Nonlinear Analysis 177 (2018) 209–237

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Strict limits of planar BV homeomorphisms

Daniel Campbell[a], Stanislav Hencl[b,*], Aapo Kauranen[c], Emanuela Radici[d]

[a] *Department of Mathematics, University of Hradec Králové, Rokitanského 62, 500 03 Hradec Králové, Czech Republic*
[b] *Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 00 Prague 8, Czech Republic*
[c] *Barcelona Graduate School of Mathematics (BGSMath), Department of Mathematics Building C Science Faculty, Universitat Autonoma de Barcelona, 08193 Bellaterra (Barcelona), Spain*
[d] *DISIM - Department of Information Engineering, Computer Science and Mathematics, University of L'Aquila, Via Vetoio 1 (Coppito), 67100 L'Aquila (AQ), Italy*

A R T I C L E   I N F O

Article history:
Received 23 February 2018
Accepted 19 March 2018
Communicated by Enzo Mitidieri

MSC:
30C65
26B30
16E35

Keywords:
Homeomorphism
Bounded variation

A B S T R A C T

We introduce a new class of planar mappings that allows for cavitations and fractures. The class is the set of strict limits of planar BV homeomorphisms. Each mapping from this class has a proper pointwise representative which is a multifunction, we show that it maps disjoint sets to essentially disjoint sets and that they have an inverse as a proper multifunction. We also characterize and study cavities and fractures of these mappings.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a domain and let $f : \Omega \to \mathbb{R}^n$ be a mapping. In his pioneering works J.M. Ball [2,3] studied mappings that can serve as a class of deformations in nonlinear elasticity. He studied the existence of energy minimizers, their continuity and invertibility. The study of mappings f in this spirit was later extended e.g. by Šverák [26] and Müller, Tang and Yan [23]. Furthermore, in the following works Müller and Spector [22] (see also [25,7]) extended this class of mappings to include also cavitations (see Example 7.1) that appear naturally in some physical deformations (see e.g. [4] and references given therein). In these papers the authors were studying properties of elastic deformations allowing or forbidding cavitation using the (INV) condition. Informally speaking this condition tells us that the image $f(B(x,r))$ lies inside $f(\partial B(x,r))$ and image of $f(\mathbb{R}^2 \setminus B(x,r))$ lies outside of $f(\partial B(x,r))$.

* Corresponding author.
E-mail addresses: campbell@math.fau.de (D. Campbell), hencl@karlin.mff.cuni.cz (S. Hencl), aapekaur@mat.uab.cat (A. Kauranen), emanuela.radici@univaq.it (E. Radici).

https://doi.org/10.1016/j.na.2018.03.013

Moreover, in his survey papers Ball [4] and [5] asks for the extension of these models to also include the model of fractures; as in many physically relevant deformations the material may break. This was done in a series of papers by Henao and Mora-Corral [12–15] where they added some energy term that corresponds to the surface energy of the fracture. Also in these models the corresponding deformation is one-to-one a.e.

We propose a different approach for the study of deformations, although we study only the planar mappings. For simplicity we assume that $f : [0,1]^2 \to [0,1]^2$ and moreover we assume that f is the identity on the boundary. This prescription of the boundary values corresponds to the fact that we hold our map on the boundary (thus prescribing the Dirichlet boundary data) and we minimize the elastic energy inside. Analogously it would be possible to study mappings $f_k : \Omega \to \Delta$ where Ω and Δ are simply connected Lipschitz planar domains with prescribed boundary values that are the same for all k. For simplicity we restrict to the case $\Omega = \Delta = (0,1)^2$ and $f_k(x) = x$ on the boundary but this is not essential for the theory.

