VISCO-ENERGETIC SOLUTIONS TO SOME RATE-INDEPENDENT SYSTEMS IN DAMAGE, DELAMINATION, AND PLASTICITY

RICCARDA ROSSI

Abstract. This paper revolves around a newly introduced weak solvability concept for rate-independent systems, alternative to the notions of *Energetic* (E) and *Balanced Viscosity* (BV) solutions. *Visco-Energetic* (VE) solutions have been recently obtained by passing to the time-continuous limit in a time-incremental scheme, akin to that for E solutions, but perturbed by a 'viscous' correction term, as in the case of BV solutions. However, for Visco-Energetic solutions this viscous correction is tuned by a fixed parameter. The resulting solution notion turns out to describe a kind of evolution in between Energetic and Balanced Viscosity evolution.

In this paper we aim to investigate the application of VE solutions to nonsmooth rate-independent processes in solid mechanics such as damage and plasticity at finite strains. We also address the limit passage, in the VE formulation, from an adhesive contact to a brittle delamination system. The analysis of these applications reveals the wide applicability of this solution concept, in particular to processes for which BV solutions are not available, and confirms its intermediate character between the E and BV notions.

Keywords: Rate-independent systems, Visco-Energetic solutions, damage, delamination, finite-strain plasticity.

1. Introduction

In this paper we explore the application of the newly introduced concept of *Visco-Energetic* solution to a rate-independent process. We address rate-independent systems in solid mechanics that can be described in terms of two variables $(u, z) \in U \times Z$. Typically, u is the displacement, or the deformation of the body, whereas z is an internal variable specific of the phenomenon under investigation, in accordance with the theory of *generalized standard materials* by Halphen & Nguyen [HN75], cf. also the modeling approach by M. Frémond [Fré02]. In the class of systems we consider here, u is governed by a *static* balance law (usually the Euler-Lagrange equation for the minimization of the elastic energy), whereas z evolves rate-independently. Indeed, when the ambient spaces U and Z have a Banach structure, the equations of interest

$$\mathrm{D}_u \mathcal{E}(t, u(t), z(t)) = 0 \quad \text{in } U^*, \ t \in (0, T), \tag{1.1a}$$

$$\partial \mathcal{R}(\dot{z}(t)) + \mathrm{D}_z \mathcal{E}(t, u(t), z(t)) \ni 0 \quad \text{in } Z^*, \ t \in (0, T), \tag{1.1b}$$

feature the derivatives w.r.t. u and z of the driving energy functional $\mathcal{E} : [0, T] \times U \times Z \to (-\infty, \infty]$, and the (convex analysis) subdifferential $\partial \mathcal{R} : Z \rightrightarrows Z^*$ of a convex, 1-positively homogeneous dissipation potential $\mathcal{R} : Z \to [0, \infty]$. System (1.1) reflects the ansatz that energy is dissipated through changes of the internal variable z only: in particular, the doubly nonlinear evolution inclusion (1.1b) balances the dissipative frictional forces from $\partial \mathcal{R}(\dot{z})$ with the restoring force $\mathrm{D}_z \mathcal{E}(t, u, z)$.

System (1.1) is most often only formally written: the very first issue attached to its analysis is the quest of a proper weak solvability notion. In fact, the energy $\mathcal{E}(t, \cdot, \cdot)$ can be nonsmooth, for instance incorporating indicator terms to ensure suitable physical constraints on the internal variable z such as, e.g., that the values of z range in a suitable interval. However, it is rate-independence that poses the most significant challenges. Since the dissipation potential $\mathcal{R}$ has linear growth at infinity, one can in general expect only BV-time regularity for z. Thus z may have jumps as a function of time and the pointwise derivative $\dot{z}$ in the subdifferential inclusion

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(1.1b) need not be defined. This has motivated the development of various weak solution concepts for system (1.1), suited to the poor time regularity of z and, at the same time, also able to properly capture the behavior of the system at jumps.

The latest of these notions, Visco-Energetic solutions, is the focus of this paper. We will illustrate it also through a detailed comparison with two other solution concepts, *Energetic* and *Balanced Viscosity* solutions. We refer to [Mie11, MR15] for a thorough survey of all the other weak solvability concepts advanced for rate-independent systems.

From now on, we will leave the Banach setting and simply assume that

- The state spaces U and Z are endowed with two topologies σ_U and σ_Z;
- Dissipative mechanisms are mathematically modeled in terms of a *dissipation distance* d_Z on Z (in fact, throughout the paper *extended, asymmetric quasi-distances* will be considered, cf. the general setup introduced in Sec. 2);
- The driving energy $\mathcal{E}(t,\cdot)$ is a $(\sigma_U{\times}\sigma_Z)$-lower semicontinuous functional.

We will often write X in place of $U\times Z$ and refer to the triple $(X,\mathcal{E},\mathsf{d}_Z)$ as a rate-independent system. On the one hand, this generalized setup comprises the Banach one of (1.1), where $\mathsf{d}_Z(z,z')=\mathcal{R}(z'{-}z)$. On the other hand, the above metric-topological setting is natural in view of the application to, e.g., finite-strain plasticity, where dissipation is described in terms of a Finsler-type distance reflecting the geometric nonlinearities of the model. The generality of this setup could also be instrumental for possible applications, still to be developed, to brittle fracture, where the state space for the crack variable only has a topological structure. Finally, working in an *abstract* setting allows us to highlight the main ideas underlying the VE approach, and to provide guidelines for its application to concrete rate-independent processes, like those addressed in this paper.

1.1. **Energetic, Balanced Viscosity, and Visco-Energetic solutions at a glance.** *Energetic* (often abbreviated as E) solutions were advanced in the late '90s by the parallel work on rate-independent systems, from different perspectives, of two groups. In particular, we refer to [MT99, MTL02, MT04] for the introduction of the Energetic concept for *abstract* rate-independent systems, and in the context of phase transformations in solids. In the realm of crack propagation, an analogous notion of evolution was pioneered in [FM98] and later further developed in [DMT02] with the concept of 'quasistatic evolution'. In the context of the abstract rate-independent system $(X,\mathcal{E},\mathsf{d}_Z)$, E solutions can be constructed by recursively solving the time-incremental minimization scheme

$$(u_\tau^n, z_\tau^n) \in \mathrm{Argmin}_{(u,z)\in X}\left(\mathsf{d}_Z(z_\tau^{n-1},z)+\mathcal{E}(t_\tau^n,u,z)\right), \quad n=1,\ldots,N_\tau\,, \tag{IM$_\mathrm{E}$}$$

where $\{t_\tau^n\}_{n=0}^{N_\tau}$ is a partition of $[0,T]$ with fineness $\tau=\max_{n=1,\ldots,N_\tau}(t_\tau^n{-}t_\tau^{n-1})$. Under suitable conditions on $\mathcal{E}$, the piecewise constant interpolants $(\overline{Z}_\tau)_\tau$ of the discrete solutions $(z_\tau^n)_{n=1}^{N_\tau}$ converge as $\tau\downarrow 0$ to an E solution of the rate-independent system $(X,\mathcal{E},\mathsf{d}_Z)$, namely a curve $z\in \mathrm{BV}_{\mathsf{d}_Z}([0,T];Z)$, together with

$$u:[0,T]\to U, \text{ an (everywhere defined) measurable selection } u(t)\in \mathrm{Argmin}_{\tilde u\in U}\mathcal{E}(t,\tilde u,z(t)), \tag{1.2}$$

fulfilling

- the *global* stability condition

$$\mathcal{E}(t,u(t),z(t))\le \mathcal{E}(t,u',z')+\mathsf{d}_Z(z(t),z') \quad \text{for all } (u',z')\in U\times Z \text{ and all } t\in[0,T]; \tag{S}$$

- the 'E energy-dissipation' balance

$$\mathcal{E}(t,u(t),z(t))+\mathrm{Var}_{\mathsf{d}_Z}(z,[0,t])=\mathcal{E}(0,u(0),z(0))+\int_0^t\partial_t\mathcal{E}(s,u(s),z(s))\,\mathrm{d}s \qquad \text{for all } t\in[0,T]. \tag{E}$$

Due to its flexibility, the Energetic concept has been successfully applied to a wide scope of problems, see e.g. [MR15] for a survey. However, it has been observed that, because of compliance with the *global* stability condition (S), E solutions driven by nonconvex energy functionals may have to jump 'too early' and 'too long', c.f., e.g., their characterization for 1-dimensional rate-independent systems obtained in [RS13], as well as the comparative analysis of the notions of solutions for crack propagation carried out in [Neg10].

This fact has motivated the introduction of an alternative weak solvability concept, pioneered in [EM06] and based on the vanishing-viscosity regularization of the rate-independent system as a selection criterion for mechanically feasible weak solutions. The vanishing-viscosity approach has in fact proved to be a robust method in manifold applications, for instance ranging from plasticity (cf., e.g., [DMDMM08, DDS11, BFM12]), to fracture [TZ09, KMZ08, LT11], and to damage [KRZ13, CL16] models. Approximating a rate-independent system by vanishing viscosity 'morally' corresponds to bringing *local stability* and, at the time-discrete level, *local minimization* into the picture. In the frame of the rate-independent system $(X, \mathcal{E}, \mathsf{d}_Z)$, this locality is achieved by perturbing the time-incremental scheme ($\mathrm{IM_E}$) by a term that penalizes the squared distance from the previous step z_τ^{n-1}, namely

$$(u_\tau^n, z_\tau^n) \in \mathrm{Argmin}_{(u,z)\in X} \left(\mathsf{d}_Z(z_\tau^{n-1}, z) + \frac{\varepsilon}{2\tau} \tilde{\mathsf{d}}_Z^2(z_\tau^{n-1}, z) + \mathcal{E}(t_\tau^n, u, z) \right) \quad \text{for } n = 1, \ldots, N_\tau \,. \tag{$\mathrm{IM_{BV}}$}$$

Here, the *viscous correction* $\tilde{\mathsf{d}}_Z^2(z_\tau^{n-1}, z)$, with $\tilde{\mathsf{d}}_Z$ a second, possibly different distance on Z, is modulated by a parameter ε, vanishing to zero with τ in such a way that $\frac{\varepsilon}{\tau} \uparrow \infty$. Under appropriate conditions on $\mathcal{E}$ (cf. e.g. [MRS16]), the approximate solutions $(\overline{Z}_\tau)_\tau$ originating from ($\mathrm{IM_{BV}}$) converge as $\tau \downarrow 0$ to the same type of solution that can be obtained via the vanishing-viscosity approach at the time-continuous level, namely a *Balanced Viscosity* (abbreviated as BV) solution of the rate-independent system $(X, \mathcal{E}, \mathsf{d}_Z)$. The latter is a curve $z \in \mathrm{BV}_{\mathsf{d}_Z}([0,T]; Z)$, with $u : [0,T] \to U$ as in (1.2), fulfilling

- the *local* stability condition

$$|\mathrm{D}_z \mathcal{E}|(t, u(t), z(t)) \leq 1 \quad \text{for every } t \in [0,T] \setminus \mathrm{J}_z, \tag{$\mathrm{S_{BV}}$}$$

  where $|\mathrm{D}_z\mathcal{E}|$ is the metric slope of $\mathcal{E}$ w.r.t. z, i.e. $|\mathrm{D}_z\mathcal{E}|(t,u,z) := \limsup_{w\to z} \frac{(\mathcal{E}(t,u,z)-\mathcal{E}(t,u,w))^+}{\mathsf{d}_Z(z,w)}$ (cf. [AGS08]), and J_z the set of jump points of z;
- the 'BV energy-dissipation' balance for all $t \in [0,T]$

$$\mathcal{E}(t, u(t), z(t)) + \mathrm{Var}_{\mathsf{d}_Z, \mathsf{v}}(z, [0,t]) = \mathcal{E}(0, u(0), z(0)) + \int_0^t \partial_t \mathcal{E}(s, u(s), z(s)) \, \mathrm{d}s \,. \tag{$\mathrm{E_{BV}}$}$$

In ($\mathrm{E_{BV}}$), $\mathrm{Var}_{\mathsf{d}_Z,\mathsf{v}}$ is an *augmented notion* of total variation, fulfilling $\mathrm{Var}_{\mathsf{d}_Z,\mathsf{v}} \geq \mathrm{Var}_{\mathsf{d}_Z}$ and measuring the energy dissipated at a jump point $t \in \mathrm{J}_u$ in terms of a Finsler-type cost $\mathsf{v}(t,\cdot,\cdot)$. While referring to [MRS16] for all details, we mention here that $\mathsf{v}(t,\cdot,\cdot)$ records the possible onset of *viscosity*, hence of rate-dependence, into the description of the system behavior at the jump point t. Because of the *local* character of the stability condition ($\mathrm{S_{BV}}$), BV solutions driven by nonconvex energies have mechanically feasible jumps, as shown by their characterization in [RS13].

Nonetheless, a crucial requirement underlying the Balanced Viscosity concept is that the energy $\mathcal{E}$, as a function of the internal variable z, complies with a chain-rule type condition, which is at the core of the energy-dissipation balance. Such chain rule is ultimately related to convexity/regularity properties of $\mathcal{E}$, and unavoidably restricts the range of applicability of BV solutions. In this respect, we may for instance mention that, for fracture the vanishing-viscosity approach has been carried out under suitable geometric restrictions on the evolving crack, cf. [LT11]; the application of the BV concept to highly nonlinear processes such as plasticity at finite strains has yet to be understood.

That is why, *Visco-Energetic* (VE) solutions have been advanced in [MS18] as a yet alternative solvability concept for the rate-independent system $(X, \mathcal{E}, \mathsf{d}_Z)$. The key idea at the core of this novel notion is to broaden the class of admissible viscous corrections of the original time-incremental scheme ($\mathrm{IM_E}$). The quadratic perturbation $\frac{\varepsilon}{2\tau}\tilde{\mathsf{d}}_Z^2(z_\tau^{n-1}, z)$ in scheme ($\mathrm{IM_{BV}}$) is in fact replaced by the term $\delta_Z(z_\tau^{n-1}, z)$, with $\delta_Z : Z \times Z \to [0,\infty]$ a general lower semicontinuous functional. This turns ($\mathrm{IM_E}$) into

$$(u_\tau^n, z_\tau^n) \in \mathrm{Argmin}_{(u,z)\in X} \left(\mathcal{E}(t_\tau^n, u, z) + \mathsf{d}_Z(z_\tau^{n-1}, z) + \delta_Z(z_\tau^{n-1}, z) \right), \ n = 1, \ldots, N_\tau \,. \tag{$\mathrm{IM_{VE}}$}$$

For simplicity, we shall confine the exposition in this Introduction to the simpler, but still significant, case in which $\delta_Z(z,z') = \frac{\mu}{2}\tilde{\mathsf{d}}_Z^2(z,z')$ with $\mu > 0$ a *fixed* parameter and $\tilde{\mathsf{d}}_Z$ a (possibly different) distance on

Z, postponing the discussion of the general case to Sec. 2. This choice gives rise to the time-incremental minimization scheme

$$(u_\tau^n, z_\tau^n) \in \mathrm{Argmin}_{(u,z)\in X} \left(\mathcal{E}(t_\tau^n, u, z) + \mathsf{d}_Z(z_\tau^{n-1}, z) + \frac{\mu}{2}\tilde{\mathsf{d}}_Z^2(z_\tau^{n-1}, z) \right), \ n = 1, \ldots, N_\tau, \ \mu > 0 \text{ fixed.} \tag{1.3}$$

In [MS18, Thm. 3.9] it has been shown that, under suitable conditions (cf. Sec. 2 ahead), the discrete solutions $(\overline{Z}_\tau)_\tau$ of (1.3) converge, as $\tau \downarrow 0$, to a VE solution of $(X, \mathcal{E}, \mathsf{d}_Z)$, i.e. a curve $z \in \mathrm{BV}_{\mathsf{d}_Z}([0,T]; Z)$, together with $u : [0,T] \to U$ as in (1.2), fulfilling

- the viscously perturbed stability condition

$$\mathcal{E}(t, u(t), z(t)) \leq \mathcal{E}(t, u', z') + \mathsf{d}_Z(z, z') + \frac{\mu}{2}\tilde{\mathsf{d}}_Z^2(z(t), z') \quad \text{for all } (u', z') \in U \times Z \text{ and all } t \in [0,T] \setminus \mathrm{J}_z; \tag{S$_{\mathrm{VE}}$}$$

- the 'VE-energy-dissipation' balance for all $t \in [0,T]$

$$\mathcal{E}(t, u(t)) + \mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}(u, [0,t]) = \mathcal{E}(0, u(0)) + \int_0^t \partial_t \mathcal{E}(s, u(s))\,\mathrm{d}s\,. \tag{E$_{\mathrm{VE}}$}$$

In (E$_{\mathrm{VE}}$), dissipation of energy is described by the total variation functional $\mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}$, which differs from the 'BV total variation' $\mathrm{Var}_{\mathsf{d}_Z,\mathsf{v}}$ in the contributions at jump points. In the VE-concept, the energy dissipated at jumps is in fact 'measured' in terms of a new cost function c, obtained by minimizing a suitable transition cost along curves connecting the two end-point $z(t-)$ and $z(t+)$ of the curve z at $t \in \mathrm{J}_z$, namely

$$\mathsf{c}(t, z(t-), z(t+)) := \inf \left\{ \mathrm{Trc}_{\mathrm{VE}}(t; \vartheta, E) \ : \ E \Subset \mathbb{R}, \vartheta \in \mathrm{C}_{\sigma_Z, \mathsf{d}_Z}(E; Z), \ \vartheta(\inf E) = z(t-), \ \vartheta(\sup E) = z(t+) \right\}. \tag{1.4}$$

The transition cost

$$\mathrm{Trc}_{\mathrm{VE}}(t; \vartheta, E) := \mathrm{Var}_{\mathsf{d}_Z}(\vartheta, E) + \mathrm{GapVar}_{\delta_Z}(\vartheta, E) + \sum_{s \in E \setminus \{\sup E\}} \mathscr{R}(t, \vartheta(s))$$

features *(i)* the d_Z-total variation of the curve ϑ; *(ii)* a quantity related to the 'gaps', or 'holes', of the set E (which is just an arbitrary compact subset of $\mathbb{R}$ and may have a more complicated structure than an interval, cf. Sec. 2.2 ahead); *(iii)* the residual function $\mathscr{R} : [0,T] \times Z \to [0,\infty)$ (defined in (2.19) ahead), which records the violation of the VE-stability condition, as it fulfills

$$\mathscr{R}(t, z) > 0 \text{ if and only if (S}_{\mathrm{VE}}\text{) does not hold.}$$

Visco-Energetic solutions are *in between* Energetic and Balanced Viscosity solutions in several respects:

(1) The structure of the solution concept: On the one hand, the stability condition (S$_{\mathrm{VE}}$), though perturbed by a viscous correction, still retains a *global* character, like for E solutions. This globality plays a key role in the argument for proving convergence of the discrete solutions of (1.3) to a VE-solution. Indeed, as shown in [MS18], once (S$_{\mathrm{VE}}$) is established for the time-continuous limit, it is sufficient to check the upper estimate $\leq$ to conclude (E$_{\mathrm{VE}}$) with an equality sign. In particular, no chain rule for $\mathcal{E}$ is needed for the energy balance. On the other hand, VE solutions provide a description of the system behavior at jumps comparable to that of BV solutions. Indeed, optimal jump transitions (i.e., transitions between the two end-points of a jump attaining the inf in (1.4)), exist at every jump point. Moreover, they turn out to solve a minimum problem akin to the time-incremental minimization scheme (IM$_{\mathrm{VE}}$), cf. (2.46) ahead. Similarly, optimal jump transitions for BV solutions solve a (possibly rate-dependent) evolutionary problem related to the scheme (IM$_{\mathrm{BV}}$) they originate from.

(2) Their characterization for 1-dimensional rate-independent systems: [Min17] has addressed the characterization of VE solutions, confining the discussion to a 1-dimensional rate-independent system; no similar result is yet known beyond that setting. Therein, it was shown that VE solutions originating from scheme (1.3) where, in addition, $\tilde{\mathsf{d}}_Z = \mathsf{d}_Z$, have a behavior strongly dependent on the parameter $\mu > 0$. If μ is above a certain threshold related to the driving energy $\mathcal{E}$, VE solutions exhibit a behavior at jumps akin to that of BV solutions, cf. [Min17]. With a 'small' μ, the behavior is intermediate between E and BV solutions.

(3) The singular limits $\mu \downarrow 0$ and $\mu \uparrow \infty$: in [RS17], in a general metric-topological setting but, again, with the special viscous correction $\delta_Z = \frac{\mu}{2}\mathsf{d}_Z^2$, VE solutions have been shown to converge to E and BV solutions as $\mu \downarrow 0$ and $\mu \uparrow \infty$, respectively.

(4) The assumptions for the existence theory: Loosely speaking, they turn out to be weaker than for BV solutions, and stronger than for E solutions. Therefore, the range of applicability of VE solutions to rate-independent processes is *intermediate* between the E and the BV concepts.

1.2. **Our results.** In this paper we are going to demonstrate the above features of VE solutions by addressing their application to carefully chosen rate-independent processes in solid mechanics. In particular,

- their in-between character concerning the assumptions for existence will be evident;
- the application to models for finite-strain plasticity and brittle delamination will show that the VE concept can be successfully implemented even in cases where BV solutions are not available;
- the analysis of the limit passage from adhesive contact to brittle delamination will provide a first instance of *Evolutionary Gamma-Convergence* for VE solutions, which in this respect seem to share the flexibility of the E concept.

Because of the wide range of applicability and of the flexibility of VE solutions, we believe that this novel solvability concept deserves to be further studied. In particular, the mechanical feasibility of its description of the system behavior at jumps needs to be assessed in well chosen examples. Let us now enter into the details of the applications developed in this paper.

In the case of the damage system studied in Section 4, the existence theory for E solutions [MR06, TM10, Tho13] (cf. also [FG06, BCGS16]) and for BV solutions [KRZ13, Neg17] seems to be well established. With **Theorem 4.1** ahead we will prove the existence of VE solutions by applying the existence result [MS18, Thm. 3.9] to a quite general damage system. Our assumptions on the constitutive functions of the model and on the problem data will *(i)* coincide with the conditions for E solutions in the case of the viscous correction $\delta_Z = \frac{\mu}{2}\mathsf{d}_Z^2$; *(ii)* turn out to be slightly stronger than those for E solutions (in particular forcing a stronger gradient regularization for the damage variable), in the case of a 'nontrivial' viscous correction δ_Z involving a distance *different* from the dissipation distance d_Z; *(iii)* be definitely weaker than those for BV solutions, cf. also Remark 4.3 ahead.

The system for rate-independent elastoplasticity at finite strains we are going to address in Section 5 has been analyzed from the viewpoint of Energetic solutions in [MM09], whereas no result on the existence of BV solutions seems to be available up to now. In fact, the corresponding, viscously regularized system has been only recently tackled in [MRS18], where an existence result has been obtained after considerable regularization of the driving energy functional to ensure the validity of the chain rule. Therefore, the application of the theory of BV solutions is seemingly possible only by means of such regularization. In contrast, we will see that, in the case of the 'trivial' viscous correction $\delta_Z = \frac{\mu}{2}\mathsf{d}_Z^2$, the existence of VE solutions to the rate-independent finite-strain plasticity system can be checked again under the same conditions as for E solutions. In turn, the 'nontrivial' case will require stronger assumptions, cf. **Theorem 5.1** and Remark 5.3 ahead.

Finally, in Sec. 6 we will tackle the application of VE solutions to a rate-independent system for *brittle* delamination, which can be thought of as a model for fracture on a prescribed surface. Due to the highly nonconvex and nonsmooth character of the underlying energy functional, the existence results for BV solutions are not applicable. It is then significant to resort to the VE concept but, in this case, the results from [MS18] do not directly apply either. In fact, in **Theorem 6.1** the existence of VE solutions will be proved by passing to the limit in an approximating system that models adhesive contact, as a certain penalization parameter tends to infinity. Our proof will rely on a careful asymptotic analysis of optimal jump transitions in the adhesive-to-brittle limit passage. In fact, limit passages in gradient systems driven by Γ-converging energy functionals and dissipation potentials have recently led to a considerable body of research, both in the gradient flow and in the rate-independent case, cf. e.g. [MRS08, SS04, Ste08, Ser11, Vis13, Bra14]. A unifying concept for such procedures goes under the name of *Evolutionary* Γ*-convergence*, see [Mie16]. In this concern, Thm. 6.1 can be understood as a first result of Evolutionary Gamma convergence for VE solutions. In a future paper, we plan to address this issue in a more systematic and comprehensive way.

Plan of the paper. In Section 2 we shall revisit the theory of VE solutions from [MS18] and slightly adapt it to processes described in terms of two variables (u, z) (while [MS18] mostly focused on rate-independent systems in the single variable z). In Section 3 we are going to test VE solutions on the benchmark example of the Prandtl-Reuss system for associative elastoplasticity, showing that, in that special context they indeed coincide with the E solutions investigated in[DMDM06] . Sections 4 & 5 will be centered on the applications to damage and finite-strain plasticity, respectively. Finally, the limit passage in the VE formulation from adhesive contact to brittle delamination will be addressed in Section 6.

Notation 1.1. Throughout the paper, we shall use the symbols c, c', C, C', etc., whose meaning may vary even within the same line, to denote various positive constants depending only on known quantities.

Given a topological space $\mathbf{X}$, we will *(i)* denote by $\mathrm{B}([0,T];\mathbf{X})$ the space of *everywhere* defined and measurable functions $v:[0,T]\to\mathbf{X}$; *(ii)* if $(\mathbf{X},\mathbf{d})$ is a metric space, denote by $\mathrm{BV}_{\mathbf{d}}([0,T];\mathbf{X})$ the space of *everywhere* defined functions $v:[0,T]\to\mathbf{X}$ with bounded variation.

Finally, if $\mathbf{X}$ is also a normed space, the symbol $\overline{B}_r^{\mathbf{X}}$ will denote the closed ball of $\mathbf{X}$ of radius $r>0$, centered at 0. We will frequently omit the symbol $\mathbf{X}$ to avoid overburdening notation. For the same reason, we will often write $\|\cdot\|_{\mathbf{X}}$ in place of $\|\cdot\|_{\mathbf{X}^d}$, and, in place of ${}_{\mathbf{X}^*}\langle\cdot,\cdot\rangle_{\mathbf{X}}$, we shall write $\langle\cdot,\cdot\rangle_{\mathbf{X}}$ (or even $\langle\cdot,\cdot\rangle$ when the duality pairing is clear from the context or has to be specified later) .

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2. Setup, definition, and existence result for Visco-Energetic solutions

In this section we recapitulate the basic assumptions and definitions underlying the notion of Visco-Energetic solutions. We draw all concepts from [MS18]. There, however, the focus was on energies depending on the sole dissipative variable z (which was in fact denoted as u in [MS18]), and the case of functionals also depending on the variable at equilibrium u was recovered through a marginal procedure, cf. [MS18, Sec. 4]. Here we will partially revisit the presentation in [MS18] by directly working with energy functionals depending on the *two* variables (u, z).

2.1. The abstract setup for Visco-Energetic solutions.

In what follows we collect the assumptions on the metric-topological setup, on the energy functional, on the dissipation (quasi-)distance, and on the viscous correction, at the core of the existence theory for VE solutions.

2.1.1. The metric-topological setting.

Throughout the paper we will denote by σ the product topology on $X=U\times Z$ induced by the two topologies σ_U and σ_Z, and by $\sigma_{\mathbb{R}}$ the product topology on $[0,T]\times X$ induced by the Euclidean and the σ-topologies. We will often write $(u_n,z_n)\overset{\sigma}{\to}(u,z)$ as $n\to\infty$ to signify convergence w.r.t. σ-topology, and we will use an analogous notation for $\sigma_{\mathbb{R}}$-, σ_Z-, and σ_U-convergence.

