AN EQUIVALENT PATH FUNCTIONAL FORMULATION OF BRANCHED TRANSPORTATION PROBLEMS

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ABSTRACT. We consider two models for branched transport: the one introduced in Bernot et al. (Publ Mat 49:417–451, 2005), which makes use of a functional defined on measures over the space of Lipschitz paths, and the path functional model presented in Brancolini et al. (J Eur Math Soc 8:415–434, 2006), where one minimizes some suitable action functional defined over the space of measure-valued Lipschitz curves, getting sort of a Riemannian metric on the space of probabilities, favouring atomic measures, with a cost depending on the masses of each of their atoms. We prove that modifying the latter model according to Brasco (Ann Mat Pura Appl 189:95–125, 2010), then the two models turn out to be equivalent.

1. INTRODUCTION

The study of variational models giving rise to branched structures of transportation as optimizers has been the object of an intensive investigation in recent times: we just cite the works of Bernot, Caselles and Morel ([3]), Maddalena, Morel and Solimini ([14]) and Xia ([17]), which are now quite standard references in this field, as leading examples.

The typical problem one has to face in this context is the following: one has some mass μ_0 that has to be transported to a destination μ_1 and wants to find the optimal way to perform this transportation. The main difference with the classical Monge-Kantorovich mass transportation problem (for which the reader can consult [1] or [16]) is that optimality should regard the type of structure used to move the mass: in particular, this transportation should be optimal with respect to some energy which has to take into account the fundamental principle that "*the more you transport mass together, the more efficient the transport is*". This is, for example, exactly what happens in many natural systems: root systems in a tree, bronchial systems and blood vessels in a human body and so on. Each of them solve the problem of transporting some "mass" (water, oxygen, blood or generic fluids) from a source to a destination, avoiding separation of masses as much as possible. This fundamental principle is translated into the energy by considering, for a mass m moving on a distance ℓ, a cost of the form $m^\alpha \ell$, with $\alpha < 1$: since the function $m \mapsto m^\alpha$ is subadditive, i.e. $(m_1 + m_2)^\alpha < m_1^\alpha + m_2^\alpha$, it is convenient to put together different masses so as to pay less. Then the typical resulting structures are trees made of bifurcating vessels.

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In the case of finitely atomic sources and destinations, that is when μ_0 and μ_1 are measures with the same mass (let's say 1, up to a renormalization) of the form

$$\mu_0 = \sum_{k=1}^{m} a_k \delta_{x_k} \quad \text{and} \quad \mu_1 = \sum_{j=1}^{n} b_j \delta_{x_j},$$

Gilbert in the '60s (see [12]) proposed to study the minimization of the energy

$$M_\alpha(\mathfrak{g}) = \sum_h m_h^\alpha \mathcal{H}^1(e_h), \tag{1.1}$$

where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure and $\mathfrak{g} = (e_h, \overrightarrow{e_h}, m_h)$ is a weighted oriented graph, with e_h standing for the edges of the graph, $\overrightarrow{e_h}$ are the orientations of these edges and the weights m_h stand for the transiting mass. Then M_α is minimized over the set of admissible weighted oriented graphs, which is given by those graphs linking μ_0 to μ_1 and satisfying Kirchhoff's Law. Observe that this can be viewed as a generalization of Steiner's problem of finding the network of minimal length connecting a set of given points, the latter corresponding to the choice $\alpha = 0$. This discrete model has been suitably extended to a continuous setting (i.e. when μ_0 and μ_1 are general probability measures) by Xia (see [17]), thanks to a relaxation procedure. This leads to the minimization of the energy

$$M^*_\alpha(\Phi) = \begin{cases} \displaystyle\int_\Sigma m^\alpha(x)\, d\mathcal{H}^1(x), & \text{if } \Phi = m\,\overrightarrow{\tau}\,\mathcal{H}^1 \llcorner \Sigma, \\ +\infty, & \text{otherwise,} \end{cases}$$

over all vector measures Φ with prescribed divergence $\nabla \cdot \Phi = \mu_0 - \mu_1$, which is finite only on those measures concentrated on a $1-$rectifiable set Σ with a vector density $m\overrightarrow{\tau}$ w.r.t. $\mathcal{H}^1$, where $\overrightarrow{\tau}$ is an orientation of Σ: this energy is obviously closely related to the original Gilbert-Steiner energy (1.1). A completely different formulation has been given to this problem by Maddalena, Morel and Solimini (the *irrigation patterns model*, [14], which is confined to the case of a single source $\mu_0 = \delta_{x_0}$) and by Bernot, Caselles and Morel (the *traffic plans model*, [3]), using some tools from fluid mechanics, such as probability measures over the set of paths: we do not discuss here these models (see Section 2), but it is remarkable to point out that in the case of $\mu_0 = \delta_{x_0}$, all the studies performed in [14], [17] and [3] lead to the same optimal structures and they all give different descriptions of the same energy, which is exactly a Gilbert-Steiner one (see Chapter 9 of [5], for these equivalences).

Among others, the paper [8] presents a possible alternative approach to these problems: actually, the model is fairly more general, as it tries to give a unified dynamic formulation of mass transportation problems both with congestion or branching effects, simply perturbing the geodesic formulation of the $p-$Wasserstein distance. The latter is actually a geodesic distance (i.e. the space of probability measures becomes a *length space* under w_p) and hence it satisfies

$$w_p(\mu_0, \mu_1) = \min \left\{ \int_0^1 |\mu'|_{w_p}\, dt \,:\, \mu \in \text{Lip},\ \mu(0) = \mu_0,\ \mu(1) = \mu_1 \right\},\ \mu_0,\, \mu_1 \in \mathcal{W}_p(\Omega).$$

Here $|\mu'|_{w_p}$ is the *metric derivative* of the curve μ with respect to the $p-$Wasserstein distance, which is defined by

$$|\mu'|_{w_p}(t) = \lim_{h \to 0} \frac{w_p(\mu(t+h), \mu_p(t))}{|h|},$$

(see Chapter 4 of [2] for more details).

One can perturb this minimal length formula studying *weighted lengths*. If suitable weight functions which allow only for very concentrated measures or very diffused ones are chosen, one can obtain models for different kind of phenomena to modelize.

For the case to study, the choice of the weight function is given by

$$g_\alpha(\mu)=\begin{cases}\displaystyle\sum_{k\in\mathbb{N}} m_k^\alpha, & \text{if } \mu=\displaystyle\sum_{k\in\mathbb{N}} m_k\delta_{y_k},\\ +\infty, & \text{otherwise,}\end{cases} \tag{1.2}$$

so that the energy under consideration in [8] is the following

$$G_{\alpha,p}(\mu)=\int_0^1 g_\alpha(\mu(t))\,|\mu'|_{w_p}(t)\;dt, \tag{1.3}$$

for every Lipschitz curve with values in $\mathcal{W}_p(\Omega)$. Observe that the term g_α is finite only on atomic measures and reproduces the energy with the masses to the power of α which is used in the other models. Moreover, this model is a purely dynamical one, as far as any optimal curve provides the evolution of the branched transportation and not just the branched structure underlying the movement: also notice that the term $g_\alpha(\mu(t))$ is local both in space and time.

This model, despite its simple description, has not received much attention (except for the recent paper [7] by Bianchini and Brancolini, where the so-called *irrigability conditions* are studied in details), after it has been discovered that it turns out not to be equivalent with the others (in the sense that the optimal structures they describe are not the same). Moreover, it shows some unnatural behaviours from a modelization point of view. These are mainly two and we try to explain them in some details, in order to provide a better understanding of the scope of this work:

(i) *energetic behaviour*: the term g_α is a function of the whole μ, which means that if some masses arrive at their destination and then stop, we continue to pay a cost for them until all the process is over.

Just to clarify, we write down a basic example: suppose you want to transport $\mu_0=\delta_{x_0}$ to $\mu_1=m\delta_{x_1}+(1-m)\delta_{x_2}$, where $|x_0-x_1|=2|x_0-x_2|$. A possible connecting curve could be

$$\mu(t)=\begin{cases} m\delta_{(1-t)\,x_0+tx_1}+(1-m)\delta_{(1-2t)\,x_0+2t\,x_2}, & t\in[0,1/2],\\ \\ m\delta_{(1-t)\,x_0+tx_1}+(1-m)\delta_{x_2}, & t\in[1/2,1],\end{cases}$$

but it is easily seen that for a path like this, the energy (1.3) will let you pay a cost for the mass $(1-m)$ also after it is stopped.

On the contrary, it would be desirable to have an energy which takes into account only the moving mass, which in this case is simply given by

$$\nu(t)=\begin{cases}\mu(t), & t\in[0,1/2],\\ m\delta_{(1-t)x_0+tx_1}, & t\in[1/2,1],\end{cases}$$

the latter being no more a curve of probability measures. This is the reason why, at a first stage, the energy (1.3) has to be modified as follows

$$\widetilde{G}_{\alpha,p}(\nu,\mu)=\int_0^1 g_\alpha(\nu(t))\,|\mu'|_{w_p}(t)\;dt,$$

where now ν is a curve of sub-probability measures, which should represent the moving mass. The curves ν and μ are linked by the condition of being an *evolution pairing*: this means that the moving part ν is always less than the total mass μ and that the mass reaching its final destination, given by the difference $\mu - \nu$, has to grow in time (see Section 3, Definition 1). Obviously, this makes sense when the starting measure $\mu_0 = \delta_{x_0}$ (which is anyway a relevant case, and it was the one studied in [14], as already said), so that at time 0 mass starts to move as a whole: on the contrary, when the starting measure is a generic probability, then one would have to take into account also the possibility that masses could start to move at different times (it is enough to think to the previous example, just exchanging the role of μ_0 and μ_1). In this case, one possibility could be that of defining an evolution pairing as a couple (ν, μ) with the property that $\mu - \nu$ must be the sum of an increasing part (the arrived mass) and a decreasing one (the mass which is still not moving). Anyway, for the sake of brevity, we will not pursue this direction in this paper and our investigation will be strictly confined to the case $\mu_0 = \delta_{x_0}$. We warn the reader from the beginning that this definition of evolution pairing, despite being quite simple and intuitive, hides some subtleties and enlarges too much the class of admissible configurations, as far as it does not take into account any constraint on the *velocity* of the moving part ν. This will be made apparent at the very beginning of Section 4 with an enlightening example (Example 4.1), in which the necessity for a more rigid class of evolution pairings (what we called *special evolution pairings*, see Definition 3) will come into play: anyway, for the ease of exposition, we have chosen to start introducing the general concept and to see how this has to be suitably modified;

(ii) *scaling behaviour*: another problem is the choice of the exponent p, which influences the energy $G_{\alpha,p}(\mu)$ through the term $|\mu'|_{w_p}$. It seems that the right choice should be $p = +\infty$, for two reasons mainly: the first is that if you rescale a curve μ to be a curve with mass m, we get

$$G_{\alpha,p}(m\,\mu) = m^{\alpha+\frac{1}{p}} G_{\alpha,p}(\mu),$$

so that the energy rescales as the power $\alpha + 1/p$, with respect to the mass. Taking $p = +\infty$ clearly settles this behaviour, giving the same scaling as a Gilbert-Steiner energy. The second reason is that the term $|\mu'|_{w_p}$ should play the role of the velocity of the particles, so that it is expected to be *mass-independent*: on the contrary, in the case $p < +\infty$ in general you would have

$$|\mu'|_{w_p} \simeq \left(\sum m\,\ell^p\right)^{\frac{1}{p}},$$

which roughly speaking means that metric velocity is a mass-weighted sum of the velocities of the particles, which strengthen the feeling that $p = +\infty$ should be the right exponent, in order to be able to compare the path functional energy with a Gilbert-Steiner one.

All in all, one is lead to the study of the modified energy given by

$$\widetilde{G}_{\alpha,\infty}(\nu, \mu) = \int_0^1 g_\alpha(\nu(t))\, |\mu'|_{w_\infty}\, dt,$$

but then we have to pay attention to another detail: observe that thanks to the subadditivy of g_α, in the standard path functional model we have

$$g_\alpha(\mu(t)) \geq 1,$$

because g_α is evaluated on probability measures: then the existence of a Lipschitz curve minimizing (1.3) under a constraint on the endpoints, is almost straightforward (see [8], Theorem 2.1), thanks to the fact that every minimizing sequence with bounded energy has equi-bounded lengths.

