Quantitative Estimates for Regular Lagrangian Flows with BV Vector Fields

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Abstract

This paper is devoted to the study of flows associated to nonsmooth vector fields. We prove the well-posedness of regular Lagrangian flows associated to vector fields $\mathbf{B} = (\mathbf{B}^1, \dots, \mathbf{B}^d) \in L^1(\mathbb{R}_+; L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d))$ satisfying $\mathbf{B}^i = \sum_{j=1}^m \mathbf{K}_j^i * b_j$, $b_j \in L^1(\mathbb{R}_+, BV(\mathbb{R}^d))$ and $\operatorname{div}(\mathbf{B}) \in L^1(\mathbb{R}_+; L^\infty(\mathbb{R}^d))$ for $d, m \geq 2$, where $(\mathbf{K}_j^i)_{i,j}$ are singular kernels in $\mathbb{R}^d$. Moreover, we also show that there exist an autonomous vector-field $\mathbf{B} \in L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$ and singular kernels $(\mathbf{K}_j^i)_{i,j}$, singular Radon measures μ_{ijk} in $\mathbb{R}^2$ satisfying $\partial_{x_k}\mathbf{B}^i = \sum_{j=1}^m \mathbf{K}_j^i \star \mu_{ijk}$ in distributional sense for some $m \geq 2$ and for $k, i = 1, 2$ such that regular Lagrangian flows associated to vector field $\mathbf{B}$ are not unique.

1 Introduction

In this paper we study the well-posedness of flows of ordinary differential equations

$$
\begin{cases} \frac{dX(t,x)}{dt} = \mathbf{B}(t, X(t,x)) & \forall t \in [0, T], \\ X(0,x) = x & \forall x \in \mathbb{R}^d, \end{cases} \tag{1.1}
$$

where $\mathbf{B}(t,x) = \mathbf{B}_t(x) \in \mathbb{R}^d$ is a function in $[0,T] \times \mathbb{R}^d$, $d \geq 2$. It is well-known that by Peano's theorem, there exists at least one solution to the problem (1.1) provided that $\mathbf{B}$ is continuous. Moreover, by the usual Cauchy-Lipschitz theorem, one also has uniqueness if $\mathbf{B}$ is a bounded smooth vector field.

The ordinary differential equation (1.1) is related to the continuity equation

$$
\begin{cases} \partial_t u(t,x) + \operatorname{div}(\mathbf{B}(t,x)u(t,x)) = G(t,x)u(t,x) + F(t,x), \\ u(0,x) = u_0(x), \end{cases} \tag{1.2}
$$

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for any $(t, x) \in [0, T] \times \mathbb{R}^d$. Indeed, assume that u_0, $\mathbf{B}$, G, and F are smooth and compactly supported. Let $X : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ be the unique solution of (1.1). It is called the flow of vector field $\mathbf{B}$. We have

$$\det(\nabla_x X(t, x)) = \exp\left(\int_0^t \operatorname{div}(\mathbf{B})(s, X(s, x))ds\right) > 0.$$

In particular, the map $X(t, \cdot)$ is a diffeomorphism from $\mathbb{R}^d$ to itself and we denote by $X^{-1}(t, \cdot)$ its inverse. A solution of (1.2) is given in term of the flow X by the following formula

$$\begin{aligned} u(t, x) &= u_0(\overline{x}) \exp\left(-\int_0^t (\operatorname{div}(\mathbf{B}) - G)(s, X(s, \overline{x}))ds\right) \\ &\quad + \int_0^t F(\tau, X(\tau, \overline{x})) \exp\left(-\int_\tau^t (\operatorname{div}(\mathbf{B}) - G)(s, X(s, \overline{x}))ds\right)d\tau, \end{aligned} \tag{1.3}$$

with $\overline{x} = X^{-1}(t, \cdot)(x)$, its proof is elementary. Therefore, we can say that the *well-posedness of* (1.1) *is equivalent to the well-posedness of* (1.2).

The continuity equations (often with nonsmooth vector fields) are important for describing various quantities in mathematical physics such as mass, energy, momentum, and electric charge. Especially, they are essential to study transport equations such as the convection-diffusion, Boltzmann, Vlasov-Poisson, Euler, and Navier-Stokes equations.

Let us start by the seminal work of Diperna and Lions [41]. They established the existence, uniqueness, and stability of distributional solutions of (1.2) for vector fields $\mathbf{B}$ in $L^1_t W^{1,1}_x$ satisfying $\operatorname{div}(\mathbf{B}) \in L^1_t L^\infty_x$ and a growth condition

$$\mathbf{B}/(1 + |x|) \in L^1_t L^1_x + L^1_t L^\infty_x.$$

Later further progress was achieved in several papers [16, 18, 32, 33, 43, 47]; finally, it was fully extended by Ambrosio [2] to *BV* vector fields. The approach by Diperna, Lions, and Ambrosio relies on the theory of renormalized solutions of (1.2). Roughly speaking, renormalized solutions are distributional solutions such that the chain rule holds for u and $\mathbf{B}$, i.e.,

$$\operatorname{div}(\mathbf{B}h(u)) = (h(u) - uh'(u)) \operatorname{div}(\mathbf{B}) + h'(u) \operatorname{div}(\mathbf{B}u)$$

for any $h \in C^1(\mathbb{R})$.

One key step in this approach consists of studying the strong convergence of the commutator

$$r_\delta := \rho_\delta \star (\operatorname{div}(\mathbf{B}u)) - (\operatorname{div}(\mathbf{B}\rho_\delta \star u))$$

to 0 in L^1_{loc} for some regularizing kernel $(\rho_\delta)_{\delta>0}$ in $\mathbb{R}^d$. In the Sobolev case, in [41], Diperna and Lions showed this convergence for any regularizing kernel $(\rho_\delta)_{\delta>0}$. The same problem in the *BV* case is much more complicated. In [2] Ambrosio took a special kernel ρ strictly depending on the structure of $\mathbf{B}$ to obtain the convergence. More precisely, first he proved that

$$|r_\delta| \rightharpoonup \sigma \quad \text{and} \quad \sigma(x) \lesssim \int |\langle M(x)z, \nabla\rho(z)\rangle| dz |D^s\mathbf{B}|(x),$$

with $M(x) = \frac{dD^s\mathbf{B}}{d|D^s\mathbf{B}|}(x)$ for any smooth kernel ρ, $\int \rho(z)dz = 1$ for any $x \in \mathbb{R}^d$, where $D^s\mathbf{B}$ is singular part of $D\mathbf{B}$ with respect to the Lebesgue measure. Then, he took ρ such that

$$\int |\langle M(x)z, \nabla\rho(z)\rangle| dz \lesssim |\text{trace}\, M(x)|.$$

Using the fact that $\text{div}(\mathbf{B}) << \mathcal{L}^d$ is equivalent to $|traceM(x)||D^s\mathbf{B}|(x) = 0$, then he proved that the "defect" measure is $\sigma = 0$.

Moreover, Diperna and Lions constructed distributional solutions to the continuity equation (1.2) with $\mathbf{B} \in W^{\alpha,1}$ ($\alpha < 1$) and $\text{div}(\mathbf{B}) = 0$ that are not renormalized. A counterexample for non-BV is provided by Depauw [40]. Further results can be found in [4, 5, 7, 9, 11–13, 21–25, 30, 31, 34, 36, 38, 44]. For a recent review on the well-posedness theories for the continuity equations (1.2) and ODE (1.1), we refer the reader the lecture notes [6] (and [3]).

In [35], C. De Lellis and G. Crippa gave an independent proof of the existence and uniqueness of the solutions of (1.1) with Sobolev vector fields, that is without exploiting the connection with the continuity equations (1.2). The main idea of [35] is to consider the following time-dependent quantity

$$\Phi_\delta(t) = \int_{B_R} \log\left(1 + \frac{|X_1(t,x) - X_2(t,x)|}{\delta}\right) dx \quad \text{with } \delta \in (0, \tfrac{1}{2}),$$

where X_1, X_2 are regular Lagrangian flows associated to the same vector field $\mathbf{B}$ and $B_R := B_R(0)$, $R > 0$. We have

$$\Phi_\delta(t) \geq \mathcal{L}^d\left(\{x \in B_R : |X_1(t,x) - X_2(t,x)| > \delta^{1/2}\}\right) \log(1 + \delta^{-1/2}). \tag{1.4}$$

However, differentiating in time, one has

$$\begin{aligned} \Phi_\delta(t) &= \int_0^t \Phi'_\delta(s) ds \\ &\leq \int_0^t\int_{B_R} \frac{|\mathbf{B}_s(X_1(s,x)) - \mathbf{B}_s(X_2(s,x))|}{\delta + |X_1(s,x) - X_2(s,x)|} dx\, ds \\ &\leq \int_0^t\int_{B_R} \min\left\{\frac{2||\mathbf{B}_s||_{L^\infty}}{\delta}, \frac{|\mathbf{B}_s(X_1(s,x)) - \mathbf{B}_s(X_2(s,x))|}{|X_1(s,x) - X_2(s,x)|}\right\} dx\, ds. \end{aligned} \tag{1.5}$$

By using the standard estimate of the Hardy-Littlewood function $\mathbf{M}$ and changing the variable along the flows, we obtain

$$\sup_{t\in[0,T]} \Phi_\delta(t) \lesssim \int_0^T\int_{B_{R_1}} \min\{\delta^{-1}, \mathbf{M}(|\nabla\mathbf{B}_s|)(x)\} dx\, ds. \tag{1.6}$$

for some $R_1 > R$. Using the boundedness of $\mathbf{M}$ from L^p to itself for $p > 1$ together with (1.4) and (1.6), we deduce that

$$\sup_{t\in[0,T]} \mathcal{L}^d\left(\left\{x \in B_R : |X_1(t,x) - X_2(t,x)| > \delta^{1/2}\right\}\right) \lesssim |\log(\delta)|^{-1} \quad \forall \delta \in (0, \tfrac{1}{2}),$$

provided $\mathbf{B} \in L^1(W^{1,p})$, $p > 1$. At this point, sending $\delta \to 0$, we get $X_1 = X_2$.

Later, in [45] P. E. Jabin successfully improved this to $\mathbf{B} \in L^1_t W^{1,1}_x$. In addition, also in [45] he extends this to $\mathbf{B} \in L^1_t SBV_x$ in any dimension, and in two dimensions to $L^1_t BV_x$ with local assumptions in the direction of flows. Furthermore, in [26] we showed that

$$\int_0^T |D^s\mathbf{B}_t|(B_{R_1})dt \lesssim \limsup_{\delta\to 0} \frac{1}{|\log(\delta)|} \int_0^T\int_{B_{R_1}} \min\left\{\delta^{-1}, \mathbf{M}(|\nabla \mathbf{B}_{s(x)}(\cdot)|)\right\}$$
$$\lesssim \int_0^T |D^s\mathbf{B}_t|(\bar{B}_{R_1})dt.$$

This explains why De Lellis and Crippa's approach is not able to deal with vector fields $\mathbf{B} \in L^1_t BV_x \setminus L^1_t W^{1,1}_x$. Moreover, in [17] F. Bouchut and G. Crippa proved the existence and uniqueness of flows for vector fields with gradients given by singular integrals of L^1 functions, i.e., $D\mathbf{B} = \mathbf{K} \star g$, $g \in L^1$, where $\mathbf{K}$ is a singular kernel of fundamental type in $\mathbb{R}^d$. Notice that this class is very natural in the study of nonlinear PDEs, such as the Euler equation and the classical Vlasov-Poisson equation; this class is not contained in BV and neither contains it. To do this, they have used the following maximal singular integral operator:

$$\mathbf{T}(\mu)(x) = \sup_{\varepsilon>0} |(\rho_\varepsilon \star \mathbf{K} \star \mu)(x)| \quad \forall x \in \mathbb{R}^d,$$

where $\rho_\varepsilon(\cdot) = \varepsilon^{-d}\rho(\cdot/\varepsilon)$ for some $\rho \in C^1_c$ such that $\int_{\mathbb{R}^d} \rho\, dx = 1$. Then, $\Phi_\delta(t) = \circ(|\log(\delta)|)$ is obtained from using the boundedness of such operator from L^1 to weak-L^1 and the fact that

$$\lambda \mathcal{L}^d(\{\mathbf{T}(\mu) > \lambda\}) \to 0 \quad \text{as } \lambda \to \infty, \tag{1.7}$$

for any $\mu \in L^1(\mathbb{R}^d)$; see the proof of Lemma 2.4. Notice that (1.7) is not true for $\mu \in \mathcal{M}_b(\mathbb{R}^d)$; indeed, it is easy to check that if $\mu = \delta_0$, then $\lambda \mathcal{L}^d(\{\mathbf{T}(\mu) > \lambda\}) \gtrsim 1, \forall \lambda > 0$ for some ρ and $\mathbf{K}$.

However, later in [14] they extended the analysis to the case where

$$D\mathbf{B} = \begin{pmatrix} D_{x_1}\mathbf{B}_1 & D_{x_1}\mathbf{B}_2 \\ D_{x_2}\mathbf{B}_1 & D_{x_2}\mathbf{B}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{K}_1 \star f_1 & \mathbf{K}_2 \star f_2 \\ \mathbf{K}_0 \star \mu & \mathbf{K}_3 \star f_3 \end{pmatrix} x = (x_1, x_2), \quad \mathbf{B} = (\mathbf{B}_1, \mathbf{B}_2),$$

where $\mathbf{K}_0$, $\mathbf{K}_1$, $\mathbf{K}_2$, and $\mathbf{K}_3$ are singular kernels of fundamental type. This is motivated from the Classical Vlasov-Poisson system associated to

$$B(x_1, x_2) = (x_2, \mathbf{P} \star \mu(x_1)), (x_1, x_2) \in \mathbb{R}^m \times \mathbb{R}^m,\ d = 2m,$$
$$\text{and } \mathbf{P}(x_1) = c\frac{x_1}{|x_1|^m},\ \mu \in \mathcal{M}_b.$$

In addition, Jabin in [27] has proven the well-posedness of this system with $\mathbf{P} \star \mu \in H^{3/4}$ (or $\mu \in H^{-1/4}$). In [37, 49] Seis has provided a quantitative theory for the continuity equation with $W^{1,1}$ vector fields via logarithmic Kantorovich-Rubinstrain distances. Recently, in [22–25], we proved sharp regularity estimates for solutions of continuity equations with $W^{1,p}(p > 1)$ vector fields.

To our knowledge, these results in [14,45] are the best results for the quantitative ODE estimates at this moment. In this paper, we give quantitative estimates for $\mathbf{K} \star BV$ vector fields with bounded divergence. Namely, we prove the following theorem: Given a vector field $\mathbf{B} = (\mathbf{B}^1, \dots, \mathbf{B}^d) \in L^1([0,T]; L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d))$, we assume that for any $R > 0$, there exist functions $b_{jR} \in L^1([0,T], BV(\mathbb{R}^d))$ for $j = 1, \dots, m$; and degree-zero homogeneous functions $(\Omega^i_{jR})_{i,j} \in L^1_{\text{loc}}(\mathbb{R}^d)$ $(i = 1, \dots, d,\ j = 1, \dots, m)$ satisfying $\int_{S^{d-1}} \Omega^i_{jR} = 0$ and $\Omega^i_{jR} \in BV(S^{d-1})$ such that

$$\mathbf{B}^i = \sum_{j=1}^{m} \left(\frac{\Omega^i_{jR}(\cdot)}{|\cdot|^d} \right) \star b_{jR} \quad \text{in } B_{2R}. \tag{1.8}$$

MAIN THEOREM. *Let* $\mathbf{B}_1, \mathbf{B}_2 \in L^1([0,T]; L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d))$ *satisfy*

$$\left\| \left(\frac{|\mathbf{B}_1|}{|x|+1}, \frac{|\mathbf{B}_2|}{|x|+1} \right) \right\|_{L^1((0,T);(L^1+L^\infty)(\mathbb{R}^d))} \le C_0,$$

and let X_1, X_2 *be regular Lagrangian flows associated to* $\mathbf{B}_1$ *and* $\mathbf{B}_2$*, respectively. Assume that* $\operatorname{div}(\mathbf{B}) \in L^1((0,T), L^1(\mathbb{R}^d))$*. Then, for any* $\kappa \in (0,1)$*,* $r > 1$*, there exist* $R_0 = R_0(d,T,r,C_0,\kappa) > 1$ *and* $\delta_0 = \delta_0(d,T,r,C_0,b_{R_0},\kappa) \in (0,1)$ *such that*

$$\begin{aligned} &\sup_{t\in[0,T]} \mathcal{L}^d\big(\{x \in B_r : |X_{1t}(x) - X_{2t}(x)| > \delta^{1/2}\}\big) \\ &\quad \lesssim \delta^{-1} \|(\mathbf{B}_1 - \mathbf{B}, \mathbf{B}_2 - \mathbf{B})\|_{L^1([0,T]\times B_{R_0})} + \kappa, \end{aligned} \tag{1.9}$$

for any $\delta \in (0, \delta_0)$.

Note that if $\mathbf{B}_1, \mathbf{B}_2 \in L^\infty_{t,x}$, we can take R_0 independently of κ. Moreover, if $\mathbf{B} \in L^1((0,T); BV_{\text{loc}}(\mathbb{R}^d))$, we can write for any $R > 0$, $\mathbf{B}^i = \sum_{j=1}^d \mathcal{R}_j^2(\chi_R \mathbf{B}^i)$ in $B_R(0)$, where $\chi_R \in C_c^\infty(\mathbb{R}^d)$ satisfies $\chi_R = 1$ in $B_{2R}(0)$ and $\chi_R = 0$ in $B_{4R}(0)^c$, $\mathcal{R}_1, \dots, \mathcal{R}_d$ are the Riesz transforms in $\mathbb{R}^d$. Thus, the assumptions in the above theorem contain the class of BV-vector fields. Consequently, this solves a main open problem posed by Luigi Ambrosio in [6].

This theorem is as a consequence of Theorem 4.3 and Corollary 4.4 in Section 4. In Section 5, we will use this to deduce the well-posedness of regular Lagrangian flows and transport, continuity equations. The following result gives an existence and uniqueness result of regular Lagrangian flows.

PROPOSITION 1.1. *Let* $\mathbf{B}$ *be as above. Assume that*

$$\left\| \frac{|\mathbf{B}|}{|x|+1} \right\|_{L^1((0,T);(L^1+L^\infty)(\mathbb{R}^d))} < \infty$$

and $\operatorname{div}(\mathbf{B}) \in L^1((0,T), L^1(\mathbb{R}^d))$. *Then, there exists a unique regular Lagrangian flow associated to vector field* $\mathbf{B}$.

Let us describe our idea to prove (1.9). For simplicity, assume that $\mathbf{B}_1(t,x) = \mathbf{B}_2(t,x) = \mathbf{B}(t,x) \equiv \mathbf{B}(x) \in (BV \cap L^\infty)(\mathbb{R}^d, \mathbb{R}^d)$. Thanks to Alberti's rank one theorem (see Section 2), there exist unit vectors $\xi(x) \in \mathbb{R}^d$ and $\eta(x) \in \mathbb{R}^d$ such that $D^s\mathbf{B}(x) = \xi(x) \otimes \eta(x)|D^s\mathbf{B}|(x)$, i.e., $D^s_{x_i}\mathbf{B}_j(x) = \xi_j(x)\eta_i(x)|D^s\mathbf{B}|(x)$ for any $i, j = 1, \dots, d$. Thus, one gets from $\operatorname{div}(\mathbf{B}) \in L^1([0,T] \times \mathbb{R}^d)$ that $|\langle \xi, \eta \rangle| = 0$ for $|D^s\mathbf{B}|$-a.e. in $\mathbb{R}^d$. We first have the following basic inequality: for any $x_1 \neq x_2 \in \mathbb{R}^d$ and $\nu \in S^{d-1}$,

$$\begin{aligned} |\langle \nu, \mathbf{B}(x_1) - \mathbf{B}(x_2) \rangle| \lesssim \sum_{l=1,2} \int \frac{\mathbf{1}_{|x_l - z| \le r}}{|x_l - z|^{d-1}} |\langle \nu, \xi(z) \rangle| \, d\mu(z) \\ + \int \frac{\mathbf{1}_{|x_l - z| \le r}}{|x_l - z|^{d-1}} d|D^a\mathbf{B}|(z); \end{aligned}$$

see Proposition 2.3, where $\mu = |D^s\mathbf{B}|$ and $r = |x_1 - x_2|$, where $D^a\mathbf{B}$ is a regular part of $D\mathbf{B}$ with respect to the Lebesgue measure. We now assume that ξ and η are smooth functions in $\mathbb{R}^d$. Then, choosing $\nu = \eta(x_1)$ and thanks to $|\langle \xi, \eta \rangle| = 0$ for $|\mu|$−a.e. in $\mathbb{R}^d$, we have

$$|\langle \nu, \xi(z) \rangle| \le \|\nabla \eta\|_{L^\infty} (|x_1 - x_2| + |x_l - z|) \quad \text{for } |\mu| - \text{a.e. } z \text{ in } \mathbb{R}^d, \ l = 1, 2.$$

This implies

$$\frac{|\langle \eta(x_1), \mathbf{B}(x_1) - \mathbf{B}(x_2) \rangle|}{|x_1 - x_2|} \lesssim \sum_{l=1,2} \|\nabla \eta\|_{L^\infty} \mathbf{I}_1(\mu)(x_l) + \mathbf{M}(|D^a\mathbf{B}|)(x_l), \tag{1.10}$$

where $\mathbf{I}_1$ is the Riesz potential with the first order in $\mathbb{R}^d$.

Let X_1, X_2 be regular Lagrangian flows associated to the same vector field $\mathbf{B}$ and $r > 0$. Thus, we derive from (1.10) that

$$\limsup_{\delta \to 0} \frac{1}{|\log(\delta)|} \int_0^T \int_{B_r} \frac{|\langle \eta(X_{1t}), \mathbf{B}(X_{1t}) - \mathbf{B}(X_{2t}) \rangle|}{\delta + |X_{1t} - X_{2t}|} \, dx \, dt = 0. \tag{1.11}$$

This suggests the following new quantity: for $\delta \in (0,1)$, $\gamma > 1$,

$$\Phi^\gamma_\delta(t) = \frac{1}{2} \int_{B_r} \log\left(1 + \frac{|X_{1t} - X_{2t}|^2 + \gamma \langle \eta(X_{1t}), X_{1t} - X_{2t}| \rangle^2}{\delta^2}\right) dx. \tag{1.12}$$

We have

$$\begin{aligned}\sup_{t\in[0,T]}\Phi_\delta^\gamma(t) &= \sup_{t_1\in[0,T]}\int_0^{t_1}\frac{d\Phi_\delta^\gamma(t)}{dt}\,dt\\ &\le \int_0^T\int_{B_r}\frac{\gamma^{1/2}|\langle\eta(X_{1t}),\mathbf{B}(X_{1t})-\mathbf{B}(X_{2t})\rangle|}{\delta+|X_{1t}-X_{2t}|}dx\,dt\\ &\quad+\int_0^T\int_{B_r}\frac{|\mathbf{B}(X_{1t})-\mathbf{B}(X_{2t})|}{\delta+|X_{1t}-X_{2t}|+\gamma^{1/2}|\langle\eta(X_{1t}),X_{1t}-X_{2t}\rangle|}\,dx\,dt\\ &\quad+\int_0^T\int_{B_r}\frac{\gamma^{1/2}|\langle\nabla\eta(X_{1t})\mathbf{B}(X_{1t}),X_{1t}-X_{2t}\rangle|}{|X_{1t}-X_{2t}|}\,dx\,dt.\end{aligned}$$

Combining this and (1.11), we get

$$\begin{aligned}&\sup_{t\in[0,T]}\mathcal{L}^d(\{x\in B_r:|X_{1t}-X_{2t}|>0\})\\ &=\limsup_{\delta\to0}\frac{1}{|\log(\delta)|}\sup_{t\in[0,T]}\Phi_\delta^\gamma(t)\\ &\le\limsup_{\delta\to0}\frac{1}{|\log(\delta)|}\int_0^T\int_{B_r}\frac{|\mathbf{B}(X_{1t})-\mathbf{B}(X_{2t})|dx\,dt}{\delta+|X_{1t}-X_{2t}|+\gamma^{1/2}|\langle\eta(X_{1t}),X_{1t}-X_{2t}\rangle|}\\ &:=\limsup_{\delta\to0}A(\delta).\end{aligned}$$

Hence, in order to get $X_{1t}=X_{2t}$ for a.e. $(x,t)\in B_r\times[0,T]$, we need to show that

$$\limsup_{\delta\to0}A(\delta)=\circ(1)\text{ as }\gamma\to\infty.$$

In fact, we use the following estimate for $\mathbf{B}(x_1)-\mathbf{B}(x_2)$:

$$\begin{aligned}&|\mathbf{B}(x_1)-\mathbf{B}(x_2)|\\ &\lesssim\varepsilon^{-d+1}|x_1-x_2|\big(\mathbf{M}(|D^a\mathbf{B}|)(x_1)+\mathbf{M}(|D^a\mathbf{B}|)(x_2)\big)\\ &\quad+\varepsilon^{-d+1}\sum_{l=1,2}\int\frac{\mathbf{1}_{|x_l-z|\le r}\mathbf{1}_{\left|\frac{x_l-z}{|x_l-z|}-e_l\right|\le\varepsilon}}{|x_l-z|^{d-1}}\frac{|\langle\eta(z),x_1-x_2\rangle|}{|x_1-x_2|}\,d|\mu|(z)\\ &\quad+\varepsilon^{-d+2}\sum_{l=1,2}\int\frac{\mathbf{1}_{|x_l-z|\le r}\mathbf{1}_{\left|\frac{x_l-z}{|x_l-z|}-e_l\right|\le\varepsilon}}{|x_l-z|^{d-1}}\,d|\mu|(z),\end{aligned}$$

for any $\varepsilon>0$ where $\mu=|D^s\mathbf{B}|$, $e_1=-e_2=\frac{x_1-x_2}{|x_1-x_2|}$, and $r=|x_1-x_2|$ for $l=1,2$ (see Proposition 2.3 and Lemma 4.5). Then, using the fact that $|\langle\eta(z),x_1-x_2\rangle|\le|\langle\eta(x_1),x_1-x_2\rangle|+2\|\nabla\eta\|_{L^\infty}r^2$ for $|z-x_1|\le r$ or $|z-x_2|\le r$ and changing the variable along the flows, we can estimate

$$A(\delta) \lesssim \frac{\gamma^{-1/2}\varepsilon^{-d+1}}{|\log(\delta)|}\int_{B_{r'}} \min\left\{\frac{\mathbf{I}_1(\mu)}{\delta}, \mathbf{M}(\mu)\right\} + \|\nabla\eta\|_{L^\infty}\frac{\varepsilon^{-d+1}}{|\log(\delta)|}\int_{B_{r'}} \mathbf{I}_1(\mu)$$
$$+ \frac{\varepsilon}{|\log(\delta)|}\int_{B_{r'}} \min\left\{\frac{\mathbf{I}_1(\mu)}{\varepsilon^{d-1}\delta}, \mathbf{M}^\varepsilon(\mu)\right\}$$
$$+ \frac{\varepsilon^{-d+1}}{|\log(\delta)|}\int_{B_{r'}} \min\left\{\frac{\mathbf{I}_1(|D^a B|)}{\delta}, \mathbf{M}(|D^a B|)\right\},$$

for some $r' > r$, where $\mathbf{M}^\varepsilon$ is the Kakeya maximal function in $\mathbb{R}^d$, i.e.,

$$\mathbf{M}^\varepsilon(\mu)(x) = \sup_{\rho\in(0,2r'), e\in S^{d-1}} \fint_{B_\rho(x)} \varepsilon^{-d+1}\mathbf{1}_{|\frac{x-z}{|x-z|}-e|\leq\varepsilon}\, d|\mu|(z).$$

We then will deduce that

$$\limsup_{\delta\to 0} A(\delta) \lesssim \gamma^{-1/2}\varepsilon^{-d+1}|\mu|(\mathbb{R}^d) + \varepsilon \limsup_{\lambda\to\infty} \lambda\mathcal{L}^d\left(\{\mathbf{M}^\varepsilon(\mu) > \lambda\}\cap B_{r'}\right)$$
$$= \varepsilon \limsup_{\lambda\to\infty} \lambda\mathcal{L}^d(\{\mathbf{M}^\varepsilon(\mu) > \lambda\}\cap B_{r'}) \quad \text{as } \gamma\to\infty.$$

So it remains to show that

$$I(\varepsilon) := \varepsilon \limsup_{\lambda\to\infty} \lambda\mathcal{L}^d(\{\mathbf{M}^\varepsilon(\mu) > \lambda\}\cap B_{r'}) = \circ(1) \quad \text{as } \varepsilon\to 0. \tag{1.13}$$

This estimate is very delicate; hence we will devote Section 3 to establish it. In order to see the key idea for proving the estimate (1.13), we only consider $\mu(x) = |D^s f|(x) \equiv |Df|(x)$ with $f \in BV(\mathbb{R}^d, \mathbb{R})$ such that $\nu = \frac{dD^s f}{d|D^s f|}(x)$ is a constant function in $B_{8r'}$. Set $H_\nu := \{x \in \mathbb{R}^d : \langle \nu, x\rangle = 0\}$ and $\widetilde{H}_\nu := \{t\nu \in \mathbb{R}^d : \forall t \in \mathbb{R}\}$. We also denote $f^\nu_{y_2} : \widetilde{H}_\nu \ni y_1 \mapsto f(y_2 + y_1)$ for any $y_2 \in H_\nu$. By assumption one has $d\mu(y) = d|Df^\nu_z|(y_1)d\mathcal{H}^{d-1}(y_2)$ for any $y_1 = \langle y, \nu\rangle\nu$, $y_2 = y - \langle y, \nu\rangle\nu$, $y \in B_{8r'}$, and $z \in H_\nu$. We can prove that

$$\mathbf{M}^\varepsilon(\mu)(x) \lesssim \mathbf{M}^1(|Df^\nu_{x_\nu}|, \widetilde{H}_\nu)(\langle x, \nu\rangle\nu) \quad \text{with } x_\nu := x - \langle x, \nu\rangle\nu, \tag{1.14}$$

where $\mathbf{M}^1(|Df^\nu_{x_\nu}|, \widetilde{H}_\nu)$ is the Hardy-Littlewood maximal function of $|Df^\nu_{x_\nu}|$ on $\widetilde{H}_\nu$. By a standard approximation argument, we only prove for case $|Df^\nu_{x_\nu}| \in L^1(\widetilde{H}_\nu, d\mathcal{H}^1)$. By changing variables, we have for any $\rho \in (0, 2r')$, $e \in S^{d-1}$, and $x \in B_{r'}$

$$
\begin{aligned}
&\fint_{B_\rho(x)} \varepsilon^{-d+1}\mathbf{1}_{|\frac{x-y}{|x-y|}-e|\le\varepsilon}\, d|\mu|(y)\\
&\quad = \rho^{-d}\varepsilon^{-d+1}\fint_{S^{d-1}}\int_0^\rho \mathbf{1}_{|\theta-e|\le\varepsilon}|Df^\nu_{x_\nu}|(\langle x,\nu\rangle\nu-\langle\theta,\nu\rangle\nu s)s^{d-1}ds\,d\mathcal{H}^{d-1}(\theta)\\
&\quad \le \varepsilon^{-d+1}\fint_{S^{d-1}}\mathbf{1}_{|\theta-e|\le\varepsilon}4\mathbf{M}^1(|Df^\nu_{x_\nu}|,\widetilde{H}_\nu)(\langle x,\nu\rangle\nu)d\mathcal{H}^{d-1}(\theta)\\
&\quad \lesssim \mathbf{M}^1(|Df^\nu_{x_\nu}|,\widetilde{H}_\nu)(\langle x,\nu\rangle\nu),
\end{aligned}
$$

which implies (1.14). Therefore, we get from (1.14) and weak type (1,1) bound of $\mathbf{M}^1(|Df^\nu_{x_\nu}|,\widetilde{H}_\nu)$ that

$$
\begin{aligned}
&\lambda\mathcal{L}^d(\{\mathbf{M}^\varepsilon(\mu)>\lambda\}\cap B_{r'})\\
&\quad \le \lambda\int_{H_\nu}\mathcal{H}^1\big(\{x_1\in\widetilde{H}_\nu:\mathbf{M}^1(|Df^\nu_{x_2}|,\widetilde{H}_\nu)(x_1)\gtrsim\lambda\}\big)d\mathcal{H}^{d-1}(x_2)\\
&\quad \lesssim \int_{H_\nu}\int_{\widetilde{H}_\nu} d|Df^\nu_{x_2}|(x_1)d\mathcal{H}^{d-1}(x_2) = |\mu|(\mathbb{R}^d).
\end{aligned}
$$

This gives (1.13). In order to prove (1.13) in the general case, we use that $\mu = |D^s\mathbf{B}|$ and the slicing theory of BV functions. Notice (1.13) is not true for any Radon measure μ, indeed, if $\mu = \delta_0$, then $\mathbf{M}^\varepsilon(\mu)(x) = \varepsilon^{-d+1}|x|^{-d}$ and so $I(\varepsilon)\sim\varepsilon^{-d+2}$.