The mechanism of energy dissipation will be described via an *extended*, possibly *asymmetric* quasi-distance

$$\mathsf{d}_Z : Z\times Z\to[0,\infty],\quad \text{l.s.c. on } Z\times Z, \text{ such that}$$

$$\begin{cases} \mathsf{d}_Z(z,z)=0 \text{ for all } z\in Z, \\ \mathsf{d}_Z(z_o,z)<\infty \text{ for some reference point } z_o\in Z \text{ and all } z\in Z, \\ \mathsf{d}_Z(z,w)\le\mathsf{d}_Z(z,\zeta)+\mathsf{d}_Z(\zeta,w) \quad \text{for all } z,\ \zeta,\ w\in Z. \end{cases} \tag{2.1}$$

We say that $W\subset Z$ is d_Z-bounded if $\sup_{w\in X}\mathsf{d}_Z(z_o,w)<\infty$, and that d_Z separates the points of W if

$$w,\ w'\in W,\quad \mathsf{d}_Z(w,w')=0 \quad\Rightarrow\quad w=w'.$$

Our first condition concerns this metric-topological setting:

Assumption $< T >$. We require that

$$\text{the topological spaces } (U,\sigma_U) \text{ and } (Z,\sigma_Z) \text{ are Hausdorff and satisfy the first axiom of countability,} \tag{2.2a}$$

$$(U,\sigma_U) \text{ is a Souslin space,} \tag{2.2b}$$

namely the image of a Polish (i.e. a separable completely metrizable) space under a continuous mapping. Furthermore, we impose that

$$\mathsf{d}_Z \text{ separates the points of } Z. \tag{2.2c}$$

Let us now recall from [MS18] the definition of (σ_Z, d_Z)-*regulated* function, encompassing a crucial property that the Visco-Energetic solution component z shall enjoy at jumps.

Definition 2.1. [MS18, Def. 2.3] *We call a curve $z : [0,T] \to Z$ (σ_Z, d_Z)-regulated if for every $t \in [0,T]$ there exist the left- and right-limits of z w.r.t. σ_Z-topology, i.e.*

$$z(t-) = \lim_{s\uparrow t} z(s) \quad \textit{in } (Z,\sigma_Z), \qquad z(t+) = \lim_{s\downarrow t} z(s) \quad \textit{in } (Z,\sigma_Z) \tag{2.3a}$$

(with the convention $z(0-) := z(0)$ and $z(T+) := z(T)$), also satisfying

$$\begin{aligned} &\lim_{s\uparrow t} \mathsf{d}_Z(z(s), z(t-)) = 0, && \lim_{s\downarrow t} \mathsf{d}_Z(z(t+), z(s)) = 0, \\ &\mathsf{d}_Z(z(t-), z(t)) = 0 \ \Rightarrow\ z(t-) = z(t), && \mathsf{d}_Z(z(t), z(t+)) = 0 \ \Rightarrow\ z(t) = z(t+). \end{aligned} \tag{2.3b}$$

We denote by $\mathrm{BV}_{\sigma_Z,\mathsf{d}_Z}([0,T];Z)$ the space of (σ_Z,d_Z)-regulated functions z with finite d_Z-total variation $\mathrm{Var}_{\mathsf{d}_Z}(z,[0,T])$, where we define, for a subset $E \subset [0,T]$,

$$\mathrm{Var}_{\mathsf{d}}(z,E) := \sup \left\{ \sum_{j=1}^{M} \mathsf{d}_Z(\vartheta_z(t_{j-1}), \vartheta_z(t_j)) \ :\ t_0 < t_1 < \ldots < t_M,\ \{t_j\}_{j=0}^M \in \mathfrak{P}_f(E) \right\} \tag{2.4}$$

with $\mathfrak{P}_f(E)$ the collection of all finite subsets of E.

If (Z, d_Z) is a complete metric space, every function $z \in \mathrm{BV}_{\mathsf{d}_Z}([0,T];Z)$ is (d_Z-)regulated, namely at every $t \in [0,T]$ there exist the left- and right-limits of u w.r.t. the metric d_Z. However, since in the present context we are not assuming completeness of (Z,d_Z), the concept of (σ_Z,d_Z)-regulated function turns out to be significant. Observe that, for every $z \in \mathrm{BV}_{\sigma_Z,\mathsf{d}_Z}([0,T];Z)$ the jump set

$$\mathrm{J}_z := \mathrm{J}_z^- \cup \mathrm{J}_z^+, \quad \text{with } \mathrm{J}_z^- := \{t \in [0,T] \ :\ z(t-) \neq z(t)\}, \qquad \mathrm{J}_z^+ := \{t \in [0,T] \ :\ z(t) \neq z(t+)\}, \tag{2.5}$$

coincides with the jump set of the real monotone function $V_z : [0,T] \to \mathbb{R},\ t \mapsto V_z(t) := \mathrm{Var}_{\mathsf{d}_Z}(z,[0,t])$. Therefore, J_z is at most countable.

Finally, as we will discuss at the beginning of Section 2.2, the u-component of a Visco-Energetic solution is in principle only an element in $\mathrm{B}([0,T];U)$ (cf. Notation 1.1). However, in qualified situations (cf. Lemma 2.10 ahead) u will additionally be a

$$\sigma_U\text{-regulated function, i.e. } \forall\, t \in [0,T] \quad \exists\, u(t-) = \lim_{s\uparrow t} u(s) \text{ in } (U,\sigma_U), \quad u(t+) = \lim_{s\downarrow t} u(s) \text{ in } (U,\sigma_U)\,. \tag{2.6}$$

2.1.2. **The energy functional.** We now recall the basic assumptions on the energy functional $\mathcal{E}$ proposed in [MS18]. In view of Proposition 3.1 ahead, differently from [MS18] we choose not to encompass lower semicontinuity and compactness requirements into a unique condition.

Assumption $< A >$**.** The rate-independent system $(X, \mathcal{E}, \mathsf{d}_Z)$ fulfills

$< A.1 >$**: Lower semicontinuity:** The proper domain $\mathrm{D}(\mathcal{E}(t,\cdot))$ does not depend on t, namely there exists $\mathrm{D} \subset X$ such that $\mathrm{D}(\mathcal{E}(t,\cdot)) \equiv \mathrm{D}$ for all $t \in [0,T]$. In what follows, we will use the notation

$$\mathrm{D}_u := \pi_1(\mathrm{D}), \qquad \mathrm{D}_z := \pi_2(\mathrm{D}) \tag{2.7}$$

with $\pi_1 : X \to U$ and $\pi_2 : X \to Z$ the projection operators. We require that

$$\begin{gathered}\text{there exists } F_0 \geq 0 \text{ such that the perturbed functional}\\ \mathcal{F} : [0,T] \times X \to (-\infty, \infty] \qquad \mathcal{F}(t,(u,z)) := \mathcal{E}(t,(u,z)) + \mathsf{d}_Z(z_o, z) + F_0\\ \text{fulfills } \mathcal{F}(t,(u,z)) \geq 0 \quad \text{for all } (t,(u,z)) \in [0,T] \times X\,,\end{gathered} \tag{2.8}$$

with z_o the reference point satisfying (2.1). In what follows, with slight abuse of notation we will write

$$\mathcal{E}(t,u,z) \text{ in place of } \mathcal{E}(t,(u,z)), \text{ and analogously for } \mathcal{F}.$$

We impose that $\mathcal{E}$ is σ-l.s.c. on the sublevels of $\mathcal{F}$.

$< A.2 >$**: Compactness:** The sublevels of $\mathcal{F}$ are $\sigma_{\mathbb{R}}$-sequentially compact in $[0,T] \times X$.

$< A.3 >$**: Power control:** The functional $t \mapsto \mathcal{E}(t,u,z)$ is differentiable for all (u,z), $\partial_t \mathcal{E} : (0,T) \times \mathrm{D} \to \mathbb{R}$ is sequentially upper semicontinuous on the sublevels of $\mathcal{F}$, and

$$\exists\, \Lambda_P,\, C_P > 0 \ \ \forall\, (t,u,z) \in (0,T) \times \mathrm{D} \ : \quad |\partial_t \mathcal{E}(t,u,z)| \leq \Lambda_P \left(\mathcal{F}(t,u,z) + C_P\right)\,. \tag{2.9}$$

Remark 2.2. A natural choice for the reference point z_o in (2.1) and (2.8) is the initial datum $z_0 \in \mathrm{D}_z$ for the rate-independent process. In fact, along the evolution there holds $\mathrm{Var}_{\mathsf{d}_Z}(z,[0,T]) < \infty$, cf. Remark 2.9 ahead, and therefore $\sup_{t\in[0,T]} \mathsf{d}_Z(z_0, z(t)) \leq C < \infty$. That is why, we may suppose without loss of generality that, for every $z \in \mathrm{D}_z$ there holds $\mathsf{d}_Z(z_0,z) < \infty$.

In [MS18] a more general version of the power-control condition was assumed, involving a generalized 'power functional' $\mathcal{P} : [0,T] \times \mathrm{D} \to \mathbb{R}$ satisfying

$$\limsup_{s \downarrow t} \frac{\mathcal{E}(s,u,z) - \mathcal{E}(t,u,z)}{s-t} \leq \mathcal{P}(t,u,z) \leq \liminf_{s \uparrow t} \frac{\mathcal{E}(t,u,z) - \mathcal{E}(s,u,z)}{t-s} \quad \text{for all } (t,u,z) \in [0,T] \times \mathrm{D}, \tag{2.10}$$

and in fact surrogating the partial time derivative $\partial_t \mathcal{E}$ whenever $\mathcal{E}$ is not differentiable w.r.t. t. This generalization was mainly motivated by the need to encompass in the theory *marginal energies*, i.e. functionals only depending on the dissipative variable z and obtained from energies depending on both variables (u,z) via minimization w.r.t. u. For simplicity, in this paper we shall not work with the power functional from (2.10).

Finally, we point out that (2.9) could be weakened by allowing for a (positive) function $\Lambda_P \in L^1(0,T)$, in place of a (positive) constant Λ_P.

A straightforward consequence of $< A.1 >$ & $< A.2 >$ is that

$$\inf_{u \in U} \mathcal{E}(t,u,z) \neq \emptyset \qquad \text{for all } (t,z) \in [0,T] \times \mathrm{D}_z. \tag{2.11}$$

In what follows, we will often work with the *reduced energy* functional

$$\mathcal{I} : [0,T] \times Z \to (-\infty, \infty] \qquad \mathcal{I}(t,z) := \begin{cases} \inf_{u\in U} \mathcal{E}(t,u,z) = \min_{u \in U} \mathcal{E}(t,u,z) & \text{if } (t,z) \in [0,T] \times \mathrm{D}_z, \\ \infty & \text{otherwise.} \end{cases} \tag{2.12}$$

Combining the power-control estimate in (2.9) with the Gronwall Lemma, we conclude that

$$\begin{gathered}\mathcal{F}(t,u,z) \leq \mathcal{F}(s,u,z) \exp\left(C_P |t-s|\right) \quad \text{for all } s,\, t \in [0,T] \text{ and all } (u,z) \in X, \qquad \text{whence}\\ \sup_{t \in [0,T]} \mathcal{F}(t,u,z) \leq \exp(C_P T) \mathcal{F}(0,u,z) \quad \text{for all } (u,z) \in X\,.\end{gathered} \tag{2.13}$$

That is why, in what follows we will direcly work with the functional

$$\mathcal{F}_0(u,z) := \mathcal{F}(0,u,z) \quad \text{for every } (u,z) \in X.$$

Finally, we highlight that the upper semicontinuity of $\partial_t\mathcal{E}$ required in $< A.3 >$ can be relaxed if d_Z enjoys an additional continuity property, stated in $< A.3' >$ below. Indeed, $< A.3' >$ can replace assumption $< A.3. >$.

$< A.3' >$**:** d_Z is left-continuous on the sublevels of $\mathcal{F}_0$, i.e. for all sequences $(u_n, z_n)_n \subset U \times Z$ s.t.

$$\mathcal{F}_0(u_n, z_n) \le C, \quad z_n \overset{\sigma_Z}{\to} z \quad \text{there holds} \quad \mathsf{d}_Z(z_n, \zeta) \to \mathsf{d}_Z(z, \zeta) \qquad \text{for all } \zeta \in Z, \tag{2.14a}$$

and the map $\partial_t\mathcal{E} : [0,T] \times X \to \mathbb{R}$ satisfies (2.9) and the *conditional* upper semicontinuity

$$(t_n, u_n, z_n) \overset{\sigma_\mathbb{R}}{\to} (t, u, z), \quad \mathcal{E}(t_n, u_n, z_n) \to \mathcal{E}(t, u, z) \;\Rightarrow\; \limsup_{n\to\infty} \partial_t\mathcal{E}(t_n, u_n, z_n) \le \partial_t\mathcal{E}(t, u, z). \tag{2.14b}$$

The condition that convergence of the energies implies convergence of the powers is often required for the analysis of rate-independent systems, cf. [MR15]. For later use, we recall here a result from [FM06], where this implication was proved in the case in which $\partial_t\mathcal{E}$ is uniformly continuous on sublevels of $\mathcal{E}$, namely

$$\begin{gathered} \forall\, C > 0 \text{ there exists a modulus of continuity } \omega_C : [0,T] \to [0,\infty) \text{ such that} \\ \forall\, (u,z) \in U \times Z \,:\, \mathcal{F}_0(u,z) \le C \;\Rightarrow\; |\partial_t\mathcal{E}(t_1,u,z) - \partial_t\mathcal{E}(t_2,u,z)| \le \omega_C(|t_1 - t_2|) \text{ for all } t_1,\, t_2 \in [0,T]\,. \end{gathered} \tag{2.15}$$

Proposition 2.3. [FM06, Prop. 3.3] *Assume* (2.15)*. Then, for every $t \in [0,T]$ the following implication holds*

$$\Big((u_n, z_n) \overset{\sigma}{\to} (u, z) \text{ in } X, \quad \mathcal{E}(t_n, u_n, z_n) \to \mathcal{E}(t, u, z)\Big) \implies \partial_t\mathcal{E}(t_n, u_n, z_n) \to \partial_t\mathcal{E}(t, u, z)\,. \tag{2.16}$$

2.1.3. **The viscous correction of the time-incremental scheme.** We consider

$$\text{a lower semicontinuous map } \delta_Z : Z \times Z \to [0,\infty] \quad \text{with } \delta_Z(z,z) = 0 \quad \text{for all } z \in Z\,.$$

We introduce the 'corrected' dissipation

$$\mathsf{D}_Z(z, z') := \mathsf{d}_Z(z, z') + \delta_Z(z, z')\,.$$

As already mentioned, the VE concept features a *global* stability condition in terms of the dissipation potential D. It is in fact useful to introduce a weaker, but still global, notion of stability, where we allow for a positive correction $Q \ge 0$ on the right-hand side of the stability inequality.

Definition 2.4. *Let $Q \ge 0$. We say that $(t,u,z) \in [0,T] \times X$ is* (D_Z, Q)-stable *if it satisfies*

$$\mathcal{E}(t,u,z) \le \mathcal{E}(t,u',z') + \mathsf{D}_Z(z,z') + Q \qquad \textit{for all } (u',z') \in X. \tag{2.17}$$

If $Q = 0$, we will simply say that (t,u,z) is D_Z-stable. We denote by $\mathscr{S}_{\mathsf{D}_Z}$ the collection of all the D_Z-stable points, and by $\mathscr{S}_{\mathsf{D}_Z}(t)$ its section at the process time $t \in [0,T]$.

In view of $< A.1 >$ & $< A.2 >$ (which guarantee (2.11)), the quasi-stability condition (2.17) is equivalent to

$$\mathcal{I}(t,z) \le \mathcal{I}(t,z') + \mathsf{D}_Z(z,z') + Q \quad \text{for all } z' \in Z \tag{2.18}$$

involving the reduced energy $\mathcal{I}$ from (2.12). That is why,

- Hereafter, we will often allow for the abuse of notation $(t,z) \in \mathscr{S}_{\mathsf{D}_Z}$ (and $z \in \mathscr{S}_{\mathsf{D}_Z}(t)$), in place of $(t,u,z) \in \mathscr{S}_{\mathsf{D}_Z}$;
- We introduce the *residual stability function* $\mathscr{R} : [0,T] \times Z \to \mathbb{R}$ directly in terms of the reduced energy $\mathcal{I}$, namely we define

$$\begin{gathered} \mathscr{R}(t,z) := \sup_{z' \in Z} \{\mathcal{I}(t,z) - \mathcal{I}(t,z') - \mathsf{D}_Z(z,z')\} = \mathcal{I}(t,z) - \mathscr{Y}(t,z) \quad \text{with} \\ \mathscr{Y}(t,z) = \inf_{z' \in Z} (\mathcal{I}(t,z') + \mathsf{D}_Z(z,z'))\,. \end{gathered} \tag{2.19}$$

Note that, as soon as the energy functional $\mathcal{E}$ complies with $< A.1 >$ and $< A.2 >$ (and we will suppose this hereafter), the inf in the definition of $\mathscr{Y}$ is attained, i.e.

$$M(t,z) := \mathrm{Argmin}_{z' \in Z} (\mathcal{I}(t,z') + \mathsf{D}_Z(z,z')) \ne \emptyset. \tag{2.20}$$

Observe that $\mathscr{R}$ in fact records the failure of the stability condition at a given point $(t,z) \in [0,T] \times Z$, since

$$\begin{aligned} &\mathscr{R}(t,z) \geq 0 \quad \text{for all } (t,z) \in [0,T]\times Z, \quad \text{with} \\ &\mathscr{R}(t,z) = 0 \quad \text{if and only if } (t,z) \in \mathscr{S}_{\mathsf{D}_Z}\,. \end{aligned} \tag{2.21}$$

Let us now specify the compatibility properties that *admissible* viscous corrections have to enjoy with respect to the driving distance d_Z.

$<B.1>$: **d_Z-compatibility:** For every $z,\, z',\, z'' \in Z$

$$\mathsf{d}_Z(z,z') = 0 \;\Rightarrow\; \delta_Z(z'',z') \leq \delta_Z(z'',z) \quad \text{and} \quad \delta_Z(z,z'') \leq \delta_Z(z',z'')\,. \tag{2.22}$$

$<B.2>$: **Left d_Z-continuity:** For every sequence $(u_n,z_n)_n$ and every $(u,z)\in X$ we have

$$\sup_n \mathcal{F}_0(u_n,z_n) < \infty, \quad z_n \overset{\sigma_Z}{\to} z, \quad \mathsf{d}_Z(z_n,z) \to 0 \;\Rightarrow\; \lim_{n\to\infty} \delta_Z(z_n,z) = 0. \tag{2.23}$$

$<B.3>$: **D_Z-stability yields local d_Z-stability:** for all $(t,u,z) \in \mathscr{S}_{\mathsf{D}_Z}$ and all $M>1$ there exist $\eta>0$ and a neighborhood $I_U \times I_Z$ of (u,z) such that

$$\begin{aligned} \mathcal{E}(s,u',z') \leq \mathcal{E}(s,u,z) + M\mathsf{d}_Z(z',z) \quad &\text{for all } (s,u',z') \in \mathscr{S}_{\mathsf{D}_Z} \\ &\text{with } s \in [t-\eta,t], \text{ for all } (u,z) \in I_U \times I_Z \text{ with } \mathsf{d}_Z(z',z) \leq \eta. \end{aligned} \tag{2.24}$$

Remark 2.5. As already observed in [MS18], (2.24) is in fact equivalent to the condition

$$\limsup_{(s,z')\overset{\simeq}{\to}(t,z)} \frac{\mathcal{I}(s,z') - \mathcal{I}(t,z)}{\mathsf{d}_Z(z',z)} \leq 1\,, \tag{2.25}$$

involving the reduced energy $\mathcal{I}$ from (2.18), where we have written $(s,z')\overset{\simeq}{\to}(t,z)$ as a place-holder for $(s \to t,\ z' \overset{\sigma_Z}{\to} z,\ \mathsf{d}_Z(z,z') \to 0,\ (s,z') \in \mathscr{S}_{\mathsf{D}_Z},\ s \leq t)$. In turn, a sufficient condition for (2.25) is

$$\limsup_{(s,z')\overset{\simeq}{\to}(t,z)} \frac{\delta_Z(z',z)}{\mathsf{d}_Z(z',z)} = 0 \qquad \text{for every } z \in \mathscr{S}_{\mathsf{D}_Z}(t) \text{ and all } t \in [0,T]. \tag{2.26}$$

In particular, any viscous correction of the form

$$\delta_Z(z,z') = h(\mathsf{d}_Z(z,z')) \qquad \text{with } h \in \mathrm{C}([0,\infty)) \text{ nondecreasing and fulfilling } \lim_{r\downarrow 0}\frac{h(r)}{r} = 0 \tag{2.27}$$

satisfies (2.26) and, in fact, the whole Assumption $<B>$.

Closedness of the (quasi-)stable set. Finally, we require

$<C>$: For every $Q \geq 0$ the (D_Z, Q)-quasistable sets have σ-closed intersections with the sublevels of $\mathcal{F}_0$.

It was proved in [MS18, Lemma 3.11] that $<C>$ holds if and only if a property akin to the *mutual recovery sequence* condition from [MRS08] holds, namely

$$\begin{aligned} &\text{for every sequence } (t_n,z_n)_n \subset [0,T]\times Z \text{ with } t_n \to t,\ z_n \overset{\sigma_Z}{\to} z,\ \sup_n \mathsf{d}_Z(z_o,z_n) < \infty \\ &\text{and } \lim_{n\to\infty} \mathcal{I}(t_n,z_n) = \mathcal{I}(t,z) + \eta,\ \eta \geq 0, \\ &\text{there exists } z' \in M(t,z) \text{ and a sequence } (z_n')_n \text{ such that} \\ &\liminf_{n\to\infty} \left(\mathcal{I}(t_n,z_n') + \mathsf{D}_Z(z_n,z_n')\right) \leq \mathcal{I}(t,z') + \mathsf{D}_Z(z,z') + \eta\,, \end{aligned} \tag{2.28}$$

(recall that $M(t,z)$ denotes the set of minimizers associated with the functional $\mathscr{Y}$ in (2.19)).

2.2. **Definition of Visco-Energetic solution.** As already mentioned in the Introduction, the concept of Visco-Energetic solution of the rate-independent system $(X, \mathcal{E}, \mathsf{d}_Z)$ (cf. Definition 2.8 ahead) consists of the D_Z-stability condition $(\mathrm{S_{VE}})$ combined with the energy-dissipation balance $(\mathrm{E_{VE}})$. In $(\mathrm{E_{VE}})$ the energy dissipated at jumps is measured in terms of a jump dissipation cost c that keeps track of the viscous correction δ_Z. This jump dissipation is obtained by minimizing a suitable transition cost over a class of continuous curves connecting the two end-points of a jump. In what follows,

(1) Firstly, we will specify what we mean by 'end-points of a jump' of a curve (u, z) enjoying the properties of a Visco-Energetic solution, viz.

$$z \in \mathrm{BV}_{\sigma_Z,\mathsf{d}_Z}([0,T]; Z) \quad \text{and } t \mapsto u(t) \text{ is a measurable selection in } \mathrm{Argmin}_{u\in U}\mathcal{E}(t, u, z(t)). \tag{2.29}$$

Namely, for a curve (u, z) as in (2.29), we will introduce *surrogate* left- and right-limits for u at a jump point $t \in \mathrm{J}_z$.

(2) Secondly, we will rigorously introduce the cost c.

1. Surrogate left- and right limits of u: given a curve (u, z) as in (2.29), we extend u in this way:

$$\text{at every } t \in \mathrm{J}_z \text{ we denote by } \begin{cases} u(t-) & \text{an element in } \mathrm{Argmin}_{u\in U}\mathcal{E}(t, u, z(t-))\,, \\ u(t+) & \text{an element in } \mathrm{Argmin}_{u\in U}\mathcal{E}(t, u, z(t+))\,, \end{cases} \tag{2.30}$$

with the convention that $u(t-) = u(t+) = u(t)$ if $t \notin \mathrm{J}_z$, such that the extended mapping, still denoted by u, continues to be measurable.

Observe that this definition is meaningful in view of (2.11). The notation $u(t-)$ and $u(t+)$ is used here in an extended sense, as the true left- and right-limits of u at t w.r.t. σ_U-topology need not exist. Nonetheless, in Lemma 2.10 ahead, we will provide some sufficient conditions, which can be verified for a reasonable class of examples, ensuring that, if (u, z) is a Visco-Energetic solution, then u is σ_U-regulated and, in that case, $u(t-)$ and $u(t+)$ defined by (2.30) are its left- and right-limits.

2. The Visco-Energetic cost c. It involves minimization of a suitable cost functional over a class of continuous curves, connecting the left- and right-limits $(u(t-), z(t-))$ and $(u(t+), z(t+))$ at a jump point $t \in \mathrm{J}_z$ (with $u(t-)$ and $u(t+)$ as in (2.30)). Such curves are in general defined on a compact subset $E \subset \mathbb{R}$ with a possibly more complicated structure than that of an interval. To describe it, we fix some notation:

$$E^- := \inf E, \quad E^+ := \sup E\,. \tag{2.31a}$$

We also introduce

$$\text{the collection } \mathfrak{h}(E) \text{ of the connected components of the set } [E^-, E^+] \setminus E. \tag{2.31b}$$

Since $[E^-, E^+] \setminus E$ is an open set, $\mathfrak{h}(E)$ consists of at most countably many open intervals, which we will often refer to as the 'holes' of E. We are now in a position to introduce the transition cost at the basis of the concept of Visco-Energetic solution, evaluated along curves $\vartheta = (\vartheta_u, \vartheta_z) \in \mathrm{B}(E; X)$ such that, in addition

$$\vartheta_z \in \mathrm{C}_{\sigma_Z,\mathsf{d}_Z}(E; Z) := \mathrm{C}_{\sigma_Z}(E; Z) \cap \mathrm{C}_{\mathsf{d}_Z}(E; Z). \tag{2.32}$$

Here, $\mathrm{C}_{\sigma_Z}(E; Z)$ is the space of functions from E to Z that are continuous with respect to the σ_Z-topology, while $\mathrm{C}_{\mathsf{d}_Z}(E; Z)$ is the space of functions $\vartheta_z : E \to Z$ satisfying the following continuity condition w.r.t. d_Z:

$$\forall\, \varepsilon > 0\ \exists\, \eta > 0\ \forall\, s_0,\ s_1 \in E \text{ with } s_0 \le s_1 \le s_0 + \eta : \qquad \mathsf{d}_Z(\vartheta_z(s_0), \vartheta_z(s_1)) \le \varepsilon\,.$$

Definition 2.6. *Let E be a compact subset of $\mathbb{R}$ and $\vartheta = (\vartheta_u, \vartheta_z) \in \mathrm{B}(E; U) \times \mathrm{C}_{\sigma_Z,\mathsf{d}_Z}(E; Z)$. For every $t \in [0, T]$ we define the* transition cost function

$$\mathrm{Trc}_{\mathrm{VE}}(t, \vartheta, E) := \mathrm{Var}_{\mathsf{d}_Z}(\vartheta_z, E) + \mathrm{GapVar}_{\delta_Z}(\vartheta_z, E) + \sum_{s\in E\setminus\{E^+\}} \mathscr{R}(t, \vartheta_z(s)) \qquad \textit{with} \tag{2.33}$$

(1) $\mathrm{Var}_{\mathsf{d}_Z}(\vartheta, E)$ *the d_Z-total variation of the curve ϑ, cf.* (2.4)*;*

(2) $\mathrm{GapVar}_{\delta_Z}(\vartheta, E) \colon = \sum_{I\in\mathfrak{h}(E)} \delta_Z(\vartheta_z(I^-), \vartheta_z(I^+))$*;*

(3) the (possibly infinite) sum

$$\sum_{s\in E\setminus\{E^+\}} \mathscr{R}(t,\vartheta_z(s)) := \begin{cases} \sup\{\sum_{s\in P} \mathscr{R}(t,\vartheta_z(s)) \ : \ P \in \mathfrak{P}_f(E\setminus\{E^+\})\} & \text{if } E\setminus\{E^+\} \neq \emptyset, \\ 0 & \text{otherwise}. \end{cases}$$

Along with [MS18], we observe that, for every fixed $t \in [0,T]$ and admissible ϑ, the transition cost fulfills the additivity property

$$\mathrm{Trc}_{\mathrm{VE}}(t,\vartheta,E\cap[a,c]) = \mathrm{Trc}_{\mathrm{VE}}(t,\vartheta,E\cap[a,b]) + \mathrm{Trc}_{\mathrm{VE}}(t,\vartheta,E\cap[b,c]) \quad \text{for all } a<b<c\,.$$

We are now in a position to define the *Visco-Energetic jump dissipation cost* $\mathsf{c} : [0,T]\times X\times X \to [0,\infty]$ between the two end-points of a jump of a curve (u,z) as in (2.29). Namely, we set

$$\begin{aligned} \mathsf{c}(t,(u_-,z_-),(u_+,z_+)) := \inf\{\mathrm{Trc}_{\mathrm{VE}}(t,\vartheta,E) \ : E \Subset \mathbb{R}, \ \vartheta = (\vartheta_u,\vartheta_z) \in \mathrm{B}(E;U)\times \mathrm{C}_{\sigma_Z,\mathsf{d}_Z}(E;Z), \\ \vartheta(E^-) = (u_-,z_-), \ \vartheta(E^+) = (u_+,z_+)\}. \end{aligned} \tag{2.34}$$

Remark 2.7. In fact, for every admissible transition curve $\vartheta = (\vartheta_u,\vartheta_z)$ between two pairs (u_-,z_-) and (u_+,z_+), all of the three contributions to the transition cost from (2.33) only depend on the ϑ_z-component. That is why, from now on with slight abuse of notation we will simply write

$$\mathsf{c}(t,z_-,z_+) \text{ in place of } \mathsf{c}(t,(u_-,z_-),(u_+,z_+))\,. \tag{2.35}$$

Accordingly, we will introduce the concept of *Optimal Jump Transition*, cf. (2.45) ahead, only in terms of the ϑ_z-component of an admissible transition curve $\vartheta = (\vartheta_u,\vartheta_z)$.