On the contrary, in the modified path functional energy you only have

$$g_\alpha(\nu(t)) \geq |\nu(t)|(\Omega)^\alpha,$$

and the last quantity can go to zero (the moving mass could decrease until it disappears). This fact completely destroys the coercivity of the energy on the space of Lipschitz curves: this means that it could be the case that the transportation process requires an infinite speed (then breaking the Lipschitz constraint), in order to bring all the mass from x_0 to μ_1 in a finite time, or equivalently, that if you want your curves to stay Lipschitz (i.e. you have an upper bound on the velocities), then you could need an infinite amount of time to complete the transportation. In other words, it may happen that you do not have an upper bound on the length of the paths that particles have to run, because of branching. Curiously enough, this fact is not a drawback, as it is in perfect accordance with the other models, where the existence of an upper bound on the lengths covered by the particles (also for optimal structures) in not known! Indeed, this is still an open problem up to some special cases (see in particular [5], Problem 15.13). We stress the fact that the only case where the answer is known - and it is *yes* - is when the irrigated measure satisfies an *Ahlfors regularity* property, i.e. when its density w.r.t. $\mathcal{H}^s$ is bounded from below for a certain $s \in [0, N]$ (N being the dimension of the ambient space): in this case, this result is just a consequence of the Hölder continuity of the so called *landscape function* proven in [15] for $s = N$ and then considerably extended in [9].

So in the end, one has to relax the requirement on the finiteness of the time interval and to take advantage of the reparametrization invariance of these weighted length functionals: to keep some compactness one can introduce a bound on the velocities (which does not affect the functional, due to reparametrization). It turns out that the kind of energy we are really interested in, as a good candidate to be equivalent to a Gilbert-Steiner energy, is of the form

$$\mathfrak{L}_\alpha(\nu, \mu) := \int_0^\infty g_\alpha(\nu(t))\, |\mu'|_{w_\infty}(t)\, dt, \tag{1.4}$$

defined for all curves μ which are $\mathcal{W}_\infty(\Omega)-$valued and Lipschitz, with a given Lipschitz constant (let us say 1, for example). It is also clear that keeping the velocity term $|\mu'|_{w_\infty}$ will not be crucial, since if one withdraws it, but keeps the bound $|\mu'|_{w_\infty} \le 1$, the only effect will be that of selecting those minimizers which move at maximal speed.

The plan of the paper is as follows: first of all (Section 2), we start recalling the basic facts about the traffic plan model of Bernot, Caselles and Morel and some of its (equivalent) variants. Then in Section 3 we introduce the concept of evolution pairing, its main features and we give an existence result for the minimization of functional (1.4) over the set of evolution pairings with prescribed endpoints. Section 4 is devoted to a deeper insight into evolution pairings, providing properties and examples which lead us to isolate a good subset (the aforementioned *special evolution pairings*) for which a complete characterization (Section 5) can be given, in terms of the Lipschitz curves of the base space. This characterization is one of the corner-stones of the paper, which finally permits us to compare, in Section 6, our energy with a Gilbert-Steiner or Bernot-Caselles-Morel one and to show equivalence between our modified path functional model and the other models, in the irrigation case.

2. A quick overview over traffic plans

Let $\Omega \subset \mathbb{R}^N$ be a compact convex set and let us indicate $I = [0, \infty)$. Moreover, as far as we are interested in studying the branched transport problem with a single Dirac mass as starting measure, in the sequel we will always refer to this configuration and in particular with μ_0 we will indicate a Dirac mass centered in some point of Ω, that is we set $\mu_0 = \delta_{x_0}$, for some $x_0 \in \Omega$.

We consider the space $\mathrm{Lip}_1(I; \Omega)$ of all $1-$Lipschitz curves over Ω, equipped with the topology of the uniform convergence on compact sets, and we call *traffic plan* every probability measure on this space. We denote with $T(\sigma)$ the *stopping time* of a curve σ, defined as

$$T(\sigma) = \inf\{t \in [0,\infty) \ : \ \sigma \text{ is constant on } [t,\infty)\},$$

and we recall that $T : \mathrm{Lip}_1(I;\Omega) \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function (see [3], Lemma 4.2). We then define the set of traffic plans with prescribed initial and final measures

$$TP(\mu_0,\mu_1) = \{Q \in \mathscr{P}(\mathrm{Lip}_1(I;\Omega)) \,:\, Q(\{T=+\infty\}) = 0,\ (e_0)_\# Q = \mu_0,\ (e_\infty)_\# Q = \mu_1\},$$

where for every $t \in I$, the function e_t is defined as

$$\begin{array}{rccc} e_t: & \mathrm{Lip}_1(I;\Omega) & \to & \Omega \\ & \sigma & \mapsto & \sigma(t); \end{array}$$

and the application e_∞ is defined on the set $\{\sigma \,:\, T(\sigma) < +\infty\}$ through $e_\infty(\sigma) = \sigma(T(\sigma))$. As the reader may easily see, e_∞ will be always applied when the space $\mathrm{Lip}_1(I;\Omega)$ is endowed with a measure Q such that $Q(\{T = +\infty\}) = 0$. The fact that the space of these measures is not closed should not worry, since the functional we will consider has coercivity properties so as to provide suitable compactness.

Given a traffic plan $Q \in \mathscr{P}(\mathrm{Lip}_1(I;\Omega))$, the *multiplicity* of Q at a point $x \in \Omega$ and time instant $t \in I$ is the quantity

$$|(x,t)|_Q = Q(\{\sigma \in \mathrm{Lip}_1(I;\Omega) \ : \ x = \sigma(t)\}),$$

which represents the quantity of mass passing from a point x at time t. Then, for any $\alpha \in (0,1)$ we define the α-energy of a traffic plan as

$$E_\alpha(Q) = \int_{\mathrm{Lip}_1(I;\Omega)} \int_0^{T(\sigma)} |(\sigma(t),t)|_Q^{\alpha-1}\ dt\, dQ(\sigma).$$

In order to avoid possible confusions, something has to be precised on the energy E_α. The reader has maybe noticed that the same energy is sometimes defined integrating $|(\sigma(t),t)|_Q^{\alpha-1}|\sigma'(t)|$, thus getting a functional which is invariant under time reparametrization. Yet if one withdraws the derivative term but only considers curves which are $1-$Lipschitz, then the minimization will give the same result but selecting one precise minimizer, the one which moves at maximal speed.

The definition of multiplicity also needs some clarifications. The one that we have chosen here is that of the so-called *synchronized traffic plan model* of Bernot and Figalli (see [6]). Other definitions are possible: the first one which had been introduced, which is only suitable for the case where μ_0 is a Dirac mass, is the multiplicity of the irrigation pattern model of Maddalena, Morel and Solimini, defined as (in the language of traffic plans)

$$[\sigma]_{t,Q} = Q(\{\eta \in \mathrm{Lip}_1(I;\Omega) \,:\, \eta(s) = \sigma(s), \forall s \in [0,t]\}).$$

The most general and widespread one, on the other hand, is the one chosen by Bernot, Caselles and Morel in [3], which reads

$$|x|_Q = Q(\{\sigma \in \mathrm{Lip}_1(I;\Omega) \,:\, x \in \sigma(I)\}).$$

Actually, the three multiplicities coincide on "single path" traffic plans (i.e. measures Q avoiding properly defined cycles) and in particular on optimal traffic plans and the optimization problem for E_α does not change if one changes the definition of the multiplicity. For the sake of our result, the synchronized one is the most practical and this justifies our choice. Anyway, we need to stress the following clarifying result (see for instance [15], Section 2.3):

Lemma 2.1. *If Q is an optimal traffic plan minimizing E_α on $TP(\mu_0,\mu_1)$, then Q is concentrated on curves σ which are parametrized by arc length (i.e. $|\sigma'(t)| = 1$ a.e. on $[0,T(\sigma)]$) and such that, for all times $t < T(\sigma)$, the equality $[\sigma]_{t,Q} = |(\sigma(t),t)|_Q$ holds.*

Remark 2.2. We recall that by means of the one-dimensional area formula (see [2], Theorem 3.4.2, for this formula, and [4], Proposition 4.8, for Equation (2.1) below), it is easy to see that E_α is strictly related to a Gilbert-Steiner energy: indeed,

$$\int_{\mathrm{Lip}_1(I;\Omega)} \int_0^{T(\sigma)} |\sigma(t)|_Q^{\alpha-1} |\sigma'(t)|_Q \; dt dQ(\sigma) \geq \int_\Omega |x|^\alpha \; d\mathcal{H}^1(x), \tag{2.1}$$

and we have equality as soon as Q in concentrated on injective curves.

3. A path functional model

We recall here the basic concepts about the space $\mathcal{W}_\infty(\Omega)$, i.e. the set of all Borel probability measures over Ω, endowed with the ∞-Wasserstein distance defined by

$$w_\infty(\mu_1, \mu_2) = \min_{\gamma \in \Pi(\mu_1,\mu_2)} \sup \big\{ |x-y| \; : \; (x,y) \in \mathrm{spt}(\gamma) \big\},$$

where $\Pi(\mu_1, \mu_2)$ is the set of all probability measures over the product space $\Omega \times \Omega$, having fixed marginals μ_1 and μ_2. The space $\mathcal{W}_\infty(\Omega)$ is a Polish (i.e. complete and separable) metric space, which is nor compact nor locally compact.

We also consider $\mathcal{M}_1^+(\Omega)$ the space of all positive Radon measures over Ω, with mass smaller than or equal to 1, metrized according to a distance inducing the $*-$weak topology, for instance

$$\mathfrak{d}(\nu_1, \nu_2) = \sum_{k \in \mathbb{N}} \frac{1}{2^k \alpha_k} \left| \int_\Omega \varphi_k(x) \; d(\nu_1(x) - \nu_2(x)) \right|, \; \nu_1, \nu_2 \in \mathcal{M}_1^+(\Omega),$$

where every function φ_k is α_k-Lipschitz and the sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ is dense in

$$\{\varphi \in C(\Omega) \; : \; \varphi \geq 0, \; ||\varphi||_{L^\infty(\Omega)} \leq 1\},$$

Let us then define the space $\mathrm{Lip}_{1,\mathfrak{d}}(I; \mathcal{W}_\infty(\Omega))$ of all 1-Lipschitz curves in the ∞-Wasserstein space $\mathcal{W}_\infty(\Omega)$, equipped with the $\mathfrak{d}-$*uniform convergence* on compact subsets, i.e., indicating with the symbol $\overset{u\mathfrak{d}}{\to}$ this convergence, we have

$$\mu_n \overset{u\mathfrak{d}}{\to} \mu \Longleftrightarrow \max_{t \in [0,k]} \mathfrak{d}(\mu_n(t), \mu(t)) \to 0, \quad \text{for every } k \in \mathbb{N}.$$

Remark 3.1. We remark that the use of this convergence is due to the lack of any kind of compactness of the space $\mathcal{W}_\infty(\Omega)$. Moreover, we recall that the topology induced by w_∞ is strictly stronger than the $*-$weak topology and we have $\mathfrak{d} \leq w_\infty$. What is worthwhile to point out here and crucial for our discussion is that w_∞ is lower semicontinuous with respect to $\mathfrak{d}$, that is

$$\left.\begin{array}{l} \mathfrak{d}(\mu_1^n, \mu_1) \to 0 \\ \\ \mathfrak{d}(\mu_2^n, \mu_2) \to 0 \end{array}\right\} \Longrightarrow w_\infty(\mu_1, \mu_2) \leq \liminf_{n\to\infty} w_\infty(\mu_n^1, \mu_n^2).$$

We also define

$$L^0(I; \mathcal{M}_1^+(\Omega)) := \{\nu : I \to \mathcal{M}_1^+(\Omega) \; : \; \nu \text{ is Borel measurable}\},$$

and on this space we will always consider the pointwise $\mathscr{L}^1-$a.e. convergence. Then in the sequel, when referring to the convergence on the product space $L^0(I; \mathcal{M}_1^+(\Omega)) \times \mathrm{Lip}_{1,\mathfrak{d}}(I; \mathcal{W}_\infty(\Omega))$, we will always mean pointwise $\mathscr{L}^1-$a.e. convergence in the first variable and $\mathfrak{d}-$uniform in the second.