To conclude this section, let us give an important remark on our result. We deduce from (1.8) that for $R>0$,

$$
\partial_l\mathbf{B}^i = \sum_{j=1}^m\left(\frac{\Omega^i_{jR}(\cdot)}{|\cdot|^d}\right)\star\mu^l_{jR}\quad\text{in } \mathcal{D}'(B_{2R}), \tag{1.15}
$$

where $\mu^l_{jR} = \partial_l b_{jR}$, $l,i=1,\dots,d$, $j=1,\dots,m$ are bounded Radon measures in $\mathbb{R}^d$. Thus: *A natural question is whether the above proposition holds for a class of vector fields* $\mathbf{B}$ *satisfying* (1.15) *with arbitrary Radon measures* μ^l_{jR} *in* $\mathbb{R}^d$.

The following proposition gives a negative answer to this question.

PROPOSITION 1.2. *There exist a vector field* $B:\mathbb{R}^2\to\mathbb{R}^2$ *and degree-zero homogeneous functions* $\Omega^i_1,\dots,\Omega^i_m\in(L^\infty\cap BV)(B_2\setminus B_1)$, $i=1,2$ *with* $\frac{|B(x)|}{|x|+1}\in L^1(\mathbb{R}^2)+L^\infty(\mathbb{R}^2)$, $\operatorname{div}(B)=0$, $\int_{S^1}\Omega_l=0$ *such that for any* $R>1$

$$
\partial_l\mathbf{B}^i = \sum_{j=1}^m\left(\frac{\Omega^i_j(\cdot)}{|\cdot|^2}\right)\star\mu^l_{jR}\quad\textit{in } \mathcal{D}'(B_R), \tag{1.16}
$$

for some $\mu^l_{jR}\in\mathcal{M}_b(\mathbb{R}^2)$, $i,l=1,2$, *and* $j=1,\dots,m$ *and the problem* (1.1) *is ill-posed with this vector field, i.e., there exist two different regular Lagrangian flows* X_1, X_2 *associated to* $\mathbf{B}$.

We will prove proposition 1.2 in the Appendix.

2 Main Notation and Preliminary Results

We begin with some notations used in this paper.

- $x \cdot y$, $\langle x, y\rangle$ denote the usual scalar product of $x, y \in \mathbb{R}^d$;
- $a \wedge b$ denotes $\min\{a, b\}$;
- $\mathbb{S}^{d-1}$ denotes the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$;
- $\mathbf{1}_E$ is the characteristic function of the set E, defined as $\mathbf{1}_E(x) = 1$ if $x \in E$ and $\mathbf{1}_E(x) = 0$ otherwise;
- $B_r(x)$ is the open ball in $\mathbb{R}^d$ with radius r and center x; B_r is the open ball in $\mathbb{R}^d$ with radius and center 0; if X is a vector subspace of $\mathbb{R}^d$, for any $x \in X$, $B_r(x, X)$ is the open ball in X with radius r and center x i.e., $B_r(x, X) = B_r(x) \cap X$;
- $\mathcal{M}_b(X)$ is a set of bounded Radon measure in a metric space X; $\mathcal{M}_b^+(X)$ is a set of positive bounded Radon measure in X;
- $|\mu|$ is the total variation of a measure μ; μ^s, μ^a are the singular component and regular component of μ with respect to the Lebesgue measure, respectively;
- $\mathcal{L}^d$ is the Lebesgue measure on $\mathbb{R}^d$ and $\mathcal{H}^k$ is the k-dimensional Hausdorff measure;
- $BV(\mathbb{R}^d, \mathbb{R}^m)$ is a set of $\mathbb{R}^m$-valued functions with bounded variation in $\mathbb{R}^d$;
- $f \star g$ is the convolution of f and g; in particular, if $f, g \in \mathbb{R}^l$, then $f \star g := \sum_{j=1}^l f_j \star g_j$; if $f \in \mathbb{R}^l$, $g \in \mathbb{R}$, then $f \star g = g \star f := (f_1 \star g, f_2 \star g, \ldots, f_l \star g)$;
- $f_\# \mu$ is the push-forward of μ via a Borel map f; more specifically, if there are a Borel map $f : \mathbb{R}^l \to \mathbb{R}^m$ and a measure μ in $\mathbb{R}^l$, then $f_\#\mu$ is a measure in $\mathbb{R}^m$ given by $f_\#\mu(B) = \mu(f^{-1}(B))$ for any Borel set $B \subset \mathbb{R}^m$; this is equivalent to $\int_{\mathbb{R}^m} \phi \, df_\#\mu = \int_{\mathbb{R}^l} \phi \circ f d\mu$ for any $\phi : \mathbb{R}^m \to [0, +\infty]$ Borel;
- $\fint_E f \, d\omega$ denotes the average of the function f over the set E with respect to the positive measure ω, i.e., $\fint_E f \, d\omega := \frac{1}{\omega(E)} \int_E f \, d\omega$;
- $\{f > \lambda\}, \{f < \lambda\}$ stand for $\{x : f(x) > \lambda\}, \{x : f(x) < \lambda\}$, respectively;
- ϱ_n is a standard sequence of mollifiers in $\mathbb{R}^d$;
- E^c is the complement of set E;
- $A \lesssim B$ denotes the estimate $A \leq CB$ for some constant $C > 0$ depending only on fixed quantities; and $A \sim B$ denotes the estimate $A \lesssim B \lesssim A$;
- $C(n, \varepsilon, \kappa, \ldots)$ is a common constant that satisfies parameters $n, \varepsilon, \kappa, \ldots$.

2.1 *BV* functions

Given $b \in BV(\mathbb{R}^d, \mathbb{R}^m)$, we have the canonical decomposition of Db as $D^a b + D^s b$, with $|D^a b| \ll \mathcal{L}^d$ and $|D^s b| \perp \mathcal{L}^d$. The following deep result of Alberti will be used in the proof of Theorem 4.3. Its proof can be found in [1, 39].

PROPOSITION 2.1 (Alberti's rank one theorem). *There exist unit vectors $\xi(x) \in \mathbb{R}^m$ and $\eta(x) \in \mathbb{R}^d$ such that $D^s b(x) = \xi(x) \otimes \eta(x) |D^s b|(x)$, i.e., $D^s_{x_i} b_j(x) = \xi_j(x)\eta_i(x)|D^s b|(x)$ for any $i = 1, \dots, d$, $j = 1, \dots, m$.*

Notice that the pair of unit vector components (ξ, η) is uniquely determined $|D^s b|$-a.e. up to a change of sign. In the case $m = d$, we can write the distributional divergence $\operatorname{div}(b)$ as $\operatorname{div}(b) = \operatorname{trace}(D^a b)\mathcal{L}^d + \langle \xi, \eta \rangle |D^s b|$; thus, $\operatorname{div}(b) \ll \mathcal{L}^d$ if and only if $\xi \perp \eta$ $|D^s b|$-a.e. in $\mathbb{R}^d$. For $e \in \mathbb{S}^{d-1}$, let us introduce the hyperplane orthogonal to e:

$$H_e := \{x \in \mathbb{R}^d : \langle e, x \rangle = 0\},$$

and the line of e:

$$\widetilde{H}_e := \{te \in \mathbb{R}^d : \forall t \in \mathbb{R}\}.$$

Given a Borel function f in $\mathbb{R}^d$, we denote $f^e_{y_1} : \widetilde{H}_e \ni z_1 \mapsto f(y_1 + z_1)$ for $y_1 \in H_e$. The following characterization of *BV* by hyperplanes will be used in the proof of Theorem 3.3.

PROPOSITION 2.2 ([8, sec. 3.11]). *Let $f \in BV(\mathbb{R}^d)$ and $e \in \mathbb{S}^{d-1}$. Then, $f^e_{y_1} \in BV(\widetilde{H}_e)$, $\mathcal{H}^{d-1}$-a.e. y_1 in H_e, and $\int_{H_e} \|f^e_{y_1}\|_{BV(\widetilde{H}_e)} d\mathcal{H}^{d-1}(y_1) < \infty$. Moreover, for any bounded Borel function $\phi : \mathbb{R}^d \to \mathbb{R}^+$, there holds*

$$\begin{aligned} \int_{H_e}\int_{\widetilde{H}_e} &\phi(t + y_1) d|D^s f^e_{y_1}|(t) d\mathcal{H}^{d-1}(y_1) \\ &= \int_{\mathbb{R}^d} \phi(x) |\langle e, \eta(x) \rangle| d|D^s f|(x), \end{aligned} \tag{2.1}$$

where $\eta(x) = \frac{dD^s f(x)}{d|D^s f|(x)}$.

We next have an extension of [27, prop. 4.2]. It is one of the main tools to be used in the proof of Theorem 4.3.

PROPOSITION 2.3. *Let* $\varepsilon \in (0, \frac{1}{100})$*,* $f \in BV_{\mathrm{loc}}(\mathbb{R}^d)$*. Then, for every* $x, y \in \mathbb{R}^d$*,* $x \neq y$*,*

$$\begin{aligned} f(x) - f(y) &= \int_{\mathbb{R}^d} \frac{\varepsilon^{-d+1}}{|x-z|^{d-1}} \Theta_1^{\varepsilon,\mathbf{e}_1}\left(\frac{x-z}{|x-y|}\right)\mathbf{e}_1 \cdot dDf(z) \\ &\quad + \int_{\mathbb{R}^d} \frac{\varepsilon^{-d+2}}{|x-z|^{d-1}} \Theta_2^{\varepsilon,\mathbf{e}_1}\left(\frac{x-z}{|x-y|}\right) \cdot dDf(z) \\ &\quad - \int_{\mathbb{R}^d} \frac{\varepsilon^{-d+1}}{|y-z|^{d-1}} \Theta_1^{\varepsilon,\mathbf{e}_2}\left(\frac{y-z}{|x-y|}\right)\mathbf{e}_2 \cdot dDf(z) \\ &\quad - \int_{\mathbb{R}^d} \frac{\varepsilon^{-d+2}}{|y-z|^{d-1}} \Theta_2^{\varepsilon,\mathbf{e}_2}\left(\frac{y-z}{|x-y|}\right) \cdot dDf(z), \end{aligned} \tag{2.2}$$

where $\mathbf{e}_1 = -\mathbf{e}_2 = \frac{x-y}{|x-y|}$ *and for* $e \in \mathbb{S}^{d-1}$*,* $\varepsilon \in (0, \frac{1}{100})$*,* $\Theta_1^{\varepsilon,e} : \mathbb{R}^d \to \mathbb{R}_+$ *and* $\Theta_2^{\varepsilon,e} : \mathbb{R}^d \to \mathbb{R}^d$ *are bounded functions satisfying* $\Theta_1^{\varepsilon,e}, \Theta_2^{\varepsilon,e} \in C^\infty(\mathbb{R}^d \setminus \{0\})$*,*

$$\begin{aligned} &\mathrm{supp}(\Theta_1^{\varepsilon,e}), \mathrm{supp}(\Theta_2^{\varepsilon,e}) \subset B_{3/4}(0) \cap \left\{x : \left|e - \frac{x}{|x|}\right| \leq \varepsilon\right\}, \\ &|\Theta_l^{\varepsilon,e}(x)| + \varepsilon|x|\|\nabla\Theta_l^{\varepsilon,e}(x)| \lesssim 1 \ \forall x \in \mathbb{R}^d, l = 1, 2, \\ &\varepsilon^{-d+1}\int_{\mathbb{R}^d} |\Theta_1^{\varepsilon,e}(x)|dx + \varepsilon^{-d+1}\int_{\mathbb{R}^d} |\Theta_2^{\varepsilon,e}(x)|dx \lesssim 1. \end{aligned} \tag{2.3}$$

PROOF OF PROPOSITION 2.3. Let $\rho : \mathbb{R} \to [0, \infty)$ be a C_c function such that $\rho \in C^\infty([0, 1])$, $\rho(t) = 1$ for $0 \leq t \leq 1/4$, $\rho(t) = 0$ for $t \geq \frac{3}{4}$ and $t < 0$, $\rho(t) + \rho(1-t) = 1$ for $0 \leq t \leq 1$. Let $\psi : (0, \infty) \to (0, \infty)$ be a C_b^∞ function such that $\psi(t) = 0$ for $t > 1$, $\psi(t) = 1$ in $(0, \varepsilon_0)$ for some $\varepsilon_0 \in (0, 1)$ and $\int_{\mathbb{R}^{d-1}} \psi(|h|)dh = 1$.
We define for $(a, b, c) \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \times (0, \infty)$,

$$\begin{aligned} \Psi_1(a, b, c) &= \frac{\rho((a.b)c)\psi\left(\frac{|a-(a.b)b|}{4(a.b)(1-(a.b)c)}\right)}{4^{d-1}(a.b)^d(1-(a.b)c)^{d-1}}, \\ \Psi_2(a, b, c) &= \Psi_1(a, b, c)\frac{(a.b)c}{1-(a.b)c}(a - (a.b)b). \end{aligned}$$

Since $\Psi_1(a,b,c) = \Psi_1(-a,-b,c)$, $\Psi_2(a,b,c) = -\Psi_2(-a,-b,c)$, thus it is not hard to obtain from the proof of [27, prop. 4.2] that

$$\begin{aligned} f(x) - f(y) = &\int_{\mathbb{R}^d} \frac{1}{|x-z|^{d-1}} \Psi_1\left(\frac{x-z}{|x-z|}, \frac{x-y}{|x-y|}, \frac{|x-z|}{|x-y|}\right) \frac{x-z}{|x-z|} \cdot dDf(z) \\ &- \int_{\mathbb{R}^d} \frac{1}{|x-z|^{d-1}} \Psi_2\left(\frac{x-z}{|x-z|}, \frac{x-y}{|x-y|}, \frac{|x-z|}{|x-y|}\right) \cdot dDf(z) \\ &- \int_{\mathbb{R}^d} \frac{1}{|y-z|^{d-1}} \Psi_1\left(\frac{y-z}{|y-z|}, \frac{y-x}{|y-x|}, \frac{|y-z|}{|x-y|}\right) \frac{y-z}{|z-y|} \cdot dDf(z) \\ &+ \int_{\mathbb{R}^d} \frac{1}{|y-z|^{d-1}} \Psi_2\left(\frac{y-z}{|y-z|}, \frac{y-x}{|y-x|}, \frac{|y-z|}{|x-y|}\right) \cdot dDf(z). \end{aligned}$$

Replacing ψ by $\frac{8^{d-1}}{\varepsilon^{d-1}}\psi(8\frac{\cdot}{\varepsilon})$, we obtain (2.2) where $\Theta_l^{\varepsilon,e}(z) = \phi_l^{\varepsilon}(z/|z|, e, |z|)$ for $(e,z) \in \mathbb{S}^{d-1} \times \mathbb{R}^d$; and

$$\begin{aligned} \phi_1^{\varepsilon}(a,b,c) &= 2^{d-1} \frac{\rho((a.b)c)\psi\left(\frac{2|a-(a\cdot b)b|}{\varepsilon(a\cdot b)(1-(a\cdot b)c)}\right)}{(a.b)^d(1-(a.b)c)^{d-1}}, \\ \phi_2^{\varepsilon}(a,b,c) &= \phi_1^{\varepsilon}(a,b,c)\frac{a-b}{\varepsilon} - \phi_1^{\varepsilon}(a,b,c)\frac{(a.b)c}{1-(a.b)c}\frac{(a-(a.b)b)}{\varepsilon}. \end{aligned}$$

Note that $\rho((a.b)c)\psi\left(\frac{2|a-(a.b)b|}{\varepsilon(a.b)(1-(a.b)c)}\right) \neq 0$ implies $|a-(a.b)b| \leq \frac{\varepsilon}{2}$ and $(a.b)c \leq 3/4$. So,

$$|a-b| = \sqrt{2(1-(a.b))} \leq \sqrt{2(1-(a.b)^2)} = \sqrt{2|a-(a.b)b|^2} \leq \varepsilon/\sqrt{2},$$

and $a \cdot b \geq 1 - \varepsilon/2 \geq 1/2$, $c \leq 3/4$. Hence, it is easy to check that $\Theta_1^{\varepsilon,e}$ and $\Theta_2^{\varepsilon,e}$ belong to $C^\infty(\mathbb{R}^d \setminus \{0\})$ and satisfy (2.3). The proof is complete. □

2.2 The Hardy-Littlewood maximal function and Riesz potential.

We recall some basic properties of the Hardy-Littlewood maximal function and Riesz potential. Let X be a vector subspace of $\mathbb{R}^d$ with $\dim(X) = k$ ($k = 1, \ldots, d$) and μ be a positive Radon measure in X. The Hardy-Littlewood maximal function of μ on X is defined by

$$\mathbf{M}^k(\mu, X)(x) = \sup_{r>0} \frac{1}{\mathcal{H}^k(B_r(x,X))} \int_{B_r(x,X)} d|\mu| \quad \forall x \in X.$$

If $X = \mathbb{R}^d$, we write $\mathbf{M}(\mu)$ instead of $\mathbf{M}^k(\mu, X)$. It is well-known that $\mathbf{M}^k(\cdot, X)$ is bounded from $L^p(X, d\mathcal{H}^k)$ to itself and $\mathcal{M}_b(X)$ to $L^{1,\infty}(X, d\mathcal{H}^k)$ for $1 < p \leq \infty$, i.e.,

$$\|\mathbf{M}^k(\mu, X)\|_{L^p(X,d\mathcal{H}^k)} \lesssim \|\mu\|_{L^p(X,d\mathcal{H}^k)} \quad \text{for any } \mu \in L^p(X, d\mathcal{H}^k), \tag{2.4}$$

$$\sup_{\lambda>0} \lambda\mathcal{H}^k\left(\{\mathbf{M}^k(\mu, X) > \lambda\}\right) \lesssim |\mu|(X) \quad \text{for any } \mu \in \mathcal{M}_b(X) \tag{2.5}$$

(see [10, 50, 51]).

The Riesz potential of μ on X is defined by

$$\mathbf{I}_\alpha^k(\mu, X)(x) = \int_X \frac{1}{|x-z|^{k-\alpha}}\, d|\mu|(z) \quad \forall x \in X,\ 0<\alpha<k. \tag{2.6}$$

If $X = \mathbb{R}^d$, we write $\mathbf{I}_\alpha(\mu)$ instead of $\mathbf{I}_\alpha^k(\mu, X)$. We have that $\mathbf{I}_\alpha^k(\cdot, X)$ is bounded from $L^p(X, d\mathcal{H}^k)$ to $L^{\frac{kp}{k-\alpha p}}(X, d\mathcal{H}^k)$ for $p>1$, $0<\alpha p<k$; and bounded from $\mathcal{M}_b^+(X)$ to $L^{\frac{k}{k-\alpha},\infty}(X, d\mathcal{H}^k)$ for $0<\alpha<k$; see [50]. It is easy to see that for $\alpha>0$,

$$\sup_{r>0} r^{-\alpha} \int_X \frac{\mathbf{1}_{|x-z|\le r}}{|x-z|^{k-\alpha}} d\mu(z) \lesssim \mathbf{M}^k(\mu, X)(x) \quad \forall x \in X. \tag{2.7}$$

Thanks to (2.5), one gets

$$\begin{aligned}
&\lambda\mathcal{H}^k\big(\{\mathbf{M}^k(\mu, X) > \lambda\}\big)\\
&\quad\le \lambda\mathcal{H}^k\big(\{\mathbf{M}^k(\mu^s, X) > \lambda/2\}\big) + \lambda\mathcal{H}^k\big(\{\mathbf{M}^k(\mu^a \mathbf{1}_{|\mu^a|\ge\lambda/4}, X) > \lambda/2\}\big)\\
&\quad\lesssim |\mu|^s(X) + \int_X \mathbf{1}_{|\mu|^a\ge\lambda/4}|\mu|^a\, dx,
\end{aligned}$$

provided $|\mu|(X)<\infty$. Thus,

$$\limsup_{\lambda\to\infty} \lambda\mathcal{H}^k(\{\mathbf{M}^k(\mu, X)>\lambda\}) \lesssim |\mu|^s(X). \tag{2.8}$$

Moreover, in [26] we showed that for any $\lambda>0$,

$$\lambda\mathcal{H}^k(\{\mathbf{M}^k(\mu, X)>\lambda\}) \gtrsim |\mu|^s(X). \tag{2.9}$$

Therefore, it follows from (2.8) and (2.9) that for any $B_R := B_R(0, X) \subset X$,

$$|\mu|^s(B_R) \lesssim \limsup_{\lambda\to\infty} \lambda\mathcal{H}^k\big(\{\mathbf{M}^k(\mu, X)>\lambda\} \cap B_R\big) \lesssim |\mu|^s(\overline{B_R}). \tag{2.10}$$

Again, (2.8) and (2.9) imply that $\mu \ll \mathcal{H}^k$ in X if any only if

$$\limsup_{\lambda\to\infty} \lambda\mathcal{H}^k(\{\mathbf{M}^k(\mu, X)>\lambda\}) = 0. \tag{2.11}$$

Next is a basic estimate of the Hardy-Littlewood maximal function, which will be used several times in this paper.

LEMMA 2.4. *Let X be a vector subspace of $\mathbb{R}^d$ with* $\dim(X) = k$ *and $q>1$. Then, for any $\mu \in \mathcal{M}_b(X)$ and ball $B_R := B_R(0, X) \subset X$ and $f \in L^q(B_R)$ there holds*

$$\limsup_{\delta\to 0} \frac{1}{|\log(\delta)|} \int_{B_R} (\delta^{-1}|f|) \wedge \mathbf{M}^k(\mu, X) d\mathcal{H}^k \lesssim |\mu|^s(\overline{B_R}). \tag{2.12}$$

Moreover, for any $0<\delta\ll 1$,

$$\frac{1}{|\log(\delta)|} \int_{B_R} (\delta^{-1}|f|) \wedge \mathbf{M}^k(\mu, X) d\mathcal{H}^k \lesssim R^k + |\mu|(X) + \|f\|_{L^q(B_R)}^q. \tag{2.13}$$

PROOF. Set $A(\lambda_1) = \sup_{\lambda>\lambda_1} \lambda \mathcal{H}^k(\{\mathbf{M}^k(\mu, X) > \lambda\} \cap B_R) \lesssim |\mu|(X)$. One has for any $0 < \delta \ll 1$ and $0 < \lambda_1 < \lambda_2 < \infty$,

$$\begin{aligned}
&\frac{1}{|\log(\delta)|} \int_{B_R} (\delta^{-1}|f|) \wedge \mathbf{M}^k(\mu, X) d\mathcal{H}^k \\
&\quad = \frac{1}{|\log(\delta)|} \int_0^\infty \mathcal{H}^k\Big(\Big\{(\delta^{-1}|f|) \wedge \mathbf{M}^k(\mu, X) > \lambda\Big\} \cap B_R\Big) d\lambda \\
&\quad \le \frac{1}{|\log(\delta)|} \int_0^{\lambda_1} \mathcal{H}^k(B_R) d\lambda + \frac{1}{|\log(\delta)|} \int_{\lambda_1}^{\lambda_2} A(\lambda_1) \frac{d\lambda}{\lambda} \\
&\qquad + \frac{1}{|\log(\delta)|} \int_{\lambda_2}^\infty \mathcal{H}^k(\{|f| > \delta\lambda\} \cap B_R) d\lambda \\
&\quad \le \frac{\lambda_1}{|\log(\delta)|} \mathcal{H}^k(B_R) + \frac{\log(\lambda_2/\lambda_1)}{|\log(\delta)|} A(\lambda_1) + \frac{1}{q|\log(\delta)|\lambda_2^{q-1}\delta^q} \|f\|_{L^q(B_R)}^q.
\end{aligned}$$

Choosing $\lambda_1 = |\log(\delta)|^{1/2}$ and $\lambda_2 = \delta^{-\frac{q}{q-1}}$, and thanks to (2.5) and (2.10), we obtain (2.12) and (2.13). The proof is complete. □

2.3 Singular integral operators with rough kernels

In this subsection, we provide some basic properties of singular integral operators with rough convolution kernels. In this paper, we consider the following general kernel in $\mathbb{R}^d$:

$$\mathbf{K}(x) = \Omega(x) K(x) \quad \forall x \in \mathbb{R}^d \setminus \{0\}, \tag{2.14}$$

where

- $K \in C^1(\mathbb{R}^d \setminus \{0\})$,

$$|K(x)| + |x||\nabla K(x)| \le \frac{1}{|x|^d} \ \forall x \in \mathbb{R}^d, \tag{2.15}$$

- $\Omega(\theta) = \Omega(r\theta)$ for any $r > 0, \theta \in \mathbb{S}^{d-1}$ and

$$\begin{aligned}
&\|\Omega\|_{W^{\alpha_0,1}(B_2\setminus B_1)} \\
&\quad := \int_{B_2\setminus B_1} |\Omega| + \int_{B_2\setminus B_1} \int_{B_2\setminus B_1} \frac{|\Omega(x) - \Omega(y)|}{|x - y|^{d+\alpha_0}} \, dx \, dy \le c_1,
\end{aligned} \tag{2.16}$$

  for some $\alpha_0 \in (0, 1)$ and $c_1 > 0$,
- the "cancellation" condition

$$\sup_{0<R_1<R_2<\infty} \left| \int_{R_1<|x|<R_2} \mathbf{K}(x) dx \right| \le c_2 \tag{2.17}$$

  for some $c_2 > 0$.

We say that the kernel $\mathbf{K}$ is a singular kernel of fundamental type in $\mathbb{R}^d$ if $\Omega \in C^1(\mathbb{S}^{d-1})$.

Remark 2.5. From (2.15) one has

$$|K(x-y)-K(x)| \leq \frac{2^{d+1}|y|}{|x|^{d+1}} \quad \forall |y|<|x|/2. \tag{2.18}$$

Remark 2.6. If $K(x)=|x|^{-d}$ for any $x \in \mathbb{R}^d \setminus \{0\}$, then (2.17) implies

$$\int_{\mathbb{S}^{d-1}} \Omega(\theta) d\mathcal{H}^{d-1}(\theta)=0.$$

Moreover, if we set

$$\Omega_n(x) := \int_0^\infty \widetilde{\Omega} \star \varrho_n\left(\frac{x}{|x|}r\right) r^{d-1}\,dr \quad \forall x \in \mathbb{R}^d, \tag{2.19}$$

where $\widetilde{\Omega}(x) := \frac{1}{\log(2)} \frac{\Omega(x)}{|x|^d} \mathbf{1}_{1\leq|x|\leq 2}$, then $\int_{\mathbb{S}^{d-1}} \Omega_n(\theta) d\mathcal{H}^{d-1}(\theta)=0$ for any n, $\Omega_n \in C_b^\infty(\mathbb{S}^{d-1})$, $\Omega_n(\theta)=\Omega_n(r\theta)$ for any $r>0, \theta \in \mathbb{S}^{d-1}$, and

$$\|\Omega_n - \Omega\|_{\dot{W}^{\frac{\alpha_0}{2},1}(B_2\setminus B_1)} \lesssim c_1 n^{-\frac{\alpha_0}{2}} \quad \forall n. \tag{2.20}$$

Remark 2.7. Since $\Omega(\theta)=\Omega(r\theta)$ for any $r>0, \theta \in \mathbb{S}^{d-1}$, so by the Sobolev inequality and condition II, one gets

$$\begin{aligned}
&\|\Omega\|_{L^q(\mathbb{S}^{d-1})} \\
&\quad + \sup_{|h|\leq 1/2} |h|^{-\alpha_0/2}\big(\|\Omega(\cdot-h)-\Omega(\cdot)\|_{L^q(B_2\setminus B_1)} \\
&\quad + \|\Omega(\cdot-h)-\Omega(\cdot)\|_{L^q(\mathbb{S}^{d-1})}\big) \\
&\quad + \left(\int_{B_2\setminus B_1} \sup_{\rho\in(0,1/2)} \fint_{B_\rho(0)} \frac{|\Omega(x-h)-\Omega(x)|^q}{|h|^{\frac{\alpha_0 q}{2}}} dh\,dx\right)^{1/q} \\
&\quad \lesssim \|\Omega\|_{W^{\alpha_0,1}(B_2\setminus B_1)} \lesssim c_1,
\end{aligned} \tag{2.21}$$

for any $1 \leq q \leq q_0 = \frac{d}{d-\alpha_0/2}$. Moreover, we also have for any $\lambda_0 \leq 2$ and $\theta \in \mathbb{S}^{d-1}$

$$\begin{aligned}
&\int_{|y|\leq\lambda_0} |\Omega(\theta-y)-\Omega(\theta)| \frac{dy}{|y|^d} \\
&\quad \lesssim \lambda_0^{\frac{\alpha_0}{2}} \sup_{r\leq 2} r^{-\frac{\alpha_0}{2}} \fint_{B_r(0)} |\Omega(\theta-y)-\Omega(\theta)|dy.
\end{aligned} \tag{2.22}$$

Remark 2.8. Thanks to (2.15) and Minkowski's inequality, one has

$$\|(\mathbf{1}_{|\cdot|>\varepsilon}\mathbf{K}(\cdot)) \star \mu\|_{L^{q_0}(\mathbb{R}^d)} \lesssim \varepsilon^{-\frac{(q_0-1)d}{q_0}} \|\Omega\|_{L^{q_0}(\mathbb{S}^{d-1})} |\mu|(\mathbb{R}^d), \tag{2.23}$$

for $\varepsilon>0$, $q_0=\frac{d}{d-\alpha_0/2}$, and $\mu \in \mathcal{M}_b(\mathbb{R}^d)$.

The following is L^p and weak-type $(1,1)$ boundedness of singular integral operators associated to the kernel $\mathbf{K}$.

PROPOSITION 2.9. *Let* $\mathbf{K}$ *be in* (2.14) *with constants* $c_1, c_2 > 0$ *and* $\alpha_0 \in (0,1)$. *Let* $\chi \in C_c(\mathbb{R}^d, [0,1])$ *be such that* $\chi = 1$ *in* $|x| > 3$ *and* $\chi = 0$ *in* $|x| < 2$. *For* $f \in C_c^\infty(\mathbb{R}^d)$, *we define for* $x \in \mathbb{R}^d$,

$$\mathbf{T}^1(f)(x) = \mathbf{K} \star f(x), \quad \mathbf{T}^2(f)(x) = \sup_{\varepsilon>0} \left| \left(\chi\left(\frac{\cdot}{\varepsilon}\right) \mathbf{K} \right) \star f(x) \right|,$$
$$\mathbf{T}^3(f)(x) = \sup_{\varepsilon>0} \left| \left(\mathbf{1}_{|\cdot|>\varepsilon} \mathbf{K} \right) \star f(x) \right|.$$

Then, $\mathbf{T}^1$, $\mathbf{T}^2$, *and* $\mathbf{T}^3$ *extend to bounded operators from* L^p *to itself* ($p > 1$) *and from* L^1 *to* $L^{1,\infty}$ *with norms*

$$\sum_{j=1,2,3} \|\mathbf{T}^j\|_{L^p \to L^p} + \|\mathbf{T}^j\|_{L^1 \to L^{1,\infty}} \lesssim c_1 + c_2. \tag{2.24}$$

Moreover, we also get

$$\sum_{j=1,2,3} \|\mathbf{T}^j\|_{\mathcal{M}_b \to L^{1,\infty}} \lesssim c_1 + c_2, \tag{2.25}$$

and for any $\mu \in \mathcal{M}_b(\mathbb{R}^d)$, *there holds*

$$\sum_{j=1,2,3} \limsup_{\lambda\to\infty} \lambda \mathcal{L}^d \left(\{ |\mathbf{T}^j(\mu)| > \lambda \} \right) \lesssim (c_1 + c_2) |\mu|^s(\mathbb{R}^d). \tag{2.26}$$

PROOF. First, we need to check that

$$\sup_{R>0} \int_{R<|x|<2R} |\mathbf{K}(x)| dx \lesssim c_1, \tag{2.27}$$

$$\sup_{y\neq 0} \int_{|x|\geq 2|y|} |\mathbf{K}(x-y) - \mathbf{K}(x)| dx \lesssim c_1. \tag{2.28}$$

Indeed, by (2.15) one has

$$\sup_{R>0} \int_{R<|x|<2R} |\mathbf{K}(x)| dx \leq \sup_{R>0} \int_{R<|x|<2R} \frac{|\Omega(x)|}{|x|^d} dx = \log(2) \int_{\mathbb{S}^{d-1}} |\Omega| \lesssim c_1,$$

which implies (2.27). Moreover, for any $y \neq 0$,

$$\begin{aligned}
&\int_{|x|\geq 2|y|} |\mathbf{K}(x-y) - \mathbf{K}(x)| dx \\
&\overset{(2.18)}{\lesssim} \int_{|x|\geq 2|y|} \frac{|\Omega(x)\|y|}{|x|^{d+1}} dx \\
&\quad + \sum_{j=1}^{\infty} \frac{1}{(2^j|y|)^d} \int_{2^j|y|<|x|<2^{j+1}|y|} |\Omega(x-y) - \Omega(x)| dx \\
&\overset{(2.18)}{\lesssim} \int_{2|y|}^{\infty} \int_{\mathbb{S}^{d-1}} |\Omega(\theta)| \frac{|y|}{r^2} d\mathcal{H}^{d-1}(\theta) dr
\end{aligned}$$

$$+ \int_0^{3/4} \sup_{|h|\le\rho} \int_{B_2\setminus B_1} |\Omega(x-h)-\Omega(x)|dx \frac{d\rho}{\rho}$$

$$\lesssim c_1,$$

which implies (2.28).