The basic function space for us is the class of mappings of bounded variation BV (see e.g. [1]) as it naturally contains fractures and cavities. To keep some notion of invertibility we assume that our f is a strict limit of BV-homeomorphisms $f_k : [0,1]^2 \to [0,1]^2$ (with $f_k(x) = x$ for every $x \in \partial[0,1]^2$), i.e.

$$f_k \to f \text{ in } L^1 \text{ and } |Df_k|([0,1]^2) \to |Df|([0,1]^2).$$

It is easy to see that this class of mappings can model both cavitations and fractures (see Examples 7.1 and 7.2). Up to a subsequence we may (and we will) also assume that $f_k \to f$ pointwise a.e. The key advantage of our planar approach comes from the fact that the inverse of a BV homeomorphism g is also a BV homeomorphism (see [17]) and the total variations are equal (see [9])

$$|Dg|((0,1)^2) = |Dg^{-1}|((0,1)^2). \tag{1.1}$$

Hence f_k^{-1} forms a bounded sequence in BV and we can select a weakly converging subsequence $f_{k_l}^{-1}$ converging to a BV mapping h. We can study properties of f using h and we can show that these mappings are in some sense inverses to each other.

The similar class of weak limits of planar mappings was previously studied by Iwaniec and Onninen in [19] and [20]. In these papers the authors characterized the class of weak limits of Sobolev homeomorphisms (for $p \geq 2$) and they showed that the class of weak and strong closures are the same. This research was extended to cover also the case $1 < p < 2$ by a recent result of De Philippis and Pratelli [8], see also Campbell, Onninen, Räbinä and Tengvall [6]. We study the more general class of limits of BV homeomorphisms to be able to incorporate fractures but we do not obtain the full characterization of this class.

Let us comment on our assumption that $f_k \to f$ strictly in BV. In variational models we study minimizers of the energy functional like

$$E(f) := \int_{(0,1)^2} W(Df(x))\, dx$$

where W satisfies some natural assumptions including the growth condition $W(A) \geq C_1|A| - C_2$. It follows that any sequence f_k such that $E(f_k) \underset{k\to\infty}{\to} \inf_f E(f)$ is a bounded sequence in BV and hence it contains a weakly converging subsequence. Moreover, with the help of lower sequential semicontinuity we obtain $\lim_{k\to\infty} E(f_k) = E(f)$ and this may imply that $|Df_k|((0,1)^2) \to |Df|((0,1)^2)$ for some special energy functionals. We have in mind the models of nonlinear elasticity and hence it is natural to study our functional only in the class of invertible mappings and thus we can restrict our situation to homeomorphisms f_k and their weak limits. However, even in the class of homeomorphisms the natural infimum of the energy is not necessarily attained by a homeomorphism because of the collapse of matter near the boundary (see e.g. [18]). This naturally leads us to the study of weak limits of homeomorphisms. For the limit mapping we would like to have some key properties that correspond to "non-interpenetration of matter" like the fact that disjoint sets are mapped to essentially disjoint sets. This may drastically fail for weak limits (see Example 7.3) but it is true once we restrict our attention to strict limits only.

We study a class of mappings

$$\mathcal{S} := \Big\{ f : (0,1)^2 \to (0,1)^2 : \text{ there are homeomorphisms } f_k : [0,1]^2 \to [0,1]^2 \text{ with } f_k(x) = x \text{ for every } x \in \partial[0,1]^2, f_k \to f \text{ strictly in } BV, \text{ strongly in } L^1 \text{ and pointwise a.e.} \Big\}. \tag{1.2}$$

Thanks to the result [24], any planar homeomorphism of bounded variation can be approximated strictly, together with its inverse, by a sequence of diffeomorphisms. Thus, it would be possible to assume that the mappings f_k of the approximating sequence are all diffeomorphisms.