With the jump dissipation cost c we associate the *incremental cost* $\Delta_{\mathsf{c}} : [0,T]\times X\times X \to [0,\infty]$ defined at all $t \in [0,T]$ and (u_-,z_-), $(u_+,z_+) \in X$ by

$$\Delta_{\mathsf{c}}(t,(u_-,z_-),(u_+,z_+)) = \Delta_{\mathsf{c}}(t,z_-,z_+) := \mathsf{c}(t,z_-,z_+) - \mathsf{d}_Z(z_-,z_+) \tag{2.36}$$

(in fact, observe that $\mathsf{c}(t,z_-,z_+) \geq \mathsf{d}_Z(z_-,z_+)$, so that $\Delta_{\mathsf{c}}(t,z_-,z_+) \geq 0$, for all $t\in[0,T]$ and $z_\pm \in Z$). We will also use the notation

$$\Delta_{\mathsf{c}}(t,z_-,z,z_+) := \Delta_{\mathsf{c}}(t,z_-,z) + \Delta_{\mathsf{c}}(t,z,z_+)\,.$$

The *augmented total variation* functional induced by c is defined, along a curve $(u,z) \in \mathrm{BV}([0,T];X)$, by

$$\mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}((u,z),[t_0,t_1]) := \mathrm{Var}_{\mathsf{d}_Z}(z,[t_0,t_1]) + \mathrm{Jmp}_{\Delta_{\mathsf{c}}}((u,z);[t_0,t_1]) \quad \text{for any sub-interval } [t_0,t_1]\subset[0,T], \tag{2.37}$$

where the *incremental jump variation* of (u,z) on $[t_0,t_1]$ is given by

$$\begin{aligned} \mathrm{Jmp}_{\Delta_{\mathsf{c}}}((u,z);[t_0,t_1]) :=& \Delta_{\mathsf{c}}(t_0,z(t_0),z(t_0+)) + \Delta_{\mathsf{c}}(t_1,z(t_1-),z(t_1)) \\ &+ \sum_{t\in \mathrm{J}_u\cap(t_0,t_1)} \Delta_{\mathsf{c}}(t,z(t-),z(t),z(t+))\,. \end{aligned} \tag{2.38}$$

Ultimately, also this jump contribution only depends on the z-component, namely

$$\mathrm{Jmp}_{\Delta_{\mathsf{c}}}((u,z);[t_0,t_1]) = \mathrm{Jmp}_{\Delta_{\mathsf{c}}}(z;[t_0,t_1])\,.$$

Therefore, hereafter we shall write

$$\mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}(z,[t_0,t_1]) \quad \text{in place of} \quad \mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}((u,z),[t_0,t_1])\,.$$

As observed in [MS18], although it is not canonically induced by a distance, the total variation functional $\mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}$ still enjoys the additivity property

$$\mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}(z,[a,c]) = \mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}(z,[a,b]) + \mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}(z,[b,c]) \qquad \text{for all } 0\leq a\leq b\leq c\leq T.$$

We are now in a position to define the concept of Visco-Energetic solution (u,z) of the rate-independent system $(X,\mathcal{E},\mathsf{d}_Z)$, featuring the D_Z-stability condition, and the energy-dissipation balance with the total variation functional $\mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}$. Let us stress in advance that, since $\mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}} \geq \mathrm{Var}_{\mathsf{d}_Z}$ only controls the z-component of the curve (u,z), it will be for z only that we shall claim $z \in \mathrm{BV}_{\mathsf{d}_Z}([0,T];Z)$ (in fact, $z\in \mathrm{BV}_{\sigma_Z,\mathsf{d}_Z}([0,T];Z)$), while for the u component only measurability will be a priori asked for.

Definition 2.8 (Visco-Energetic solution)**.** *A curve* $(u,z) : [0,T] \to X$*, with* $u \in \mathrm{B}([0,T];U)$ *and* $z \in \mathrm{BV}_{\sigma_Z,\mathsf{d}_Z}([0,T];Z)$*, is a Visco-Energetic (*VE*) solution of the rate-independent system* $(X,\mathcal{E},\mathsf{d}_Z)$ *with the viscous correction* δ_Z*, if it satisfies*

- *the minimality condition*

$$u(t) \in \mathrm{Argmin}_{u\in U}\mathcal{E}(t,u,z(t)) \qquad \text{for all } t \in [0,T]; \tag{2.39}$$

- *the* D_Z*-stability condition*

$$\begin{aligned}\mathcal{E}(t,u(t),z(t)) &\le \mathcal{E}(t,u',z') + \mathsf{D}_Z(z(t),z') \\ &= \mathcal{E}(t,u',z') + \mathsf{d}_Z(z(t),z') + \delta_Z(z(t),z') \quad \text{for all } (u',z') \in X \text{ and all } t \in [0,T]\setminus \mathrm{J}_z,\end{aligned} \tag{S$_{\mathrm{VE}}$}$$

- *the* $(\mathsf{d}_Z,\mathsf{c})$*-energy-dissipation balance*

$$\mathcal{E}(t,u(t),z(t)) + \mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}(z,[0,t]) = \mathcal{E}(0,u(0),z(0)) + \int_0^t \partial_t\mathcal{E}(s,u(s),z(s))\,\mathrm{d}s \quad \text{for all } t\in[0,T]\,. \tag{E$_{\mathrm{VE}}$}$$

Remark 2.9. From the energy-dissipation balance, exploiting the power-control condition (2.9) to estimate the power term on the right-hand side of (E$_{\mathrm{VE}}$), we easily deduce that

$$\begin{cases}\sup_{t\in[0,T]}|\mathcal{E}(t,u(t),z(t))| \le \sup_{t\in[0,T]}\mathcal{F}(t,u(t),z(t)) \le C_0, \\ \mathrm{Var}_{\mathsf{d}_Z}(z,[0,T]) \le \mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}(z,[0,T]) \le C_0\end{cases} \tag{2.40}$$

for a constant $C_0 > 0$ only depending on $(u(0),z(0)) \in \mathrm{D}$.

Observe that the D_Z-stability condition, tested with $(u',z') = (u',z(t))$ and u' arbitrary in U, in particular ensures that $u(t) \in \mathrm{Argmin}_{u\in U}\mathcal{E}(t,u,z(t))$ for all $t \in [0,T]\setminus \mathrm{J}_z$. We want to claim this property at *all* $t\in[0,T]$, though. That is why, (2.39) is required, as a separate property, at all $t \in [0,T]$.

2.3. **Characterization, properties, and main existence result for Visco-Energetic solutions.** In the following statements we will implicitly assume that the rate-independent system $(X,\mathcal{E},\mathsf{d}_Z)$ satisfies conditions $< T >$, $< A >$, $< B >$, and $< C >$ from Sec. 2.1, and impose them explicitly only for Theorem 2.12.

Lemma 2.10. *Suppose that*

$$\mathrm{Argmin}_{u\in U}\mathcal{E}(t,u,z) \quad \text{is a singleton for every } (t,z) \in [0,T]\times \mathrm{D}_z. \tag{2.41}$$

Let (u,z) *be a* VE *solution to* $(X,\mathcal{E},\mathsf{d}_Z)$*. Then,* u *is* σ_U*-regulated, with left- and right-limits given by* (2.30)*.*

Proof. Let us fix $t \in [0,T)$. In order to show that the only element $u(t+)$ in $\mathrm{Argmin}_{u\in U}\mathcal{E}(t,u,z(t+))$ is the right-limit of u w.r.t. the σ_U-topology, it is sufficient to show that, for all $(s_n)_n \subset (0,T)$ with $s_n \downarrow t$, there holds $u(s_n) \to u(t+)$ in (U,σ_U). Since J_z is at most countable, we may suppose that $(s_n)_n \subset (0,T)\setminus \mathrm{J}_z$. It follows from (2.40) and $< A.2 >$ that there exists some $u^* \in U$ such that, up to a (not relabeled) subsequence, $u(s_n) \to u^*$ in (U,σ_U) as $n\to\infty$. Clearly, $z(s_n) \to z(t+)$ in (Z,σ_Z). By the closure of the stable set $\mathscr{S}_{\mathsf{D}_Z}$, we conclude that $(t,u^*,z(t+)) \in \mathscr{S}_{\mathsf{D}_Z}$. Then, $u^* \in \mathrm{Argmin}_{u\in U}\mathcal{E}(t,u,z(t+))$, which yields $u^* = u(t+)$.

The argument for the existence of the left-limit $u(t-)$ at all $t \in (0,T]$ is completely analogous. □

A characterization of VE solutions can be given either in terms of the sole upper estimate in the $(\mathsf{d}_Z,\mathsf{c})$-energy-dissipation balance (E$_{\mathrm{VE}}$), or in terms of the energy-dissipation upper estimate for E solutions, combined with jump conditions where the release of energy at a jump point is balanced by the VE jump cost c.

Proposition 2.11. [MS18, Prop. 3.8] *A curve* $(u,z) \in \mathrm{B}([0,T];U)\times \mathrm{BV}_{\sigma_Z,\mathsf{d}_Z}([0,T];Z)$ *satisfying the* D_Z*-stability condition* (S$_{\mathrm{VE}}$) *is a* VE *solution of the rate-independent system* $(X,\mathcal{E},\mathsf{d}_Z)$ *with the viscous correction* δ_Z *if and only if* z *satisfies, in addition,*

(1) either the $(\mathsf{d}_Z,\mathsf{c})$*-energy-dissipation upper estimate*

$$\mathcal{E}(T,u(T),z(T)) + \mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}(z,[0,T]) \le \mathcal{E}(0,u(0),z(0)) + \int_0^T \partial_t\mathcal{E}(s,u(s),z(s))\,\mathrm{d}s; \tag{2.42}$$

(2) or the d_Z-energy-dissipation upper estimate

$$\mathcal{E}(T,u(T),z(T)) + \mathrm{Var}_{\mathsf{d}_Z}(z,[0,T]) \leq \mathcal{E}(0,u(0),z(0)) + \int_0^T \partial_t\mathcal{E}(s,u(s),z(s))\,\mathrm{d}s, \tag{2.43}$$

joint with the following jump conditions at every jump point $t \in \mathrm{J}_z$:

$$\begin{aligned} \mathcal{E}(t,u(t-),z(t-)) - \mathcal{E}(t,u(t),z(t)) &= \mathsf{c}(t,z(t-),z(t)) \\ \mathcal{E}(t,u(t),z(t)) - \mathcal{E}(t,u(t+),z(t+)) &= \mathsf{c}(t,z(t),z(t+)) \\ \mathcal{E}(t,u(t-),z(t-)) - \mathcal{E}(t,u(t+),z(t+)) &= \mathsf{c}(t,z(t-),z(t+))\,. \end{aligned} \tag{2.44}$$

Let us now gain further insight into the description of the system behavior at jumps provided by the VE concept, via the properties of Optimal Jump Transitions. We recall that (cf. [MS18, Def. 3.13]), given $t \in [0,T]$ and z_-, $z+ \in Z$, an admissible transition curve $\vartheta_z \in \mathrm{C}_{\sigma_Z,\mathsf{d}_Z}(E;Z)$, with $E \Subset \mathbb{R}$, is an optimal transition between z_- and z_+ at time $t \in [0,T]$ if it is a minimizer for $\mathsf{c}(t,z_-,z_+)$, namely

$$\vartheta_z(E^-) = z_-, \quad \vartheta_z(E^+) = z_+, \quad \mathrm{Trc}_{\mathrm{VE}}(t,\vartheta_z,E) = \mathsf{c}(t,z_-,z_+)\,. \tag{2.45}$$

Furthermore, we say that ϑ_z is a

(1) *sliding transition*, if $\mathscr{R}(t,\vartheta_z(s)) = 0$ for all $s \in E$;
(2) *viscous transition*, if $\mathscr{R}(t,\vartheta_z(s)) > 0$ for all $s \in E \setminus \{E^-,E^+\}$.

It has been shown in [MS18, Rmk. 3.15, Cor. 3.17] that, for a viscous transition ϑ_z between z_- and z_+ the compact set $E \setminus \{E^-,E^+\}$ is discrete, i.e. all of its points are isolated: namely, ϑ_z is a *pure jump* transition. In fact, ϑ_z may be represented as a finite, or countable, sequence $(\vartheta_n^z)_{n\in O}$, with $O \subset \mathbb{Z}$ satisfying (recall the definition (2.20) of the set $M(t,z)$)

$$\vartheta_n^z \in M(t,\vartheta_{n-1}^z) = \mathrm{Argmin}_{z'\in Z}\left(\mathcal{I}(t,z') + \mathsf{D}_Z(\vartheta_{n-1}^z,z')\right) \quad \text{for all } n \in O \setminus \{O^-\}. \tag{2.46}$$

Furthermore, it has been proved in [MS18, Prop. 3.18] that any optimal jump transition can be canonically decomposed into an (at most) countable collection of sliding and viscous, pure jump transitions. Finally, it has been shown in [MS18, Thm. 3.14] that, at every jump point t of a VE solution z there exists an optimal jump transition ϑ_z between $z(t-)$ and $z(t+)$ such that $\vartheta_z(s) = z(t)$ for some $s \in E$.

We conclude this section by giving an existence result for VE solutions, proved in [MS18, Thm. 4.7]. For completeness, in the statement below we also encompass the convergence result (cf. [MS18, Thm. 7.2]) for the (left-continuous) piecewise constant interpolants

$$\overline{Z}_\tau : [0,T] \to U, \quad \overline{Z}_\tau(0) := z_0, \quad \overline{Z}_\tau(t) := z^n \quad \text{for } t \in (t_\tau^{n-1}, t_\tau^n], \quad n = 1,\ldots,N_\tau \tag{2.47}$$

associated with the discrete solutions $(z_\tau^n)_{n=1}^{N_\tau}$ of the time-incremental minimization problem (IM$_{\mathrm{VE}}$). We shall discuss the convergence of the interpolants $(\overline{U}_\tau)_\tau$ of the elements $(u_\tau^n)_{n=1}^{N_\tau}$, with u_τ^n minimizers for time-incremental minimization problem (IM$_{\mathrm{VE}}$), right after the statement of Thm. 2.12.

Theorem 2.12. [MS18, Thm. 4.7] *Under Assumptions $< T >$, $< A >$, $< B >$, and $< C >$, let $z_0 \in \mathrm{D}_z$. Then, for every sequence $(\tau_k)_k$ of time steps with $\tau_k \downarrow 0$ as $k \to \infty$ there exist a (not relabeled) subsequence $(\overline{Z}_{\tau_k})_k$ and $z \in \mathrm{BV}_{\sigma_Z,\mathsf{d}_Z}([0,T];Z)$ such that*

(1) $z(0) = z_0$, and

$$\overline{Z}_{\tau_k}(t) \overset{\sigma_Z}{\to} z(t) \qquad \text{in } Z \text{ for all } t \in [0,T]; \tag{2.48}$$

(2) there exists $u \in \mathrm{B}([0,T];U)$ such that (u,z) is a VE *solution to the rate-independent system $(X,\mathcal{E},\mathsf{d}_Z)$, with the viscous correction δ_Z.*

In fact, the curve u in the above statement is obtained as a measurable selection in $\mathrm{Argmin}_{u\in U}\mathcal{E}(t,u,z(t))$. It is not, in general, related to the limit of the piecewise constant interpolants $(\overline{U}_{\tau_k})_k$. However, if, in addition, property (2.41) holds, and the functional $\mathcal{E}$ fulfills the following condition

$$\begin{aligned}&\text{for all } (t_k)_k \subset [0,T] \text{ and } (z_k)_k \subset Z \text{ with } t_k \to t,\ z_k \overset{\sigma_Z}{\to} z \text{ in } Z, \text{ then}\\ &\text{for all } v \in U \text{ there exists } (v_k)_k \subset U \text{ such that } \limsup_{k\to\infty} \mathcal{E}(t_k, v_k, z_k) \le \mathcal{E}(t,v,z),\end{aligned} \tag{2.49}$$

then it is possible to prove convergence to the curve u.

Proposition 2.13. *Under the same hypotheses as Thm. 2.12, assume in addition conditions* (2.41) *and* (2.49)*. Then, the curve* $u \in \mathrm{B}([0,T];U)$ *from the statement of Thm. 2.12, such that* (u,z) *is a* VE *solution to the rate-independent system* $(X, \mathcal{E}, \mathsf{d}_Z)$*, is unique, and along the same sequence as in* (2.48) *there holds*

$$\overline{U}_{\tau_k}(t) \overset{\sigma_U}{\to} u(t) \qquad \text{in } U \text{ for all } t \in [0,T]. \tag{2.50}$$

Proof. To start with, we may observe that from ($\mathrm{IM_{VE}}$) it follows that

$$\overline{U}_{\tau_k}(t) \in \mathrm{Argmin}_{u\in U}\, \mathcal{E}(\overline{t}_{\tau_k}(t), u, \overline{Z}_{\tau_k}(t)) \qquad \text{for all } t \in (0,T] \tag{2.51}$$

(with $\overline{t}_{\tau_k}$ the left-continuous piecewise constant interpolant associated with the partition of $[0,T]$). From the energy bound $\mathcal{F}_0(\overline{U}_{\tau_k}(t), \overline{Z}_{\tau_k}(t)) \le C$ for a constant independent of $k \in \mathbb{N}$ and $t \in [0,T]$, cf. [MS18, Thm. 7.1], combined with Assumption $< A.2 >$, we infer that there exists a compact subset $\mathbf{U} \Subset U$ such that $\overline{U}_{\tau_k}(t) \in \mathbf{U}$ for all $t \in [0,T]$ and $k \in \mathbb{N}$. Then, for all $t \in [0,T]$ there exists $u_*(t) \in U$ such that, along a (not relabeled) subsequence possibly depending on t, there holds

$$\overline{U}_{\tau_k}(t) \overset{\sigma_U}{\to} u_*(t). \tag{2.52}$$

Combining (2.51) and (2.52) with (2.48) and taking into account the lower semicontinuity $< A.1 >$ we find that $\mathcal{E}(t, u_*(t), z(t)) \le \liminf_{k\to\infty} \mathcal{E}(\overline{t}_{\tau_k}(t), \overline{U}_{\tau_k}(t), \overline{Z}_{\tau_k}(t)) \le \liminf_{k\to\infty} \mathcal{E}(\overline{t}_{\tau_k}(t), v, \overline{Z}_{\tau_k}(t))$ for all $v \in U$ and all $t \in [0,T]$. Exploiting (2.49), we conclude that $u_*(t) \in \mathrm{Argmin}_{u\in U}\, \mathcal{E}(t, u(t), z(t))$. Since the latter set is a singleton by (2.41), convergence (2.52) holds along the *whole* sequence $(\tau_k)_k$, whence the desired (2.50). □

3. When Visco-Energetic solutions are Energetic

The following result characterizes the situation in which VE solutions turn out to be E solutions as well. Note that it holds under the sole conditions $< A.1 >$ and $< A.3 >$.

Proposition 3.1. *Assume* $< T >$, $< A.1 >$*, and* $< A.3 >$*. Then, a Visco-Energetic solution* (u,z) *of the rate-independent system* $(X, \mathcal{E}, \mathsf{d}_Z)$ *is an Energetic solution if and only if it satisfies the global stability condition* (S) *at every* $t \in [0,T] \setminus \mathrm{J}_z$ *and at* $t = 0$*. In that case, at every jump point* $t \in \mathrm{J}_z$ *the curves* (u,z) *fulfill the jump conditions*

$$\begin{aligned}\mathcal{E}(t, u(t-), z(t-)) - \mathcal{E}(t, u(t), z(t)) &= \mathsf{d}_Z(z(t-), z(t)),\\ \mathcal{E}(t, u(t), z(t)) - \mathcal{E}(t, u(t+), z(t+)) &= \mathsf{d}_Z(z(t), z(t+)).\end{aligned} \tag{3.1}$$

Proof. Clearly, if (u,z) is an E solution, then (S) holds at all $t \in [0,T]$.

Conversely, let (u,z) be a VE solution complying with (S) at every $t \in [0,T] \setminus \mathrm{J}_z$ and at $t = 0$. First of all, we show that (u,z) fulfills (S) at every $t \in \mathrm{J}_z$. Indeed, passing to the limit in (S) we may conclude that it holds for the left and right limits $(u(t-), z(t-))$ and $(u(t+), z(t+))$, namely

$$\mathcal{E}(t, u(t\pm), z(t\pm)) \le \mathcal{E}(t, u', z') + \mathcal{R}(z' - z(t\pm)) \quad \text{for all } (u', z') \in U \times Z \text{ and all } t \in \mathrm{J}_z\,. \tag{3.2}$$

We now recall that (u,z) fulfills the Energetic energy-dissipation upper estimate (2.43) on $[0,t]$, cf. Prop. 2.11. With the very same 'localization' argument as in the proof of [RS17, Thm. 1], from (2.43) we then deduce that

$$\mathcal{E}(t, u(t), z(t)) + \mathcal{R}(z(t) - z(t-)) \le \mathcal{E}(t, z(t-), z(t-)) \qquad \text{for all } t \in \mathrm{J}_z\,.$$

Combining this with (3.2) and the triangle inequality for $\mathcal{R}$, delivers the stability (S) at all $t \in \mathrm{J}_z$ as desired.

The validity of the Energetic energy-dissipation upper estimate (2.43) and of (S) at all $t \in [0,T]$ entails that (u,z) is an E solution: this follows from either [MR15, Prop. 2.1.23] or [MS18, Lemma 6.2] (mimicking the argument of the proof of Thm. 6.5 therein).

Hence, comparing (E_{VE}) and (E) we ultimately find

$$\mathrm{Var}_{\mathsf{d}_Z}(z,[0,t]) = \mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}(z,[0,t]) \stackrel{(2.37)}{=} \mathrm{Var}_{\mathsf{d}_Z}(z,[0,t]) + \mathrm{Jmp}_{\Delta_{\mathsf{c}}}(z;[0,t]) \quad \text{for all } t \in [0,T]\,.$$

Therefore, at every $t \in \mathrm{Jump}_z$ there holds $\Delta_{\mathsf{c}}(t,z(t-),z(t)) = \Delta_{\mathsf{c}}(t,z(t),z(t+)) = 0$, i.e.

$$\mathsf{c}(t,z(t-),z(t)) = \mathsf{d}_Z(z(t-),z(t)) \quad \text{and} \quad \mathsf{c}(t,z(t),z(t+)) = \mathsf{d}_Z(z(t),z(t+)). \tag{3.3}$$

Combining (3.3) with the Visco-Energetic jump conditions (2.44) we immediately deduce (3.1). □

VE **solutions in perfect plasticity coincide with** E **solutions.** Small-strain associative elastoplasticity, with the Prandtl-Reuss flow rule (without hardening) for the plastic strain, provides an example of a rate-independent system to which Proposition 3.1 applies, cf. Corollary 3.5 ahead. Before entering into details, we fix the following notation, also useful for Section 5.

Notation 3.2. We will use the symbol $\mathbb{M}^{d\times d}$ for the space of $d\times d$ matrices, endowed with the Frobenius inner product $\eta : \xi := \sum_{ij} \eta_{ij}\xi_{ij}$ for two matrices $\eta = (\eta_{ij})$ and $\xi = (\xi_{ij})$. We will denote by $|\cdot|$ the induced the matrix norm and, in accordance with Notation 1.1, by $\overline{B}_r$ the closed ball with radius r centered at 0 in $\mathbb{M}^{d\times d}_{\mathrm{sym}}$. The latter symbol denotes the subspace of symmetric matrices, while $\mathbb{M}^{d\times d}_{\mathrm{dev}}$ stands for the subspace of symmetric matrices with null trace. In fact, every $\eta \in \mathbb{M}^{d\times d}_{\mathrm{sym}}$ can be written as $\eta = \eta_{\mathrm{dev}} + \frac{\mathrm{tr}(\eta)}{d} I$ with η_{dev} the orthogonal projection of η into $\mathbb{M}^{d\times d}_{\mathrm{dev}}$. We will refer to η_{dev} as the deviatoric part of η. With the symbol $\odot$ we will denote the symmetrized tensor product of two vectors a, $b \in \mathbb{R}^d$, defined as the symmetric matrix with entries $\frac{a_i b_j + a_j b_i}{2}$. Finally,

$$\mathrm{BD}(\Omega;\mathbb{R}^d) := \{\tilde{u} \in L^1(\Omega;\mathbb{R}^d) \,:\, \varepsilon(\tilde{u}) \in \mathrm{M}(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}})\}$$

is the space of functions with bounded deformation, such that the (distributional) strain tensor $\varepsilon(\tilde{u})$ is a Radon measure on Ω, valued in $\mathbb{M}^{d\times d}_{\mathrm{sym}}$, and

$$\mathrm{M}(\Omega\cup\Gamma_{\mathrm{D}};\mathbb{M}^{d\times d}_{\mathrm{dev}}) \text{ is the space of } (\mathbb{M}^{d\times d}_{\mathrm{dev}}\text{-valued) Radon measures on } \Omega\cup\Gamma_{\mathrm{D}}.$$

The PDE system governing perfect plasticity, formulated in a (bounded, Lipschitz) domain $\Omega \subset \mathbb{R}^d$ (the reference configuration) consists of

- the equilibrium equation

$$-\operatorname{div}(\mathbb{C}e) = f \quad \text{in } \Omega \times (0,T), \tag{3.4a}$$

  where f is a time-dependent body force, $\mathbb{C}$ is the (symmetric, positive definite) elasticity tensor, e the elastic strain, which enters into the *additive* decomposition of the (symmetric) linearized strain tensor $\varepsilon(\tilde{u}) = \frac{1}{2}(\nabla\tilde{u} + \nabla\tilde{u}^{\mathsf{T}})$ (with $\tilde{u} : \Omega \to \mathbb{R}^d$ the displacement and A^{T} the transpose of a matrix A), into an elastic and a plastic part, i.e.

$$\varepsilon(\tilde{u}) = e + p \quad \text{in } \Omega \times (0,T); \tag{3.4b}$$

- the flow rule for the plastic tensor p

$$\partial\mathrm{R}(\dot{p}) \ni \sigma_{\mathrm{dev}} \quad \text{in } \Omega \times (0,T), \tag{3.4c}$$

  where σ_{dev} is the deviatoric part of the stress $\sigma := \mathbb{C}e$, the 1-homogeneous dissipation potential R is the support function of the closed convex subset $K \subset \mathbb{M}^{d\times d}_{\mathrm{dev}}$ to which the (deviatoric part of the) stress is constrained to belong, and $\partial\mathrm{R} : \mathbb{M}^{d\times d}_{\mathrm{dev}} \rightrightarrows \mathbb{M}^{d\times d}_{\mathrm{dev}}$ is the convex analysis subdifferential of R.