Therefore, $\mathbf{K}$ satisfies (2.27), (2.28), and (2.17), so by [42, theorems 5.4.1, 5.4.5, and 5.3.5], we obtain (2.24).

We now prove (2.25). Let $\mu \in \mathcal{M}_b(\mathbb{R}^d)$. Thanks to (2.23), one has $(\chi(\cdot/\varepsilon)\mathbf{K}) \star \mu \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ for any $\varepsilon > 0$. Thus, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} (\chi(./\varepsilon)\mathbf{K}) \star (\varrho_n \star \mu) = (\chi(./\varepsilon)\mathbf{K}) \star \mu \text{ a.e in } \mathbb{R}^d.$$

This implies that $\mathbf{1}_{|\mathbf{T}^2(\mu)|>\lambda} \le \liminf_{n\to\infty} \mathbf{1}_{|\mathbf{T}^2(\varrho_n\star\mu)|>\lambda}$ a.e. in $\mathbb{R}^d$ for any $\lambda > 0$. On the other hand, by (2.24),

$$\sup_{\lambda>0} \lambda\mathcal{L}^d\big(\{|\mathbf{T}^2(\varrho_n\star\mu)|>\lambda\}\big) \lesssim (c_1+c_2)\|\varrho_n\star\mu\|_{L^1(\mathbb{R}^d)} \lesssim (c_1+c_2)|\mu|(\mathbb{R}^d).$$

By applying Fatou's lemma, we find

$$\sup_{\lambda>0} \lambda\mathcal{L}^d\big(\{|\mathbf{T}^2(\mu)|>\lambda\}\big) \lesssim (c_1+c_2)|\mu|(\mathbb{R}^d).$$

Similarly, we also get

$$\sup_{\lambda>0} \lambda\mathcal{L}^d\big(\{|\mathbf{T}^3(\mu)|>\lambda\}\big) \lesssim (c_1+c_2)|\mu|(\mathbb{R}^d).$$

Hence, we conclude (2.25) since $|\mathbf{T}^1(\mu)| \le |\mathbf{T}^3(\mu)|$. To get (2.26), one has for $R > 1$ and $\gamma > 1$

$$\begin{aligned}
&\lambda\mathcal{L}^d\big(\{|\mathbf{T}^j(\mu)|>\lambda\}\big)\\
&\quad\le \lambda\mathcal{L}^d\big(\{|\mathbf{T}^j(\mu^s)|>\lambda/4\}\big) + \lambda\mathcal{L}^d\big(\{|\mathbf{T}^j(\mu^{\mathrm{a}}\mathbf{1}_{B_R^c})|>\lambda/4\}\big)\\
&\qquad + \lambda\mathcal{L}^d\big(\{|\mathbf{T}^j(\mu^{\mathrm{a}}\mathbf{1}_{|\mu|^a>\gamma}\mathbf{1}_{B_R})|>\lambda/4\}\big) + \lambda\mathcal{L}^d\big(\{|\mathbf{T}(\mu^{\mathrm{a}}1_{|\mu|^a\le\gamma}\mathbf{1}_{B_R})|>\lambda/4\}\big).
\end{aligned}$$

Using the boundedness of $\mathbf{T}$ from $\mathcal{M}_b(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$ for first three terms and the boundedness of $\mathbf{T}$ from $L^2(\mathbb{R}^d)$ to itself for last term yields

$$\begin{aligned}
&\lambda\mathcal{L}^d\big(\{|\mathbf{T}^j(\mu)|>\lambda\}\big)\\
&\quad\lesssim (c_1+c_2)\bigg(|\mu|^s(\mathbb{R}^d) + \int_{B_R^c} |\mu|^a + \int_{B_R} \mathbf{1}_{|\mu|^a>\gamma}|\mu|^a\\
&\qquad\qquad + \lambda^{-1}\int_{B_R} \mathbf{1}_{|\mu|^a\le\gamma}(|\mu|^a)^2\bigg).
\end{aligned}$$

This implies (2.26) by letting $\lambda \to \infty$ and then $\gamma \to \infty$, $R \to \infty$. The proof is complete. □

Remark 2.10. Since $\mathbf{T}^j(\mathbf{1}_{B^c_{R+\varepsilon}}\mu) \in L^1(B_R)$ for any $B_R \subset \mathbb{R}^d$ and $\varepsilon > 0$, so

$$\limsup_{\lambda\to\infty} \lambda\mathcal{L}^d\big(\{|\mathbf{T}^j(\mu)| > \lambda\} \cap B_R\big) \\ \leq \limsup_{\lambda\to\infty} \lambda\mathcal{L}^d\big(\{|\mathbf{T}^j(\mathbf{1}_{B_{R+\varepsilon}}\mu)| > \lambda/2\}\big) \quad \forall \varepsilon > 0.$$

Applying (2.26) to $\mathbf{1}_{B_{R+\varepsilon}}\mu$ and then letting $\varepsilon \to 0$, we find that

$$\sum_{j=1,2,3} \limsup_{\lambda\to\infty} \lambda\mathcal{L}^d\big(\{|\mathbf{T}^j(\mu)| > \lambda\} \cap B_R\big) \lesssim (c_1 + c_2)|\mu|^s(\bar{B}_R).$$

Remark 2.11. It is unknown when $\mathbf{T}^3$ is bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$ where $\Omega \in W^{\alpha_0,1}(B_2 \setminus B_1)$ is replaced by $\Omega \in L^q(\mathbb{S}^{d-1})$ for $q > 1$. This is an interesting open problem posed by A. Seeger.

In this paper, we also need a boundedness of the following singular maximal operator: for $f \in L^1(\mathbb{R}^d)$,

$$\mathbf{M}^{\widetilde{\Omega}} f(x) = \sup_{\rho>0} \fint_{B_\rho(x)} |\widetilde{\Omega}(y/|y|)||f(x-y)|dy \quad \forall x \in \mathbb{R}^d,$$

where $\widetilde{\Omega} \in L^1(\mathbb{S}^{d-1})$.

PROPOSITION 2.12 ([28,29,51]). *There hold for $p > 1$ and $q > 1$,*

$$\|\mathbf{M}^{\widetilde{\Omega}}\|_{L^p\to L^p} \lesssim \|\widetilde{\Omega}\|_{L^1(\mathbb{S}^{d-1})}, \ \|\mathbf{M}^{\widetilde{\Omega}}\|_{L^1\to L^{1,\infty}} \lesssim \|\widetilde{\Omega}\|_{L^q(\mathbb{S}^{d-1})}. \tag{2.29}$$

By a standard approximation, we obtain from (2.29) that for $q > 1$

$$\|\mathbf{M}^{\widetilde{\Omega}}\|_{\mathcal{M}_b\to L^{1,\infty}} \lesssim \|\widetilde{\Omega}\|_{L^q(\mathbb{S}^{d-1})}. \tag{2.30}$$

PROPOSITION 2.13. *Let $\mathbf{K}$ be in (2.14) with constants $c_1, c_2 > 0$ and $\alpha_0 \in (0,1)$. Let $\{\phi^e\}_e \subset C^1(\mathbb{R}^d \setminus \{0\}) \cap L^\infty(\mathbb{R}^d)$ be a family of kernels such that*

$$\operatorname{supp}(\phi^e) \subset B_1, \qquad \sup_{x\in\mathbb{R}^d, e} |\phi^e(x)| + |x||\nabla\phi^e(x)| \leq c_0.$$

For $\alpha \in (0,d)$ and $f \in C_c^\infty(\mathbb{R}^d)$ we define

$$\mathbf{T}(f)(x) = \sup_e \sup_{\rho>0} \left| \left(\frac{\rho^{-\alpha}}{|\cdot|^{d-\alpha}} \phi^e\Big(\frac{\cdot}{\rho}\Big) \right) \star \mathbf{K} \star f(x) \right| \ \forall x \in \mathbb{R}^d.$$

Then, $\mathbf{T}$ extends to bounded operator from $L^p(\mathbb{R}^d)$ to itself ($p > 1$) and $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$ with norms

$$\|\mathbf{T}\|_{L^p\to L^p} + \|\mathbf{T}\|_{\mathcal{M}_b\to L^{1,\infty}} \lesssim c_0(c_1 + c_2). \tag{2.31}$$

Moreover, for any $\mu \in \mathcal{M}_b(\mathbb{R}^d)$

$$\limsup_{\lambda\to\infty} \lambda\mathcal{L}^d(\{\mathbf{T}(\mu) > \lambda\}) \lesssim c_0(c_1 + c_2)|\mu|^s(\mathbb{R}^d), \tag{2.32}$$

In particular, for any $B_R \subset \mathbb{R}^d$ *and* $f \in L^q(B_R)$ *for* $q > 1$,

$$\limsup_{\delta\to 0} \frac{1}{|\log(\delta)|} \int_{B_R} \min\{\delta^{-1}|f|, \mathbf{T}(\mu)\} \lesssim c_0(c_1 + c_2)|\mu|^s(\bar{B}_R). \tag{2.33}$$

Proposition 2.13 is still true for any $\alpha \geq d$. This was proven in [17] for the smooth kernel case (i.e., $\Omega \in C_b^1(\mathbb{S}^{d-1})$).

PROOF OF PROPOSITION 2.13. For $f \in C_c^\infty(\mathbb{R}^d)$, we set

$$\mathbf{T}_1(f)(x) = \sup_{\rho>0}\left|\int_{|x-z|>2\rho} \mathbf{K}(x-z)f(z)dz\right|,$$
$$\mathbf{T}_2(f)(x) = \sup_{e}\sup_{\rho>0}\left|\mathbf{K}_{e,\rho} \star f(x)\right|,$$

for any $x \in \mathbb{R}^d$, where

$$\mathbf{K}_{e,\rho}(x) = \int_{\mathbb{R}^d} \frac{\rho^{-\alpha}}{|y|^{d-\alpha}}\phi^e\left(\frac{y}{\rho}\right)\mathbf{K}(x-y)dy - \int_{\mathbb{R}^d} \frac{1}{|y|^{d-\alpha}}\phi^e(y)dy\mathbf{1}_{|x|>2\rho}\mathbf{K}(x).$$

We show that

$$|\mathbf{K}_{e,\rho}(x)| \lesssim \frac{c_0}{|x|^{d-\alpha}} \min\left\{\frac{1}{\rho^\alpha}, \frac{\rho^{\frac{\alpha_0}{2}}}{|x|^{\frac{\alpha_0}{2}+\alpha}}\right\}\Omega_1(x/|x|) \quad \forall x \in \mathbb{R}^d, \tag{2.34}$$

where

$$\Omega_1(\theta) = c_1 + c_2 + |\Omega(\theta)| + \sup_{r\in(0,2)} r^{-\frac{\alpha_0}{2}} \fint_{B_r(0)} |\Omega(\theta - z) - \Omega(\theta)|dz \quad \forall \theta \in \mathbb{S}^{d-1}.$$

It is enough to prove that

$$\left|\int_{\mathbb{R}^d} \mathbf{K}(x-y)\frac{1}{|y|^{d-\alpha}}\phi^e\left(\frac{y}{\rho}\right)dy\right| \lesssim \frac{c_0}{|x|^{d-\alpha}}\Omega_1(x/|x|) \quad \text{if } |x| \leq 2\rho, \tag{2.35}$$

$$|\mathbf{K}_{e,\rho}(x)| \lesssim \frac{c_0\rho^{\frac{\alpha_0}{2}}}{|x|^{d+\frac{\alpha_0}{2}}}\Omega_1(x/|x|) \quad \text{if } |x| \geq 2\rho. \tag{2.36}$$

Case 1. $|x| \leq 2\rho$. Assume that $2^{-j_0}\rho < |x| \leq 2^{-j_0+1}\rho$ for $j_0 \in \mathbb{N}$. Let χ be a smooth function in $\mathbb{R}^d$ such that $\chi(y) = 1$ if $|y| \leq 1$ and $\chi(y) = 0$ if $|y| > \frac{11}{10}$. Set

$$\mathbf{K}_0(x) = \int_{\mathbb{R}^d} \mathbf{K}(x-y)\frac{1}{|y|^{d-\alpha}}\phi^e\left(\frac{y}{\rho}\right)\left(\chi(2^{j_0-2}\rho^{-1}y) - \chi(2^{j_0+1}\rho^{-1}y)\right)dy.$$

By definition of $\mathbf{K}$ and ϕ^e, we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} \mathbf{K}(x-y)\frac{1}{|y|^{d-\alpha}}\phi^e\left(\frac{y}{\rho}\right)dy\right| \\
&\quad\lesssim \mathbf{K}_0(x) + c_0\int_{|y|\geq 2|x|}\frac{|\Omega(x-y)|}{|x-y|^d|y|^{d-\alpha}}dy \\
&\qquad + c_0\int_{|y|\leq |x|/2}\frac{|\Omega(x-y)|}{|x-y|^d|y|^{d-\alpha}}dy \\
&\quad\lesssim \mathbf{K}_0(x) + c_0\int_{|x-y|\geq |x|}\frac{|\Omega(x-y)|}{|y-x|^{2d-\alpha}}dy \\
&\qquad + \frac{c_0}{|x|^{d-\alpha}}\int_{|y|\leq |x|/2}|\Omega(x-y)-\Omega(x)|\frac{dy}{|y|^d} + \frac{c_0|\Omega(x)|}{|x|^{d-\alpha}} \\
&\quad\lesssim \mathbf{K}_0(x) + c_0\frac{\|\Omega\|_{L^1(\mathbb{S}^{d-1})}+|\Omega(x)|}{|x|^{d-\alpha}} \\
&\qquad + \frac{c_0}{|x|^{d-\alpha}}\int_{|y|\leq 1/2}\left|\Omega\left(\frac{x}{|x|}-y\right)-\Omega\left(\frac{x}{|x|}\right)\right|\frac{dy}{|y|^d} \\
&\quad\lesssim \mathbf{K}_0(x) + \frac{c_0}{|x|^{d-\alpha}}\Omega_1(x/|x|).
\end{aligned}\tag{2.37}$$

Here we have used (2.22) in the last inequality.

Thanks to the Gagliardo-Nirenberg interpolation inequality, we find

$$\sup_{|y|\sim|x|}|\mathbf{K}_0(y)| \lesssim |x|^{1/2}\|\nabla \mathbf{K}_0\|_{L^{2d}(\mathbb{R}^d)} + \frac{1}{|x|^{d/2}}\left(\int_{\mathbb{R}^d}|\mathbf{K}_0|^2\,dy\right)^{1/2}.$$

By the boundedness of operator $f\mapsto \mathbf{K}\star f$ in L^2 and $\dot{W}^{1,2d}$ (see Proposition 2.9), one obtains

$$\begin{aligned}
&\sup_{|y|\sim|x|}|\mathbf{K}_0(y)| \\
&\quad\lesssim (c_1+c_2)|x|^{1/2}\left(\int_{\mathbb{R}^d}\left|\nabla\left(\frac{\phi^e(\frac{y}{\rho})}{|y|^{d-\alpha}}\big(\chi(2^j\rho^{-1}y)-\chi(2^{j+1}\rho^{-1}y)\big)\right)\right|^{2d}dy\right)^{c\frac{1}{2d}} \\
&\qquad + (c_1+c_2)\frac{1}{|x|^{d/2}}\left(\int_{\mathbb{R}^d}\left|\frac{\phi^e(\frac{y}{\rho})}{|y|^{d-\alpha}}\big(\chi(2^j\rho^{-1}y)-\chi(2^{j+1}\rho^{-1}y)\big)\right|^2dy\right)^{1/2} \\
&\quad\lesssim c_0(c_1+c_2)\left(|x|^{1/2}(2^{-j_0}\rho)^{-d+\alpha-\frac{1}{2}} + \frac{1}{|x|^{d/2}}(2^{-j_0}\rho)^{-d/2+\alpha}\right) \\
&\quad\lesssim \frac{c_0(c_1+c_2)}{|x|^{d-\alpha}}.
\end{aligned}\tag{2.38}$$

Thus, it follows (2.35) from (2.37) and (2.38).

Case 2. $|x|>2\rho$. By (2.15) and (2.18) we have

$$
\begin{aligned}
\left|\mathbf{K}_{e,\rho}(x)\right| &\le \left|\int_{\mathbb{R}^d}\frac{\rho^{-\alpha}}{|y|^{d-\alpha}}\phi^e\left(\frac{y}{\rho}\right)(\mathbf{K}(x-y)-\mathbf{K}(x))dy\right|\\
&\lesssim c_0|\Omega(x)|\int_{|y|<\rho}\frac{\rho^{-\alpha}}{|y|^{d-\alpha}}\frac{|y|}{|x|^{d+1}}dy\\
&\quad+\frac{c_0}{|x|^d}\int_{|y|<\rho}\frac{\rho^{-\alpha}}{|y|^{d-\alpha}}|\Omega(x-y)-\Omega(x)|dy \qquad (2.39)\\
&\lesssim \frac{c_0|\Omega(x)|\rho}{|x|^{d+1}}+\frac{c_0}{|x|^d}\int_{|y|<\rho/|x|}\left|\Omega\left(\frac{x}{|x|}-y\right)-\Omega\left(\frac{x}{|x|}\right)\right|\frac{dy}{|y|^d}\\
&\overset{(2.22)}{\lesssim} \frac{c_0\rho}{|x|^{d+1}}\Omega_1(x/|x|)+\frac{c_0\rho^{\frac{\alpha_0}{2}}}{|x|^{d+\frac{\alpha_0}{2}}}\Omega_1(x/|x|),
\end{aligned}
$$

which implies (2.36).

Then, (2.34) gives

$$
|\mathbf{T}(f)|\lesssim c_0\mathbf{T}_1(f)+\mathbf{T}_2(f)\lesssim c_0\mathbf{T}_1(f)+c_0\mathbf{M}^{\Omega_1}(f). \tag{2.40}
$$

Thanks to Proposition 2.9 and 2.12 and using the fact that $\|\Omega_1\|_{L^q(\mathbb{S}^{d-1})}\overset{(2.21)}{\lesssim} c_1+c_2$, we get

$$
\left\|\left(\mathbf{T}^1,\mathbf{M}^{\Omega_1}\right)\right\|_{L^p\to L^p}+\left\|\left(\mathbf{T}^1,\mathbf{M}^{\Omega_1}\right)\right\|_{L^1\to L^{1,\infty}}\lesssim c_0(c_1+c_2).
$$

This gives (2.31). Then, similar to the proof of (2.26) and (2.12), we obtain (2.32) and (2.33) from (2.31). The proof is complete. □

LEMMA 2.14. *We denote for $\rho_0>0$, $\alpha_1\in(0,\alpha]$, and $\mu\in\mathcal{M}_b(\mathbb{R}^d)$*

$$
\mathbf{T}^{\alpha_1}(\mu)(x)=\sup_e\sup_{\rho\in(0,\rho_0)}\left|\left(\frac{\rho^{\alpha_1-\alpha}}{|\cdot|^{d-\alpha}}\phi^e\left(\frac{\cdot}{\rho}\right)\right)\star\mathbf{K}\star\mu(x)\right|\quad\forall x\in\mathbb{R}^d.
$$

Then,

$$
\begin{aligned}
&\|\mathbf{T}^{\alpha_1}(\mu)\|_{L^{q_0}(\mathbb{R}^d)}\lesssim(c_1+c_2)|\mu|(\mathbb{R}^d),\\
&q_0=\frac{d}{d-\frac{1}{4}\min\{\alpha,\alpha_0,\alpha_1\}}>1.
\end{aligned}\tag{2.41}
$$

PROOF. We deduce from (2.35) and (2.36) that for any $x\in\mathbb{R}^d$,

$$
\begin{aligned}
\left|\left(\frac{\rho^{\alpha_1-\alpha}}{|\cdot|^{d-\alpha}}\phi^e\left(\frac{\cdot}{\rho}\right)\right)\star\mathbf{K}(x)\right|&\lesssim\left(\frac{\rho^{\alpha_1-\alpha}}{|x|^{d-\alpha}}\wedge\frac{\rho^{\alpha_1}}{|x|^d}\right)\Omega_1\left(\frac{x}{|x|}\right)\\
&\lesssim(1+\rho^{\alpha_1})P(x),
\end{aligned}
$$

with

$$
P(x)=\left(\frac{1}{|x|^{d-\alpha_1}}\wedge\frac{1}{|x|^d}\right)\Omega_1\left(\frac{x}{|x|}\right).
$$

Thus,

$$
\mathbf{T}^{\alpha_1}(\mu)(x)\lesssim(1+\rho_0^{\alpha_1})P\star|\mu|(x).
$$

Then, by Minkowski's inequality, one has

$$\|\mathbf{T}^{\alpha_1}(\mu)\|_{L^{q_0}(\mathbb{R}^d)} \lesssim (1+\rho_0^{\alpha_1})\|\Omega_1\|_{L^{q_0}(\mathbb{S}^{d-1})}|\mu|(\mathbb{R}^d),$$

which implies (2.41). □

Remark 2.15. As done for Lemma (2.14), we also show that for $\rho_0 > 0$ and $\mu \in \mathcal{M}_b(\mathbb{R}^d)$,

$$\begin{aligned}\mathbf{P}(\mu)(x) = \sup_e \sup_{\rho\in(0,\rho_0)} &\left| \left(\frac{\rho^{-\alpha}}{|\cdot|^{d-\alpha}} \phi^e\left(\frac{\cdot}{\rho}\right)\right) \star \mathbf{K} \star ((\psi(\cdot) - \psi(x))\mu)(x)\right| \\ &\in L^{q_0}_{\text{loc}}(\mathbb{R}^d),\end{aligned} \tag{2.42}$$

for some $q_0 > 1$, with $\psi \in W^{1,\infty}(\mathbb{R}^d)$; one has

$$\|\mathbf{P}(\mu)\|_{L^{q_0}(B_R(0))} \lesssim_R \|\psi\|_{W^{1,\infty}(\mathbb{R}^d)}|\mu|(\mathbb{R}^d) \quad \forall R > 0. \tag{2.43}$$

Furthermore, if $\Omega \in C^1_b(\mathbb{S}^{d-1})$, then $\mathbf{P}(\mu)(x) \lesssim \mathbf{I}_1(\mu)(x)$ for any $x \in \mathbb{R}$.

Remark 2.16. If

$$\mu_t(x) = \mu(t,x) \in L^1([0,T],\mathcal{M}_b(\mathbb{R}^d)) \quad \text{and} \quad f \in L^1((0,T),L^q(B_R))$$

for $q > 1$, it follows from (2.33) and the dominated convergence theorem that

$$\begin{aligned}\limsup_{\delta\to 0} \frac{1}{|\log(\delta)|} \int_0^T \int_{B_R} &\min\{\delta^{-1}|f(x,t)|, \mathbf{T}(\mu_t)(x)\} dx\, dt \\ &\lesssim \int_0^T |\mu_t|^s(\bar{B}_R) dt.\end{aligned} \tag{2.44}$$

Remark 2.17. We do not know how to prove Proposition 2.13 when $\alpha_0 = 0$.

3 Kakeya Singular Integral Operators

In this section we introduce the Kakeya singular integral operators and establish a strong version of (2.32) for this operator. It is a main tool of this paper.

Assume that $\{\phi^{e,\varepsilon}\}_{\varepsilon,e} \subset C^1(\mathbb{R}^d\backslash\{0\},\mathbb{R}^d)\cap L^\infty_c(\mathbb{R}^d,\mathbb{R}^d)$ is a family of kernels such that

$$\begin{aligned}\text{supp}(\phi^{e,\varepsilon}) \subset B_1(0) \cap &\left\{x : \left|e - \frac{x}{|x|}\right| \le \varepsilon\right\}, \\ |\phi^{e,\varepsilon}(x)| + \varepsilon|x||\nabla\phi^{e,\varepsilon}(x)| &\le c_0,\end{aligned} \tag{3.1}$$

for any $x \in \mathbb{R}^d$, $\varepsilon \in (0,1/10)$, and $e \in \mathbb{S}^{d-1}$. Let $\mathbf{K}$ be in (2.14) with constants $c_1, c_2 > 0$ and $\alpha_0 \in (0,1)$. Assume that there exists a sequence of $\Omega_n \in C^2_b(\mathbb{S}^{d-1})$ such that $\Omega_n(\theta) = \Omega_n(r\theta)$ for any $r > 0$, $\theta \in \mathbb{S}^{d-1}$, and

$$\|\Omega_n\|_{W^{\alpha_0,1}(B_2\backslash B_1)} \le 2c_1, \qquad \lim_{n\to\infty} \|\Omega_n - \Omega\|_{W^{\alpha_0,1}(B_2\backslash B_1)} = 0, \tag{3.2}$$

and $\mathbf{K}_n(x) := \Omega_n(x)K(x)$ satisfies (2.17) i.e.,

$$\sup_{0<R_1<R_2<0}\left|\int_{R_1<|x|<R_2}\mathbf{K}_n(x)dx\right| \leq c_3 \tag{3.3}$$

for some $c_3 > 0$. Moreover,

$$\lim_{n\to\infty}\sup_{0<R_1<R_2<0}\left|\int_{R_1<|x|<R_2}(\mathbf{K}_n(x)-\mathbf{K}(x))dx\right| = 0. \tag{3.4}$$

For any $\mu \in C_c^\infty(\mathbb{R}^d,\mathbb{R}^d)$ and $\rho_0 > 0$, the Kakeya singular integral operator $\mathbf{T}_\varepsilon$ is given by

$$\begin{aligned}\mathbf{T}_\varepsilon(\mu)(x)\\ &:= \mathbf{T}_\varepsilon^{\mathbf{K}}(\mu)(x)\\ &= \sup_{\rho\in(0,\rho_0),e\in S^{d-1}}\frac{\varepsilon^{-d+1}}{\rho^\alpha}\left|\left(\frac{1}{|\cdot|^{d-\alpha}}\phi_\rho^{e,\varepsilon}(\cdot)\right)\star\mathbf{K}\star\mu(x)\right| \quad \forall x\in\mathbb{R}^d,\end{aligned} \tag{3.5}$$

for some $\alpha \in (0,d)$, where $\phi_\rho^{e,\varepsilon}(\cdot) = \phi^{e,\varepsilon}(\frac{\cdot}{\rho})$. Set

$$\mathbf{T}_\varepsilon^{1,n} := \mathbf{T}_\varepsilon^{\mathbf{K}_n}, \quad \mathbf{T}_\varepsilon^{2,n} := \mathbf{T}_\varepsilon^{\mathbf{K}_n-\mathbf{K}}.$$

Thanks to Proposition 2.13 and conditions (3.2), (3.3), and (3.4), we have for any $\mu \in \mathcal{M}_b(\mathbb{R}^d)$

$$\begin{aligned}&\limsup_{\lambda\to\infty}\lambda\mathcal{L}^d(\{\mathbf{T}_\varepsilon(\mu)>\lambda\}) + \lambda\mathcal{L}^d\left(\{\mathbf{T}_\varepsilon^{1,n}(\mu)>\lambda\}\right)\\ &\quad\lesssim \varepsilon^{-d+1}(c_1+c_2)|\mu|^s(\mathbb{R}^d),\end{aligned} \tag{3.6}$$

$$\limsup_{\lambda\to\infty}\lambda\mathcal{L}^d\left(\{\mathbf{T}_\varepsilon^{2,n}(\mu)>\lambda\}\right) \lesssim \varepsilon^{-d+1}c_n|\mu|^s(\mathbb{R}^d), \tag{3.7}$$

for any $\varepsilon \in (0,1/10)$ $\forall n$, where $c_n = 0$ as $n \to \infty$.

Remark 3.1. Ω_n in Remark 2.6 satisfies (3.2), (3.3), and (3.4).

Remark 3.2. If $\mathbf{K} = \sum_{j=1}^d \mathcal{R}_j^2 = \delta_0$ where $\mathcal{R}_1,\ldots,\mathcal{R}_d$ are the Riesz transforms in $\mathbb{R}^d$, we thus get $\mathbf{T}_\varepsilon(\mu) \lesssim \mathbf{M}^\varepsilon(\mu)$, where $\mathbf{M}^\varepsilon$ is the Kakeya maximal function in $\mathbb{R}^d$, i.e.,

$$\mathbf{M}^\varepsilon(\mu)(x) = \sup_{\rho>0,e\in\mathbb{S}^{d-1}}\fint_{B_\rho(x)}\varepsilon^{-d+1}\mathbf{1}_{\left|\frac{z-x}{|z-x|}-e\right|\leq\varepsilon}d|\mu|(z) \quad \forall x\in\mathbb{R}^d.$$

Our main result is the following:

THEOREM 3.3. *Assume that $\mu = Df$, $f \in BV(\mathbb{R}^d)$. Then, we have*

$$\limsup_{\lambda\to\infty}\lambda\mathcal{L}^d(\{\mathbf{T}_\varepsilon(\mu)>\lambda\}) \lesssim |\log(\varepsilon)|\|\mu|^s(\mathbb{R}^d) \tag{3.8}$$

for any $\varepsilon \in (0, \frac{1}{10})$. *In particular, for any* $B_R \subset \mathbb{R}^d$ *and* $f \in L^q(B_R)$ *for* $q > 1$,

$$\limsup_{\delta\to 0} \frac{1}{|\log(\delta)|} \int_{B_R} \min\{\delta^{-1}|f|, \mathbf{T}_\varepsilon(\mu)\}dx \lesssim |\log(\varepsilon)||\mu|^s(\bar{B}_R) \tag{3.9}$$

for any $\varepsilon \in (0, \frac{1}{10})$.

Remark 3.4. Estimate (3.8) is not true for all $\mu \in \mathcal{M}_b(\mathbb{R}^d, \mathbb{R}^d)$. Indeed, let $d\mu = d\delta_{\{0\}}$ and $|\phi^{e,\varepsilon}(e)| \geq 1$ for any $e \in \mathbb{S}^{d-1}$ and $\varepsilon > 0$; let $\mathbf{T}_{j,\varepsilon}$ be $\mathbf{T}_\varepsilon$ associated to

$$\mathbf{K}(x) = \mathbf{K}_j(x) = \frac{|x|^2 - x_j^2 d}{|x|^{d+2}}.$$

One has

$$\sum_{j=1}^d \mathbf{T}_{j,\varepsilon}(\mu)(x) \gtrsim \frac{\varepsilon^{-d+1}}{|x|^d}|\phi^{x/|x|,\varepsilon}(x/|x|)| \gtrsim \frac{\varepsilon^{-d+1}}{|x|^d}.$$

Thus, for any $\lambda > 1$

$$\lambda\mathcal{L}^d\big(\{\mathbf{T}_{1,\varepsilon}(\mu) > \lambda\}\big) \geq d^{-1}\mathcal{L}^d\left(\left\{\sum_{j=1}^d \mathbf{T}_{j,\varepsilon}(\mu) > d\lambda\right\}\right) \gtrsim \varepsilon^{-d+1}.$$

It is well-known that

$$\{D_{x_1} f \in \mathcal{M}_b(\mathbb{R}^d) : f \in BV(\mathbb{R}^d)\}$$

is not dense in $L^1(\mathbb{R}^{d-1}, \mathcal{M}_{b,x_1}(\mathbb{R}))$. So, a natural question is *whether* (3.8) holds for any $\mu \in L^1(\mathbb{R}^{d-1}, \mathcal{M}_b(\mathbb{R}))$.

