewhere.} \end{cases} \tag{3.11}$$

In order to conclude the proof, by the compactness of $\mathcal{D}(\mathbb{R}^N)$ it suffices to show that every convergent subsequence of $\big(d_{\varphi_n}\big)_n$ has $d_{\varphi_k^\lambda}$ as limit.

Let us then fix such a subsequence, denoted by (d_n) to ease notations, and let d be its limit. In this case, the metric length functional (2.1) associated with $d_{\varphi_k^\lambda}$ agrees on $\Gamma(\mathbb{R}^N)$ with $\mathbb{L}_{\underline{\varphi}_k^\lambda}$, where $\underline{\varphi}_k^\lambda$ denotes the lower semicontinuous envelope of φ_k^λ. Thus, to prove that $d = d_{\varphi_k^\lambda}$, it suffices to show that

$$\int_0^1 \varphi_d(\gamma,\dot\gamma)\,\mathrm{d}t = \int_0^1 \underline{\varphi}_k^\lambda(\gamma,\dot\gamma)\,\mathrm{d}t \qquad \text{for every } \gamma \in \Gamma(\mathbb{R}^N). \tag{3.12}$$

Let $x \in Q_i$ for some $i \in \mathbb{N}$. Then, for a sufficiently small $h_x > 0$, all minimal curves for L_{d_n} with end–points within $B(x,h_x)$ are entirely contained in Q_i, for each $n \in \mathbb{N}$. By (3.10), we deduce that

$$\varphi_d(x,\xi) = \varphi(x_i^k,\xi) = \underline{\varphi}_k^\lambda(x,\xi) \qquad \text{on } Q_i \times \mathbb{R}^N, \text{ for any } i \in \mathbb{N}. \tag{3.13}$$

It is furthermore easy to see that $\varphi_d(x,\xi) = \beta|\xi| = \underline{\varphi}_k^\lambda(x,\xi)$ for every $\xi \in \mathbb{R}^N$ when $x \notin \bigcup_{i\in\mathbb{N}} \overline{Q_i}$. Now, pick up a point x belonging to ∂Q_i for some $i \in \mathbb{N}$, and let ξ be a vector in $\mathbb{R}^N$ tangential to ∂Q_i at x (which exists whenever x is not a vertex of Q_i, hence at $\mathcal{H}^1$–almost every x). Then there exists a sequence of points $x_n \in Q_i$ and $h_x > 0$ such that $x_n \to x$, and $x_n + h\xi \in Q_i$ for every $h \in [0,h_x]$ and $n \in \mathbb{N}$. By (3.13) we get

$$d(x,x+h\xi) = \lim_{n\to+\infty} d(x_n,x_n+h\xi) \le \varphi(x_i^k,h\xi) \quad \text{for every } h \in [0,h_x],$$

from which we infer $\varphi_d(x,\xi) \le \varphi(x_i^k,\xi)$. To prove the opposite inequality, we note that by (3.11) $\varphi_n \ge \phi_n^i$ in a tubular neighborhood of $\overline{Q}_i \times \mathbb{R}^N$, so, for h_x small enough, we have

$$d(x, x+h\xi) = \lim_{n\to+\infty} d_n(x, x+h\xi) \ge \lim_{n\to+\infty} d_{\phi_n^i}(x, x+h\xi) = \varphi(x_i^k, h\xi)$$

for every $h \in [0, h_x]$. We conclude that, for $\mathcal{H}^1$–a.e. $x \in \partial Q_i$ and for each $i \in \mathbb{N}$,

$$\varphi_d(x,\xi) = \varphi(x_i^k,\xi) = \underline{\varphi}_k^\lambda(x,\xi) \qquad \text{for every } \xi \in \mathbb{R}^N \text{ tangential to } \partial Q_i \text{ at } x. \tag{3.14}$$

We are now ready to prove (3.12). Let $\gamma \in \Gamma(\mathbb{R}^N)$. Set

$$I_i := \{t \in [0,1] \,:\, \gamma(t) \in \partial Q_i\}, \qquad J := [0,1] \setminus \cup_{i\in\mathbb{N}} I_i.$$

For a.e. $t \in I_i$, the curve $\gamma(t)$ is tangent to ∂Q_i. In particular, by (3.14) we get

$$\varphi_d(\gamma,\dot\gamma) = \underline{\varphi}_k^\lambda(\gamma,\dot\gamma) \qquad \text{a.e. on } I_i$$

for each $i \in \mathbb{N}$. Since an analogous equality holds on J too, (3.12) follows and the proof is complete. □

4. Homogenization with prescribed volume fraction

An interesting problem in view of applications to composites is the study of the asymptotic behavior of two–phase metrics with prescribed *volume fraction* of its phases. More precisely, for a fixed $0 \le \theta \le 1$ we will denote by $\mathcal{M}_\theta^{\alpha,\beta}$ the class of metrics $\varphi \in \mathcal{M}_p^{\alpha,\beta}$ defined on the unit cube Q by

$$\varphi(x,\xi) := \begin{cases} \alpha|\xi| & \text{if } x \in E_\alpha; \\ \beta|\xi| & \text{if } x \in E_\beta, \end{cases}$$

with $|E_\alpha| = \theta$. Let us set

$$Cl(\mathcal{M}_\theta^{\alpha,\beta}) := \{\phi \in \mathcal{N} :\ \text{there exists } \varphi_n \in \mathcal{M}_\theta^{\alpha,\beta} \text{ with } \varphi_n \rightrightarrows \phi\}.$$

4.1. Some qualitative properties of $Cl(\mathcal{M}_\theta^{\alpha,\beta})$.

Lemma 4.1. *The set $Cl(\mathcal{M}_\theta^{\alpha,\beta})$ coincides with the class of all $\phi \in \mathcal{N}$ such that there exists a sequence $\varphi_n \in \mathcal{M}_{\theta_n}^{\alpha,\beta}$ with $\theta_n \to \theta$ such that $\varphi_n \rightrightarrows \phi$.*

Proof. We have to prove that if $\theta_n \to \theta$, $\varphi_n \in \mathcal{M}_{\theta_n}^{\alpha,\beta}$ and $\varphi_n \rightrightarrows \phi$ for some $\phi \in \mathcal{N}$, then $\phi \in Cl(\mathcal{M}_\theta^{\alpha,\beta})$. Let us consider for simplicity the case of θ_n increasing, the general case being very similar. By Theorem 3.3, and using a diagonal argument, we can find a sequence $\varepsilon_n \to 0$ such that, denoted by $\psi_n(x,\cdot) := \varphi_n(\frac{x}{\varepsilon_n},\cdot)$, we have $\psi_n \rightrightarrows \phi$. Moreover we can assume that $1/\varepsilon_n \in \mathbb{N}$ for every n, so that ψ_n are 1-periodic. Therefore on the unit cube Q the metric ψ_n is of the type

$$\psi_n(x,\xi) := \begin{cases} \alpha|\xi| & \text{if } x \in E_\alpha^n; \\ \beta|\xi| & \text{if } x \in E_\beta^n. \end{cases}$$

We have to suitably modify the sequence ψ_n in order to achieve the right volume fraction θ. By construction for every open subset U of Q we have that

$$(|E_\beta^n \cap U|)/|U| \to 1-\theta \qquad \text{as } n \to \infty.$$

Therefore, we can find a sequence of concentric balls B_n with vanishing radius, and a sequence of measurable sets $F^n \subset B_n \cap E^n_\beta$ such that (for n big enough) $|F^n| = \theta - \theta_n$. Then we set

$$\tilde{\psi}_n(x,\xi) := \begin{cases} \alpha|\xi| & \text{if } x \in E^n_\alpha \cup F^n; \\ \beta|\xi| & \text{if } x \in E^n_\beta \setminus F^n, \end{cases}$$

and we extend it by periodicity on $\mathbb{R}^N$. It is easily seen that $d_{\psi_n} - d_{\tilde{\psi}_n}$ uniformly converges to 0 on compact subsets of $\mathbb{R}^N \times \mathbb{R}^N$, therefore $\tilde{\psi}_n \rightrightarrows \phi$ as claimed. □

For every ϕ_1, ϕ_2 in $\mathcal{N}$, we will write $\phi_1 \le \phi_2$ if $\phi_1(\xi) \le \phi_2(\xi)$ for every $\xi \in \mathbb{R}^N$.

