# Metric and geometric relaxations of self-contracted curves

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**Abstract** Self-contractedness (or self-expandedness, depending on the orientation) is hereby extended in two natural ways giving rise, for any $\lambda \in [-1, 1)$, to the metric notion of $\lambda$-curve and the (weaker) geometric notion of $\lambda$-cone property ($\lambda$-eel). In the Euclidean space $\mathbb{R}^d$ it is established that for $\lambda \in [-1, 1/d)$ bounded $\lambda$-curves have finite length. For $\lambda \geq 1/\sqrt{5}$ it is always possible to construct bounded curves of infinite length in $\mathbb{R}^3$ which do satisfy the $\lambda$-cone property. This can never happen in $\mathbb{R}^2$ though: it is shown that all bounded planar curves with the $\lambda$-cone property have finite length.

**Key words** Self-contracted curve, self-expanded curve, rectifiability, length, $\lambda$-curve, $\lambda$-cone property.

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Contents



## 1. Introduction

Self-contracted curves have been introduced in [2]. They attract a lot of interest, since they are intimately linked to convex foliations ([1], [7], [8]), to the proximal algorithm of a convex function and the gradient flow of a quasiconvex potential in a Euclidean space ([2], [3]) and recently to generalized flows in CAT(0) spaces ([9]). The main feature of this notion is its simple purely metric definition, which inspires developments in more general settings:

**Definition 1.1.** Let $(M, d)$ be a metric space and $I \subset \mathbb{R}$ be an interval. A curve $\gamma : I \to M$ is called self-contracted, if for all $\tau \in I$, the map $t \mapsto d(\gamma(t), \gamma(\tau))$ is non-increasing on $I \cap (-\infty, \tau]$.

The length of a curve $\gamma$ is defined as

$$\ell(\gamma) := \sup \Big\{ \sum_{i=0}^{m-1} d(\gamma(t_i), \gamma(t_{i+1})) \Big\},$$

where the supremum is taken over all finite increasing sequences $t_0 < t_1 < \cdots < t_m$ lying in $I$. The curve $\gamma$ is called rectifiable, if its total variation is locally bounded around any $t \in I$, that is, its length is locally finite.

Rectifiability and asymptotic behaviour are central questions in the study of self-contracted curves. It is shown in [2] that self-contracted curves (are rectifiable and) have finite length whenever $M$ is a bounded subset of the 2-dimensional Euclidean space. Based on ideas of [8], the aforementionned result was extended in [3], and independently in [7], to any finite dimensional Euclidean space. In [4] a further extension has been established encompassing the case where $M$ is a compact subset of a Riemannian manifold. In [6] the result of [2] has been generalized for 2-dimensional spaces equipped with other (smooth) norms. This has been the first result of this type outside a Euclidean/Riemannian setting. An important breakthrough is eventually achieved in [10] by establishing (rectifiability and) finite length for all self-contracted curves contained on a bounded subset of any finite dimensional normed space. Finally, rectifiability of self-contracted curves in Hadamard manifolds and CAT(0) spaces is established in [9].

The aforementioned results remain valid if we replace the assumption "$\gamma$ self-contracted" by the assumption "$\gamma$ self-expanded". A curve $\gamma$ is called *self-expanded* if for all $\tau \in I$, the map $t \mapsto d(\gamma(t), \gamma(\tau))$ is non decreasing on $I \cap [\tau, +\infty)$, or equivalently, when the curve $\overline{\gamma} : -I \to M$ given by $\overline{\gamma}(t) = \gamma(-t)$ is self-contracted. Thus, $\gamma : I \to \mathbb{R}^d$ is self-expanded if for every $t_1 \leq t_2 \leq t_3$ in $I$ we have

$$d(\gamma(t_1), \gamma(t_2)) \leq d(\gamma(t_1), \gamma(t_3)).$$

In the Euclidean setting, there is a nice geometric interpretation of self-expandedness (see [3, Lemma 2.8]). A differentiable curve is self-expanded if and only if

$$\langle \gamma'(t), \gamma(u) - \gamma(t) \rangle \leq 0 \quad \text{for all } u \in I \text{ such that } u < t,$$

which geometrically means that the tail of the curve (the past) is always contained in half-space (cone of aperture $\pi$). The notion of self-expandedness therefore admits the following two natural generalizations. Let us fix $-1 \leq \lambda < 1$. A curve $\gamma : I \to \mathbb{R}^d$ is called $\lambda$-curve if for every $t_1 \leq t_2 \leq t_3$ in $I$ we have

$$d(\gamma(t_1), \gamma(t_2)) \leq d(\gamma(t_1), \gamma(t_3)) + \lambda d(\gamma(t_2), \gamma(t_3)). \tag{1.1}$$

If $\gamma$ is continuous and admits right derivative at each point, we say that $\gamma$ has the $\lambda$-cone-property if, for every $t < \tau$ in $I$, we have, denoting $\gamma'(\tau)$ the right derivative,

$$\langle \gamma'(\tau), \gamma(t) - \gamma(\tau) \rangle \leq \lambda \, ||\gamma'(\tau)|| \; ||\gamma(t) - \gamma(\tau)|| \, .$$

As a matter of the fact, the $\lambda$-cone property will be defined more generally, for merely continuous curves using (forward) secants, see Definition 2.5 and it will be shown that every $\lambda$-curve has the $\lambda$-cone property (*c.f.* Proposition 2.6). However there exist smooth curves satisfying the latter property for some $\lambda_0 < 1$ without being $\lambda$-curves for any $\lambda \in [-1, 1)$ (*c.f.* Example 2.7).

In this work we establish the following results:

- if $|\cdot|$ is an equivalent norm to the Euclidean norm $||\cdot||$, then there exists $\lambda \in [0, 1)$ such that every $|\cdot|$-self-expanded curve is a $||\cdot||$-$\lambda$-curve (Proposition 2.2);
- for $\lambda < 1/d$ every bounded $\lambda$-curve (is rectifiable and) has finite length (Theorem 3.5);
- for $\lambda \geq 1/\sqrt{5}$ there exists a bounded curve in $\mathbb{R}^3$ with infinite length satisfying the $\lambda$-cone property (Theorem 4.2).

Nonetheless due to topological obstructions for $d = 2$ we have:

- for any $\lambda < 1$, bounded planar curves with the $\lambda$-cone property (and a fortiori $\lambda$-curves) have finite length (Theorem 5.3).

Combining the first and the last statement, we readily obtain that all bounded planar self-contracted curves (under any norm) are rectifiable and have finite length. This clearly generalizes the result of [6], but it is contained in the result of [10] that asserts that the same holds in any dimension. Notice that the asymptotic behaviour of both $\lambda$-curves and curves with the $\lambda$-cone property remains unknown in $\mathbb{R}^d$ for $d \geq 3$ and $\lambda \in [1/d, 1/\sqrt{5})$.

**Notation.** Let us fix our notation. Throughout this work $\mathbb{R}^d$ will denote the $d$-dimensional Euclidean space endowed with the Euclidean norm $||\cdot||$ and the scalar product $\langle \cdot, \cdot \rangle$. We denote by $\mathbb{S}^{d-1}$ the unit sphere of $\mathbb{R}^d$, and by $B(x,r)$ (respectively, $\overline{B}(x,r)$) the open (respectively, closed) ball of radius $r > 0$ and center $x \in \mathbb{R}^d$. A (convex) subset $C$ of $\mathbb{R}^d$ is called a (convex) cone, if for every $x \in C$ and $r > 0$ it holds $rx \in C$. If $A$ is a nonempty subset of $\mathbb{R}^d$, we denote by $\mathrm{int}(A)$ its interior, by $\mathrm{conv}\,(A)$ its convex hull and by $\mathrm{diam}\, A := \sup\{d(x,y) : x, y \in A\}$ its diameter.

Given a closed convex subset $K$ of $\mathbb{R}^d$, the normal cone $N_K(u_0)$ of $K$ at $u_0 \in K$ is the following closed convex cone (see [11] *e.g.*):

$$N_K(u_0) = \{v \in \mathbb{R}^n : \langle v, u - u_0 \rangle \leq 0, \forall u \in K\}.$$

Notice that $u_0 \in K$ is the projection onto $K$ of all elements of the form $u_0 + tv$, where $t \geq 0$ and $v \in N_K(u_0)$. In the particular case that $K$ is a closed convex pointed cone (that is, $K$ contains no lines), then its polar (or dual) cone

$$K^o := N_K(0) = \{v \in \mathbb{R}^n : \langle v, u \rangle \leq 0, \forall u \in K\}$$

has nonempty interior and the bipolar theorem holds: $K^{oo} = K$. For $\delta > 0$ sufficiently small, we denote by $K_\delta$ the $\delta$-enlargement of the cone $K$, that is, the closed convex cone generated by the set $\left(K \cap \mathbb{S}^{d-1}\right) + B_\delta$, where $B_\delta := B(0, \delta)$. Notice that

$$\Big((K_\delta)^o \cap \mathbb{S}^{d-1}\Big) + B_\delta \ \subset K^o. \tag{1.2}$$

We define the aperture $A(S)$ of a nonempty subset $S \subset \mathbb{S}^{d-1}$ by

$$A(S) := \inf\{\, \langle u_1, u_2 \rangle \,:\, u_1, u_2 \in S \,\}. \tag{1.3}$$

Based on the above notion, we define the aperture $\mathcal{A}(C)$ of a nontrivial convex pointed cone $C$ as follows:

$$\mathcal{A}(C) = \arccos\Big(A(C \cap \mathbb{S}^{d-1})\Big).$$

Given $v \in \mathbb{S}^{d-1}$ and $\alpha \in [0, \pi)$, we define the "open" cone directed by $v$ as follows:

$$C(v, \alpha) = \Big\{u \in \mathbb{R}^d : \ \langle u, v \rangle > \ ||u|| \,\cos\alpha\Big\} \cup \{0\}. \tag{1.4}$$

Notice that if $\alpha < \pi/2$, the above cone is convex and has aperture $2\alpha$. Given $x \in \mathbb{R}^d$, we adopt the notation

$$C_x(v, \alpha) := x + C(v, \alpha). \tag{1.5}$$

A mapping $\gamma : I = [0, T_\infty) \to \mathbb{R}^d$, where $T_\infty \in \mathbb{R} \cup \{+\infty\}$ is referred in the sequel as a curve. Although the usual definition of a curve comes along with continuity and injectivity requirements for the map $\gamma$, we do not make these prior assumptions here. By the term continuous (respectively, absolutely continuous, Lipschitz, smooth) curve we shall refer to the corresponding properties of the mapping $\gamma : I \to \mathbb{R}^d$. A curve $\gamma$ is said to be bounded if its image, denoted by $\Gamma = \gamma(I)$, is a bounded set of $\mathbb{R}^d$.