Along the footsteps of [DMDM06], we will suppose hereafter that

$$\overline{B}_{r_k} \subset K \subset \overline{B}_{R_K} \quad \text{for some } 0 < r_K \le R_K. \tag{3.5a}$$

Furthermore, we will have $\partial\Omega = \Gamma_{\rm D} \cup \Gamma_{\rm N} \cup \partial\Gamma$, with $\Gamma_{\rm D}$ and $\Gamma_{\rm N}$ disjoint open sets and $\partial\Gamma$ their common boundary, and we will denote by ν the external unit normal to $\partial\Omega$. We will assume that

$$\mathscr{H}^{d-1}(\Gamma_{\rm D}) > 0 \ \text{ and } \ \partial\Omega,\, \partial\Gamma \text{ are of class } \mathrm{C}^2 \tag{3.5b}$$

(with $\mathscr{H}^{d-1}$ the $(d{-}1)$-dimensional Hausdorff measure). On the Dirichlet part of the boundary $\Gamma_{\rm D}$ we will prescribe a Dirichlet condition through an assigned function

$$w_{\rm D} \in \mathrm{C}^1([0,T]; H^1(\Omega;\mathbb{R}^d)), \tag{3.5c}$$

with trace on $\Gamma_{\rm D}$ still denoted by $w_{\rm D}$. On the Neumann part $\Gamma_{\rm N}$ we will apply a non-zero traction g. A standard condition in perfect plasticity is that the body and surface forces

$$f \in \mathrm{C}^1([0,T]; L^d(\Omega;\mathbb{R}^d)),\ g \in \mathrm{C}^1([0,T]; L^\infty(\Gamma_{\rm N};\mathbb{R}^d)) \text{ satisfy the } \textit{safe-load} \text{ condition,} \tag{3.5d}$$

cf. [DMDM06, (2.17)–(2.19)]. With f and g we associate the total load function

$$\ell : [0,T] \to \mathrm{BD}(\Omega;\mathbb{R}^d)^*, \qquad \langle \ell(t), v\rangle_{\mathrm{BD}(\Omega;\mathbb{R}^d)} := \int_\Omega f(t) v \,\mathrm{d}x + \int_{\Gamma_{\rm N}} g(t) v \,\mathrm{d}\mathscr{H}^{d-1}(x). \tag{3.6}$$

Indeed, the above integrals are well defined for any $v \in \mathrm{BD}(\Omega;\mathbb{R}^d)$ due to the embedding and trace properties of $\mathrm{BD}(\Omega;\mathbb{R}^d)$. Clearly, $\ell(t)$ is also an element of $H^1(\Omega;\mathbb{R}^d)^*$ for every $t \in [0,T]$; in what follows, to avoid overburdening notation, we will often omit to specify the spaces when writing the duality pairing $\langle \ell(t), v\rangle$.

With the boundary datum $w_{\rm D}$ we associate the set $\mathcal{A}(w_{\rm D})$ of the kinematically admissible states $(\tilde u, p)$, viz.

$$\begin{aligned} (\tilde u, p) \in \mathcal{A}(w_{\rm D}) \text{ if and only if } \ &\text{(i)} \quad \tilde u \in \mathrm{BD}(\Omega;\mathbb{R}^d),\ p \in \mathrm{M}(\Omega\cup\Gamma_{\rm D}; \mathbb{M}^{d\times d}_{\rm dev}),\\ &\text{(ii)} \quad e = \varepsilon(\tilde u) - p \in L^2(\Omega;\mathbb{M}^{d\times d}_{\rm sym}),\\ &\text{(iii)} \quad p = (w_{\rm D} - \tilde u) \odot \nu \mathscr{H}^{d-1} \text{ on } \Gamma_{\rm D}. \\ \text{We set } \mathcal{A} := \mathcal{A}(0). \end{aligned} \tag{3.7}$$

Indeed, an admissible $\tilde u$ may have jumps (i.e., the measure $\varepsilon(\tilde u)$ can concentrate on) $\partial\Omega$. Hence, the boundary condition $\tilde u = w_{\rm D}$ on $\Gamma_{\rm D}$ has to be relaxed in terms of (3.7)(iii) (to be understood as an equality between measures on $\Gamma_{\rm D}$), which expresses the fact that any jump of $\tilde u$ violating the Dirichlet condition $\tilde u = w_{\rm D}$ is due to a localized plastic deformation. From now on, we will use the splitting

$$\tilde u = u + w_{\rm D} \tag{3.8}$$

and work with the state variables (u,p).

The Energetic formulation (cf. [DMDM06]) of the perfectly plastic system (3.4) is given in this setup:
Ambient space:

$$X = U \times Z \quad \text{with } U = \mathrm{BD}(\Omega;\mathbb{R}^d),\ Z = \mathrm{M}(\Omega\cup\Gamma_{\rm D}; \mathbb{M}^{d\times d}_{\rm dev}) \tag{3.9a}$$

and (1) σ_Z is the weak*-topology on $\mathrm{M}(\Omega\cup\Gamma_{\rm D}; \mathbb{M}^{d\times d}_{\rm dev})$, identified with the dual of the space of ($\mathbb{M}^{d\times d}_{\rm dev}$-valued) continuous functions with compact support on $\Omega\cup\Gamma_{\rm D}$; (2) σ_U is the weak* topology on $\mathrm{BD}(\Omega;\mathbb{R}^d)$ (which has in fact a predual, cf. e.g. [TS80]), inducing the following notion of weak*-convergence: $u_k \rightharpoonup^* u$ in $\mathrm{BD}(\Omega;\mathbb{R}^d)$ if and only if $u_k \rightharpoonup u$ in $L^1(\Omega;\mathbb{R}^d)$ and $\varepsilon(u_k) \rightharpoonup^* \varepsilon(u)$ in $\mathrm{M}(\Omega;\mathbb{M}^{d\times d}_{\rm sym})$.
Energy functional:

$$\mathcal{E}(t,u,p) := \frac{1}{2}\int_\Omega \mathbb{C}(\varepsilon(u{+}w_{\rm D}(t)){-}p) : (\varepsilon(u{+}w_{\rm D}(t)){-}p)\,\mathrm{d}x + I_{\mathcal{A}}(u,p) - \langle \ell(t), u{+}w_{\rm D}(t)\rangle_{\mathrm{BD}(\Omega;\mathbb{R}^d)}\,. \tag{3.9b}$$

Here, the indicator function $I_{\mathcal{A}}$ forces the constraint $(u,p) \in \mathcal{A}$, so that $\tilde u = u + w_{\rm D} \in \mathcal{A}(w_{\rm D})$;

Dissipation distance: it is defined in terms of the support function of the set K from (3.5a), i.e.

$$\mathrm{R} : \mathbb{M}^{d\times d}_{\mathrm{dev}} \to [0,\infty), \qquad \mathrm{R}(\pi) := \sup_{\omega\in K} \omega : \pi \qquad \text{whence}$$

$$\mathsf{d}_Z(p,\tilde{p}) := \mathcal{R}(\tilde{p}-p) \quad \text{with } \mathcal{R}(\pi) := \int_{\Omega\cup\Gamma_{\mathrm{D}}} \mathrm{R}\left(\frac{\pi}{|\pi|}\right) |\pi|(\mathrm{d}x) \text{ for all } \pi \in \mathrm{M}(\Omega\cup\Gamma_{\mathrm{D}}; \mathbb{M}^{d\times d}_{\mathrm{dev}}), \tag{3.9c}$$

where $|\pi|$ is the variation of π and $\frac{\pi}{|\pi|}$ its Radon-Nykodím derivative w.r.t. $|\pi|$.

It is straightforward to check that in the above metric-topological setting $< T >$ is fulfilled. In fact, adapting some arguments from [DMDM06, Lemmas 3.1, 3.2], it is possible to show that the energy functional $\mathcal{E}$ from (3.9b) fulfills $< A.1 >, < A.2 >, < A.3 >$.

The viscous correction: Let us now consider the family of viscous corrections

$$\delta_Z(p,\tilde{p}) := h(\mathsf{d}_Z(p,\tilde{p})) = h(\mathcal{R}(\tilde{p}-p)) \quad \text{for all } p,\, \tilde{p} \in Z \text{ and } h \text{ as in (2.27)} \tag{3.10}$$

(cf. Remark 3.4 for a discussion on more general viscous corrections).

We now show that, in the frame of the rate-independent system $(X,\mathcal{E},\mathsf{d}_Z)$ given by (3.9) and with this choice of δ_Z, the Visco-Energetic stability condition (S_{VE}) in indeed *equivalent* to the Energetic stability (S).

Proposition 3.3. *Assume* (3.5) *and let* $(u,p) \in \mathrm{B}([0,T];\mathrm{BD}(\Omega;\mathbb{R}^d)) \times \mathrm{BV}([0,T];\mathrm{M}(\Omega\cup\Gamma_{\mathrm{D}};\mathbb{M}^{d\times d}_{\mathrm{dev}}))$ *fulfill* $(u(t),p(t)) \in \mathcal{A}$ *for all* $t\in[0,T]$. *Then, the following conditions are* equivalent *at a given* $t \in [0,T]$*:*

(1) (u,p) *fulfill the stability condition* (S_{VE}) *for the rate-independent system* $(X,\mathcal{E},\mathsf{d}_Z)$ (3.9)*, with the viscous correction* δ_Z *from* (3.10)*;*

(2) there holds

$$\sigma(t) = \mathbb{C}(\varepsilon(u(t)+w_{\mathrm{D}}(t)) - p(t)) \in \Sigma(\Omega)\cap\mathcal{K}(\Omega) \text{ with}$$
$$\begin{cases} \Sigma(\Omega) := \{\sigma \in L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}) \,:\, \operatorname{div}(\sigma)\in L^d(\Omega;\mathbb{R}^d),\ \sigma_{\mathrm{dev}} \in L^\infty(\Omega;\mathbb{M}^{d\times d}_{\mathrm{dev}})\}, \\ \mathcal{K}(\Omega) := \{\sigma \in L^2(\Omega;\mathbb{M}^{d\times d}_{\mathrm{sym}}) \,:\, \sigma_{\mathrm{dev}}(x) \in K \text{ for a.a. } x\in\Omega\}, \end{cases} \tag{3.11a}$$

and

$$-\operatorname{div}(\sigma(t)) = f(t) \text{ a.e. in } \Omega, \qquad \sigma(t)\nu = g(t) \text{ on } \Gamma_{\mathrm{N}}; \tag{3.11b}$$

(3) (u,p) *fulfill the stability condition* (S) *for the rate-independent system* $(X,\mathcal{E},\mathsf{d}_Z)$ *from* (3.9).

Proof. First of all, we show that **(1)** $\Rightarrow$ **(2)**. Indeed, in the stability condition (S_{VE}), i.e.

$$\begin{aligned} &\frac12 \int_\Omega \mathbb{C}(\varepsilon(u{+}w_{\mathrm{D}}(t)){-}p) : (\varepsilon(u{+}w_{\mathrm{D}}(t)){-}p)\,\mathrm{d}x - \langle \ell(t), u(t)\rangle_{\mathrm{BD}(\Omega;\mathbb{R}^d)} \\ &\leq \frac12 \int_\Omega \mathbb{C}(\varepsilon(u'{+}w_{\mathrm{D}}(t)){-}p') : (\varepsilon(u'{+}w_{\mathrm{D}}(t)){-}p')\,\mathrm{d}x - \langle \ell(t), u'\rangle_{\mathrm{BD}(\Omega;\mathbb{R}^d)} + \mathcal{R}(p'{-}p(t)) + h(\mathcal{R}(p'{-}p(t))) \end{aligned}$$

for all $(u',p') \in \mathcal{A}$, we choose $(u',p') := (u(t)+\eta v, p(t)+\eta q)$, with arbitrary $\eta\in\mathbb{R}$ and $(v,q)\in\mathcal{A}$. With straightforward calculations we find

$$\begin{aligned} 0 \leq \frac12 \int_\Omega (\eta\varepsilon(v){-}\eta q) : (\eta\varepsilon(v){-}\eta q)\,\mathrm{d}x + \int_\Omega \mathbb{C}(\varepsilon(u{+}w_{\mathrm{D}}(t)){-}p) : (\eta\varepsilon(v){-}\eta q)\,\mathrm{d}x - \langle \ell(t), \eta v\rangle_{\mathrm{BD}(\Omega;\mathbb{R}^d)} \\ + \mathcal{R}(\eta q) + h(\mathcal{R}(\eta q)). \end{aligned}$$

Hence, by the positive homogeneity of $\mathcal{R}$ we conclude

$$\begin{aligned} 0 \leq \eta^2 \frac12 \int_\Omega \mathbb{C}(\pm\varepsilon(v)\mp q) : (\pm\varepsilon(v)\mp q)\,\mathrm{d}x + \eta \int_\Omega \mathbb{C}(\varepsilon(u{+}w_{\mathrm{D}}(t)){-}p) : (\pm\varepsilon(v)\mp \eta q)\,\mathrm{d}x - \eta\,\langle \ell(t), \pm v\rangle_{\mathrm{BD}(\Omega;\mathbb{R}^d)} \\ + \eta\mathcal{R}(\pm q) + h(\mathcal{R}(\eta(\pm q))) \quad \text{for all } \eta>0. \end{aligned}$$

Dividing by η and letting $\eta\downarrow 0$, and using that

$$\lim_{\eta\downarrow0} \frac{h(\mathcal{R}(\eta(\pm q)))}{\eta} = \lim_{\eta\downarrow0} \frac{h(\mathcal{R}(\eta(\pm q)))}{\mathcal{R}(\eta(\pm q))} \frac{\mathcal{R}(\eta(\pm q))}{\eta} = 0 \tag{3.12}$$

thanks to property (2.27), we find that

$$\begin{cases} -\mathcal{R}(q) \le \int_\Omega \sigma(t) : (\varepsilon(v){-}q)\,\mathrm{d}x - \ \langle \ell(t), v\rangle_{\mathrm{BD}(\Omega;\mathbb{R}^d)}, \\ \int_\Omega \sigma(t) : (\varepsilon(v){-}q)\,\mathrm{d}x - \ \langle \ell(t), v\rangle_{\mathrm{BD}(\Omega;\mathbb{R}^d)} \le \mathcal{R}(q) \end{cases} \qquad \text{for all } (v,q) \in \mathcal{A}. \tag{3.13}$$

It has been shown in [DMDM06, Prop. 3.5] that (3.13) is equivalent to (3.11). This shows **(2)**.

In turn, **(2)** ⇔ **(3)** by [DMDM06, Thm. 3.6]. Finally, we clearly have that **(3)** ⇒ **(1)**. This concludes the proof. □

Remark 3.4. Proposition 3.3 (and then, Corollary 3.5 ahead) carries over to VE solutions of the perfectly plastic system with a more general viscous correction $\delta_Z : Z \times Z \to [0,\infty]$, provided that it fulfills the compatibility condition

$$\lim_{\tilde p \to p \text{ strongly in } Z} \frac{\delta_Z(p,\tilde p)}{\mathcal{R}(\tilde p - p)} = 0 \qquad \text{for all } p \in Z\,. \tag{3.14}$$

Note that (3.14) is a strengthened version of (2.26), in turn implying $< B.3 >$. As a matter of fact, (3.14) guarantees the analogue of (3.12), and then the proof of Proposition 3.3 still goes through.

We are now in a position to deduce

Corollary 3.5. *Assume* (3.5) *and let* $(u,p) \in \mathrm{B}([0,T];\mathrm{BD}(\Omega;\mathbb{R}^d)) \times \mathrm{BV}([0,T];\mathrm{M}(\Omega\cup\Gamma_{\mathrm{D}};\mathbb{M}^{d\times d}_{\mathrm{dev}}))$ *be a* VE *solution of the rate-independent system* $(X,\mathcal{E},\mathsf{d}_Z)$ *from* (3.9)*, with the viscous correction* δ_Z *from* (3.10)*. Suppose that* (u,p) *fulfills at* $t=0$ *the stability condition*

$$\mathcal{E}(t,u(0),p(0)) \le \mathcal{E}(t,u',p') + \mathcal{R}(p'{-}p(0)) \quad \textit{for all } (u',p') \in \mathcal{A}. \tag{3.15}$$

Then, (u,p) *is an Energetic solution of the rate-independent system* $(X,\mathcal{E},\mathsf{d}_Z)$ (3.9)*.*

Proof. We have that (u,p) fulfills the stability condition (S) at $t=0$ and, by Prop. 3.3, whenever it fulfills ($\mathrm{S_{VE}}$), i.e. at every $t \in [0,T]\setminus \mathrm{J}_z$. We may then apply Prop. 3.1 and conclude that (u,p) is an E solution. □

Perfect plasticity is indeed a special example because, in turn, it has been shown in [DMDM06] that, any E solution (u,p) to the perfectly plastic system fulfills $(u,p) \in \mathrm{AC}([0,T];(\mathrm{BD}(\Omega;\mathbb{R}^d)\times\mathrm{M}(\Omega\cup\Gamma_{\mathrm{D}};\mathbb{M}^{d\times d}_{\mathrm{dev}})))$. Therefore, $t \mapsto p(t)$ has no jumps, and from this we can easily conclude that (u,p) is a VE solution as well.

4. Visco-Energetic solutions for a damage system

We consider a rate-independent damage process in a nonlinearly elastic material, located in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$. The body is subject to a time-dependent external force and it is clamped on a portion Γ_{D} of its boundary $\partial\Omega$, fulfilling $\mathscr{H}^{d-1}(\Gamma_{\mathrm{D}}) > 0$. Hence, on Γ_{D} the displacement field $\tilde u : (0,T)\times\Omega \to \mathbb{R}^d$ is prescribed by the time-dependent Dirichlet condition

$$\tilde u(t) = w_{\mathrm{D}}(t) \quad \text{on } \Gamma_{\mathrm{D}},\ t \in (0,T). \tag{4.1}$$

From now on, as in Sec. 3 we will use the splitting $\tilde u = u + w_{\mathrm{D}}$ with $u = 0$ on Γ_{D} and, with slight abuse of notation, w_{D} the extension of the Dirichlet datum into the domain Ω. The state variables of the damage process will thus be u and a scalar damage variable $z : (0,T)\times\Omega \to \mathbb{R}$, with values in the interval $[0,1]$, such that $z(t,x) = 1$ means no damage and $z(t,x) = 0$ means maximal damage in the neighborhood of the point $x \in \Omega$, at the process time $t \in [0,T]$.

We will confine the discussion to a *gradient theory* for damage, thus accounting for an internal length scale. Namely, we allow for the gradient regularizing contribution $\int_\Omega |\nabla z|^r\,\mathrm{d}x$ to the driving energy, along the footsteps of [MR06, TM10, Tho13] analyzing Energetic solutions. More precisely, the condition $r > d$ imposed in [MR06] was weakened to $r > 1$ in [TM10] and, further, to $r = 1$ (i.e. a BV-gradient) in [Tho13]. Here we will stay with the case $r > 1$, possibly strengthening this condition to $r > d$ when considering a viscous correction that involves a norm different from that of the rate-independent dissipation potential, cf. Thm. 4.1 ahead.

All in all, we consider the rate-independent PDE system for damage

$$\begin{aligned} & -\operatorname{div}(\mathrm{D}_e W(x,\varepsilon(u{+}w_{\mathrm{D}}),z)) = f && \text{in } \Omega\times(0,T),\\ & \partial\mathrm{R}(x,\dot z) - \Delta_r z + \partial I_{[0,1]}(z) \ni -\mathrm{D}_z W(x,\varepsilon(u{+}w_{\mathrm{D}}),z) && \text{in } \Omega\times(0,T), \end{aligned} \tag{4.2}$$

supplemented with the homogeneous Dirichlet condition $u = 0$ on Γ_{D}, with the Neumann boundary conditions $\varepsilon(u{+}w_{\mathrm{D}})\nu = g$ on $\Gamma_{\mathrm{N}} = \partial\Omega\setminus\Gamma_{\mathrm{D}}$ (where ν is the exterior unit normal to $\partial\Omega$), and $\partial_\nu z = 0$ on $\partial\Omega$. The conditions on the elastic energy density $W = W(x,e,z)$ (whose Gâteau derivatives w.r.t. e and z are denoted by D_e and D_z, respectively), and on the body and surface forces f, g will be specified in (4.5) and (4.6) ahead; $-\Delta_r$ is the r-Laplacian operator and $\partial I_{[0,1]} : \mathbb{R} \rightrightarrows \mathbb{R}$ is the subdifferential of the indicator function $I_{[0,1]}$, enforcing the constraint $0 \le z \le 1$ a.e. in Ω. The dissipation potential $\mathrm{R} : \Omega\times\mathbb{R} \to [0,\infty]$ is given by

$$\mathrm{R}(x,v) := \begin{cases} \kappa(x)|v| & \text{if } v \le 0,\\ \infty & \text{otherwise} \end{cases} \quad \text{with } \kappa \in L^\infty(\Omega),\ 0 < \kappa_0 < \kappa(x) \text{ for a.a. } x \in \Omega\,. \tag{4.3}$$

The Energetic formulation of the damage system (4.2) is given in the following setup:
Ambient space: $X = U \times Z$ with

$$U = W^{1,p}_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d) := \{u \in W^{1,p}(\Omega;\mathbb{R}^d)\ :\ u = 0 \text{ on } \Gamma_{\mathrm{D}}\} \text{ and } Z = W^{1,r}(\Omega)\,. \tag{4.4a}$$

Here, p is as in (4.5c) below, and $r > 1$. The topology σ_U on the space of admissible displacements is the weak topology of $W^{1,p}(\Omega;\mathbb{R}^d)$; analogously, σ_Z is the weak $W^{1,r}(\Omega)$-topology.
Energy functional: $\mathcal{E} : [0,T]\times X \to (-\infty,\infty]$ is given by the sum of (1) the stored elastic energy $\mathcal{W}$; (2) a term $\mathcal{J}$ encompassing the gradient regularization and the indicator term $I_{[0,1]}(z)$; (3) the power of the external loadings, with the force term ℓ comprising volume and surface forces f and g via

$$\langle \ell(t), v\rangle :=\ \langle f(t), v\rangle + \langle g(t), v\rangle,$$

where the duality pairings involving the forces f and g are nor specified for simplicity, and the duality pairing between ℓ and $u + w_{\mathrm{D}}(t)$ will be settled below, cf.(4.6). Namely, $\mathcal{E}$ is defined by

$$\mathcal{E}(t,u,z) := \mathcal{W}(t,u,z) + \mathcal{J}(z) - \langle \ell(t), u + w_{\mathrm{D}}(t)\rangle \text{ with } \begin{cases} \mathcal{W}(t,u,z) := \int_\Omega W(x,\varepsilon(u)+\varepsilon(w_{\mathrm{D}}(t)),z)\,\mathrm{d}x,\\ \mathcal{J}(z) := \int_\Omega \left(\frac{1}{r}|\nabla z|^r + I_{[0,1]}(z)\right)\mathrm{d}x, \quad r > 1. \end{cases} \tag{4.4b}$$

Then,

$$\mathrm{D}_z := \{z \in W^{1,r}(\Omega)\ :\ z(x) \in [0,1] \text{ for a.a. } x \in \Omega\}\,.$$

Dissipation distance: We consider the asymmetric extended quasi-distance $\mathsf{d}_Z : Z\times Z \to [0,\infty]$ defined by

$$\mathsf{d}_Z(z,z') := \mathcal{R}(z'-z) \quad \text{ with } \mathcal{R} : L^1(\Omega) \to [0,\infty],\ \mathcal{R}(\zeta) := \int_\Omega \mathrm{R}(x,\zeta(x))\,\mathrm{d}x\,. \tag{4.4c}$$

Along the footsteps of [TM10], for the elastic energy density W we assume

$$W(x,\cdot,\cdot) \in \mathrm{C}^0(\mathbb{M}^{d\times d}_{\mathrm{sym}}\times\mathbb{R}) \quad \text{for a.a. } x \in \Omega, \quad W(\cdot,e,z) \text{ measurable on } \Omega \quad \text{for all } (e,z) \in \mathbb{M}^{d\times d}_{\mathrm{sym}}\times\mathbb{R}; \tag{4.5a}$$

$$W(x,\cdot,z) \text{ is convex for every } (x,z) \in \Omega\times\mathbb{R}; \tag{4.5b}$$

$$\exists\, c_1,\, C_1 > 0\ \exists\, p \in (1,\infty)\ \forall\, (x,e,z) \in \Omega\times\mathbb{M}^{d\times d}_{\mathrm{sym}}\times\mathbb{R}\ : \qquad W(x,e,z) \ge c_1|e|^p - C_1; \tag{4.5c}$$

$$\begin{aligned} & \text{for all } (x,z) \in \Omega\times[0,1] \text{ we have } W(x,\cdot,z) \in \mathrm{C}^1(\mathbb{M}^{d\times d}_{\mathrm{sym}}) \text{ and}\\ & \qquad \exists\, c_2,\, C_2 > 0\ \forall\, (x,e,z) \in \Omega\times\mathbb{M}^{d\times d}_{\mathrm{sym}}\times\mathbb{R}\ : \qquad |\mathrm{D}_e W(x,e,z)| \le c_2(W(x,e,z)+C_2); \end{aligned} \tag{4.5d}$$

$$\begin{aligned} & \exists\, c_3,\, C_3 > 0\ \forall\, (x,e,z),\, (x,e,\tilde z) \in \Omega\times\mathbb{M}^{d\times d}_{\mathrm{sym}}\times\mathbb{R} \text{ with } \tilde z \le z \text{ there holds}\\ & \qquad W(x,e,z) \le W(x,e,\tilde z) \le c_3(W(x,e,z)+C_3)\,. \end{aligned} \tag{4.5e}$$

While referring to [TM10, Sec. 3] for all details, here we may comment that (4.5d) enters in the proof of the power-control condition $< A.3 >$ for the energy functional $\mathcal{E}$ (4.4b), whereas the 'monotonicity' type requirement (4.5e) is helpful for the closedness condition $< C >$. As for the data ℓ and u_{D}, we require

$$u_{\mathrm{D}} \in \mathrm{C}^1([0,T]; W^{1,\infty}(\Omega;\mathbb{R}^d)); \tag{4.6a}$$

$$\ell \in \mathrm{C}^1([0,T]; W^{-1,p'}(\Omega;\mathbb{R}^d)), \tag{4.6b}$$

so that the power of the external loadings features the duality pairing between $W^{-1,p'}_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d)$ and $W^{1,p}_{\Gamma_{\mathrm{D}}}(\Omega;\mathbb{R}^d)$.
The viscous correction: We will either take a viscous correction of the form

$$\delta_Z(z,z') = h(\mathcal{R}(z'-z)), \tag{4.7a}$$

with h as in (2.27), or consider the viscous correction

$$\delta_Z : Z \times Z \to [0,\infty] \text{ defined by } \delta_Z(z,z') := \begin{cases} \frac{1}{q}\|z-z'\|^q_{L^q(\Omega)} & \text{if } z,\, z' \in L^q(\Omega), \\ \infty & \text{otherwise,} \end{cases} \quad \text{and } q>1 \tag{4.7b}$$

(cf. also Remark 4.2 ahead).

The main result of this section guarantees the existence of VE solutions of the rate-independent damage system $(X, \mathcal{E}, \mathsf{d}_Z)$ given by (4.4).

Theorem 4.1. *Assume* (4.3)*,* (4.5)*, and* (4.6)*. If the viscous correction δ_Z is given by* (4.7b)*, suppose in addition that*

$$r > d\,. \tag{4.8}$$

Then, for every $z_0 \in \mathrm{D}_z$ there exists a VE *solution (u,z) of the rate-independent damage system $(X,\mathcal{E},\mathsf{d}_Z)$* (4.4) *with the viscous correction δ_Z from* (4.7)*, such that $z(0) = z_0$ and*

$$u \in L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^d)), \quad z \in L^\infty(0,T;W^{1,r}(\Omega)) \cap \mathrm{BV}([0,T];L^1(\Omega))\,. \tag{4.9}$$

The *proof* will be carried out in Sec. 4.1 below.