Lemma 4.2. *Let $\phi \in Cl(\mathcal{M}^{\alpha,\beta}_\theta)$. Then $Cl(\mathcal{M}^{\alpha,\beta}_\theta)$ contains every norm $\tilde{\phi} \in \mathcal{N}$ such that $\tilde{\phi} \le \phi$. In particular if $0 \le \theta_1 \le \theta_2 \le 1$ then $Cl(\mathcal{M}^{\alpha,\beta}_{\theta_2}) \subseteq Cl(\mathcal{M}^{\alpha,\beta}_{\theta_1})$.*

Proof. The first part of the Lemma will be proved if, for every $\varepsilon > 0$, we exhibit a norm $\varphi \in Cl(\mathcal{M}^{\alpha,\beta}_{\theta^\varepsilon})$ such that

$$|\theta^\varepsilon - \theta| < \varepsilon \qquad \text{and} \qquad |\varphi(\xi) - \tilde{\phi}(\xi)| < C\varepsilon \quad \text{for every } \xi \in \mathbb{S}^{N-1}, \tag{4.1}$$

for some constant C independent of ε. To this end, take a sequence $(\xi_i)_{i\in\mathbb{N}}$ of unit vectors with rational directions and dense in $\mathbb{S}^{N-1}$, and let ϕ_M be the metric defined in (3.2) with $\tilde{\phi}$ in place of ϕ, with $M \in \mathbb{N}$ large enough so that

$$\inf_{1\le i\le M} |\xi - \xi_i| < \varepsilon \qquad \text{for any } \xi \in \mathbb{S}^{N-1}.$$

For every $k \in \mathbb{N}$, let

$$a_k(x) := \begin{cases} \alpha & \text{if } x \in Z^z_i(1/k), \text{ for some } 1 \le i \le M \text{ and } z \in \mathbb{Z}^N \\ \beta & \text{elsewhere,} \end{cases}$$

where $Z^z_i(1/k)$ are the zig–zag polygonal curves introduced in the proof of Theorem 3.1. Arguing as in that proof, we get that $a_k \rightrightarrows \phi_M$ as $k \to +\infty$, in particular $a_k^{hom} \rightrightarrows \phi_M^{hom}$ by Lemma 2.21. Via a diagonal argument we infer that there exists a diverging sequence of integer numbers $(m_k)_k$ such that $a_k(m_k x) \rightrightarrows \phi_M^{hom}$. We recall that, by construction,

$$\phi_M^{hom} \ge \tilde{\phi} \qquad \text{and} \qquad \phi_M^{hom}(\xi_i) = \tilde{\phi}(\xi_i) \quad \text{for each } 1 \le i \le M. \tag{4.2}$$

Now pick up a sequence $(\varphi_n)_n$ in $\mathcal{M}^{\alpha,\beta}_\theta$ such that $\varphi_n \rightrightarrows \phi$. For each i let us denote by $R^z_i(\delta)$ the $\delta-$neighborhood of the sets R^z_i introduced in the proof of Theorem 3.1. For every $k \in \mathbb{N}$, let

$$\nu^\delta_k(x,\xi) := \begin{cases} a_k(m_k x)|\xi| & \text{if } x \in R^z_i(\delta), \text{ for some } 1 \le i \le M \text{ and } z \in \mathbb{Z}^N \\ \varphi_k(x,\xi) & \text{elsewhere.} \end{cases}$$

By construction, $\nu^\delta_k \in \mathcal{M}^{\alpha,\beta}_{\theta^\delta_k}$ with $\theta^\delta_k \to \theta^\delta$ as $k \to +\infty$ for some θ^δ satisfying $|\theta^\delta - \theta| \to 0$ as $\delta \to 0$. Furthermore, $\nu^\delta_k \rightrightarrows \nu^\delta$ for $k \to +\infty$, where

$$\nu^\delta(x,\xi) := \begin{cases} \phi_M^{hom}(\xi) & \text{if } x \in R^z_i(\delta), \text{ for some } 1 \le i \le M \text{ and } z \in \mathbb{Z}^N \\ \phi(\xi) & \text{elsewhere.} \end{cases}$$

In particular, by Lemma 4.1 and Lemma 2.21 we derive that $\left(\nu^\delta\right)^{hom} \in Cl(\mathcal{M}^{\alpha,\beta}_{\theta^\delta})$. Moreover, by (4.2) and from the fact that $\phi \ge \tilde{\phi}$ we infer that

$$\left(\nu^\delta\right)^{hom}(\xi_i) = \tilde{\phi}(\xi_i) \qquad \text{for each } 1 \le i \le M,$$

so

$$\inf_{1\le i\le M} \left|\left(\nu^{\delta}\right)^{hom}(\xi) - \tilde{\phi}(\xi)\right| \le \inf_{1\le i\le M} \left\{\left|\left(\nu^{\delta}\right)^{hom}(\xi-\xi_i)\right| + |\tilde{\phi}(\xi_i-\xi)|\right\} < 2\beta\,\varepsilon$$

for every $\xi \in \mathbb{S}^{n-1}$. This immediately implies (4.1) with $\varphi := (\nu^\delta)^{hom}$ for a suitably small $\delta > 0$.

The second part of the assertion is an easy consequence of the first one. Let in fact $\tilde{\phi} \in Cl(\mathcal{M}^{\alpha,\beta}_{\theta_2})$. Then there exists a sequence $\tilde{\varphi}_n \in \mathcal{M}^{\alpha,\beta}_{\theta_2}$ such that $\tilde{\varphi}_n \rightrightarrows \tilde{\phi}$. It is now very easy to construct a sequence $\varphi_n \in Cl(\mathcal{M}^{\alpha,\beta}_{\theta_1})$ with $\varphi_n \ge \tilde{\varphi}_n$ for every n (simply by switching enough phase α to phase β for each φ_n). Up to a subsequence we have that $\varphi_n \rightrightarrows \phi$ for some ϕ which by construction belongs to $Cl(\mathcal{M}^{\alpha,\beta}_{\theta_1})$ and satisfies $\phi \ge \tilde{\phi}$. We deduce that also $\tilde{\phi}$ belongs to $Cl(\mathcal{M}^{\alpha,\beta}_{\theta_1})$. □

The next Proposition clarifies the dependence of $Cl(\mathcal{M}^{\alpha,\beta}_{\theta})$ on θ.

Proposition 4.3. *The following properties hold.*

i) $Cl(\mathcal{M}^{\alpha,\beta}_{\theta}) = \bigcap_{s<\theta} Cl(\mathcal{M}^{\alpha,\beta}_{s})$ *for every* $0 < \theta \le 1$*;*

ii) $Cl(\mathcal{M}^{\alpha,\beta}_{\theta}) = \overline{\bigcup_{s>\theta} Cl(\mathcal{M}^{\alpha,\beta}_{s})}$ *for every* $0 \le \theta < 1$*.*

Proof. Let us prove property *i)*. By Proposition 4.2 we have $Cl(\mathcal{M}^{\alpha,\beta}_{\theta}) \subset Cl(\mathcal{M}^{\alpha,\beta}_{s})$ for every $s < \theta$, so that

$$Cl(\mathcal{M}^{\alpha,\beta}_{\theta}) \subseteq \bigcap_{s<\theta} Cl(\mathcal{M}^{\alpha,\beta}_{s}).$$

To prove the opposite inclusion let $\phi \in \bigcap_{s<\theta} Cl(\mathcal{M}^{\alpha,\beta}_{s})$. Using a diagonal argument we can find a sequence $\varphi_k \in \mathcal{M}^{\alpha,\beta}_{s_k}$, with $s_k \to \theta$, such that $\varphi_k \rightrightarrows \phi$. By Lemma 4.1 we deduce that $\phi \in \mathcal{M}^{\alpha,\beta}_{\theta}$, and this concludes the proof of i).