To prove Theorem 3.3, we need the following lemmas:

LEMMA 3.5. *Let* $\omega \in \mathcal{M}_b^+(\mathbb{R})$ *and* $a : \mathbb{R}^d \to \mathbb{R}^+$ *be a Borel function. Then, for any* $\rho > 0$,

$$\begin{aligned}&\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}} \mathbf{1}_{\rho<|y_1+y_2|\leq 2\rho} a(y_1 + y_2)d\omega(y_1)d\mathcal{H}^{d-1}(y_2)\\ &\quad\lesssim \rho^d\left[\int_{\mathbb{S}^{d-1}} \sup_{r\in[\rho,2\rho]} a(r\theta)d\mathcal{H}^{d-1}(\theta)\right]\mathbf{M}^1(\omega, \mathbb{R})(0).\end{aligned} \tag{3.10}$$

PROOF OF LEMMA 3.5. Let $d\omega_\kappa(y) = \mathbf{1}_{|y|>\kappa}d\omega(y)$ for $\kappa \in (0, \rho/100)$. Let ϱ_m be a standard sequence of mollifiers in $\mathbb{R}$. For any $m > 4/\kappa$, we have $\operatorname{supp}(\varrho_m \star \omega_\kappa) \subset \{z : |z| > \kappa/2\}$ and

$$\begin{aligned}&\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}} \mathbf{1}_{\rho<|y_1+y_2|\leq 2\rho} a(y_1 + y_2)(\varrho_m \star \omega_\kappa)(y_1)d\mathcal{H}^1(y_1)d\mathcal{H}^{d-1}(y_2)\\ &\quad= \int_{\mathbb{S}^{d-1}}\int_\rho^{2\rho} r^{d-1}a(r\theta)\mathbf{1}_{|r\theta_1|>\kappa/2}(\varrho_m \star \omega_\kappa)(r\theta_1)dr\, d\mathcal{H}^{d-1}(\theta)\\ &\quad\leq (2\rho)^{d-1}\int_{\mathbb{S}^{d-1}}\Big(\sup_{r'\in[\rho,2\rho]} a(r'\theta)\Big)\int_\rho^{2\rho} \mathbf{1}_{|r\theta_1|>\kappa/2}(\varrho_m \star \omega_\kappa)(r\theta_1)dr\, d\mathcal{H}^{d-1}(\theta).\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_\rho^{2\rho} \mathbf{1}_{|r\theta_1|>\kappa/2}(\varrho_m \star \omega_\kappa)(r\theta_1)dr &\le \int_{\mathbb{R}}\int_\rho^{2\rho} \mathbf{1}_{|r\theta_1|>\kappa/2}\varrho_m(r\theta_1 - z)dr\, d\omega(z)\\ &\le \frac{\mathbf{1}_{|\theta_1|>\frac{\kappa}{4\rho}}}{|\theta_1|}\int_{\mathbb{R}}\int_{-2|\theta_1|\rho}^{2|\theta_1|\rho}\varrho_m(r-z)drd\omega(z) \le \frac{\mathbf{1}_{|\theta_1|>\frac{\kappa}{4\rho}}}{|\theta_1|}\int_{\mathbb{R}}\mathbf{1}_{|z|<2|\theta_1|\rho+\frac{2}{m}}d\omega(z)\\ &\le \frac{\mathbf{1}_{|\theta_1|>\frac{\kappa}{4\rho}}}{|\theta_1|}\int_{-4|\theta_1|\rho}^{4|\theta_1|\rho}d\omega(z) \le 8\rho\mathbf{M}^1(\omega,\mathbb{R})(0).\end{aligned}$$

Thus, by Fatou's lemma, letting $m \to \infty$ and then $\kappa \to \infty$, we get (3.10). The proof is complete. □

Remark 3.6. From proof of Lemma 3.5 we can see that for any $e_0 \in \mathbb{S}^{d-1}$ and $\mu \in \mathcal{M}_b^+(\mathbb{R}^d)$ and $\omega \in \mathcal{M}_b^+(\widetilde{H}_{e_0})$, if $\mu \le \omega \otimes \mathcal{H}^{d-1}$ then

$$\mathbf{M}^\varepsilon(\mu)(x) \lesssim \mathbf{M}^1(\omega, \widetilde{H}_{e_0})(\langle e_0, x\rangle e_0) \quad \forall x \in \mathbb{R}^d,\ \varepsilon > 0. \tag{3.11}$$

LEMMA 3.7. *Let $\{e_1, \dots, e_d\}$ be an orthonormal basis in $\mathbb{R}^d$. Let $y_{0i} \in \widetilde{H}_{e_i}$ $i = 1, \dots, d$ and $\varepsilon \in (0,1)$. For any $x_i \in \widetilde{H}_{e_i}$, $i = 1, \dots, d$, we denote $\nu^1_{k,\sum_{i=d-k+1}^d x_i}, \nu^2_{k,\sum_{i=d-k+1}^d x_i} \in \mathcal{M}^+(\bigotimes_{i=1}^{d-k}\widetilde{H}_{e_i})$ for $k = 1, \dots, d$, by*

$$\begin{aligned}&d\nu^1_{k,\sum_{i=d-k+1}^d x_i}(y_{d-k}, \dots, y_1)\\ &\qquad = d\left|Df^{e_{d-k}}_{\sum_{i=1}^{d-k-1} y_i + \sum_{i=d-k+1}^d x_i}\right|(y_{d-k})d\mathcal{H}^1(y_{d-k-1})\cdots d\mathcal{H}^1(y_1),\end{aligned}$$

$$\begin{aligned}&d\nu^2_{k,\sum_{i=d-k+1}^d x_i}(y_{d-k}, \dots, y_1)\\ &\qquad = \mathbf{1}_{|\sum_{i=1}^{d-k} y_{0i} - \sum_{i=1}^{d-k} y_i| \le 2\varepsilon}d\nu^1_{k,\sum_{i=d-k+1}^d x_i}(y_{d-k}, \cdots, y_1).\end{aligned}$$

Then, for any $x_i \in \widetilde{H}_{e_i}$, $i = 1, \dots, d$,

$$\begin{aligned}&\int_{\widetilde{H}_{e_1}}\cdots\int_{\widetilde{H}_{e_d}}\left(1 \wedge \left(\frac{\rho}{|\sum_{i=1}^d (x_i - y_i)|}\right)^{d+2}\right)\mathbf{1}_{|\sum_{i=1}^d (y_{0i}-y_i)|\le\varepsilon}\\ &\quad \times \left|f\left(\sum_{i=1}^d y_i\right) - f\left(y_1 + \sum_{i=2}^d x_i\right)\right| d\mathcal{H}^1(y_d)\cdots d\mathcal{H}^1(y_1)\\ &\quad \lesssim \sum_{k=0}^{d-2}\frac{\rho^{d+5/4}}{\varepsilon}\mathbf{I}_{3/4}^{d-k}\left(\nu^1_{k,\sum_{i=d-k+1}^d x_i}, \bigotimes_{i=1}^{d-k}\widetilde{H}_{e_i}\right)\left(\sum_{i=1}^{d-k} x_i\right)\\ &\qquad + \sum_{k=0}^{d-2}\rho^{d+1}\mathbf{1}_{\left|\sum_{i=1}^d (y_{0i}-x_i)\right|\le 2\varepsilon}\mathbf{M}^{d-k}\left(\nu^2_{k,\sum_{i=d-k+1}^d x_i}, \bigotimes_{i=1}^{d-k}\widetilde{H}_{e_i}\right)\left(\sum_{i=1}^{d-k} x_i\right),\end{aligned}$$

and

$$\int_{\widetilde{H}_{e_1}}\cdots\int_{\widetilde{H}_{e_d}}\left(1\wedge\left(\frac{\rho}{|\sum_{i=1}^{d}(x_i-y_i)|}\right)^{d+1}\right)$$
$$\times\left|f\left(\sum_{i=1}^{d}y_i\right)-f\left(y_1+\sum_{i=2}^{d}x_i\right)\right|d\mathcal{H}^1(y_d)\cdots d\mathcal{H}^1(y_1)$$
$$\lesssim\sum_{k=0}^{d-2}\rho^{d+\frac{1}{4}}\mathbf{I}_{3/4}^{d-k}\left(\nu^1_{k,\sum_{i=d-k+1}^{d}x_i},\bigotimes_{i=1}^{d-k}\widetilde{H}_{e_i}\right)\left(\sum_{i=1}^{d-k}x_i\right).$$

We will prove Lemma 3.7 in the Appendix. Now, we are ready to prove Theorem 3.3.

PROOF OF THEOREM 3.3.

Step 1. We prove that

$$\limsup_{\lambda\to\infty}\lambda\mathcal{L}^d\left(\left\{\mathbf{T}_\varepsilon^{1,n}(\mu)>\lambda\right\}\cap B_R\right)\lesssim|\log(\varepsilon)\||\mu|^s(\mathbb{R}^d)\tag{3.12}$$

for any $R>0$, $\varepsilon\in(0,\frac{1}{10})$, and $n\in\mathbb{N}$. We now assume that (3.12) is proven. Let $\chi\in C_c^\infty(\mathbb{R}^d)$ be such that $\chi=1$ in $B_{R/4}$ and $\chi=0$ in $B_{R/2}^c$. Thanks to (3.12) and using the fact that $\mathbf{T}_\varepsilon^{1,n}(D(\chi f))\in L^\infty(B_R^c)$, one gets

$$\begin{aligned}&\limsup_{\lambda\to\infty}\lambda\mathcal{L}^d\left(\left\{\mathbf{T}_\varepsilon^{1,n}(\mu)>\lambda\right\}\right)\\&\quad\le\limsup_{\lambda\to\infty}\lambda\mathcal{L}^d\left(\left\{\mathbf{T}_\varepsilon^{1,n}(\mu)>\lambda\right\}\cap B_R\right)+\limsup_{\lambda\to\infty}\lambda\mathcal{L}^d\left(\left\{\mathbf{T}_\varepsilon^{1,n}(\mu)>\lambda\right\}\cap B_R^c\right)\\&\quad\lesssim|\log(\varepsilon)\||\mu|^s(\mathbb{R}^d)+\limsup_{\lambda\to\infty}\lambda\mathcal{L}^d\left(\left\{\mathbf{T}_\varepsilon^{1,n}(D((1-\chi)f))>\lambda/2\right\}\right).\end{aligned}$$

So, by (3.6) and (3.7) and using the fact that $\mathbf{T}_\varepsilon(\mu)\le\mathbf{T}_\varepsilon^{1,n}(\mu)+\mathbf{T}_\varepsilon^{2,n}(\mu)$, we have

$$\begin{aligned}&\limsup_{\lambda\to\infty}\lambda\mathcal{L}^d(\{\mathbf{T}_\varepsilon(\mu)>\lambda\})\\&\quad\lesssim|\log(\varepsilon)\||\mu|^s(\mathbb{R}^d)+C(\varepsilon)|D^s((1-\chi)f)|(\mathbb{R}^d)+C(\varepsilon)c_n|\mu|^s(\mathbb{R}^d)\\&\quad\lesssim|\log(\varepsilon)\||\mu|^s(\mathbb{R}^d)+C(\varepsilon)|\mu|^s(B_{R/4}^c)+C(\varepsilon)c_n|\mu|^s(\mathbb{R}^d).\end{aligned}$$

This implies (3.8) by letting $R\to\infty$, $n\to\infty$. Moreover, as proof of Lemma 2.4 we also get (3.9).

We are now going to prove (3.12) in several steps.

Step 2. Let $\operatorname{supp}(\mu^s)$ be the support of μ^s. Let $\eta:\mathbb{R}^d\to\mathbb{S}^{d-1}$ be such that $\eta(x)=\frac{d\mu^s(x)}{d|\mu|^s(x)}$ if $x\in\operatorname{supp}(\mu^s)$ and $\eta(x)=(1,\dots,0)\in\mathbb{S}^{d-1}$ if $x\notin\operatorname{supp}(\mu^s)$. Let $\eta^\kappa:\mathbb{R}^d\to\mathbb{S}^{d-1}$ be smooth functions such that $\eta^\kappa\to\eta$ $|\mu|^s$-a.e. in $\mathbb{R}^d$ and $\lim_{\kappa\to0}\int_{\mathbb{R}^d}|\eta^\kappa-\eta|d|\mu|^s=0$. Let $\varphi_r\in C_b^\infty(\mathbb{R}^d)$ be such that $\varphi_r(z)=1$ if $|z|>2r$ and $\varphi_r(z)=0$ if $|z|\le r$ and $\|\nabla\varphi_r\|_{L^\infty(\mathbb{R}^d)}\le Cr^{-1}$.

Let us define $S_\tau = \{y \in 2\tau\mathbb{Z}^d : y \in B_{R+4\rho_0}\}$ for $\tau \in (0, \rho_0/100)$. There exists a sequence of smooth functions $\{\chi^\tau_{y_\tau}\}_{y_\tau \in S_\tau}$ such that $0 \le \chi^\tau_{y_\tau}(y) \le 1$,

$$\sum_{y_\tau \in S_\tau} \chi^\tau_{y_\tau}(y) = 1 \quad \forall y \in B_{R+4\rho_0},$$

and $\chi^\tau_{y_\tau} = 1$ in $B_\tau(y_\tau)$, $\operatorname{supp}(\chi^\tau_{y_\tau}) \subset B_{2\tau}(y_\tau)$, $|\nabla \chi^\tau_{y_\tau}(y)| \le C\tau^{-1}$ $\forall y \in \mathbb{R}^d$. Note that $\operatorname{Card}(S_\tau) \sim \left(\frac{R+\rho_0}{\tau}\right)^d$, $B_\tau(y_\tau) \cap B_\tau(y'_\tau) = \varnothing$ for $y_\tau, y'_\tau \in S_\tau$, $y_\tau \neq y'_\tau$, and

$$\sum_{y_\tau \in S_\tau} \mathbf{1}_{B_{100\tau}(y_\tau)}(y) \lesssim \mathbf{1}_{B_{R+6\rho_0}}(y) \quad \forall y \in \mathbb{R}^d. \tag{3.13}$$

Set $\chi_0 = \sum_{y_\tau \in S_\tau} \chi^\tau_{y_\tau}$. For any $y_\tau \in S_\tau$, we denote $\eta^\kappa_{y_\tau} = \eta^\kappa(y_\tau)$.

Because of $\mu^s = \eta\langle \eta, \mu^s\rangle$, one has

$$\mu = (1-\chi_0)\mu + \chi_0\mu^a + \chi_0(\eta-\eta^\kappa)\langle\eta,\mu^s\rangle + \chi_0\eta^\kappa\langle(\eta-\eta^\kappa),\mu^s\rangle + \chi_0\eta^\kappa\langle\eta^\kappa,\mu^s\rangle,$$

and

$$\begin{aligned}\chi_0\eta^\kappa\langle\eta^\kappa,\mu^s\rangle &= \left(\sum_{y_\tau \in S_\tau} \chi^\tau_{y_\tau}\right)\eta^\kappa\langle\eta^\kappa,\mu^s\rangle \\ &= \sum_{y_\tau \in S_\tau} \chi^\tau_{y_\tau}\eta^\kappa\big\langle\big(\eta^\kappa - \eta^\kappa_{y_\tau}\big),\mu^s\big\rangle + \sum_{y_\tau \in S_\tau} \chi^\tau_{y_\tau}(\eta^\kappa - \eta^\kappa_{y_\tau})\langle\eta^\kappa_{y_\tau},\mu^s\rangle \\ &\quad + \sum_{y_\tau \in S_\tau} \chi^\tau_{y_\tau}\eta^\kappa_{y_\tau}\langle\eta^\kappa_{y_\tau},\mu\rangle - \sum_{y_\tau \in S_\tau} \chi^\tau_{y_\tau}\eta^\kappa_{y_\tau}\langle\eta^\kappa_{y_\tau},\mu^a\rangle.\end{aligned}$$

Hence, with

$$\widetilde{\mathbf{K}}^n = \frac{\varepsilon^{-d+1}}{\rho^\alpha}\left(\frac{1}{|\cdot|^{d-\alpha}}\phi^{e,\varepsilon}_\rho(\cdot)\right) \star \mathbf{K}_n$$

and $\zeta \in (0, \frac{1}{10})$, we write

$$\begin{aligned}\widetilde{\mathbf{K}}^n \star \mu &= \widetilde{\mathbf{K}}^n \star ((1-\chi_0)\mu) + \widetilde{\mathbf{K}}^n \star (\chi_0\mu^a) + \widetilde{\mathbf{K}}^n \star (\chi_0(\eta-\eta^\kappa)\langle\eta,\mu^s\rangle) \\ &\quad + \widetilde{\mathbf{K}}^n \star (\chi_0\eta^\kappa\langle(\eta-\eta^\kappa),\mu^s\rangle) + \sum_{y_\tau \in S_\tau} \widetilde{\mathbf{K}}^n \star (\chi^\tau_{y_\tau}\eta^\kappa\langle(\eta^\kappa-\eta^\kappa_{y_\tau}),\mu^s\rangle) \\ &\quad + \sum_{y_\tau \in S_\tau} \widetilde{\mathbf{K}}^n \star (\chi^\tau_{y_\tau}(\eta^\kappa-\eta^\kappa_{y_\tau})\langle\eta^\kappa_{y_\tau},\mu^s\rangle) - \sum_{y_\tau \in S_\tau} \widetilde{\mathbf{K}}^n \star (\chi^\tau_{y_\tau}\eta^\kappa_{y_\tau}\langle\eta^\kappa_{y_\tau},\mu^a\rangle) \\ &\quad + \sum_{y_\tau \in S_\tau} \frac{\varepsilon^{-d+1}}{\rho^\alpha}\left(\frac{(1-\varphi_{\zeta\rho})}{|\cdot|^{d-\alpha}}\phi^{e,\varepsilon}_\rho(\cdot)\right) \star \mathbf{K}_n \star (\chi^\tau_{y_\tau}\eta^\kappa_{y_\tau}\langle\eta^\kappa_{y_\tau},\mu\rangle) \\ &\quad + \sum_{y_\tau \in S_\tau} \frac{\varepsilon^{-d+1}}{\rho^\alpha}\left(\frac{\varphi_{\zeta\rho}}{|\cdot|^{d-\alpha}}\phi^{e,\varepsilon}_\rho(\cdot)\right) \star \mathbf{K}_n \star (\chi^\tau_{y_\tau}\eta^\kappa_{y_\tau}\langle\eta^\kappa_{y_\tau},\mu\rangle) := \sum_{i=1}^{9} I^{e,\rho}_{i,\varepsilon}.\end{aligned}$$

Step 3. In this proof, we denote

$$A_i(\lambda,\varepsilon) := \lambda\mathcal{L}^d\Big(\Big\{\sup_{\rho\in(0,\rho_0),e\in\mathbb{S}^{d-1}}\big|I^{e,\rho}_{i,\varepsilon}\big| > \lambda\Big\}\cap B_R\Big).$$

Thus, for $\lambda > 1$,

$$\lambda\mathcal{L}^d\big(\big\{\mathbf{T}^{1,n}_\varepsilon(\mu) > \lambda\big\}\cap B_R\big) \leq 9\sum_{i=1}^{9} A_i(\lambda/9,\varepsilon). \tag{3.14}$$

Thanks to (3.6) we have

$$\limsup_{\lambda\to\infty}\sum_{i=2,7} A_i(\lambda,\varepsilon) = 0, \tag{3.15}$$

$$\limsup_{\lambda\to\infty}\sum_{i=3,4} A_i(\lambda,\varepsilon) \leq C(\varepsilon)\||\eta-\eta^\kappa||\mu^s|\|_{\mathcal{M}(\mathbb{R}^d)}, \tag{3.16}$$

$$\begin{aligned}\limsup_{\lambda\to\infty}\sum_{i=5,6} A_i(\lambda,\varepsilon) &\leq C(\varepsilon)\Big|\sum_{y_\tau\in S_\tau}\chi^\tau_{y_\tau}|\eta^\kappa-\eta^\kappa_{y_\tau}||\mu^s|\Big|_{\mathcal{M}(\mathbb{R}^d)}\\ &\leq C(\varepsilon,\kappa)\tau|\mu|^s(\mathbb{R}^d).\end{aligned} \tag{3.17}$$

Here in the last inequality we have used the fact that

$$\begin{aligned}\sum_{y_\tau\in S_\tau}\chi^\tau_{y_\tau}(x)|\eta^\kappa(x)-\eta^\kappa_{y_\tau}| &\lesssim \|\nabla\eta^\kappa\|_{L^\infty(\mathbb{R}^d)}\sum_{y_\tau\in S_\tau}\mathbf{1}_{B_{2\tau}(y_\tau)}(x)|x-y_\tau|\\ &\overset{(3.13)}{\lesssim} \|\nabla\eta^\kappa\|_{L^\infty(\mathbb{R}^d)}\tau\mathbf{1}_{B_{R+6\rho_0}}(x) \quad \forall x\in\mathbb{R}^d.\end{aligned}$$

Again, applying (3.6) (where ρ is replaced by $\zeta\rho$) yields

$$\limsup_{\lambda\to\infty} A_8(\lambda,\varepsilon) \leq C(\varepsilon)\zeta^\alpha\Big\|\sum_{y_\tau\in S_\tau}\chi^\tau_{y_\tau}|\mu^s|\Big\|_{\mathcal{M}(\mathbb{R}^d)} \leq C(\varepsilon)\zeta^\alpha|\mu|^s(\mathbb{R}^d). \tag{3.18}$$

On the other hand, it is easy to see that $\sup_{\rho\in(0,\rho_0),e\in\mathbb{S}^{d-1}}|I^{e,\rho}_{1,\varepsilon}(\cdot)| \in L^\infty(B_R)$, so

$$\limsup_{\lambda\to\infty} A_1(\lambda,\varepsilon) = 0. \tag{3.19}$$

Therefore, we deduce from (3.14) and (3.15)–(3.19) that

$$\begin{aligned}&\limsup_{\lambda\to\infty}\lambda\mathcal{L}^d\big(\big\{\mathbf{T}^{1,n}_\varepsilon(\mu) > \lambda\big\}\cap B_R\big)\\ &\quad\lesssim C(n,\varepsilon)\||\eta-\eta^\kappa||\mu|^s\|_{\mathcal{M}(\mathbb{R}^d)} + C(n,\varepsilon,\kappa)\tau|\mu|^s(\mathbb{R}^d)\\ &\qquad + C(n,\varepsilon)\zeta^\alpha|\mu|^s(\mathbb{R}^d) + \limsup_{\lambda\to\infty} A_9(\lambda,\varepsilon).\end{aligned} \tag{3.20}$$

In the next steps, we will deal with $A_9(\lambda,\varepsilon)$.

Step 4. One has

$$\begin{aligned} I_{9,\varepsilon}^{e,\rho}&(x) \\ &= \sum_{y_\tau \in S_\tau} \int_{\mathbb{R}^d} \mathbf{K}_{e,\rho}^{\varepsilon,n}(x-y)\chi_{y_\tau}^{\tau}(y)\, d\langle \eta_{y_\tau}^{\kappa}, \mu(y)\rangle \\ &\qquad + \mathbf{c}(\varepsilon,\kappa,\tau,\zeta) \sum_{y_\tau\in S_\tau} (\varphi_\rho \mathbf{K}_n) \star (\chi_{y_\tau}^{\tau}\langle \eta_{y_\tau}^{\kappa},\mu\rangle)(x) \\ &=: \mathrm{I}_{10,\varepsilon}^{e,\rho}(x) + \mathrm{I}_{11,\varepsilon}^{e,\rho}(x), \end{aligned}$$

where

$$\begin{aligned} \mathbf{K}_{e,\rho}^{\varepsilon,n}(z') = \frac{\varepsilon^{-d+1}}{\rho^{\alpha}} \int_{\mathbb{R}^d} \mathbf{K}_n(z)\varphi_{\zeta\rho}(z'-z) \frac{\langle \phi_\rho^{e,\varepsilon}(z'-z), \eta_{y_\tau}^{\kappa}\rangle}{|z'-z|^{d-\alpha}}\, dz \\ - \mathbf{c}(\varepsilon,\kappa,\tau,\zeta)\varphi_\rho(z')\mathbf{K}_n(z') \quad \forall z' \in \mathbb{R}^d, \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} \mathbf{c}(\varepsilon,\kappa,\tau,\zeta) &= \frac{\varepsilon^{-d+1}}{\rho^{\alpha}} \int_{\mathbb{R}^d} \varphi_{\zeta\rho}(z'-z) \frac{\langle \phi^{e,\varepsilon}((z'-z)/\rho), \eta_{y_\tau}^{\kappa}\rangle}{|z'-z|^{d-\alpha}}\, dz \\ &= \varepsilon^{-d+1} \int_{\mathbb{R}^d} \varphi_\zeta(z) \frac{\langle \phi^{e,\varepsilon}(z), \eta_{y_\tau}^{\kappa}\rangle}{|z|^{d-\alpha}}\, dz. \end{aligned} \tag{3.22}$$

Note that $|\mathbf{c}(\varepsilon,\kappa,\tau,\zeta)| \lesssim 1$ for all $\kappa,\varepsilon,\zeta > 0$, $e \in \mathbb{S}^{d-1}$, and by (2.34) in the proof of Proposition 2.13, we have for any $x \in \mathbb{R}^d \setminus \{0\}$,

$$\left|\mathbf{K}_{e,\rho}^{\varepsilon,n}(x)\right| \le C(n,\varepsilon,\zeta)\frac{1}{|x|^{d-\alpha}} \min\left\{\frac{1}{\rho^{\alpha}}, \frac{\rho}{|x|^{1+\alpha}}\right\}.$$

Similarly, we also have for any $x \in \mathbb{R}^d \setminus \{0\}$,

$$\left|\nabla\mathbf{K}_{e,\rho}^{\varepsilon,n}(x)\right| \le C(n,\varepsilon,\zeta)\frac{1}{|x|^{d-\alpha+1}} \min\left\{\frac{1}{\rho^{\alpha}}, \frac{\rho}{|x|^{1+\alpha}}\right\}.$$

Moreover, since $|\varphi_{\zeta\rho}(z)| \le C1_{|z|>\zeta\rho}$, so we have, for any $|x| \le \zeta\rho/4$,

$$\left|\mathbf{K}_{e,\rho}^{\varepsilon,n}(x)\right| + \rho|\nabla\mathbf{K}_{e,\rho}^{\varepsilon,n}(x)| \le C(n,\varepsilon,\zeta)\frac{1}{\rho^{d}}.$$

Thus,

$$\begin{aligned} \left|\mathbf{K}_{e,\rho}^{\varepsilon,n}(x)\right| &\le C(n,\varepsilon,\zeta)\min\left\{\frac{1}{\rho^{d}}, \frac{\rho}{|x|^{d+1}}\right\}, \\ |\nabla\mathbf{K}_{e,\rho}^{\varepsilon,n}(x)| &\le C(n,\varepsilon,\zeta)\min\left\{\frac{1}{\rho^{d+1}}, \frac{\rho}{|x|^{d+2}}\right\}. \end{aligned} \tag{3.23}$$

Thanks to Proposition 2.9, we get

$$\limsup_{\lambda\to\infty} A_{11}(\lambda,\varepsilon) \lesssim \left\| \sum_{y_\tau\in S_\tau} \chi_{y_\tau}^{\tau}|\mu|^s \right\|_{\mathcal{M}(\mathbb{R}^d)} \lesssim |\mu|^s(\mathbb{R}^d). \tag{3.24}$$

Using integration by parts, we have

$$\begin{aligned}&\int_{\mathbb{R}^d} \mathbf{K}^{\varepsilon,n}_{e,\rho}(x-y)\chi^{\tau}_{y_\tau}(y)\,d\langle \eta^{\kappa}_{y_\tau},\mu(y)\rangle\\&\quad = -\int_{\mathbb{R}^d} \eta^{\kappa}_{y_\tau}.\nabla_y\big[\mathbf{K}^{\varepsilon,n}_{e,\rho}(x-y)\chi^{\tau}_{y_\tau}(y)\big]f(y)dy,\\&\int_{H_{\eta^{\kappa}_{y_\tau}}}\int_{\widetilde{H}_{\eta^{\kappa}_{y_\tau}}} \mathbf{K}^{\varepsilon,n}_{e,\rho}(x-y_1-y_2)\chi^{\tau}_{y_\tau}(y_1+y_2)\,dDf^{\eta^{\kappa}_{y_\tau}}_{\widetilde{x}_{\kappa,y_\tau}}(y_1)\,d\mathcal{H}^{d-1}(y_2)\\&\quad = -\int_{\mathbb{R}^d} \eta^{\kappa}_{y_\tau}.\nabla_y\big[\mathbf{K}^{\varepsilon,n}_{e,\rho}(x-y)\chi^{\tau}_{y_\tau}(y)\big]f(\langle y,\eta^{\kappa}_{y_\tau}\rangle\eta^{\kappa}_{y_\tau}+\widetilde{x}_{\kappa,y_\tau})dy.\end{aligned}$$

So,

$$\begin{aligned}&\mathrm{I}^{e,\rho}_{10,\varepsilon}(x)\\&\quad = -\sum_{y_\tau\in S_\tau}\int_{\mathbb{R}^d} \eta^{\kappa}_{y_\tau}.\nabla_y\big[\mathbf{K}^{\varepsilon,n}_{e,\rho}(x-y)\chi^{\tau}_{y_\tau}(y)\big]\big[f(y)-f(\langle y,\eta^{\kappa}_{y_\tau}\rangle\eta^{\kappa}_{y_\tau}+\widetilde{x}_{\kappa,y_\tau})\big]dy\\&\qquad + \sum_{y_\tau\in S_\tau}\int_{H_{\eta^{\kappa}_{y_\tau}}}\int_{\widetilde{H}_{\eta^{\kappa}_{y_\tau}}} \mathbf{K}^{\varepsilon,n}_{e,\rho}(x-y_1-y_2)\chi^{\tau}_{y_\tau}(y_1+y_2)\,dD^s f^{\eta^{\kappa}_{y_\tau}}_{\widetilde{x}_{\kappa,y_\tau}}(y_1)\,d\mathcal{H}^{d-1}(y_2)\\&\qquad + \sum_{y_\tau\in S_\tau}\int_{H_{\eta^{\kappa}_{y_\tau}}}\int_{\widetilde{H}_{\eta^{\kappa}_{y_\tau}}} \mathbf{K}^{\varepsilon,n}_{e,\rho}(x-y_1-y_2)\chi^{\tau}_{y_\tau}(y_1+y_2)\\&\qquad\qquad \times\langle \eta^{\kappa}_{y_\tau}, D^a f(y_1+\widetilde{x}_{\kappa,y_\tau})\rangle\,d\mathcal{H}^1(y_1)\,d\mathcal{H}^{d-1}(y_2)\\&\quad =: \sum_{i=12}^{14} I^{e,\rho}_{i,\varepsilon}(x),\end{aligned}$$

where throughout this proof we denote $\widetilde{x}_{\kappa,y_\tau} = x-\langle x,\eta^{\kappa}_{y_\tau}\rangle\eta^{\kappa}_{y_\tau}$.
Thus,

$$A_9(\lambda,\varepsilon)\le 4\sum_{i=11}^{14} A_i(\lambda/4,\varepsilon). \tag{3.25}$$

Step 5. To treat $A_{13}(\lambda,\varepsilon)$ and $A_{14}(\lambda,\varepsilon)$, we need to show the following inequality:

$$\begin{aligned}&\left|\int_{H_{\eta^{\kappa}_{y_\tau}}}\int_{\widetilde{H}_{\eta^{\kappa}_{y_\tau}}} \mathbf{K}^{\varepsilon,n}_{e,\rho}(x-y_1-y_2)\chi^{\tau}_{y_\tau}(y_1+y_2)\,d\nu(y_1)\,d\mathcal{H}^{d-1}(y_2)\right|\\&\quad\lesssim |\log(\varepsilon)|\mathbf{1}_{B_{4\tau}(y_\tau)}(x)\mathbf{M}^1(1_{B_{2\tau}(y_{\tau,1})}\nu,\widetilde{H}_{\eta^{\kappa}_{y_\tau}})(x_1)\\&\qquad + C(n,\varepsilon,\zeta,\tau)\rho\mathbf{1}_{B_{4\tau}(y_\tau)^c}(x)\int_{H_{\eta^{\kappa}_{y_\tau}}}\int_{\widetilde{H}_{\eta^{\kappa}_{y_\tau}}}\mathbf{1}_{B_{2\tau}(y_\tau)}(y_1+y_2)\,d\nu(y_1)\,d\mathcal{H}^{d-1}(y_2),\end{aligned}\tag{3.26}$$

for any $\nu\in\mathcal{M}_b(\widetilde{H}_{\eta^{\kappa}_{y_\tau}})$ and $x\in\mathbb{R}^d$ where

$$x_1 = \langle x,\eta^{\kappa}_{y_\tau}\rangle\eta^{\kappa}_{y_\tau},\quad y_{\tau,1} = \langle y_\tau,\eta^{\kappa}_{y_\tau}\rangle\eta^{\kappa}_{y_\tau}.$$

Now set $\tau_{x_1}(z) = x_1 - z$ for any $z \in \widetilde{H}_{\eta^\kappa_{y_\tau}}$. By Lemma 3.5 with $a = |\mathbf{K}^{\varepsilon,n}_{e,\rho}(\cdot)|$, $\omega = (\tau_{x_1})_\#(\mathbf{1}_{B_{2\tau}(y_{\tau,1},\widetilde{H}_{\eta^\kappa_{y_\tau}})}|\nu|)$, we have

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty}\int_{H_{\eta^\kappa_{y_\tau}}}\int_{\widetilde{H}_{\eta^\kappa_{y_\tau}}} \mathbf{1}_{2^j\rho<|x-y_1-y_2|\le 2^{j+1}\rho}\, a(x-y_1-y_2)\\
&\qquad\qquad \times \chi^\tau_{y_\tau}(y_1+y_2)\,d|\nu|(y_1)\,d\mathcal{H}^{d-1}(y_2)\\
&\quad\lesssim \sum_{j=-\infty}^{\infty}\int_{H_{\eta^\kappa_{y_\tau}}}\int_{\widetilde{H}_{\eta^\kappa_{y_\tau}}} \mathbf{1}_{2^j\rho<|y_1+y_2|\le 2^{j+1}\rho}\, a(y_1+y_2)\,d\omega(y_1)\,d\mathcal{H}^{d-1}(y_2) \lesssim\\
&\quad\lesssim \sum_{j=-\infty}^{\infty}\Big[(2^j\rho)^d\int_{\mathbb{S}^{d-1}}\sup_{r\in[2^j\rho,2^{j+1}\rho]} a(r\theta)\,d\mathcal{H}^{d-1}(\theta)\Big]\mathbf{M}^1(\omega,\widetilde{H}_{\eta^\kappa_{y_\tau}})(0)\\
&\quad= \Big[\sum_{j=-\infty}^{\infty}(2^j\rho)^d\int_{\mathbb{S}^{d-1}}\sup_{r\in[2^j\rho,2^{j+1}\rho]} a(r\theta)\,d\mathcal{H}^{d-1}(\theta)\Big]\\
&\qquad \times \mathbf{M}^1(\mathbf{1}_{B_{2\tau}(y_{\tau,1},\widetilde{H}_{\eta^\kappa_{y_\tau}})}|\nu|,\widetilde{H}_{\eta^\kappa_{y_\tau}})(x_1).
\end{aligned}$$

So, by (3.32) in Lemma 3.8 below, we obtain that

$$\begin{aligned}
&\left|\int_{H_{\eta^\kappa_{y_\tau}}}\int_{\widetilde{H}_{\eta^\kappa_{y_\tau}}} \mathbf{K}^{\varepsilon,n}_{e,\rho}(x-y_1-y_2)\chi^\tau_{y_\tau}(y_1+y_2)\,d\nu(y_1)\,d\mathcal{H}^{d-1}(y_2)\right|\\
&\qquad \lesssim |\log(\varepsilon)|\mathbf{M}^1(1_{B_{2\tau}(y_{\tau,1})}\nu,\widetilde{H}_{\eta^\kappa_{y_\tau}})(x_1).
\end{aligned}\tag{3.27}$$

On the other hand, for any $x \notin B_{4\tau}(y_\tau)$,

$$\begin{aligned}
&\left|\int_{H_{\eta^\kappa_{y_\tau}}}\int_{\widetilde{H}_{\eta^\kappa_{y_\tau}}} \mathbf{K}^{\varepsilon,n}_{e,\rho}(x-y_1-y_2)\chi^\tau_{y_\tau}(y_1+y_2)\,d\nu(y_1)\,d\mathcal{H}^{d-1}(y_2)\right|\\
&\quad\lesssim \int_{H_{\eta^\kappa_{y_\tau}}}\int_{\widetilde{H}_{\eta^\kappa_{y_\tau}}} \mathbf{1}_{|x-y_1-y_2|>2\tau}\mathbf{1}_{|y_1+y_2-y_\tau|<2\tau}\\
&\qquad\qquad\qquad \times |\mathbf{K}^{\varepsilon,n}_{e,\rho}(x-y_1-y_2)|\,d|\nu|(y_1)\,d\mathcal{H}^{d-1}(y_2)\\
&\quad\overset{(3.23)}{\le} C(n,\varepsilon,\zeta,\tau)\rho\int_{H_{\eta^\kappa_{y_\tau}}}\int_{\widetilde{H}_{\eta^\kappa_{y_\tau}}} \mathbf{1}_{|y_1+y_2-y_\tau|<2\tau}\,d|\nu|(y_1)\,d\mathcal{H}^{d-1}(y_2).
\end{aligned}$$

From this and (3.27), we find (3.26).