For $t \in I$ we denote by $\Gamma(t) := \{\gamma(t') \in \Gamma : t' \leq t\}$ the initial part of the curve and by

$$K(t) = \overline{\mathrm{cone}}\,(\Gamma(t) - \gamma(t)) \tag{1.6}$$

the closed convex cone generated by $\Gamma(t)$. In particular

$$\Gamma(t) \subset \gamma(t) + K(t) \tag{1.7}$$

Notice further that $K(t)$ contains the set $\sec^-(t)$ of (all possible limits of) backward secants at $\gamma(\tau)$ which is defined as follows (see [3]):

$$\sec^-(t) := \left\{ q \in \mathbb{S}^{d-1} : q = \lim_{t_k \nearrow t^-} \frac{\gamma(t_k) - \gamma(t)}{||\gamma(t_k) - \gamma(t)||} \right\},$$

where the notation $\{t_k\}_k \nearrow t^-$ indicates that $\{t_k\}_k \to t$ and $t_k < t$ for all $k$.

The set $\sec^+(t)$ of all possible limits of forward secants at $\gamma(t)$ is defined analogously:

$$\sec^+(t) := \left\{ q \in \mathbb{S}^{d-1} : q = \lim_{t_k \searrow t^+} \frac{\gamma(t_k) - \gamma(t)}{||\gamma(t_k) - \gamma(t)||} \right\},$$

where the notation $\{t_k\}_k \searrow t^+$ indicates that $\{t_k\}_k \to t$ and $t < t_k$ for all $k$. Compactness of $\mathbb{S}^{d-1}$ guarantees that both $\sec^-(t)$ and $\sec^+(t)$ are nonempty. If $\gamma : I \to \mathbb{R}^d$ is differentiable at $t \in I$ and $\gamma'(t) \neq 0$, then $\sec^+(t) = \left\{ \frac{\gamma'(t)}{||\gamma'(t)||} \right\}$.

In this work we introduce two new notions, depending on a parameter $\lambda \in [-1, 1)$. For each value of $\lambda$ we obtain the class of $\lambda$-curves and the class of curves with the $\lambda$-cone property. We associate to these classes an angle $\alpha \in (0, \pi]$ via the relation

$$\alpha = \arccos(\lambda). \tag{1.8}$$

As we shall see, the above classes enjoy interesting geometric properties which can be described in terms of the angle $\alpha$. (For $\lambda = 0$, which corresponds to the angle $\alpha = \pi/2$, the above classes coincide and yield the class of self-expanded curves.)

## 2. $\lambda$-curves and curves with the $\lambda$-cone property

**Definition 2.1** ($\lambda$-curve)**.** A curve $\gamma : I \to \mathbb{R}^d$ is called $\lambda$-curve ($-1 \leq \lambda < 1$) if for every $t_1 \leq t_2 \leq t_3$ in $I$ we have

$$d(\gamma(t_1), \gamma(t_2)) \leq d(\gamma(t_1), \gamma(t_3)) + \lambda d(\gamma(t_2), \gamma(t_3)). \tag{2.1}$$

The above definition yields that every $\lambda$-curve is necessarily injective and cannot admit more than one accumulation point. Based on this, one can easily see that every $\lambda$-curve has at most countable discontinuities. Setting $\lambda = 1$ to (2.1) yields the triangle inequality of the distance (hence no restriction) while $\lambda = -1$ corresponds to segments. On the other hand, for $\lambda = 0$ we recover the definition of a self-expanded curve. The following result shows that the study of self-contracted/self-expanded curves with respect to a non-Euclidean norm can be shifted to the study of $\lambda$-curves in the Euclidean setting.

**Proposition 2.2** (self-expanded vs $\lambda$-curve)**.** *Let $\|\cdot\|$ be an Euclidean norm in $\mathbb{R}^d$ and $|\cdot|$ be another norm in $\mathbb{R}^d$. Then there exists $\lambda < 1$ (depending on the equivalence constant of the norms) such that every $|\cdot|$-self-expanded curve is a $\|\cdot\|$-$\lambda$-curve.*

*Proof.* Since the norms $|\cdot|$ and $\|\cdot\|$ are equivalent and since the properties of being self-expanded or being a $\lambda$-curve are invariant by homothetic transformation, we may assume that there exists $\delta > 0$ such that for all $x \in \mathbb{R}^d$, $\delta\|x\| \leq |x| \leq \|x\|$. Let $t_0 < t_1 < t_2$ in $I$ and set $x_0 = \gamma(t_0)$, $x = \gamma(t_1)$ and $y = \gamma(t_2)$. It follows by assumption that

$$|x - x_0| \leq |y - x_0|. \tag{2.2}$$

To establish the result it is sufficient to prove that there exists $\lambda < 1$ such that for all choices of $x_0, x, y$ satisfying (2.2), we have

$$\|x - x_0\| \leq \|y - x_0\| + \lambda \|x - y\|.$$

By translation, we may, and do assume, that $x_0 = 0$. Moreover, by homogeneity, we can assume $|y| = 1$. Set

$$B = \{z \in \mathbb{R}^d;\ |z| \leq \delta\} \quad \text{and} \quad C_y = \{y + t(y - z);\ \|z\| < \delta,\ t > 0\}.$$

We claim that $B \cap C_y = \emptyset$. Indeed, fix $z$ such that $\|z\| < \delta$. The function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(t) = |y + t(y - z)|$ is convex, $\varphi(-1) < 1$ and $\varphi(0) = 1$, hence $\varphi(t) > 1$ whenever $t > 0$, that is, $y + t(y - z) \notin B$. Since this is true for all $z$ satisfying $\|z\| < \delta$, the claim is proved.

Therefore Proposition 2.2 is a consequence of the following lemma.

**Lemma 2.3.** *There exists $\lambda < 1$ such that, whenever $1 \leq \|y\| \leq 1/\delta$, $\|x\| \leq 1/\delta$ and $x \notin C_y$, then $\|x\| - \|y\| \leq \lambda \|x - y\|$.*

*Proof.* Set $u = x - y$ and $\Gamma_y := \{t(y - z);\ ||z|| < \delta,\ t > 0\}$. We claim that there exists $\rho < 1$ such that, whenever $1 \leq ||y|| \leq 1/\delta$ and $u \in \mathbb{R}^d \diagdown \Gamma_y$, then

$$\langle u, y \rangle \leq \rho\, ||u|| \cdot ||y||. \tag{2.3}$$

Indeed, since $\mathbb{R}^d \diagdown \Gamma_y$ is a cone, it is enough to establish (2.3) when $\|u\| = \|y\|$. Let us denote by $c(u, y)$ the cosine of the angle of the two vectors $u$ and $y$. The condition $u \notin \Gamma_y$ yields $\|u - y\| \geq \delta$. Then we obtain $\|u - y\|^2 = \|y\|^2(2 - 2c(u, y)) \geq 1$, which yields

$$c(u, y) \ \leq\ 1 - \frac{\delta^2}{2\|y\|^2} \ \leq\ 1 - \frac{\delta^4}{2} := \rho \ <\ 1.$$

This proves the claim.

Since $||y + u||^2 \leqslant ||y||^2 + ||u||^2 + 2||y||\, ||u||\, \rho$, we deduce from (2.3) that

$$||x|| - ||y|| = ||y + u|| - ||y|| \ \leq\ ||y|| \left( \sqrt{1 + \frac{2\rho||u||}{||y||} + \frac{||u||^2}{||y||^2}} - 1 \right).$$

Since $||u|| = ||x - y|| \leq ||x|| + ||y|| \leq 2/\delta$ and $||y|| \geq |y| = 1$, we have $t = \frac{||u||}{||y||} \in [0, 2/\delta]$. Notice that taking $\lambda < 1$ sufficiently close to 1, we ensure that for all $t \in [0, 2/\delta]$ it holds

$$\sqrt{1 + 2\rho t + t^2} - 1 \leq \lambda t.$$

Therefore we conclude that

$$||x|| - ||y|| \leq \lambda ||u|| = \lambda ||x - y||.$$

The proof is complete. □

From now on we consider exclusively a Euclidean setting. An important feature of the notion of $\lambda$-curve is the following property:

**Proposition 2.4** (uniform non-collinearity)**.** *Let $\gamma : I \to \mathbb{R}^d$ be a $\lambda$-curve. Then, $\gamma$ is $\lambda$-uniformly non-collinear, that is, for every $s, u, t \in I$ such that $s, u \leq t$ we have*

$$\left\langle \frac{\gamma(u) - \gamma(t)}{\|\gamma(u) - \gamma(t)\|}, \frac{\gamma(s) - \gamma(t)}{\|\gamma(s) - \gamma(t)\|} \right\rangle \ >\ -\lambda \quad (> -1)\ . \tag{2.4}$$

*Proof.* Assume that $u < s < t$. Because $\gamma$ is $\lambda$-curve we have that

$$d(\gamma(u), \gamma(s)) \le d(\gamma(u), \gamma(t)) + \lambda\, d(\gamma(s), \gamma(t))$$

Consider the triangle of vertices $\gamma(t)$, $\gamma(u)$ and $\gamma(s)$ and set $c = d(\gamma(u), \gamma(s))$, $a = d(\gamma(u), \gamma(t))$ and $b = d(\gamma(s), \gamma(t))$. The previous equation now reads $c \le a + \lambda b$, and after squaring both sides we get

$$c^2 \le a^2 + \lambda^2 b^2 + 2\lambda ab. \tag{2.5}$$

Evoking the law of cosine $c^2 = a^2 + b^2 - 2ab\cos\varphi$ we deduce

$$\cos\varphi = \frac{a^2 + b^2 - c^2}{2ab} \overset{(2.5)}{\ge} \frac{(1-\lambda^2)b^2 - 2\lambda ab}{2ab} > -\lambda,$$

that is, the angle $\varphi$ between the vectors

$$\frac{\gamma(u) - \gamma(t)}{\|\gamma(u) - \gamma(t)\|} \quad \text{and} \quad \frac{\gamma(s) - \gamma(t)}{\|\gamma(s) - \gamma(t)\|}$$

is strictly less than $\pi - \alpha$ ($\alpha = \arccos(\lambda)$ is given by (1.8)). ☐

Before we proceed, we give the following definition.