Let us pass to the proof of *ii)*. By Proposition 4.2 we deduce that $Cl(\mathcal{M}^{\alpha,\beta}_{s}) \subseteq Cl(\mathcal{M}^{\alpha,\beta}_{\theta})$ for every $s > \theta$, so that

$$\overline{\bigcup_{s>\theta} Cl(\mathcal{M}^{\alpha,\beta}_{s})} \subseteq Cl(\mathcal{M}^{\alpha,\beta}_{\theta}).$$

To prove the opposite inclusion we will use a perturbation argument similar to that used in the proof of Lemma 4.1. Let $\phi \in Cl(\mathcal{M}^{\alpha,\beta}_{\theta})$, and let φ_k be a sequence in $\mathcal{M}^{\alpha,\beta}_{\theta}$ such that $\varphi_k \rightrightarrows \phi$. We can find a sequence of balls $B_{r_k}(x_k)$ with vanishing radius such that the metrics $\tilde{\varphi}_k$ defined by

$$\tilde{\varphi}_k(x,\xi) := \begin{cases} \alpha|\xi| & \text{if } x \in B_{r_k}(x_k) + \mathbb{Z}^N; \\ \varphi_k(x,\xi) & \text{elsewhere,} \end{cases}$$

are in $\mathcal{M}^{\alpha,\beta}_{s_k}$, with $s_k > \theta$, and $s_k \to \theta$. It is very easy to see that $\tilde{\varphi}_k \rightrightarrows \phi$, and this concludes the proof of the proposition. □

Remark 4.4. Proposition 4.3 implies in particular that the multifunction $\theta \mapsto Cl(\mathcal{M}^{\alpha,\beta}_{\theta})$ is continuous with respect to the Hausdorff convergence of compact subsets of $\mathcal{N}$ (see for instance [24]).

We conclude the paragraph with a lemma which cannot be derived directly from Proposition 4.3, and which will be used in the proofs of our next results.

Lemma 4.5. *Let $0 \le \tilde\theta < \theta \le 1$, and let $\phi \in Cl(\mathcal{M}_\theta^{\alpha,\beta})$. Then ϕ is internal to $Cl(\mathcal{M}_{\tilde\theta}^{\alpha,\beta})$; i.e., there exists $r > 0$ such that*

$$\{\psi \in \mathcal{N} : \|\psi - \phi\| < r\} \subset Cl(\mathcal{M}_{\tilde\theta}^{\alpha,\beta}).$$

Proof. Let $\varphi_k \in \mathcal{M}_\theta^{\alpha,\beta}$ with $\varphi_k \rightrightarrows \phi$. Let moreover U be an open neighborhood of ∂Q with respect to the relative topology of Q, such that $0 < |U| < 1 - \tilde\theta/\theta$. Let $M_k \to \infty$, and let us denote by $\tilde\varphi_k$ the metrics in $\mathcal{M}_p^{\alpha,\beta}$ defined on Q by

$$\tilde\varphi_k(x,\xi) := \begin{cases} \beta|\xi| & \text{if } x \in U; \\ \varphi_k(M_k x, \xi) & \text{elsewhere.} \end{cases}$$

Up to a subsequence, we have that $\tilde\varphi_k \rightrightarrows \tilde\varphi$ for some $\tilde\varphi \in \mathcal{M}_p$ satisfying

$$\begin{aligned} \tilde\varphi(x,\xi) &= \beta|\xi| && \text{if } x \in U; \\ \tilde\varphi(x,\xi) &= \phi(\xi) && \text{if } x \in Q \setminus \overline{U}; \\ \tilde\varphi(x,\xi) &\ge \phi(\xi) && \text{if } x \in \partial U. \end{aligned}$$

Moreover by construction, the volume fraction relative to the metric $\tilde\varphi_k$ is definitively bigger then $\tilde\theta$. Therefore $\tilde\varphi^{hom} \in Cl(\mathcal{M}_{\tilde\theta}^{\alpha,\beta})$. Applying the homogenization formula (2.14) we immediately deduce that

$$\tilde\varphi^{hom}(\xi) > \phi(\xi) + r|\xi|,$$

for some r depending on U. In view of Lemma 4.2 the proof is now complete. □

4.2. **Bounds for $Cl(\mathcal{M}_\theta^{\alpha,\beta})$.** Here we provide upper and lower bounds for the set $Cl(\mathcal{M}_\theta^{\alpha,\beta})$.

Theorem 4.6. *The following bounds for $Cl(\mathcal{M}_\theta^{\alpha,\beta})$ hold for every $0 \le \theta \le 1$.*

i) The class $Cl(\mathcal{M}_\theta^{\alpha,\beta})$ contains every norm $\phi \in \mathcal{N}$ satisfying

$$\phi(\xi) \le (\theta^{1/N}\alpha + (1-\theta^{1/N})\beta)|\xi| \quad \text{for every } \xi \in \mathbb{R}^N;$$

ii) every norm $\phi \in Cl(\mathcal{M}_\theta^{\alpha,\beta})$ satisfies

$$\phi(\xi) \le (\theta\alpha + (1-\theta)\beta)|\xi| \qquad \text{for every } \xi \in \mathbb{R}^N.$$

Proof. Let us start by proving *i*). In view of Lemma 4.2, it is enough to exhibit a geometry (E_α, E_β) with volume fraction θ, such that the norm $\varphi^{hom} \in \mathcal{N}$ associated with the metric

$$\varphi(x,\xi) := \begin{cases} \alpha|\xi| & \text{if } x \in E_\alpha; \\ \beta|\xi| & \text{if } x \in E_\beta, \end{cases}$$

satisfies the following inequality.

$$\varphi(\xi) \ge \left(\theta^{1/N}\alpha + (1-\theta^{1/N})\beta\right)|\xi| \quad \text{for every } \xi \in \mathbb{R}^N. \tag{4.3}$$

To this aim, let B_i be a family of disjoint balls in the unit cube Q, with center x_i and radius r_i, which cover Q in measure (i.e. such that $Q \setminus \cup_i B_i$ has zero Lebesgue measure). For every i we denote by B_i^α the ball contained in B_i, with the same center x_i and with radius $r_i^\alpha := \theta^{1/N} r_i$. We set $E_\alpha := \cup_i B_i^\alpha$, and $E_\beta := Q \setminus E_\alpha$. By the fact that $\frac{r_i^\alpha}{r_i} = \theta^{1/N}$ it easily follows that the volume fraction of the phase E_α is equal to θ.

To prove (4.3) let us fix p, $q \in \mathbb{R}^N$, and let γ_ε be the geodesic curve joining p and q with respect to the metric $\varphi_\varepsilon(x,\xi) := \varphi(x/\varepsilon,\xi)$. Using the rough idea that the curve γ_ε has to pass through β to achieve the α phase, it is easy to prove that

$$\lim_{\varepsilon\to 0^+} \frac{\mathcal{H}^1\big(\gamma_\varepsilon\left([0,1]\right)\cap E_\alpha\big)}{\mathcal{H}^1(\gamma_\varepsilon)} \le \theta^{1/N},$$

from which, in view of the homogenization formula (2.14), we deduce (4.3).

Now let us pass to the proof of *ii*). We have to prove that any norm $\phi \in \mathcal{M}^{\alpha,\beta}_p$ is less than or equal to $\theta\alpha + (1-\theta)\beta$ times the Euclidean norm. This inequality can be easily deduced from Theorem 4.2 in [5] with $\Omega := \mathbb{R}^N$ and $F(x,s) := \chi_Q(x)s$. For the reader's convenience, we give a proof in our setting. In view of Theorem 3.3 it is enough to prove that for every $\xi \in \mathbb{S}^{N-1}$ with rational direction, and for every $\varphi \in Cl(\mathcal{M}^{\alpha,\beta}_\theta)$ we have

$$\varphi^{hom}(\xi) \le \big(\theta\alpha + (1-\theta)\beta\big)|\xi|. \tag{4.4}$$

By Fubini's Theorem there exists a vector η orthogonal to ξ such that

$$\lim_{L\to\infty} \frac{\mathcal{H}^1\big(\{l\xi+\eta \,:\, l\in[-L,L]\}\cap\big(E_\alpha+\mathbb{Z}^N\big)\big)}{L} \ge \theta.$$

Applying the homogenization formula (2.14), we immediately deduce that (4.4) holds, and this concludes the proof of the theorem. □

4.3. **The localization Theorems.** For every $\varphi \in \mathcal{M}(\mathbb{R}^N)$ we denote by $\theta_\varphi : \mathbb{R}^N \to [0,1]$ the function defined as

$$\theta_\varphi(x) := \max\{\theta\in[0,1] :\, \varphi(x,\cdot)\in Cl(\mathcal{M}^{\alpha,\beta}_\theta)\} \qquad \text{for every } x\in\mathbb{R}^N.$$

When $\varphi \in \mathcal{M}^{\alpha,\beta}$, $\theta_\varphi(\cdot) = \chi_{B_\alpha}(\cdot)$, where, we recall, $B_\alpha := \{x\in\mathbb{R}^N \,:\, \varphi(x,\cdot) = \alpha|\cdot|\,\}$. We will identify θ_ϕ with its constant value whenever ϕ belongs to $\mathcal{N}$.