Step 6. Estimate $A_{13}(\lambda,\varepsilon)$ and $A_{14}(\lambda,\varepsilon)$.
We set

$$\omega^\tau_{y_\tau,z_2} := \mathbf{1}_{B_{2\tau}(y_{\tau,1},\widetilde{H}_{\eta^\kappa_{y_\tau}})}\left|D^s f^{\eta^\kappa_{y_\tau}}_{z_2}\right| \quad \forall z_2 \in H_{\eta^\kappa_{y_\tau}}.$$

We then apply (3.26) for $\nu(y_1) = D^s f^{\eta^\kappa_{y_\tau}}_{\widetilde{x}_{\kappa,y_\tau}}(y_1)$ to get that

$$\begin{aligned} I^{e,\rho}_{13,\varepsilon}(x) &\lesssim |\log(\varepsilon)| \sum_{y_\tau \in S_\tau} \mathbf{1}_{B_{4\tau}(y_\tau)}(x)\mathbf{M}^1(\omega^\tau_{y_\tau,\widetilde{x}_{\kappa,y_\tau}}, \widetilde{H}_{\eta^\kappa_{y_\tau}})(\langle x, \eta^\kappa_{y_\tau}\rangle \eta^\kappa_{y_\tau}) \\ &\quad + \sum_{y_\tau \in S_\tau} C(\varepsilon,\zeta,\tau)\rho \int_{H_{\eta^\kappa_{y_\tau}}} \int_{\widetilde{H}_{\eta^\kappa_{y_\tau}}} \mathbf{1}_{B_{2\tau}(y_\tau)}(y_1 + y_2) d \\ &\qquad \times \left| D^s f^{\eta^\kappa_{y_\tau}}_{\widetilde{x}_{\kappa,y_\tau}} \right|(y_1) d\mathcal{H}^{d-1}(y_2). \end{aligned}$$

By (2.1) in Proposition 2.2 and (3.13), we have

$$\begin{aligned} I^{e,\rho}_{13,\varepsilon}(x) &\lesssim |\log(\varepsilon)| \sum_{y_\tau \in S_\tau} \mathbf{1}_{B_{4\tau}(y_\tau)}(x)\mathbf{M}^1(\omega^\tau_{y_\tau,\widetilde{x}_{\kappa,y_\tau}}, \widetilde{H}_{\eta^\kappa_{y_\tau}})(\langle x, \eta^\kappa_{y_\tau}\rangle \eta^\kappa_{y_\tau}) \\ &\quad + \sum_{y_\tau \in S_\tau} C(\varepsilon,\zeta,\tau)\rho \int_{\mathbb{R}^d} \mathbf{1}_{B_{2\tau}(y_\tau)}(y) d|\mu|^s(y) \\ &\lesssim |\log(\varepsilon)| \sum_{y_\tau \in S_\tau} \mathbf{1}_{B_{4\tau}(y_\tau)}(x)\mathbf{M}^1\big(\omega^\tau_{y_\tau,\widetilde{x}_{\kappa,y_\tau}}, \widetilde{H}_{\eta^\kappa_{y_\tau}}\big)\big(\langle x, \eta^\kappa_{y_\tau}\rangle \eta^\kappa_{y_\tau}\big) \\ &\quad + C(\varepsilon,\zeta,\tau)\rho|\mu|^s(\mathbb{R}^d). \end{aligned}$$

Thus, for $\lambda \gg 1$

$$\begin{aligned} &A_{13}(\lambda,\varepsilon) \\ &\quad \le \sum_{y'_\tau \in S_\tau} \lambda \mathcal{L}^d\Bigg(\Bigg\{x \in B_{2\tau}(y'_\tau) : \sum_{y_\tau \in S_\tau} \mathbf{1}_{B_{4\tau}(y_\tau)}(x) \\ &\qquad \times \mathbf{M}^1(\omega^\tau_{y_\tau,\widetilde{x}_{\kappa,y_\tau}}, \widetilde{H}_{\eta^\kappa_{y_\tau}})(\langle x, \eta^\kappa_{y_\tau}\rangle \eta^\kappa_{y_\tau}) \gtrsim \frac{\lambda}{|\log(\varepsilon)|}\Bigg\}\Bigg) \\ &\overset{(3.13)}{\lesssim} \sum_{y_\tau \in S_\tau} \lambda \mathcal{L}^d\Bigg(\Bigg\{x \in B_{8\tau}(y_\tau) : \mathbf{M}^1(\omega^\tau_{y_\tau,\widetilde{x}_{\kappa,y_\tau}}, \widetilde{H}_{\eta^\kappa_{y_\tau}})(\langle x, \eta^\kappa_{y_\tau}\rangle \eta^\kappa_{y_\tau}) \gtrsim \frac{\lambda}{|\log(\varepsilon)|}\Bigg\}\Bigg). \end{aligned}$$

Thanks to the boundedness of $\mathbf{M}^1(\cdot, \widetilde{H}_{\eta^\kappa_{y_\tau}})$ from $\mathcal{M}(\widetilde{H}_{\eta^\kappa_{y_\tau}})$ to $L^{1,\infty}(\widetilde{H}_{\eta^\kappa_{y_\tau}})$ yields for $\lambda \gg 1$

$$\begin{aligned} A_{13}(\lambda,\varepsilon) &\lesssim \sum_{y_\tau \in S_\tau} \lambda \int_{H_{\eta^\kappa_{y_\tau}}} \mathbf{1}_{|z_2 - (y_\tau - y_{\tau,1})| \le 8\tau} \\ &\qquad \times \mathcal{H}^1\Bigg(\Bigg\{\mathbf{M}^1\Big(\omega^\tau_{y_\tau,z_2}, \widetilde{H}_{\eta^\kappa_{y_\tau}}\Big) \gtrsim \frac{\lambda}{|\log(\varepsilon)|}\Bigg\}\Bigg) d\mathcal{H}^{d-1}(z_2) \\ &\lesssim |\log(\varepsilon)| \sum_{y_\tau \in S_\tau} \int_{H_{\eta^\kappa_{y_\tau}}} \mathbf{1}_{|z_2 - (y_\tau - y_{\tau,1})| \le 8\tau} \end{aligned}$$

$$\times \int_{\widetilde{H}_{\eta^\kappa_{y_\tau}}} \mathbf{1}_{B_{2\tau}(y_{1,\tau},\widetilde{H}_{\eta^\kappa_{y_\tau}})}(z_1) d|D^s f_{z_2}^{\eta^\kappa_{y_\tau}}|(z_1) d\mathcal{H}^{d-1}(z_2)$$

$$\lesssim |\log(\varepsilon)| \sum_{y_\tau \in S_\tau} \int_{B_{10\tau}(y_\tau)} d|D^s f|(z) \lesssim |\log(\varepsilon)||\mu|^s(\mathbb{R}^d).$$

Here we have used (2.1) in Proposition 2.2 for the third inequality and (3.13) for the last one.

Thus,

$$\limsup_{\lambda\to\infty} A_{13}(\lambda,\varepsilon) \lesssim |\log(\varepsilon)||\mu|^s(\mathbb{R}^d). \tag{3.28}$$

Similarly, we also have

$$A_{14}(\lambda,\varepsilon) \lesssim |\log(\varepsilon)\||\mu^a\|_{L^1(B_{R+6\rho_0})} \quad \forall \lambda \gg 1.$$

Since

$$\limsup_{\lambda\to\infty} \lambda \mathcal{H}^1\big(\big\{\mathbf{M}^1\big(1_{B_{2\tau}(y_{1,\tau},\widetilde{H}_{\eta^\kappa_{y_\tau}})}|D^a f(\cdot + z_2)|, \widetilde{H}_{\eta^\kappa_{y_\tau}}\big) > \lambda\big\}\big) = 0$$

for $\mathcal{H}^{d-1}$-a.e. z_2 in $H_{\eta^\kappa_{y_\tau}}$, so by the dominated convergence theorem we get

$$\limsup_{\lambda\to\infty} A_{14}(\lambda,\varepsilon) = 0. \tag{3.29}$$

Step 7. We will prove that

$$\begin{aligned}&\limsup_{\lambda\to\infty} A_{12}(\lambda,\varepsilon)\\ &\quad\lesssim C(n,\varepsilon,\zeta)\|(\eta-\eta^\kappa)|\mu|^s\|_{\mathcal{M}(\mathbb{R}^d)} + C(n,\varepsilon,\zeta,\kappa)\tau|\mu|^s(\mathbb{R}^d).\end{aligned} \tag{3.30}$$

Let $\{\eta^\kappa_1(y_\tau), \eta^\kappa_2(y_\tau), \dots, \eta^\kappa_d(y_\tau)\}$ be an orthonormal basis in $\mathbb{R}^d$ such that $\eta^\kappa_1(y_\tau) = \eta^\kappa_{y_\tau}$. So, for any $x \in \mathbb{R}^d$, throughout this proof we denote

$$x_{\eta^\kappa_i(y_\tau)} = \langle x, \eta^\kappa_i(y_\tau)\rangle \eta^\kappa_i(y_\tau),\ x^{1,j}_{\eta^\kappa_i(y_\tau)} = \sum_{i=1}^{j} x_{\eta^\kappa_i(y_\tau)},\ x^{2,j}_{\eta^\kappa_i(y_\tau)} = \sum_{i=j+1}^{d} x_{\eta^\kappa_i(y_\tau)}.$$

$$\begin{aligned}\big|I^{e,\rho}_{12,\varepsilon}(x)\big| \le{}& C(n,\varepsilon,\zeta)\frac{1}{\rho^{d+1}} \sum_{y_\tau\in S_\tau} \int_{\mathbb{R}^d} \bigg(1 \wedge \bigg(\frac{\rho}{|x-y|}\bigg)^{d+2}\bigg)\mathbf{1}_{|y_\tau-y|\le 2\tau}\\ &\times \bigg|f(y) - f(y_{\eta^\kappa_{y_\tau}} + \sum_{i=2}^{d} x_{\eta^\kappa_i(y_\tau)})\bigg| dy\\ &+ C(n,\varepsilon,\tau,\zeta)\frac{1}{\rho^d} \sum_{y_\tau\in S_\tau} \int_{\mathbb{R}^d} \bigg(1 \wedge \bigg(\frac{\rho}{|x-y|}\bigg)^{d+1}\bigg)\\ &\times \bigg|f(y) - f(y_{\eta^\kappa_{y_\tau}} \sum_{y_\tau\in S_\tau} \int_{\mathbb{R}^d} + \sum_{i=2}^{d} x_{\eta^\kappa_i(y_\tau)})\bigg| dy.\end{aligned}$$

Applying Lemma 3.7 to $\{e_1, \dots, e_d\} = \{\eta_1^\kappa(y_\tau), \eta_2^\kappa(y_\tau), \dots, \eta_d^\kappa(y_\tau)\}$ and $x_i = x_{\eta_i^\kappa(y_\tau)}$ for $i = 1, \dots, d$ and $\varepsilon = 2\tau$, we find that

$$\begin{aligned} &I_{12,\varepsilon}^{e,\rho}(x) \\ &\quad \le C(n,\varepsilon,\tau,\zeta)\rho^{\frac{1}{4}} \sum_{k=0}^{d-2} \sum_{y_\tau \in S_\tau} \mathbf{I}_{\frac{3}{4}}^{d-k}\Big(\nu^1_{k,x^{2,d-k}_{\eta_i^\kappa(y_\tau)}}, \bigotimes_{i=1}^{d-k} \widetilde{H}_{\eta_i^\kappa(y_\tau)}\Big)(x^{1,d-k}_{\eta_i^\kappa(y_\tau)}) \\ &\qquad + C(n,\varepsilon,\zeta) \sum_{k=0}^{d-2} \sum_{y_\tau \in S_\tau} 1_{B_{4\tau}(y_\tau)}(x) \mathbf{M}^{d-k}\Bigg(\nu^2_{k,x^{2,d-k}_{\eta_i^\kappa(y_\tau)}}, \bigotimes_{i=1}^{d-k} \widetilde{H}_{\eta_i^\kappa(y_\tau)}\Bigg)(x^{1,d-k}_{\eta_i^\kappa(y_\tau)}), \end{aligned}$$

where

$$\begin{aligned} d\nu^1_{k,z}(y_{d-k}, \dots, y_1) &= d\left|Df^{\eta^\kappa_{d-k}(y_\tau)}_{\sum_{i=1}^{d-k-1} y_i + z}\right|(y_{d-k}) d\mathcal{H}^1(y_{d-k-1}) \cdots d\mathcal{H}^1(y_1), \\ d\nu^2_{k,z}(y_{d-k}, \dots, y_1) &= \mathbf{1}_{\left|\sum_{i=1}^{d-k} (y_\tau)_{\eta_i^\kappa(y_\tau)} - \sum_{i=1}^{d-k} y_i\right| \le 4\tau} d\nu^1_{k,z}(y_{d-k}, \dots, y_1), \end{aligned}$$

for any $z \in \bigotimes_{i=d-k+1}^{d} \widetilde{H}_{\eta_i^\kappa(y_\tau)}$. Hence

$$\begin{aligned} &\limsup_{\lambda\to\infty} A_{12}(\lambda,\varepsilon) \\ &\quad \le \limsup_{\lambda\to\infty} \lambda \mathcal{L}^d\Bigg(\Bigg\{x \in B_R : C(n,\varepsilon,\tau,\zeta)\rho_0^{\frac{1}{4}} \\ &\qquad\qquad \times \sum_{k=0}^{d-2} \sum_{y_\tau \in S_\tau} \mathbf{I}_{\frac{3}{4}}^{d-k}\Big(\nu^1_{k,x^{2,d-k}_{\eta_i^\kappa(y_\tau)}}, \bigotimes_{i=1}^{d-k} \widetilde{H}_{\eta_i^\kappa(y_\tau)}\Big)(x^{1,d-k}_{\eta_i^\kappa(y_\tau)}) > \lambda\Bigg\}\Bigg) \\ &\qquad + \limsup_{\lambda\to\infty} \lambda \mathcal{L}^d\Bigg(\Bigg\{x \in B_R : C(n,\varepsilon,\zeta) \sum_{k=0}^{d-2} \sum_{y_\tau \in S_\tau} \mathbf{1}_{B_{4\tau}(y_\tau)}(x) \\ &\qquad\qquad \times \mathbf{M}^{d-k}\Big(\nu^2_{k,x^{2,d-k}_{\eta_i^\kappa(y_\tau)}}, \bigotimes_{i=1}^{d-k} \widetilde{H}_{\eta_i^\kappa(y_\tau)}\Big)(x^{1,d-k}_{\eta_i^\kappa(y_\tau)}) > \lambda\Bigg\}\Bigg). \end{aligned} \tag{3.31}$$

We easily derive from the boundedness of $\mathbf{I}_{3/4}^{d-k}(\cdot, X)$ from

$$\mathcal{M}_b \quad \text{to} \quad L^{\frac{d-k}{d-k-\frac{3}{4}},\infty}$$

with $X = \bigotimes_{i=1}^{d-k} \widetilde{H}_{\eta_i^\kappa(y_\tau)}$ that the first term in the right-hand side of (3.31) equals 0. Thanks to (2.8), we get that the second term in the right-hand side of

(3.31) is bounded by

$$\begin{aligned}
&\limsup_{\lambda\to\infty}\sum_{k=0}^{d-2}\sum_{y_\tau\in S_\tau}\lambda\mathcal{L}^d\Bigg(\Bigg\{x\in B_{8\tau}(y_\tau):\\
&\qquad\qquad C(n,\varepsilon,\zeta)\mathbf{M}^{d-k}\Bigg(\nu^2_{k,x^{2,d-k}_{\eta_i^\kappa(y_\tau)}},\bigotimes_{i=1}^{d-k}\widetilde{H}_{\eta_i^\kappa(y_\tau)}\Bigg)\big(x^{1,d-k}_{\eta_i^\kappa(y_\tau)}\big)>\lambda\Bigg\}\Bigg)\\
&\quad\le C(n,\varepsilon,\zeta)\sum_{k=0}^{d-2}\sum_{y_\tau\in S_\tau}\int_{\widetilde{H}_{\eta_d^\kappa(y_\tau)}}\cdots\int_{\widetilde{H}_{\eta_1^\kappa(y_\tau)}}\mathbf{1}_{|\sum_{i=d-k+1}^d((y_\tau)_{\eta_i^\kappa(y_\tau)}-x_i)|\le 8\tau}\\
&\qquad\qquad\times d\nu^{2,s}_{k,\sum_{i=d-k+1}^d x_i}(x_1,\dots,x_{d-k})\,d\mathcal{H}^1(x_{d-k+1})\cdots d\mathcal{H}^1(x_d)\\
&\quad\le C(n,\varepsilon,\zeta)\sum_{k=0}^{d-2}\sum_{y_\tau\in S_\tau}\int_{H_{\eta^\kappa_{d-k}(y_\tau)}}\int_{\widetilde{H}_{\eta^\kappa_{d-k}(y_\tau)}}\mathbf{1}_{|y_\tau-(z_1+z_2)|\le 16\tau}\\
&\qquad\qquad\times d|D^s f_{z_2}^{\eta^\kappa_{d-k}(y_\tau)}|(z_1)\,d\mathcal{H}^{d-1}(z_2),
\end{aligned}$$

where $\nu^{2,s}_{k,\sum_{i=d-k+1}^d x_i}$ is the singular part of

$$\nu^2_{k,\sum_{i=d-k+1}^d x_i}.$$

Thanks to (2.1) in Proposition 2.2 and the definition of η, one has

$$\begin{aligned}
&\limsup_{\lambda\to\infty}A_{12}(\lambda,\varepsilon)\\
&\quad\le C(n,\varepsilon,\zeta)\sum_{k=0}^{d-2}\sum_{y_\tau\in S_\tau}\int_{\mathbb{R}^d}\mathbf{1}_{B_{20\tau}(y_\tau)}(x)|\langle\eta^\kappa_{d-k}(y_\tau),\eta(x)\rangle|\,d|\mu|^s(x).
\end{aligned}$$

Because of $\langle\eta^\kappa_{d-k}(y_\tau),\eta^\kappa(y_\tau)\rangle=0$ for any $k=0,1,\dots,d-2$, so

$$\begin{aligned}
&|\langle\eta^\kappa_{d-k}(y_\tau),\eta(x)\rangle|\\
&\quad\le|\langle\eta^\kappa_{d-k}(y_\tau),\eta(x)-\eta^\kappa(x)\rangle|+\langle\eta^\kappa_{d-k}(y_\tau),\eta^\kappa(x)-\eta^\kappa(y_\tau)\rangle|\\
&\quad\le|(\eta-\eta^\kappa)(x)|+\|\nabla\eta^\kappa\|_{L^\infty(\mathbb{R}^d)}|x-y_\tau|,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\limsup_{\lambda\to\infty}A_{12}(\lambda,\varepsilon)\\
&\quad\le C(n,\varepsilon,\zeta)\sum_{y_\tau\in S_\tau}\int_{\mathbb{R}^d}\mathbf{1}_{B_{20\tau}(y_\tau)}(x)\big[|(\eta-\eta^\kappa)(x)|+\|\nabla\eta^\kappa\|_{L^\infty(\mathbb{R}^d)}\tau\big]d|\mu|^s(x)\\
&\quad\overset{(3.13)}{\le}C(n,\varepsilon,\zeta)\|(\eta-\eta^\kappa)|\mu|^s\|_{\mathcal{M}(\mathbb{R}^d)}+C(n,\varepsilon,\zeta,\kappa)\tau|\mu|^s(\mathbb{R}^d).
\end{aligned}$$

Therefore, we get (3.30).

Step 8. Estimate $A_9(\lambda,\varepsilon)$ and finish the proof.
Hence, we derive from (3.25) and (3.24), (3.28), (3.29), (3.30) that

$$\limsup_{\lambda\to\infty} A_9(\lambda,\varepsilon) \lesssim |\log(\varepsilon)\|\mu|^s(\mathbb{R}^d) \\ + C(n,\varepsilon,\zeta)\||\eta-\eta^\kappa\|\mu|^s\|_{\mathcal{M}(\mathbb{R}^d)} + C(n,\varepsilon,\kappa,\zeta)\tau|\mu|^s(\mathbb{R}^d).$$

Combining this with (3.20) yields

$$\limsup_{\lambda\to\infty} \lambda\mathcal{L}^d\big(\{\mathbf{T}_\varepsilon^{1,n}(\mu)>\lambda\}\cap B_R\big) \\ \lesssim |\log(\varepsilon)\|\mu|^s(\mathbb{R}^d) + C(n,\varepsilon,\zeta)\||\eta-\eta^\kappa\|\mu|^s\|_{\mathcal{M}(\mathbb{R}^d)} \\ + C(n,\varepsilon,\kappa,\zeta)\tau|\mu|^s(\mathbb{R}^d) + C(n,\varepsilon)\zeta^\alpha|\mu|^s(\mathbb{R}^d).$$

At this point, sending $\tau\to0$, then $\kappa\to0$ and $\zeta\to0$, we obtain (3.12). The proof is complete. □

LEMMA 3.8. *Let $\mathbf{K}_{e,\rho}^{\varepsilon,n}$ be in* (3.21). *Then, for any $e\in\mathbb{S}^{d-1}$ there holds*

$$\sum_{j=-\infty}^{\infty}(2^j\rho)^d\int_{\mathbb{S}^{d-1}}\sup_{r\in[2^j\rho,2^{j+1}\rho]}\big|\mathbf{K}_{e,\rho}^{\varepsilon,n}(r\theta)|\,d\mathcal{H}^{d-1}(\theta)\lesssim|\log(\varepsilon)|. \tag{3.32}$$

PROOF.

Case 1. $j\geq1$. For any $r\in[2^j\rho,2^{j+1}\rho]$, $\theta\in\mathbb{S}^{d-1}$, we can estimate

$$\big|\mathbf{K}_{e,\rho}^{\varepsilon,n}(r\theta)\big| = \bigg|\frac{\varepsilon^{-d+1}}{\rho^\alpha}\int_{\mathbb{R}^d}[\mathbf{K}_n(y)-\mathbf{K}_n(r\theta)]\varphi_{\zeta\rho}(r\theta-y) \\ \times\frac{\langle\phi^{e,\varepsilon}((r\theta-y)/\rho),\eta_{y_\tau}^\kappa\rangle}{|r\theta-y|^{d-\alpha}}\,dy\bigg| \\ \lesssim\frac{\varepsilon^{-d+1}}{\rho^\alpha}\int_{\mathbb{R}^d}|\mathbf{K}_n(r\theta-y)-\mathbf{K}_n(r\theta)|\frac{\mathbf{1}_{|y|\leq\rho}\mathbf{1}_{|\frac{y}{|y|}-e|\leq\varepsilon}}{|y|^{d-\alpha}}\,dy.$$

By (2.18), one has for $|y|<r/2$,

$$|\mathbf{K}_n(r\theta-y)-\mathbf{K}_n(r\theta)|\lesssim\frac{|\Omega_n(\theta)\|y|}{r^{d+1}}+\frac{1}{r^d}|\Omega_n(r\theta-y)-\Omega_n(r\theta)|.$$

So, for any $r\in[2^j\rho,2^{j+1}\rho]$,

$$\big|\mathbf{K}_{e,\rho}^{\varepsilon,n}(r\theta)\big|\lesssim\frac{\varepsilon^{-d+1}|\Omega_n(\theta)|}{\rho^\alpha r^{d+1}}\int_{\mathbb{R}^d}\frac{\mathbf{1}_{|y|\leq\rho}\mathbf{1}_{|\frac{y}{|y|}-e|\leq\varepsilon}}{|y|^{d-\alpha-1}}\,dy \\ +\frac{\varepsilon^{-d+1}}{\rho^\alpha r^d}\int_{\mathbb{R}^d}|\Omega_n(r\theta-y)-\Omega_n(r\theta)|\frac{\mathbf{1}_{|y|\leq\rho}\mathbf{1}_{|\frac{y}{|y|}-e|\leq\varepsilon}}{|y|^{d-\alpha}}\,dy \\ \lesssim\frac{2^{-j}|\Omega_n(\theta)|}{(2^j\rho)^d}$$

$$+\frac{2^{j\alpha}\varepsilon^{-d+1}}{(2^j\rho)^d}\int_{\mathbb{R}^d}|\Omega_n(\theta-y)-\Omega_n(\theta)|\frac{\mathbf{1}_{|y|\le 2^{-j}}\mathbf{1}_{|\frac{y}{|y|}-e|\le\varepsilon}}{|y|^{d-\alpha}}\,dy.$$

Thus,

$$\begin{aligned}
&(2^j\rho)^d\int_{\mathbb{S}^{d-1}}\sup_{r\in[2^j\rho,2^{j+1}\rho]}|\mathbf{K}^{\varepsilon,n}_{e,\rho}(r\theta)|\,d\mathcal{H}^{d-1}(\theta)\lesssim 2^{-j}\|\Omega_n\|_{L^1(\mathbb{S}^{d-1})}\\
&\quad+\varepsilon^{-d+1}2^{j\alpha}\int_{\mathbb{R}^d}\frac{\mathbf{1}_{|y|\le 2^{-j}}\mathbf{1}_{\left|y/|y|-e\right|\le\varepsilon}}{|y|^{d-\alpha-\alpha_0/2}}dy\sup_{|h|\le 1/2}\frac{\|\Omega_n(\cdot-h)-\Omega_n(\cdot)\|_{L^1(\mathbb{S}^{d-1})}}{|h|^{\alpha_0/2}}\\
&\overset{(3.2),(2.21)}{\lesssim} 2^{-j}+2^{-j\alpha_0/2}\lesssim 2^{-j\alpha_0/2},
\end{aligned}$$

which implies

$$\sum_{j=1}^{\infty}(2^j\rho)^d\int_{\mathbb{S}^{d-1}}\sup_{r\in[2^j\rho,2^{j+1}\rho]}\left|\mathbf{K}^{\varepsilon,n}_{e,\rho}(r\theta)\right|d\mathcal{H}^{d-1}(\theta)\lesssim 1. \tag{3.33}$$

Case 2. $j\le 0$. We prove that

$$A_j:=(2^j\rho)^d\int_{\mathbb{S}^{d-1}}\sup_{r\in[2^j\rho,2^{j+1}\rho]}\left|\mathbf{K}^{\varepsilon,n}_{e,\rho}(r\theta)\right|\le C|\log(\varepsilon)|2^{j\frac{1}{2}\min\{\alpha,1\}}. \tag{3.34}$$

Indeed, let ψ be a smooth function in $\mathbb{R}^d$ such that $\psi(x)=1$ if $|x|\le 1$ and $\psi(x)=0$ if $|x|>2$. Assume $r\in(2^j\rho,2^{j+1}\rho]$. One has for any $\theta\in\mathbb{S}^{d-1}$,

$$A_j\lesssim 2^{jd}+\sum_{i=-\infty}^{1}(2^j\rho)^d\int_{\mathbb{S}^{d-1}}\sup_{r\in[2^j\rho,2^{j+1}\rho]}|\mathbf{K}_{i,n}(r\theta)|, \tag{3.35}$$

where

$$\begin{aligned}
\mathbf{K}_{i,n}(r\theta)=\frac{\varepsilon^{-d+1}}{\rho^{\alpha}}\int_{\mathbb{R}^d}\mathbf{K}_n(r\theta-y)\varphi_{\zeta\rho}(y)\frac{\langle\phi^{e,\varepsilon}(y/\rho),\eta^{\kappa}_{y_\tau}\rangle}{|y|^{d-\alpha}}\\
\times\left(\psi(2^{-i}\rho^{-1}y)-\psi(2^{-i+1}\rho^{-1}y)\right)dy.
\end{aligned}$$

We now estimate

$$\left(\sum_{i=-\infty}^{j-3}+\sum_{i=j+3}^{1}\right)(2^j\rho)^d\int_{\mathbb{S}^{d-1}}\sup_{r\in[2^j\rho,2^{j+1}\rho]}|\mathbf{K}_{i,n}(r\theta)|.$$

To do this, we have

$$\begin{aligned}
&\left(\sum_{i=-\infty}^{j-3}+\sum_{i=j+3}^{1}\right)|\mathbf{K}_{i,n}(r\theta)| \\
&\quad\lesssim \left(\sum_{i=-\infty}^{j-3}+\sum_{i=j+3}^{1}\right)\frac{\varepsilon^{-d+1}}{\rho^{\alpha}}\int_{\mathbb{R}^d}\frac{|\Omega_n(r\theta-y)|}{|r\theta-y|^d}\frac{\mathbf{1}_{|\frac{y}{|y|}-e|\le\varepsilon}}{|y|^{d-\alpha}}\mathbf{1}_{2^{i-1}\rho<|y|<2^{i+1}\rho}\,dy \\
&\quad\lesssim \left(\sum_{i=-\infty}^{j-3}+\sum_{i=j+3}^{1}\right)\frac{\varepsilon^{-d+1}}{\rho^{d}2^{(d-\alpha)j}}\int_{\mathbb{R}^d}\frac{|\Omega_n(\theta-y)|}{1+|y|^d}\frac{\mathbf{1}_{|\frac{y}{|y|}-e|\le\varepsilon}}{|y|^{d-\alpha}}\mathbf{1}_{2^{i-j-2}<|y|<2^{i-j+1}}\,dy \\
&\quad\lesssim \sum_{i=-\infty}^{j-3}\frac{2^{(i-j)\alpha}}{\rho^d 2^{j(d-\alpha)}}F_{2^{i-j+1}}(\theta)+\sum_{i=j+3}^{1}\frac{1}{\rho^d 2^{i(d-\alpha)}}F_{2^{i-j+1}}(\theta),
\end{aligned}$$

where $F_{\vartheta}(\theta)=\frac{\varepsilon^{-d+1}}{\vartheta^d}\int_{\mathbb{R}^d}|\Omega_n(\theta-y)|\mathbf{1}_{|\frac{y}{|y|}-e|\le\varepsilon}\mathbf{1}_{\vartheta/8<|y|<\vartheta}\,dy$.