We want to consider the following energy defined on $L^0(I;\mathcal{M}_1^+(\Omega))\times \mathrm{Lip}_{1,\mathfrak{d}}(I;\mathcal{W}_\infty(\Omega))$

$$\mathfrak{L}_\alpha(\nu,\mu)=\int_0^\infty g_\alpha(\nu(t))|\mu'|_{w_\infty}(t)\,dt, \tag{3.1}$$

where $g_\alpha:\mathcal{M}_1^+(\Omega)\to\mathbb{R}\cup\{+\infty\}$ is the lower semicontinuous function defined in (1.2). For the rest of the paper, the exponent α will always be considered as belonging to the open interval $(0,1)$.

Lemma 3.2. *The functional* $\mathfrak{L}_\alpha$ *defined by* (3.1) *is lower semicontinuous on the product space* $L^0(I;\mathcal{M}_1^+(\Omega))\times \mathrm{Lip}_{1,\mathfrak{d}}(I;\mathcal{W}_\infty(\Omega))$.

Proof. The functional under consideration can be written as

$$\mathfrak{L}_\alpha(\nu,\mu)=\sup_{k\in\mathbb{N}}\mathfrak{L}_\alpha^k(\nu,\mu):=\sup_{k\in\mathbb{N}}\int_0^k g_\alpha(\nu(t))|\mu'|_{w_\infty}(t)\,dt,$$

and thanks to the semicontinuity of g_α and of w_∞ with respect to $\mathfrak{d}$, we get that each $\mathfrak{L}_\alpha^k$ is lower semicontinuous with respect to the desired convergence, by means of Theorem 7 of [10]. It is only left to observe that the supremum of a sequence of lower semicontinuous functions is still a lower semicontinuous function. □

We now need to formalize the idea that the curves ν that we want actually to consider, should represent the moving mass. In order to do this, we recall the concept of *evolution pairing*, introduced in [10].

Definition 1. Let $(\nu,\mu)\in L^0(I;\mathcal{M}_1^+(\Omega))\times \mathrm{Lip}_{1,\mathfrak{d}}(I;\mathcal{W}_\infty(\Omega))$ be two curves of measures, such that the following are satisfied:

- (E1) $\nu(t)\le\mu(t)$, for every $t\in I$;
- (E2) $\rho(t):=\mu(t)-\nu(t)$ is monotone non-decreasing and $\mathfrak{d}-$left continuous, that is:

$$\rho(s)\le\rho(t),\quad \text{for every } s,t\in I, \text{ with } s<t \text{ and } \lim_{s\nearrow t}\mathfrak{d}(\rho(s),\rho(t))=0;$$

Then we say that (ν,μ) is an *evolution pairing* and we write $\nu\preceq\mu$.

Notice that, being ρ non-decreasing, the condition of left continuity is non-crucial, since one can always modify $\rho(t)$ for t in a negligible set of times and get a left-continuous curve. It is mainly imposed to give a precise and unambiguous pointwise meaning to $\rho(t)$ for every t, and also to get more easily some of our proofs. Moreover, as far as μ is Lipschitz and ρ is left-continuous, we get that, up to modify also ν for a $\mathscr{L}^1-$negligible set of times, ν can be thought as being left-continuous and defined pointwisely.

Remark 3.3. Observe that property (E2) above implies that the quantity

$$t\mapsto|\nu(t)|(\Omega),$$

is non-increasing, while it does not imply that ν has a monotone decreasing (in the sense of measures) behaviour.

Given two Borel probability measures μ_0 and μ_1 over Ω, we define the set of admissible evolution pairings

$$EP(\mu_0,\mu_1)=\{\nu\preceq\mu\ :\ \mu(0)=\mu_0,\ \mu(\infty)=\mu_1\},$$

where the condition $\mu(\infty)=\mu_1$ has to be intended in the sense $\lim_{t\to+\infty}\mathfrak{d}(\mu(t),\mu_1)=0$, or equivalently, $\mu(t)\rightharpoonup\mu_1$ as t goes to $+\infty$.

Definition 2. An evolution pairing $(\nu, \mu) \in EP(\mu_0, \mu_1)$ is said to be *normal* if the following conditions hold:

(i) $|\mu'|_{w_\infty}(t) = 1$, for a.e. $t \in [0, T(\mu)]$;
(ii) $\nu(t) = 0$, for $t \in (T(\mu), +\infty)$, where if $T(\mu) = +\infty$ this condition must intended in the strong sense that $\lim_{t\to\infty} |\nu(t)|(\Omega) = 0$.

In the sequel, with the term *cutting at time* T, we will simply mean the operation that to every ν assigns the product $\nu \cdot 1_{[0,T]}$ of ν for the characteristic function of some time interval $[0, T]$.

We have the following basic result:

Lemma 3.4. *Every* $(\nu, \mu) \in EP(\mu_0, \mu_1)$ *with* $\mathfrak{L}_\alpha(\nu, \mu) < +\infty$ *is normal, up to a reparametrization of* μ *and a cutting of* ν *at the stopping time of* μ.

Proof. Let us take an evolution pairing $(\nu, \mu) \in EP(\mu_0, \mu_1)$ and reparametrize the $1-$Lipschitz curve μ by arc-length, that is we take $\varphi : [0, +\infty) \to [0, +\infty)$ given by

$$\mathfrak{t}(s) = \inf \left\{ \tau \in I \ : \ s = \int_0^\tau |\mu'|_{w_\infty}(\varrho)\, d\varrho \right\}, \ s \in I, \tag{3.2}$$

and we set $\tilde{\mu} = \mu \circ \varphi$, then this is a reparametrization of μ (see [2], Theorem 4.2.1) and

$$|\tilde{\mu}'|_{w_\infty}(t) = 1, \ t \in I.$$

Moreover, setting $\tilde{\nu} = \nu \circ \varphi$, we clearly get that $(\tilde{\nu}, \tilde{\mu})$ is still an evolution pairing contained in $EP(\mu_0, \mu_1)$, for which

$$\mathfrak{L}_\alpha(\tilde{\nu}, \tilde{\mu}) = \int_0^\infty g_\alpha(\tilde{\nu}(t))\ dt = \int_0^\infty g_\alpha(\nu(t))\, |\mu'|_{w_\infty}(t)\ dt < +\infty.$$

Using the subadditivity of g_α, the previous in turn implies that the integral

$$\int_0^\infty (|\tilde{\nu}(t)|(\Omega))^\alpha\ dt,$$

must be finite: as far as we are integrating a positive non-increasing function over $[0, \infty)$, we obtain that the integrand must tend to 0, as t tends to ∞.

If $T(\tilde{\mu}) = +\infty$ we have already obtained a normal evolution pairing, otherwise it is sufficient to cut $\tilde{\nu}$ at the time $t = T(\tilde{\mu})$. □

The following lemma is useful for proving the closedness of the set $EP(\mu_0, \mu_1)$ of evolution pairings joining two given measures, but we will state it in the case where the second measure is not fixed, so as to use it later on in its generality.

Lemma 3.5. *Let* $\{(\nu^n, \mu^n)\} \subset EP(\mu_0, \mu_1^n)$ *be a sequence of normal evolution pairings such that* $(\nu^n, \mu^n) \to (\nu, \mu)$ *in* $L^0(I; \mathcal{M}_1^+(\Omega)) \times \mathrm{Lip}_{1,\mathfrak{d}}(I; \mathcal{W}_\infty(\Omega))$*. Suppose moreover that* $\mu_1^n \rightharpoonup \mu_1$ *and that*

$$\sup_{n\in\mathbb{N}} \mathfrak{L}_\alpha(\nu^n, \mu^n) < +\infty.$$

Then, up to changing the representant of ν *on a negligible set of times* $t \in I$*,* $(\nu, \mu) \in EP(\mu_0, \mu_1)$*.*

Proof. We first show that (ν, μ) is an evolution pairing and that $\mu \in \mathrm{Lip}_{1,\mathfrak{d}}(I; \mathcal{W}_\infty(\Omega))$: this can be done as in Lemma 13 of [10], since (E1) and (E2) easily pass to limit. Moreover, observe that if $\{\nu^n\}_{n\in\mathbb{N}}$ converges to ν $\mathscr{L}^1-$a.e., the same is true for ρ^n to $\rho := \mu - \nu$. In particular, the nondecreasing behaviour of ρ^n easily passes to the limit, up to the negligible set of non-convergence.

Up to replacing ρ with its left-continuous representant (which means that we only change $\rho(t)$ on a negligible set of times), we get a function which is both monotone and left-continuous.

It remains to show that (ν, μ) still verifies the conditions on the endpoints: the fact that $\mu(0) = \mu_0$ is trivial, so that the only thing to verify is the condition on the final point, that is $\mu(\infty) = \mu_1$ in the sense precised before.

In the case that

$$\sup_{n\in\mathbb{N}} T(\mu^n) = T < +\infty, \tag{3.3}$$

then we have also $T(\mu) \leq T$, using the lower semicontinuity of T. It is now sufficient to use the uniform converge of $\{\mu^n\}_{n\in\mathbb{N}}$ on the interval $[0, T]$ to obtain that

$$\mu(T) = \mu_1,$$

which proves the thesis, under the additional hypothesis (3.3), by means of the fact that $T(\mu) \leq T$.

We now remove assumption (3.3), exploiting the concept of evolution pairing. First observe that using property (E2) we have that

$$\rho^n(t) \leq \rho^n(s) \leq \mu^n(s), \quad \text{for every } t, s \in I, \text{ with } t < s,$$

and using the fact that $\mu_n(\infty) = \mu_1^n$ we obtain

$$\rho_n(t) \leq \mu_1^n, \tag{3.4}$$

and, at the limit as $n \to \infty$, we easily deduce from (3.4) that we have $\rho(t) \leq \mu_1$. Moreover, the curve ρ is non-decreasing and

$$|\rho(t)|(\Omega) = 1 - |\nu(t)|(\Omega), \quad \text{for } t \in I,$$

so that, if we are able to prove that $|\nu(t)|(\Omega) \to 0$ as $t \to \infty$, we can conclude

$$\lim_{t\to\infty} |\rho(t) - \mu_1|(\Omega) = 0, \quad \text{and hence} \quad \mu(\infty) = \rho(\infty) + \nu(\infty) = \mu_1,$$

giving the thesis. At this end we observe that

$$\int_0^\infty g_\alpha(\nu(t))\, dt \leq \liminf_{n\to\infty} \int_0^\infty g_\alpha(\nu^n(t))\, dt = \liminf_{n\to\infty} \mathfrak{L}_\alpha(\nu^n, \mu^n) < +\infty,$$

where the first inequality is just a consequence of Fatou Lemma, while the equality right after is a consequence of the normality of each (ν^n, μ^n), so that $\int_0^\infty g_\alpha(\nu^n(t))\, dt = \int_0^\infty g_\alpha(\nu^n(t))|\mu'|_{w_\infty}(t)\, dt = \mathfrak{L}_\alpha(\nu^n, \mu^n)$. Using again $g_\alpha(\nu(t)) \geq |\nu(t)|(\Omega)^\alpha$ and the monotone behaviour of $|\nu(t)|(\Omega)$ as in Lemma 3.4, the latter implies that

$$\lim_{t\to\infty} |\nu(t)|(\Omega) = 0,$$

which concludes the proof. □

We are now ready to state and prove a result, about the existence of a minimal evolution pairing connecting two given measures.