Remark 4.2. The condition $r > d$ can be weakened to the requirement

$$r > \frac{qd}{q+d}\,, \tag{4.10}$$

on r, q, and the space dimension d, provided that we replace the viscous correction (4.7b) by

$$\tilde{\delta}_Z : Z \times Z \to [0,\infty] \text{ given by } \tilde{\delta}_Z(z,z') := \begin{cases} \frac{1}{q}\|z-z'\|^\gamma_{L^q(\Omega)} & \text{if } z,\, z' \in L^q(\Omega), \\ \infty & \text{otherwise} \end{cases} \tag{4.11}$$

and $\gamma > 1$ satisfying a further compatibility condition with r and q, cf. (4.20) in Remark 4.4 ahead.

Remark 4.3 (VE solutions are in between E and BV solutions (I))**.** The application of the VE concept to damage well shows that this weak solvability notion has an *intermediate* character between Energetic and Balanced Viscosity solutions. Indeed,

- When the viscous correction is given by (4.7a), then the existence theory for VE-solutions works under the same conditions as for E solutions, cf. [TM10]. In particular, it is possible to consider a gradient regularization with an *arbitrary* exponent $r > 1$; the restriction $r > d$ (or (4.10)) comes into play only upon choosing the viscous correction (4.7b) (or (4.11)).
- Balanced Viscosity solutions to the rate-independent system (4.2) have been in turn addressed in [KRZ18], with a *quadratic* viscous regularization (modulated by a vanishing parameter). The vanishing-viscosity analysis developed in [KRZ18] crucially relies on the requirement $r > d$ and, additionally, on the *quadratic* character of the elastic energy density W, as well as on smoothness requirements on the reference domain Ω (the smoothness of Ω can be dropped if the nonlinear r-Laplacian is replaced by a less standard fractional Laplacian regularization, cf. [KRZ13]). Here, instead, we can allow for an energy density W of arbitrary p-growth and we do not need to restrict to smooth domains.

4.1. **Proof of Theorem 4.1.** In what follows, we are going to check that the rate-independent damage system $(X, \mathcal{E}, \mathsf{d}_Z)$ given by (4.4) complies with Assumptions $<A>$, $<B>$, and $<C>$ of Theorem 2.12 (it is immediate to see that $<T>$ is satisfied). As it will be clear from the ensuing proof, $<A>$ and $<C>$ can be checked under the sole condition that the exponent r is strictly bigger than 1. It is in the proof of $<B>$, in the case the viscous correction δ is given by (4.7b), that the restriction $r > d$ comes into play.

$\triangleright$ **Validity of Assumption $<A>$:** It was shown in [TM10, Lemma 3.3] that $\mathcal{E}$ satisfies the coercivity estimate

$$\exists\, c_4,\, C_4 > 0\ \forall\, (t,u,z) \in [0,T] \times U \times Z : \quad \mathcal{E}(t,u,z) \geq c_4(\|u\|^p_{W^{1,p}(\Omega;\mathbb{R}^d)} + \|z\|^r_{W^{1,r}(\Omega)}) - C_4 \,. \tag{4.12}$$

Hence, the sublevels of $\mathcal{E}(t,\cdot,\cdot)$ are bounded in $W^{1,p}(\Omega;\mathbb{R}^d) \times W^{1,r}(\Omega)$, uniformly w.r.t. $t \in [0,T]$. In [TM10, Lemma 3.4] it was proved that $\mathcal{E}(t,\cdot,\cdot)$ is sequentially lower semicontinuous w.r.t. the weak topology on $W^{1,p}(\Omega;\mathbb{R}^d) \times W^{1,r}(\Omega)$. In view of (4.6), a standard modification of that argument yields the lower semicontinuity of $\mathcal{E}$, hence $<A.1>$. Therefore, its sublevels are (sequentially) compact in $[0,T] \times U \times Z$ w.r.t. to the $\sigma_{\mathbb{R}}$-topology. This ensures the validity of $<A.2>$.

It was shown in [TM10, Thm. 3.7] that there exist constants $c_5,\, C_5 > 0$ such that for all $(u,z) \in [0,T] \times U \times \mathrm{D}_z$ the function $t \mapsto \mathcal{E}(t,u,z)$ belongs to $\mathrm{C}^1([0,T])$, with

$$\begin{aligned} &\partial_t \mathcal{E}(t,u,z) = \int_\Omega \mathrm{D}_e W(x, \varepsilon(u + u_{\mathrm{D}}(t)), z) \colon \varepsilon(\dot{u}_{\mathrm{D}}(t))\,\mathrm{d}x - \langle \dot{\ell}(t), u + u_{\mathrm{D}}(t)\rangle - \langle \ell(t), \dot{u}_{\mathrm{D}}(t)\rangle \quad \text{and} \\ &|\partial_t \mathcal{E}(t,u,z)| \leq c_5(\mathcal{E}(t,u,z) + C_5) \quad \text{for all } (t,u,z) \in (0,T) \times \mathrm{D}, \end{aligned}$$

whence (2.9). We now check $<A.3'>$: observe that d_Z is left-continuous on the sublevels of $\mathcal{F}_0$ since the latter subsets are bounded in $W^{1,r}(\Omega)$ by (4.12) and $W^{1,r}(\Omega) \Subset L^1(\Omega)$. It remains to prove the conditional upper semicontinuity (2.14b) of $\partial_t \mathcal{E}$. For this, we apply [TM10, Lemma 3.11], ensuring that $\partial_t \mathcal{E}$ complies with (2.15). Then, we are in a position to apply Proposition 2.3 and conclude the validity of property (2.14b).

$\triangleright$ **Validity of Assumption $<C>$:** We will verify property (2.28) in the case of the viscous correction (4.7b) (the case (4.7a) can be handled with similar calculations). Let $(t_n, u_n, z_n)_n$, (t,u,z) fulfill the conditions of (2.28), and let (u',z') be *any* element in $M(t,z)$. Preliminarily, from $\sup_{n\in\mathbb{N}} \mathcal{E}(t_n,u_n,z_n) < \infty$ we deduce, via (4.12), that the sequence $(z_n)_n$ is bounded in $W^{1,r}(\Omega)$ and, thus, that $z_n \rightharpoonup z$ in $W^{1,r}(\Omega)$ as $n \to \infty$. Since $0 \leq z_n \leq 1$ a.e. in Ω, we then infer that $z_n \to z$ in $L^s(\Omega)$ for all $s \in [1,\infty)$. For the sequence $(u'_n, z'_n)_n$ we borrow the construction for the *mutual recovery sequence* devised in the proof of [TM10, Thm. 3.14]. Note that this construction is in fact applicable to *any* $(u',z') \in U \times \mathrm{D}_z$ such that $\mathcal{R}(z'-z) < \infty$. In particular, we pick $u' \in \mathrm{Argmin}_{u\in U} \mathcal{E}(t,u,z')$. Namely, we set for every $n \in \mathbb{N}$

$$\begin{aligned} &u'_n := u' \\ &z'_n := \min\{(z'-\delta_n)^+, z_n\} = \begin{cases} (z'-\delta_n)^+ & \text{if } (z'-\delta_n)^+ \leq z_n, \\ z_n & \text{if } (z'-\delta_n)^+ > z_n, \end{cases} \quad \text{with } \delta_n := \|z_n - z\|^{1/r}_{L^r(\Omega)} \to 0 \text{ as } n \to \infty. \end{aligned} \tag{4.13}$$

Observe that this construction gives $z'_n \in W^{1,r}(\Omega)$ as well as $0 \leq z'_n \leq z_n \leq 1$ a.e. in Ω, so that $\mathcal{R}(z'_n - z_n) < \infty$. In the proof of [TM10, Thm. 3.14] it is shown that

$$z'_n \rightharpoonup z' \quad \text{in } W^{1,r}(\Omega) \quad \text{as } n \to \infty \,. \tag{4.14}$$

Slightly adapting the argument from [TM10, Thm. 3.14] to allow for a sequence $(t_n)_n$ of times converging to t, we find that

$$\limsup_{n\to\infty} \left(\mathcal{E}(t_n, u'_n, z'_n) + \mathsf{d}_Z(z_n, z'_n)\right) \leq \mathcal{E}(t,u',z') + \mathsf{d}_Z(z,z') \,.$$

Therefore, for the reduced energy $\mathcal{I}(t,z) = \min_{u\in U} \mathcal{E}(t,u,z)$ we deduce

$$\limsup_{n\to\infty} \left(\mathcal{I}(t_n, z'_n) + \mathsf{d}_Z(z_n, z'_n)\right) \leq \mathcal{I}(t,z') + \mathsf{d}_Z(z,z'), \tag{4.15}$$

where we have used that $\mathcal{I}(t_n, z'_n) \le \mathcal{E}(t_n, u'_n, z'_n)$ and that $\mathcal{I}(t, z') = \mathcal{E}(t, u', z')$ by our choice of u'. On the other hand, again using that $0 \le z'_n \le 1$ a.e. in Ω, from (4.14) we infer that $z'_n \to z'$ in $L^s(\Omega)$ for every $s \in [1, \infty)$. All in all, we gather that $z_n \to z$ and $z'_n \to z'$ in $L^q(\Omega)$. Therefore,

$$\lim_{n\to\infty} \delta_Z(z_n, z'_n) = \delta_Z(z, z') \tag{4.16}$$

which, combined with (4.15), finishes the proof of property (2.28).

$\triangleright$ **Validity of Assumption** $< B >$**:** The viscous correction δ from (4.7b) clearly complies with $< B.1 >$ and $< B.2 >$. To check $< B.3 >$, we verify property (2.26). Preliminarily, observe that the Gagliardo-Nirenberg inequality gives

$$\frac{\delta_Z(z', z)}{\mathsf{d}_Z(z', z)} = \frac{1}{q}\frac{\|z-z'\|^q_{L^q(\Omega)}}{\|z-z'\|_{L^1(\Omega)}} \le C\frac{\|z-z'\|^{\theta q}_{W^{1,r}(\Omega)}\|z-z'\|^{(1-\theta)q}_{L^1(\Omega)}}{\|z-z'\|_{L^1(\Omega)}} \qquad \text{with} \tag{4.17}$$

$$\frac{1}{q} = \theta\left(\frac{1}{r} - \frac{1}{d}\right) + 1 - \theta\,. \tag{4.18}$$

Since $r > d$, there exists $\theta \in (0, 1)$ complying with (4.18).

Let us now consider $(t, z) \in \mathscr{S}_{\mathsf{D}}$ and a sequence $(t_n, z_n) \overset{\mathcal{S}}{\rightharpoonup} (t, z)$, namely $(t_n, z_n)_n \subset \mathscr{S}_{\mathsf{D}}$, $t_n \uparrow t$, $z_n \rightharpoonup z$ in $W^{1,r}(\Omega)$, $\mathcal{R}(z_n - z) \to 0$. Then,

$$\sup_{n\in\mathbb{N}} \|z_n\|_{W^{1,r}(\Omega)} \le C, \qquad \text{and thus} \tag{4.19}$$

$$\limsup_{(t_n,z_n)\overset{\mathcal{S}}{\rightharpoonup}(t,z)} \frac{\delta_Z(z_n, z)}{\mathsf{d}_Z(z_n, z)} \overset{(1)}{\le} C \limsup_{(t_n,z_n)\overset{\mathcal{S}}{\rightharpoonup}(t,z)} \frac{\|z-z_n\|^{\theta q}_{W^{1,r}(\Omega)}\|z-z_n\|^{(1-\theta)q}_{L^1(\Omega)}}{\|z-z_n\|_{L^1(\Omega)}} \overset{(2)}{\le} C' \limsup_{(t_n,z_n)\overset{\mathcal{S}}{\rightharpoonup}(t,z)} \|z-z_n\|^{(1-\theta)q-1}_{L^1(\Omega)} \overset{(3)}{=} 0,$$

where (1) follows from (4.17), (2) from (4.19), and (3) from the fact that, since $r > d$, the exponent θ in (4.18) fulfills $(1-\theta)q > 1$. This finishes the proof of (2.26).

Conclusion of the proof: Theorem 2.12 applies, yielding the existence of a VE solution. The summability properties (4.9) for u and z follow from combining the coercivity property (4.12) with the energy bound $\sup_{t\in[0,T]} |\mathcal{E}(t, u(t), z(t))| \le C$, cf. (2.40) in Remark 2.9. ∎

Remark 4.4. Observe that (4.10) is the sharpest condition ensuring that θ given by (4.18) is in $(0, 1)$. The requirement $r > d$ can be weakened to (4.10), provided that we replace the viscous correction δ from (4.7b) by that in (4.11), with $\gamma > 1$ chosen in such a way that θ from (4.18) fulfills $(1 - \theta)\gamma > 1$. This amounts to imposing the following condition on γ

$$\gamma\left(\frac{1}{q} - \left(1 - \frac{1}{q}\right)\frac{d-r}{dr+r-d}\right) > 1. \tag{4.20}$$

For instance, if $d = 3$ and $r = 2$ (i.e. we consider the standard Laplacian regularization), then $q = 2$ complies with the compatibility condition (4.10). An admissible viscous correction would then be

$$\delta_Z(z, z') := \frac{1}{2}\|z'-z\|^\gamma_{L^2(\Omega)} \quad \text{with } \gamma > \frac{5}{2}.$$

5. Visco-Energetic solutions for plasticity at finite strains

We consider a model for elastoplasticity at finite strains in a bounded body $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary. Finite plasticity is based on the multiplicative decomposition of the gradient of the elastic deformation $\varphi : \Omega \to \mathbb{R}^d$ into an elastic and a plastic part, i.e. $\nabla\varphi = F_{\mathrm{el}}P$ with $P \in \mathbb{R}^{d\times d}$ the plastic tensor, usually assumed with determinant $\det(P) = 1$. While the elastic part $F_{\mathrm{el}} = \nabla\varphi P^{-1}$ contributes to energy storage and is at elastic equilibrium, energy is dissipated through changes of the plastic tensor, which thus plays the role of a (dissipative) internal variable.

The model for rate-independent finite-strain plasticity we address was first analyzed in [MM09] within the framework of energetic solutions. The PDE system in the unknowns (φ, P) can be formally written as

$$
\begin{aligned}
&\varphi(t) \in \operatorname{Argmin}\left(\int_\Omega W(x, \nabla\hat\varphi P^{-1}(t))\,\mathrm{d}x - \langle \ell(t), \hat\varphi\rangle \,:\, \hat\varphi \in \mathscr{F}\right), && t \in (0,T),\\
&\partial \mathrm{R}(\dot P P^{-1})P^{-\mathsf{T}} + (\nabla\varphi P^{-1})^{\mathsf{T}} \mathrm{D}_F W(x, \nabla\varphi P^{-1})P^{-\mathsf{T}} \\
&\qquad + \mathrm{D}_P H(x, P, \nabla P) - \mathrm{div}(\mathrm{D}_{\nabla P} H(x, P, \nabla P)) = 0, && (x,t) \in \Omega \times (0,T)\,.
\end{aligned}
\tag{5.1a}
$$

Here, $W = W(x, F)$ is the elastic energy density, ℓ is a time-dependent loading, e.g. associated with an applied body force f and a traction g on the Neumann part Γ_{N} of $\partial\Omega$, $\mathscr{F}$ is the set of admissible deformations (cf. (5.1b) below), the dissipation potential $\mathrm{R}(x,\cdot)$ is 1-homogeneous, and the energy density H encompasses hardening and regularizing effects through the term $\int_\Omega |\nabla P|^r\,\mathrm{d}x$, for some $r > 1$ specified later. System (5.1) is further supplemented with a time-dependent Dirichlet condition for φ

$$
\varphi(t,x) = \phi_{\mathrm{D}}(t,x) \quad (t,x) \in [0,T] \times \Gamma_{\mathrm{D}},
\tag{5.1b}
$$

with $\phi_{\mathrm{D}} : [0,T] \times \Gamma_{\mathrm{D}} \to \mathbb{R}^d$ given on the Dirichlet boundary $\Gamma_{\mathrm{D}} \subset \partial\Omega$ such that $\mathscr{H}^{d-1}(\Gamma_{\mathrm{D}}) > 0$. Following [FM06, MM09], to treat (5.1b) compatibly with the multiplicative decomposition of $\nabla\varphi$, we will seek for φ in the form of a composition

$$
\varphi(t,x) = \phi_{\mathrm{D}}(t, y(t,x)) \quad \text{with } y(t,\cdot) \text{ fulfilling } y = \mathrm{Id} \text{ on } \Gamma_{\mathrm{D}},
\tag{5.2}
$$

where we have denoted by the same symbol the extension of ϕ_{D} to $[0,T] \times \mathbb{R}^d$, cf. (5.6a) below.

Therefore, we consider the pair (y, P) as state variables and, accordingly, the Energetic formulation of system (5.1) is given in the following setup:

Ambient space: we take $X = U \times Z$, with

$$
\begin{aligned}
U &:= \left\{ y \in W^{1,q_Y}(\Omega;\mathbb{R}^d) \,:\, y = \mathrm{Id} \text{ on } \Gamma_{\mathrm{D}} \right\} \quad \text{for } q_Y > 1 \text{ to be specified later, and}\\
Z &= \{ P \in W^{1,r}(\Omega;\mathbb{R}^{d\times d}) \cap L^{q_P}(\Omega;\mathbb{R}^{d\times d}) \,:\, P(x) \in G \quad \text{for a.a. } x \in \Omega\}, \quad q_P,\, r > 1 \text{ specified below.}
\end{aligned}
\tag{5.3a}
$$

Here, G is a Lie subgroup of $\mathrm{GL}^+(d) := \{P \in \mathbb{R}^{d\times d}) \,:\, \det(P) > 0\}$. From now on, we will focus on the case

$$
G = \mathrm{SL}(d) := \{ P \in \mathbb{R}^{d\times d}) \,:\, \det(P) = 1\}
$$

cf. [Mie02] for other examples of G. We take σ_U as the weak topology of $W^{1,q_Y}(\Omega;\mathbb{R}^d)$ and σ_Z as the weak topology of $W^{1,r}(\Omega;\mathbb{R}^{d\times d}) \cap L^{q_P}(\Omega;\mathbb{R}^{d\times d})$.

Energy functional: $\mathcal{E} : [0,T] \times X \to (-\infty, \infty]$ is given by

$$
\mathcal{E}(t, y, P) := \mathcal{E}_1(P) + \mathcal{E}_2(t, y, P).
\tag{5.3b}
$$

The functional $\mathcal{E}_1 : [0,T] \times Z \to \mathbb{R}$ includes the hardening and gradient regularizing terms, i.e.

$$
\mathcal{E}_1(P) = \int_\Omega H(x, P(x), \nabla P(x))\,\mathrm{d}x \text{ with } H : \Omega \times \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d\times d} \to \mathbb{R} \text{ fulfilling (5.4) below.}
$$

The stored elastic energy $\mathcal{E}_2$ reflects the multiplicative split for the deformation gradient $\nabla\varphi = \nabla\phi_{\mathrm{D}}(t,y)\nabla y$ due to (5.2), and it is thus of the form

$$
\mathcal{E}_2(t, y, P) := \int_\Omega W(x, \nabla\phi_{\mathrm{D}}(t,y)\nabla y P^{-1})\,\mathrm{d}x - \langle \ell(t), \phi_{\mathrm{D}}(t,y)\rangle_{W^{1,q_Y}},
$$

with the elastic energy density W specified ahead and $\nabla\phi_{\mathrm{D}}$ the gradient of ϕ_{D} w.r.t. the variable y.

Dissipation distance: Along the footsteps of [MM09] (cf. also [Mie02, HMM03]), we consider on X dissipation distances of the form

$$\mathsf{d}_Z(P_0, P_1) := \int_\Omega \mathcal{R}(P_1(x)P_0^{-1}(x))\,\mathrm{d}x, \tag{5.3c}$$

where the functional $\mathcal{R} : \mathrm{SL}(d) \to [0,\infty)$ (for simplicity, we omit the possible x-dependence of $\mathcal{R}$) is generated by a norm-like function R, cf. (5.5) below, on the Lie-algebra $T_{\mathbf{1}}\mathrm{SL}(d)$ via the formula

$$\mathcal{R}(\Sigma) := \inf\left\{ \int_0^1 \mathrm{R}(\dot\Xi(s)\Xi(s)^{-1})\,\mathrm{d}s \;:\; \Xi \in \mathrm{C}^1([0,1];G),\ \Xi(0) = \mathbf{1},\ \Xi(1) = \Sigma \right\}.$$

Let us now detail our assumptions on the constitutive functions H and W, on R, and on the problem data. The hardening function H satisfies

$$\begin{aligned}
&H : \Omega \times \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d\times d} \to \mathbb{R} \text{ is a normal integrand, } H(x,P,\cdot) \text{ convex for all } (x,P) \in \Omega \times \mathbb{R}^{d\times d},\\
&\exists\, c_1 > 0\ \exists\, h \in L^1(\Omega)\ \exists\ q_P > 1,\ r > 1 \text{ for a.a. } x \in \Omega\ \forall\,(P,A) \in \mathbb{R}^{d\times d} \times \mathbb{R}^{d\times d\times d} :\\
&\qquad H(x,P,A) \geq h(x) + c_1(|P|^{q_P} + |A|^r)\,,
\end{aligned} \tag{5.4a}$$

while we require the following conditions on the elastic energy density $W : \Omega \times \mathbb{R}^{d\times d} \to [0,\infty]$: Firstly,

$$\mathrm{dom}(W) = \Omega \times \mathrm{GL}^+(d), \text{ i.e. } W(x,F) = \infty \text{ for } \det F \leq 0 \text{ for all } x \in \Omega, \tag{5.4b}$$

$$\exists\, c_2 > 0\ \exists\, j \in L^1(\Omega)\ \exists\, q_F > d\ \ \forall (x,F) \in \mathrm{dom}(W) : \quad W(x,F) \geq j(x) + c_2|F|^{q_F}, \tag{5.4c}$$

and we impose a further compatibility condition between the integrability powers q_Y, q_F, q_P, i.e.

$$\frac{1}{q_F} + \frac{1}{q_P} = \frac{1}{q_Y} < \frac{1}{d}. \tag{5.4d}$$

Secondly, $W(x,\cdot) : \mathbb{R}^{d\times d} \to (-\infty,\infty]$ is *polyconvex* for all $x \in \Omega$, i.e. it is a convex function of its minors:

$$\begin{aligned}
&\exists\, \mathbb{W} : \Omega \times \mathbb{R}^{\mu_d} \to (-\infty,\infty] \text{ such that}\\
&\text{(i) } \mathbb{W} \text{ is a normal integrand,}\\
&\text{(ii) } \forall\,(x,F) \in \Omega \times \mathbb{R}^{d\times d} : \quad W(x,F) = \mathbb{W}(x,\mathbb{M}(F)),\\
&\text{(iii) } \forall x \in \Omega : \quad \mathbb{W}(x,\cdot) : \mathbb{R}^{\mu_d} \to (-\infty,\infty] \text{ is convex,}
\end{aligned} \tag{5.4e}$$

where $\mathbb{M} : \mathbb{R}^{d\times d} \to R^{\mu_d}$ is the function which maps a matrix to all its minors, with $\mu_d := \sum_{s=1}^d \binom{d}{s}^2$. Thirdly, W satisfies the multiplicative stress control conditions

$$\begin{aligned}
&\exists\, \delta > 0\ \ \exists\, c_3,\ c_4 > 0\ \ \forall\,(x,F) \in \mathrm{dom}(W)\ \forall\, N \in \mathcal{N}_\delta :\\
&\text{(i) } W(x,\cdot) : \mathrm{GL}^+(d) \to \mathbb{R} \text{ is differentiable,}\\
&\text{(ii) } |\mathrm{D}_F W(x,F)F^\mathsf{T}| \leq c_3(W(x,F)+1),\\
&\text{(iii) } |\mathrm{D}_F W(x,F)F^\mathsf{T} - \mathrm{D}_F W(x,NF)(NF)^\mathsf{T}| \leq c_4|N - \mathbf{1}|(W(x,F)+1),
\end{aligned} \tag{5.4f}$$

with $\mathcal{N}_\delta := \left\{ N \in \mathbb{R}^{d\times d} \;:\; |N - \mathbf{1}| < \delta \right\}$. We refer to [MM09] for examples of functionals H and W complying with (5.4). Finally, the functional (whose possible dependence on x is neglected by simplicity)

$$\begin{aligned}
&\mathrm{R} : T_{\mathbf{1}}\mathrm{SL}(d) \to [0,\infty) \text{ is 1-positively homogeneous and fulfills}\\
&\exists\, c_R,\, C_R > 0\ \forall\, \Sigma \in T_{\mathbf{1}}\mathrm{SL}(d) : \quad c_R|\Sigma| \leq \mathrm{R}(\Sigma) \leq C_R|\Sigma|\,,
\end{aligned} \tag{5.5}$$

cf. [HMM03] for examples in von-Mises and single-crystal plasticity. For the Dirichlet loading ϕ_{D} we require

$$\begin{aligned}
&\phi_{\mathrm{D}} \in \mathrm{C}^1([0,T] \times \mathbb{R}^d; \mathbb{R}^d), \quad \nabla\phi_{\mathrm{D}} \in \mathrm{BC}^1([0,T] \times \mathbb{R}^d; \mathbb{R}^{d\times d}),\\
&\exists\, c_5 > 0\ \ \forall\,(t,x) \in [0,T] \times \mathbb{R}^d : \quad |\nabla\phi_{\mathrm{D}}(t,x)^{-1}| \leq c_5,
\end{aligned} \tag{5.6a}$$

where BC stands for *bounded continuous*. Finally, on the external load ℓ we impose

$$\ell \in \mathrm{C}^1([0,T];W^{1,q_Y}(\Omega;\mathbb{R}^d)^*)\,. \tag{5.6b}$$

The viscous correction: We will take viscous corrections

(1) either of the form

$$\delta_Z(P_0,P_1) = h(\mathrm{d}_Z(P_0,P_1)) \quad \text{with } h \text{ as in (2.27)}, \tag{5.7a}$$

(2) or we define $\delta_Z : Z\times Z \to [0,\infty]$ by

$$\delta_Z(P_0,P_1) := \begin{cases} \int_\Omega \mathrm{R}_q((P_1(x)-P_0(x))P_0(x)^{-1})\,\mathrm{d}x = \int_\Omega \mathrm{R}_q(P_1(x)P_0(x)^{-1}-\mathbf{1})\,\mathrm{d}x & \text{if } \mathrm{R}_q(P_1P_0^{-1}-\mathbf{1})\in L^1(\Omega), \\ \infty & \text{otherwise,} \end{cases} \tag{5.7b}$$

for a given convex lower semicontinuous functional $\mathrm{R}_q : T_{\mathbf{1}}\mathrm{SL}(d)\to[0,\infty)$ fulfilling

$$\mathrm{R}_q(\Sigma) = \mathrm{R}_q(-\Sigma) \text{ for all } \Sigma\in T_{\mathbf{1}}\mathrm{SL}(d) \quad \text{and} \quad \lim_{\Sigma\to 0}\frac{\mathrm{R}_q(\Sigma)}{|\Sigma|^q} = C_q \in (0,\infty) \text{ for some } q>1\,. \tag{5.8}$$

For our existence result of VE solutions to the rate-independent system $(X,\mathcal{E},\mathsf{d}_Z)$ from (5.3), like for the damage system in Sec. 4 we shall strengthen the condition $r>1$ to $r>d$ when addressing the non-trivial viscous correction (5.7b).

Theorem 5.1. *Assume* (5.4)*,* (5.5)*, and* (5.6)*. Furthermore, if the viscous correction δ_Z is given by* (5.7b)*, suppose in addition that $r>d$. Then, for every $P_0\in Z$ there exists a* VE *solution (y,P) of the rate-independent finite-plasticity system $(X,\mathcal{E},\mathsf{d}_Z)$* (5.3)*, with the viscous correction δ_Z from* (5.7)*, such that $P(0)=P_0$ and*

$$y\in L^\infty(0,T;W^{1,q_Y}(\Omega;\mathbb{R}^d)), \quad P\in L^\infty(0,T;W^{1,r}(\Omega;\mathbb{R}^{d\times d}))\cap \mathrm{BV}([0,T];L^1(\Omega;\mathbb{R}^{d\times d}))\,. \tag{5.9}$$

The *proof* will be carried out in Section 5.1 ahead.