We claim that

$$\int_{\mathbb{S}^{d-1}}F_{\vartheta}(\theta)\lesssim 1+\vartheta^{d-1}\text{ if }\vartheta\ge 16\text{ or }\vartheta\le 1/2.$$

In fact:
If $\vartheta\le\frac{1}{2}$, thanks to $\Omega_n(\theta)=\Omega_n(\varsigma\theta)$ for any $\varsigma>0$, $\theta\in\mathbb{S}^{d-1}$, we have

$$\begin{aligned}
\int_{\mathbb{S}^{d-1}}F_{\vartheta}(\theta)&=\frac{5}{2}\frac{\varepsilon^{-d+1}}{\vartheta^d}\int_{\mathbb{R}^d}\int_{4/5<|x|<6/5}|\Omega_n(x-|x|y)|dx\mathbf{1}_{|\frac{y}{|y|}-e|\le\varepsilon}\mathbf{1}_{\vartheta/8<|y|<\vartheta}\,dy \\
&\lesssim \|\Omega_n\|_{L^1(\mathbb{S}^{d-1})}\frac{\varepsilon^{-d+1}}{\vartheta^d}\int_{\mathbb{R}^d}\mathbf{1}_{|\frac{y}{|y|}-e|\le\varepsilon}\mathbf{1}_{\vartheta/16<|y|<2\vartheta}\,dy\overset{(2.21)}{\lesssim}1.
\end{aligned}$$

If $\vartheta\ge 16$. We have

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}}F_{\vartheta}(\theta) \\
&\quad=\frac{5}{2}\frac{\varepsilon^{-d+1}}{\vartheta^d}\int_{\mathbb{R}^d}\int_{4/5<|x|<6/5}|\Omega_n(x-|x|y)|dx\mathbf{1}_{|y/|y|-e|\le\varepsilon}\mathbf{1}_{\vartheta/16<|y|<2\vartheta}\,dy \\
&\quad\lesssim\frac{\varepsilon^{-d+1}}{\vartheta^d}\int_{\mathbb{R}^d}\int_{|y|-2<|x|<2+|y|}|\Omega_n(x)|dx\,\mathbf{1}_{|y/|y|-e|\le\varepsilon}\mathbf{1}_{\vartheta/32<|y|<4\vartheta}\,dy\,dh \\
&\quad\lesssim\frac{\varepsilon^{-d+1}}{\vartheta^d}\int_{\mathbb{R}^d}|y|^{d-1}\|\Omega_n\|_{L^1(\mathbb{S}^{d-1})}\mathbf{1}_{|\frac{y}{|y|}-e|\le\varepsilon}\mathbf{1}_{\vartheta/32<|y|<4\vartheta}\,dy\,dh\lesssim\vartheta^{d-1}.
\end{aligned}$$

Therefore,

$$\left(\sum_{i=-\infty}^{j-3}+\sum_{i=j+3}^{1}\right)(2^j\rho)^d\int_{\mathbb{S}^{d-1}}\sup_{r\in[2^j\rho,2^{j+1}\rho]}|\mathbf{K}_{i,n}(r\theta)|$$
$$\lesssim\sum_{i=-\infty}^{j-3}\frac{2^{jd}2^{(i-j)\alpha}}{2^{j(d-\alpha)}}\int_{\mathbb{S}^{d-1}}F_{2^{i-j+1}}(\theta)+\sum_{i=j+3}^{1}\frac{2^{jd}}{2^{i(d-\alpha)}}\int_{\mathbb{S}^{d-1}}F_{2^{i-j+1}}(\theta)$$
$$\lesssim\sum_{i=-\infty}^{j-3}2^{i\alpha}+\sum_{i=j+3}^{1}2^j2^{i(\alpha-1)}\lesssim 2^{j\frac{1}{2}\min\{\alpha,1\}}.$$

Here we have used the fact that $j\leq 0$ in the last inequality.

Next, we estimate

$$\sum_{i=j-2}^{j+2}(2^j\rho)^d\int_{\mathbb{S}^{d-1}}\sup_{r\in[2^j\rho,2^{j+1}\rho]}|\mathbf{K}_{i,n}(r\theta)|.$$

We can decompose

$$\mathbf{K}_{i,n}(r\theta)=\sum_{l=-4}^{\infty}\mathbf{K}_{i,n,l}(r\theta),\quad i=j-2,\dots,j+2\leq 2,$$

where

$$\mathbf{K}_{i,n,l}(r\theta)=\frac{\varepsilon^{-d+1}}{\rho^\alpha}\int_{\mathbb{R}^d}\mathbf{1}_{2^{i-l-1}\rho<|r\theta-y|\leq 2^{i-l}\rho}\mathbf{K}_n(r\theta-y)\varphi_{\zeta\rho}(y)$$
$$\times\frac{\langle\phi^{e,\varepsilon}(y/\rho),\eta^{\kappa}_{y_\tau}\rangle}{|y|^{d-\alpha}}\big(\psi(2^{-i}\rho^{-1}y)-\psi(2^{-i+1}\rho^{-1}y)\big)dy.$$

First we will show that

$$\sum_{i=j-2}^{j+2}(2^j\rho)^d\int_{\mathbb{S}^{d-1}}\sup_{r\in[2^j\rho,2^{j+1}\rho]}|\mathbf{K}_{i,n,l}(r\theta)|\lesssim 2^{j\alpha}\quad\forall l\geq -4.\tag{3.36}$$

In fact, one has

$$(2^j\rho)^d\int_{\mathbb{S}^{d-1}}\sup_{r\in[2^j\rho,2^{j+1}\rho]}|\mathbf{K}_{i,n,l}(r\theta)|$$
$$\lesssim(2^j\rho)^d\frac{\varepsilon^{-d+1}}{\rho^\alpha}\frac{1}{(2^{i-l}\rho)^d(2^i\rho)^{d-\alpha}}$$
$$\times\int_{\mathbb{S}^{d-1}}\sup_{r\in[2^j\rho,2^{j+1}\rho]}$$
$$\int_{\mathbb{R}^d}|\Omega_n(r\theta-y)|\mathbf{1}_{|\frac{y}{|y|}-e|\leq\varepsilon}\mathbf{1}_{|r\theta-y|\sim 2^{i-l}\rho}\mathbf{1}_{|y|\sim 2^i\rho}\,dy\,d\mathcal{H}^{d-1}(\theta).$$

We change the variable to get that

$$(2^j\rho)^d\int_{\mathbb{S}^{d-1}}\sup_{r\in[2^j\rho,2^{j+1}\rho]}|\mathbf{K}_{i,n,l}(r\theta)| \lesssim \varepsilon^{-d+1}2^{j\alpha}2^{ld}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}|\Omega_n(\theta-y)|\mathbf{1}_{|\frac{y}{|y|}-e|\leq\varepsilon}\mathbf{1}_{|\theta-y|\sim 2^{-l}}\,dy\,d\mathcal{H}^{d-1}(\theta).$$

On the other hand,

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}^d}|\Omega_n(\theta-y)|\mathbf{1}_{|\frac{y}{|y|}-e|\leq\varepsilon}\mathbf{1}_{|\theta-y|\sim 2^{-l}}\,dy\,d\mathcal{H}^{d-1}(\theta)\\
&\quad\lesssim 2^l\int_{||h|-1|\leq 2^{-l-10}}\int_{\mathbb{R}^d}|\Omega_n(h-y)|\mathbf{1}_{|\frac{y}{|y|}-e|\leq\varepsilon}\mathbf{1}_{|h-y|\sim 2^{-l}}\mathbf{1}_{||y|-1|\lesssim 2^{-l}}\,dy\,dh\\
&\quad\lesssim 2^l\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}|\Omega_n(h-y)|\mathbf{1}_{|h-y|\sim 2^{-l}}\,dh\right]\mathbf{1}_{|\frac{y}{|y|}-e|\leq\varepsilon}\mathbf{1}_{||y|-1|\lesssim 2^{-l}}\,dy\\
&\quad\lesssim 2^{-(d-1)l}\int_{\mathbb{R}^d}\mathbf{1}_{|\frac{y}{|y|}-e|\leq\varepsilon}\mathbf{1}_{||y|-1|\lesssim 2^{-l}}\,dy\lesssim 2^{-dl}\varepsilon^{d-1}.
\end{aligned}$$

Consequently,

$$(2^j\rho)^d\int_{\mathbb{S}^{d-1}}\sup_{r\in[2^j\rho,2^{j+1}\rho]}|\mathbf{K}_{i,n,l}(r\theta)|\lesssim 2^{j\alpha};$$

this implies (3.36).

Next, thanks to 3.3, we have for $l_0>100$,

$$\begin{aligned}
\sum_{l=l_0}^{\infty}|\mathbf{K}_{i,n,l}(r\theta)| \lesssim \sum_{l=l_0}^{\infty}\frac{\varepsilon^{-d+1}}{\rho^{\alpha}}\int_{\mathbb{R}^d}\mathbf{1}_{|y|\sim 2^{i-l}\rho}|\mathbf{K}_n(y)||\Theta(r\theta-y)-\Theta(r\theta)|dy\\
+\frac{\varepsilon^{-d+1}}{\rho^{\alpha}}|\Theta(r\theta)|,
\end{aligned}$$

where $\Theta(y)=\varphi_{\zeta\rho}(y)\frac{\langle\phi^{e,\varepsilon}(y/\rho),\eta^{\kappa}_{y_\tau}\rangle}{|y|^{d-\alpha}}\big(\psi(2^{-i}\rho^{-1}y)-\psi(2^{-i+1}\rho^{-1}y)\big)$.

Since

$$|\varphi_{\zeta\rho}(y)|\leq C\mathbf{1}_{|y|>\zeta\rho},\ |\nabla\varphi_{\zeta\rho}(y)|\leq\frac{C\mathbf{1}_{\zeta\rho<|y|\leq 2\zeta\rho}}{|y|},$$

we easily see that for any $l>100$, $2^{i-l-1}\rho<|r\theta-y|\leq 2^{i-l}\rho$, and $r\in[2^j\rho,2^{j+1}\rho]$,

$$|\Theta(r\theta)|\lesssim\frac{1_{|\theta-e|\leq\varepsilon}}{(2^i\rho)^{d-\alpha}},\quad |\Theta(r\theta-y)-\Theta(r\theta)|\lesssim\frac{|y|}{\varepsilon}\frac{1}{(2^i\rho)^{d-\alpha+1}}.$$

Thus, we get

$$
\begin{aligned}
&\sum_{l=l_0}^{\infty}\sum_{i=j-2}^{j+2}(2^j\rho)^d\int_{\mathbb{S}^{d-1}}\sup_{r\in[2^j\rho,2^{j+1}\rho]}|\mathbf{K}_{i,n,l}(r\theta)|d\mathcal{H}^{d-1}(\theta)\\
&\quad\lesssim\sum_{l=l_0}^{\infty}\sum_{i=j-2}^{j+2}(2^j\rho)^d\int_{\mathbb{S}^{d-1}}\frac{\varepsilon^{-d+1}}{\rho^\alpha}\int_{|y|\sim 2^{i-l}\rho}\frac{|\Omega_n(y)|}{\varepsilon(2^i\rho)^{d-\alpha+1}|y|^{d-1}}\,dy\,d\mathcal{H}^{d-1}(\theta)\\
&\qquad+\sum_{i=j-2}^{j+2}(2^j\rho)^d\int_{\mathbb{S}^{d-1}}\frac{\varepsilon^{-d+1}}{\rho^\alpha}\frac{\mathbf{1}_{|\theta-e|\le\varepsilon}}{(2^i\rho)^{d-\alpha}}d\mathcal{H}^{d-1}(\theta)\\
&\quad\lesssim\sum_{l=l_0}^{\infty}\sum_{i=j-2}^{j+2}(2^j\rho)^d\frac{\varepsilon^{-d+1}}{\rho^\alpha}\frac{2^{i-l}\rho}{\varepsilon(2^i\rho)^{d-\alpha+1}}+2^{j\alpha}\lesssim 2^{j\alpha}\big(\varepsilon^{-d}2^{-l_0}+1\big).
\end{aligned}
\tag{3.37}
$$

Therefore, it follows from (3.36) and (3.37) that

$$
\sum_{i=j-2}^{j+2}(2^j\rho)^d\int_{\mathbb{S}^{d-1}}\sup_{r\in[2^j\rho,2^{j+1}\rho]}|\mathbf{K}_{i,n}(r\theta)|d\mathcal{H}^{d-1}(\theta)\le C2^{j\alpha}(1+l_0+\varepsilon^{-d}2^{-l_0}).
$$

At this point we take $2^{l_0}\sim\varepsilon^{-d}$ and obtain that

$$
\sum_{i=j-2}^{j+2}(2^j\rho)^d\int_{\mathbb{S}^{d-1}}\sup_{r\in[2^j\rho,2^{j+1}\rho]}|\mathbf{K}_{i,n}(r\theta)|d\mathcal{H}^{d-1}(\theta)\le C2^{j\alpha}|\log(\varepsilon)|.
$$

From this and (3.35), we get (3.34).

Then, (3.32) follows from (3.33) and (3.34). The proof is complete. □

4 Regular Lagrangian Flows and Quantitative Estimates with BV Vector Fields

We first recall some definitions and properties of Regular Lagrangian flows introduced in [17]. Given a vector field $\mathbf{B}(t,x):(0,T)\times\mathbb{R}^d\to\mathbb{R}^d$, we assume the following growth condition:

(R1) The vector field $\mathbf{B}(t,x)$ can be decomposed as

$$
\frac{\mathbf{B}(t,x)}{1+|x|}=\widetilde{B}_1(t,x)+\widetilde{B}_2(t,x),
$$

$$
\text{with } \widetilde{B}_1\in L^1((0,T);L^1(\mathbb{R}^d)) \text{ and } \widetilde{B}_2\in L^1((0,T);L^\infty(\mathbb{R}^d)).
$$

We denote by L^0_{loc} the space of measurable functions endowed with local convergence in measure, and $\mathcal{B}(E_1;E_2)$ the space of bounded functions between the sets E_1 and E_2, $\log L_{\text{loc}}(\mathbb{R}^d)$ the space of measurable functions $u:\mathbb{R}^d\to\mathbb{R}$ such that $\int_{B_r}\log(1+|u(x)|)dx$ is finite for any $r>0$. The following is definition of Regular Lagrangian flow:

DEFINITION 4.1. If $\mathbf{B}$ is a vector field satisfying **(R1)**, then for fixed $t_0 \in [0, T)$, a map

$$X \in C([t_0, T]; L^0_{\text{loc}}(\mathbb{R}^d)) \cap \mathcal{B}([t_0, T]; \log L_{\text{loc}}(\mathbb{R}^d))$$

is a regular Lagrangian flow in the renormalized sense relative to $\mathbf{B}$ starting at t_0 if we have the following:

(i) The equation

$$\partial_t(h(X(t,x))) = (\nabla h)(X(t,x))\mathbf{B}(t, X(t,x))$$

holds in $\mathcal{D}'((t_0, T) \times \mathbb{R}^d)$ for every function $h \in C^1(\mathbb{R}^d, \mathbb{R})$ that satisfies $|h(z)| \le C(1 + \log(1 + |z|))$ and $|\nabla h(z)| \le \frac{C}{1+|z|}$ for all $z \in \mathbb{R}^d$.

(ii) $X(t_0, x) = x$ for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$.

(iii) There exists a constant $L > 0$ such that $X(t, \cdot)_{\#}\mathcal{L}^d \le L\mathcal{L}^d$ for any $t \in [t_0, T]$, i.e.,

$$\int_{\mathbb{R}^d} \varphi(X(t,x))dx \le L \int_{\mathbb{R}^d} \varphi(x)dx,$$

for all measurable $\varphi : \mathbb{R}^d \to [0, \infty)$. The constant L here will be called the compressibility constant of X.

We define the sub-level of the flow as

$$G_R = \{x \in \mathbb{R}^d : |X(t,x)| \le R \text{ for almost all } t \in [t_0, T]\}.$$

The following lemma gives a basic estimate for the decay of the super-levels of a regular Lagrangian flow. This lemma was proven in [17].

LEMMA 4.2. *Let $\mathbf{B}$ be a vector field satisfying* **(R1)** *and let X be a regular Lagrangian flow relative to $\mathbf{B}$ starting at time t_0, with compressibility constant L. Then for all $r, R > 0$ we have $\mathcal{L}^d(B_r \setminus G_R) \le g(r, R)$ where the function g depends only on L, $\|\widetilde{B}_1\|_{L^1((0,T);L^1(\mathbb{R}^d))}$ and $\|\widetilde{B}_2\|_{L^1((0,T);L^\infty(\mathbb{R}^d))}$ and satisfies $g(r, R) \downarrow 0$ for r fixed and $R \uparrow \infty$.*

The following is our main theorem.

THEOREM 4.3. *Let $\mathbf{B} \in L^1([0, T]; L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d))$ and $R > 1$. Assume that*

$$\mathbf{B}^i = \sum_{j=1}^m \mathbf{K}^i_j \star b_j \quad \text{in } B_{2R}, \text{ with } b_j \in L^1([0, T], BV(\mathbb{R}^d)), \tag{4.1}$$

where $(\mathbf{K}^i_j)_{i,j}$ are singular kernels in $\mathbb{R}^d$ satisfying conditions of singular kernel $\mathbf{K}$ in Theorem 3.3 with constants $c_1, c_2 > 0$. Let $t_0 \in [0, T)$, $\mathbf{B}_1, \mathbf{B}_2 \in L^1([0, T]; L^1_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d))$ and let X_1, X_2 be regular Lagrangian flows starting at time t_0 associated to $\mathbf{B}_1, \mathbf{B}_2$ resp. with compression constants $L_1, L_2 \le L_0$ for some $L_0 > 0$. Assume that $\|(\mathbf{B}_1, \mathbf{B}_2)\|_{L^1([0,T]\times B_R)} \le c_R$.

Then, if

$$\operatorname{div}(\mathbf{B}) \in L^1((0, T), \mathcal{M}_b(B_{2R})) \quad \text{and} \quad (\operatorname{div}(\mathbf{B}))^+ \in L^1((0, T), L^1(B_{2R}))$$

for any $\kappa \in (0,1), r > 1$ *there exists* $\delta_0 = \delta_0(d,T,r,R,c_R,c_1,c_2,L_0,b,\kappa) \in (0,\frac{1}{100})$ *such that*

$$\begin{aligned}&\sup_{t_1\in[t_0,T]} \mathcal{L}^d\big(\{x \in B_r : |X_{1t_1}(x) - X_{2t_1}(x)| > \delta^{1/2}\}\big)\\ &\quad\lesssim \mathcal{L}^d\big(B_r \setminus G_{1,R}\big) + \mathcal{L}^d\big(B_r \setminus G_{2,R}\big)\\ &\qquad + \frac{L_0}{\delta}\|(\mathbf{B}_1 - \mathbf{B}, \mathbf{B}_2 - \mathbf{B})\|_{L^1([0,T]\times B_R)} + \kappa \quad \textit{for any } \delta \in (0,\delta_0).\end{aligned} \tag{4.2}$$

where $G_{i,R} = \{x \in \mathbb{R}^N : |X_i(s,x)| \le R \textit{ for almost all } s \in [t_0,T]\}$ *for* $i = 1,2$.

We derive the following from Theorem 4.3 and Lemma 4.2:

COROLLARY 4.4. *Let* $\mathbf{B} \in L^1([0,T]; L^1_{\text{loc}}(\mathbb{R}^d,\mathbb{R}^d))$. *Assume that for any* $R > 0$, *there exist singular kernels* $(\mathbf{K}^i_{jR})_{i,j}$ $(i = 1,\dots,d,\ j = 1,\dots,m(R))$ *in* $\mathbb{R}^d$ *satisfying conditions of singular kernel* $\mathbf{K}$ *in Theorem* 3.3 *with constants* $c_{1R}, c_{2R} > 0$; *and* $b_{jR} \in L^1([0,T], BV(\mathbb{R}^d))$ *such that*

$$\mathbf{B}^i = \sum_{j=1}^m \mathbf{K}^i_{jR} \star b_{jR} \quad \textit{in } B_{2R}. \tag{4.3}$$

Let $t_0 \in [0,T)$, $\mathbf{B}_1, \mathbf{B}_2 \in L^1([0,T]; L^1_{\text{loc}}(\mathbb{R}^d,\mathbb{R}^d))$ *and let* X_1, X_2 *be regular Lagrangian flows starting at time* t_0 *associated to* $\mathbf{B}_1, \mathbf{B}_2$ *resp. with compression constants* $L_1, L_2 \le L_0$ *for some* $L_0 > 0$. *Assume that* $\mathbf{B}_1, \mathbf{B}_2$ *satisfy* $(\mathbf{R}_1)$ *i.e.,* $\frac{\mathbf{B}_l(t,x)}{|x|+1} = \widetilde{B}_{1l}(t,x) + \widetilde{B}_{2l}(t,x)$, $l = 1,2$ *with*

$$\sum_{l=1,2} \|\widetilde{B}_{1l}\|_{L^1((0,T);L^1(\mathbb{R}^d))} + \|\widetilde{B}_{2l}\|_{L^1((0,T);L^\infty(\mathbb{R}^d))} \le C_0.$$

Then, if $\operatorname{div}(\mathbf{B}) \in L^1((0,T),\mathcal{M}_{\text{loc}}(\mathbb{R}^d))$ *and* $(\operatorname{div}(\mathbf{B}))^+ \in L^1((0,T),L^1_{\text{loc}}(\mathbb{R}^d))$, *for any* $\kappa \in (0,1), r > 1$ *there exist* $R_0 = R_0(d,T,r,C_0,L_0,\kappa) > 1$, $\delta_0 = \delta_0(d,T,r,C_0,c_{1R_0},c_{2R_0},L_0,b_{R_0},\kappa) \in (0,1/100)$ *such that*

$$\begin{aligned}&\sup_{t_1\in[t_0,T]} \mathcal{L}^d\Big(\Big\{x \in B_r : |X_{1t_1}(x) - X_{2t_1}(x)| > \delta^{1/2}\Big\}\Big)\\ &\qquad\lesssim \frac{L_0}{\delta}\|(\mathbf{B}_1 - \mathbf{B}, \mathbf{B}_2 - \mathbf{B})\|_{L^1([0,T]\times B_{R_0})} + \kappa,\end{aligned} \tag{4.4}$$

for any $\delta \in (0,\delta_0)$.

PROOF OF THEOREM 4.3. Without loss generality, we assume $t_0 = 0$.

Step 1. By Proposition 2.1, there exist unit vectors $\xi_t(x) \in \mathbb{R}^m, \eta_t(x) \in \mathbb{R}^d$ such that $D^s b_t(x) = \xi_t(x) \otimes \eta_t(x)|D^s b_t|(x)$ i.e.,

$$D^s_{x_j} b_{tk}(x) = \xi_{tk}(x)\eta_{tj}(x)|D^s b_t|(x)$$

for any $k = 1, \dots, m$, $j = 1, \dots, d$. Let $\eta_t^\varepsilon \in C^\infty((0,T) \times \mathbb{R}^d, \mathbb{R}^d)$, $\xi_t^\varepsilon \in C^\infty((0,T) \times \mathbb{R}^d, \mathbb{R}^m)$ be such that $|\eta_t^\varepsilon| = |\xi_t^\varepsilon| = 1$ and

$$\lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{R}^d} |\eta_t - \eta_t^\varepsilon| d|D^s b_t| dt + \lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{R}^d} |\xi_t - \xi_t^\varepsilon| d|D^s b_t| dt = 0.$$

For $\delta \in (0, \frac{1}{100})$, $1 < \gamma < |\log(\delta)|$, $\varepsilon > 0$, and $t \in [0,T]$, let us define the quantity

$$\begin{aligned}\Phi_\delta^{\gamma,\varepsilon}(t) \\ = \frac{1}{2} \int_D \log\left(1 + \frac{|X_{1t}(x) - X_{2t}(x)|^2 + \gamma \langle \eta_t^\varepsilon(X_{1t}(x)), X_{1t}(x) - X_{2t}(x)\rangle^2}{\delta^2}\right) dx.\end{aligned} \tag{4.5}$$

where $D = B_r \cap G_{1,R} \cap G_{2,R}$. Since $\partial_t X_{jt} = \mathbf{B}_{jt}(X_{jt})$, 1 has for any $t_1 \in [0,T]$

$$\begin{aligned}
\sup_{t_1 \in [0,T]} \Phi_\delta^{\gamma,\varepsilon}(t_1) &= \sup_{t_1 \in [0,T]} \int_0^{t_1} \frac{d\Phi_\delta^{\gamma,\varepsilon}(t)}{dt} dt \\
&\le \sup_{t_1 \in [0,T]} \int_0^{t_1} \int_D \frac{\langle X_{1t} - X_{2t}, \mathbf{B}_{1t}(X_{1t}) - \mathbf{B}_{2t}(X_{2t})\rangle}{\delta^2 + |X_{1t} - X_{2t}|^2 + \gamma \langle \eta_t^\varepsilon(X_{1t}), X_{1t} - X_{2t}\rangle^2} dx\, dt \\
&\quad + \sup_{t_1 \in [0,T]} \int_0^{t_1} \int_D \frac{\gamma \langle \eta^\varepsilon(X_{1t}), X_{1t} - X_{2t}\rangle \langle \eta_t^\varepsilon(X_{1t}), \mathbf{B}_{1t}(X_{1t}) - \mathbf{B}_{2t}(X_{2t})\rangle}{\delta^2 + |X_{1t} - X_{2t}|^2 + \gamma \langle \eta_t^\varepsilon(X_{1t}), X_{1t} - X_{2t}\rangle^2} dx\, dt \\
&\quad + \sup_{t_1 \in [0,T]} \int_0^{t_1} \int_D \frac{\gamma \langle \eta^\varepsilon(X_{1t}), X_{1t} - X_{2t}\rangle \langle (\nabla \eta_t^\varepsilon)(X_{1t}) \mathbf{B}_{1t}(X_{1t}), X_{1t} - X_{2t}\rangle}{\delta^2 + |X_{1t} - X_{2t}|^2 + \gamma \langle \eta_t^\varepsilon(X_{1t}), X_{1t} - X_{2t}\rangle^2} dx\, dt \\
&\quad + \sup_{t_1 \in [0,T]} \int_0^{t_1} \int_D \frac{\gamma \langle \eta^\varepsilon(X_{1t}), X_{1t} - X_{2t}\rangle \langle (\partial_t \eta_t^\varepsilon)(X_{1t}), X_{1t} - X_{2t}\rangle}{\delta^2 + |X_{1t} - X_{2t}|^2 + \gamma \langle \eta_t^\varepsilon(X_{1t}), X_{1t} - X_{2t}\rangle^2} dx\, dt \\
&= I_1(\delta,\varepsilon,\gamma) + I_2(\delta,\varepsilon,\gamma) + I_3(\delta,\varepsilon,\gamma) + I_4(\delta,\varepsilon,\gamma).
\end{aligned} \tag{4.6}$$

By $\|\eta^\varepsilon\|_{L^\infty((0,T)\times\mathbb{R}^d)} \le 1$, and changing variable along the flows with $(X_{jt})_\# \mathcal{L}^d \le L_0 \mathcal{L}^d$ for all $t \in [0,T]$ and $j = 1,2$, we get

$$\sum_{i=1,2} I_i(\delta,\varepsilon,\gamma) \lesssim \frac{L_0 \gamma^{1/2}}{\delta} \|(\mathbf{B}_1 - \mathbf{B}, \mathbf{B}_2 - \mathbf{B})\|_{L^1([0,T]\times B_R)} + \sum_{i=5,6} I_i(\delta,\varepsilon,\gamma) \tag{4.7}$$

and

$$|I_3(\delta,\varepsilon,\gamma)| \lesssim L_0 \gamma^{1/2} \|\nabla \eta^\varepsilon\|_{L^\infty((0,T)\times\mathbb{R}^d)} \|\mathbf{B}_1\|_{L^1([0,T]\times B_R)}, \tag{4.8}$$

$$|I_4(\delta,\varepsilon,\gamma)| \lesssim \gamma^{1/2} r^d T \|\partial_t \eta^\varepsilon\|_{L^\infty((0,T)\times\mathbb{R}^d)}, \tag{4.9}$$

where

$$\begin{aligned}I_5(\delta,\varepsilon,\gamma) \\ = \sup_{t_1 \in [0,T]} \int_0^{t_1} \int_D \frac{\langle X_{1t} - X_{2t}, \mathbf{B}_t(X_{1t}) - \mathbf{B}_t(X_{2t})\rangle}{\delta^2 + |X_{1t} - X_{2t}|^2 + \gamma \langle \eta_t^\varepsilon(X_{1t}), X_{1t} - X_{2t}\rangle^2} dx\, dt,\end{aligned}$$

$$\begin{aligned}I_6(\delta,\varepsilon,\gamma) \\ = \sup_{t_1 \in [0,T]} \int_0^{t_1} \int_D \frac{\gamma \langle \eta_t^\varepsilon(X_{1t}), X_{1t} - X_{2t}\rangle \langle \eta_t^\varepsilon(X_{1t}), \mathbf{B}_t(X_{1t}) - \mathbf{B}_t(X_{2t})\rangle}{\delta^2 + |X_{1t} - X_{2t}|^2 + \gamma \langle \eta_t^\varepsilon(X_{1t}), X_{1t} - X_{2t}\rangle^2} dx\, dt.\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sup_{t_1\in[0,T]} \Phi_\delta^{\gamma,\varepsilon}(t_1) \\
&\quad\geq \frac{1}{2}|\log(\delta)| \sup_{t_1\in[0,T]} \mathcal{L}^d\big(\{x\in D : |X_{1t_1}(x)-X_{2t_1}(x)|>\delta^{1/2}\}\big) \\
&\quad\geq \frac{1}{2}|\log(\delta)| \sup_{t_1\in[0,T]} \mathcal{L}^d\big(\{x\in B_r : |X_{1t_1}(x)-X_{2t_1}(x)|>\delta^{1/2}\}\big) \\
&\qquad - \frac{1}{2}|\log(\delta)|\big(\mathcal{L}^d(B_r\setminus G_{1,R})+\mathcal{L}^d(B_r\setminus G_{2,R})\big).
\end{aligned} \tag{4.10}$$

It follows from (4.6), (4.7), (4.8), and (4.10) and $\gamma < |\log(\delta)|$ that for any $t_1\in[0,T]$

$$\begin{aligned}
&\sup_{t_1\in[0,T]} \mathcal{L}^d\big(\{x\in D : |X_{1t_1}(x)-X_{2t_1}(x)|>\delta^{1/2}\}\big) \\
&\quad\lesssim \mathcal{L}^d(B_r\setminus G_{1,R})+\mathcal{L}^d(B_r\setminus G_{2,R}) \\
&\qquad + \frac{C(\varepsilon,\gamma,r,T)}{|\log(\delta)|}\big(L_0\|B_1\|_{L^1([0,T]\times B_R)}+1\big) \\
&\qquad + \frac{L_0}{\delta}\|(\mathbf{B}_1-\mathbf{B},\mathbf{B}_2-\mathbf{B})\|_{L^1([0,T]\times B_R)} \\
&\qquad + \frac{I_5(\delta,\varepsilon,\gamma)}{|\log(\delta)|}+\frac{I_6(\delta,\varepsilon,\gamma)}{|\log(\delta)|}.
\end{aligned} \tag{4.11}$$

Step 2. We prove that for any $\varepsilon_1\in(0,\frac{1}{100})$,

$$\begin{aligned}
&\limsup_{\delta\to 0} \frac{I_5(\delta,\varepsilon,\gamma)}{|\log(\delta)|} \\
&\quad\leq C(\varepsilon_1)\int_0^T\int_{\mathbb{R}^d} |\eta_t-\eta_t^\varepsilon|\,d|D^s b_t|\,dt \\
&\qquad + C(L_0)\varepsilon_1|\log(\varepsilon_1)|\int_0^T\int_{\mathbb{R}^d} d|D^s b_t|\,dt \\
&\qquad + C(L_0,\varepsilon_1)\gamma^{-1/2}\int_0^T\int_{\mathbb{R}^d} d|D^s b_t|\,dt.
\end{aligned} \tag{4.12}$$

Indeed, thanks to (4.19) in Lemma 4.6 below with $x_1 = X_{1t}, x_2 = X_{2t}\in B_R$ and changing variable along the flows with $(X_{lt})_\#\mathcal{L}^d \leq L_0\mathcal{L}^d$ for all $t\in[0,T]$ and

$l = 1, 2$, we find that

$$
\begin{aligned}
&\limsup_{\delta\to 0} \frac{I_5(\delta,\varepsilon,\gamma)}{|\log(\delta)|} \\
&\quad\lesssim \limsup_{\delta\to 0} \frac{L_0}{|\log(\delta)|} \sum_{i,j} \int_0^T\int_{B_R} \frac{\mathbf{P}_1(Db)}{\delta} \wedge \mathbf{T}^1_{\varepsilon_1,i,j}(D^a b_j)dx\,dt \\
&\qquad + \limsup_{\delta\to 0} \frac{L_0}{|\log(\delta)|} \sum_{i,j} \int_0^T\int_{B_R} \frac{\mathbf{P}_1(Db)}{\delta} \wedge \mathbf{T}^1_{\varepsilon_1,i,j}(\omega^\varepsilon_{tij})dx\,dt \\
&\qquad + \limsup_{\delta\to 0} \frac{L_0}{|\log(\delta)|} \int_0^T\int_{B_R} \mathbf{P}_1(Db)dx\,dt \\
&\qquad + \limsup_{\delta\to 0} \frac{L_0\varepsilon_1}{|\log(\delta)|} \sum_{i,j} \int_0^T\int_{B_R} \frac{\mathbf{P}_1(Db)}{\delta} \wedge \mathbf{T}^2_{\varepsilon_1,i,j}(Db_{tj})dx\,dt \\
&\qquad + \limsup_{\delta\to 0} \frac{L_0\gamma^{-1/2}}{|\log(\delta)|} \sum_{i,j} \int_0^T\int_{B_R} \frac{\mathbf{P}_1(Db)}{\delta} \wedge \mathbf{T}^1_{\varepsilon_1,i,j}(\xi_{tj}|D^s b_{tj}|)dx\,dt \\
&\qquad = (1) + (2) + (3) + (4) + (5),
\end{aligned}
\tag{4.13}
$$

where $\sum_{i,j} := \sum_{i=1}^d \sum_{j=1}^m$, $\omega^\varepsilon_{tij} := (\eta_t - \eta^\varepsilon_t)\xi_{tj}|D^s b_{tj}|$ and $\mathbf{T}^1_{\varepsilon_1,i,j}, \mathbf{T}^2_{\varepsilon_1,i,j}$ are defined in Lemma 4.6 and $\mathbf{P}_1(Db) \in L^1((0,T), L^{q_0}_{\text{loc}}(\mathbb{R}^d))$ for some $q_0 > 1$.