Proposition 3.6. *The minimization problem*

$$\inf_{(\nu,\mu)\in EP(\mu_0,\mu_1)} \mathfrak{L}_\alpha(\nu, \mu), \tag{3.5}$$

admits a solution, provided that there exists an admissible evolution pairing $(\overline{\nu}, \overline{\mu})$ having finite $\mathfrak{L}_\alpha$.

Proof. Let $\mathfrak{L}_\alpha(\overline{\nu},\overline{\mu}) = L$ and let us take a minimizing sequence $\{(\nu^n,\mu^n)\}_{n\in\mathbb{N}} \subset EP(\mu_0,\mu_1)$, we can assume that

$$\sup_{n\in\mathbb{N}} \mathfrak{L}_\alpha(\nu^n,\mu^n) \le L+1.$$

Observe that thanks to Lemma 3.4, we can think of every (ν^n,μ^n) as being normal. It is straightforward to see that (up to a subsequence) this minimizing sequence converges in $L^0(I;\mathcal{M}_1^+(\Omega)) \times \mathrm{Lip}_{1,\mathfrak{d}}(I;\mathcal{W}_\infty(\Omega))$ to an evolution pairing (ν,μ): the convergence of $\{\mu^n\}_{n\in\mathbb{N}}$ is just a consequence of the compactness of the space $\mathrm{Lip}_{1,\mathfrak{d}}(I;\mathcal{W}_\infty(\Omega))$, while the convergence of $\{\nu^n\}_{n\in\mathbb{N}}$ follows with a slight modification of the argument in [10], Theorem 12.

Moreover (ν,μ) is still admissible, thanks to Lemma 3.5, and the thesis follows straightforwardly using the semicontinuity of $\mathfrak{L}_\alpha$ (Lemma 3.2). □

4. Further properties of evolution pairings

We start this section with a counter-example, which shows that the class $EP(\mu_0,\mu_1)$ is not the right one in which problem (3.5) has to be posed, in order to obtain equivalence with Xia, Bernot-Caselles-Morel and Maddalena-Morel-Solimini models.

Example 4.1. Let $\mu_0 = \delta_0$ and $\mu_1 = \mathscr{L}^1 \llcorner [-1/2,1/2]$, we define an evolution pairing (ν,μ) as follows:

$$\mu(t) = \begin{cases} \mathscr{L}^1\llcorner[-t,t] + (1-2t)\delta_t, & t\in[0,1/2],\\ \mathscr{L}^1\llcorner\left[-\frac{1}{2},\frac{1}{2}\right], & t\in(1/2,+\infty) \end{cases}$$

$$\nu(t) = \begin{cases} (1-2t)\delta_t, & t\in[0,1/2],\\ 0, & t\in(1/2,+\infty). \end{cases}$$

Observe that (ν,μ) is normal and it connects μ_0 to μ_1. Computing its energy, we have that

$$\mathfrak{L}_\alpha(\nu,\mu) = \int_0^{\frac{1}{2}} g_\alpha(\nu(t))|\mu'|_{w_\infty}(t)\,dt = \int_0^{\frac{1}{2}} (1-2t)^\alpha\,dt = \frac{1}{2(\alpha+1)},$$

while the minimal E_α energy is given by

$$2\int_0^{\frac{1}{2}} \left(\frac{1}{2}-t\right)^\alpha dt = \frac{1}{2^\alpha(\alpha+1)},$$

which is strictly greater than the previous one. We observe that the latter is realized by the traffic plan given by the image measure

$$Q = (\Psi)_\#\mu_1,$$

of μ_1 through the application Ψ that sends every $x\in[-1/2,1/2]$ to the $1-$Lipschitz curve Ψ_x defined by (if $x\ge 0$)

$$\Psi_x(t) = \begin{cases} t, & t\in[0,x],\\ x, & t\in(x,\infty), \end{cases}$$

and by (if $x<0$)

$$\Psi_x(t) = \begin{cases} -t, & t\in[0,-x],\\ x, & t\in(-x,\infty). \end{cases}$$

Observe that the movement induced by Q is the following: the mass starts to move from the center of the segment, instantaneously splitting in two branches, one going on the right, the other going on the left and continuously disseminating particles on the segment, in an uniform way. This is better visualized by looking at the corresponding evolution pairing, given by

$$\widetilde{\mu}(t) = (e_t)_\# Q = \begin{cases} \mathscr{L}^1 \llcorner [-t, t] + \frac{1-2t}{2}\delta_{-t} + \frac{1-2t}{2}\delta_t, & t \in [0, 1/2], \\ \mathscr{L}^1 \llcorner \left[-\frac{1}{2}, \frac{1}{2}\right], & t \in (1/2, +\infty), \end{cases}$$

$$\widetilde{\nu}(t) = \begin{cases} \frac{1-2t}{2}\delta_{-t} + \frac{1-2t}{2}\delta_t, & t \in [0, 1/2], \\ 0, & t \in (1/2, +\infty), \end{cases}$$

for which $E_\alpha(Q) = \mathfrak{L}_\alpha(\widetilde{\nu}, \widetilde{\mu}) > \mathfrak{L}_\alpha(\nu, \mu)$.

The previous example tells us that in general the elements of $EP(\mu_0, \mu_1)$ (even the minimizers of $\mathfrak{L}_\alpha$, actually) can have strange properties, which has little to do with real physical phenomena of branched transportation: in fact, in Example 4.1 what seems to go wrong is the fact that ν, which is supposed to represent the moving mass, operates a sort of *teleport* from an endpoint of the segment $[-t, t]$ to the opposite.

Then we have to restrict the class of admissible evolution pairings, isolating those with some good *traveling properties*. In order to do this, we start investigating a property which holds true for a curve having a fixed atomic part. This is a sort of Lipschitz-invariance under mass subtraction, which tells us that once some mass is stopped, then this is no more involved in the transportation process.

Lemma 4.2. *Let* $\mu \in \mathrm{Lip}(I; \mathcal{W}_\infty(\Omega))$ *be given and suppose that there exists an atomic measure* $\rho_0 = \sum_{i=1}^\infty m_i \delta_{x_i}$ *and* $t_0 \in I$ *such that*

$$\rho_0 \leq \mu(t), \quad \textit{for every } t \in [t_0, \infty).$$

Then the curve $[t_0, \infty) \ni t \mapsto \mu(t) - \rho_0$ *has the same metric derivative of the curve* μ *and hence satisfies the following Lipschitz estimate*

$$w_\infty(\mu(t) - \rho_0, \mu(t+h) - \rho_0) \leq \int_t^{t+h} |\mu'|_{w_\infty}(s)\, ds, \quad \textit{for every } h \geq 0. \tag{4.1}$$

Proof. The proof may be achieved if one thinks at the characterization of absolutely continuous curves in Wasserstein spaces in terms of solutions of the *continuity equation*. It is well known (see for instance [1], Theorem 8.3.1) that if μ is a Lipschitz curve defined on a time interval $[0, T]$ and valued in the space $\mathcal{W}_p(\Omega)$, then there exists a Borel vector field $v : (x, t) \mapsto v_t(x)$ such that

$$v_t \in L^p(\mu(t)), \quad \|v_t\|_{L^p(\mu(t))} = |\mu'|_{w_p}(t), \quad \text{for } \mathscr{L}^1\text{-a.e. } t \in I, \tag{4.2}$$

and such that the continuity equation holds

$$\frac{\partial \mu}{\partial t} + \nabla \cdot (v_t \mu(t)) = 0. \tag{4.3}$$

This result is true for any $p \in (1, \infty]$ (the case $p = \infty$ is less studied, but one can easily get it as a limit as $p \to \infty$).

Another important point is the so called *superposition principle* or *probabilistic representation* (see [1], Theorem 8.2.1), which says that any absolutely continuous curve $t \mapsto \mu(t)$ solving (4.3) may be obtained as $(e_t)_\# Q$, for a probability measure Q on the space of absolutely continuous curves which is concentrated on the solutions of the equation $\sigma'(t) = v_t(\sigma(t))$, in the sense that

$$\int_C \left| \sigma(t) - \sigma(0) - \int_0^t v_s(\sigma(s))\, ds \right| dQ(\sigma) = 0, \quad \text{for every } t.$$

For this representation to hold, some integrability conditions on v is needed, but (4.2) is widely sufficient.

In our case, since μ is Lipschitz in the $w_\infty-$distance, one knows the existence of a vector field v such that for almost any t the inequality $|v_t(x)| \leq 1$ is satisfied (actually, it would be satisfied $\mu(t)-$a.e. but one can choose a representant which is everywhere smaller than 1). This implies that the solutions σ of the ODE are Lipschitz continuous curves: as a consequence, they are regular enough to say that, thanks to the one-dimensional area formula, if S is a countable set, then σ' vanishes almost everywhere on the set $\sigma^{-1}(S)$. This implies

$$Q \otimes \mathscr{L}^1\big(\{(\sigma,t)\ :\ \sigma(t) \in S,\, \sigma'(t) \text{ exists and } \sigma'(t) \neq 0\}\big) = 0.$$

and since the curves are solutions of $\sigma'(t) = v_t(\sigma(t))$, using the fact that $\mu(t) = (e_t)_\# Q$, this means

$$\int_0^T \mu_t(S \cap \{v_t \neq 0\})dt = Q \otimes \mathscr{L}^1\big(\{(\sigma,t)\ :\ \sigma(t) \in S,\, v_t(\sigma(t)) \neq 0\}\big) = 0.$$

If one chooses as S the set of atoms of ρ_0, the previous implies that ρ_0-a.e. we have $v_t = 0$, at least for almost any time. Now observe that the continuity equation may obviously be rewritten as

$$\frac{\partial(\mu - \rho_0)}{\partial t} + \nabla \cdot (v_t(\mu(t) - \rho_0)) + \nabla \cdot (v_t \rho_0) = 0.$$

and the last term vanishes as a consequence of $v_t\rho_0 = 0$: hence one gets that $\mu - \rho_0$ is a solution of the continuity equation with the same velocity field v_t. In particular, since $|v_t| \leq |\mu'|_{w_\infty}(t)$, one gets that $\mu - \rho_0$ is Lipschitz according to the $w_\infty-$distance (the latter being easily adapted to the framework of measures with the same mass, instead of probability measures) and its metric derivative with respect to the distance w_∞ does not exceed that of μ. Since it is a straightforward fact to see that there holds

$$w_\infty(\mu(t) - \rho_0, \mu(t+h) - \rho_0) \geq w_\infty(\mu(t), \mu(t+h)),$$

one can also see the opposite inequality and conclude

$$|(\mu - \rho_0)'|_{w_\infty}(t) = |\mu'|_{w_\infty}(t), \quad \text{for } \mathscr{L}^1\text{−a. e. } t \in I,$$

which gives the thesis. □

Remark 4.3. It is not difficult to see that the same conclusions of the previous Lemma hold, if we take ρ_0 to be a positive Borel measure concentrated on some $\mathcal{H}^1-$negligible Borel set S.

The properties proved in the previous Lemma roughly says that, in certain cases, the speed of a curve μ coincides with the speed of its moving part. This seems a general fact, but we will check that the evolution pairing in Example 4.1 is far from satisfying this property. We want hence to introduce a new class of evolution pairings. Thanks to the reparametrization invariance of the functional $\mathfrak{L}_\alpha$, we are more interested in the case where the evolution pairings are normal, i.e. when $|\mu'|_{w_\infty} = 1$ and we give the following definition.

Definition 3. Let $(\nu,\mu) \in EP(\mu_0,\mu_1)$ be an evolution pairing. If there holds

$$w_\infty(\mu(t) - \rho(t), \mu(t+h) - \rho(t)) \leq h, \text{ for } t \in I,\ h > 0, \tag{4.4}$$

then we say that (ν,μ) is a *special evolution pairing* and we denote by $SEP(\mu_0,\mu_1)$ the set of all special evolution pairings contained in $EP(\mu_0,\mu_1)$.