Let us briefly comment the content of this paper. In Section 2 we recall some preliminaries about BV mappings. In Section 3 we define our f pointwise as a multifunction

$$\tilde{f}(x) = \big\{ y \in [0,1]^2 : \text{ there is } x_n \to x \text{ and } k_n \to \infty \text{ such that } f_{k_n}(x_n) \to y \big\}. \tag{1.3}$$

In this way the image of a cavitation is a whole ball and the image of each point on the fracture corresponds to a segment in the image (see Section 7). In previous works (see [22] or [12]) an analogous multifunction was defined using the topological degree. Unfortunately this is not available for us as our function is essentially discontinuous around the fractures. Therefore we have decided to use a different definition, (1.3) and in Section 3 we show also other equivalent definitions. We show some basic properties of $\tilde{f}$ and f like differentiability a.e. and the fact that the weak limit of f_k^{-1} is the inverse multifunction to f. Further we show there that $\tilde{f}(x)$ is connected for each x and that it equals to $\{f(x)\}$ for a.e. x.

In Section 4 we show that $\tilde{f}(x)$ is well-defined as this set does not depend on the sequence f_k and further we show that images of disjoint balls by $\tilde{f}$ are essentially disjoint but as pointed out before it is necessary to have strict convergence (and not only weak convergence) here.

In Section 5 we study cavitations in detail. We show that we have at most countably many points $x \in (0,1)^2$ such that $|\tilde{f}(x)| > 0$. We prove that for these points we have

$$\lim_{r\to 0+} \lim_{k\to\infty} |f_k(B(x,r))| = |\tilde{f}(x)|$$

and we also show that this corresponds to the singularities of the so called distributional Jacobian (see Section 5 for its definition). Further in Section 6 we study fractures of f defined as

$$Frac := \{x : \ \mathcal{H}^1(\tilde{f}(\{x\})) > 0\}.$$

We show that the set of fractures has $\sigma-$finite $\mathcal{H}^1$-measure and that for $\mathcal{H}^1$-a.e. point in $Frac$ we know that $\tilde{f}(x)$ is a line segment. Moreover, we estimate the size of fractures with $|Df|$.

Finally in the last section we give some examples to show which mappings belong to our class (models of cavitation, models of fracture). Further we explain there why we need to have strict limits and not only weak limits and we give an example to show that the strict convergence $f_k \to f$ gives only weak convergence of $f_k^{-1} \to h$ but not strict convergence in general.

2. Preliminaries

Given a point $x \in \mathbb{R}^n$ and $r > 0$ we denote the open ball centered at x with radius r by $B(x,r)$ and $S(x,r) := \partial B(x,r)$. We also denote the square centered at x with sidelength $2r$ by $Q(x,r)$. For simplicity we use notation $Q := [0,1]^2$ sometimes.

For future reference we recall Stepanov's theorem.

Theorem 2.1 (*Stepanov*). *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and $g : \Omega \to \mathbb{R}^m$ be a map such that*

$$\limsup_{y \to x} \frac{|g(x) - g(y)|}{|x - y|} < \infty \quad \text{for a.e. } x \in \Omega.$$

Then g is differentiable almost everywhere in Ω.

2.1. Strict convergence on subsets

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^n$ and $f_k \in BV(\Omega, \mathbb{R}^2)$ be a sequence of BV mappings converging strictly to f in BV. Let $A \subset \Omega$ with $|Df|(\partial A) = 0$. Then the restriction of f_k to A converges strictly and the restriction of f_k to $\Omega \setminus \overline{A}$ converges strictly. Moreover, for almost every $t \in [0, 1]$ we have that the restriction of f_k to $\Omega \cap \{y > t\}$ converges strictly and the restriction of f_k to $\Omega \cap \{y < t\}$ converges strictly.*

Proof. The second claim follows from the first easily after noticing that $|Df|(\{y = t\}) = 0$ for almost every t (see [1, Theorem 3.103]).

Denote the set $\Omega \setminus \overline{A}$ by B. First, we notice that $f_k \to f$ in L^1 on A and B. To prove the strict convergence it is enough to prove the convergence of total variation measures on sets A and B.

Consider the sequences $|Df_k|(A)$ and $|Df_k|(B)$. It is clear that both sequences are bounded and have converging subsequences. Choose any subsequence $j(k)$ so that

$$|Df_{j(k)}|(A) \to a \text{ and } |Df_{j(k)}|(B) \to b$$

for some numbers a and b.