Lemma 4.7. *The following properties hold.*

- *i) If $\phi_n \to \phi$ in $\mathcal{N}$, then $\theta_{\phi_n} \to \theta_\phi$.*
- *ii) If $\varphi \in \mathcal{M}$ is continuous, then θ_φ is a continuous function from $\mathbb{R}^N$ to $[0,1]$.*
- *iii) Let $\varphi, \varphi_n \in \mathcal{M}$ be such that $\varphi_n(x,\cdot)$ converge pointwise to $\varphi(x,\cdot)$ for a.e. $x\in\mathbb{R}^N$. Then θ_{φ_n} converge to θ_φ almost everywhere on $\mathbb{R}^N$.*
- *iv) For every $\varphi\in\mathcal{M}$, the function θ_φ is measurable.*

Proof. Let us prove property *i*). Up to a subsequence, we have $\theta_{\phi_n} \to \tilde\theta$ for some $\tilde\theta\in[0,1]$. By Proposition 4.3 we have that $\phi \in Cl(\mathcal{M}^{\alpha,\beta}_{\tilde\theta})$, so that $\tilde\theta \le \theta_\phi$. Let us assume by contradiction that $\tilde\theta < \theta < \theta_\phi$ for some $\theta\in(0,1)$. By Lemma 4.5, there exists a neighborhood of ϕ contained in $Cl(\mathcal{M}^{\alpha,\beta}_\theta)$. Therefore, for n big enough, we deduce that ϕ_n belong to $Cl(\mathcal{M}^{\alpha,\beta}_\theta)$, hence $\theta \le \theta_{\phi_n}$, which is in contradiction with $\theta_{\phi_n} \to \tilde\theta$.

Properties *ii*) and *iii*) are a direct consequence of (*i*). To prove (*iv*), it is enough to observe that the metrics $\varphi_n := \rho_n * \varphi$, where ρ_n is a sequence of convolution kernels, are continuous and converge to φ, almost everywhere with respect to x. In view of property *iii*), we deduce that θ_φ is a.e. limit of a sequence of continuous functions, so it is measurable. □

The weak–star convergence in $L^\infty(\mathbb{R}^N)$ of a sequence of measurable functions $\theta_n : \mathbb{R}^N \to [0,1]$ to θ will be hereafter denoted by $\theta_n \overset{*}{\rightharpoonup} \theta$.

Now we are in a position to give our main localization theorems.

Theorem 4.8. *Let $\varphi \in \mathcal{M}(\mathbb{R}^N)$ such that $\varphi(x,\cdot) \in \mathcal{M}^{\alpha,\beta}_{\theta(x)}$ for a.e. $x \in \mathbb{R}^N$, for some measurable function $\theta : \mathbb{R}^N \to [0,1]$. Then there exists a sequence of upper semicontinuous metrics $\varphi_n \in \mathcal{M}^{\alpha,\beta}$ such that $\varphi_n \rightrightarrows \varphi$ and $\theta_{\varphi_n} \overset{*}{\rightharpoonup} \theta$.*

Proof. The proof is divided in three steps.

Step 1: φ is continuous and $\theta = \theta_\varphi$.
The result is achieved by suitably modifying the contruction provided in the proof of Theorem 3.4. For each $k \in \mathbb{N}$ and $\lambda \in (0,1)$, let φ_k^λ be the upper semicontinuous metric in $\mathcal{M}$ defined as in (3.9). We notice that

$$\theta_{\varphi_k^\lambda}(x) = \begin{cases} \theta_\varphi(x_i^k) & \text{if } x \in x_i^k + (-\frac{\lambda}{2k}, \frac{\lambda}{2k})^N \text{ for some } i \in \mathbb{N} \\ \beta & \text{otherwise.} \end{cases}$$

Thanks to the continuity of θ_φ (cf. Lemma 4.7), we have that $\varphi_{k_n}^{\lambda_n} \rightrightarrows \varphi$ and $\theta_{\varphi_{k_n}^{\lambda_n}} \overset{*}{\rightharpoonup} \theta_\varphi$ as $k_n \to +\infty$ and $\lambda_n \nearrow 1$. On the other hand, any metric φ_k^λ can be approximated by a sequence of upper semicontinuous metrics $\psi_n \in \mathcal{M}^{\alpha,\beta}$ such that $\psi_n \rightrightarrows \varphi_k^\lambda$ and $\theta_{\psi_n} \overset{*}{\rightharpoonup} \theta_{\varphi_k^\lambda}$: it suffices to define each ψ_n as in (3.11) for some $\psi_n^i \in \mathcal{M}^{\alpha,\beta}_{\theta_\varphi(x_i^k)}$ satisfying (3.10), for each $i \in \mathbb{N}$. The assertion now follows via a diagonal argument.

Step 2: $d_\varphi = d^\varphi$ and $\theta = \theta_\varphi$.
Let $(\rho_n)_n$ be a sequence of standard mollifiers, and set $\varphi_n := \rho_n * \varphi$ for each $n \in \mathbb{N}$. As $\varphi_n(x,\cdot)$ converge pointwise to $\varphi(x,\cdot)$ for a.e. $x \in \mathbb{R}^N$, by Lemma 4.7 θ_{φ_n} converge to θ_φ almost everywhere on $\mathbb{R}^N$, in particular $\theta_{\varphi_n} \overset{*}{\rightharpoonup} \theta_\varphi$. Furthermore, $\varphi_n \rightrightarrows \varphi$ by Proposition 2.19. Since each φ_n is continuous, the assertion follows from Step 1 via a diagonal argument.

Step 3: the general case.
Pick up a metric $\tilde\varphi \in \mathcal{M}(\mathbb{R}^N)$ with $\tilde\varphi(x,\cdot) = \varphi(x,\cdot)$ for almost every $x \in \mathbb{R}^N$ such to satisfy $d_{\tilde\varphi} = d^{\tilde\varphi}$ (which does exist in force of Theorem 3.1 in [8]). By Step 2, there exists a sequence of upper semicontinuous metrics $\tilde\varphi_n \in \mathcal{M}^{\alpha,\beta}$ such that $\tilde\varphi_n \rightrightarrows \tilde\varphi$ and $\theta_{\tilde\varphi_n} \overset{*}{\rightharpoonup} \theta_{\tilde\varphi} = \theta_\varphi$. Since by definition $\theta \le \theta_{\tilde\varphi} = \theta_\varphi$, we can modify the sequence $\tilde\varphi_n$ by suitably adding phase β somewhere, obtaining a new sequence $\eta_n \in \mathcal{M}^{\alpha,\beta}$ converging (up to a subsequence) to some metric $\eta \ge \tilde\varphi$, such that $\theta_{\eta_n} \overset{*}{\rightharpoonup} \theta$. We want now to modify the metrics η_n in order to get convergence to the metric φ, keeping the convergence of the volume fractions. As in [15], the idea is to modify the metric along geodesics for the distance d.