The intuitive idea behind evolution pairings is that, in going from $\mu(t)$ to $\mu(t+h)$, the mass which is moving is (or should be) essentially that given by $\nu(t)$, which distributes over the difference between the total mass at the time $t+h$ and the mass which was already arrived at time t: property (4.4) expresses exactly the requirement that it is this mass that must move at most with unitary speed. Roughly speaking, this means that we can think of the quantity

$$\lim_{h\to 0^+}\frac{w_\infty(\mu(t)-\rho(t),\mu(t+h)-\rho(t))}{h},\ t\in I,$$

as a kind of velocity of the moving mass ν: the elements of $SEP(\mu_0,\mu_1)$ are exactly those for which ν is $1-$Lipschitz, in this sense.

What is remarkable is that, thanks to Lemma 4.2, we can assure that the class $SEP(\mu_0,\mu_1)$ is large enough to contain all the finitely atomic curves. Even more, it is sufficient that μ_1 is atomic to ensure it. Indeed, we have the following:

Proposition 4.4. *Let us take $(\nu,\mu)\in EP(\mu_0,\mu_1)$, with μ_1 atomic. Then $(\nu,\mu)\in SEP(\mu_0,\mu_1)$.*

Proof. It is enough to apply, for any given t, the estimate (4.1) with $\rho_0=\rho(t)$. Since $\rho(t)\le\mu_1$ and μ_1 is atomic, the same is true for ρ_0 and hence we get

$$w_\infty(\mu(t)-\rho(t),\mu(t+h)-\rho(t))\le\int_t^{t+h}|\mu'|_{w_\infty}(s)\,ds\le h,\ \text{ for every } h\ge 0.\qquad\square$$

Example 4.5. Let us go back to the evolution pairing (ν,μ) given by Example 4.1. It is easily seen that this is not an element of $SEP(\mu_0,\mu_1)$: indeed, taking $t<1/2$ we have for every $0<h<1/2-t$

$$\mu(t+h)-\rho(t)=\mathscr{L}^1\llcorner[-t+h,-t]+\mathscr{L}^1\llcorner[t,t+h]+(1-2t-2h)\delta_{t+h},$$

so it is not difficult to see that $\Pi(\mu(t)-\rho(t),\mu(t+h)-\rho(t))$ contains only one element and

$$\frac{w_\infty(\mu(t)-\rho(t),\mu(t+h)-\rho(t))}{h}=\frac{t2t+h}{h},$$

which goes to $+\infty$ as h approaches to 0, while $|\mu'|_{w_\infty}\equiv 1$.

If we see the previous example from the point of view of the continuity equation, we may observe that the problem is that the velocity field v_t associated to this curve does not vanish on the part of $\mu(t)$ which is supposed to be at rest, i.e. on $\rho(t)$. This is what allows for the teleport phenomenon and this is why the moving measure ν does not satisfy the same Lipschitz estimate as μ. We can also provide another example, that we will not develop in details, where the vector field v_t actually vanishes outside the support of ν_t, but its L^∞ norm is not the same if we consider μ or $\mu-\rho$ in the continuity equation, so that the Lipschitz constant increases (without blowing-up) while passing from μ to ν.

Example 4.6. Consider the measures $\rho_0=\frac{9}{10}\mathscr{L}^1\llcorner[0,1]$ and $\nu(t)=\frac{9}{10}\mathscr{L}^1\llcorner[t,t+1/9]$ for $t\in[0,8/9]$. Set $\mu(t)=\rho_0+\nu(t)$ and then consider the vector field

$$v_t(x)=\overrightarrow{e_1}\cdot 1_{[t,t+\frac{9}{10}]}(x),$$

which at every time t moves rightwards the particles of the interval $[t,t+9/10]$. We have

$$\frac{\partial\nu}{\partial t}+\nabla\cdot(v_t\nu(t))=0,$$

and it is quite easy to see that $\|v_t\|_{L^\infty(\nu_t)} = 1 = |\nu'|_{w_\infty}(t)$. On the other hand one can see that μ is a solution of the continuity equation with velocity field given by $1/2\, v_t(x)$, that is we have

$$\frac{\partial \mu}{\partial t} + \nabla \cdot \left(\frac{1}{2} v_t \mu(t) \right) = 0,$$

as a consequence of $v_t \nu(t) = 1/2\, v_t \mu(t)$. This shows that $|\mu'|_{w_\infty}(t) \leq \|1/2\, v_t\|_{L^\infty(\mu_t)} = 1/2$, i.e. the speed of the two curves μ and ν is finite in both cases but different.

We then turn our attention to the functional

$$\overline{\mathfrak{L}}_\alpha(\nu, \mu) = \int_0^\infty g_\alpha(\nu(t))\, dt, \ \ (\nu, \mu) \in SEP(\mu_0, \mu_1),$$

for which the following existence result is almost straightforward. As we said, to give a cleaner definition of the class SEP and of the functional, we decided to stick to the case where the velocity $|\mu'|_{w_\infty}$ (nor, in any sense, $|\nu'|$) does not appear explicitly in the criterion to be minimized, but only in the constraints.

Theorem 4.7. *The minimization problem*

$$\inf_{(\nu,\mu) \in SEP(\mu_0,\mu_1)} \overline{\mathfrak{L}}_\alpha(\nu, \mu), \tag{4.5}$$

admits a solution, provided that there exists an admissible special evolution pairing $(\overline{\nu}, \overline{\mu})$ having finite $\overline{\mathfrak{L}}_\alpha$.

Proof. It should be clear that it is enough to show that $SEP(\mu_0, \mu_1)$ is closed: then one has to simply reproduce step by step the proof of Proposition 3.6, taking into account the fact that every special evolution pairing (ν, μ) having finite $\overline{\mathfrak{L}}_\alpha$, has to satisfy

$$\lim_{t \to \infty} |\nu(t)|(\Omega) = 0.$$

Concerning the closedness of $SEP(\mu_0, \mu_1)$, it is enough to use the fact that the distance w_∞ is lower semicontinuous with respect to the $*-$weak convergence of measures, as already pointed out, so that property (4.4) easily pass to the limit. □

Remark 4.8. If one wants Theorem 4.7 to be interesting, one has to provide conditions for the existence of special evolution pairings with finite energy. The idea is the following: suppose that μ_1 is a probability measure which is irrigable in the sense of Xia, Solimini et al. This means (see the Introduction)

$$\min\{M^*_\alpha(\Phi) \,:\, \nabla \cdot \Phi = \mu_0 - \mu_1\} < +\infty$$

and thanks to the relaxed definition by Xia, there exists a sequence of finite graphs $\mathfrak{g}_n$, corresponding to traffic plans Q_n, such that $\sup_n E_\alpha(Q_n) < +\infty$ and $(e_\infty)_\# Q_n = \mu_1^n \rightharpoonup \mu_1$, where the measures μ_1^n are atomic. Then one uses the results of Section 6 to see that these traffic plans give raise to some evolution pairings (ν^n, μ^n) which are actually special evolution pairings in $SEP(\mu_0, \mu_1^n)$ and whose energy is the same as $E_\alpha(Q_n)$. Up to subsequences, thanks to the semicontinuity of $\overline{\mathfrak{L}}_\alpha$ and to the closedness result of Lemma 3.5, one can get $(\nu_n, \mu_n) \to (\nu, \mu)$ with $(\nu, \mu) \in SEP(\mu_0, \mu_1)$ and $\overline{\mathfrak{L}}_\alpha(\nu, \mu) < +\infty$.

5. Characterization of $SEP(\mu_0,\mu_1)$

The main tool in order to compare the energy $\overline{\mathfrak{L}}_\alpha$ with a Gilbert-Steiner energy, will be a complete characterization of the special evolution pairings, in terms of the Lipschitz curves of the base space. So our aim now is to give a refinement to the case of SEP of a result by Lisini (see [13], Theorems 4 and 5), characterizing $p-$absolutely continuous curves in the Wasserstein space $\mathcal{W}_p(\Omega)$ in terms of the $p-$absolutely continuous curves of the ambient space Ω: the main difference (apart from the fact that we explicitly refer to the case $p=+\infty$) is the characterization of the moving part ν in terms of the $1-$Lipschitz curves in Ω which at every fixed time t are still moving.

In order to achieve our scope, we have to start with a couple of technical Lemmas: they are nothing but *ad hoc* adaptations of the Gluing Lemma (see [16], Lemma 7.6). First of all we prove the existence of the composition of two transport plans, that takes into account the fact that the mass which arrives at destination must stay in place: at this level, this sentence could sound mysterious, but in the proof of Theorem 5.4 it should become clearer. We point out that in the following, given two positive Borel measures $\nu_1,\nu_2\in\mathcal{M}^+(\Omega)$ with the same mass, by $\Pi(\nu_1,\nu_2)$ we will denote the set of all positive Borel measures over the product space $\Omega\times\Omega$, having fixed marginals ν_1 and ν_2.

Lemma 5.1 (Modified Gluing Lemma)**.** *Let $(\mu_1,\mu_2,\mu_3)\in\mathscr{P}(\Omega)$ and $(\nu_1,\nu_2,\nu_3)\in\mathcal{M}^+_1(\Omega)$ such that*

$$\nu_i\le\mu_i,\ i=1,2,3,$$

and suppose that, setting $\rho_i=\mu_i-\nu_i$, we have $\rho_1\le\rho_2\le\rho_3$. For every $\gamma_{1,2}\in\Pi(\mu_1-\rho_1,\mu_2-\rho_1)$ and $\gamma_{2,3}\in\Pi(\mu_2-\rho_2,\mu_3-\rho_2)$, there exists $\gamma\in\mathscr{P}(\Omega\times\Omega\times\Omega)$ with the following properties:

(i) $(\pi_{i,i+1})_\#\gamma=\gamma_{i,i+1}+(\mathrm{Id}\times\mathrm{Id})_\#(\rho_i)$, *for $i=1,2$;*
(ii) $(\pi_i)_\#(\gamma 1_{S_i})\ge\rho_i$, *for $i=1,2$, where the set S_i is given by*

$$S_i=\{(x_1,x_2,x_3)\in\Omega\times\Omega\times\Omega\ :\ x_j=x_i,\quad \textit{for } j\ge i\}.$$

Proof. We will use the so called Disintegration Theorem (see [11], Chapter III). First of all, we define

$$\tilde\gamma_{1,2}=\gamma_{1,2}+\gamma^0_{1,2}=\gamma_{1,2}+(\mathrm{Id}\times\mathrm{Id})_\#(\rho_1),$$

and

$$\tilde\gamma_{2,3}=\gamma_{2,3}+\gamma^0_{2,3}=\gamma_{2,3}+(\mathrm{Id}\times\mathrm{Id})_\#(\rho_2),$$

which are actually elements of $\Pi(\mu_1,\mu_2)$ and $\Pi(\mu_2,\mu_3)$, respectively. Then we disintegrate $\gamma_{1,2}$ with respect to the x_2 variable, that is

$$\gamma_{1,2}=\int\xi^1_{x_2}\,d(\mu_2-\rho_1)(x_2)=\int\xi^1_{x_2}\,d(\mu_2-\rho_2)(x_2)+\int\xi^1_{x_2}\,d(\rho_2-\rho_1)(x_2),$$

where for $(\mu_2-\rho_1)-$a.e. $x_2\in\Omega$, $\xi^1_{x_2}$ is a Borel probability measure on Ω and equally for $\gamma^0_{1,2}$, thus obtaining

$$\gamma^0_{1,2}=\int\eta^1_{x_2}\,d\rho_1(x_2).$$

On the other hand, we disintegrate $\gamma_{2,3}$ and $\gamma^0_{2,3}$ with respect to the x_1 variable, that is

$$\gamma_{2,3}=\int\xi^3_{x_2}\,d(\mu_2-\rho_2)(x_2),$$

$$\gamma^0_{2,3}=\int\eta^3_{x_2}\,d\rho_2(x_2)=\int\eta^3_{x_2}\,d(\rho_2-\rho_1)(x_2)+\int\eta^3_{x_2}\rho_1(x_2).$$

Observe that actually, by construction we have

$$\eta^1_{x_2} = \delta_{x_2}, \quad \text{for } \rho_1\text{–a.e. } x_2 \in \Omega, \tag{5.1}$$

and

$$\eta^3_{x_2} = \delta_{x_2}, \quad \text{for } \rho_2\text{–a.e. } x_2 \in \Omega, \tag{5.2}$$