By the definition of the strict convergence we have

$$\lim_{j\to\infty} |Df_{j(k)}|(A) + \lim_{j\to\infty} |Df_{j(k)}|(B) \le \lim_{j\to\infty} |Df_{j(k)}|(\Omega) = |Df|(\Omega).$$

By the lower semi-continuity of the total variation (see [11, 5.2.1., Theorem 1]) we see that

$$a \ge |Df|(A) \text{ and } b \ge |Df|(B). \tag{2.1}$$

If we had $a > |Df|(A)$ then we have

$$|Df|(B) = |Df|(\Omega) - |Df|(A) > a + b - a = b. \tag{2.2}$$

Notice that here we used the assumption $|Df|(\partial A) = 0$. Eq. (2.2) contradicts (2.1). Thus the only possibility is

$$|Df_k|(A) \to |Df|(A) \text{ and } |Df_k|(B) \to |Df|(B).$$

That is, the total variations converge as required and this concludes the proof. □

Proposition 2.3. *Let $f_k \in BV(Q, \mathbb{R}^2)$ be a sequence of BV mappings of Q converging strictly to f in BV. Then there exists a f_{k_m} a subsequence of f_k and an $\mathcal{L}^1$-zero-measure set $N \subset [0, 1]$ such that for any $t \in [0, 1] \setminus N$ mappings f_{k_m} and f are one dimensional mappings of bounded variation on $Q \cap \{y = t\}$ and the restriction of f_{k_m} converges strictly to the restriction of f on $Q \cap \{y = t\}$ as one dimensional BV functions.*

Proof. We take a subsequence, which we refer to again as f_k such that f_k converge pointwise almost everywhere to f. From L^1-convergence we know that, for almost every t, that the restriction of f_k to $Q \cap \{y = t\}$ converges to f in $L^1(Q \cap \{y = t\})$. Since f_k converge to f strictly then for almost every $t \in [0, 1]$

the restriction of f_k and f are one dimensional mappings of bounded variation on $Q \cap \{y = t\}$. Further we know that (the total variation converges which in conjunction with the lower semi-continuity of $|D_i f|$ gives

$$\int_0^1 |D_1 f|(\{y = t\})dt = \lim_{k\to\infty} \int_0^1 |D_1 f_k|(\{y = t\})dt. \tag{2.3}$$

For those t such that $f_k \to f$ in $L^1(Q \cap \{y = t\})$ we have

$$|D_1 f|(\{y = t\}) \leq \liminf |D_1 f_k|(\{y = t\}) \tag{2.4}$$

simply because for $\varphi \in \mathcal{D}(0, 1)$, $\|\varphi\|_\infty \leq 1$ we have

$$\liminf_{k\to\infty} |D_1 f_k|(\{y = t\}) \geq \liminf_{k\to\infty} \int_{\{y=t\}} f_k D_1\varphi d\mathcal{L}^1 = \int_{\{y=t\}} f D_1\varphi d\mathcal{L}^1$$

and taking the supremum over φ.

Now we use the Fatou Lemma together with (2.3) and (2.4) to see that for almost every t we have $|D_1 f|(\{y = t\}) = \liminf |D_1 f_k|(\{y = t\})$. For every m there exists a k_0 such that for all $k \geq k_0$ we have by (2.3)

$$\Big|\int_0^1 |D_1 f_k|(\{y = t\}) - |D_1 f|(\{y = t\})dt\Big| < \frac{|D_1 f|(Q)}{m}. \tag{2.5}$$

Call $\eta_m > 0$ the number such that $\int_A |D_1 f|(\{y = t\})dt < |D_1 f|(Q)/m$ whenever $\mathcal{L}^1(A) < \eta_m$. Assuming k_0 is large enough, we have that

$$|D_1 f_k|(\{y = t\}) \geq (1 - m^{-1})|D_1 f|(\{y = t\})$$

for all $k \geq k_0$ and all $t \in [0, 1] \setminus S$ where $\mathcal{L}^1(S) \leq \eta_m$. The proof of this simply mirrors the proof that a point-wise limit is a uniform limit up to a set of arbitrarily small measure. Also using $|D_1 f_k|(\{y = t\}) \geq 0$ and so

$$\int_0^1 \Big(|D_1 f|(\{y = t\}) - |D_1 f_k|(\{y = t\})\Big)^+ dt < 2\frac{|D_1 f|(Q)}{m}. \tag{2.6}$$