**Definition 2.5** ($\lambda$-cone property)**.** Let $\lambda \in [-1, 1)$ and $\alpha = \arccos(\lambda)$. We say that a continuous curve $\gamma : I \to \mathbb{R}^d$ satisfies the $\lambda$-cone property if for every $t \in I$ and for every $q_t^+ \in \sec^+(t)$ it holds

$$\left\langle q_t^+, \frac{\gamma(u) - \gamma(t)}{\|\gamma(u) - \gamma(t)\|} \right\rangle \le \lambda, \quad \text{for all } u < t. \tag{2.6}$$

In other words, recalling (1.4), the set $\Gamma(t) - \gamma(t)$ does not intersect the cone $C\left(q_t^+, \alpha\right)$ directed by $q_t^+$ and of aperture $2\alpha$ expect at 0, that is, for every $t \in I$

$$\left(\gamma(t) + \bigcup_{q_t^+ \in \sec^+(t)} C\left(q_t^+, \alpha\right)\right) \bigcap \Gamma(t) = \{\gamma(t)\}. \tag{2.7}$$

We shall now consider a second important feature of the class of (continuous) $\lambda$-curves.

**Proposition 2.6** ($\lambda$-curve $\Longrightarrow$ $\lambda$-cone property)**.** *Every continuous $\lambda$-curve has the $\lambda$-cone property.*

*Proof.* Fix $t \in I$, let $u < t$, $q_t^+ \in \sec^+(t)$ and choose $\{t_k\}_k \searrow t$ such that

$$\frac{\gamma(t_k) - \gamma(t)}{\|\gamma(t_k) - \gamma(t)\|} \longrightarrow q_t^+.$$

Since $\gamma$ is a $\lambda$-curve we have

$$\|\gamma(t) - \gamma(u)\| \le \|\gamma(t_k) - \gamma(u)\| + \lambda\|\gamma(t_k) - \gamma(t)\|,$$

yielding

$$\frac{\|\gamma(t_k) - \gamma(u)\| - \|\gamma(t) - \gamma(u)\|}{||\gamma(t_k) - \gamma(t)||} \ge -\lambda$$

Set $\Phi(X) = ||X||$, $X_k = \gamma(t_k) - \gamma(u)$ and $X = \gamma(t) - \gamma(u)$. Then the above inequality reads

$$\frac{\Phi(X_k) - \Phi(X)}{||X_k - X||} \ge -\lambda.$$

Since the norm is differentiable around the segment $[X, X_k] := \{tX + (1-t)X_k : t \in [0,1]\}$, applying the Mean Value theorem we obtain $\theta_k \in [0,1)$ such that

$$\Phi(X_k) - \Phi(X) = D\Phi(X + \theta_k(X_k - X))(X_k - X) = \left\langle \frac{X + \theta_k(X_k - X)}{\|X + \theta_k(X_k - X)\|}, X_k - X \right\rangle.$$

Combining the above formulas and taking the limit as $k \to \infty$ we get

$$\left\langle \frac{\gamma(t) - \gamma(u)}{\|\gamma(t) - \gamma(u)\|}, q_t^+ \right\rangle \geqslant -\lambda.$$

The above is equivalent to (2.6) and the proof is complete. □

The following example reveals that there exist $C^1$curves satisfying the $\lambda$-cone property but failing to satisfy the non-collinearity property. Therefore these curves cannot be $\lambda$-curves for any value of the parameter $\lambda \in [-1, 1)$.

**Example 2.7.** Let $\gamma : [-3\pi/2, 1+\pi] \to \mathbb{R}^3$ be defined by

$$\gamma(t) = \begin{cases} (0, -\sin t, -\cos t), & \text{if } t \in [-3\pi/2, -\pi/2], \\ (-\frac{1}{2}(1 + \cos 2t), 1, \frac{1}{2}\sin 2t), & \text{if } t \in [-\pi/2, 0], \\ (-1, 1, t), & \text{if } t \in [0, 1], \\ (-1, \frac{1}{2}(1 + \cos 2(t-1)), 1 + \frac{1}{2}\sin 2(t-1)), & \text{if } t \in [1, 1+\pi/2], \\ (-\sin(t-1), 0, 1 + \cos(t-1)), & \text{if } t \in [1+\pi/2, 1+\pi]. \end{cases}$$

It is easy to check that $\gamma$ is $C^1$-smooth. Moreover, $\gamma$ fails to satisfy the non-collinearity property: indeed, $\gamma(1+\pi) = (0,0,0)$ is the midpoint of the segment $[\gamma(-3\pi/2), \gamma(-\pi/2)]$. Hence, by Proposition 2.4, $\gamma$ cannot be a $\lambda$-curve for any value of the parameter $\lambda < 1$. On the other hand, any tangent line

$$\{\gamma(t) + s\gamma'(t);\ s \in \mathbb{R}\}$$

meets the curve $\{\gamma(\tau);\ \tau \in [-3\pi/2, 1+\pi]\}$ only at the point $\gamma(t)$. Therefore, by a simple compactness argument, there exists $\lambda_0 < 1$ for which $\gamma$ satisfies the $\lambda_0$-cone property.

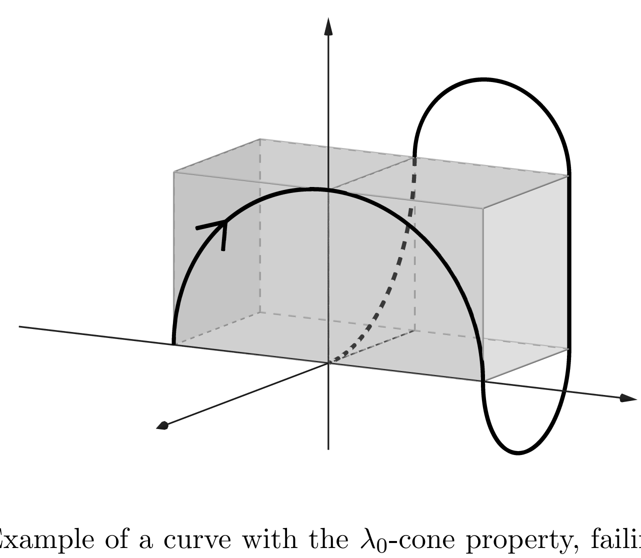

FIGURE 1. Example of a curve with the $\lambda_0$-cone property, failing to be $\lambda$-curve for any $\lambda < 1$.

## 3. Length of $\lambda$-curves

Before we proceed we recall from [3] the following result (we provide a proof for completeness).

**Lemma 3.1.** *Let $\Sigma \subset \mathbb{S}^{d-1}$ (the unit sphere of $\mathbb{R}^d$, $d > 1$) and assume that for $\lambda < 1/d$ it holds*

$$\langle x, x' \rangle \geq -\lambda, \quad \text{for all } x, x' \in \Sigma.$$

*Then $\Sigma$ is contained in a half-sphere (therefore it generates a closed convex pointed cone).*

*Proof.* Notice that the conclusion holds if and only if $0 \notin \operatorname{conv}(\Sigma)$. Let us assume that $0 \in \operatorname{conv}(\Sigma)$. Then by Caratheodory theorem, there exist $\alpha_0, \alpha_1, \cdots, \alpha_d \geq 0$ and $x_0, x_1, \cdots, x_d \in \mathbb{S}^{d-1}$ such that

$$\sum_{i=0}^{d} \alpha_i = 1 \quad \text{and} \quad \sum_{i=0}^{d} \alpha_i x_i = \mathbf{0}.$$

It follows that for $j \in \{0, 1, \ldots, d\}$,

$$0 = \langle \mathbf{0}, x_j \rangle = \sum_{i=0}^{d} \alpha_i \langle x_i, x_j \rangle \geq \alpha_j - \lambda \sum_{i \neq j} \alpha_i = \alpha_j - \lambda(1 - \alpha_j).$$

Summing up for all $j \in \{0, 1, \ldots, d\}$ we get $0 \geq 1 - \lambda(d+1-1)$, which contradicts the assumption $\lambda < 1/d$. $\square$

Recalling the notation of (1.3), (1.7) and (1.8), and assuming $\lambda < 1/d$ we obtain the following result (as a straightforward combination of Lemma 3.1 with Proposition 2.4).

**Corollary 3.2** (conical control of the initial part)**.** *Let $-1 \leq \lambda < 1/d$ and $\alpha = \arccos(\lambda)$. Then for every $t \in I$, the initial part $\Gamma(t)$ of a $\lambda$-curve $\gamma$ is contained in a closed convex cone $K(t)$ of aperture at most $\pi - \alpha$ centered at $\gamma(t)$. In other words,*

$$\Gamma(t) \subset \gamma(t) + K(t) \quad \text{and} \quad \mathcal{A}(K(t)) \leq \pi - \alpha. \tag{3.1}$$

To sum up, given a continuous $\lambda$-curve $\gamma$, Proposition 2.6 ensures that its initial part $\Gamma(t)$ avoids the union of all cones centered at $\gamma(t)$ and directed by forward secants of $\gamma$ at $t$, see (2.7), while Corollary 3.2 asserts that, provided $\lambda < 1/d$, the initial part of the curve $\Gamma(t)$ is itself contained in the closed convex pointed cone $\gamma(t) + K(t)$, centered at $\gamma(t)$. The following proposition asserts that an even stronger property is satisfied.