We can rewrite everything as follows

$$\tilde\gamma_{1,2} = \int \xi^1_{x_2}\, d(\mu_2 - \rho_2)(x_2) + \int \xi^1_{x_2}\, d(\rho_2 - \rho_1)(x_2) + \int \eta^1_{x_2}\, d\rho_1(x_2),$$

$$\tilde\gamma_{2,3} = \int \xi^3_{x_2}\, d(\mu_2 - \rho_2)(x_2) + \int \eta^3_{x_2}\, d(\rho_2 - \rho_1)(x_2) + \int \eta^3_{x_2}\, d\rho_1(x_2),$$

that is we have "piecewise" disintegrated with respect to their common marginals our transport plans. Then it is natural to glue this two decompositions as follows

$$\gamma = \int \xi^1_{x_2} \otimes \xi^3_{x_2}\, d(\mu_2 - \rho_2)(x_2) + \int \xi^1_{x_2} \otimes \eta^3_{x_2}\, (\rho_2 - \rho_1)(x_2) + \int \eta^1_{x_2} \otimes \eta^3_{x_2}\, d\rho_1(x_2), \tag{5.3}$$

and it is straightforward to verify that γ has the desired properties: (i) is trivially satisfied, while concerning (ii) let us observe that for every Borel set $A \subset \Omega$, we have

$$\begin{aligned}(\pi_1)_\#(\gamma 1_{S_1})(A) = \gamma\left(\{(a,a,a) \ : \ a \in A\}\right) &\geq \int_A \eta^1_{x_2}(\{x_2\})\eta^3_{x_2}(\{x_2\})\ d\rho_1(x_2)\\ &= \int_A d\rho_1(x_2) = \rho_1(A),\end{aligned}$$

where we have used (5.1) and (5.2) and the fact that $\rho_1 \leq \rho_2$. In the end, we have proved property (ii) for $i = 1$, while for $i = 2$ the proof is straightforward. □

Remark 5.2. Observe that the probability measure γ given by (5.3) can also be written (with the convention $\rho_0 = 0$) as

$$\gamma = \int \xi_{x_3}\, d(\mu_3 - \rho_3)(x_3) + \sum_{i=1}^{3} \int \eta^i_{x_3}\, d(\rho_i - \rho_{i-1})(x_3), \tag{5.4}$$

for suitable Borel families of probability measures on $\Omega \times \Omega$ $\{\xi_{x_3}\}_{x_3 \in \Omega}$ and $\{\eta^i_{x_3}\}_{x_3 \in \Omega}$ such that $\eta^3_{x_3} = \xi_{x_3}$ for ρ_3–a.e. $x \in \Omega$ and

$$\eta^1_{x_3} = \delta_{(x_3,x_3)}, \quad \text{for } \rho_1\text{-a.e. } x_3 \in \Omega,$$

and

$$(\pi_2)_\# \eta^2_{x_3} = \delta_{x_3}, \quad \text{for } \rho_2\text{-a.e. } x_3 \in \Omega.$$

Indeed, it is sufficient to observe that by construction

$$(\pi_3)_\# \left(\int \xi^1_{x_2} \otimes \xi^3_{x_2}\, d(\mu_2 - \rho_2)(x_2) \right) = (\mu_3 - \rho_3) + (\rho_3 - \rho_2),$$

then we can disintegrate this measure with respect to the x_3 variable, thus obtaining the existence of a Borel family of probability measures $\{\xi_{x_3}\}_{x_3 \in \Omega}$ on the product space $\Omega \times \Omega$ such that

$$\int_\Omega \xi^1_{x_2} \otimes \xi^3_{x_2}\, d(\mu_2 - \rho_2)(x_2) = \int_\Omega \xi_{x_3}\, d(\mu_3 - \rho_3)(x_3) + \int_\Omega \xi_{x_3}\, d(\rho_3 - \rho_2)(x_3).$$

Equally, taking into account

$$(\pi_3)_\# \left(\int \xi^1_{x_2} \otimes \eta^3_{x_2}\, d(\rho_2 - \rho_1)(x_2) \right) = \rho_2 - \rho_1,$$

$$(\pi_3)_\# \left(\int \eta^1_{x_2} \otimes \eta^3_{x_2}\, d\rho_1(x_2) \right) = \rho_1,$$

and disintegrating with respect to the x_3 variable, we obtain the desired representation (5.4), keeping in mind (5.1) and (5.2).

The Modified Gluing Lemma can be easily generalized to every n-uple of probability measures. More precisely, we have the following:

Lemma 5.3. *For $n \geq 3$, let $\{\mu_i\}_{i=1}^n \subset \mathscr{P}(\Omega)$ and $\{\nu_i\}_{i=1}^n \subset \mathcal{M}_1^+(\Omega)$ be such that*

$$\nu_i \leq \mu_i, \quad \text{for every } i = 1, \dots, n,$$

and suppose that, setting $\rho_i = \mu_i - \nu_i$, we have $\rho_i \leq \rho_{i+1}$. For every $\gamma_{i,i+1} \in \Pi(\nu_i, \mu_{i+1} - \rho_i)$, with $i = 1, \dots, n-1$, there exists $\gamma \in \mathscr{P}(\Omega^n)$ with the following properties:

(i) $(\pi_{i,i+1})_\# \gamma = \gamma_{i,i+1} + (\mathrm{Id} \times \mathrm{Id})_\#(\rho_i)$, *for $i = 1, \dots, n-1$;*

(ii) $(\pi_i)_\#(\gamma 1_{S_i}) \geq \rho_i$, *for $i = 1, \dots, n-1$, where the set S_i is given by*

$$S_i = \{(x_1, \dots, x_n) \in \Omega^n \ : \ x_j = x_i, \quad \text{for } j \geq i\}.$$

Moreover γ can be written as

$$\gamma = \int \xi_{x_n}\, d(\mu_n - \rho_n)(x_n) + \sum_{i=1}^n \int \eta^i_{x_n}\, d(\rho_i - \rho_{i-1})(x_n), \tag{5.5}$$

where $\xi_{x_n}, \eta^i_{x_n} \in \mathscr{P}(\Omega^{n-1})$ and every $\eta^i_{x_n}$ is such that

$$(\pi_{i,\dots,n-1})_\# \eta^i_{x_n} = \delta_{(x_n,\dots,x_n)}, \quad \text{for } \rho_i\text{-a.e. } x_n \in \Omega, \tag{5.6}$$

the function $\pi_{i,\dots,n-1}$ being the projection on the $(x_i, \dots, x_{n-1})$ coordinates.

Proof. We proceed by induction on n, the thesis being true for $n = 3$ thanks to Lemma 5.1 and Remark 5.2.

Suppose now that the assertion is true for n, that is there exists a probability measure $\gamma \in \mathscr{P}(\Omega^n)$ with the required properties and consider the case $n+1$. As in the proof of Lemma 5.1, we can define

$$\tilde{\gamma}_{n,n+1} = \gamma_{n,n+1} + \gamma^0_{n,n+1} = \gamma_{n,n+1} + (\mathrm{Id} \times \mathrm{Id})_\# \rho_n,$$

and then we disintegrate $\gamma_{n,n+1}$ and $\gamma^0_{n,n+1}$ with respect to x_n, thus getting

$$\begin{aligned} \tilde{\gamma}_{n,n+1} &= \int \xi^{n+1}_{x_n}\, d(\mu_n - \rho_n)(x_n) + \int \eta^{n+1}_{x_n}\, d\rho_n(x_n) \\ &= \int \xi^{n+1}_{x_n}\, d(\mu_n - \rho_n)(x_n) + \sum_{i=1}^n \int \eta^{n+1}_{x_n}\, d(\rho_i - \rho_{i-1})(x_n) \end{aligned}$$

where $\eta^{n+1}_{x_n} = \delta_{x_n}$ for ρ_n-a.e. $x_n \in \Omega$. Then using the decomposition (5.5) for γ, we can define

$$\widehat{\gamma} = \int \xi_{x_n} \otimes \xi^{n+1}_{x_n}\, d(\mu_n - \rho_n)(x_n) + \sum_{i=1}^n \int \eta^i_{x_n} \otimes \eta^{n+1}_{x_n}\, d(\rho_i - \rho_{i-1})(x_n),$$

which is an element of $\mathscr{P}(\Omega^{n+1})$. It is straightforward to see that $\widehat{\gamma}$ satisfies property (i), so let us show that also (ii) holds true: for every Borel subset $A \subset \Omega$ we get

$$\begin{aligned}(\pi_j)_\#(\widehat{\gamma}1_{S_j})(A) &= \widehat{\gamma}(\{(x_1,\ldots,x_{j-1},a\ldots,a) \ : \ x_1,\ldots,x_{j-1}\in\Omega,\ a\in A\}) \\ &\geq \sum_{i=1}^{j}\int_A (\pi_{j,\ldots,n-1})_\#\eta^i_{x_n}(\{(x_n,\ldots,x_n)\})\eta^{n+1}_{x_n}(\{x_n\})\,d(\rho_i-\rho_{i-1})(x_n) \\ &= \sum_{i=1}^{j}(\rho_i-\rho_{i-1})(A) = \rho_j(A), \quad \text{for every } j=1,\ldots,n,\end{aligned}$$

where we have used property (5.6). To conclude the proof, it remains to show that $\widehat{\gamma}$ can be decomposed as in (5.5): observe that by construction we have

$$(\pi_{n+1})_\#\left(\int \xi_{x_n}\otimes\xi^{n+1}_{x_n}\,d(\mu_n-\rho_n)(x_n)\right) = \mu_{n+1}-\rho_n = (\mu_{n+1}-\rho_{n+1})+(\rho_{n+1}-\rho_n),$$

so that as in Remark 5.2 we can disintegrate $\int \xi_{x_N}\otimes\xi^{N+1}_{x_n}\,d(\mu_n-\rho_n)$ with respect to the x_{n+1} variable, thus obtaining

$$\int \xi_{x_n}\otimes\xi^{n+1}_{x_n}\,d(\mu_n-\rho_n)(x_n) = \int \xi_{x_{n+1}}\,d(\mu_{n+1}-\rho_{n+1})(x_{n+1}) + \int \xi_{x_{n+1}}\,d(\rho_{n+1}-\rho_n)(x_{n+1}),$$

where for $(\mu_{n+1}-\rho_n)$−a.e. $x_{n+1}\in\Omega$, $\xi_{x_{n+1}}$ is a Borel probability measure over the space Ω^n. The same can be done for each term

$$\int \eta^i_{x_n}\otimes\eta^{n+1}_{x_n}\,d(\rho_i-\rho_{i-1})(x_n),$$

then taking into account that $\eta^{n+1}_{x_n}=\delta_{x_n}$ for ρ_n−a.e. $x\in\Omega$ and that $\eta^i_{x_n}$ satisfies (5.6) by hypothesis, we can conclude. □

We now have all the elements in order to prove the first main result of this section.

Theorem 5.4. *Let $(\nu,\mu)\in SEP(\mu_0,\mu_1)$. Then there exists $Q\in\mathscr{P}(\mathrm{Lip}_1(I;\Omega))$ such that*

$$(e_t)_\# Q = \mu(t), \quad (e_t)_\# Q^t \leq \nu(t), \qquad t\in I,$$

where $Q^t = Q\llcorner\{\sigma\in\mathrm{Lip}_1(I;\Omega) \ : \ T(\sigma)\geq t\}$.