Remark 5.2 (Extensions)**.** The model for finite plasticity considered in [MM09] is actually more general than that addressed here, as it features a further internal variable $p\in\mathbb{R}^m$, $m\geq 1$, besides the plastic tensor P. The vector p possibly encompasses hardening variables/slip strains and, like P, it is subject to a gradient regularization. Under the very same conditions as in [MM09, Thm. 3.1], it is possible to show that the energy functional comprising p complies with condition $<A>$ in the metric topological setup where

$$Z = \left(L^{q_P}(\Omega;\mathbb{R}^{d\times d})\cap W^{1,r}(\Omega;;\mathbb{R}^{d\times d})\right)\times\left(L^{q_P}(\Omega;\mathbb{R}^m)\cap W^{1,r}(\Omega;;\mathbb{R}^m)\right)\,.$$

A typical example where the additional variable p comes into play is *isotropic hardening*, cf. [MM09, Example 3.3]. There, the scalar $p\in\mathbb{R}$ measures the amount of hardening and the variables (P,p) are subject to some constraint. The relevant dissipation distance accounts for such constraint and takes ∞ as a value.

Actually, our analysis could be extended to dissipation distances with values in $[0,\infty]$ under the very same conditions enucleated in [MM09, formula (3.4)]. In particular, if we take the 'trivial' viscous correction δ_Z from (5.7a), then the same argument as in [MM09, Sec. 5.3] allows us to check condition (2.28), whence the validity of assumption $<C>$ of the general existence Thm. 2.12. With the viscous correction in (5.7a) we can generalize our existence Thm. 5.1 for VE solutions also in the other directions outlined in [MM09, Sec. 6].

Remark 5.3 (VE solutions are in between E and BV solutions (II))**.** The statement of Thm. 5.1, as well as Remark 5.2, highlight the fact that, in the case of the viscous correction (5.7a), the existence theory for VE solutions to the finite-strain plasticity system works under the very same conditions as for E solutions. Nonetheless, when bringing into play a different viscous correction such as that in (5.7b), like for the damage system in Sec. 4 we need to strengthen our conditions on the gradient regularization and in fact impose $r>d$. For E solutions to the finite plasticity system, this requirement was made only in the cases in which the dissipation distance took values in $[0,\infty]$, cf. [MM09]. Instead, in the case of the viscous correction from (5.7b) we cannot weaken this condition even when d_Z is valued in $[0,\infty)$, cf. also Remark 5.6 ahead.

At any rate, the existence of VE solutions is proved here under weaker conditions than for BV solutions. Although the latter have not yet been addressed in the context of finite plasticity, we may observe that a prerequisite for tackling them is the existence of solutions to the corresponding viscously regularized problem, which has been recently proved in [MRS18]. Such viscous solutions have to fulfill an energy-dissipation balance that, in turn, relies on the validity of a suitable chain rule for the driving energy. Actually, this chain rule is at the very core of the existence argument. In [MRS18] it has been possible to prove this condition, and to ultimately conclude the existence of solutions to the viscoplastic finite-strain system, only for a considerably regularized version of the energy functional $\mathcal{E}$ from (5.3b).

5.1. **Proof of Theorem 5.1.** Preliminarily, we collect the properties of $\mathcal{R}_1$ in the following result.

Lemma 5.4. *Assume* (5.5). *Then, the functional* $\mathcal{R}_1 : \mathrm{SL}(d) \to [0,\infty)$ *is continuous, strictly positive for* $\Sigma \neq \mathbf{1}$*, satisfies the triangle inequality* $\mathcal{R}_1(\Sigma_1\Sigma_0) \le \mathcal{R}_1(\Sigma_0) + \mathcal{R}_1(\Sigma_1)$ *for all* $\Sigma_0,\, \Sigma_1 \in T_{\mathbf{1}}\mathrm{SL}(d)$*, as well as the estimate*

$$\exists\, C_1 > 0 \ \ \exists\, q_\gamma \in [1, q_P) \ \ \forall\, \Sigma_0,\, \Sigma_1 \in \mathrm{SL}(d) : \qquad \mathcal{R}_1(\Sigma_1\Sigma_0^{-1}) \le C_1(1+|\Sigma_0|^{q_\gamma}+|\Sigma_1|^{q_\gamma})\,. \tag{5.10}$$

Moreover,

$$\forall\, M > 0 \ \exists\, c_M > 0 \ \forall\, \Sigma \in \mathrm{SL}(d) : \qquad \mathcal{R}_1(\Sigma) \le M \ \Rightarrow\ \mathcal{R}_1(\Sigma) \ge c_M|\Sigma - \mathbf{1}|\,. \tag{5.11}$$

Proof. In order to check (5.11) (we refer to [MM09, Sec. 3] for the proof of all the other properties of $\mathcal{R}_1$), let Σ fulfill $\mathcal{R}_1(\Sigma) \le M$: we choose an infimizing sequence $(\Xi_n)_n \subset \mathrm{C}^1([0,1];G)$ such that $\Xi_n(0) = \mathbf{1}$ and $\Xi_n(1) = \Sigma$, fulfilling $\lim_{n\to\infty}\int_0^1 \mathrm{R}_1(\dot\Xi_n(s)\Xi_n(s)^{-1})\,\mathrm{d}s = \mathcal{R}_1(\Sigma)$. We define

$$\mathsf{s}_n : [0,1] \to [0,1] \quad \text{by } \mathsf{s}_n(t) := c_n \int_0^t \Big(1{+}\mathrm{R}_1(\dot\Xi_n\Xi_n^{-1})\Big)\,\mathrm{d}s,$$

with the normalization constant $c_n := \Big(1 + \int_0^1 \mathrm{R}_1(\dot\Xi_n(s)\Xi_n(s)^{-1})\,\mathrm{d}s\Big)^{-1}$, and set

$$\mathsf{t}_n := \mathsf{s}_n^{-1}, \qquad \widetilde\Xi_n := \Xi_n \circ \mathsf{t}_n\,.$$

Therefore, for n sufficiently big we have

$$\begin{aligned} 2 + M \ge 1 + \int_0^1 \mathrm{R}_1(\dot\Xi_n(s)\Xi_n(s)^{-1})\,\mathrm{d}s = \frac{1}{c_n} &\ge \frac{1}{c_n}\frac{\mathrm{R}_1(\dot\Xi_n(\mathsf{t}_n(s))\Xi_n(\mathsf{t}_n(s))^{-1})}{(1 + \mathrm{R}_1(\dot\Xi_n(\mathsf{t}_n(s))\Xi_n(\mathsf{t}_n(s))^{-1}))} \\ &= \mathrm{R}_1(\dot{\widetilde\Xi}_n(s)\widetilde\Xi_n(s)^{-1}) \ge c_R|\dot{\widetilde\Xi}_n(s)\widetilde\Xi_n(s)^{-1}|\,, \end{aligned}$$

for all $s \in [0,1]$, where the latter estimate ensues from (5.5). Hence the function $s \mapsto \Lambda_n(s) := \dot{\widetilde\Xi}_n(s)\widetilde\Xi_n(s)^{-1}$ is uniformly bounded in $L^\infty(0,1;\mathbb{R}^{d\times d})$. Writing $\widetilde\Xi_n(s) := \mathbf{1} + \int_0^s \Lambda_n(r)\widetilde\Xi_n(r)\,\mathrm{d}r$ we conclude, via the Gronwall Lemma, that

$$\exists\, \tilde c_M > 0 \ \forall\, n \in \mathbb{N} : \qquad \|\widetilde\Xi_n\|_{L^\infty(0,1;\mathbb{R}^{d\times d})} \le \tilde c_M, \qquad \text{whence}$$

$$\begin{aligned} \mathcal{R}_1(\Sigma) = \lim_{n\to\infty}\int_0^1 \mathrm{R}_1(\dot{\widetilde\Xi}_n(s)\widetilde\Xi_n(s)^{-1})\,\mathrm{d}s &\ge c_R \liminf_{n\to\infty}\int_0^1 |\dot{\widetilde\Xi}_n(s)\widetilde\Xi_n(s)^{-1}|\,\mathrm{d}s \\ &\ge \frac{c_R}{\tilde c_M}\liminf_{n\to\infty}\int_0^1 |\dot{\widetilde\Xi}_n(s)|\,\mathrm{d}s \ge \frac{c_R}{\tilde c_M}|\Sigma - \mathbf{1}| \end{aligned}$$

where we have used the estimate $|AB^{-1}| \ge \frac{|A|}{|B|}$. This gives (5.11) and concludes the proof. □

Corollary 5.5. *Assume* (5.5) *and* (5.8). *Then,* d_Z *from* (5.3c) *is a (possibly asymmetric) quasi-distance separating the points of* Z*, and fulfilling*

$$\begin{aligned} &\forall\, M > 0 \ \exists\, \tilde c_M > 0 \ \forall\, P_0, P_1 \in Z : \\ &\|P_0\|_{L^\infty(\Omega)} + \|P_1\|_{L^\infty(\Omega)} \le M \ \Rightarrow\ \mathsf{d}_Z(P_0,P_1) \ge \tilde c_M \int_\Omega |P_1(x)P_0(x)^{-1} - \mathbf{1}|\,\mathrm{d}x\,. \end{aligned} \tag{5.12}$$

Furthermore, the viscous correction δ_Z *from* (5.7b) *is* σ_Z*-lower semicontinuous on* $Z \times Z$*.*

Proof. To check that d_Z separates the points of Z, we observe that

$$\mathrm{d}_Z(P_0,P_1)=0 \;\Rightarrow\; \mathcal{R}_1(P_1(x)P_0^{-1}(x))=0 \text{ for a.a. } x\in\Omega \;\Rightarrow\; P_0(x)=P_1(x) \text{ for a.a. } x\in\Omega$$

since $\mathcal{R}_1(\Sigma)>0$ if $\Sigma\neq\mathbf{1}$.

Let us now show how (5.12) derives from (5.11). From $\|P_0\|_{L^\infty}+\|P_1\|_{L^\infty}\leq M$ it follows that $\|P_0^{-1}\|_{L^\infty}+\|P_1\|_{L^\infty}\leq\widetilde{M}$. To check this, we use that

$$P_0^{-1}=\frac{1}{\det(P_0)}\mathrm{cof}(P_0)^\mathsf{T}=\mathrm{cof}(P_0)^\mathsf{T} \tag{5.13}$$

($\mathrm{cof}(P_0)$ denoting cofactor matrix of P_0), as $P_0\in\mathrm{SL}(d)$. Since $\mathcal{R}_1$ is continuous, $\sup_{x\in\Omega}\mathcal{R}_1(P_1(x)P_0^{-1}(x))\leq\widetilde{M}'$ for some $\widetilde{M}'>0$, so that (5.11) yields $\tilde{c}_M>0$ such that

$$\mathcal{R}_1(P_1(x)P_0^{-1}(x))\geq\tilde{c}_M|P_1(x)P_0^{-1}(x)-\mathbf{1}| \quad \text{for almost all } x\in\Omega\,.$$

Then, (5.12) follows.

Finally, let $(P_i^n)_n\subset Z$ fulfill $P_i^n\rightharpoonup P_i$ as $n\to\infty$ in $L^{q_P}(\Omega;\mathbb{R}^{d\times d})\cap W^{1,r}(\Omega;;\mathbb{R}^{d\times d})$, for $i=0,1$. Therefore, $P_i^n\to P_i$ in $L^r(\Omega;;\mathbb{R}^{d\times d})\cap L^{q_P-\epsilon}(\Omega;;\mathbb{R}^{d\times d})$ for every $\epsilon\in(0,q_P-1]$. This implies that

$$P_i^n(x)\to P_i(x), \text{ whence } (P_i^n(x))^{-1}\overset{(5.13)}{=}\mathrm{cof}(P_i^n(x))^\mathsf{T}\to\mathrm{cof}(P_i(x))^\mathsf{T}\overset{(5.13)}{=}(P_i(x))^{-1} \text{ for a.a. } x\in\Omega,\ i=\{0,1\},$$

as $\det(P_i^n(x))=1$ for a.a. $x\in\Omega$, and hence $\det(P_i)\equiv 1$ a.e. in Ω. All in all, we conclude that

$$P_1^n(x)(P_0^n(x))^{-1}\to P_1(x)(P_0(x))^{-1} \qquad \text{for a.a. } x\in\Omega\,. \tag{5.14}$$

Therefore, if $\liminf_{n\to\infty}\delta_Z(P_0^n,P_1^n)<\infty$, we easily conclude that $\delta_Z(P_0,P_1)<\infty$ and

$$\lim_{n\to\infty}\delta_Z(P_0^n,P_1^n)=\liminf_{n\to\infty}\int_\Omega \mathrm{R}_q(P_1^n(x)(P_0^n(x))^{-1}-\mathbf{1})\,\mathrm{d}x\geq\int_\Omega \mathrm{R}_q(P_1(x)(P_0(x))^{-1}-\mathbf{1})\,\mathrm{d}x=\delta_Z(P_0,P_1),$$

i.e. the claimed lower semicontinuity of δ_Z. □

We are now in a position to carry out the **proof of Theorem 5.1** by verifying the validity of the conditions of Theorem 2.12. As we will see, the requirement $r>d$ enters in the proof of $<B>$ & $<C>$, only in the case the viscous correction is given by (5.7b).

▷ **Validity of Assumption** $<T>$**:** It follows from Corollary 5.5.

▷ **Validity of Assumption** $<A>$**:** In the proof of [MM09, Thm. 3.1] it was shown that

$$\begin{aligned}&\exists\, C_2,\, C_3>0\ \forall\,(t,y,P)\in[0,T]\times U\times Z\ :\\ &\quad\mathcal{E}(t,y,P)\geq C_2(\|\nabla y\|_{L^{q_Y}(\Omega)}^{q_Y}+\|P\|_{L^{q_P}(\Omega)}^{q_P}+\|\nabla P\|_{L^r(\Omega)}^r)-C_3\,.\end{aligned} \tag{5.15}$$

In view of Korn's inequality, this yields that the sublevels of $\mathcal{E}(t,\cdot,\cdot)$ are bounded in the space $V:=W^{1,q_Y}(\Omega;\mathbb{R}^d)\times W^{1,r}(\Omega;\mathbb{R}^{d\times d})$, uniformly w.r.t. $t\in[0,T]$, i.e.

$$\forall\,S>0\ \exists\,R_S>0\ \forall\,(t,y,P)\in[0,T]\times U\times Z\ :\quad |\mathcal{E}(t,y,P)|\leq S\ \Rightarrow\ (y,P)\in\overline{B}_{R_S}^V \tag{5.16}$$

(cf. Notation 1.1). We will now show that

$$\begin{aligned}&\Big(t_n\to t \text{ in } [0,T],\ y_n\rightharpoonup y \text{ in } W^{1,q_Y}(\Omega;\mathbb{R}^d),\ P_n\rightharpoonup P \text{ in } W^{1,r}(\Omega;\mathbb{R}^{d\times d})\cap L^{q_P}(\Omega;\mathbb{R}^{d\times d})\Big)\\ &\Rightarrow\ \liminf_{n\to\infty}\mathcal{E}(t_n,y_n,P_n)\geq\mathcal{E}(t,y,P)\,.\end{aligned} \tag{5.17}$$

The (sequential) lower semicontinuity of the functional $\mathcal{E}_1$ w.r.t. σ_Z follows from [MM09, Thm. 5.2]. We adapt the arguments from the latter result to show the lower semicontinuity of $\mathcal{E}_2$. First of all, since $q_Y>d$ by (5.4d), from $y_n\rightharpoonup y$ in $W^{1,q_Y}(\Omega;\mathbb{R}^d)$ we deduce that $y_n\to y$ in $\mathrm{C}^0(\overline{\Omega};\mathbb{R}^d)$. Therefore, by (5.6a) we deduce that

$$\nabla\phi_{\mathrm{D}}(t_n,y_n)\to\nabla\phi_{\mathrm{D}}(t,y) \quad \text{ in } \mathrm{C}^0(\overline{\Omega};\mathbb{R}^d). \tag{5.18}$$

All in all, we conclude that $\phi_{\mathrm{D}}(t_n, y_n) \rightharpoonup \phi_{\mathrm{D}}(t, y)$ in $W^{1,q_Y}(\Omega;\mathbb{R}^d)$ so that, since $\ell(t_n) \to \ell(t)$ in $W^{1,q_Y}(\Omega;\mathbb{R}^d)^*$ by (5.6b), we ultimately find $\langle \ell(t_n), \phi_{\mathrm{D}}(t_n, y_n)\rangle_{W^{1,q_Y}(\Omega;\mathbb{R}^d)} \to \langle \ell(t), \phi_{\mathrm{D}}(t, y)\rangle_{W^{1,q_Y}(\Omega;\mathbb{R}^d)}$ as $n \to \infty$. To conclude (5.17), it remains to check that

$$\liminf_{n\to\infty} \int_\Omega W(x, \nabla\phi_{\mathrm{D}}(t_n, y_n(x))\nabla y_n(x)P_n(x)^{-1})\,\mathrm{d}x \geq \int_\Omega W(x, \nabla\phi_{\mathrm{D}}(t, y(x))\nabla y(x)P(x)^{-1})\,\mathrm{d}x\,.$$

For this, we follow the very same arguments as in the proof of [MM09, Thm. 5.2], also exploiting (5.18). Clearly, (5.15) and (5.17) ensure the validity of $< A.1 >$ and $< A.2 >$.

It was shown in [MRS18, Lemma 6.1] that for every $(y, P) \in U\times Z$ the mapping $t \mapsto \mathcal{E}(t, y, P)$ is differentiable on $[0, T]$, with

$$\partial_t\mathcal{E}(t, y, P) = \int_\Omega \mathbb{K}(x, \nabla\phi_{\mathrm{D}}(t, y(x))\nabla y(x)P(x)^{-1}) : V(t, y(x))\,\mathrm{d}x - \langle \dot\ell(t), \phi_{\mathrm{D}}(t, y)\rangle_{W^{1,q_Y i}} - \langle \ell(t), \dot g_{\mathrm{D}}(t, y)\rangle_{W^{1,q_Y}}\,,$$

with the short-hand notation $\mathbb{K}(x, F) := \mathrm{D}_F W(x, F)F^\top$ for the (multiplicative) Kirchhoff stress tensor, and $V(t, y) := \nabla\dot g_{\mathrm{D}}(t, y)(\nabla\phi_{\mathrm{D}}(t, y))^{-1}$. The power-control estimate (2.9) holds too, cf. again [MRS18, Lemma 6.1].

Now, for all $\Xi \in Z$ the functional $\mathsf{d}_Z(\cdot, \Xi)$ is left-continuous on (Z, σ_Z) in the sense of (2.14a). Indeed, from $P_n \rightharpoonup P$ in $W^{1,r}(\Omega;\mathbb{R}^{d\times d}) \cap L^{q_P}(\Omega;\mathbb{R}^{d\times d})$ as $n \to \infty$ we have that $P_n \to P$ in $L^{q_P-\epsilon}(\Omega;\mathbb{R}^{d\times d})$ for all $\epsilon \in (0, q_P - 1]$. Combining the growth condition (5.10) of $\mathcal{R}_1$ and the dominated convergence theorem we deduce that

$$\mathsf{d}_Z(P_n, \Xi) = \int_\Omega \mathcal{R}_1(\Xi(x)P_n^{-1}(x))\,\mathrm{d}x \to \int_\Omega \mathcal{R}_1(\Xi(x)P^{-1}(x))\,\mathrm{d}x = \mathsf{d}_Z(P, \Xi)\,. \tag{5.19}$$

Therefore, we can check $< A.3' >$, namely the conditional upper semicontinuity (2.14b). This has been done in [MM09, Prop. 4.4] by resorting to Prop. 2.3.

$\triangleright$ **Validity of Assumption** $< C >$**:** We will in fact check (2.28). Let $(t_n, y_n, P_n)_n$, converging to (t, y, P), be a sequence as in (2.28): with the very same arguments used for $< A.3' >$, from $\sup_{n\in\mathbb{N}} \mathcal{E}(t_n, y_n, P_n) \leq C$ we deduce that $P_n \to P$ in $L^{q_P-\epsilon}(\Omega;\mathbb{R}^{d\times d})$ for all $\epsilon \in (0, q_P - 1]$. Let us now pick *any* $(y', P') \in U \times Z$ with $y' \in \mathrm{Argmin}_{y\in U}\mathcal{E}(t, y, P)$ and take the *constant* recovery sequence $(y'_n, P'_n) := (y', P')$ for all $n \in \mathbb{N}$. Clearly, $\lim_{n\to\infty} \mathcal{E}(t, y'_n, P'_n) = \mathcal{E}(t, y', P')$, which entails $\limsup_{n\to\infty} \mathcal{I}(t, P'_n) \leq \mathcal{I}(t, P')$ for the reduced energy. Arguing as in the above lines, we also find $\mathsf{d}_Z(P_n, P'_n) = \mathsf{d}_Z(P_n, P') \to \mathsf{d}_Z(P, P')$ as $n \to \infty$, which concludes the proof of (2.28) in the case the viscous correction δ_Z is the 'trivial' one, as in (5.7a).

When δ_Z is instead given by (5.7b), we rely on the compact embedding $W^{1,r}(\Omega;\mathbb{R}^{d\times d}) \Subset \mathrm{C}^0(\overline{\Omega};\mathbb{R}^{d\times d})$ due to $r > d$. This guarantees that the sequence $(P_n)_n$, bounded in $W^{1,r}(\Omega;\mathbb{R}^{d\times d})$, in fact fulfills $P_n \to P$ in $\mathrm{C}^0(\overline{\Omega};\mathbb{R}^{d\times d})$. Therefore, $\mathrm{cof}(P_n)^\top \to \mathrm{cof}(P)^\top$ in $\mathrm{C}^0(\overline{\Omega};\mathbb{R}^{d\times d})$ and thus we find

$$P'_nP_n^{-1} = P'\mathrm{cof}(P_n)^\top \to P'\mathrm{cof}(P)^\top = P'P^{-1} \quad \text{in } \mathrm{C}^0(\overline{\Omega};\mathbb{R}^{d\times d}) \tag{5.20}$$

(here we have again used that $P_n^{-1} = \mathrm{cof}(P_n)^\top$, and analogously for P', in view of (5.13) and of the fact that $\det(P') = \det(P_n) = 1$ for every $n \in \mathbb{N}$). Thus, by the continuity of R_q we have that $\sup_{x\in\Omega} \mathrm{R}_q(P'(x)P_n(x)^{-1} - \mathbf{1}) \leq C$. The dominated convergence theorem we yields that $\delta_Z(P_n, P'_n) \to \delta_Z(P, P')$, which establishes (2.28).

$\triangleright$ **Validity of Assumption** $< B >$**:** Since $\mathrm{R}_q(0) = 0$ by (5.8), we easily check that the viscous correction δ_Z from (5.7b) complies with $< B.1 >$. Condition $< B.2 >$ follows from the very same arguments as in the above lines. We will prove $< B.3 >$ through (2.26). Let us now consider $(t, P) \in \mathscr{S}_{\mathsf{D}}$ and a sequence $(t_n, P_n) \overset{\mathscr{S}}{\rightharpoonup} (t, P)$, i.e. $(t_n, P_n)_n \subset \mathscr{S}_{\mathsf{D}}$, $t_n \uparrow t$, $P_n \rightharpoonup P$ in $W^{1,r}(\Omega)$, $\mathsf{d}_Z(P_n, P) \to 0$. Since $P_n \to P$ in $\mathrm{C}^0(\overline{\Omega};\mathbb{R}^{d\times d})$, we may use that

$$\exists\,\bar c > 0\ \forall\, n \in \mathbb{N}\ :\ \mathsf{d}_Z(P_n, P) \geq \bar c\|PP_n^{-1} - 1\|_{L^1(\Omega;\mathbb{R}^{d\times d})} \tag{5.21}$$

thanks to (5.11). Moreover, observing that, indeed, we even have that

$$PP_n^{-1} \to \mathbf{1} \quad \text{in } \mathrm{C}^0(\overline{\Omega};\mathbb{R}^{d\times d}) \tag{5.22}$$

(cf. (5.20)), in view of (5.8) we find, for n sufficiently big,

$$\mathrm{R}_q(P(x)P_n^{-1}(x)) \leq \left(C_q+\frac{1}{2}\right)|P(x)P_n^{-1}(x)-\mathbf{1}|^q \quad \text{for all } x \in \Omega, \qquad \text{and thus}$$

$$\delta_Z(P_n,P) \leq \left(C_q + \frac{1}{2}\right) \|PP_n^{-1} - \mathbf{1}\|^q_{L^q(\Omega;\mathbb{R}^{d\times d})}\,. \tag{5.23}$$

Ultimately, we conclude

$$\begin{aligned}
\lim_{(t_n,P_n)\simeq(t,P)} \frac{\delta_Z(P_n,P)}{\mathsf{d}_Z(P_n,P)} &\leq C \lim_{(t_n,P_n)\simeq(t,P)} \frac{\|PP_n^{-1}-\mathbf{1}\|^q_{L^q(\Omega)}}{\|PP_n^{-1}-1\|_{L^1(\Omega)}} \\
&\leq C \lim_{(t_n,P_n)\simeq(t,P)} \frac{\|PP_n^{-1}-\mathbf{1}\|^{\theta q}_{W^{1,r}(\Omega)} \|PP_n^{-1}-\mathbf{1}\|^{(1-\theta)q}_{L^1(\Omega)}}{\|PP_n^{-1}-\mathbf{1}\|_{L^1(\Omega)}} \\
&\leq C \lim_{(t_n,P_n)\simeq(t,P)} \|PP_n^{-1}-\mathbf{1}\|^{(1-\theta)q-1}_{L^1(\Omega)} = 0\,.
\end{aligned}$$

Here we have used the Gagliardo-Nirenberg inequality in the very same way as in the proof of Thm. 4.1, and the previously established convergence (5.22). Hence, we conclude condition (2.26), yielding $< B.3 >$.

Thus, we are in a position to apply Thm. 2.12 and conclude the existence of VE solutions. The summability properties (5.9) follows from the energy bound $\sup_{t\in(0,T)} |\mathcal{E}(t,y(t),P(t))| \leq C$, cf. (2.40), combined with the coercivity estimate (5.15). We have thus finished the proof of Thm. 5.1. ∎

Remark 5.6. A close perusal of the proof of the validity of conditions $< B >$ and $< C >$, in the case of the non-trivial viscous regularization δ_Z from (5.7b), reveals the key role played by the condition $r > d$ (which has been for instance used in the proof of (5.21)). Unlike for the damage system tackled in Sec. 4, it is not clear how to weaken this requirement.

6. Passing from adhesive contact to brittle delamination with Visco-Energetic solutions

The main result of this section, Theorem 6.1, provides the existence of VE solutions to a rate-independent system for brittle delamination between two elastic bodies, by passing to the limit in the VE formulation of an approximating system for adhesive contact.