**Proposition 3.3** (conical split at each $t$)**.** *Let $\gamma : I \to \mathbb{R}^d$ be a continuous $\lambda$-curve, with $\lambda \in [-1, 1/d)$ and $\alpha = \arccos(\lambda)$. Then it holds:*

$$\left( \bigcup_{q_t^+ \in \sec^+(t)} C\left(q_t^+, \alpha\right) \right) \bigcap \; K(t) \;=\; \{0\}, \qquad \text{for all } \; t \in I. \tag{3.2}$$

*Proof.* Assume towards a contradiction that for some $q_t^+ \in \sec^+(t)$ there exists $q \in C\left(q_t^+, \alpha\right) \cap K(t)$, $q \neq 0$. This yields, in view of Proposition 2.6, that $\operatorname{int} K(t)$ is nonempty. Therefore, since $q$ satisfies the open condition

$$\langle q_t^+, q \rangle \;>\; \lambda = \cos \alpha,$$

there is no loss of generality to assume that $q \in \operatorname{int} K(t)$. Therefore, there exist $t_1 < t_2 < \ldots < t_d < t$ and $\{\mu_i\}_{i=1}^d \subset \mathbb{R}_+$ such that

$$u_i := \frac{\gamma(t_i) - \gamma(t)}{\|\gamma(t_i) - \gamma(t)\|} \quad \text{and} \quad q = \sum_{i=1}^{d} \mu_i \, u_i.$$

Fix $\varepsilon > 0$ such that $\langle q_t^+, q\rangle > \lambda + 3\varepsilon$. By continuity, there exists $\delta > 0$ such that for all $s \in (t, t+\delta)$ the vectors

$$\tilde{u}_i := \frac{\gamma(t_i) - \gamma(s)}{\|\gamma(t_i) - \gamma(s)\|}, \quad i \in \{1, \dots, d\},$$

are sufficiently close to $\{u_i\}_{i=1}^d$ to ensure that

$$\langle q_t^+, \tilde{q}\rangle > \lambda + 2\varepsilon, \qquad \text{where} \quad \tilde{q} = \sum_{i=1}^n \mu_i \, \tilde{u}_i.$$

Take now $s \in (t, t+\delta)$ in a way that the vector $\hat{q} = (\|\gamma(s) - \gamma(t)\|)^{-1} (\gamma(s) - \gamma(t))$ is sufficiently close to the secant $q_t^+$ so that $\langle \hat{q}, \tilde{q}\rangle > \lambda + \varepsilon$ or equivalently, $\langle -\hat{q}, \tilde{q}\rangle < -\lambda - \varepsilon$. Since $\tilde{q}, -\hat{q} \in K(s) \cap \mathbb{S}^{d-1}$, we deduce that $\mathcal{A}(K(s)) > \pi - \alpha$, which contradicts Corollary 3.2 for $s = t$. □

We shall finally need the following lemma.

**Lemma 3.4.** *Let $\gamma : I \to \mathbb{R}^d$ be a continuous $\lambda$-curve, with $\lambda \in [-1, 1/d)$ and $\alpha = \arccos(\lambda)$. Then there exists $\rho > 0$ such that for every $t \in I$ and $q_t^+ \in \sec^+(t)$, there exists $\xi_t \in \mathbb{S}^{d-1}$ satisfying*

$$\langle \xi_t, u\rangle \le -\rho < 0, \quad \textit{for all } u \in K(t) \tag{3.3}$$

*and*

$$\langle \xi_t, q_t^+\rangle \ \ge\ \rho > 0\,. \tag{3.4}$$

*Proof.* Let $\delta \le \sqrt{2(1-\lambda)}$ and $\rho = \delta/2$. Then for every $t \in I$ and $q_t^+ \in \sec^+(t)$, we have $\mathbb{S}^{d-1} \cap B(q_t^+, \delta) \subset C(q_t^+, \alpha)$. We deduce from Proposition 3.3 that the $\delta$-enlargement of the cone $K(t)$ satisfies:

$$K(t)_\delta \cap \sec^+(t) = \emptyset.$$

Setting $\tilde{N}(t) = (K(t)_\delta)^o$ and $N(t) = K(t)^o$ (the polar of $K(t)_\delta$ and $K(t)$ respectively), we deduce by (1.2) that

$$\bar{B}(\xi, \delta) \cap \mathbb{S}^{d-1} \subset N(t), \quad \text{for every } \xi \in \tilde{N}(t) \cap \mathbb{S}^{d-1}. \tag{3.5}$$

Let us now fix $q_t^+ \in \sec^+(t)$. Then by the bipolar theorem we get $q_t^+ \notin \tilde{N}(t)^o = K(t)_\delta$, that is, there exists $\tilde{\xi} \in \tilde{N}(t) \cap \mathbb{S}^{d-1}$ such that $\langle \tilde{\xi}, q_t^+\rangle > 0$. Maximizing the functional $q_t^+$ over the closed ball $\bar{B}(\tilde{\xi}, \rho)$ we obtain $\xi_t \in \mathbb{S}^{d-1}$ such that (3.4) holds. Since $B(\xi_t, \rho) \subset B(\tilde{\xi}, \delta) \subset N(t)$, we easily deduce that (3.3) also holds. □

We are now ready to prove the main result of this section.

**Theorem 3.5** (rectifiability)**.** *Every continuous $\lambda$-curve $\gamma : I \to \mathbb{R}^d$ with $\lambda < 1/d$ is rectifiable. In particular, bounded $\lambda$-curves with $\lambda < 1/d$ have finite length.*

*Proof.* We may assume that $I = [0, +\infty)$ and that $\gamma$ is bounded. Set $\eta = \rho/3$, where $\rho$ is given by Lemma 3.4. Since $\mathbb{S}^{d-1}$ is compact, there exists an $\eta$-net $\mathcal{F} := \{\xi_1, \cdots, \xi_N\}$, satisfying that for every $v \in \mathbb{S}^{d-1}$, there exists $i \in \{1, \cdots, N\}$ such that $\langle v, \xi_i\rangle > \eta$ (that is, $v$ is $\eta$-close to some $\xi_i \in \mathcal{F}$). Then we deduce from Lemma 3.4 that for every $t \in I$ and $q_t^+ \in \sec^+(t)$, there exists $\xi_i \in \mathcal{F}$ such that

$$\langle \xi_i, q^+\rangle > 2\eta \qquad \text{and} \qquad \langle \xi_i, u\rangle \le -2\eta < 0, \quad \text{for all } u \in K(t). \tag{3.6}$$

Reasoning by contradiction we can prove the existence of some $\delta_t > 0$ such that for every $s \in [t, t+\delta_t)$ there exists $q_{t,s}^+ \in \sec^+(t)$ such that

$$\left\| \frac{\gamma(s) - \gamma(t)}{||\gamma(s) - \gamma(t)||} - q_{t,s}^+ \right\| < \eta. \tag{3.7}$$

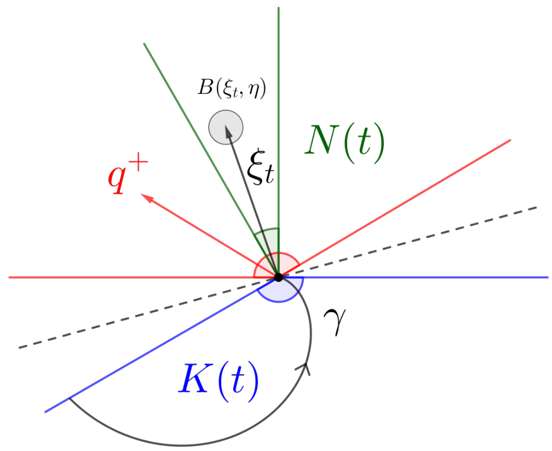


FIGURE 2. The initial part of the curve generates the cone $K(t)$ (in blue) with aperture $\mathcal{A}(K(t)) \leq \pi - \alpha$ and avoids the cone generated by the positive secants (in red).

Combining the above we deduce that for every $t \in I$ and $s \in [t, t+\delta_t)$, there exists $\xi_i \in \mathcal{F}$ such that

$$\langle \xi_i, \gamma(s) - \gamma(t) \rangle \geq \eta \, \|\gamma(s) - \gamma(t)\|. \tag{3.8}$$

On the other hand, it follows directly from (3.6) that for every $\tau \in [0, t)$

$$\langle \xi_i, \gamma(t) - \gamma(\tau) \rangle \geq \eta \, \|\gamma(t) - \gamma(\tau)\|. \tag{3.9}$$

Considering for $i \in \{1, \ldots, N\}$ the projection operator

$$\begin{cases} \pi_i : \mathbb{R}^d \to \mathbb{R}\xi_i \\ \pi_i(x) = \langle \xi_i, x \rangle \, \xi_i \end{cases}$$

we define $W_i(t)$ to be the width of the projection of the initial part of the curve $\Gamma(t)$ onto $\mathbb{R}\xi_i$, that is,

$$W_i(t) := \mathcal{H}^1(\pi_i(\Gamma(t))), \quad t \in I,$$

where $\mathcal{H}^1$ denotes the 1-dimensional Lebesgue measure. Notice that $\mathcal{H}^1(\pi_i(\Gamma(t)))$ is simply the length of the bounded interval $\pi_i(\Gamma(t))$ of $\mathbb{R}\xi_i$. It follows readily that for every $i \in \{1, \ldots, N\}$ the function $t \mapsto W_i(\tau)$ is non-decreasing on $[0, T_\infty)$ and bounded above by $r := \operatorname{diam}(\gamma(I))$. Therefore, the function

$$W_{\mathcal{F}}(t) := \sum_{i=1}^{N} W_i(t),$$

is non-decreasing on $I$ and bounded above by $Nr$. We now deduce from (3.8) and (3.9) that for every $t \in I$ there exists $\delta_t > 0$ such that for all $s \in [t, t+\delta_t)$ we have

$$W_{\mathcal{F}}(s) - W_{\mathcal{F}}(t) \geq \eta \, \|\gamma(s) - \gamma(t)\|. \tag{3.10}$$

The result follows via a standard argument if we establish that for any $a, b \in I$ with $a < b$ it holds:

$$W_{\mathcal{F}}(b) - W_{\mathcal{F}}(a) \geq \eta \, \|\gamma(b) - \gamma(a)\|. \tag{3.11}$$

Let us assume, towards a contradiction, that (3.11) does not hold, that is,

$$W_{\mathcal{F}}(b) - W_{\mathcal{F}}(a) + \varepsilon < \eta \, \|\gamma(b) - \gamma(a)\|, \quad \text{for some } \varepsilon > 0.$$