Let $S := \{(x_i, y_i)\}_{i\in\mathbb{N}}$ be a dense subset of $\mathbb{R}^N \times \mathbb{R}^N$. For each $i \in \mathbb{N}$, let γ_i be a geodesic for φ_d connecting x_i with y_i (which does exist by Theorem 2.2 and Proposition 2.6) and set $\Gamma_i := \gamma_i([0,1])$. For every $\delta > 0$ and $M \in \mathbb{N}$, let

$$T_M^\delta := \{x \in \mathbb{R}^N : \operatorname{dist}\left(x, \cup_{i=1}^M \Gamma_i\right) < \delta\}.$$

Clearly, $\lim_{\delta\to 0^+} |T_M^\delta| = 0$ for any fixed $M \in \mathbb{N}$. For each $M \in \mathbb{N}$, let $\delta = \delta(M)$ be such that $|T_M^{2\delta(M)}| < 1/M$ and define a sequence of upper semicontinuous metrics $(\varphi_n^M)_n$ as

$$\varphi_n^M(x,\xi) := \begin{cases} \psi_n(x,\xi) & \text{if } x \in T_M^{\delta(M)} \\ \beta|\xi| & \text{if } x \in \overline{T_M^{2\delta(M)}} \setminus T_M^{\delta(M)} \\ \eta_n(x,\xi) & \text{elsewhere,} \end{cases}$$

where ψ_n is a sequence of upper semicontinuous metrics in $\mathcal{M}^{\alpha,\beta}$ such that $\psi_n \rightrightarrows \varphi_d$, chosen according to Theorem 3.4. Up to subsequences, we have that

$$\theta_{\varphi_n^M} \overset{*}{\rightharpoonup} \theta^M \qquad \text{as } n \to +\infty \tag{4.5}$$

for some measurable function $\theta^M : \mathbb{R}^N \to [0,1]$, and since $\lim\limits_{M\to+\infty} |T_M^{2\delta(M)}| = 0$, it is easy to see that

$$\theta^M \overset{*}{\rightharpoonup} \theta \qquad \text{when } M \to +\infty. \tag{4.6}$$

We now claim that there is a diverging sequence $(k_n)_n$ such that $\varphi_{k_n}^n \rightrightarrows \varphi$, which is enough to conclude in view of (4.5) and (4.6). In fact, let d^M be an accumulation point for $(d_{\varphi_n^M})_n$. It is fairly easy to show that

$$\begin{array}{rcll} \varphi_{d^M}(x,\cdot) & = & \varphi_d(x,\cdot) & \quad \text{for every } x \in T_M^{\delta(M)}, \\ \varphi_{d^M}(x,\cdot) & \geq & \varphi_d(x,\cdot) & \quad \text{elsewhere,} \end{array}$$

hence $d^M \geq d$. On the other hand, we have that

$$d^M(x_i,y_i) \leq d(x_i,y_i) \qquad \text{for every } i \leq M,$$

which is apparent by choosing as a competitor curve for $d^M(x_i,y_i)$ the d–geodesic γ_i connecting x_i to y_i. We derive

$$\lim_{M\to+\infty} d^M(x_i,y_i) = d(x_i,y_i) \qquad \text{for every } (x_i,y_i) \in S,$$

which in fact yields $d^M \rightrightarrows d$ since $\mathcal{D}(\mathbb{R}^N)$ is compact and S is dense in $\mathbb{R}^N \times \mathbb{R}^N$. The conclusion follows via a diagonal argument. □

Remark 4.9. By Theorem 4.8 we deduce in particular that, given a metric $\varphi \in \mathcal{M}(\mathbb{R}^N)$, there exists a sequence of upper semicontinuous metrics $\varphi_n \in \mathcal{M}^{\alpha,\beta}$ such that $\varphi_n \rightrightarrows \varphi$ and $\theta_{\varphi_n} \overset{*}{\rightharpoonup} \theta_\varphi$.

We record for later use:

Remark 4.10. Let $\varphi \in \mathcal{M}(\mathbb{R}^N)$ be such that d_φ is intrinsic and $\theta = \theta_\varphi$. If φ is constantly equal to $\beta\,|\cdot|$ outside an open subset Ω of $\mathbb{R}^N$, the approximating metrics $\varphi_n \in \mathcal{M}^{\alpha,\beta}$ in the statement of Theorem 4.8 can be taken of the same form.

Indeed, let $\Omega_k := \{x \in \mathbb{R}^N : \mathrm{dist}(x, \mathbb{R}^N \setminus \Omega) > 1/k\}$ for each $k \in \mathbb{N}$ and set

$$\psi_k(x,\xi) = \chi_{\Omega_k}(x)\,\varphi(x,\xi) + \chi_{\mathbb{R}^N\setminus\Omega_k}(x)\,\beta|\xi| \qquad \text{for any } (x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N.$$

Then by Proposition 2.8$\psi_k(x,\cdot) \in \mathcal{M}^{\alpha,\beta}_{\theta(x)}$ for a.e. $x \in \mathbb{R}^N$, and $\psi_k \rightrightarrows \varphi$; so it is suffices to prove the assertion for each ψ_k. Let $(\rho_h)_h$ be a sequence of standard mollifiers. Then the metrics $\rho_h * \psi_k \rightrightarrows \psi_k$ in $\mathcal{M}(\mathbb{R}^N)$, and each $\rho_h * \psi_k$ is continuous and constantly equal to $\beta|\cdot|$ outside $\Omega_{k(h)}$ for some $k(h) > k$. Looking back at the proof of Theorem 4.8, it is now easy to see that, for each $\rho_h * \psi_k$, the approximating metrics $(\varphi_n^{k,h})_n \subset \mathcal{M}^{\alpha,\beta}$ can be chosen identically equal to $\beta|\cdot|$ outside Ω. The assertion last follows via a diagonal argument.

Theorem 4.11. *Let $(\varphi_k) \subset \mathcal{M}^{\alpha,\beta}$ such that $d_{\varphi_k} \rightrightarrows d$ for some $d \in \mathcal{D}(\mathbb{R}^N)$ and $\theta_{\varphi_k} \overset{*}{\rightharpoonup} \theta$ for some $\theta \in L^\infty(\mathbb{R}^N)$. Then the norm $\varphi_d(x,\cdot)$ belongs to $Cl(\mathcal{M}^{\alpha,\beta}_{\theta(x)})$ for a.e. $x \in \mathbb{R}^N$.*

Proof. Let $(\xi)_{i\in\mathbb{N}}$ be a sequence of unit vectors with rational direction, and dense in $\mathbb{S}^{N-1}$. We denote by E the set of Lebesgue points shared by θ and by the functions $\varphi(\cdot,\xi_i)$, and we assume without loss of generality that this functions take their Lebesgue value at each Lebesgue point; i.e.,

$$\lim_{r\to 0^+} \fint_{B_r(x)} \theta(y)\,\mathrm{d}y = \theta(x) \qquad \text{for every } x \in E, \tag{4.7}$$

$$\lim_{r\to 0^+} \fint_{B_r(x)} \varphi(y,\xi_i)\,\mathrm{d}y = \varphi(x,\xi_i) \qquad \text{for every } x \in E \text{ and } i \in \mathbb{N}. \tag{4.8}$$

Since $\mathbb{R}^N \setminus E$ is negligible, the theorem will be proved if for every fixed $x \in E$ we show that $\varphi_d(x,\cdot)$ belongs to $Cl(\mathcal{M}^{\alpha,\beta}_{\theta(x)})$.

Let us fix such $\bar{x} \in E$. For every ε, $\delta > 0$, in view of (4.7), (4.8) there exist a radius $r = r(\varepsilon,\delta)$ and an open set $A = A_{\varepsilon,\delta}$, with $|A| \le \delta\, r^N$, such that

$$\varphi_d(y,\xi) > \varphi_d(\bar{x},\xi) - \varepsilon|\xi| \qquad \text{for every } y \in Q_r(\bar{x}) \setminus A \text{ and } \xi \in \mathbb{S}^{N-1}, \tag{4.9}$$

where $Q_r(\bar{x})$ denotes the closed square of center $\bar{x}$ and side r. Let U be an open neighborhood of $\partial Q_r(\bar{x})$ with respect to the relative topology of $Q_r(\bar{x})$, such that $0 < |U| < \delta\, r^N$. Let us consider the metrics $\varphi_k^{\varepsilon,\delta}$, defined on $Q_r(\bar{x})$ by

$$\varphi_k^{\varepsilon,\delta}(x,\xi) := \begin{cases} \beta|\xi| & \text{if } x \in A \cup U \\ \varphi_k(x,\xi) & \text{otherwise in } Q_r(\bar{x}), \end{cases} \tag{4.10}$$

and replicate them by r–periodicity on the whole $\mathbb{R}^N$. Without loss of generality, we can also assume they are 1–periodic. Denoting by $\theta_k^{\varepsilon,\delta}$ the volume fraction associated with $\varphi_k^{\varepsilon,\delta}$, we have that (up to a subsequence) $\theta_k^{\varepsilon,\delta} \to \theta^{\varepsilon,\delta}$, for some $\theta^{\varepsilon,\delta}$ satisfying

$$|\theta^{\varepsilon,\delta} - \theta(\bar{x})| \to 0 \quad \text{as } \delta \to 0, \text{ uniformly with respect to } \varepsilon. \tag{4.11}$$