Clearly, $(3) = 0$. We can apply (2.33) in Proposition 2.13 (and Remark 2.16) to $\mathbf{T}^1_{\varepsilon_1,i,j}$ and $f = \mathbf{P}_1(Db)$ to get that

$$
\begin{aligned}
(1) = 0, \quad (2) &\le C(\varepsilon_1) \int_0^T\int_{\mathbb{R}^d} |\eta_t - \eta^\varepsilon_t| d|D^s b_t|dt, \\
(5) &\le C(\varepsilon_1)\gamma^{-1/2} \int_0^T\int_{\mathbb{R}^d} d|D^s b_t|dt.
\end{aligned}
$$

On the other hand, it is clear to see that $\mathbf{K}^i_j$ and $\Theta^{\varepsilon_1,e}_2$ satisfy Theorem 3.3. So, we can apply (3.9) in Theorem 3.3 to $\mathbf{T}^2_{\varepsilon_1,i,j}$ and $f = \mathbf{P}_1(Db)$, (with $\alpha = 1, \varepsilon = \varepsilon_1$) and obtain that

$$
(4) \lesssim \varepsilon_1|\log(\varepsilon_1)| \int_0^T\int_{\mathbb{R}^d} d|D^s b_t|dt.
$$

Plugging above estimates into (4.13) gives (4.12).

Step 3. We prove that for any $\varepsilon_2 \in (0, \frac{1}{100})$

$$
\begin{aligned}
\limsup_{\delta\to 0} \frac{I_6(\delta,\varepsilon,\gamma)}{|\log(\delta)|} &\lesssim C(\varepsilon_2)\gamma^{1/2} \int_0^T\int_{\mathbb{R}^d} |\eta_t - \eta^\varepsilon_t| d|D^s b_t|dt \\
&\quad + \gamma^{1/2}\varepsilon_2|\log(\varepsilon_2)| \int_0^T\int_{\mathbb{R}^d} d|D^s b_t|dt.
\end{aligned}
\tag{4.14}
$$

Thanks to (4.20) in Lemma 4.6 below with $x_1 = X_{1t}, x_2 = X_{2t} \in B_R, \varepsilon_1 = \varepsilon_2$, and changing variable along the flows with $(X_{lt})_{\#}\mathcal{L}^d \leq L_0\mathcal{L}^d$ for all $t \in [0, T]$ and $l = 1, 2$, we find that

$$\begin{aligned}
&\limsup_{\delta\to 0} \frac{I_6(\delta,\varepsilon,\gamma)}{|\log(\delta)|} \\
&\quad\lesssim \limsup_{\delta\to 0} \frac{\gamma^{1/2}L_0}{|\log(\delta)|}\sum_{i,j}\int_0^T\int_{B_R} \frac{\mathbf{P}_2(Db)}{\delta} \wedge \mathbf{T}^1_{\varepsilon_2,i,j}(D^a b_j)\,dx\,dt \\
&\qquad + \limsup_{\delta\to 0} \frac{\gamma^{1/2}L_0}{|\log(\delta)|}\sum_{i,j}\int_0^T\int_{B_R} \frac{\mathbf{P}_2(Db)}{\delta} \wedge \mathbf{T}^1_{\varepsilon_2,i,j}(\omega^{\varepsilon}_{tij})\,dx\,dt \\
&\qquad + \limsup_{\delta\to 0} \frac{L_0\gamma^{1/2}}{|\log(\delta)|}\sum_{i,j}\int_0^T\int_{B_R} \mathbf{P}_2(Db)\,dx\,dt \\
&\qquad + \limsup_{\delta\to 0} \frac{L_0\gamma^{1/2}\varepsilon_2}{|\log(\delta)|}\sum_{i,j}\int_0^T\int_{B_R} \frac{\mathbf{P}_2(Db)}{\delta} \wedge \mathbf{T}^2_{\varepsilon_2,i,j}(Db_j)\,dx\,dt \\
&\qquad + \limsup_{\delta\to 0} \frac{C(\varepsilon_2,\gamma)}{|\log(\delta)|}\int_0^T\int_{B_R} \frac{\mathbf{I}_1(\mathbf{1}_{B_{4R}}(\operatorname{div}^a(B_t))^+)}{\delta} \wedge \mathbf{M}(\mathbf{1}_{B_{4R}}(\operatorname{div}^a(B_t))^+)\,dx\,dt \\
&\quad= (6) + (7) + (8) + (9) + (10),
\end{aligned}$$

where $\omega^{\varepsilon}_{tij} := (\eta_t - \eta^{\varepsilon}_t)\xi_{tj}|D^s b_{tj}|$ and $\mathbf{P}_2(Db) \in L^1((0,T), L^{q_0}_{\mathrm{loc}}(\mathbb{R}^d))$ for some $q_0 > 1$. Similarly, we also obtain that (6) + (8) = 0 and

$$\begin{aligned}
(7) &\leq C(\varepsilon_2)\gamma^{1/2}\int_0^T\int_{\mathbb{R}^d} |\eta_t - \eta^{\varepsilon}_t|\,d|D^s b_t|\,dt, \\
(9) &\lesssim \gamma^{1/2}\varepsilon_2|\log(\varepsilon_2)|\int_0^T\int_{\mathbb{R}^d} d|D^s b_t|\,dt.
\end{aligned}$$

Moreover, by 2.12 in Lemma (2.4), one has (10) = 0. Thus, we get (4.14). Therefore, we derive from (4.11) and (4.12), (4.14) that

$$\begin{aligned}
&\sup_{t_1\in[0,T]} \mathcal{L}^d\big(\{x \in D : |X_{1t_1}(x) - X_{2t_1}(x)| > \delta^{1/2}\}\big) \\
&\quad\lesssim \mathcal{L}^d(B_r \setminus G_{1,R}) + \mathcal{L}^d(B_r \setminus G_{2,R}) \\
&\qquad + \frac{L_0}{\delta}\|(\mathbf{B}_1 - \mathbf{B}, \mathbf{B}_2 - \mathbf{B})\|_{L^1([0,T]\times B_R)} + A(\delta),
\end{aligned}\tag{4.15}$$

and

$$\begin{aligned}
&\limsup_{\delta\to 0} A(\delta)\\
&\quad\lesssim \limsup_{\delta\to 0}\frac{I_5(\delta,\varepsilon,\gamma)}{|\log(\delta)|}+\limsup_{\delta\to 0}\frac{I_6(\delta,\varepsilon,\gamma)}{|\log(\delta)|}\\
&\quad\lesssim \big(C(\varepsilon_1)+C(\varepsilon_2)\gamma^{1/2}\big)\int_0^T\int_{\mathbb{R}^d}|\eta_t-\eta_t^\varepsilon|d|D^s b_t|dt\\
&\qquad+\big(\gamma^{1/2}\varepsilon_2|\log(\varepsilon_2)|+C(\varepsilon_1)\gamma^{-1/2}+\varepsilon_1\big|\log(\varepsilon_1)\big|\big)\int_0^T\int_{\mathbb{R}^d}d|D^s b_t|dt.
\end{aligned}\tag{4.16}$$

In the right-hand side of (4.16), we let $\varepsilon\to 0$, then $\varepsilon_2\to 0$, $\gamma\to\infty$ and $\varepsilon_1\to 0$ to get that $\limsup_{\delta\to 0} A(\delta)\le 0$. Combining this and (4.15) yields (4.2). The proof is complete. □

Let $\Theta_1^{\varepsilon,e}, \Theta_2^{\varepsilon,e}$ be in Lemma 2.3. For any $i=1,\dots,d$, $x_1\neq x_2\in B_R(0)$, and $\varepsilon_1\in(0,\frac{1}{100})$, we define $\mathbf{e}_1=-\mathbf{e}_2=\frac{x_1-x_2}{|x_1-x_2|}$, $r=|x_1-x_2|$, and

$$\Theta_{l,r}^{\varepsilon_1,e}(\cdot)=\Theta_l^{\varepsilon_1,e}\left(\frac{\cdot}{r}\right),\quad \widetilde{\Theta}_{l,r}^{\varepsilon_1,e}\left(\frac{\cdot}{r}\right)=\frac{1}{r}\frac{\varepsilon_1^{-d+1}}{|\cdot|^{d-1}}\Theta_{l,r}^{\varepsilon_1,e}(\cdot),\quad l=1,2,$$

$$A_{i1}^{\mathrm{reg}}:=\sum_{k=1,2}\sum_{j=1}^m\big[\mathbf{K}_j^i\star\widetilde{\Theta}_{1,r}^{\varepsilon_1,\mathbf{e}_k}\star(\mathbf{e}_1.D^a b_j)\big](x_k),$$

$$A_{i1}^{\mathrm{appro}}:=\sum_{k=1,2}\sum_{j=1}^m\big[\mathbf{K}_j^i\star\widetilde{\Theta}_{1,r}^{\varepsilon_1,\mathbf{e}_k}\star\big((\mathbf{e}_1.(\eta_t-\eta_t^\varepsilon))\xi_{tj}|D^s b_{tj}|\big)\big](x_k),$$

$$A_{i1}^{\text{diff-1}}:=\sum_{k=1,2}\sum_{j=1}^m\big[\mathbf{K}_j^i\star\widetilde{\Theta}_{1,r}^{\varepsilon_1,\mathbf{e}_k}\star\big((\mathbf{e}_1.(\eta_t^\varepsilon-\eta_t^\varepsilon(x_k)))\xi_{tj}|D^s b_{tj}|\big)\big](x_k),$$

$$A_{i1}^{\text{diff-2}}:=\sum_{j=1}^m\big(\mathbf{e}_1\cdot(\eta_t^\varepsilon(x_2)-\eta_t^\varepsilon(x_1))\big)\big[\mathbf{K}_j^i\star\widetilde{\Theta}_{1,r}^{\varepsilon_1,\mathbf{e}_2}\star\big(\xi_{tj}|D^s b_{tj}|\big)\big](x_2),$$

$$A_{i1}^{\mathrm{sing}}:=\sum_{k=1,2}\sum_{j=1}^m\big[\mathbf{K}_j^i\star\widetilde{\Theta}_{1,r}^{\varepsilon_1,\mathbf{e}_k}\star\big(\xi_{tj}|D^s b_{tj}|\big)\big](x_k),$$

$$A_{i2}=\sum_{j=1}^m\big[\mathbf{K}_j^i\star\widetilde{\Theta}_{2,r}^{\varepsilon_1,\mathbf{e}_1}\star Db_j\big](x_1)-\big[\mathbf{K}_j^i\star\widetilde{\Theta}_{2,r}^{\varepsilon_1,\mathbf{e}_2}\star Db_j\big](x_2),$$

$$E^{\mathrm{reg}}:=-\sum_{k=1,2}\sum_{i=1}^d\sum_{j=1}^m\big[\mathbf{K}_j^i\star\widetilde{\Theta}_{1,r}^{\varepsilon_1,\mathbf{e}_k}\star\big(D_{x_i}^a b_{tj}\big)\big](x_k),$$

$$E^{\mathrm{appro}}:=\sum_{k=1,2}\sum_{i=1}^d\sum_{j=1}^m\Big[\mathbf{K}_j^i\star\widetilde{\Theta}_{1,r}^{\varepsilon_1,\mathbf{e}_k}\star\big((\eta_{ti}^\varepsilon-\eta_{ti})\xi_{tj}|D^s b_{tj}|\big)\Big](x_k),$$

$$E^{\text{diff-1}} := \sum_{k=1,2}\sum_{i=1}^{d}\sum_{j=1}^{m}\big[\mathbf{K}_j^i \star \widetilde{\Theta}_{1,r}^{\varepsilon_1,\mathbf{e}_k} \star \big((\eta_{ti}^{\varepsilon}(x_1)-\eta_{ti}^{\varepsilon})\xi_{tj}|D^s b_{tj}|\big)\big](x_k),$$

$$E^{\text{diff-2}} := \sum_{i=1}^{d}\sum_{j=1}^{m}(\eta_{ti}^{\varepsilon}(x_1)-\eta_{ti}^{\varepsilon}(x_2))\big[\mathbf{K}_j^i \star \widetilde{\Theta}_{1,r}^{\varepsilon_1,\mathbf{e}_2} \star \big(\xi_{tj}|D^s b_{tj}|\big)\big](x_2).$$

Then, we have the following identities:

LEMMA 4.5. *There holds*

$$\begin{aligned}\mathbf{B}_t^i(x_1)-\mathbf{B}_t^i(x_2) &= rA_{i1}^{\text{reg}} + rA_{i1}^{\text{appro}} + rA_{i1}^{\text{diff-1}} + rA_{i1}^{\text{diff-2}}\\ &\quad + r\varepsilon_1 A_{i2} + r(\mathbf{e}_1.\eta_t^{\varepsilon}(x_1))A_{i1}^{\text{sing}},\end{aligned} \tag{4.17}$$

and [1]

$$\begin{aligned}\langle \eta_t^{\varepsilon}(x_1), A_1^{\text{sing}}\rangle &= E^{\text{reg}} + E^{\text{appro}} + E^{\text{diff-1}} + E^{\text{diff-2}}\\ &\quad + \sum_{i=1,2}\widetilde{\Theta}_{1,r}^{\varepsilon_1,\mathbf{e}_i} \star [\operatorname{div}(\mathbf{B}_t)](x_i),\end{aligned} \tag{4.18}$$

PROOF. By Proposition 2.3 with $\varepsilon=\varepsilon_1$ we have

$$\begin{aligned}&b_{tj}(x_1-z)-b_{tj}(x_2-z)\\ &\quad = r\widetilde{\Theta}_{1,r}^{\varepsilon_1,\mathbf{e}_1} \star (\mathbf{e}_1.Db_{tj})(x_1-z) + \varepsilon_1 r\widetilde{\Theta}_{2,r}^{\varepsilon_1,\mathbf{e}_1} \star Db_{tj}(x_1-z)\\ &\qquad - r\widetilde{\Theta}_{1,r}^{\varepsilon_1,\mathbf{e}_2} \star (\mathbf{e}_2.Db_{tj})(x_2-z) - \varepsilon_1 r\widetilde{\Theta}_{2,r}^{\varepsilon_1,\mathbf{e}_2} \star Db_{tj}(x_2-z),\end{aligned}$$

for any $z\in\mathbb{R}^d$. So, by (4.1), we get

$$\begin{aligned}&\mathbf{B}_t^i(x_1)-\mathbf{B}_t^i(x_2)\\ &\quad = \sum_{j=1}^{m}\big(\mathbf{K}_j^i \star b_{tj}(x_1) - \mathbf{K}_j^i \star b_{tj}(x_2)\big)\\ &\quad = \sum_{j=1}^{m} r\big[\mathbf{K}_j^i \star \widetilde{\Theta}_{1,r}^{\varepsilon_1,\mathbf{e}_1} \star (\mathbf{e}_1.Db_{tj})\big](x_1) + r\varepsilon_1\big[\mathbf{K}_j^i \star \Theta_{2,r}^{\varepsilon_1,\mathbf{e}_1} \star Db_{tj}\big](x_1)\\ &\qquad\quad - r\big[\mathbf{K}_j^i \star \widetilde{\Theta}_{1,r}^{\varepsilon_1,\mathbf{e}_2} \star (\mathbf{e}_2.Db_{tj})\big](x_2) - r\varepsilon_1\big[\mathbf{K}_j^i \star \widetilde{\Theta}_{2,r}^{\varepsilon_1,\mathbf{e}_2} \star Db_{tj}\big](x_2).\end{aligned}$$

Using $Db_{tj} = D^a b_{tj} + \xi_{tj}\eta_t|D^s b_t|$ yields

$$\begin{aligned}\mathbf{B}_t^i(x_1)-\mathbf{B}_t^i(x_2) &= rA_{i1}^{\text{reg}} + r\varepsilon_1 A_{i2}\\ &\qquad + \sum_{k=1,2}\sum_{j=1}^{m} r\big[\mathbf{K}_j^i \star \widetilde{\Theta}_{1,r}^{\varepsilon_1,\mathbf{e}_k} \star \big((\mathbf{e}_1.\eta_t)\xi_{tj}|D^s b_{tj}|\big)\big](x_k).\end{aligned}$$

[1] Here $A_1^{\text{sing}} = (A_{11}^{\text{sing}}, A_{21}^{\text{sing}}, \ldots, A_{d1}^{\text{sing}})$.

Since $\mathbf{e}_1.\eta_t = \mathbf{e}_1.(\eta_t - \eta_t^\varepsilon) + \mathbf{e}_1.(\eta_t^\varepsilon - \eta_t^\varepsilon(x_1)) + \mathbf{e}_1.\eta_t^\varepsilon(x_1)$,

$$\begin{aligned}\mathbf{B}_t^i(x_1) - \mathbf{B}_t^i(x_2) &= rA_{i1}^{\text{reg}} + r\varepsilon_1 A_{i2} + rA_{i1}^{\text{appro}} + rA_{i1}^{\text{diff-1}} + rA_{i1}^{\text{diff-2}} \\ &\quad + \sum_{k=1,2}\sum_{j=1}^{m} r\big[\mathbf{K}_j^i \star \widetilde{\Theta}_{1,r}^{\varepsilon_1,\mathbf{e}_k} \star \big((\mathbf{e}_1.\eta_t^\varepsilon(x_1))\xi_{tj}|D^s b_{tj}|\big)\big](x_k),\end{aligned}$$

which implies (4.17).

We have

$$\langle \eta_t^\varepsilon(x_1), A_1^{\text{sing}}\rangle = \sum_{k=1,2}\sum_{i=1}^{d}\sum_{j=1}^{m}\big[\mathbf{K}_j^i \star \widetilde{\Theta}_{1,r}^{\varepsilon_1,\mathbf{e}_k} \star \big(\eta_{ti}^\varepsilon(x_1)\xi_{tj}|D^s b_{tj}|\big)\big](x_k).$$

Since $\eta_{ti}^\varepsilon(x_1) = (\eta_{ti}^\varepsilon(x_1) - \eta_{ti}^\varepsilon) + (\eta_{ti}^\varepsilon - \eta_{ti}) + \eta_{ti}$ and $\eta_{ti}\xi_{tj}|D^s b_{tj}| = -D_{x_i}^a b_{tj} + D_{x_i} b_{tj}$, thus

$$\begin{aligned}\langle \eta_t^\varepsilon(x_1), A_1^{\text{sing}}\rangle &= E^{\text{diff-1}} + E^{\text{diff-2}} + E^{\text{appro}} + E^{\text{reg}} \\ &\quad + \sum_{k=1}^{2}\sum_{i=1}^{d}\sum_{j=1}^{m}\mathbf{K}_j^i \star \widetilde{\Theta}_{1,r}^{\varepsilon_1,\mathbf{e}_k} \star D_{x_i} b_{tj}(x_k).\end{aligned}$$

Using associative and commutativity properties of convolution and

$$\sum_{i=1}^{d}\sum_{j=1}^{m}\mathbf{K}_j^i \star D_{x_i} b_{tj} = \sum_{i=1}^{d} D_{x_i}\left(\sum_{j=1}^{m}\mathbf{K}_j^i \star b_{tj}\right) = \operatorname{div}(\mathbf{B}_t)$$

yields

$$\begin{aligned}\langle \eta_t^\varepsilon(x_1), A_1^{\text{sing}}\rangle &= E^{\text{reg}} + E^{\text{appro}} + E^{\text{diff-1}} + E^{\text{diff-2}} \\ &\quad + \sum_{k=1,2}\widetilde{\Theta}_{1,r}^{\varepsilon_1,\mathbf{e}_k} \star [\operatorname{div}(\mathbf{B}_t)](x_k).\end{aligned}$$

This gives (4.18). The proof is complete. □

Lemma 4.5 implies the following:

LEMMA 4.6. *We define for* $\varepsilon_1 \in (0, 1/100)$ *and* $R > 0$

$$\begin{aligned}&\mathbf{T}_{\varepsilon_1,i,j}^{l}(\mu_l)(x) \\ &\quad = \sup_{\rho\in(0,2R),e\in\mathbb{S}^{d-1}} \frac{\varepsilon_1^{-d+1}}{\rho}\left|\left(\frac{1}{|\cdot|^{d-1}}\Theta_{l,\rho}^{\varepsilon_1,e}(\cdot)\right) \star \mathbf{K}_j^i \star \mu_l(x)\right| \quad \forall x \in \mathbb{R}^d,\end{aligned}$$

with $\mu_2 \in \mathcal{M}_b(\mathbb{R}^d, \mathbb{R}^d)$, $\mu_1 \in \mathcal{M}_b(\mathbb{R}^d)$ *or* $\mu_1 \in \mathcal{M}_b(\mathbb{R}^d, \mathbb{R}^d)$.

There exist $\mathbf{P}_1(Db), \mathbf{P}_2(Db) \in L^1((0,T), L^{q_0}_{\text{loc}}(\mathbb{R}^d))$ *for some* $q_0 > 1$ *such that*

$$\sum_{k=1,2}\|\mathbf{P}_k(Db)\|_{L^1((0,T),L^{q_0}(B_R(0)))} \leq C(R,\varepsilon_1,\varepsilon)\|b\|_{L^1((0,T),BV(\mathbb{R}^d))}$$

for any $x_1 \neq x_2 \in B_R$, we have

$$
\begin{aligned}
A_1 &:= \frac{|\langle x_1 - x_2, \mathbf{B}_t(x_1) - \mathbf{B}_t(x_2)\rangle|}{\delta^2 + |x_1 - x_2|^2 + \gamma\langle \eta_t^\varepsilon(x_1), x_1 - x_2\rangle^2} \\
&\lesssim \sum_{l,i,j} \frac{\mathbf{P}_1(Db)(x_l,t)}{\delta} \wedge \mathbf{T}^1_{\varepsilon_1,i,j}(D^a b_j)(x_l) \\
&\quad + \sum_{l,i,j} \frac{\mathbf{P}_1(Db)(x_l,t)}{\delta} \wedge \mathbf{T}^1_{\varepsilon_1,i,j}(\omega^\varepsilon_{tij})(x_l) \\
&\quad + \mathbf{P}_1(Db)(x_l,t) + \varepsilon_1 \sum_{l,i,j} \frac{\mathbf{P}_1(Db)(x_l,t)}{\delta} \wedge \mathbf{T}^2_{\varepsilon_1,i,j}(Db_{tj})(x_l) \\
&\quad + \gamma^{-1/2} \sum_{l,i,j} \frac{\mathbf{P}_1(Db)(x_l,t)}{\delta} \wedge \mathbf{T}^1_{\varepsilon_1,i,j}(\xi_{tj}|D^s b_{tj}|)(x_l),
\end{aligned} \tag{4.19}
$$

$$
\begin{aligned}
A_2 &:= \frac{\gamma\langle \eta_t^\varepsilon(x_1), x_1 - x_2\rangle\langle \eta_t^\varepsilon(x_1), \mathbf{B}_t(x_1) - \mathbf{B}_t(x_2)\rangle}{\delta^2 + |x_1 - x_2|^2 + \gamma\langle \eta_t^\varepsilon(x_1), x_1 - x_2\rangle^2} \\
&\lesssim \gamma^{1/2} \sum_{l,i,j} \frac{\mathbf{P}_2(Db)(x_l,t)}{\delta} \wedge \mathbf{T}^1_{\varepsilon_1,i,j}(D^a b_j)(x_l) \\
&\quad + \gamma^{1/2} \sum_{l,i,j} \frac{\mathbf{P}_2(Db)(x_l,t)}{\delta} \wedge \mathbf{T}^1_{\varepsilon_1,i,j}(\omega^\varepsilon_{tij})(x_l) \\
&\quad + \gamma^{1/2} \sum_{l} \mathbf{P}_2(Db)(x_l,t) \\
&\quad + \gamma^{1/2}\varepsilon_1 \sum_{l,i,j} \frac{\mathbf{P}_2(Db)(x_l,t)}{\delta} \wedge \mathbf{T}^2_{\varepsilon_1,i,j}(Db_{tj})(x_l) + \\
&\quad + C(\varepsilon_1,\gamma) \sum_{l=1}^{2} \frac{\mathbf{I}_1(\mathbf{1}_{B_{4R}}(\operatorname{div}^a(B_t))^+)(x_l)}{\delta} \\
&\qquad \wedge \mathbf{M}(\mathbf{1}_{B_{4R}}(\operatorname{div}^a(B_t))^+)(x_l)
\end{aligned} \tag{4.20}
$$

where $\sum_{l,i,j} := \sum_{l=1}^2 \sum_{i=1}^d \sum_{j=1}^m$, $\omega^\varepsilon_{tij} := (\eta_t - \eta_t^\varepsilon)\xi_{tj}|D^s b_{tj}|$.

Set

$$
\mathbf{T}^{l,1}_{\varepsilon_1,i,j}(\mu_l)(x) = \sup_{\rho\in(0,2R), e\in\mathbb{S}^{d-1}} \varepsilon_1^{-d+1} \left| \left(\frac{1}{|\cdot|^{d-1}} \Theta^{\varepsilon_1,e}_{l,\rho}(\cdot) \right) \star \mathbf{K}^i_j \star \mu_l(x) \right| \quad \forall x \in \mathbb{R}^d.
$$

PROOF. *Step* 1. Thanks to (4.17), we obtain that

$$
\begin{aligned}
A_1 &= \frac{r\langle \mathbf{e}_1, \mathbf{B}_t(x_1) - \mathbf{B}_t(x_2)\rangle}{W} \\
&= \frac{r^2\langle \mathbf{e}_1, A_1^{\text{reg}}\rangle}{W} + \frac{r^2\langle \mathbf{e}_1, A_1^{\text{appro}}\rangle}{W} + \frac{r^2\langle \mathbf{e}_1, A_1^{\text{diff-1}}\rangle}{W} \\
&\quad + \frac{r^2(\mathbf{e}_1.\eta_t^\varepsilon(x_1))\langle \mathbf{e}_1, A_1^{\text{sing}}\rangle}{W} + \frac{r^2\langle \mathbf{e}_1, A_1^{\text{diff-2}}\rangle}{W} + \frac{r^2\varepsilon_1\langle \mathbf{e}_1, A_2\rangle}{W} \\
&= (1) + (2) + (3) + (6) + (4) + (5),
\end{aligned} \tag{4.21}
$$

with $r = |x_1 - x_2|$ and $W = \delta^2 + r^2 + \gamma r^2(\mathbf{e}_1.\eta_t^\varepsilon(x_1))^2$.

By definition of $\mathbf{T}^l_{\varepsilon_1,i,j}$ and $\mathbf{T}^{l,1}_{\varepsilon_1,i,j}$, we can estimate that

$$
\begin{aligned}
|(1)| &\le \sum_{l,i,j} \frac{\mathbf{T}^{1,1}_{\varepsilon_1,i,j}(D^a b_j)(x_l)}{\delta} \wedge \mathbf{T}^1_{\varepsilon_1,i,j}(D^a b_j)(x_l), \\
|(2)| &\le \sum_{l,i,j} \frac{\mathbf{T}^{1,1}_{\varepsilon_1,i,j}((\eta_t - \eta_t^\varepsilon)\xi_{tj}|D^s b_{tj}|)(x_l)}{\delta} \\
&\qquad \wedge \mathbf{T}^1_{\varepsilon_1,i,j}((\eta_t - \eta_t^\varepsilon)\xi_{tj}|D^s b_{tj}|)(x_l), \\
|(3)| &\le \sum_{l,i,j} \mathbf{T}^1_{\varepsilon_1,i,j}((\eta_t^\varepsilon - \eta_t^\varepsilon(x_l))\xi_{tj}|D^s b_{tj}|)(x_l), \\
|(4)| &\le \|\nabla \eta_t^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \sum_{l,i,j} \mathbf{T}^{1,1}_{\varepsilon_1,i,j}(\xi_{tj}|D^s b_{tj}|)(x_l), \\
|(5)| &\le \varepsilon_1 \sum_{l,i,j} \frac{\mathbf{T}^{2,1}_{\varepsilon_1,i,j}(Db_{tj})(x_l)}{\delta} \wedge \mathbf{T}^2_{\varepsilon_1,i,j}(Db_{tj})(x_l), \\
|(6)| &\le \gamma^{-1/2} \sum_{l,i,j} \frac{\mathbf{T}^{1,1}_{\varepsilon_1,i,j}(\xi_{tj}|D^s b_{tj}|)(x_l)}{\delta} \wedge \mathbf{T}^1_{\varepsilon_1,i,j}(\xi_{tj}|D^s b_{tj}|)(x_l).
\end{aligned}
$$

Set

$$
\begin{aligned}
&\mathbf{P}_1(Db)(x,t) \\
&\quad = \sum_{i,j} \mathbf{T}^{1,1}_{\varepsilon_1,i,j}(D^a b_j)(x) + \mathbf{T}^{1,1}_{\varepsilon_1,i,j}((\eta_t - \eta_t^\varepsilon)\xi_{tj}|D^s b_{tj}|)(x) \\
&\qquad + \mathbf{T}^1_{\varepsilon_1,i,j}((\eta_t^\varepsilon - \eta_t^\varepsilon(x))\xi_{tj}|D^s b_{tj}|)(x) + \\
&\qquad + \|\nabla \eta_t^\varepsilon\|_{L^\infty(\mathbb{R}^d)} \mathbf{T}^{1,1}_{\varepsilon_1,i,j}(\xi_{tj}|D^s b_{tj}|)(x) \\
&\qquad + \mathbf{T}^{2,1}_{\varepsilon_1,i,j}(Db_{tj})(x) + \mathbf{T}^{1,1}_{\varepsilon_1,i,j}(\xi_{tj}|D^s b_{tj}|)(x).
\end{aligned}
$$

By Lemma 2.14 and Remark 2.15, there exists $q_0 > 1$ such that

$$\|\mathbf{P}_1(Db)\|_{L^1((0,T),L^{q_0}(B_R(0)))} \le C(R,\varepsilon_1,\varepsilon)\|b\|_{L^1((0,T),BV(\mathbb{R}^d))},$$

for any $R > 0$. Combining these with (4.21) yields (4.19).