First of all, let us briefly sketch the brittle model. We consider delamination between two bodies Ω_+, $\Omega_- \subset \mathbb{R}^d$, $d \in \{2,3\}$ along their common boundary Γ_{C}, the prescribed $(d-1)$-dimensional delamination surface. More precisely, throughout this section we shall suppose that

$$\Omega_\pm, \quad \Omega := \Omega_+ \cup \Gamma_{\mathrm{C}} \cup \Omega_- \ \text{ are Lipschitz domains,} \quad \partial\Omega = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}} \text{ with } \begin{cases} \mathscr{H}^{d-1}(\partial\Omega_\pm \cap \Gamma_{\mathrm{D}}) > 0, \\ \overline{\Gamma_{\mathrm{C}}} \cap \overline{\Gamma_{\mathrm{D}}} = \emptyset\,. \end{cases} \tag{6.1}$$

The process is modeled with the aid of an internal, damage-like, delamination variable $z : [0;T] \to \Gamma_{\mathrm{C}}$, $0 \leq z \leq 1$ on Γ_{C}, which describes the state of the adhesive material located on Γ_{C} during a time interval $[0,T]$. In particular, in our notation $z(x,t) = 1$, resp. $z(x,t) = 0$, shall indicate that the glue is fully intact, resp. broken, at the point $x \in \Gamma_{\mathrm{C}}$ and at the process time $t \in [0,T]$. Within the assumption of small strains, we also consider the displacement variable $\tilde{u} : \Omega \to \mathbb{R}^d$. *Brittle* delamination is characterized by the

$$\textit{brittle constraint} \qquad z(x,t)\,[\![\tilde{u}(x,t)]\!] = 0 \quad \text{ on } \Gamma_{\mathrm{C}} \times (0,T), \tag{6.2}$$

where $[\![\tilde{u}]\!] := \tilde{u}^+|_{\Gamma_{\mathrm{C}}} - \tilde{u}^-|_{\Gamma_{\mathrm{C}}}$ is the difference of the traces on Γ_{C} of $\tilde{u}^\pm = \tilde{u}|_{\Omega_\pm}$. This condition allows for displacement jumps only at points $x \in \Gamma_{\mathrm{C}}$ where the bonding is completely broken, i.e. $z(x,t) = 0$; at points where $z(x,t) > 0$ it ensures $[\![\tilde{u}(x,t)]\!] = 0$, i.e. the continuity of the displacements. Therefore, (6.2) distinguishes between the crack set, where the displacements may jump, and the complementary set with active bonding, where it imposes a transmission condition on the displacements.

The (formally written) rate-independent system for brittle delamination reads

$$-\operatorname{div}(\mathbb{C}\varepsilon(\tilde u)) = f \qquad \text{in } \Omega\times(0,T), \tag{6.3a}$$

$$\tilde u = w_{\rm D} \qquad \text{on } \Gamma_{\rm D}\times(0,T), \qquad \mathbb{C}\varepsilon(\tilde u)|_{\Gamma_{\rm N}}\nu = g \qquad \text{on } \Gamma_{\rm N}\times(0,T), \tag{6.3b}$$

$$\mathbb{C}\varepsilon(\tilde u)|_{\Gamma_{\rm C}} n + \partial_u I_{\mathsf{C}}(\tilde u, z) + \partial I_{\mathsf{U}(x)}(\llbracket \tilde u\rrbracket) \ni 0 \qquad \text{on } \Gamma_{\rm C}\times(0,T), \tag{6.3c}$$

$$\partial \mathrm{R}(x,\dot z) + \partial_z I_{\mathsf{C}}(\llbracket \tilde u\rrbracket, z) + \partial I_{[0,1]}(z) \ni a_0 \qquad \text{on } \Gamma_{\rm C}\times(0,T). \tag{6.3d}$$

The static momentum balance (6.3a), where $\mathbb{C}$ is the (positive definite, symmetric) elasticity tensor and f a body force, is coupled with a time-dependent Dirichlet condition on the Dirichlet portion $\Gamma_{\rm D}$ of the boundary $\partial\Omega$, with outward unit normal ν (cf. (6.1) below). On the Neumann part $\Gamma_{\rm N}$ a surface force g is assigned. The evolutions of $\tilde u$ and z are coupled by the Robin-type boundary condition (6.3c) on the contact surface $\Gamma_{\rm C}$, where $\partial_u I_{\mathsf{C}} : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is the (convex analysis) subdifferential w.r.t. u of the indicator function of the set

$$\mathsf{C} := \{(v,z)\in\mathbb{R}^d\times\mathbb{R} : \ \llbracket v\rrbracket z = 0\},$$

while $\partial I_{\mathsf{U}(x)} : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is the subdifferential of the indicator of

$$\mathsf{U}(x) = \{v\in\mathbb{R}^d \, : v\cdot n(x)\geq 0\}, \quad x\in\Gamma_{\rm C},$$

with n the unit normal to $\Gamma_{\rm C}$, oriented from Ω_+ to Ω_-. Hence, besides (6.2), we are also imposing the *non-penetration* constraint $\llbracket\tilde u\rrbracket\cdot n\geq 0$ in Ω between Ω_+ and Ω_-. Finally, the flow rule (6.3d) for the delamination parameter z involves the very same dissipation density R from (4.3), the subdifferential w.r.t. z of I_{C}, and the coefficient a_0, i.e. the phenomenological specific energy per area which is stored by disintegrating the adhesive.

From now on, as in Secs. 3 and 5 we will use the splitting $\tilde u = u + w_{\rm D}$, with $w_{\rm D}$ an extension of the Dirichlet datum to the whole of Ω. Thus, we shall work with the variable u. In view of (6.1), without loss of generality we may assume that this extension fulfills

$$w_{\rm D}|_{\Gamma_{\rm C}} \equiv 0 \text{ on } \Gamma_{\rm C}, \text{ so that } \llbracket\tilde u\rrbracket = \llbracket u + w_{\rm D}\rrbracket = \llbracket u\rrbracket, \tag{6.4}$$

so that the brittle constraint turns into $z\llbracket u\rrbracket = 0$ on $\Gamma_{\rm C}\times(0,T)$. The Energetic formulation of the brittle system (6.3) thus involves the following:

Ambient space: $X = U\times Z$ with

$$U = H^1_{\Gamma_{\rm D}}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d) := \{u\in H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d) \, : \, u=0 \text{ on } \Gamma_{\rm D}\}, \quad Z := \{z\in L^\infty(\Gamma_{\rm C}) \, : \, 0\leq z\leq 1 \text{ on } \Gamma_{\rm C}\}, \tag{6.5a}$$

endowed with the weak topology σ_U of $H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)$ and with the weak*-topology σ_Z of $L^\infty(\Gamma_{\rm C})$, respectively.

Energy functional: $\mathcal{E} : [0,T]\times X\to(-\infty,\infty]$ is given by

$$\begin{aligned}\mathcal{E}(t,u,z) :=& \frac12\int_{\Omega\backslash\Gamma_{\rm C}} \mathbb{C}\varepsilon(u{+}w_{\rm D}) : \varepsilon(u{+}w_{\rm D})\,\mathrm{d}x \\ &+\int_{\Gamma_{\rm C}} \big(I_{\mathsf{U}(x)}(\llbracket u\rrbracket){+}I_{\mathsf{C}}(\llbracket u\rrbracket, z){+}I_{[0,1]}(z){-}a_0 z\big)\,\mathrm{d}\mathscr{H}^{d-1}(x) - \langle \ell(t), u + w_{\rm D}(t)\rangle_{H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)},\end{aligned} \tag{6.5b}$$

where the function $\ell : [0,T]\to H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)^*$ subsumes the body and surface forces f and g. Observe that the domain of $\mathcal{E}$ does not depend on the time variable, i.e.

$$\mathrm{D}(\mathcal{E}(t,\cdot)) = \{(u,z)\in U\times Z \, : \, \llbracket u(x)\rrbracket\in\mathsf{U}(x), \ z(x)\llbracket u(x)\rrbracket = 0, \ z(x)\in[0,1] \text{ for a.a. } x\in\Gamma_{\rm C}\} \quad \text{for all } t\in[0,T].$$

Dissipation distance: We consider the extended asymmetric quasi-distance $\mathsf{d}_Z : Z\times Z\to[0,\infty]$ defined by

$$\mathsf{d}_Z(z,z') := \mathcal{R}(z'-z) \quad \text{with } \mathcal{R} : L^1(\Gamma_{\rm C})\to[0,\infty], \quad \mathcal{R}(\zeta) := \int_{\Gamma_{\rm C}} \mathrm{R}(x,\zeta(x))\,\mathrm{d}\mathscr{H}^{d-1}(x) \tag{6.5c}$$

and the dissipation density R from (4.3). Due to the highly nonconvex character of the brittle constraint (6.2), the existence of Energetic solutions to the rate-independent system $(X,\mathcal{E},\mathsf{d}_Z)$ from (6.5) cannot be proved by directly passing to the time-continuous limit in the associated time-incremental minimization scheme. Indeed, an existence result was obtained in [RSZ09] by passing to the limit in the Energetic formulation for a penalized version of system (6.3). The resulting system is in fact a model for *adhesive contact*. The relevant energy functional, in the very same displacement and delamination variables, is given by

$$\begin{aligned}\mathcal{E}_k(t,u,z) :=& \frac{1}{2}\int_{\Omega\backslash\Gamma_{\rm C}} \mathbb{C}\varepsilon(u{+}w_{\rm D}) : \varepsilon(u{+}w_{\rm D})\,{\rm d}x \\ &+\int_{\Gamma_{\rm C}} \left(I_{\mathsf{U}(x)}([\![u]\!]){+}\tfrac{k}{2}z|[\![u]\!]|^2{+}I_{[0,1]}(z){-}a_0 z\right)\,{\rm d}\mathscr{H}^{d-1}(x) - \langle \ell(t), u+w_{\rm D}(t)\rangle_{H^1}, \quad k>0.\end{aligned} \tag{6.6}$$

Note that the brittle constraint (6.2) is penalized by the term $\frac{k}{2}z|[\![u]\!]|^2$. Via the Evolutionary Gamma-convergence theory from [MRS08], in [RSZ09] it was shown that E solutions to the adhesive contact system $(X,\mathcal{E}_k,\mathsf{d}_Z)$ converge as $k\to\infty$ to E solutions to the brittle delamination system $(X,\mathcal{E},\mathsf{d}_Z)$.

We aim to extend this approach, in order to prove the existence of VE solutions of the brittle system. In fact, VE solutions of the adhesive contact system were tackled in [MS18, Example 4.5] with the
Viscous correction: $\delta_Z : Z\times Z \to [0,\infty]$ of the form

$$\delta_Z(z,z') := h(\mathsf{d}_Z(z,z')) \quad \text{with } h \text{ as in (2.27)}, \tag{6.7}$$

cf. also Remark 6.2 below. Under the condition that

$$w_{\rm D}\in {\rm C}^1([0,T];H^1(\Omega;\mathbb{R}^d))\,, \qquad \ell\in {\rm C}^1([0,T];H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)^*)\,, \tag{6.8}$$

the existence of VE solutions $(u_k,z_k)\in L^\infty(0,T;H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d))\times(L^\infty(\Gamma_{\rm C}{\times}(0,T))\cap{\rm BV}([0,T];L^1(\Gamma_{\rm C})))$ to the adhesive contact system $(X,\mathcal{E}_k,\mathsf{d}_Z)$, with the viscous correction from (6.7), was derived in [MS18] (again, observe that the summability $u\in L^\infty(0,T;H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d))$ derives from the bound $\sup_{t\in(0,T)}|\mathcal{E}_k(t,u(t),z(t))|\leq C$, cf. (2.40), and the coercivity properties of $\mathcal{E}$, cf. (6.12) below).

We now address the limit passage in the VE formulation of $(X,\mathcal{E}_k,\mathsf{d}_Z)$ as $k\to\infty$. From now on, we will assume for simplicity that $k\in\mathbb{N}$. The *proof* of Theorem 6.1 below will be carried out throughout Sec. 6.1.

Theorem 6.1. *Assume* (6.1)*,* (6.4)*, and* (6.8)*.*
Let $(u_k,z_k)_{k\in\mathbb{N}}\subset L^\infty(0,T;H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d))\times(L^\infty(\Gamma_{\rm C}{\times}(0,T))\cap{\rm BV}([0,T];L^1(\Gamma_{\rm C})))$ *be a sequence of* VE *solutions to the rate-independent systems* $(X,\mathcal{E}_k,\mathsf{d}_Z)$*, with* δ_Z *from* (6.7) *and initial datum* $z_0\in{\rm D}_z$*.*

Then, for any sequence $(k_j)_{j\in\mathbb{N}}$ *with* $k_j\to\infty$ *as* $j\to\infty$ *there exist a (not relabeled) subsequence* $(u_{k_j},z_{k_j})_{j\in\mathbb{N}}$ *and* $(u,z)\in L^\infty(0,T;H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d))\times(L^\infty(\Gamma_{\rm C}{\times}(0,T))\cap{\rm BV}([0,T];L^1(\Gamma_{\rm C})))$ *such that* $z(0)=z_0$ *and*

(1) the following convergences hold as $j\to\infty$

$$u_{k_j}(t)\rightharpoonup u(t) \qquad \text{in } H^1(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d) \qquad \text{for all } t\in[0,T], \tag{6.9a}$$

$$z_{k_j}(t)\rightharpoonup^* z(t) \qquad \text{in } L^\infty(\Gamma_{\rm C}) \qquad \text{for all } t\in[0,T]; \tag{6.9b}$$

(2) (u,z) *is a* VE *solution of the brittle delamination system* $(X,\mathcal{E},\mathsf{d}_Z)$ (6.5)*, with* δ_Z *from* (6.7)*, such that the minimality property* (2.39) *holds at all* $t\in[0,T]\setminus\bar{\rm J}$*, with* $\bar{\rm J}$ *a negligible subset of* $(0,T]$*.*

Furthermore, we have the additional convergences as $j\to\infty$

$$\mathcal{E}_{k_j}(t,u_{k_j}(t),z_{k_j}(t))\to\mathcal{E}(t,u(t),z(t)) \text{ and } {\rm Var}_{\mathsf{d}_Z,\mathsf{c}}(z_{k_j},[0,t])\to{\rm Var}_{\mathsf{d}_Z,\mathsf{c}}(z,[0,t]) \qquad \text{for all } t\in[0,T]. \tag{6.10}$$

Observe that we are able to recover the minimality property (2.39) only almost everywhere in $(0,T)$. In fact, our argument for (2.39) is based on the limit passage as $k\to\infty$ in the VE-stability condition, which holds outside the jump sets of the curves $(z_k)_k$.

Remark 6.2. The existence of VE solutions to the adhesive contact system $(X,\mathcal{E}_k,\mathsf{d}_Z)$ could be extended to the case of the 'non-trivial' viscous correction

$$\delta_Z(z,z') := \frac{1}{q}\|z'{-}z\|_{L^q(\Gamma_{\rm C})}^\gamma, \quad q,\,\gamma>1, \tag{6.11}$$

as soon as a gradient regularizing term of the type $|\nabla z|^r$ is added to the energy functional $\mathcal{E}_k$ (under the additional, technical condition that $\Gamma_{\rm C}$ is a 'flat' $(d{-}1)$-dimensional surface, so that Laplace-Beltrami operators can be avoided). The exponents r, q, γ should satisfy the compatibility condition (4.10). For instance, in the case $\Omega \subset \mathbb{R}^3$ (so that $\Gamma_{\rm C} \subset \mathbb{R}^2$), with $r = 2$ and $q = 2$ one would have to take $\gamma > 2$.

We could perform the adhesive-to-brittle limit passage with δ_Z from (6.11) by straightforwardly adapting the arguments in the proof of Thm. 6.1. Anyhow, we have preferred not to do so in order to simplify the discussion and highlight the analytical difficulties related to the limit passage in the notion of VE solution.

6.1. **Proof of Theorem 6.1.** In what follows we will use that

$$\exists\, c_1, c_2 > 0 \ \ \forall\, k \in \mathbb{N} \cup \{\infty\} \ \ \forall\, (t, u, z) \in [0, T] \times U \times Z \, : \qquad \mathcal{E}_k(t, u, z) \geq c_1 \|u\|^2_{H^1(\Omega\backslash\Gamma_{\rm C})} - c_2 \tag{6.12}$$

(where, with slight abuse of notation, the index $k = \infty$ refers to the limiting energy functional $\mathcal{E}$ (6.5b)), due to the positive definiteness of $\mathbb{C}$, Korn's inequality and (6.8). As a consequence of this coercivity property and of the fact that $u \mapsto \mathcal{E}_k(t, u, z)$ is lower semicontinuous w.r.t. H^1-weak convergence and uniformly convex, we have that $\mathrm{Argmin}_{u\in U}\, \mathcal{E}_k(t, u, z) \neq \emptyset$ is non-empty, and consists of a unique minimizer, for all $k \in \mathbb{N} \cup \{\infty\}$.

Now, let us recall the Γ-convergence properties of the adhesive contact energies $(\mathcal{E}_k)_k$. Such properties are at the core of the proof of Thm. 6.1.

Lemma 6.3. [RSZ09, Corollary 3.2] *Assume* (6.1), (6.4), *and* (6.8). *Then the functionals* $\mathcal{E}_k$ *from* (6.6) Γ-*converge as* $k \to \infty$ *to* $\mathcal{E}$ *w.r.t. to the* $\sigma_{\mathbb{R}}$-*topology of* $[0, T] \times H^1(\Omega\backslash\Gamma_{\rm C}; \mathbb{R}^d) \times L^\infty(\Gamma_{\rm C})$ *(i.e., the weak*-topology), namely there hold the*

$$\begin{aligned} &\Gamma\text{-}\liminf \ \textit{estimate: } (t_k, u_k, z_k) \overset{\sigma_{\mathbb{R}}}{\to} (t, u, z) \ \Rightarrow && \liminf_{k\to\infty} \mathcal{E}_k(t_k, u_k, z_k) \geq \mathcal{E}(t, u, z), \\ &\Gamma\text{-}\limsup \ \textit{estimate: } \forall\, (t, u, z)\ \exists\, (t_k, u_k, z_k)_k \, : (t_k, u_k, z_k) \overset{\sigma_{\mathbb{R}}}{\to} (t, u, z), && \limsup_{k\to\infty} \mathcal{E}_k(t_k, u_k, z_k) \leq \mathcal{E}(t, u, z)\,. \end{aligned} \tag{6.13}$$

In order to pass to the limit in the VE-formulation, we also need to investigate the closure, as $k \to \infty$, of the stable (in the Visco-Energetic sense) sets

$$\begin{aligned} \mathscr{S}^k_{\mathsf{D}_Z} := \{(t, u, z) \in [0, T] \times U \times Z \, : \, &\mathcal{E}_k(t, u, z) \leq \mathcal{E}_k(t, u', z') + \mathsf{d}_Z(z, z') + h(\mathsf{d}_Z(z, z')) \\ &\text{for all } (u', z') \in U \times Z\}, \ k \in \mathbb{N}, \end{aligned}$$

with h as (6.7) (while we will denote by $\mathscr{S}_{\mathsf{D}}$ the stable set for the brittle delamination system). More precisely, we will study the *Kuratowski limit inferior*

$$\mathrm{Li}_{k\to\infty} \mathscr{S}^k_{\mathsf{D}_Z} := \{(t, u, z) \in [0, T] \times U \times Z \, : \, \exists\, (t_k, z_k, u_k) \in \mathscr{S}^k_{\mathsf{D}_Z} \text{ such that } (t_k, z_k, u_k) \overset{\sigma_{\mathbb{R}}}{\to} (t, u, z)\}\,.$$

Recall that, by (2.21) $\mathscr{S}^k_{\mathsf{D}_Z}$ is the zero set of the residual stability function

$$\mathscr{R}_k(t, z) := \sup_{z'\in Z} \{\mathcal{I}_k(t, z) - \mathcal{I}_k(t, z') - \mathsf{d}_Z(z, z') - h(\mathsf{d}_Z(z, z'))\} \quad \text{with } \mathcal{I}_k(t, z) = \min_{u\in H^1_{\Gamma_{\rm D}}(\Omega\backslash\Gamma_{\rm C};\mathbb{R}^d)} \mathcal{E}_k(t, u, z).$$

In fact, the study of $\mathrm{Li}_{k\to\infty} \mathscr{S}^k_{\mathsf{D}_Z}$ is related to the analysis of Γ-$\liminf$ (w.r.t. $\sigma_{\mathbb{R}}$-topology) of the functionals $(\mathscr{R}_k)_k$. That is why, we will first obtain the $\liminf$-inequality (6.14) below. Such estimate will also play a crucial role for the limit passage in the Visco-Energetic energy-dissipation balance as $k \to \infty$.

Lemma 6.4. *Assume* (6.1), (6.4), (6.8). *Then, for all* $(t_k, z_k)_k \subset [0, T] \times Z$ *there holds*

$$(t_k, z_k) \to (t, z) \ \textit{in } [0, T] \times Z \ \Rightarrow \ \liminf_{k\to\infty} \mathscr{R}_k(t_k, z_k) \geq \mathscr{R}(t, z)\,. \tag{6.14}$$

Moreover,

$$\mathrm{Li}_{k\to\infty} \mathscr{S}^k_{\mathsf{D}_Z} \subset \mathscr{S}_{\mathsf{D}_Z}\,. \tag{6.15}$$

Proof. We start by showing (6.14). We use that

$$\mathscr{R}_k(t_k, z_k) = \sup_{z'\in Z} \left(\mathcal{I}_k(t_k, z_k) - \mathcal{I}_k(t_k, z') - \mathsf{d}_Z(z_k, z') - h(\mathsf{d}_Z(z_k, z'))\right), \tag{6.16a}$$

$$\mathscr{R}(t, z) = \sup_{z'\in Z} \left(\mathcal{I}(t, z) - \mathcal{I}(t, z') - \mathsf{d}_Z(z, z') - h(\mathsf{d}_Z(z, z'))\right) \tag{6.16b}$$

(with $\mathcal{I}$ the reduced energy associated with $\mathcal{E}$). To prove (6.14) it is thus sufficient to exhibit, for any fixed $z' \in Z$ with $\mathsf{d}_Z(z,z') < \infty$ (i.e., $z' \leq z$ a.e. in Γ_{C}) and $\mathcal{I}_k(t,z') < \infty$, a *recovery* sequence $(z_k')_k \subset Z$ such that

$$\limsup_{k\to\infty} (\mathcal{I}_k(t_k,z_k') + \mathsf{d}_Z(z_k,z_k') + h(\mathsf{d}_Z(z_k,z_k'))) \leq (\mathcal{I}(t,z') + \mathsf{d}_Z(z,z') + h(\mathsf{d}_Z(z,z'))) \,. \tag{6.17}$$

Then, we will have

$$\begin{aligned}\liminf_{k\to\infty} \mathscr{R}_k(t_k,z_k) &\overset{(6.16a)}{\geq} \liminf_{k\to\infty} (\mathcal{I}_k(t_k,z_k) - \mathcal{I}_k(t_k,z_k') - \mathsf{d}_Z(z_k,z_k') - h(\mathsf{d}_Z(z_k,z_k'))) \\ &\overset{(6.17)}{\geq} (\mathcal{I}(t,z) - \mathcal{I}(t,z') - \mathsf{d}_Z(z,z') - h(\mathsf{d}_Z(z,z'))) \,,\end{aligned}$$

where we have also exploited the Γ-lim inf-estimate in (6.13). Then, (6.14) shall follow from the arbitrariness of z'. We borrow the definition of the sequence $(z_k')_k$ from the proof of [RSZ09, Thm. 3.3], letting

$$z_k' := \begin{cases} z_k \dfrac{z'}{z} & \text{if } z' > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6.18}$$

Since $z' \leq z$ a.e. in Γ_{C}, it is immediate to verify that $0 \leq z_k' \leq z_k \leq 1$ a.e. in Γ_{C}. Furthermore, $z_k \rightharpoonup^* z$ in $L^\infty(\Gamma_{\mathrm{C}})$ gives $z_k' \rightharpoonup^* z'$ in $L^\infty(\Gamma_{\mathrm{C}})$. Therefore,

$$\begin{gathered}\lim_{k\to\infty} \mathsf{d}_Z(z_k,z_k') = \lim_{k\to\infty} \int_{\Gamma_{\mathrm{C}}} \kappa(z_k(x) - z_k'(x)) \,\mathrm{d}\mathscr{H}^{d-1}(x) = \int_{\Gamma_{\mathrm{C}}} \kappa(z(x) - z'(x)) \,\mathrm{d}\mathscr{H}^{d-1}(x) = \mathsf{d}_Z(z,z') \\ \text{whence } \lim_{k\to\infty} h(\mathsf{d}_Z(z_k,z_k')) = h(\mathsf{d}_Z(z,z')),\end{gathered} \tag{6.19}$$

too. Let us now consider the (unique) minimizer $u' \in U$ for $\mathcal{E}(t,\cdot,z')$. We have

$$\limsup_{k\to\infty} \mathcal{I}_k(t_k,z_k') \leq \limsup_{k\to\infty} \mathcal{E}_k(t_k,u',z_k') = \lim_{k\to\infty} \mathcal{E}_k(t_k,u',z_k') \overset{(1)}{=} \mathcal{E}(t,u',z') = \mathcal{I}(t,z'). \tag{6.20}$$

Indeed, for (1) we have used the fact that $z_k'[\![u']\!] = 0$ a.e. in Γ_{C}, which follows from $z'[\![u']\!] = 0$ and from the definition (6.18) of z_k'. From (6.19) and (6.20) we clearly conclude (6.17), whence (6.14).

In order to show that every element (t,u,z) in $\mathrm{Li}_{k\to\infty}\mathscr{S}^k_{\mathsf{D}_Z}$ fulfills the D_Z-stability condition with the brittle energy functional, for every (u',z') we need to exhibit a recovery sequence $(u_k',z_k')_k$ such that

$$\limsup_{k\to\infty} (\mathcal{E}_k(t_k,u_k',z_k') + \mathsf{d}_Z(z_k,z_k') + h(\mathsf{d}_Z(z_k,z_k'))) \leq (\mathcal{E}(t,u',z') + \mathsf{d}_Z(z,z') + h(\mathsf{d}_Z(z,z'))) \,.$$

The sequence $(u_k',z_k')_k := (u',z_k')_k$ with $(z_k')_k$ from (6.18), does the job. This finishes the proof. □

The **proof of Thm. 6.1** will be carried out in the following steps:

(1) First of all, we will show that the sequence $(z_{k_j})_{j\in\mathbb{N}}$ of VE solutions in the statement of the theorem does admit a subsequence converging in the sense of (6.9b) to z;
(2) Secondly, we will prove that z complies with the stability condition (S_{VE}) for the brittle system $(X,\mathcal{E},\mathsf{d}_Z)$ from (6.5) and, as a byproduct, obtain convergence (6.9a) for $(u_{k_j})_{j\in\mathbb{N}}$;
(3) Thirdly, we will show that (u,z) fulfills the upper energy-dissipation estimate (2.42) for the brittle system also relying on Proposition 6.5 ahead;
(4) We shall thus conclude that (u,z) is a VE solution to the brittle system $(X,\mathcal{E},\mathsf{d}_Z)$ (6.5).

▷ **Step 1:** Since the constant C_0 in (2.40) only depends on the initial data (u_0,z_0), which in turn do not depend on k_j, for the VE solutions $(u_{k_j},z_{k_j})_j$ to the adhesive contact system the following bounds are valid

$$\exists\, C>0\ \forall\, j\in\mathbb{N}\ \forall\, t\in[0,T]:\quad \sup_{t\in[0,T]} |\mathcal{E}_{k_j}(t,u_{k_j}(t),z_{k_j}(t))| + \mathrm{Var}_{\mathsf{d}_Z}(z_{k_j},[0,T]) \leq C\,.$$

In view of (6.12), and taking into account that $z\in[0,1]$ a.e. on $\Gamma_{\mathrm{C}}\times(0,T)$, we conclude that the sequences $(u_{k_j})_j$ and $(z_{k_j})_j$ are bounded in $L^\infty(0,T;H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d))$ and in $L^\infty(\Gamma_{\mathrm{C}}\times(0,T))\cap\mathrm{BV}([0,T];L^1(\Gamma_{\mathrm{C}}))$, respectively. An infinite-dimensional version of Helly's compactness theorem (cf., e.g., [MM05, Thm. 3.2]) yields that, up to a not relabeled subsequence, convergence (6.9b) for $(z_{k_j})_j$ holds. As for $(u_{k_j})_j$, for every $t \in (0,T]$ there exist a subsequence (k_j^t), possibly depending on t, and $\tilde{u}(t) \in H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d)$ such that

$$u_{k_j^t}(t) \rightharpoonup \tilde{u}(t) \quad \text{in } H^1_{\mathrm{D}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d). \tag{6.21}$$

Furthermore, mimicking the arguments in the proof of [MS18, Thm. 7.2], we also find a finer approximation property at every t in the jump set J_z of z, namely

$$\forall\, t \in \mathrm{J}_z \cap (0,T)\ \exists\, (\alpha_{k_j})_j,\ (\beta_{k_j})_j \subset [0,T] \text{ such that } \begin{cases} \alpha_{k_j} \uparrow t \text{ and } z_{k_j}(\alpha_{k_j}) \rightharpoonup^* z(t-) \text{ in } L^\infty(\Gamma_{\mathrm{C}}), \\ \beta_{k_j} \downarrow t \text{ and } z_{k_j}(\beta_{k_j}) \rightharpoonup^* z(t+) \text{ in } L^\infty(\Gamma_{\mathrm{C}}), \end{cases} \tag{6.22}$$

with obvious modifications at $t \in \mathrm{J}_z \cap \{0,T\}$.