Up to a subsequence, we have that the distances d_k associated with $\varphi_k^{\varepsilon,\delta}$ converge to a distance $\tilde{d}$ whose derivative, denoted by $\varphi^{\varepsilon,\delta}$, satisfies the following inequality, in view of (4.9), (4.10):

$$\varphi^{\varepsilon,\delta}(x,\xi) \ge \varphi_d(\bar{x},\xi) - \varepsilon|\xi| \qquad \text{for every } x,\ \xi \in \mathbb{R}^N.$$

We deduce the following inequality for the stable norm $(\varphi^{\varepsilon,\delta})^{hom}$ associated with $\varphi^{\varepsilon,\delta}$:

$$(\varphi^{\varepsilon,\delta})^{hom}(\xi) \ge \max\{\varphi_d(\bar{x},\xi) - \varepsilon|\xi|, \alpha|\xi|\} \qquad \text{for every } \xi \in \mathbb{R}^N.$$

By Lemma 4.2 we have that the norm $\xi \mapsto \max\{\varphi_d(\bar{x},\xi) - \varepsilon|\xi|, \alpha|\xi|\}$ belongs to $Cl(\mathcal{M}^{\alpha,\beta}_{\theta^{\varepsilon,\delta}})$. Letting ε, $\delta \to 0$, by (4.11) and by Lemma 4.1 we conclude via a diagonal argument that $\varphi_d(\bar{x},\cdot) \in Cl(\mathcal{M}^{\alpha,\beta}_{\theta(\bar{x})})$. ∎

Remark 4.12. Consider a sequence $(\varphi_k)_k$ of metrics in $\mathcal{M}^{\alpha,\beta}$ with $\theta_{\varphi_k} \stackrel{*}{\rightharpoonup} \theta$ for some $\theta \in L^\infty(\mathbb{R}^N)$, and $\varphi_k \rightrightarrows \varphi$ for some $\varphi \in \mathcal{M}^{\alpha,\beta}$. In general, if $\varphi \neq \varphi_{d_\varphi}$, we can not conclude that $\varphi(x,\cdot) \in Cl(\mathcal{M}^{\alpha,\beta}_{\theta(x)})$ for a.e. $x \in \mathbb{R}^N$. In fact, let us consider the constant sequence $\varphi_k(x,\xi) := \alpha|\xi|$, and let φ be a metric defined by

$$\varphi(x,\xi) := \begin{cases} \alpha|\xi| & \text{if } x \in \Sigma \\ \beta|\xi| & \text{elsewhere,} \end{cases} \tag{4.12}$$

where Σ is the (countable) union of segments whose extreme points belong to $\mathbb{Q}^N$. It is clear that φ induces the same distance as each φ_k. On the other hand $\varphi(x,\cdot) \notin \mathcal{M}^{\alpha,\beta}_1$ for a.e. $x \in \mathbb{R}^N$.

Let us also observe that, in general, θ can be strictly lower than θ_{φ_d}. In fact, let φ_k be the constant sequence defined as in (4.12). In this case φ_d is identically equal to $\alpha|\xi|$, so that $\theta_{\varphi_d} = 1$, while $\theta_{\varphi_k} \stackrel{*}{\rightharpoonup} 0$.

5. Applications to two–phase gradient constraint functionals

In this section we give an application of our results to the problem of the Γ–convergence of two–phase gradient constraint functionals. In what follows, Ω will denote a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. The space $W^{1,\infty}(\Omega)$ is endowed with the metric of uniform convergence in Ω, and Γ–convergence of functionals on $W^{1,\infty}(\Omega)$ will be always meant with respect to this topology.

5.1. **Two–phase gradient constraints and supremal functionals.**

Let $C : \Omega \to \mathbb{R}$ be a two–phase function of the form

$$C(x) := \begin{cases} \alpha & \text{if } x \in B_\alpha \\ \beta & \text{if } x \in B_\beta, \end{cases} \tag{5.1}$$

where B_α, B_α are disjoint Borel sets whose union is Ω. The constraint functional associated with $C(\cdot)$ is the functional $G : W^{1,\infty}(\Omega) \to \mathbb{R}$ defined by

$$G(u) := \begin{cases} 0 & \text{if } |\nabla u(x)| \le C(x) \text{ for a.e. } x \in \Omega; \\ +\infty & \text{otherwise.} \end{cases} \tag{5.2}$$

To any such $C(\cdot)$ we can associate the supremal functional

$$F(u) := \operatorname*{ess\,sup}_{\Omega} \frac{1}{C(x)}\,|\nabla u(x)| \qquad \text{for every } u \in W^{1,\infty}(\Omega). \tag{5.3}$$

The relation between G and F is the following:

$$G(u) = 0 \quad \text{if and only if} \quad F(u) \le 1. \tag{5.4}$$

The following fact is a trivial consequence of our definitions:

Proposition 5.1. *Let $(C_n)_n$ be a sequence of functions of the type* (5.1)*, and let G_n and F_n be the corresponding functionals defined by* (5.2) *and* (5.4)*, respectively. If F_n Γ-converge in $W^{1,\infty}(\Omega)$ to some functional F, then G_n Γ-converge in $W^{1,\infty}(\Omega)$ to the functional G defined by*

$$G(u) := \begin{cases} 0 & \textit{if } F(u) \leq 1; \\ +\infty & \textit{otherwise.} \end{cases}$$

Therefore the asymptotic behavior of constraint functionals can be studied in the equivalent setting of Γ-convergence of supremal functionals.

To every supremal functional F of the kind considered above we can associate the distance $d_F \in \mathcal{D}(\Omega)$ defined as

$$d_F(x,y) := \sup\{u(y) - u(x) \,:\, u \in W^{1,\infty}(\Omega),\, F(u) \leq 1\} \tag{5.5}$$

for every $x, y \in \Omega$. The distance d_F is an intrinsic distance on Ω, according to Definition 2.15. Indeed, d_F is nothing but the distance d^φ given by (2.11), where $\varphi(x,\xi) := C(x)|\xi|$.

To study the asymptotic behavior of supremal functionals it is convenient to use an alternative representation in terms of *difference quotient functionals* given in [25]: any supremal functional F of the type (5.3) coincides with the difference quotient functional R^{d_F} defined by

$$R^{d_F}(u) := \sup_{x,\, y\in\Omega,\, x\neq y} \frac{u(y) - u(x)}{d_F(x,y)} \qquad \text{for every } u \in W^{1,\infty}(\Omega). \tag{5.6}$$

On the other hand, if $d \in \mathcal{D}(\Omega)$ and φ_d is its derivative obtained through (2.5), the quotient functional R^d can be expressed in a supremal form whenever d is intrinsic. More precisely we have

$$R^d(u) = \operatorname*{ess\,sup}_{\Omega} \ \varphi_d^*(x, \nabla u(x)) \qquad \text{for every } u \in W^{1,\infty}(\Omega), \tag{5.7}$$

where φ_d^* is the metric associated to φ_d by duality according to (2.10).

Remark 5.2. Note that, for every $\varphi \in \mathcal{M}^{\alpha,\beta}(\Omega)$, the function φ^* is given by

$$\varphi^*(x,\xi) := \begin{cases} \beta'\,|\xi| & \text{if } x \in B_\alpha; \\ \alpha'\,|\xi| & \text{if } x \in B_\beta, \end{cases}$$

where $\beta' = 1/\alpha$ and $\alpha' = 1/\beta$.

The following proposition is a particular case of [26, Proposition 4.1].

Proposition 5.3. *Let F_n be a sequence of two–phase supremal functionals of the kind* (5.3)*, and let d_{F_n} be the associated distances, defined via* (5.5)*. If d_{F_n} converge to some $d \in \mathcal{D}(\overline{\Omega})$, the functionals F_n Γ-converge in $W^{1,\infty}(\Omega)$ to the difference quotient functional R^d defined according to* (5.6)*.*

5.2. **Γ–closure of two–phase constraint functionals.** In this paragraph we study the Γ–closure of two–phase constraint functionals. For a function $C(\cdot)$ of the type (5.1) we denote by $\theta_C(\cdot) := \chi_{B_\alpha}(\cdot)$. The first result we give in this direction is a consequence of Proposition 5.1, Proposition 5.3 and Theorem 4.11.