Set $\sigma(t) = \sup\{s > t : \text{ (3.10) holds}\}$, for $t \in [a, b)$. Then our assumption yields that for every $t \in [a, b)$ we have $a \le t + \delta_t \le \sigma(t) < b$. Using transfinite induction we construct a (necessarily) countable set $\Lambda = \{t_\mu\}_{\mu \le \hat{\varsigma}}$ by setting $t_1 = a$, $t_\mu = \sigma(t_{\mu^-})$ if $\mu = \mu^- + 1$ is a successor ordinal, and $t_\mu = \sup\{t_\nu : \nu < \mu\}$ if $\mu$ is a limit ordinal and we stop when $t_{\hat{\varsigma}} = b$. Let now $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0, \varepsilon)$ with $\sum_{n\in N} \varepsilon_n = \varepsilon\eta^{-1}$. Let $i : \Lambda \to \mathbb{N}$ be an injection of $\Lambda$ into $\mathbb{N}$. Then denoting by $\mu^+$ the successor of $\mu$, we obtain by continuity, that for each ordinal $\mu$ there exists $t_\mu \le s_\mu < \sigma(t_\mu) := t_{\mu^+}$ such that $||\gamma(s_\mu) - \gamma(t_{\mu^+})\| < \varepsilon_{i(\mu)}$. We deduce by (3.10):

$$\begin{aligned}\|\gamma(b) - \gamma(a)\| &\le \sum_{\mu\in\Lambda} \|\gamma(t_{\mu^+}) - \gamma(t_\mu)\| \le \sum_{\mu\in\Lambda} \left(\|\gamma(s_\mu) - \gamma(t_\mu)\| + \varepsilon_{i(\mu)}\right) \\ &\le \frac{1}{\eta}\left(\sum_{\mu\in\Lambda} \left(W_{\mathcal{F}}(s_\mu) - W_{\mathcal{F}}(t_\mu)\right) + \varepsilon\right) \le \frac{1}{\eta}\left(W_{\mathcal{F}}(b) - W_{\mathcal{F}}(a) + \varepsilon\right),\end{aligned}$$

which contradicts (3.11). □

**Remark 3.6** (universal constant)**.** The above proof reveals that the length $\ell(\gamma)$ of any $\lambda$-curve lying in a set of diameter $r$ is bounded by the quantity $N \cdot \eta^{-1} \cdot r$. Since the constant $\eta > 0$ is determined in Lemma 3.4, it only depends on $\lambda$ and the dimension $d$ of the space (in particular, it is independent of the specific $\lambda$-curve $\gamma$). Since $N$ (the cardinality of the net $\mathcal{F}$) also depends exclusively on $\eta$ and the dimension $d$, we conclude that for a given $\lambda \in [-1, 1/d)$ there exists a prior bound for the lengths of all $\lambda$-curves $\gamma$ lying inside a prescribed bounded subset of $\mathbb{R}^d$.

**Remark 3.7** (Double cone property)**.** A close inspection of Theorem 3.5 shows that the proof depends exclusively on (3.3)–(3.4) which in turn depend on (3.2). Therefore, every bounded continuous curve $\gamma$ satisfying (3.2) has finite length.

## 4. A bounded curve with the $\lambda$-cone property and infinite length

In this section we consider continuous right differentiable curves $\gamma : I \to \mathbb{R}^d$ satisfying the $\lambda$-cone property (Definition 2.5). In the sequel we denote by $\gamma'(\tau)$ the right derivative of $\gamma$ at the point $\tau$ and we assume this derivative is nonzero. Observe that in this case we have

$$\sec^+(t) = \left\{\frac{\gamma'(t)}{\|\gamma'(t)\|}\right\}.$$

So $\gamma$ satisfies the $\lambda$-cone property if, for all $t, \tau \in I$ with $t < \tau$, (2.6) holds, or equivalently:

$$\langle \gamma'(\tau), \gamma(t) - \gamma(\tau)\rangle \le \lambda\, ||\gamma'(\tau)||\, ||\gamma(t) - \gamma(\tau)||.$$

This means that the angle between the vectors $\gamma'(\tau)$ and $\gamma(t) - \gamma(\tau)$ is greater or equal to $\alpha$, where $\alpha = \arccos(\lambda)$. We simplify the notation by setting

$$C(t, \alpha) := \gamma(t) + C\left(\frac{\gamma'(t)}{\|\gamma'(t)\|}, \alpha\right). \tag{4.1}$$

A curve $\gamma$ satisfying the above property will be also called a $\lambda$-eel. The reason is as follows: the set $\Gamma(\tau) := \{\gamma(t);\, t \in I,\, t < \tau\}$ is the apparent body (or tail) of a $\lambda$-eel at time $\tau$ going out of a hole. The cone $C(\tau, \alpha)$ represents what the $\lambda$-eel can see at time $\tau$. The $\lambda$-cone property just says that the $\lambda$-eel never sees its apparent tail. Notice that $\pi/2$-eels correspond to self-expanded curves. Therefore, if the range of $\gamma$ is bounded and $\gamma$ is a $\pi/2$-eel, then its length is finite ([3], [7]).

Recall from the introduction that a curve $\gamma$ is self-expanded if for all $\tau \in I$, the map $t \mapsto d(\gamma(t), \gamma(\tau))$ is non decreasing on $I \cap [\tau, +\infty)$. The following lemma illustrates that one can also associate a Lyapunov function to $\lambda$-eels.

**Lemma 4.1.** *If $\gamma : I \to \mathbb{R}^d$ is a $\lambda-$eel, then the function*

$$t \mapsto \|\gamma(t_1) - \gamma(t)\| + \lambda \ell(\gamma_{|[t_1,t]})$$

*is non-decreasing on $I \cap [t_1, \infty)$.*

*Proof.* By definition,

$$\frac{d}{d\tau}(\|\gamma(\tau) - \gamma(t)\|) = \left\langle \gamma'(\tau), \frac{\gamma(\tau) - \gamma(t)}{\|\gamma(\tau) - \gamma(t)\|} \right\rangle \geq -\lambda \|\gamma'(\tau)\| \quad \forall\, t < \tau.$$

For $t < t_1 < t_2$, integrating for $\tau \in [t_2, t_3]$ we obtain

$$\int_{t_2}^{t_3} \frac{d}{d\tau}(\|\gamma(\tau) - \gamma(t)\|) d\tau \geq -\lambda \int_{t_2}^{t_3} \|\gamma'(s)\| ds \quad \forall\, t < \tau,$$

which implies

$$\|\gamma(t_3) - \gamma(t)\| - \|\gamma(t_2) - \gamma(t)\| \geq -\lambda \ell(\gamma_{|[t_2,t_3]}).$$

Since $\ell(\gamma_{|[t_2,t_3]}) = \ell(\gamma_{|[t_1,t_3]}) - \ell(\gamma_{|[t_1,t_2]})$ the conclusion follows. $\square$

Our main aim now is to prove the following result.

**Theorem 4.2** ($\lambda$-eel of infinite length)**.** *Assume $\lambda = \frac{1}{\sqrt{5}}$ (i.e. $\alpha = \arccos \frac{1}{\sqrt{5}}$), and let $B = \overline{B}(0, 1)$ the unit ball of $\mathbb{R}^3$. Then, there exists a $\lambda$-eel $\gamma : [0, +\infty) \to B$ of infinite length. Moreover $\lim_{t\to\infty} \gamma(t)$ exists.*

The proof of Theorem 4.2 is constructive: the construction will be carried out in three steps organized in subsections. Let us mention that the result remains true if we require $\gamma$ to be $\mathcal{C}^1$-smooth (and probably even $\mathcal{C}^\infty$-smooth), but the construction would then become less transparent. Before we proceed, let us make the following remark.

**Remark 4.3.** Let us denote by $\lambda_*$ the infimum of all $\lambda$ for which there exists a bounded $\lambda$-eel of infinite length inside the unit ball of $\mathbb{R}^3$. Since for $\lambda = 0$ we obtain a self-expanded curve, it follows from the above theorem that $0 \leq \lambda_* \leq \frac{1}{\sqrt{5}}$. Notice that we cannot readily conclude that $\lambda_*$ is strictly greater than 0. (Nonetheless, according to [8] or [3], for $\lambda = 0$ bounded $\lambda$-eels have finite length.)

4.1. **Helicoidal maps.** Let us start by constructing a helicoidal curve along the $z$-axis, which is self-expanded.

**Lemma 4.4.** *There exists a positive constant $\mu < 1/2$ such that, if $\gamma : \mathbb{R} \to \mathbb{R}^3$ is a spiral of the form*

$$\gamma(t) = (r \cos t,\, r \sin t,\, \mu r t)\,, \quad t \in \mathbb{R}, \tag{4.2}$$

*then $\gamma$ is self-expanded (hence $\gamma$ satisfies the $\lambda$-cone property for all $\lambda \in [0, 1)$).*

*Proof.* Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a spiral along a cylinder of radius $r > 0$ of the form (4.2) and let us show that $\gamma$ is a self-expanded curve. By symmetry, this amounts to verify that

$$a(t) := \langle \gamma'(0), \gamma(t) - \gamma(0) \rangle \leq 0, \quad \text{for all } t < 0.$$

We check easily that $\gamma(0) = (r, 0, 0)$ and $\dot{\gamma}(0) = (0, r, \mu r)$, so that $a(t) = r^2(\sin t + \mu^2 t)$. Since $\sup\{-t^{-1}\sin t : t < 0\} < 1/4$, we deduce that there exists $\mu < \frac{1}{2}$ such that the curve $\gamma$ is self-expanded. □

**Notation.** Throughout this subsection, $\gamma$ will refer to the curve given in Lemma 4.4 and $\mu < 1/2$ will be the constant fixed there.

The following lemma says that the curve $\gamma$ constructed in the previous lemma satisfies that for each $\tau$, the associated cone $C(t,\alpha)$, $\alpha = \arccos(1/\sqrt{5})$ does not meet the $z$-axis, that is, the axis of evolution of the spiral curve.