But then (2.5) and (2.6) together give

$$\int_0^1 \Big(|D_1 f|(\{y = t\}) - |D_1 f_k|(\{y = t\})\Big)^- dt < 3\frac{|D_1 f|(Q)}{m}$$

and so

$$\int_0^1 \Big||D_1 f|(\{y = t\}) - |D_1 f_k|(\{y = t\})\Big| < 5\frac{|D_1 f|(Q)}{m}.$$

Therefore since $|D_1 f_k|(\{y = t\})$ converges to $|D_1 f|(\{y = t\})$ in L^1 (as a function of t) we can find a subsequence that converges point-wise almost everywhere. For this subsequence the restriction of f_k to $(0, 1) \times \{t\}$ converges strictly to the restriction of f for a.e. t. $\square$

2.2. Fine properties of BV functions

Definition 2.4. Let $u \in L^1(\Omega, \mathbb{R}^m)$ and $x \in \Omega$. The point x is said to be an *approximate jump point of* u if there exist distinct points $a_+, a_- \in \mathbb{R}^m$ and $\xi \in S^{n-1}$ such that

$$\lim_{r\to 0} \fint_{B_+(x,r,\xi)} |u(x) - a_+|dx = 0 \text{ and } \lim_{r\to 0} \fint_{B_-(x,r,\xi)} |u(x) - a_-|dx = 0,$$

where $B_+(x, r, \xi) = B(x, r) \cap \{x : x\cdot\xi > 0\}$ is the half ball and similarly $B_-(x, r, \xi) = B(x, r) \cap \{x : x\cdot\xi < 0\}$. The set of all approximate jump points of u is denoted by $Jump_u$.

Remark 2.5. Let $f \in BV(\Omega, \mathbb{R}^2)$. Then

(1) The jump set $Jump_f$ is a subset of NL_f, the set of non-Lebesgue points of f [1, Prop. 3.69].
(2) By the Theorem of Federer and Vol'pert [1, Thm. 3.78] the set NL_f is (countably) 1-rectifiable and $\mathcal{H}^1(NL_f \setminus Jump_f) = 0$.
(3) Let us define

$$\Theta_f = \{x \in \Omega : \liminf_{r\to 0} r^{-1}|Df|(B(x,r)) > 0\}.$$

Then by [1, Prop. 3.92] we have $\mathcal{H}^1(\Theta_f \setminus Jump_f) = 0$.

It follows that for every $BV(\Omega, \mathbb{R}^2)$ we have

$$\lim_{\delta\to 0^+} \frac{|Df|(B(x,\delta))}{\delta} = 0 \quad \text{for a.e. } x \in [0,1]^2. \tag{2.7}$$

2.3. Properties of BV homeomorphisms

For planar BV homeomorphisms, we also need the following estimate of the distortion of the preimage of a ball.

Lemma 2.6 (*Lemma 1 [17]*). *Let $\Omega \subset \mathbb{R}^2$ be a domain, $g \in BV_{loc}(\Omega, \mathbb{R}^2)$ be a homeomorphism and let $B(y, 2r) \subset g(\Omega)$ be a ball. Then*

$$\operatorname{diam} g^{-1}(B(y,r)) \le \frac{C}{r}|Dg|\big(g^{-1}(B(y,2r))\big).$$

Recalling that the inverse of a planar homeomorphism of bounded variation has bounded variation (see [17,9] and (1.1)), we obtain the following corollary.