$\rhd$ **Step 2:** Let us consider the lim sup of the jump sets $(\mathrm{J}_{z_{k_j}})_j$, i.e. $\tilde{\mathrm{J}} := \cap_{m\in N} \cup_{j\geq m} \mathrm{J}_{z_{k_j}}$. For every $t \in [0,T]\setminus\tilde{\mathrm{J}}$ there exists $m_t \in \mathbb{N}$ such that for every $j \geq m_t$ we have $t \in [0,T] \setminus \mathrm{J}_{z_{k_j}}$. Therefore, up to taking a bigger m_t if necessary, we have $(t, u_{k_j^t}(t), z_{k_j^t}(t)) \in \mathscr{S}_{\mathsf{D}}^{k_j^t}$ for all $j \geq m_t$. By virtue of (6.15), we conclude that

$$(t, \tilde{u}(t), z(t)) \in \mathscr{S}_{\mathsf{D}} \quad \text{for all } t \in [0,T] \setminus \tilde{\mathrm{J}}. \tag{6.23}$$

From (6.23) we gather, in particular, that $\tilde{u}(t)$ is the *unique* element in $\mathrm{Argmin}_{u'\in U}\mathcal{E}(t,u',z(t))$. Therefore, convergence (6.21) holds at every $t \in [0,T] \setminus \tilde{\mathrm{J}}$ along the *whole* sequence $(k_j)_j$. This shows (6.9a) at all $t \in [0,T] \setminus \tilde{\mathrm{J}}$.

Finally, we conclude the validity of (6.9a) at *every* $t \in [0,T]$ by observing that, at every t in the *countable* set $\tilde{\mathrm{J}}$ we can extract a subsequence of $(k_j)_j$ such that (6.21) holds. With a diagonal procedure we thus construct a subsequence fitting all $t \in \tilde{\mathrm{J}}$ and (6.9a) follows.

We now show that

$$z(t-),\, z(t+) \in \mathscr{S}_{\mathsf{D}}(t) \quad \text{for all } t \in (0,T), \quad z(0+) \in \mathscr{S}_{\mathsf{D}}(0), \quad z(T-) \in \mathscr{S}_{\mathsf{D}}(T)\,. \tag{6.24}$$

In order to prove the assert at $t \in (0,T)$ and, e.g., for $z(t-)$, we pick a sequence $(t_n)_n \subset [0,T] \setminus \tilde{\mathrm{J}}$ with $t_n \uparrow t$ as $n \to \infty$, so that $z(t_n) \rightharpoonup^* z(t-)$ in $L^\infty(\Gamma_{\mathrm{C}})$ (cf. Definition 2.1). From (6.23) we have that $\mathscr{R}(t_n, z(t_n)) = 0$ for all $n \in \mathbb{N}$. With the very same arguments as in the proof of Lemma 6.4, it can be shown that $\mathscr{R}$ is lower semicontinuous w.r.t. the weak*-topology of $[0,T] \times Z$. Thus, we conclude that $\mathscr{R}(t, z(t)) = 0$.

From (6.24) we clearly conclude that (u,z) fulfills the stability condition $(\mathrm{S_{VE}})$ at all $t \in [0,T]\setminus\mathrm{J}_z$, which in particular yields the minimality property (2.39) at all $t \in [0,T]\setminus\mathrm{J}_z$. All in all, (2.39) holds at every $t \in [0,T]\setminus\bar{\mathrm{J}}$ with $\bar{\mathrm{J}} = \tilde{\mathrm{J}} \cap \mathrm{J}_z$.

$\rhd$ **Step 3:** Let us now take the lim inf as $k \to \infty$ in the (upper) energy-dissipation estimate (2.42) for the adhesive contact system. We handle the terms on the left-hand side by observing that

$$\liminf_{j\to\infty} \mathcal{E}_{k_j}(t, u_{k_j}(t), z_{k_j}(t)) \geq \mathcal{E}(t,u(t),z(t)) \text{ and } \liminf_{j\to\infty} \mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}(z_{k_j},[0,t]) \geq \mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}(z,[0,t]) \quad \text{for all } t \in [0,T],$$

where the first inequality is due to the Γ-lim inf estimate (6.13), and the second one follows from Proposition 6.5 ahead. As for the right-hand side, we observe that

$$\begin{aligned}\partial_t \mathcal{E}_{k_j}(t, u_{k_j}(t), z_{k_j}(t)) = -\langle \dot{\ell}(t), u_{k_j}(t) + w_{\mathrm{D}}(t)\rangle_{H^1} - \langle \ell(t), \dot{w}_{\mathrm{D}}(t)\rangle_{H^1} &\to -\ \langle \dot{\ell}(t), u(t) + w_{\mathrm{D}}(t)\rangle_{H^1} - \langle \ell(t), \dot{w}_{\mathrm{D}}(t)\rangle_{H^1} \\ &= \partial_t \mathcal{E}(t,u(t),z(t))\end{aligned}$$

for every $t \in [0,T]$, with $|\partial_t \mathcal{E}_{k_j}(t, u_{k_j}(t), z_{k_j}(t))| \leq C$ by (6.8) and the previously obtained bound for $(u_{k_j})_j$ in $L^\infty(0,T; H^1_{\Gamma_{\mathrm{D}}}(\Omega\backslash\Gamma_{\mathrm{C}};\mathbb{R}^d))$. Then,

$$\lim_{j\to\infty} \int_0^t \partial_t \mathcal{E}_{k_j}(s, u_{k_j}(s), z_{k_j}(s))\,\mathrm{d}s = \int_0^t \partial_t \mathcal{E}(s,u(s),z(s))\,\mathrm{d}s \quad \text{for all } t \in [0,T], \tag{6.25}$$

and we thus conclude the upper energy-dissipation estimate (2.42) for the brittle system.

$\triangleright$ **Step 4, conclusion of the proof:** Since we have proved the stability condition (S_{VE}) and the upper energy-dissipation estimate (2.42), thanks to Proposition 2.11 we conclude that (u,z) is a VE solution of the brittle system. The energy convergence (6.10) ensues from the following standard argument:

$$\begin{aligned}\limsup_{j\to\infty}\left(\mathcal{E}_{k_j}(t,u_{k_j}(t),z_{k_j}(t))+\mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}(z_{k_j},[0,t])\right) &\stackrel{(1)}{\leq} \mathcal{E}(0,u_0,z_0)+\lim_{j\to\infty}\int_0^t \partial_t\mathcal{E}_{k_j}(s,u_{k_j}(s),z_{k_j}(s))\,\mathrm{d}s\\ &\stackrel{(2)}{=} \mathcal{E}(0,u_0,z_0)+\int_0^t \partial_t\mathcal{E}(s,u(s),z(s))\,\mathrm{d}s\\ &\stackrel{(3)}{=} \mathcal{E}(t,u(t),z(t))+\mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}(z,[0,t])\,,\end{aligned}$$

with (1) due to (E_{VE}) for the adhesive system, (2) due to (6.25), and (3) following from the energy balance (E_{VE}) for the brittle system. This finishes the proof of Thm. 6.1. ∎

With the following result we obtain the key lower semicontinuity estimate for the total variation functionals exploited in Step 3 of the proof of Thm. 6.1.

Proposition 6.5. *Assume* (6.1), (6.4), *and* (6.8). *Let* $(z_k)_k,\ z\subset L^\infty(\Gamma_{\mathrm{C}}\times(0,T))\cap\mathrm{BV}([0,T];L^1(\Omega))$ *fulfill*

$$z_k(t)\rightharpoonup^* z(t) \text{ in } L^\infty(\Gamma_{\mathrm{C}}) \text{ for all } t\in[0,T], \tag{6.26a}$$

$$\forall\, t\in \mathrm{J}_z\ \exists\,(\alpha_k)_k,\ (\beta_k)_k\subset[0,T] \text{ such that } \begin{cases}\alpha_k\uparrow t \text{ and } z_k(\alpha_k)\rightharpoonup^* z(t-) \text{ in } L^\infty(\Gamma_{\mathrm{C}}),\\ \beta_k\downarrow t \text{ and } z_k(\beta_k)\rightharpoonup^* z(t+) \text{ in } L^\infty(\Gamma_{\mathrm{C}}).\end{cases} \tag{6.26b}$$

Then,

$$\liminf_{k\to\infty}\mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}(z_k,[a,b])\geq \mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}(z,[a,b]) \quad \text{for all } [a,b]\subset[0,T]. \tag{6.27}$$

The *proof* follows the very same lines as the argument for [RS17, Thm. 4], to which we shall refer for all details. Let us just outline it: up to an extraction we may suppose that $\sup_{k\in\mathbb{N}}\mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}(z_k,[0,T])\leq C$. Therefore, the non-negative and bounded Borel measures η_k on $[0,T]$ defined by $\eta_k([a,b]):=\mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}(z_k,[a,b])$ for all $[a,b]\subset[0,T]$ weakly* converge (in the duality with $\mathrm{C}^0([0,T])$) to a measure η. We observe that

$$\eta([a,b])\stackrel{(1)}{\geq}\limsup_{k\to\infty}\eta_k([a,b])\stackrel{(2)}{\geq}\limsup_{k\to\infty}\mathrm{Var}_{\mathsf{d}_Z}(z_k,[a,b])\stackrel{(3)}{\geq}\mathrm{Var}_{\mathsf{d}_Z}(z,[a,b]), \tag{6.28}$$

with (1) due to the upper semicontinuity of the weak* convergence of measures on closed sets, (2) due to the fact that $\mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}\geq\mathrm{Var}_{\mathsf{d}_Z}$, and (3) due to (6.26a). It follows from Lemma 6.6 ahead that, at any $t\in\mathrm{J}_z$ and for all sequences $(\alpha_k)_k,\ (\beta_k)_k\subset[0,T]$ fulfilling (6.26b) there holds

$$\eta(\{t\})\geq\limsup_{k\to\infty}\eta_k([\alpha_k,\beta_k])\geq\mathsf{c}(t,z(t-),z(t+))\,. \tag{6.29}$$

Combining (6.28) and (6.29) and arguing as in the proof of [RS17, Thm. 4] (cf. also [MRS16, Prop. 7.3]), we establish (6.27). ∎

We conclude this section by stating a crucial lower estimate for the Visco-Energetic total variation of a sequence $(z_k)_k$ of solutions to the adhesive contact system (for notational simplicity, we drop the subsequence $(k_j)_j$ and revert to the original sequence of indexes (k)). The total variation of the curves z_k is considered on a sequence of intervals shrinking as $k\to\infty$ to a jump point of the limit curve z.

Lemma 6.6. *Assume* (6.1), (6.4), *and* (6.8). *Let* $(z_k)_k,\ z\subset L^\infty(\Gamma_{\mathrm{C}}\times(0,T))\cap\mathrm{BV}([0,T];L^1(\Omega))$ *fulfill* (6.26). *For any* $t\in\mathrm{J}_z$ *pick two sequences* $(\alpha_k)_k$ *and* $(\beta_k)_k$ *converging to* t *and fulfilling* (6.26). *Then,*

$$\liminf_{k\to\infty}\mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}(z_k,[\alpha_k,\beta_k])\geq\mathsf{c}(t,z(t-),z(t+))\,. \tag{6.30}$$

The *proof* will be given in Sec. 6.2.

6.2. **Proof of Lemma 6.6.** Let us briefly outline the proof, partially borrowed from that of [RS17, Prop. 3]:

(1) for every $k \in \mathbb{N}$, the curve z_k has countably many jump points $(t^k_m)_{m\in M_k}$ between α_k and β_k. As in [RS17], we will suitably reparameterize both the continuous pieces of the trajectory z_k and the optimal transitions $\vartheta^k_{z,m}$ connecting the left and right limits $z_k(t^k_m-)$ and $z_k(t^k_m+)$ at a jump point t^k_m. We will then glue the (reparameterized) continuous pieces and the (reparameterized) jump transitions together.

(2) In this way, we shall obtain a sequence of curves $(\zeta_k)_k$, defined on compact sets $(\mathfrak{C}_k)_k$, to which we will apply a refined compactness argument from [MS18], yielding the existence of a limiting Lipschitz curve ζ, defined on a compact set $\mathfrak{C} \Subset \mathbb{R}$, connecting the left and the right limits $z(t-)$ and $z(t+)$.

(3) We will then show that

$$\liminf_{k\to\infty} \mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}(z_k,[\alpha_k,\beta_k]) \geq \mathrm{Trc}_{\mathrm{VE}}(t,\zeta,\mathfrak{C})\,. \tag{6.31}$$

(4) From (6.31) we shall conclude (6.30).

$\triangleright$ **Step 1 (reparameterization):** We set

$$\mathfrak{m}_k := \beta_k - \alpha_k + \mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}(z_k,[\alpha_k,\beta_k]) + \sum_{m\in M_k} 2^{-m}$$

and define the rescaling function $\mathfrak{s}_k : [\alpha_k,\beta_k] \to [0,\mathfrak{m}_k]$ by

$$\mathfrak{s}_k(t) := t - \alpha_k + \mathrm{Var}_{\mathsf{d}_Z,\mathsf{c}}(z_k,[\alpha_k,t]) + \sum_{\{m\in M_k:\, t^k_m\leq t\}} 2^{-m}\,.$$

Observe that $\mathfrak{s}_k$ is strictly increasing, with jump set $\mathrm{J}_{\mathfrak{s}_k} = (t^k_m)_{m\in M_k}$. We set

$$I^k_m := (\mathfrak{s}_k(t^k_m-),\mathfrak{s}_k(t^k_m+)),\quad I_k := \cup_{m\in M_k} I^k_m,\quad \Lambda_k := [\mathfrak{s}_k(\alpha_k),\mathfrak{s}_k(\beta_k)].$$

On $\Lambda_k \setminus I_k$ the inverse $\mathsf{t}_k : \Lambda_k\setminus I_k \to [\alpha_k,\beta_k]$ of $\mathfrak{s}_k$ is well defined and Lipschitz continuous. We introduce

$$\zeta_k(s) := (u_k \circ \mathsf{t}_k)(s) \qquad \text{for all } s \in \Lambda_k \setminus I_k \tag{6.32}$$

and observe that ζ_k is Lipschitz as well.

We now reparameterize the 'jump pieces' of the trajectory. Recall that at every jump point t^k_m there exists an optimal jump transition $\vartheta^k_{z,m} \in \mathrm{C}_{\sigma_Z,\mathsf{d}_Z}(E^k_m;Z)$, fulfilling

$$\begin{aligned} &z(t^k_m-) = \vartheta^k_{z,m}((E^k_m)^-), \qquad z(t^k_m+) = \vartheta^k_{z,m}((E^k_m)^+), \qquad z(t^k_m) \in \vartheta^k_{z,m}(E^k_m),\\ &\mathcal{E}_k(t^k_m,u(t^k_m-),z(t^k_m-)) - \mathcal{E}_k(t^k_m,u(t^k_m+),z(t^k_m+)) = \mathsf{c}(t^k_m,z(t^k_m-),z(t^k_m+)) = \mathrm{Trc}_{\mathrm{VE}}(t^k_m,\vartheta^k_{z,m},E^k_m)\,. \end{aligned} \tag{6.33}$$

We define the rescaling function σ^k_m on E^k_m by

$$\begin{aligned} \sigma^k_m(t) :=& \frac{1}{2^m}\frac{t-(E^k_m)^-}{(E^k_m)^+-(E^k_m)^-} + \mathrm{Var}_{\mathsf{d}_Z}(\vartheta^k_{z,m},E^k_m\cap[(E^k_m)^-,t])\\ &+ \mathrm{GapVar}_{\delta_Z}(\vartheta^k_{z,m},E^k_m\cap[(E^k_m)^-,t]) + \sum_{r\in[(E^k_m)^-,t]\setminus(E^k_m)^+} \mathscr{R}_k(t^k_m,\vartheta^k_{z,m}(r)) + \mathfrak{s}_k(t^k_m-) \quad \text{for all } t\in E^k_m. \end{aligned}$$

It can be checked that σ^k_m is continuous and strictly increasing, with image a compact set $S^k_m \subset I^k_m$ such that $(S^k_m)^\pm = \sigma^k_m((E^k_m)^\pm) = \mathfrak{s}_k(t^k_m\pm)$. Its inverse function $\tau^k_m : S^k_m \to E^k_m$ is Lipschitz continuous.

Finally, we introduce the *compact* set

$$\mathfrak{C}_k := (\Lambda_k\setminus I_k) \cup (\cup_{m\in M_k} S^k_m) \subset \Lambda_k \subset [0,\mathfrak{m}_k]$$

and extend the functions t_k and ζ_k, so far defined on $\Lambda_k\setminus I_k$, only, to the set $\mathfrak{C}_k$ by setting

$$\mathsf{t}_k(s) \equiv t^k_m \quad \text{and} \quad \zeta_k(s) := \vartheta^k_m(\tau^k_m(s)) \qquad \text{whenever } s\in S^k_m \text{ for some } m\in M_k.$$

It has been checked in [RS17] that the extended curve ζ_k is in $\mathrm{C}_{\sigma_Z,\mathsf{d}_Z}(\mathfrak{C}_k;X)\cup\mathrm{BV}_{\mathsf{d}_Z}(\mathfrak{C}_k;X)$, with

$$\begin{aligned} &\mathrm{Var}_{\mathsf{d}_Z}(\zeta_k,[s_0,s_1]) \leq \mathrm{Var}_{\mathsf{d}_Z}(z_k,[\mathsf{t}_k(s_0),\mathsf{t}_k(s_1)]) + (\mathsf{t}_k(s_1){-}\mathsf{t}_k(s_0)) \quad \text{for all } s_0,\, s_1\in\Lambda_k\setminus I_k \text{ with } s_0<s_1,\\ &\mathrm{Var}_{\mathsf{d}_Z}(\zeta_k,S_m^k) = \mathrm{Var}_{\mathsf{d}_Z}(\vartheta_{z,m}^k,E_m^k), \qquad \mathrm{GapVar}_{\delta_Z}(\zeta_k,S_m^k) = \mathrm{GapVar}_{\delta_Z}(\vartheta_{z,m}^k,E_m^k),\\ &\sum_{s\in S_m^k\setminus\{(S_m^k)^+\}} \mathscr{R}_k(t_m^k,\zeta_k(s)) = \sum_{r\in E_m^k\setminus\{(E_m^k)^+\}} \mathscr{R}_k(t_m^k,\vartheta_{z,m}^k(r)). \end{aligned} \tag{6.34}$$

$\triangleright$ **Step 2 (a priori estimates and compactness):** We refer to the proof of [RS17, Prop. 3] for the calculations leading to these a priori estimates:

$$\exists\,\overline{C}>0\ \forall\,k\in\mathbb{N}\ : \qquad \begin{cases} \mathfrak{C}_k^+ \leq \overline{C},\\ \mathrm{Var}_{\mathsf{d}_Z}(\zeta_k,\mathfrak{C}_k)\leq \overline{C},\\ \mathrm{Var}_{\mathsf{d}_Z}(\zeta_k,\mathfrak{C}_k\cap[s_0,s_1]) \leq (s_1{-}s_0) \quad \text{for all } s_0,s_1\in\mathfrak{C}_k \text{ with } s_0<s_1,\\ \sup_{s\in\mathfrak{C}_k}\mathcal{F}_{0,k}(\mathfrak{u}_k(s),\zeta_k(s))\leq\overline{C}, \end{cases} \tag{6.35}$$

where $\mathfrak{u}_k(s)$ is the unique element in $\mathrm{Argmin}_{u\in U}\mathcal{E}_k(s,u,\zeta_k(s))$ and $\mathcal{F}_{0,k}$ is the perturbed functional associated with $\mathcal{E}_k$ via (2.8).

Therefore, we are in a position to apply the compactness result from [MS18, Thm. 5.4] and conclude that there exist a (not relabeled) subsequence, a compact set $\mathfrak{C}\subset[0,\overline{C}]$ with $\overline{C}$ as in (6.35), and a function $\zeta\in\mathrm{C}_{\sigma_Z,\mathsf{d}_Z}(\mathfrak{C};X)$ such that, as $k\to\infty$, there hold

(1) $\mathfrak{C}_k\to\mathfrak{C}$ à la Kuratowski, namely $\mathrm{Li}_{k\to\infty}\mathfrak{C}_k = \mathrm{Ls}_{k\to\infty}\mathfrak{C}_k = \mathfrak{C}$ with

$$\begin{aligned} &\mathrm{Li}_{k\to\infty}\mathfrak{C}_k := \{t\in[0,\infty)\,:\,\exists\,t_k\in\mathfrak{C}_k \text{ s.t. } t_k\to t\},\\ &\mathrm{Ls}_{k\to\infty}\mathfrak{C}_k := \{t\in[0,\infty)\,:\,\exists\,j\mapsto k_j \text{ increasing and } t_{k_j}\in\mathfrak{C}_{k_j} \text{ s.t. } t_{k_j}\to t\}; \end{aligned}$$

(2) for every $s\in\mathfrak{C}$ there exists a sequence $(s_k)_k$, with $s_k\in\mathfrak{C}_k$ for all $k\in\mathbb{N}$, such that $s_k\to s$ and $\zeta_k(s_k)\overset{\sigma_Z}{\to}\zeta(s)$ in Z as $k\to\infty$;

(3) whenever $s_k\in\mathfrak{C}_k$ converge to $s\in\mathfrak{C}$, then $\zeta_k(s_k)\overset{\sigma_Z}{\to}\zeta(s)$ in Z;

(4) $\zeta_k((\mathfrak{C}_k)^\pm)\overset{\sigma_Z}{\to}\zeta(\mathfrak{C}^\pm)$;

(5) for every $I\in\mathfrak{h}(\mathfrak{C})$ (recall (2.31b)) there exists a sequence $(J_k)_k$ with

$$J_k\in\mathfrak{h}(\mathfrak{C}_k) \text{ for all } k\in\mathbb{N} \text{ and } J_k^+\to I^+,\ J_k^-\to I^-\,. \tag{6.36}$$

Therefore, $\zeta(\mathfrak{C}^-)=z(t-)$, and $\zeta(\mathfrak{C}^+)=z(t+)$. Finally, for later use we observe that

$$\lim_{k\to\infty}\sup_{s\in\mathfrak{C}_k}|\mathsf{t}_k(s)-t| = 0, \tag{6.37}$$

since the functions t_k take values in the intervals $[\alpha_k,\beta_k]$ shrinking to the singleton $\{t\}$.

$\triangleright$ **Step 3 (proof of** (6.31)**):** Repeating the very same arguments as in the proof of [MS18, Thm. 5.3], from the above convergence properties we conclude

$$\mathrm{Var}_{\mathsf{d}_Z}(\zeta,\mathfrak{C}) \leq \liminf_{k\to\infty}\mathrm{Var}_{\mathsf{d}_Z}(\zeta_k,\mathfrak{C}_k)\,. \tag{6.38}$$

Let us now address the term in the transition cost involving the residual stability function. To this end, we fix a finite set $\{\mathfrak{C}^- =: \sigma^0 < \sigma^1 < \ldots < \sigma^N := \mathfrak{C}^+\}\subset\mathfrak{C}$ such that $\mathscr{R}(\sigma^n,\zeta(\sigma^n))>0$ for all $n=1,\ldots,N-1$. We use that for every $n\in\{1,\ldots,N\}$ there exists a sequence $(\sigma_k^n)_k$ with $\sigma_k^n\in\mathfrak{C}_k$ for all $k\in\mathbb{N}$, $\sigma_k^n\to\sigma^n$ and $\zeta_k(\sigma_k^n)\overset{\sigma_Z}{\to}\zeta(\sigma^n)$ as $k\to\infty$. Furthermore, in view of (6.37) we have that $\mathsf{t}_k(\sigma_k^n)\to t$ as $k\to\infty$ for all $n\in\{0,\ldots,N\}$. By the Γ-$\liminf$ estimate (6.14), we infer that

$$\liminf_{k\to\infty}\mathscr{R}_k(\mathsf{t}_k(\sigma_k^n),\zeta_k(\sigma_k^n)) \geq \mathscr{R}(t,\zeta(\sigma^n)) \quad \text{for all } n\in\{0,\ldots,N\},$$

therefore there exist $c>0$ and an index $\bar{k}\in\mathbb{N}$ such that

$$\mathscr{R}_k(\mathsf{t}_k(\sigma_k^n),\zeta_k(\sigma_k^n)) \geq c > 0 \qquad \text{for all } k\geq\bar{k}.$$

This entails that for every $n \in \{1, \ldots, N-1\}$ and $k \geq \bar{k}$ there exists $m_k^n \in M_k$ (the countable set of jump points of z_k between α_k and β_k) such that $\mathsf{t}_k(\sigma_k^n) = t_{m_k^n}$. All in all, we conclude that

$$
\begin{aligned}
\sum_{n=1}^{N-1} \mathscr{R}(t, \zeta(\sigma^n)) \leq \sum_{n=1}^{N-1} \liminf_{k\to\infty} \mathscr{R}_k(t_{m_k^n}, \zeta_k(\sigma_k^n)) &\leq \liminf_{k\to\infty} \sum_{n=1}^{N-1} \mathscr{R}_k(t_{m_k^n}, \zeta_k(\sigma_k^n)) \\
&\leq \liminf_{k\to\infty} \sum_{m\in M_k} \sum_{s\in S_m^k\setminus\{(S_m^k)^+\}} \mathscr{R}_k(t_m^k, \zeta_k(s)) \\
&= \liminf_{k\to\infty} \sum_{m\in M_k} \sum_{r\in E_m^k\setminus\{(E_m^k)^+\}} \mathscr{R}_k(t_m^k, \vartheta_{z,m}^k(r)),
\end{aligned}
$$

the latter identity due to (6.34). Taking the supremum of the left-hand side over all finite subsets of $\mathfrak{C} \setminus \{\mathfrak{C}^+\}$, we then conclude that

$$
\sum_{\sigma\in\mathfrak{C}\setminus\{\mathfrak{C}^+\}} \mathscr{R}(t, \zeta(\sigma)) \leq \liminf_{k\to\infty} \sum_{m\in M_k} \sum_{r\in E_m^k\setminus\{(E_m^k)^+\}} \mathscr{R}_k(t_m^k, \vartheta_{z,m}^k(r)) \,. \tag{6.39}
$$

Finally, (6.36) and, again, the very same arguments as in the proof of [MS18, Thm. 5.3] yield that

$$
\begin{aligned}
\mathrm{GapVar}_{\mathsf{d}_Z}(\zeta, C) = \sum_{I\in\mathfrak{h}(\mathfrak{C})} \delta_Z(\zeta(I^-), \zeta(I^+)) &\leq \liminf_{k\to\infty} \sum_{J\in\mathfrak{h}(\mathfrak{C}_k)} \delta_Z(\zeta_k(J^-), \zeta_k(J^+)) \\
&= \liminf_{k\to\infty} \sum_{m\in M_k} \mathrm{GapVar}_{\delta_Z}(\zeta_k, S_m^k) \\
&\overset{(1)}{=} \liminf_{k\to\infty} \sum_{m\in M_k} \mathrm{GapVar}_{\delta_Z}(\vartheta_{z,m}^k, E_m^k)
\end{aligned}
\tag{6.40}
$$

with (1) due to (6.34). Combining (6.38), (6.39) and (6.40), we deduce (6.31).

▷ **Step** 4 **(conclusion):** Observe that $\mathsf{c}(t, z(t-), z(t+)) \leq \mathrm{Trc}_{\mathrm{VE}}(t, \zeta, \mathfrak{C})$. Therefore, (6.30) follows from (6.31). This finishes the proof of Lemma 6.6. ■

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DIMI, Università di Brescia, via Branze 38, I–25133 Brescia, Italy.
E-mail address: riccarda.rossi@unibs.it