Theorem 5.4. *Let $(C_n)_n$ be a sequence of functions of the kind* (5.1)*, and denote by G_n the associated constraint functionals defined via* (5.2)*, and d^{C_n} the distances defined via* (5.3)*,* (5.5). *Assume that* d^{C_n} *converge to some* $d \in \mathcal{D}(\overline{\Omega})$. *Then the functionals* G_n Γ*-converge in* $W^{1,\infty}(\Omega)$ *to the functional* G *defined by*

$$G(u) := \begin{cases} 0 & \text{if } R^d(u) \leq 1; \\ +\infty & \text{otherwise,} \end{cases}$$

where R^d *is the difference quotient functional defined according to* (5.6).

Moreover, if $\theta_{C_n} \overset{*}{\rightharpoonup} \theta$, *then* $\varphi_d(x, \cdot) \in Cl(\mathcal{M}^{\alpha,\beta}_{\theta(x)})$ *for a.e.* $x \in \Omega$.

Proof. The Γ-convergence of G_n to G follows immediately from Proposition 5.1 and Proposition 5.3. To prove that $\varphi_d(x, \cdot) \in Cl(\mathcal{M}^{\alpha,\beta}_{\theta(x)})$, note that up to modifying C_n on a negligible subset of Ω, we can always assume that $d_{C_n} = d^{C_n}$ in $\mathcal{D}(\Omega)$ for each $n \in \mathbb{N}$ (cf. Theorem 2.16 and Remark 2.17). Next we extend each C_n to $\mathbb{R}^N$ by setting $\overline{C}_n(x) = C_n(x)\,\chi_\Omega(x) + \beta\,\chi_{\mathbb{R}^N\setminus\Omega}(x)$ for every $x \in \mathbb{R}^N$. Clearly $\theta_{\overline{C}_n} \overset{*}{\rightharpoonup} \theta\chi_\Omega$ in $L^\infty(\mathbb{R}^N)$. Up to subsequences, the associated distances $d_{\overline{C}_n} \in \mathcal{D}(\mathbb{R}^N)$ converge to some $\delta \in \mathcal{D}(\mathbb{R}^N)$, hence, by Theorem 4.11,

$$\varphi_\delta(x, \cdot) \in Cl(\mathcal{M}^{\alpha,\beta}_{\theta(x)}) \qquad \text{for a.e. } x \in \Omega. \tag{5.8}$$

Now note that δ locally coincides with d in Ω; i.e., for every $x_0 \in \Omega$ there exists $r > 0$ such that

$$d(x, y) = \delta(x, y) \qquad \text{for every } x, y \in B_r(x_0).$$

That follows by the fact that, for r suitably small, any minimizing sequence of curves for $\delta(x, y)$ is definitively contained in Ω, where $\overline{C}_n$ and C_n agree. We infer that $\varphi_d(x, \cdot) = \varphi_\delta(x, \cdot)$ for every $x \in \Omega$, and we conclude by (5.8). $\square$

The converse is established in the following theorem.

Theorem 5.5. *Let* $d \in \mathcal{D}(\overline{\Omega})$ *and let* R^d *be the associated different quotient functional given by* (5.6). *Then there exists a sequence* $(C_n)_n$ *of functions of the kind* (5.1) *such that the associated two–phase constraint functionals* G_n Γ*-converge in* $W^{1,\infty}(\Omega)$ *to the functional* G *defined as*

$$G(u) := \begin{cases} 0 & \text{if } R^d(u) \leq 1; \\ +\infty & \text{otherwise.} \end{cases} \tag{5.9}$$

Remark 5.6. Theorem 5.4 and Theorem 5.5 yields that the closure in terms of Γ-convergence of two–phase constrain functionals of the form (5.2) is given by the class of functional of the type (5.9), where d varies into the class $\mathcal{D}(\overline{\Omega})$. In particular the Γ-closure contains any functional G of the form

$$G(u) := \begin{cases} 0 & \text{if } \operatorname{ess\,sup}_\Omega \ f(x, \nabla u(x)) \leq 1; \\ +\infty & \text{otherwise,} \end{cases}$$

were $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is any Caratéodory function satisfying

$$\alpha'|\xi| \leq f(x, \xi) \leq \beta'|\xi| \qquad \text{for every } \xi \in \mathbb{R}^N \tag{5.10}$$

$$f(x, \lambda\,\xi) = |\lambda|\, f(x, \xi) \qquad \text{for every } \xi \in \mathbb{R}^N \text{ and } \lambda \in \mathbb{R}, \tag{5.11}$$

for a.e. $x \in \Omega$. In fact, any such G corresponds to a functional of the form (5.9) with $d = d^{f^*} \in \mathcal{D}(\Omega)$ intrinsic (cf. [26]).

Proof. In view of Theorem 2.10 and of (2.9), there exists a sequence of continuous metrics $\varphi_n \in \mathcal{M}(\overline{\Omega})$ such that $d_{\varphi_n} \rightrightarrows d$ in $\mathcal{D}(\overline{\Omega})$. Therefore, by a diagonal argument, we can assume $\varphi \in \mathcal{M}(\overline{\Omega})$ continuous.

Denote by d the distance in $\mathcal{D}(\overline{\Omega})$ associated to φ through (2.3), and let $(\Omega_k)_k$ be an increasing sequence of open sets well contained in Ω such that $|\Omega \setminus \Omega_k| \searrow 0$ for $k \to +\infty$. Moreover we assume that each $\overline{\Omega}_k$ consists of a finite disjoint union of closed balls. The metrics $\psi^k \in \mathcal{M}(\Omega)$ defined as

$$\psi^k(x,\xi) := \varphi(x,\xi)\,\chi_{\Omega_k}(x) + \beta|\xi|\,\chi_{\Omega\setminus\Omega_k}(x) \qquad \text{for every } (x,\xi) \in \Omega \times \mathbb{R}^N$$

are such that $\varphi = \inf_k \psi^k$ on $\big(\Omega \setminus F\big) \times \mathbb{R}^N$, where F is a negligible subset of Ω, so the associated distances $d_{\psi^k} \in \mathcal{D}(\Omega)$ defined according to (2.3) satisfy

$$d(x,y) = \inf_k d_{\psi^k}(x,y) \qquad \text{for every } x, y \in \Omega$$

as d is intrinsic, and the convergence is in fact uniform on $\Omega\times\Omega$ for the d_{ψ^k} are equi–Lipschitz. By a diagonal argument, it is enough to prove the statement for each ψ^k. Fix $k \in \mathbb{N}$ and set $\varphi := \psi^k$. We extend it to a metric $\overline{\varphi}$ on $\mathbb{R}^N$ defined as

$$\overline{\varphi}(x,\xi) := \varphi(x,\xi)\,\chi_{\Omega_k}(x) + \beta|\xi|\,\chi_{\mathbb{R}^N\setminus\Omega_k}(x) \quad \text{for every } (x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N.$$

We denote by $d_{\overline{\varphi}} \in \mathcal{D}(\mathbb{R}^N)$ and $d_\varphi \in \mathcal{D}(\Omega)$ the distances associated trough (2.3) to $\overline{\varphi}$ and φ, respectively. Hereafter d_φ will be identified with its unique continuous extension to $\overline{\Omega}\times\overline{\Omega}$, according to Proposition 2.11. By Theorem 4.8 there exists a sequence of upper semicontinuous metrics $\overline{\varphi}_n \in \mathcal{M}^{\alpha,\beta}$, namely of the form $\overline{\varphi}_n(x,\xi) = \overline{C}_n(x)\,|\xi|$, such that $d_{\overline{\varphi}_n} \rightrightarrows d_{\overline{\varphi}}$ in $\mathbb{R}^N \times \mathbb{R}^N$. We also assume that $\overline{\varphi}_n(x,\cdot) = \beta|\cdot|$ when $x \notin \Omega_k$, by Remark 4.10. Denote by C_n the restriction of $\overline{C}_n$ to Ω, and by d^{C_n} and d_{C_n} the distances in $\mathcal{D}(\Omega)$ associated with C_n through (2.11) and (2.3), respectively. Note that $d_{C_n} = d^{C_n}$ by upper semicontinuity of C_n, in view of Proposition 2.18. We claim that

$$C_n \rightrightarrows \varphi \qquad \text{in } \mathcal{M}(\Omega),$$

which is all we need to conclude in view of Remark 2.12 and Theorem 5.4.