Step 2. Again, thanks to (4.17) we obtain that

$$\begin{aligned}
A_2 &= \frac{\gamma r U\langle \eta_t^\varepsilon(x_1), \mathbf{B}_t(x_1) - \mathbf{B}_t(x_2)\rangle}{W} \\
&= \frac{\gamma r^2 U\langle \eta_t^\varepsilon(x_1), A_1^{\text{reg}}\rangle}{W} + \frac{\gamma r^2 U\langle \eta_t^\varepsilon(x_1), A_1^{\text{appro}}\rangle}{W} + \frac{\gamma r^2 U\langle \eta_t^\varepsilon(x_1), A_1^{\text{diff-1}}\rangle}{W} \\
&\quad + \frac{\gamma r^2 U\langle \eta_t^\varepsilon(x_1), A_1^{\text{diff-2}}\rangle}{W} + \frac{\gamma r^2 \varepsilon_1 U\langle \eta_t^\varepsilon(x_1), A_2\rangle}{W} \\
&\quad + \frac{\gamma r^2 U^2\langle \eta_t^\varepsilon(x_1), A_1^{\text{sing}}\rangle}{W} \\
&= (7) + (8) + (9) + (10) + (11) + \frac{\gamma r^2 U^2\langle \eta_t^\varepsilon(x_1), A_1^{\text{sing}}\rangle}{W},
\end{aligned} \tag{4.22}$$

with $U = (\mathbf{e}_1.\eta_t^\varepsilon(x_1))$.

Plugging (4.18) into (4.22) gives

$$\begin{aligned}
A_2 &= (7) + (8) + (9) + (10) + (11) + \frac{\gamma r^2 U^2 E^{\text{reg}}}{W} + \frac{\gamma r^2 U^2 E^{\text{appro}}}{W} \\
&\quad + \frac{\gamma r^2 U^2 E^{\text{diff-1}}}{W} + \frac{\gamma r^2 U^2 E^{\text{diff-2}}}{W} \\
&\quad + \frac{\gamma r^2 U^2 \sum_{k=1,2} \widetilde{\Theta}_{1,r}^{\varepsilon_1,\mathbf{e}_k} \star [\operatorname{div}(\mathbf{B}_t)](x_k)}{W} \\
&= (7) + (8) + \cdots + (16).
\end{aligned} \tag{4.23}$$

As above, there exists $\mathbf{P}_2(Db)(x,t) \in L^1((0,T), L^{q_0}_{\text{loc}}(\mathbb{R}^d))$ for $q_0 > 1$ such that

$$\|\mathbf{P}_2(Db)\|_{L^1((0,T),L^{q_0}(B_R(0)))} \le C(R,\varepsilon_1,\varepsilon)\|b\|_{L^1((0,T),BV(\mathbb{R}^d))},$$

for any $R > 0$ and

$$|(7)| + |(12)| \lesssim \gamma^{1/2} \sum_{l,i,j} \frac{\mathbf{P}_2(Db)(x_l,t)}{\delta} \wedge \mathbf{T}^1_{\varepsilon_1,i,j}(D^a b_j)(x_l),$$

$$|(8)| + |(13)| \lesssim \gamma^{1/2} \sum_{l,i,j} \frac{\mathbf{P}_2(Db)(x_l,t)}{\delta} \wedge \mathbf{T}^1_{\varepsilon_1,i,j}((\eta_t - \eta_t^\varepsilon)\xi_{tj}|D^s b_{tj}|)(x_l),$$

$$|(9)| + |(10)| + |(14)| + |(15)| \le \gamma^{1/2} \sum_l \mathbf{P}_2(Db)(x_l,t),$$

$$|(11)| \le \varepsilon_1 \gamma^{1/2} \sum_{l,i,j} \frac{\mathbf{P}_2(Db)(x_l,t)}{\delta} \wedge \mathbf{T}^2_{\varepsilon_1,i,j}(Db_{tj})(x_l).$$

For (16), thanks to $\Theta_1^{\varepsilon,e} \geq 0$ and $\operatorname{div}^s(B_t) \leq 0$, we can estimate

$$(16) \leq \frac{\gamma r^2 U^2 \sum_{k=1,2} \widetilde{\Theta}_{1,r}^{\varepsilon_1,\mathbf{e}_k} \star \left[(\operatorname{div}^a(\mathbf{B}_t))^+\right](x_k)}{W}$$

$$\leq C(\varepsilon_1,\gamma) \sum_{l=1}^{2} \frac{\mathbf{I}_1(\mathbf{1}_{B_{4R}}(\operatorname{div}^a(\mathbf{B}_t))^+)(x_l)}{\delta} \wedge \mathbf{M}(\mathbf{1}_{B_{4R}}(\operatorname{div}^a(\mathbf{B}_t))^+)(x_l).$$

Combining above inequalities together yields (4.20). The proof is complete. □

Remark 4.7. In this remark, we would like to discuss another conjecture of Bressan. Let $B_n \in C_b^1((0,\infty)\times\mathbb{R}^d)$ be such that

$$\|B_n\|_{L^1\cap L^\infty} + \|DB_n\|_{L^1} \leq C \quad \forall n.$$

Bressan's compactness conjecture: If X_n solves $\frac{d}{dt}X_n(t,x) = B_n(t,X_n(t,x))$, $X_n(0) = \mathrm{Id}\ \mathbb{R}^d \times (0,\infty)$ and satisfies

$$C_1 \leq JX_n(t,x) \leq C_2 \quad \forall n. \tag{4.24}$$

Then X_n is locally compact in $L^1((0,\infty)\times\mathbb{R}^d)$. This conjecture was proven in [11] via the well-posedness of continuity equations in the class of nearly incompressible *BV* vector fields. Note that (4.24) implies

$$\sup_{t_1,t_2}\left|\int_{t_1}^{t_2} \operatorname{div}(B^n)(t,X_n(t,x))dt\right| \leq C \quad \forall n. \tag{4.25}$$

We *hope* that thanks to our estimates in Theorem 4.3 and the assumption (4.25), we can obtain this conjecture.

5 Well-Posedness of Regular Lagrangian Flows and Transport, Continuity Equations

5.1 Well-posedness of regular Lagrangian flows

The following results are obtained from Theorem 4.3, Corollary 4.4, and Lemma 4.2. The proofs are very similar to proofs in Sections 6 and 7 in [17].

PROPOSITION 5.1 (Uniqueness). *Let* $\mathbf{B}$ *be a vector field as in Corollary* 4.4 *satisfying the assumption* $(\mathbf{R_1})$. *Assume that* $\operatorname{div}(\mathbf{B}) \in L^1((0,T),\mathcal{M}_{\mathrm{loc}}(\mathbb{R}^d))$ *and* $(\operatorname{div}(\mathbf{B}))^+ \in L^1((0,T),L^1_{\mathrm{loc}}(\mathbb{R}^d))$. *If there exist the regular Lagrangian flows* X_1, X_2 *associated to* $\mathbf{B}$ *starting at time t, then we have* $X_1 \equiv X_2$.

PROPOSITION 5.2 (Stability). *Let* $\mathbf{B}_n$ *be a sequence of vector fields satisfying the assumption* $(\mathbf{R_1})$ *converging in* $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^d)$ *to a vector field* $\mathbf{B}$ *which satisfies the assumptions of* $\mathbf{B}$ *in Proposition* 5.1. *Assume that there exist* X_n *and* X *regular Lagrangian flows starting at time t associated to* $\mathbf{B}_n$ *and* $\mathbf{B}$ *resp. and denote by* L_n *and L the compression constants of the flows. Assume that for some decomposition* $\frac{\mathbf{B}_n}{1+|x|} = \widetilde{B}_{n,1} + \widetilde{B}_{n,2}$ *as in the assumption* $(\mathbf{R_1})$, *we have*

$$L_n + \|\widetilde{B}_{n,1}\|_{L^1((0,T),L^1(\mathbb{R}^d))} + \|\widetilde{B}_{n,2}\|_{L^1((0,T),L^\infty(\mathbb{R}^d))} \lesssim 1 \quad \forall n \in \mathbb{N}.$$

Then, for any compact set K,

$$\lim_{n\to\infty} \sup_{s\in[t,T]} \int_K |X_n(s,x) - X(s,x)| \wedge 1\, dx = 0. \tag{5.1}$$

PROPOSITION 5.3 (Compactness). *Let $\mathbf{B}_n \in C^1_b([0,T]\times\mathbb{R}^d, \mathbb{R}^d)$ converge in $L^1_{\mathrm{loc}}([0,T]\times\mathbb{R}^d)$ to a vector field $\mathbf{B}$ which satisfies the assumptions of $\mathbf{B}$ in Proposition* 5.1. *Let X_n be the flow starting at time t associated to $\mathbf{B}_n$ and denote by L_n the compression constants of the flow. Assume that for some decomposition $\frac{\mathbf{B}_n}{1+|x|} = \widetilde{B}_{n,1} + \widetilde{B}_{n,2}$ as in the assumption* ($\mathbf{R_1}$)*, we have*

$$L_n + \|\widetilde{B}_{n,1}\|_{L^1((0,T),L^1(\mathbb{R}^d))} + \|\widetilde{B}_{n,2}\|_{L^1((0,T),L^\infty(\mathbb{R}^d))} \lesssim 1 \quad \forall n \in \mathbb{N}.$$

Then, there exists a regular Lagrangian flow X starting at time t associated to B such that for any compact set K,

$$\lim_{n\to\infty} \sup_{s\in[t,T]} \int_K |X_n(s,x) - X(s,x)| \wedge 1\, dx = 0. \tag{5.2}$$

PROPOSITION 5.4 (Existence). *Let $\mathbf{B}$ be as in Proposition* 5.1*, and assume that* $\mathrm{div}(\mathbf{B}) \geq a(t)$ *in $(0,T)\times\mathbb{R}^d$ with $a \in L^1((0,T))$. Then, for all $t \in [0,T)$ there exists a regular Lagrangian flow $X := X(.,t,\cdot)$ associated to $\mathbf{B}$ starting at time t. Moreover, the flow X satisfies $X \in C(D_T; L^0_{\mathrm{loc}}(\mathbb{R}^d)) \cap \mathcal{B}(D_T; \log L_{\mathrm{loc}}(\mathbb{R}^d))$ where $D_T = \{(s,t) : 0 \leq t \leq s \leq T\}$ and for every $0 \leq t \leq \tau \leq s \leq T$, there holds $X(s,\tau,X(\tau,t,x)) = X(s,t,x)$ for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$.*

In the previous proposition we assume the condition $\mathrm{div}(\mathbf{B}) \geq a(t)$ in order to be sure to have a smooth approximating sequence with equibounded compression constants.

PROPOSITION 5.5 (Properties of the Jacobian). *Let $\mathbf{B}$ be as in Proposition* (5.4)*, $X(.,t,\cdot)$ the regular Lagrangian flow associated to $\mathbf{B}$ starting at time t. Assume that* $\mathrm{div}(\mathbf{B}) \in L^1((0,T), L^\infty(\mathbb{R}^d))$. *Then, the function*

$$JX(s,t,x) = \exp\left(\int_t^s \mathrm{div}(\mathbf{B})(\tau, X(\tau,t,x))\,d\tau\right)$$

satisfies

$$\int_{\mathbb{R}^d} \phi(x)\,dx = \int_{\mathbb{R}^d} \phi(X(s,t,x))\,JX(s,t,x)\,dx \quad \forall \phi \in L^1(\mathbb{R}^d) \tag{5.3}$$

and $\partial_s JX(s,t,x) = JX(s,t,x)\,\mathrm{div}(\mathbf{B})(\tau, X(s,t,x))$ for all $s \in (t,T)$. Moreover,

$$\exp(-L) \leq JX(s,t,x) \leq \exp(L),$$

with $L = \|\mathrm{div}(\mathbf{B})\|_{L^1((0,T),L^\infty(\mathbb{R}^d))}$. In addition, for any $0 \leq t \leq s \leq T$, $X^{-1}(t,s,\cdot)(x)$ exists almost everywhere $x \in \mathbb{R}^d$. The function JX is called the Jacobian of the flow X.

5.2 Well-posedness of transport and continuity equations

Next, we will connect the regular Lagrangian flows to the transport and continuity equations. We first recall the definition of a renormalized solution of (1.2), first introduced in [41].

DEFINITION 5.6. Let $u_0 : \mathbb{R}^d \to \mathbb{R}$ be a measurable function, let $\mathbf{B} \in L^1_{\text{loc}}((0,T)\times \mathbb{R}^d;\mathbb{R}^d)$ be a vector field such that $\text{div}(\mathbf{B}) \in L^1_{\text{loc}}((0,T)\times\mathbb{R}^d)$ and let $G, F \in L^1_{\text{loc}}((0,T)\times\mathbb{R}^d)$. A measure function $u : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ is a renormalized solution of (1.2) if for every function $\beta : \mathbb{R} \to \mathbb{R}$ satisfying $\beta \in C^1_b(\mathbb{R})$ and $\beta'(z)z \in L^\infty(\mathbb{R})$, $\beta(0) = 0$ we have that

$$\partial_t\beta(u) + \text{div}(\mathbf{B}\beta(u)) + \text{div}(\mathbf{B})\big(u\beta'(u) - \beta(u)\big) = Gu\beta'(u) + F\beta'(u)$$

and $\beta(u)(t=0) = \beta(u_0)$ in the sense of distributions.

We have the following proposition:

PROPOSITION 5.7. *Let* $\mathbf{B}$ *be in Proposition* (5.4)*,* X *be the regular Lagrangian flow associated to* $\mathbf{B}$ *starting at time* 0 *in Proposition* (5.4)*. Assume that* $\text{div}(\mathbf{B}) \in L^1((0,T), L^\infty(\mathbb{R}^d))$*. Let* $G, F \in L^1((0,T)\times\mathbb{R}^d)$ *and let* $u_0 : \mathbb{R}^d \to \mathbb{R}$ *be a measurable function. Then, there exists a unique renormalized solution* $u : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ *of* (1.2) *starting from* u_0*. Furthermore, for any* $(t,x) \in [0,T]\times\mathbb{R}^d$ *we have*

$$\begin{aligned} u(t,x) = {} & \frac{u_0(\overline{x})}{JX(t,\overline{x})}\exp\left(\int_0^t G(s,X(s,\overline{x}))ds\right) \\ & + \frac{1}{JX(t,\overline{x})}\int_0^t f(\tau,X(\tau,\overline{x}))\exp\left(\int_\tau^t G(s,X(s,\overline{x}))ds\right)JX(\tau,\overline{x})d\tau, \end{aligned} \tag{5.4}$$

with $\overline{x} = X^{-1}(t,\cdot)(x)$*,* $JX(t,\overline{x}) := JX(t,0,\overline{x})$*.*

Proof of previous proposition is very similar to [34][proof of theorem 2.7]. It is left to the reader.

Appendix

PROOF OF PROPOSITION 1.2. First we set

$$b_1(x) = -\,\text{sign}(x_2)\frac{x_1}{|x_2|^2}\mathbf{1}_{|x_1|\le|x_2|},\ b_2(x) = -\,\text{sign}(x_2)\mathbf{1}_{|x_1|\le|x_2|},$$

$$b_3(x) = -\frac{1}{|x_2|}\mathbf{1}_{|x_1|\le|x_2|},\ b_4(x) = -\mathbf{1}_{|x_1|\le|x_2|}.$$

By [41], there exist two different regular Lagrangian flows X_1, X_2 associated to the following vector field $\mathbf{B}(x) = (b_1(x)+b_2(x), b_3(x)+b_4(x))$ such that for any $x \in \mathbb{R}^2$, $X_1, X_2 \in W^{1,p}_{\text{loc}}(\mathbb{R}^2)$ for any $1 < p < 2$, $X_1, X_2 \in L^\infty_{\text{loc}}(\mathbb{R}^2; C(\mathbb{R}))\cap C(\mathbb{R}^2; L^q_{\text{loc}}(\mathbb{R}))$ for any $q < \infty$, and $X(t,\cdot)_\#\mathcal{L}^d = \mathcal{L}^d$ for any $t \in [0,T]$,

$$X_j(t+s,\cdot) = X_j(t, X(s,\cdot))\text{a.e. on }\mathbb{R}^2\text{for all } t,s \in \mathbb{R}^d \text{ and } j = 1,2.$$

Clearly, $\frac{|\mathbf{B}(x)|}{|x|+1} \in L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$ and $\operatorname{div}(\mathbf{B}) = 0$.

We show that $\mathbf{B}$ satisfies (1.16). Since $b_2, b_4 \in BV_{\mathrm{loc}}(\mathbb{R}^2)$, so it is enough to show that there exist functions $\Omega_1, \dots, \Omega_m \in (L^\infty \cap BV)(S^1)$ such that $\Omega_l(\theta) = \Omega_l(t\theta)$ for $\theta \in S^1$, $t > 0$, $\int_{S^1} \Omega_j = 0$, and

$$\partial_{x_l} b_i = \sum_{j=1}^m \left(\frac{\Omega_j^i(\cdot)}{|\cdot|^2}\right) \star \mu_j^l + \sigma_{li} \quad \text{in } \mathcal{D}'(\mathbb{R}^2), \tag{A.1}$$

for any $i = 1, 3$, $l = 1, 2$, and for some $\mu_j^l \in \mathcal{M}_b(\mathbb{R}^2)$, $\sigma_{li} \in \mathcal{M}(\mathbb{R}^2)$ and $j = 1, \dots, m$.

Let

$$K_1(x_1, x_2) = c\frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2}, \quad K_2(x_1, x_2) = c\frac{x_1^2 - x_2^2}{(x_1^2 + x_2^2)^2}$$

be kernels of operators $\mathcal{R}_1^2, \mathcal{R}_2^2$, where $\mathcal{R}_1, \mathcal{R}_2$ are the Riesz transforms in $\mathbb{R}^2$. We have

$$\begin{aligned} \partial_{x_1} b_1(x) &= -\frac{\operatorname{sign}(x_2)}{|x_2|^2}\mathbf{1}_{|x_1|\le|x_2|} + \frac{\operatorname{sign}(x_2)|x_1|}{|x_2|^2} d\delta_{|x_1|=|x_2|}(x_1) d\mathcal{L}^1(x_2) \\ &= \frac{\Omega_1(x)}{|x|^2} + \sum_{j=1,2} \mathcal{R}_j^2(\nu)(x) + \sigma(x) \quad \text{in } \mathcal{D}'(\mathbb{R}^2). \end{aligned} \tag{A.2}$$

for some $\sigma \in \mathcal{M}(\mathbb{R}^2)$, where

$$\Omega_1(y) = -\frac{\operatorname{sign}(y_2)|y|^2}{y_2^2}\mathbf{1}_{|y_1|\le|y_2|} \in (L^\infty \cap BV)(B_2 \setminus B_1),$$

with $\int_{S^1} \Omega_1 = 0$, and $\nu(x_1, x_2) = \frac{\operatorname{sign}(x_2)|x_1|}{|x_2|^2} d\delta_{|x_1|=|x_2|}(x_1) d\mathcal{L}^1(x_2)$.

By definition, we can write

$$\begin{aligned} \mathcal{R}_j^2(\nu)(x) &= \int_{\mathbb{R}} \big(K_j(x_1 - |y_2|, x_2 - y_2) + K_j(x_1 + |y_2|, x_2 - y_2)\big)\frac{dy_2}{y_2} \\ &= \frac{1}{|x|^2}\int_{\mathbb{R}} \big(K_j(\theta_1 - |y_2|, \theta_2 - y_2) + K_j(\theta_1 + |y_2|, \theta_2 - y_2)\big)\frac{dy_2}{y_2} \\ &:= \frac{\widetilde{\Omega}_j(\theta)}{|x|^2}, \end{aligned}$$

with $\theta = x/|x|$.

Clearly, $\widetilde{\Omega}_j(\theta_1, -\theta_2) = -\widetilde{\Omega}_j(\theta_1, \theta_2)$ and $\widetilde{\Omega}_j(-\theta_1, \theta_2) = \widetilde{\Omega}_j(\theta_1, \theta_2)$, so therefore $\int_{S^1} \widetilde{\Omega}_j = 0$.

Now we need to show $\widetilde{\Omega}_j(\theta) \in (L^\infty \cap W^{1,1})(B_{11/10} \setminus B_{9/10})$. Indeed, let $\chi \in C^\infty(\mathbb{R})$ be such that $\chi = 1$ in $[0, 1)$ and $\chi = 0$ in $(\frac{3}{2}, \infty)$ and set $\chi_r(x) = \chi(\frac{x}{r})$.

It is clear to see that

$$\begin{aligned}&\int_{\mathbb{R}} K_j(\theta_1-|y_2|,\theta_2-y_2)(1-\chi_2)(|y_2|)\frac{dy_2}{y_2},\\ &\int_{\mathbb{R}} K_j(\theta_1-|y_2|,\theta_2-y_2)\chi_{1/2}(|y_2|)\frac{dy_2}{y_2},\end{aligned}\tag{A.3}$$

$$\begin{aligned}&\int_{\frac{1}{2}\le|y_2|\le 3} K_j(\theta_1-|y_2|,\theta_2-y_2)dy_2,\\ &\int_{\frac{1}{2}\le|y_2|\le 3} K_j(\theta_1-|y_2|,\theta_2-y_2)(\theta_2-y_2)dy_2,\end{aligned}\tag{A.4}$$

belong to $(L^\infty\cap W^{1,1})(B_{11/10}\setminus B_{9/10})$. Thanks to

$$\begin{gathered}|\widetilde{f}(y_2)-\widetilde{f}(\theta_2)-\widetilde{f}'(\theta_2)(y_2-\theta_2)|\lesssim|y_2-\theta_2|^2,\\ \widetilde{f}(y_2)=y_2^{-1}(\chi_2(|y_2|)-\chi_{1/2}(|y_2|)),\end{gathered}$$

we find that

$$\int_{\frac{1}{2}\le|y_2|\le 3} K_j(\theta_1-|y_2|,\theta_2-y_2)\Big(\widetilde{f}(y_2)-\widetilde{f}(\theta_2)-\widetilde{f}'(\theta_2)(y_2-\theta_2)\Big)dy_2$$

belongs to $(L^\infty\cap W^{1,1})(B_{\frac{11}{10}}\setminus B_{\frac{9}{10}})$. Combining this and (A.3) and (A.4), we conclude

$$\int_{\mathbb{R}} K_j(\theta_1-|y_2|,\theta_2-y_2)\frac{dy_2}{y_2}\in(L^\infty\cap W^{1,1})(B_{\frac{11}{10}}\setminus B_{\frac{9}{10}}).$$

This implies that $\widetilde{\Omega}_j(\theta)\in(L^\infty\cap W^{1,1})(B_{11/10}\setminus B_{9/10})$. Therefore, (A.2) leads to (A.1) with $l=1$, $i=1$. Similarly, we can do this for $\partial_{x_2}b_1$, $\partial_{x_1}b_2$, and $\partial_{x_2}b_2$. The proof is complete. □

To prove Lemma 3.7, we need the following result:

LEMMA A.1. *Let $e\in\mathbb{S}^{d-1}$. For any $z_1,z_1'\in\widetilde{H}_e$, $y_2,y_2',z_2,z_2'\in H_e$, $\varepsilon>0$, and $\rho>0$, there holds for $z'=z_1'+z_2'$ and $\overline{z}=z_1+z_2$*

$$\begin{aligned}M:=&\int_{\widetilde{H}_e}|f(y_2'+y_1)-f(y_2'+z_1')|\\ &\qquad\times\mathbf{1}_{|\overline{z}-(y_1+y_2)|\le\varepsilon}\left(1\wedge\frac{\rho}{|z'-(y_1+y_2)|}\right)^{d+2}d\mathcal{H}^1(y_1)\\ &\lesssim\frac{\rho^2}{\varepsilon}\int_{\widetilde{H}_e}1\wedge\left(\frac{\rho}{|z'-(z+y_2)|}\right)^{d-\frac{1}{2}}d|Df^e_{y_2'}|(z)\\ &\quad+\rho\int_{\widetilde{H}_e}\mathbf{1}_{|\overline{z}-(z+y_2)|\le 4\varepsilon}\,1\wedge\left(\frac{\rho}{|z'-(z+y_2)|}\right)^{d+\frac{1}{2}}d|Df^e_{y_2'}|(z).\end{aligned}\tag{A.5}$$

PROOF. Since $|f(y_2'+y_1)-f(y_2'+z_1')|\le\int_{\widetilde{H}_e}\mathbf{1}_{|z-z_1'|\le 2|z_1'-y_1|}d|Df^e_{y_2'}|(z)$, so

$$M\le\int_{\widetilde{H}_e}Vd\big|Df^e_{y_2'}\big|(z),$$

where

$$V=\int_{\widetilde{H}_e}\mathbf{1}_{|z-z_1'|\le 2|z_1'-y_1|}\mathbf{1}_{|\bar{z}-(y_1+y_2)|\le\varepsilon}\,1\wedge\left(\frac{\rho}{|z'-(y_1+y_2)|}\right)^{d+2}d\mathcal{H}^1(y_1).$$

Note that if $|z-z_1'|\le 2|z_1'-y_1|$ with $y_1\in H_e$, then

$$|z'-(z+y_2)|\le 4|z'-(y_1+y_2)|,$$
$$|\bar{z}-(z+y_2)|\le|\bar{z}-(y_1+y_2)|+3|z'-(y_1+y_2)|.$$

Thus, we can estimate

$$\begin{aligned}V&=\int_{\widetilde{H}_e}\mathbf{1}_{|z'-(y_1+y_2)|\le\varepsilon}\cdots+\int_{\widetilde{H}_e}\mathbf{1}_{|z'-(y_1+y_2)|>\varepsilon}\cdots\\&\lesssim\mathbf{1}_{|\bar{z}-(z+y_2)|\le 4\varepsilon}\,1\wedge\left(\frac{\rho}{|z'-(z+y_2)|}\right)^{d+\frac12}\int_{\widetilde{H}_e}1\wedge\left(\frac{\rho}{|z_1'-y_1|}\right)^{\frac32}d\mathcal{H}^1(y_1)\\&\quad+\frac{\rho}{\varepsilon}\,1\wedge\left(\frac{\rho}{|z'-(z+y_2)|}\right)^{d-\frac12}\int_{\widetilde{H}_e}1\wedge\left(\frac{\rho}{|z_1'-y_1|}\right)^{\frac32}d\mathcal{H}^1(y_1)\\&\lesssim\rho\mathbf{1}_{|\bar{z}-(z+y_2)|\le 4\varepsilon}\,1\wedge\left(\frac{\rho}{|z'-(z+y_2)|}\right)^{d+\frac12}\\&\quad+\frac{\rho^2}{\varepsilon}\,1\wedge\left(\frac{\rho}{|z'-(z+y_2)|}\right)^{d-\frac12}\end{aligned}$$

which implies (A.5). The proof is complete. □

PROOF OF LEMMA 3.7. We first observe that

$$\begin{aligned}M_0&:=\int_{\widetilde{H}_{e_1}}\cdots\int_{\widetilde{H}_{e_d}}\left(1\wedge\frac{\rho}{|\sum_{i=1}^d(x_i-y_i)|}\right)^{d+2}\mathbf{1}_{|\sum_{i=1}^d(y_{0i}-y_i)|\le\varepsilon}\\&\qquad\times\left|f\left(\sum_{i=1}^d y_i\right)-f(y_1+\sum_{i=2}^d x_i)\right|d\mathcal{H}^1(y_d)\cdots d\mathcal{H}^1(y_1)\\&\le\sum_{k=0}^{d-2}\int_{\widetilde{H}_{e_1}}\cdots\int_{\widetilde{H}_{e_d}}\left(1\wedge\frac{\rho}{|\sum_{i=1}^d(x_i-y_i)|}\right)^{d+2}\mathbf{1}_{|\sum_{i=1}^d(y_{0i}-y_i)|\le\varepsilon}\\&\qquad\times\left|f\left(\sum_{i=1}^{d-k-1}y_i+\sum_{i=d-k+1}^d x_i+y_{d-k}\right)\right.\end{aligned}$$

$$- f\left(\sum_{i=1}^{d-k-1} y_i + \sum_{i=d-k}^{d} x_i\right)\Bigg| d\mathcal{H}^1(y_d d)\cdots\mathcal{H}^1(y_1).$$

Applying Lemma A.1 to $e = e_{d-k}$, $y_2' = \sum_{i=1}^{d-k-1} y_i + \sum_{i=d-k+1}^{d} x_i$, $y_2 = \sum_{i\neq d-k} y_i$, $z_1 = y_{0,d-k}$, $z_2 = \sum_{i\neq d-k} y_{0,i}$, and $z_1' = x_{d-k}$, $z_2' = \sum_{i\neq d-k} x_i$ yields

$$\begin{aligned} M_0 \lesssim{}& \sum_{k=0}^{d-2} \frac{\rho^2}{\varepsilon} \int_{\widetilde{H}_{e_1}} \cdots \int_{\widetilde{H}_{e_d}} 1 \wedge \left(\frac{\rho}{|\sum_{i=0}^{d}(x_i - y_i)|}\right)^{d-\frac{1}{2}} d\mathcal{H}^{d-k}(y_d)\cdots d\mathcal{H}^1(y_{d-k+1}) \\ &\times d|Df^{e_{d-k}}_{\sum_{i=1}^{d-k-1} y_i + \sum_{i=d-k+1}^{d} x_i}|(y_{d-k})d\mathcal{H}^1(y_{d-k-1})\cdots d\mathcal{H}^1(y_1) \\ &+ \sum_{k=0}^{d-2} \rho \int_{\widetilde{H}_{e_1}} \cdots \int_{\widetilde{H}_{e_d}} 1 \wedge \left(\frac{\rho}{|\sum_{i=0}^{d}(x_i - y_i)|}\right)^{d+\frac{1}{2}} \mathbf{1}_{|\sum_{i=1}^{d}(y_{0i}-y_i)|\le\varepsilon} d\mathcal{H}^1(y_d) \\ &\cdots d\mathcal{H}^1(y_{d-k+1}) d|Df^{e_{d-k}}_{\sum_{i=1}^{d-k-1} y_i + \sum_{i=d-k+1}^{d} x_i}|(y_{d-k})d\mathcal{H}^1(y_{d-k-1})\cdots d\mathcal{H}^1(y_1). \end{aligned} \tag{A.6}$$

It is clear to see that for $A = \rho/|\sum_{i=1}^{d-k}(x_i - y_i)|$

$$\int_{\widetilde{H}_{e_{d-k+1}}} \cdots \int_{\widetilde{H}_{e_d}} \left(1 \wedge \frac{\rho}{|\sum_{i=0}^{d}(x_i - y_i)|}\right)^{d-\frac{1}{2}} \lesssim \rho^k\ (1\wedge A)^{d-k-\frac{3}{4}},$$

$$\begin{aligned} &\int_{\widetilde{H}_{e_{d-k+1}}} \cdots \int_{\widetilde{H}_{e_d}} 1 \wedge \left(\frac{\rho}{|\sum_{i=0}^{d}(x_i - y_i)|}\right)^{d+\frac{1}{2}} \mathbf{1}_{|\sum_{i=1}^{d}(y_{0i}-y_i)|\le\varepsilon} \\ &\quad\lesssim \rho^k \mathbf{1}_{|\sum_{i=1}^{d}(y_{0i}-x_i)|\le 2\varepsilon} \mathbf{1}_{|\sum_{i=1}^{d-k}(y_{0i}-y_i)|\le 2\varepsilon} (1\wedge A)^{d-k+\frac{1}{4}} + \frac{\rho^{k+1}}{\varepsilon}\ (1\wedge A)^{d-k-\frac{3}{4}}. \end{aligned}$$

Combining these with (A.6) we find that

$$\begin{aligned} M_0 \lesssim{}& \sum_{k=0}^{d-2} \frac{\rho^{k+2}}{\varepsilon} \int_{\widetilde{H}_{e_1}} \cdots \int_{\widetilde{H}_{e_{d-k}}} \left(1 \wedge \frac{\rho}{|\sum_{i=1}^{d-k}(x_i - y_i)|}\right)^{d-k-\frac{3}{4}} d\nu^1_{k,\sum_{i=d-k+1}^{d} x_i}(y_{d-k},\dots,y_1) \\ &+ \sum_{k=0}^{d-2} \rho^{k+1} \mathbf{1}_{|\sum_{i=1}^{d}(y_{0i}-x_i)|\le 2\varepsilon} \\ &\times \int_{\widetilde{H}_{e_1}} \cdots \int_{\widetilde{H}_{e_{d-k}}} \left(1 \wedge \frac{\rho}{|\sum_{i=1}^{d-k}(x_i - y_i)|}\right)^{d-k+\frac{1}{4}} d\nu^2_{k,\sum_{i=d-k+1}^{d} x_i}(y_{d-k},\dots,y_1). \end{aligned}$$

Hence, using the fact that for any $\omega \in \mathcal{M}^+(\bigotimes_{i=1}^{d-k} \widetilde{H}_{e_i})$, we obtain

$$\begin{aligned} &\int_{\widetilde{H}_{e_1}} \cdots \int_{\widetilde{H}_{e_{d-k}}} 1 \wedge \left(\frac{\rho}{|\sum_{i=1}^{d-k}(x_i - y_i)|}\right)^{d-k+\frac{1}{4}} d\omega(y_{d-k},\dots,y_1) \\ &\quad\lesssim \rho^{d-k}\mathbf{M}\left(\omega, \bigotimes_{i=1}^{d-k} \widetilde{H}_{e_i}\right)\left(\sum_{i=1}^{d-k} x_i\right). \end{aligned}$$

Then, one gets the first inequality of Lemma 3.7. Similarly, we also have the second one. □

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