Proof. We fix $M\in\mathbb{N}$ and then for every $n\in\mathbb{N}$ we take a dyadic partition

$$t_{i,n} = \frac{M}{2^n}\,i, \ \ i=0,1,\ldots,2^n,$$

of the interval $[0,M]$. Indicating as always

$$\rho(t) := \mu(t)-\nu(t),$$

we take $\tilde{\gamma}_{i,i+1}\in\Pi(\nu(t_{i,n}),\mu(t_{i+1,n})-\rho(t_{i,n}))$ to be optimal for w_∞, that is

$$w_\infty(\nu(t_{i,n}),\mu(t_{i+1,n})-\rho(t_{i,n})) = \sup\{|x-y| \ : \ (x,y)\in\mathrm{spt}(\tilde{\gamma}_{i,i+1})\} \leq \frac{M}{2^n},$$

and we define $\gamma_{i,i+1}\in\Pi(\mu(t_{i,n}),\mu(t_{i+1,n}))$ by

$$\gamma_{i,i+1} = \tilde{\gamma}_{i,i+1} + (\mathrm{Id}\times\mathrm{Id})_\#\rho(t_{i,n}).$$

Let $\gamma^n_M\in\mathscr{P}(\Omega^{2^n+1})$ be the multi-transport plan given by Lemma 5.3 such that:

(i) $(\pi_{i,i+1})_\#\gamma^n_M = \gamma_{i,i+1}$;
(ii) $(\pi_i)_\#(\gamma^n_M 1_{S_i})\geq\rho(t_{i,n})$, where $S_i=\{\mathbf{x}=(x_0,\ldots,x_{2^n})\in\Omega^{2n+1} \ : \ x_j=x_i, \text{ for } j\geq i\}$.

We now define the application

$$\begin{array}{rccc}\Theta^n: & \Omega^{2^n+1} & \to & \mathrm{Lip}([0,M];\Omega) \\ & \mathbf{x} & \mapsto & \Theta^n_{\mathbf{x}},\end{array}$$

where for every $\mathbf{x}=(x_0,\dots,x_{2^n})\in\Omega^{2^n+1}$, the curve $\Theta^n_{\mathbf{x}}$ is given by

$$\Theta^n_{\mathbf{x}}(t)=\frac{t_{i+1,n}-t}{t_{i+1,n}-t_{i,n}}x_i+\frac{t-t_{i,n}}{t_{i+1,n}-t_{i,n}}x_{i+1},\ t\in[t_{i,n},t_{i+1,n}],\ i\in\{0,\dots,2^n-1\},$$

and we further set

$$Q^n_M=(\Theta^n)_\sharp\gamma^n_M\in\mathscr{P}(C([0,M];\Omega)).$$

By construction, it is almost straightforward to see that every Q^n_M is concentrated on $\mathrm{Lip}_1([0,M];\Omega)$, the latter being a compact space. This in turn implies that the sequence $\{Q^n_M\}_{n\in\mathbb{N}}$ narrowly converges (up to subsequences) to an element Q_M of $\mathscr{P}(\mathrm{Lip}_1([0,M];\Omega))$, by means of Prokhorov Theorem (see [11], Chapter III).

We now show that $(e_t)_\# Q_M=\mu(t)$ for every $t\in[0,M]$: first observe that by its very definition, the sequence $\{Q^n_M\}_{n\in\mathbb{N}}$ satisfies

$$\mu(t_{i,n})=(e_{t_{i,n}})_\# Q^n_M,\ i=0,1,\dots,2^n.$$

On the other hand, thanks to the fact that we are considering dyadic partitions of $[0,M]$, we have for every $k<n$

$$\{t_{j,k}\}_{j=0}^{2^k}\subset\{t_{i,n}\}_{i=0}^{2^n},$$

so that for every $k<n$

$$\mu(t_{i,k})=(e_{t_{i,k}})_\# Q^n_M,\ i=0,1,\dots,2^k.$$

Letting n go to ∞, we then obtain, for every k, the following equalities

$$\mu(t_{i,k})=(e_{t_{i,k}})_\# Q_M,\ i=0,1,\dots,2^k.$$

We have proven that the two uniformly continuous functions $\mu(\cdot)$ and $(e_{(\cdot)})_\# Q_M$ coincide on the points $\{t_{j,k}\}_{j=0}^{2^k}$, for every $k\in\mathbb{N}$, thus giving the equality on $[0,M]$ of these functions.

Before going on, we define the following subset of $[0,M]$

$$\mathcal{N}:=\left\{t\in[0,M]\,:\,\text{either } Q_M(T^{-1}(\{t\}))>0\ \text{ or there exists } n \text{ such that } Q^n_M(T^{-1}(\{t\}))>0\right\},$$

that is $\mathcal{N}$ is the set of times such that $\{\sigma\in\mathrm{Lip}_1([0,M];\Omega)\,:\,T(\sigma)=t\}$ is charged by at least one of the measures Q_M or Q^n_M. Due to the fact that as t varies in $[0,M]$ these sets $T^{-1}(\{t\})$ constitute a partition of the whole space, we observe that $\mathcal{N}$ must be at most countable. We now set

$$Q^t_M=Q_M\llcorner\{\sigma\,:\,T(\sigma)\ge t\},\ Q^{n,t}_M=Q^n_M\llcorner\{\sigma\,:\,T(\sigma)\ge t\},\ n\in\mathbb{N},$$

and we first notice that if $t\notin\mathcal{N}$, we have $\liminf_{n\to\infty}Q^{n,t}_M\ge Q^t_M$, in the sense that for every continuous and positive test function φ there holds

$$\liminf_{n\to\infty}\int\varphi\,dQ^{n,t}_M\ge\int\varphi\,dQ^t_M.$$

This is the same as saying that any possible limit measure $\tilde{Q}$ of a subsequence of $Q^{n,t}_M$ must be larger than Q^t_M. To prove such a property, it is sufficient to notice that this is true if $Q^{n,t}_M$ and Q^t_M are replaced with $1_{\{T>t\}}\cdot Q^n_M$ and $1_{\{T>t\}}\cdot Q_M$, respectively, since the function $1_{\{T>t\}}$ is l.s.c. and this modification may be performed for free if the set $T^{-1}(t)$ is negligible for all these measures.

In order to prove that $(e_t)_\# Q^t_M \le \nu(t)$, we first observe that using property (ii) of $\{\gamma^n_M\}$ we get that

$$(e_{t_{i,n}})_\# Q^{n,t_{i,n}}_M \le \nu(t_{i,n}). \tag{5.7}$$

Let us give a brief justification of (5.7): indeed, we have

$$\int_{C([0,M];\Omega)} \varphi(\sigma(t_{i,n}))\, dQ^{n,t_{i,n}}_M(\sigma) \ge \int_{C([0,M];\Omega)} \varphi(\sigma(t_{i,n}))\, dQ^n_M(\sigma) - \int_{\{\sigma\,:\,T(\sigma)\le t_{i,n}\}} \varphi(\sigma(t_{i,n}))\, dQ^n_M(\sigma),$$

and the first integral in the right-hand side is just the integral of φ with respect to the measure $\mu(t_{i,n})$, while for the second we observe that

$$\int_{\{\sigma\,:\,T(\sigma)\le t_{i,n}\}} \varphi(\sigma(t_{i,n}))\, dQ^n_M(\sigma) = \int_{\{\mathbf{x}\,:\,x_j=x_i,\ \text{ for } j\ge i\}} \varphi(\Theta^n_{\mathbf{x}}(t_{i,n}))\, d\gamma^n_M(\mathbf{x}) \ge \int_\Omega \varphi(x)\, d\rho(t_{i,n})(x),$$

having used the definition of Θ^n and property (ii) in the last inequality. In conclusion, using $\nu(t_{i,n}) = \mu(t_{i,n}) - \rho(t_{i,n})$ we have shown the validity of (5.7).

Observe moreover that we have $Q^{n,t}_M \le Q^{n,t_{i,n}}_M$ for every $t \ge t_{i,n}$, and using again the fact that the partition under consideration is dyadic, in the end we get

$$(e_{t_{i,k}})_\# Q^{n,t}_M \le \nu(t_{i,k}), \quad \text{for every } t \ge t_{i,k},$$

for every $k < n$. Taking the limit as n goes to ∞, and using the "semicontinuity" we addressed before, i.e. the fact $\liminf_{n\to\infty} Q^{n,t}_M \ge Q^t_M$, which is true for $t \notin \mathcal{N}$, we get for every i and k

$$(e_{t_{i,k}})_\# Q^t_M \le \nu(t_{i,k}), \quad \text{for every } t \notin \mathcal{N},\ t \ge t_{i,k}.$$

The condition $t \notin \mathcal{N}$ may be withdrawn, if for $t > t_{i,k}$ one takes $s \in (t_{i,k}, t) \setminus \mathcal{N}$ and uses the inequality $Q^t_M \le Q^s_M$, which gives

$$(e_{t_{i,k}})_\# Q^t_M \le (e_{t_{i,k}})_\# Q^s_M \le \nu(t_{i,k}), \quad \text{for every } t > t_{i,k}.$$

It is then sufficient to consider a sequence of dyadic numbers $t_{i,k}$ converging to t from the left: we then have $\nu(t_{i,k}) \to \nu(t)$ because of the assumption of left continuity of ν and $(e_{t_{i,k}})_\# Q^t_M \to (e_t)_\# Q^t_M$ because Q^t_M is a fixed measure on $\mathrm{Lip}_1([0,M];\Omega)$ and the maps $e_{t_{i,k}}$ converge uniformly to e_t on this set. Actually, we can also say

$$w_\infty((e_{t_{i,k}})_\# Q^t_M, (e_t)_\# Q^t_M) \le |t_{i,k} - t|,$$

thanks to the Lipschitz property of the curves in $\mathrm{Lip}_1([0,M];\Omega)$. This gives

$$(e_t)_\# Q^t_M \le \nu(t), \quad \text{for every } t \in [0,M]. \tag{5.8}$$

Finally, we have to take the limit as $M \to +\infty$: defining the continuous mapping

$$\Phi_M : \mathrm{Lip}_1([0,M];\Omega) \to \mathrm{Lip}_1(I;\Omega),$$

such that for every $\sigma \in \mathrm{Lip}_1([0,M];\Omega)$, the curve $\Phi_M(\sigma)$ is given by

$$\Phi_M(\sigma)(t) = \begin{cases} \sigma(t), & \text{if } t \le M, \\ \sigma(M), & \text{if } t > M, \end{cases}$$

we set $\widetilde{Q}_M = (\Phi_M)_\# Q_M \in \mathrm{Lip}_1(I;\Omega)$; then the sequence $\{Q_M\}_{M\in\mathbb{N}}$ is narrowly converging (up to subsequences), again thanks to the compactness of the space $\mathrm{Lip}_1(I;\Omega)$. If we call Q its limit, it is not difficult to see that we have $(e_t)_\#\widetilde{Q}_M = \mu(t)$ on $[0,M]$ and passing to the limit, we obtain that the same holds true for Q on I. Moreover, if $\widetilde{Q}^t_M = \widetilde{Q}_M \llcorner \{\sigma \, : \, T(\sigma)\geq t\}$, then using the fact

$$(e_t)_\#\widetilde{Q}^t_M \leq \nu(t), \quad \text{for } t\in[0,M],$$

which is actually equivalent to (5.8), and taking again the limit as M goes to $+\infty$, we can show that

$$(e_t)_\# Q^t \leq \nu(t), \quad \text{for } t\in I\setminus\widetilde{\mathcal{N}},$$

where the negligible set $\widetilde{\mathcal{N}}$ where the inequality could not hold is, as before, the countable set of times such that $T^{-1}(\{t\})$ is charged by at least one of the measures $\widetilde{Q}_M$ or Q. After that, we consider a general t and $s<t$ with $s\notin\widetilde{\mathcal{N}}$: we have

$$(e_s)_\# Q^t \leq (e_s)_\# Q^s \leq \nu(s).$$

Taking the limit $s\nearrow t$ we get, as before, $(e_t)_\# Q^t\leq\nu(t)$, which concludes the proof. □

The next result of this section states that the previous Theorem can be reverted, thus giving a nice correspondence between $SEP(\mu_0,\mu_1)$ and the $1-$Lipschitz curves of Ω: this has to be compared with Theorem 4 of [13].

Theorem 5.5. *Let $Q\in TP(\mu_0,\mu_1)$ be a traffic plan. For every $t\in I$, we set $Q^t = Q\llcorner\{T(\sigma)\geq t\}$ and we define*

$$\mu(t) = (e_t)_\# Q, \quad \nu(t) = (e_t)_\# Q^t,$$

then $(\nu,\mu)\in SEP(\mu_0,\mu_1)$.