**Lemma 4.5.** *Let $\gamma : \mathbb{R} \to \mathbb{R}^3$ be a spiral of the form (4.2). If $\lambda = 1/\sqrt{5}$ and $\alpha = \arccos(\lambda)$, then the cone $C(t,\alpha)$ does not intersect the line parametrized by $\ell(z) = (0, 0, z)$.*

*Proof.* Under the notation of the previous lemma, it is enough to verify that for all $z \in \mathbb{R}$

$$\langle \gamma'(0), \ell(z) - \gamma(0)\rangle \leq \frac{1}{\sqrt{5}} \|\dot{\gamma}(0)\| \, \|\ell(z) - \gamma(0)\| .$$

The above condition reads

$$\mu r z \leq \frac{1}{\sqrt{5}} \sqrt{r^2(1+\mu^2)}\, \sqrt{r^2 + z^2}, \qquad \text{for all } z \in \mathbb{R}, \tag{4.3}$$

or equivalently,

$$\frac{(z/r)}{\sqrt{1 + (z/r)^2}} \leq \frac{\sqrt{(1+\mu^2)}}{\mu\sqrt{5}}, \qquad \text{for all } z \in \mathbb{R}. \tag{4.4}$$

Since $t \mapsto t^{-1}\sqrt{1+t^2}$ is decreasing for $t > 0$ and $\mu < 1/2$, we have $\mu^{-1}\sqrt{1+\mu^2} > \sqrt{5}$, therefore (4.4) is satisfied. □

We shall now enhance in the above construction to deduce that the cone $C(\tau, \alpha)$ avoids a thin (infinite) cylinder

$$\mathrm{Cyl}\,(r_0) = \{(x, y, z) \in \mathbb{R}^3;\, x^2 + y^2 = r_0^2,\, z \in \mathbb{R}\}$$

containing the $z$-axis. Indeed, taking $r_0 << r$ the above cylinder is very close to the $z$-axis, therefore we obtain (almost) the same result as before. This is formulated in the next lemma.

**Lemma 4.6.** *There exists an integer $N \geq 2$ such that whenever $r = Nr_0$ and $\alpha = \arccos 1/\sqrt{5}$, we have:*

$$C(\alpha, \tau) \cap \mathrm{Cyl}\,(r_0) = \emptyset, \quad \textit{for all } \tau \geq 0.$$

*Proof.* We consider again the curve $\gamma$ given by (4.2). Thanks to the symmetry, it is enough to check the assertion for $\tau = 0$. Therefore, for $\sigma(\theta, z) = (r_0\cos\theta, r_0\sin\theta, z)$, it is enough to verify

$$\langle \gamma'(0), \sigma(\theta, z) - \gamma(0)\rangle \leq \cos\alpha \left\|\gamma'(0)\right\| \|\sigma(\theta, z) - \gamma(0)\| \qquad \forall \theta \in [0, 2\pi],\, \forall z \geq 0,$$

where $\gamma(0) = (r, 0, 0)$ and $\gamma'(0) = (0, r, \mu r)$ .The above condition reads

$$r r_0 \sin\theta + \mu r z \leq \cos\alpha \sqrt{r^2(1+\mu^2)} \sqrt{\left(r_0\cos\theta - r\right)^2 + r_0^2\sin^2\theta + z^2} \quad \forall \theta \in [0, 2\pi],\, \forall z \geq 0.$$

Dividing by $rr_0$, setting $w = z/r_0$, and since $\cos\alpha = 1/\sqrt{5}$, we deduce

$$\sin\theta + \mu w \leq \frac{1}{\sqrt{5}} \sqrt{1+\mu^2} \sqrt{\left(\frac{r}{r_0} - 1\right)^2 + 2\frac{r}{r_0}(1-\cos\theta) + w^2}.$$

Setting $r = Nr_0$ we obtain the condition

$$\frac{1}{\sqrt{5}} \geq \frac{1}{\sqrt{1+\mu^2}} \sup_{\theta\in[0,2\pi], w\in\mathbb{R}} \left\{ \frac{\sin\theta + \mu|w|}{\sqrt{(N-1)^2 + 2N(1-\cos\theta) + w^2}} \right\}.$$

But for any $\theta \in [0, 2\pi]$, $u = |w| \geq 0$

$$\frac{\sin\theta + \mu u}{\sqrt{(N-1)^2 + 2N(1-\cos\theta) + u^2}} \leq \frac{1+\mu u}{\sqrt{(N-1)^2 + u^2}}$$

and

$$\sup_{u\geq 0} \left\{ \frac{1+\mu u}{\sqrt{(N-1)^2 + u^2}} \right\} = \frac{1}{N-1} \sqrt{1+\mu^2 (N-1)^2} \longrightarrow \mu \quad \text{as} \quad N \to +\infty.$$

Since $\mu \left(1+\mu^2\right)^{-1/2} < \left(\sqrt{5}\right)^{-1}$, we can choose $N$ large enough such that

$$\frac{\sqrt{1+\mu^2(N-1)^2}}{(N-1)\sqrt{1+\mu^2}} < \frac{1}{\sqrt{5}}.$$

Therefore, for this choice of $N$, we get $C(0, \alpha) \cap \mathrm{Cyl}\,(r_0) = \emptyset$. □

Let $\gamma$ be given by (4.2). We shall now include a further restriction. We shall show that the cone $C(\tau, \alpha)$ associated to $\gamma$ also avoids radial segments $S$ of the form:

$$S = \{(x, 0, 0);\ 0 \leq x \leq r\}.$$

This is the aim of the following lemma.

**Lemma 4.7.** *If* $\lambda = \frac{1}{\sqrt{5}}$ *and* $\alpha = \arccos(\lambda)$, *then* $C(\tau, \alpha) \cap S = \emptyset$ *for all* $\tau \geq 0$.

*Proof.* It is enough to verify

$$\langle \gamma'(\tau), (x, 0, 0) - \gamma(\tau) \rangle \leq \cos\alpha \; \|\dot\gamma(\tau)\| \; \|(x, 0, 0) - \gamma(\tau)\|, \qquad \text{for all } 0 \leq x \leq r,$$

where

$$\gamma(\tau) = (r\,\cos\tau,\ r\,\sin\tau,\ \mu r\tau) \quad \text{and} \quad \dot\gamma(\tau) = (-r\sin\tau,\ r\cos\tau,\ \mu r).$$

Setting $\lambda = \cos\alpha$ and simplifying by $r$, we obtain for all $0 \leq x \leq r$

$$-x\sin\tau - \mu^2 r\tau \;\leq\; \lambda\sqrt{1+\mu^2}\,\sqrt{(x - r\cos\tau)^2 + r^2\sin^2\tau + \mu^2 r^2\tau^2}, \tag{4.5}$$

Notice that $\mu$ satisfies $\sin\tau + \mu^2\tau > 0$ for every $\tau \geq 0$. Therefore,

$$-x\sin\tau - \mu^2 r\tau \leq -x(\sin\tau + \mu^2\tau) \leq 0,$$

so (4.5) is clearly satisfied. □

4.2. **Arbitrary long eels inside a bounded cylinder.** We are now ready to construct arbitrarily long $\lambda$-eels lying inside the following bounded cylinder:

$$\mathrm{Cyl}\,(r, [a, a+2\pi\mu r]) := \{(x, y, z) \in \mathbb{R}^3;\ x^2 + y^2 = r^2,\ a \leq z \leq a + 2\pi\mu r\}. \tag{4.6}$$

Indeed we have the following result.

**Proposition 4.8.** *Let* $\lambda \geq 1/\sqrt{5}$ *and let* $\mathrm{Cyl}\,(r, [a, a+2\pi\mu r])$ *be the bounded cylinder defined in (4.6). Then there exists a* $\lambda$*-eel*

$$\gamma : I \longmapsto \mathrm{Cyl}\,(r, [a, a+2\pi\mu r])$$

*whose length is greater than* 1. *Moreover, the initial point of* $\gamma$ *lies in the upper part of the cylinder* ($z = a + 2\pi\mu r$) *while the last point lies at the bottom* ($z = a$).

*Proof.* Without loss of generality, we assume $a = 0$. Below, $N$ is a fixed integer given by Lemma 4.6. Let us fix an odd integer $n$ such that $2\pi\mu r n > 1$. Then for $1 \leq k \leq n$, we define internal cylinders

$$C_k := \mathrm{Cyl}\,(\frac{r}{N^{n-k}}, [0, 2\pi\mu r]) = \{(x, y, z) \in \mathbb{R}^3;\, x^2 + y^2 = \left(\frac{r}{N^{n-k}}\right)^2, 0 \leq z \leq 2\pi\mu r\}.$$

For $k = 2\ell + 1 \leq n$ (odd) we define a downward spiral curve $\gamma_k^{\downarrow}$ as follows:

$$\gamma_k^{\downarrow}(t) = \frac{r}{N^{n-k}}\big(\cos(t),\, \sin(t),\, \mu(2\pi N^{n-k} - t)\big), \qquad \text{for } 0 \leq t \leq 2\pi N^{n-k}.$$

while for $k = 2\ell \leq n$ (even) we define an upward spiral curve $\gamma_k^{\uparrow}$ as follows:

$$\gamma_k^{\uparrow}(t) = \frac{r}{N^{n-k}}\big(\cos(t),\, \sin(t),\, \mu t\big), \qquad \text{for } 0 \leq t \leq 2\pi N^{n-k}.$$

Notice that if $k$ odd,

$$\gamma_k^{\downarrow}(0) = (\frac{r}{N^{n-k}}, 0, 2\pi\mu r) \quad \text{and} \quad \gamma_k^{\downarrow}(2\pi N^{n-k}) = (\frac{r}{N^{n-k}}, 0, 0),$$

while for $k$ even

$$\gamma_k^{\uparrow}(0) = (\frac{r}{N^{n-k}}, 0, 0) \quad \text{and} \quad \gamma_k^{\uparrow}(2\pi N^{n-k}) = (\frac{r}{N^{n-k}}, 0, 2\pi\mu r).$$

Each spiral $\gamma_k$ lies on the surface of the cylinder $C_k$ and makes $N^{n-k}$ loops to reach the upper part of the cylinder starting from the bottom and going upwards if $k$ is even (respectively, to reach the bottom, starting from the upper part and going downward, if $k$ is odd). We finally define parametrized segments $e_k^+$ joining the end point of $\gamma_k^{\downarrow}$ to the initial point of $\gamma_k^{\uparrow}$ (for $k = 2\ell + 1$), and respectively $e_k^-$ joining the end point of $\gamma_k^{\uparrow}$ to the initial point of $\gamma_{k+1}^{\downarrow}$ (for $k = 2\ell$), that is:

$$e_k^+(t) = \left(\frac{r}{N^{n-k}}(1 + t(N-1)), 0, 2\pi\mu r\right) \text{ and } e_k^-(t) = \left(\frac{r}{N^{n-k}}(1 + t(N-1)), 0, 0\right), \quad t \in [0, 1].$$