Corollary 2.7. *Let $\Omega \subset \mathbb{R}^2$ be a domain and suppose that $g \in BV_{loc}(\Omega, \mathbb{R}^2)$ is a homeomorphism and $B(x, 2r) \subset \Omega$. Then*

$$\operatorname{osc}_{B(x,r)} g \le C\frac{|Dg^{-1}|(g(B(x,2r)))}{r} = C\frac{|Dg|(B(x,2r))}{r}. \tag{2.8}$$

3. Definition of $\tilde{f}(x)$ and basic properties

In this section we prove some basic properties of our class of strict limits of BV homeomorphisms $\mathcal{S}$ (see (1.2)). We present some geometric properties of the limiting mappings f and h. Recall that h is the weak* limit of sequence of inverse mappings f_k^{-1}. In order to investigate the geometry of the limit function, it is convenient to introduce a complete description of the pointwise limits of f_k and f_k^{-1} via multifunctions. In detail, we define for every $x \in [0,1]^2$ the multifunction $\tilde{f}$ as

$$\tilde{f}(x) := \big\{y \in [0,1]^2 : \text{ there is } x_n \to x \text{ and } k_n \to \infty \text{ such that } f_{k_n}(x_n) \to y\big\}. \tag{3.1}$$

Let us note that in principle this definition depends on the approximating sequence f_k. In Section 4 we show that this is not the case and that $\tilde{f}(x)$ is the same if we define it using different g_k that converges to f strictly.

The aim of this section is to study some basic properties of $\tilde{f}(x)$. Let us start with the following equivalent characterization.

Lemma 3.1. *Let $f \in \mathcal{S}$ and let f_k be the corresponding sequence of BV homeomorphisms. For every $x \in [0,1]^2$ one has*

$$\begin{aligned} \tilde{f}(x) &= \bigcap_{\delta>0}\bigcap_{k=1}^{\infty}\overline{\bigcup_{m=k}^{\infty}\Big[f_m(B(x,\delta))+B(0,\delta)\Big]} && (I) \\ &= \bigcap_{\delta>0}\bigcap_{k=1}^{\infty}\overline{\bigcup_{m=k}^{\infty} f_m(B(x,\delta))} && (II) \\ &= \big\{y \in [0,1]^2 : \text{ there is } x_n \to x \text{ such that } f_{k_n}(x_n) \to y\big\}. && (III) \end{aligned} \tag{3.2}$$

Proof. We analyze the inclusions independently.

$II \subseteq I$. This follows immediately from the inclusion

$$f_m(B(x,\delta)) \subseteq f_m(B(x,\delta)) + B(0,\delta).$$

$I \subseteq III$. Consider first $y \in \bigcap_{\delta>0}\bigcap_{k=1}^{\infty}\overline{\bigcup_{m=k}^{\infty}\Big[f_m(B(x,\delta))+B(0,\delta)\Big]}$. For every fixed value of $\delta > 0$ and $k \geq 1$, we have that

$$y \in A_k = \overline{\bigcup_{m=k}^{\infty}\Big[f_m(B(x,\delta))+B(0,\delta)\Big]}$$

which is a closed set in $[0,1]^2$ not containing isolated points. Indeed, every set $f_m(B(x,\delta))$ has to be connected since f_m is continuous, thus also $f_m(B(x,\delta))+B(0,\delta)$ is connected and it cannot be a singleton because f_m are homeomorphisms. As a consequence every point of A_k is an accumulation point for A_k and hence for every $\varepsilon > 0$ there is a number $m \geq k$ and a point $z_m \in f_m(B(x,\delta))$ such that $|y - z_m| < \varepsilon + \delta$. Moreover, being $z_m \in f_m(B(x,\delta))$, it clearly exists $t_m \in B(x,\delta)$ for which $z_m = f_m(t_m)$. Recalling that we can always choose $\varepsilon = \delta = 1/k$, we obtain that for every $k \geq 1$ there exist $m_k \geq k$ and $t_{m_k} \in B(x,1/k)$ such that $|y - f_{m_k}(t_{m_k})| < 2/k$. Clearly, as $k \to \infty$, $t_{m_k} \to x$ and $f_{m_k}(t_{m_k}) \to y$ thus implying that $y \in \tilde{f}(x)$.