To this aim, denote by φ_n the restriction of $\overline{\varphi}_n$ to $\overline{\Omega}\times\mathbb{R}^N$. The sequence of distances $(d_n)_n$ in $\mathcal{D}(\overline{\Omega})$ accordingly associated through (2.7) is precompact, hence uniformly converges along a subsequence (not relabelled) to a distance $\delta \in \mathcal{D}(\overline{\Omega})$. We want to show that $\delta = d_\varphi$ on $\overline{\Omega}\times\overline{\Omega}$, which in particular implies that the whole sequence $(d_n)_n$ converges to d_φ, by compactness of $\mathcal{D}(\overline{\Omega})$.

Indeed, it is easy to see that φ_δ agrees with $\overline{\varphi}$ in $\big(\overline{\Omega}\setminus\partial\Omega_k\big)\times\mathbb{R}^N$. Now pick up a point x in $\partial\Omega_k$ and chose $r > 0$ small enough such that $B_r(x)$ intersects only one connected component of Ω_k. All quasi–minimal curves for $d_n(y,z)$ are definitively contained in $B_\rho(x)$ for any $y, z \in B_\rho(x)$ whenever $\rho < r\,\alpha/(3\beta)$. Since $\Omega_k \cap B_r(x)$ is convex, we can apply Lemma 2.13 to infer that

$$d_n(y,z) = d_{\overline{\varphi}_n}(y,z) \quad \text{for any } y, z \in \Omega_k \cap B_\rho(x),\ n \in \mathbb{N},$$

and sending n to $+\infty$ we get $\delta(y,z) = d(y,z)$ for any $y, z \in \Omega_k \cap B_\rho(x)$. We derive that $\varphi_\delta(x,\xi) = \varphi(x,\xi)$ for every vector ξ tangent to $\partial\Omega_k$ at x, hence

$$\mathbb{L}_{\varphi_\delta}(\gamma) = \mathbb{L}_\varphi(\gamma) \qquad \text{for every } \gamma \in \Gamma(\overline{\Omega}).$$

This gives $\delta = d_\varphi$ in $\mathcal{D}(\overline{\Omega})$, and $d_{\varphi_n} \rightrightarrows d_\varphi$ in $\overline{\Omega}\times\overline{\Omega}$. Now note that each $\varphi_n(x,\cdot)$ is equal to $\beta|\cdot|$ on $\partial\Omega$, hence by (2.8) we infer that $\varphi_{n|\Omega} \rightrightarrows \varphi$ in $\mathcal{M}(\Omega)$, where $\varphi_{n|\Omega}$ denotes the restriction of φ_n to $\Omega \times \mathbb{R}^N$. The claim follows since $\varphi_n(x,\xi) = C_n(x)|\xi|$ on $\Omega \times \mathbb{R}^N$ by construction. □

Remark 5.7. Up to subsequences, we can always assume that $\theta_{C_n} \overset{*}{\rightharpoonup} \theta$ for some $\theta \in L^\infty(\Omega)$, by compactness of weak–star convergence, so, by Theorem 5.4, we conclude that $\varphi_d(x,\cdot) \in Cl(\mathcal{M}^{\alpha,\beta}_{\theta(x)})$ for a.e. $x \in \Omega$. A natural question is whether we can choose $(C_n)_n$ such that $\theta_{C_n} \overset{*}{\rightharpoonup} \theta_{\varphi_d}$. This fact can be easily proved if $d \in \mathcal{D}(\Omega)$ is intrinsic. However we believe that this is always true, for instance by extending the results of Section 4.3 to the case when $\mathbb{R}^N$ is replaced by Ω, but for this it seems necessary to carry out a specific analysis.

5.3. **The periodic case.** Here we consider the problem of Γ–convergence of two–phase periodic constraints on $W^{1,\infty}(\mathbb{R}^N)$, endowed with the metric induced by the local uniform convergence on $\mathbb{R}^N$. More precisely, let $C : \mathbb{R}^n \to \{\alpha,\beta\}$ be a two–phase periodic function, defined on the unit cube Q by

$$C(x) := \begin{cases} \alpha & \text{if } x \in B_\alpha \\ \beta & \text{if } x \in B_\beta, \end{cases}$$

where B_α is a Borel subset of Q, and $B_\beta = Q \setminus B_\alpha$. We set $\theta := |B_\alpha|$. For every $n \in \mathbb{N}$, consider the function $C_n : \Omega \to \mathbb{R}$ defined by $C_n(x) := C(nx)$ for every $x \in \Omega$, and denote by G_n and d^{C_n} the constraint functional and distance function associated with C_n through (5.2) and (5.5), respectively. By periodicity, we have that $d^{C_n} \rightrightarrows d_\phi$, where $\phi \in \mathcal{N}$ is the stable norm of the metric $C(x)\,|\xi|$. The dual norm ϕ^* clearly satisfies (5.10) and (5.11). We have:

Theorem 5.8. *The functionals G_n Γ-converge in $W^{1,\infty}(\mathbb{R}^N)$ to the functional G^{hom} defined by*

$$G^{hom}(u) := \begin{cases} 0 & \textit{if } \operatorname{ess\,sup}_{\mathbb{R}^N} \phi^*(\nabla u(x)) \le 1 \\ +\infty & \textit{otherwise.} \end{cases}$$

Moreover $\phi \in Cl(\mathcal{M}^{\alpha,\beta}_\theta)$ and its dual norm satisfies the following bounds:

$$\phi^*(\xi) \ge \frac{\alpha'\beta'}{\theta\alpha' + (1-\theta)\beta'} \qquad \textit{for every } \xi \in \mathbb{S}^{N-1}. \tag{5.12}$$

Proof. The Γ-convergence result is a then a consequence of Theorem 5.4 and of (5.7). Inequality (5.12) comes directly from Theorem 4.6–(ii). □

In next theorem a sort of converse of the previous result is established.

Theorem 5.9. *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a function satisfying* (5.10), (5.11), *and assume that $f^* \in Cl(\mathcal{M}^{\alpha,\beta}_\theta)$ for some $\theta \in (0,1)$. Then there exists a sequence of* 1*-periodic functions $C_n : \mathbb{R}^N \to \{\alpha,\beta\}$, with volume fraction $\theta_{C_n} \equiv \theta$, such that the associated constraint functionals G_n Γ-converge in $W^{1,\infty}(\mathbb{R}^N)$ to the functional G^{hom} defined by*

$$G^{hom}(u) := \begin{cases} 0 & \textit{if } \operatorname{ess\,sup}_{\mathbb{R}^N} f(\nabla u) \le 1; \\ +\infty & \textit{otherwise.} \end{cases} \tag{5.13}$$

In particular this holds whenever $f : \mathbb{R}^N \to \mathbb{R}$ *satisfies conditions* (5.10), (5.11) *and*

$$f(\xi) \geq \frac{\alpha'\beta'}{\theta^{\frac{1}{N}}\alpha' + (1-\theta^{\frac{1}{N}})\beta'} \qquad \text{for every } \xi \in \mathbb{S}^{N-1}. \tag{5.14}$$

Proof. The Γ-convergence result is a consequence of Theorem 5.5. Moreover by Theorem 4.6 we have that inequality (5.14) implies $f^* \in Cl(\mathcal{M}_\theta^{\alpha,\beta})$, and this concludes the proof. □

Remark 5.10. Note that any function $f : \mathbb{R}^N \to \mathbb{R}$ satisfying (5.10), (5.11), satisfies also (5.12) and (5.14) for suitable θ_1, $\theta_2 \in [0, 1]$. We deduce that the class of functionals G^{hom} associated through (5.13) to such functions f is the closure, in terms of Γ-convergence in $W^{1,\infty}(\mathbb{R}^N)$, of two–phase periodic constraint functionals with arbitrary volume fraction.

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