The curve $\gamma$ will now be defined concatenating the above curves: we start with $k = 1$ and the downward spiral $\gamma_1^{\downarrow}$ and we concatenate with the segment $e_1^-$. We continue with the upward spiral $\gamma_1^{\uparrow}$ and the segment $e_1^+$ and concatenate with $\gamma_2^{\downarrow}$ ($k = 2$), then the segment $e_2^-$ and so on, up to the final downward spiral $\gamma_n^{\downarrow}$. The resulting curve is clearly continuous. Applying Lemma 4.6 and Lemma 4.7 we deduce that $\gamma$ is a $\lambda$-eel. The length of $\gamma$ is clearly greater than 1 since we cross $n$ times the cylinder of length $2\pi\mu r$ (and we have taken $n \geq 2$ such that $2\pi\mu r n > 1$). □

**Remark 4.9.** Proposition 4.8 ensures, by rescaling, that we can construct arbitrarily long $\lambda$-eels inside arbitrarily small cylinders. The $\lambda$-eel $\gamma$ inside the cylinder $\mathrm{Cyl}\,(r, [a, b])$, where $b = a + 2\pi\mu r$, is obtained by concatenating pieces of three different types:

- Type 1: a spiral going downward: $\gamma_i^{\downarrow}(t) = (\rho\cos(t), \rho\sin(t), b - \rho\mu t)$,
- Type 2: a spiral going upward: $\gamma_i^{\uparrow}(t) = (\rho\cos(t), \rho\sin(t), a + \rho\mu t)$,
- Type 3: a segment parametrized by $e_i^-(t) = (t, 0, a)$ or by $e_i^+(t) = (t, 0, b)$.

**Remark 4.10.** It is possible to modify slightly the above construction to get a $\lambda$-ell (with $\lambda = 1/\sqrt{5}$) $\gamma : (-\infty, 0] \to \mathrm{Cyl}\,(r, [a, a + 2\pi\mu r])$, with infinite length, and such that $\lim_{t\to-\infty} \gamma(t)$ does not exist. Since the curve constructed $\gamma$ above depends on the parameter $n$, let us denote it $\gamma_n$. We can assume, without loss of generality, by choosing a suitable parametrization, that $\gamma$ is defined on $[-n, 0]$ and that for each $n$, the restriction of $\gamma_{n+1}$ to $[-n, 0]$ coincides with $\gamma_n$.

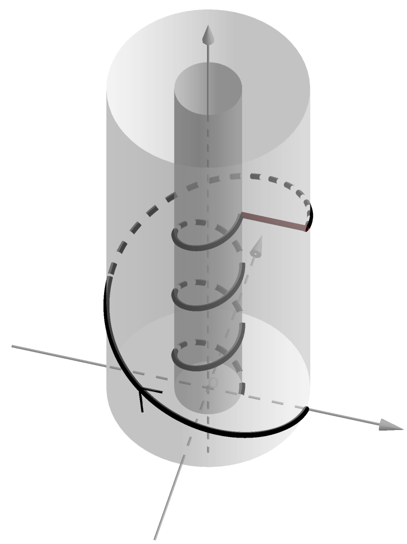

Figure 3. A block of the construction

Now we define $\gamma$ on $(-\infty, 0]$, satisfying, for each $n$, $\gamma_{|[-n,0]} = \gamma_n$. Since each $\gamma_n$ is a $\lambda$-ell, it is clear that $\gamma$ is a $\lambda$-ell. Morover, the $z$-coordinate of $\gamma(t)$ oscillates infinitely many times between $a$ and $a + 2\pi\mu r$. This shows both that $\gamma$ has infinite length and that $\lim_{t\to-\infty} \gamma(t)$ does not exist.

4.3. **Constructing bounded eels of infinite length in 3D.** To construct a bounded $\lambda$-eel with infinite length, we need to glue together curves of length greater than 1 (constructed in the previous subsection) that lie each time in prescribed disjoint bounded cylinders, all taken along the $z$-axis, of the form $C_n := \mathrm{Cyl}\,(r_n, [a_n, b_n])$ with $a_n > b_{n+1}$ and $r_n \searrow 0^+$. To construct efficiently such a curve, and to establish that it is a $\lambda$-eel, we shall need the following result, asserting that a $\lambda$-eel lying in a small cylinder does not see a bigger remote cylinder of the same axis.

**Lemma 4.11.** *Let $\lambda = 1/\sqrt{5}$, $\alpha = \arccos(\lambda)$, and let us set*

$$\mathrm{Cyl}\,(R, [a, b]) := \{(x, y, z) \in \mathbb{R}^3;\ x^2 + y^2 \le R,\ a \le z \le b\}.$$

*Then there exists $M > 1$ such that, for every $r \in (0, R/2)$ and $a', b' \in \mathbb{R}$ such that*

$$a' < b' < a < b \quad \text{and} \quad b' - a' \ge MR$$

*the curve $\gamma : [a', b'] \to \mathbb{R}^3$ with equation $\gamma(t) = (r\,\cos t,\ r\,\sin t,\ \mu r t)$, satisfies*

$$C(\tau, \alpha) \cap \mathrm{Cyl}\,(R, [a, b]) = \emptyset.$$

*Proof.* Without loss of generality, we can assume $b' = 0$. The equation of the spiral $\gamma : [0, \infty) \to \mathbb{R}^3$ is of the form

$$\gamma(t) = (r\,\cos t,\ r\,\sin t,\ \mu r t)\,, \qquad a' \le t \le 0,$$

where $\mu < 1/2$ is given by Lemma 4.4. Set

$$\sigma(\theta, z, u) = (u\cos\theta, u\sin\theta, z)\,, \quad \gamma(0) = (r, 0, 0) \quad \text{and} \quad \gamma'(0) = (0, r, \mu r)\,.$$

It is enough to check that

$$\langle \gamma'(0), \sigma(\theta,z,u)-\gamma(0)\rangle \le \cos\alpha \, \|\gamma'(0)\| \, \|\sigma(\theta,z,u)-\gamma(0)\| \qquad \forall\theta\in[0,2\pi],\ \forall z\in[a,b],\ \forall u\in[0,R].$$

The above condition reads, for all $\theta \in [0, 2\pi]$, for all $z \in [a, b]$ and for all $u \in [0, R]$,

$$ru\sin\theta + \mu rz \le \frac{1}{\sqrt{5}}\sqrt{r^2(1+\mu^2)}\sqrt{(u\cos\theta - r)^2 + u^2\sin^2\theta + z^2}.$$

So it is enough to check that for all $z \in [a, b]$ and $u \in [0, R]$ it holds

$$u + \mu z \le \frac{1}{\sqrt{5}} z\sqrt{1+\mu^2}.$$

In order to do that, let us fix the value of $M$. For $\mu < 1/2$, we have $\sqrt{1+\mu^2} > \sqrt{5}\mu$. Therefore, we can choose $M > 0$ such that $\sqrt{1+\mu^2} > \left(\mu + \frac{1}{M}\right)\sqrt{5}$. Now for all $u \le R$ and $z \ge a$ we have

$$\frac{u+\mu z}{z} \le \frac{R}{a} + \mu \le \frac{1}{M} + \mu < \frac{\sqrt{1+\mu^2}}{\sqrt{5}},$$

This completes the proof of the lemma. □

We are now ready to prove Theorem 4.2, that is, given $\lambda = \frac{1}{\sqrt{5}}$, we construct a continuous curve $\gamma : [0, +\infty] \to \mathbb{R}^3$ of infinite length, lying in the unit ball, with nonzero right derivative at each point and satisfying the $\lambda$-cone property ($\lambda$-eel).

*Proof of Theorem 4.2.* We claim that we can construct a sequence of disjoint bounded cylinders

$$C_n = \mathrm{Cyl}\,(r_n, [a_n, a_n + 2\pi\mu r_n]), \quad n \ge 1$$

along the $z$-axis, such that $a_n \in [0, 1)$, $r_{n+1} \le r_n/2$, $\ell_n := a_n - (a_{n+1} + 2\pi\mu r_{n+1}) > 0$ and $\ell_n/r_n$ is sufficiently big to ensure that the cylinder $C_n$ is not seen by any $\lambda$-eel lying in a (smaller) cylinder $C_m$ for $m > n$ (*c.f.* Lemma 4.11). More precisely, we define $a_0 = 0$, and, for $n \ge 1$,

$$a_n = 2^{-n} \quad \text{and} \quad r_n = \frac{1}{2^{n+1}(\pi\mu + M)},$$

where $M > 0$ is given by Lemma 4.11. Let us check that the conditions of Lemma 4.11 are fulfilled for the cylinders $C_n$ (big remote cylinder) and $C_{n+1}$ (small cylinder):

$$\ell_n = a_n - (a_{n+1} + 2\pi\mu r_{n+1}) = \frac{1}{2^{n+1}} - \frac{2\pi\mu}{2^{n+2}(\pi\mu + M)} = \frac{M}{2^{n+1}(\pi\mu + M)} \ge Mr_n.$$