Proof. The fact that $\mu\in\mathrm{Lip}_1(I;\mathcal{W}_\infty(\Omega))$ is straightforward, since for every (t,s) the measure $(e_t,e_s)_\# Q$ is a transport plan between $\mu(t)$ and $\mu(s)$, providing a cost smaller than $|t-s|$. It is interesting to compare with Theorem 4 of [13], where the case of $\mathcal{W}_q(\Omega)$ with $q<\infty$ is treated in connection with AC^q curves (absolutely continuous curves having $q-$summable metric derivative).

We then observe that the set

$$\{\sigma\in\mathrm{Lip}_1(I;\Omega) \; : \; T(\sigma)\geq t\},$$

is Borel measurable for every $t\in I$, thanks to the lower semicontinuity of T, so that ν is well-defined. We have to show that $\rho(t)=\mu(t)-\nu(t)=(e_t)_\#\left(Q\llcorner\{\sigma\in\mathrm{Lip}_1(I;\Omega)\,:\,T(\sigma)<t\}\right)$ is nondecreasing and left continuous. To see the monotonicity property, consider a positive test function $\varphi\in C(\Omega)$ and $s\leq t$:

$$\begin{aligned}\int_\Omega \varphi(x)\, d\rho(s)(x) &= \int_{\{\sigma\,:\,T(\sigma)<s\}} \varphi(\sigma(s))\, dQ(\sigma)\\ &= \int_{\{\sigma:T(\sigma)<s\}} \varphi(\sigma(t))\, dQ(\sigma)\\ &\leq \int_{\{\sigma:T(\sigma)<t\}} \varphi(\sigma(t))\, dQ(\sigma) = \int_\Omega \varphi(x)\, d\rho(t)(x).\end{aligned}$$

Once one has the monotonicity, weak continuity is the same as strong continuity and we can turn to prove that $\lim_{s\nearrow t}|\rho(s)-\rho(t)|(\Omega)=\lim_{s\nearrow t}|\rho(t)|(\Omega)-|\rho(s)|(\Omega)=0$. It is hence sufficient to prove that the mass of ρ is left continuous, which is the same as looking at the mass of $Q\llcorner\{\sigma \in \mathrm{Lip}_1(I;\Omega) : T(\sigma) < t\}$. This corresponds to saying that

$$\{\sigma \in \mathrm{Lip}_1(I;\Omega) : T(\sigma) < t\} = \bigcup_{s<t} \{\sigma \in \mathrm{Lip}_1(I;\Omega) : T(\sigma) < s\},$$

which is evident.

In order to check that $\nu \preceq \mu$ we notice that property (E1) is evidently verified, since $Q^t \le Q$, while property (E2) has already been verified when we proved that ρ is increasing. Hence (ν, μ) is an evolution pairing, which clearly connects μ_0 and μ_1. We have to verify that actually it is a special evolution pairing: fixed $h > 0$, let us call

$$\gamma = (e_t, e_{t+h})_\# Q^t.$$

It is easy to check that this is a transport plan between $\mu(t) - \rho(t)$ and $\mu(t+h) - \rho(t)$ (just check that $(\pi_2)_\#\gamma = \mu(t+h) - \mu(t) + \nu(t)$). Using the definition of w_∞ and the fact that Q^t is a measure over $\mathrm{Lip}_1(I;\Omega)$, we get

$$w_\infty(\nu(t), \mu(t+h) - \mu(t) + \nu(t)) \le \gamma\text{-ess}\sup_{(x,y)\in\Omega\times\Omega} |x - y| = Q^t\text{-ess sup}\, |\sigma(t) - \sigma(t+h)| \le h,$$

which finally gives $(\nu, \mu) \in SEP(\mu_0, \mu_1)$. □

6. Equivalence between the models

Up to now, we have collected enough elements to compare our energy $\overline{\mathfrak{L}}_\alpha$ with a Gilbert-Steiner one. Then the main result of the paper is the following.

Theorem 6.1. *Given $\mu_0 = \delta_{x_0}$ and $\mu_1 \in \mathscr{P}(\Omega)$, we get*

$$\min_{SEP(\mu_0,\mu_1)} \overline{\mathfrak{L}}_\alpha = \min_{TP(\mu_0,\mu_1)} E_\alpha.$$

Moreover, given any optimal traffic plan $Q \in TP(\mu_0, \mu_1)$, the special evolution pairing provided by Theorem 5.5 is optimal, and conversely, given an optimal special evolution pairing $(\nu, \mu) \in SEP(\mu_0, \mu_1)$, the construction of Theorem 5.4 provides an optimal traffic plan.

Proof. Let us take $Q \in TP(\mu_0, \mu_1)$ optimal for the traffic plan problem and suppose that it has finite energy. We will use the following fact, as a consequence of Lemma 2.1: for every t, the following equality is satisfied Q^t−a.e.

$$|(\sigma(t), t)|_Q = Q^t\left(\{\eta : \eta(t) = \sigma(t)\}\right). \tag{6.1}$$

This is true since we know $|(\sigma(t), t)|_Q = [\sigma]_{t,Q}$, which means that we can restrict our attention to those curves η who stayed together with σ for all the times between 0 and t. Moreover, we can assume that σ is parametrized by arc length on $[0, T(\sigma)]$: this implies that, if σ is still moving, i.e. $T(\sigma) \ge t$, this is the case for all the curves η such that $\eta = \sigma$ on $[0, t]$ and proves that we can further restrict our attention to those curves η with $T(\eta) \ge t$, i.e. switching from Q to Q^t, thus proving assertion (6.1).

Exchanging the order of integration, we can write

$$E_\alpha(Q) = \int_{\mathrm{Lip}_1(I;\Omega)} \int_0^{T(\sigma)} |(\sigma(t), t)|_Q^{\alpha-1}\, dt\, dQ(\sigma) = \int_0^\infty \int_{\mathrm{Lip}_1(I;\Omega)} |(\sigma(t), t)|_Q^{\alpha-1} dQ^t(\sigma)\, dt.$$

Then define the equivalence classes $\Sigma_{t,x} = \{\sigma \in \mathrm{Lip}_1(I;\Omega) : \sigma(t) = x\}$ and notice that, by finiteness of the energy, for $\mathscr{L}^1$−a.e. t the measure Q^t must be concentrated on those classes $\Sigma_{t,x}$ such that

$Q(\Sigma_{t,x}) > 0$. Since they have all positive mass, these classes are no more than a countable number and one can restrict the integral over them:

$$\int_{\mathrm{Lip}_1(I;\Omega)} |(\sigma(t),t)|_Q^{\alpha-1} dQ^t(\sigma) = \sum_i \int_{\Sigma_{t,x_i}} |(\sigma(t),t)|_Q^{\alpha-1} dQ^t(\sigma).$$

Yet, for all the curves $\sigma \in \Sigma_{t,x_i}$ we have $|(\sigma(t),t)|_Q = Q(\Sigma_{t,x_i}) = Q^t(\Sigma_{t,x_i})$, thanks to the fact that the class is non negligible and to the condition (6.1). Hence we may go on with

$$\int_{\mathrm{Lip}_1(I;\Omega)} |(\sigma(t),t)|_Q^{\alpha-1} dQ^t(\sigma) = \sum_i \int_{\Sigma_{t,x_i}} Q^t(\Sigma_{t,x_i})^{\alpha-1} dQ^t(\sigma) = \sum_i Q^t(\Sigma_{t,x_i})^{\alpha}.$$

Moreover, if one constructs the measures μ and ν associated to Q thanks to Theorem 5.5, $\nu(t)$ must be atomic and equal to $\sum_i Q^t(\Sigma_{t,x_i})\delta_{x_i}$, which gives in the end

$$\int_{\mathrm{Lip}_1(I;\Omega)} |(\sigma(t),t)|_Q^{\alpha-1} dQ^t(\sigma) = g_\alpha(\nu(t)).$$

Hence, if we compare the energy of the special evolution pairing given by Theorem 5.5 with the energy of Q, we get

$$\overline{\mathfrak{L}}_\alpha(\nu,\mu) = \int_0^\infty g_\alpha(\nu(t))\, dt = E_\alpha(Q),$$

which shows that

$$\min_{SEP(\mu_0,\mu_1)} \overline{\mathfrak{L}}_\alpha \le \min_{TP(\mu_0,\mu_1)} E_\alpha,$$

using the minimality of Q.

Conversely, let us take $(\nu,\mu) \in SEP(\mu_0,\mu_1)$ optimal and construct the traffic plan $Q \in TP(\mu_0,\mu_1)$ given by Theorem 5.4. As before, we revert the order of the integration in the definition of E_α. Let us set $\widehat{\nu}(t) = (e_t)_\# Q^t$, and consider

$$\int_{\mathrm{Lip}_1(I;\Omega)} |(\sigma(t),t)|_Q^{\alpha-1}\, dQ^t(\sigma) = \int_\Omega |(x,t)|_Q^{\alpha-1}\, d\widehat{\nu}(t)(x).$$

Moreover

$$\begin{aligned} |(x,t)|_Q = Q(\{\sigma \in \mathrm{Lip}_1(I;\Omega)\,:\,\sigma(t)=x\}) &\ge Q^t(\{\sigma \in \mathrm{Lip}_1(I;\Omega)\,:\,\sigma(t)=x\}) \\ &= Q^t(\{e_t^{-1}(x)\}) = \widehat{\nu}(t)(\{x\}), \end{aligned}$$

which in turn implies

$$\int_{\mathrm{Lip}_1(I;\Omega)} |(\sigma(t),t)|_Q^{\alpha-1}\, dQ^t(\sigma) \le \int_\Omega \widehat{\nu}(t)(\{x\})^{\alpha-1}\, d\widehat{\nu}(t).$$

It is only left to observe that

$$\int_\Omega \widehat{\nu}(t)(\{x\})^{\alpha-1}\, d\widehat{\nu}(t) = g_\alpha(\widehat{\nu}(t)), \quad \text{for } \mathscr{L}^1\text{−a.e. } t \in I,$$

thanks to the fact that $\widehat{\nu}$ is atomic: indeed, $\nu(t)$ is atomic for a.e. t and we have $\widehat{\nu} \le \nu$. Therefore collecting all this estimates, we end up with

$$E_\alpha(Q) \le \int_0^\infty \left(\int_\Omega \widehat{\nu}(t)(\{x\})^{\alpha-1}\, d\widehat{\nu}(t)\right) dt = \int_0^\infty g_\alpha(\widehat{\nu}(t))\, dt \le \overline{\mathfrak{L}}_\alpha(\nu,\mu),$$

thus concluding the proof. □

The case of μ_1 atomic is interesting and deserves some words more: indeed, in this case thanks to Proposition 4.4 we obtain $SEP(\mu_0,\mu_1) = EP(\mu_0,\mu_1)$, so that

$$\min_{SEP(\mu_0,\mu_1)} \overline{\mathfrak{L}}_\alpha = \min_{EP(\mu_0,\mu_1)} \mathfrak{L}_\alpha.$$

It is then sufficient to note that by means of Theorem 6.1, the left-hand side is equal to the minimum of E_α over the set $TP(\mu_0,\mu_1)$: summarizing, we have shown the following important fact.

Corollary 6.2. *Suppose that μ_1 is a purely atomic probability and $\mu_0 = \delta_{x_0}$. Then*

$$\min_{EP(\mu_0,\mu_1)} \mathfrak{L}_\alpha = \min_{TP(\mu_0,\mu_1)} E_\alpha.$$

This last connection with atomic measures suggests the following question, somehow in the spirit of Xia's relaxation procedure (see [17]): if one considers the functional defined as $\mathfrak{L}_\alpha$ on those evolution pairings (ν,μ) with μ_1 which is finitely atomic and $+\infty$ on the other evolution pairings, what is its relaxation $\mathfrak{L}^*_\alpha$? Is the relaxed functional related to $\mathfrak{L}_\alpha$ on $SEP(\mu_0,\mu_1)$?

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