Now the construction is as follows. For each $n$, let $\gamma_n$ be the $\lambda$-eel given by Proposition 4.8, of length greater than 1 lying inside the cylinder $C_n$, entering this cylinder from the upper part ($z = a_n + 2\pi\mu r_n$) and having its endpoint at the bottom ($z = a_n$). Let $\tilde{e}_n$ be the oriented segment going from the endpoint of the curve $\gamma_n$ (bottom of the cylinder $C_n$) to the starting point of $\gamma_{n+1}$ (upper part of the cylinder $C_{n+1}$). We now define $\gamma : [0, +\infty) \to \mathbb{R}^3$ by concatenation of the following curves : $\gamma_1$, $\tilde{e}_1$, $\gamma_2$, $\tilde{e}_2$, and so on. It is clear that $\gamma$ is continuous and has right derivative at each point. Morever, $\gamma$ is contained in the unit ball of $\mathbb{R}^3$ and its length $\ell(\gamma)$ is greater than $\ell(\gamma_1) + \ell(\gamma_2) + \cdots + \ell(\gamma_n) \ge n$ for every $n$, therefore it is infinite. Observe that $\gamma(t)$ has limit 0 as $t \to +\infty$. It remains to prove that $\gamma$ is a $\lambda$-eel, that is, it satisfies the $\lambda$-cone condition. Notice that each curve $\gamma_n$, $\tilde{e}_n$ is individually a $\lambda$-eel (that is, it satisfies the $\lambda$-cone property with respect to itself). Provided $M$ is sufficiently big, the segment $\tilde{e}_n$ is almost parallel to the $z$-axis and it is oriented to the opposite direction of the previous curves $\gamma_1$, $\tilde{e}_1, \cdots$, $\gamma_n$. Therefore, if the $\lambda$-cone $C(t, \alpha)$, given in (4.1), has its origin onto a segment $\tilde{e}_n$, then it does not meet the union of the ranges of $\gamma_1$, $\tilde{e}_1, \cdots$, $\gamma_n$. It remains to treat the case where $C(t,\alpha)$ has its origin to a curve of the form $\gamma_n$. These curves are constructed (for each $n$) by concatenating pieces of the form $\gamma_i^{\downarrow}$ (of type 1), $\gamma_i^{\uparrow}$ (of type 2) and $e_i^+$ or $e_i^-$ (of type 3) (*c.f.* Remark 4.9). If the $\lambda$-cone lies on a piece of type 1 or of type 3 of $\gamma_n$, then it is oriented to the opposite directions of all of the previous pieces $\gamma_1$, $\tilde{e}_1, \cdots$, $\gamma_{n-1}$, $\tilde{e}_{n-1}$ of $\gamma$, therefore it does not meet the union of their ranges. If now the cone $C(t,\alpha)$ has its origin on an upward piece $\gamma_i^{\uparrow}$ (type 2) of the curve $\gamma_n$, then the result follows from Lemma 4.11. The proof is complete. □

## 5. Curves with the $\lambda$-cone property in 2 dimensions

It is remarkable that there is no analogue of the construction in Theorem 4.2 in dimension 2. Indeed, we shall show that for any value of the parameter $\lambda \in [-1,1)$, any bounded planar $\lambda$-eel (that is, continuous curve with right derivative at each point that satisfies the $\lambda$-cone property) is rectifiable and has finite length. We shall need the following lemmas. (Recall $\alpha = \arccos(\lambda)$.)

**Lemma 5.1.** *Let $\gamma : I \to \mathbb{R}^2$ be a planar $\lambda$-eel and $t_1 < t_2 < t_3$ in $I$. Then*

$$\gamma(t_3) \notin [\gamma(t_1), \gamma(t_2)].$$

*Proof.* Set $A = \gamma(t_1)$, $B = \gamma(t_2)$, $C = \gamma(t_3)$ and assume towards a contradiction that $C \in [A,B]$. Choosing adequate coordinates in $\mathbb{R}^2$ we may assume that $A = (0,0)$, $B = (1,0)$ and $C = (c,0)$ with $c \in (0,1)$. In the sequel, we shall write $\gamma = (\gamma_1, \gamma_2)$ in these coordinates.

Before we proceed, notice that we may assume

$$\gamma(t) \notin (A,C) \quad \text{for all } t \in (t_1,t_2]. \tag{5.1}$$

Indeed, set $N_1 = \{t \in [t_1,t_2) : \gamma(t) \in [A,C]\} = \{t \in [t_1,t_2) : \gamma_1(t) \in [0,c], \gamma_2(t) = 0\}$ and $\alpha_1 := \sup\{\gamma_1(t) : t \in N_1\}$. Then $\alpha_1 < c$ (since $\gamma$ is continuous and injective) and consequently, there exists $t_1 \le \tilde{t}_1 < t_2$ with $\gamma(\tilde{t}_1) = (\alpha_1, 0) = \tilde{A}$. In this case we can replace $A$ by $\tilde{A}$ and $t_1$ by $\tilde{t}_1$ and get (5.1).

We set $\tilde{t}_2 = \inf\{t \in [t_1,t_2] : \gamma_1(t) \ge c, \gamma_2(t) = 0\}$. There is no loss of generality to assume $t_2 = \tilde{t}_2$, since we can always replace $B$ by $\tilde{B} = (\gamma_1(\tilde{t}_2), 0)$ (notice that $\gamma_1(\tilde{t}_2) > c$ by injectivity).

Therefore for all $t \in (t_1,t_2)$ we have $\gamma(t) \notin (A,B)$. Setting $\Gamma_{AB} = \{\gamma(t) : t \in [t_1,t_2]\}$ we deduce that $\Gamma_{AB} \cup (B,A]$ is a Jordan curve which separates $\mathbb{R}^2$ in two regions, exactly one of them being bounded. Call $\mathcal{R}$ this bounded region, set $H^+ = \{x = (x_1,x_2) : x_2 > 0\}$, $H^- = \{x = (x_1,x_2) : x_2 < 0\}$ and let $\varepsilon > 0$ be such that $B(C,\varepsilon) \cap \Gamma_{AB} = \emptyset$. Then at least one of the sets $B(C,\varepsilon) \cap H^+$ and $B(C,\varepsilon) \cap H^-$ has nonempty intersection with $\mathcal{R}$. Assume, with no loss of generality, that

$$B(C,\varepsilon) \cap H^- \cap \mathcal{R} \neq \emptyset.$$

Then for every $x \in H^- \cap \operatorname{int} \mathcal{R}$ and every direction $d = (d_1,d_2) \in \mathbb{S}^1$ (the unit sphere of $\mathbb{R}^2$) with $d_2 \le 0$ it holds $\ell_{x,d} \cap \Gamma_{AB} \neq \emptyset$, where $\ell_{x,d} := \{x + \mu d : \mu \ge 0\}$ is the half-line emanating from $x$ with direction $d$. In particular, shrinking $\varepsilon > 0$ if necessary, and recalling notation (1.5) we deduce that

$$C_x(d,\alpha) \cap \Gamma_{AB} \neq \emptyset, \quad \text{for all } x \in B(C,\varepsilon) \cap H^- \cap \mathcal{R} \text{ and all } d = (d_1,d_2) \text{ with } d_2 \le 0. \tag{5.2}$$

Let $\tau_3 \in (t_2,t_3)$ be such that for all $t \in (\tau_3,t_3]$ we have $\gamma(t) \in B(C,\varepsilon)$ (such $\tau_3$ exists by continuity). Then it follows by (5.2) and the $\lambda$-eel property that $\gamma_2'(t) > 0$, and consequently, $\gamma_2(t) < 0$ (since $\gamma_2(t_3) = 0$). Let further $\tau \in [t_2,t_3]$ be such that

$$\gamma_2(\tau) = \min_{t \in [t_2,t_3]} \gamma_2(t) \; ( < 0 \, ).$$

Then since $\gamma_2(t_2) = 0$, there exists $\tilde{t} \in [t_2, \tau]$ with ($\gamma(t) \in \mathcal{R}$ and) $\gamma_2'(t) < 0$ which together with (5.2) contradicts the $\lambda$-eel property. $\square$

For the next statement, recall notation (4.1) and (1.5).

**Lemma 5.2.** *Under the assumptions of the previous lemma we have:*

$$C(\gamma'(t), \alpha) \cap K(t) = \{0\}, \quad \textit{for all } t \in I.$$

*Proof.* Fix $t \in I$ and assume with no loss of generality (by translation) that $\gamma(t) = 0$. Then $K(t) = \overline{\text{cone}}\,(\Gamma(t) - \gamma(t)) = \overline{\text{cone}}\,\Gamma(t)$, where $\Gamma(t) = \{\gamma(\tau) : t \in [0, t]\}$. Let assume that there exists $x \in C(\gamma'(t), \alpha) \cap K(t)$, $x \neq 0$. Then by Caratheodory theorem, there exist $x_i = \gamma(\tau_i)$, $i \in \{1, 2, 3\}$ with $\tau_1 \leq \tau_2 \leq \tau_3 < t$ and $x \in \text{conv}\{x_1, x_2, x_3\}$ (convex envelope). Set $\ell_1 := \{x_1 + \mu(x - x_1) : \mu \geq 0\}$ and $\ell_2 = \{x_2 + \mu(x - x_2) : \mu \geq 0\}$. If $\ell_1 \cap \Gamma(t) = \ell_2 \cap \Gamma(t) = \emptyset$, then for $\mu_1$, $\mu_2$ sufficiently big, the point $x_3$ should belong to the triangle defined by the points $\ell_1(\mu_1) := x_1 + \mu_1(x - x_1)$, $x$ and $\ell_2(\mu_2) = x_2 + \mu_2(x - x_2)$. Then by connectedness of $\gamma([\tau_1, \tau_3])$, we deduce that for some $s < \tau_3 < t$ it holds $\gamma(s) \in \ell_1 \cup \ell_2$. We deduce that $x$ is a convex combination of two points of $\Gamma(t)$, that is, $x \in [\gamma(s_1), \gamma(s_2)]$ for some $s_1 < s_2 < t$. Set $\Gamma_{12} := \{\gamma(\tau\} : \tau \in [s_1, s_2]\}$. Since $\gamma$ is a $\lambda$-eel, we have $C(t, \alpha) \cap \Gamma_{12} = \emptyset$. Then $[\gamma(s_2), \gamma(s_1)] \cup \Gamma_{12}$ is a Jordan curve and $\gamma(t) = 0 \in \mathcal{R}$ where $\mathcal{R}$ is the bounded region delimited by the Jordan curve. This yields that for some $t_1 < t_2 < t$, $\gamma(t) = 0$ is a convex combination of $\gamma(t_1)$ and $\gamma(t_2)$, which contradicts Lemma 5.1. $\square$

In view of Lemma 5.2 and Remark 3.7 we obtain our main result.

**Theorem 5.3** (bounded planer eels have finite length)**.** *Let $\gamma : I \to \mathbb{R}^2$ be a bounded $\lambda$-eel. Then $\gamma$ is rectifiable and has finite length.* $\square$

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