$III \subseteq II$. Let $y \in \tilde{f}(x)$, then for every $\delta > 0$ there is $\bar{N} = \bar{N}(\delta) \in \mathbb{N}$ and a subsequence $k_n = k_n(\delta)$ such that for every $n \geq \bar{N}$ and $k_n \geq \bar{N}$ one has $|x_n - x| < \delta$ and $|f_{k_n}(x_n) - y| < \delta$. In particular, $x_n \in B(x,\delta)$ and $f_{k_n}(x_n) \in f_{k_n}(B(x,\delta)) \cap B(y,\delta)$. Now, let us fix for a moment two values $\delta > 0$ and $k \geq 1$. Then we can find $N \geq \max\{\bar{N}; k\}$ and a subsequence k_n so that

$$f_{k_n}(x_n) \in \bigcup_{n=N}^{\infty}\big(B(y,\delta) \cap f_{k_n}(B(x,\delta))\big) \quad \text{for every } n, k_n \geq N.$$

Clearly, being $N \geq k$, one has

$$\bigcup_{n=N}^{\infty}\big(B(y,\delta) \cap f_{k_n}(B(x,\delta))\big) \subseteq \bigcup_{n=k}^{\infty} f_{k_n}(B(x,\delta))$$

thus for every $n, k_n \geq N$ the element $f_{k_n}(x_n)$ is contained in the set

$$\bigcup_{n=k}^{\infty} f_{k_n}(B(x,\delta)).$$

Finally, since $f_{k_n}(x_n) \to y$, we deduce that

$$y \in \overline{\bigcup_{n=k}^{\infty} f_{k_n}(B(x,\delta))}$$

and from the arbitrary choice of δ and k we conclude that

$$y \in \bigcap_{\delta>0} \bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} f_{k_n}(B(x,\delta))}. \quad \square$$

Lemma 3.2. *Let $f \in \mathcal{S}$. Then for every $x \in [0,1]^2$ the set $\tilde{f}(x)$ is nonempty, compact and connected.*

Proof. For every point $x \in [0,1]^2$ it is always possible to construct a sequence $x_n \to x$, then $(f_n(x_n))_n$ is bounded in $[0,1]^2$ and hence it admits a converging subsequence. In particular, being $[0,1]^2$ closed, there exists $y \in [0,1]^2$ and a subsequence k_n such that $f_{k_n}(x_n) \to y$, so y is an element of $\tilde{f}(x)$ and it is thus nonempty.

The compactness follows straightforward from the characterization (3.2). Indeed, $\tilde{f}(x) \subset [0,1]^2$ so it is clearly bounded, and, on the other hand, it is closed because the intersection of closed sets is still closed.

We would like to show that a set of the form $D_\delta = \bigcap_k \overline{A_k^\delta}$ is connected for every choice of δ, where $A_k^\delta = \bigcup_{m=k}^{\infty} f_m(B(x,\delta)) + B(0,\delta)$. Let us assume for a moment that this is true, then $D_{\delta_1} \subseteq D_{\delta_2}$ whenever $\delta_1 > \delta_2$. Then the connectedness of $\tilde{f}(x)$ follows immediately thanks to Lemma 3.1 and the fact that the intersection of nested connected compact sets is a connected compact set (see [10, Theorem 6.1.18]).

Therefore, it remains to show that D_δ is always connected. As we already noticed in Lemma 3.1, the sets $f_m(B(x,\delta)) + B(0,\delta)$ are connected. Moreover, for almost all $z \in B(x,\delta)$ we know that $f_m(z)$ is converging to an element $y = f(z) \in [0,1]^2$. Then for every $\delta > 0$ fixed, we can find $\bar{m} = \bar{m}(\delta)$ such that for every $m \geq \bar{m}$ we have $|f_m(z) - y| < \delta$. This implies that $y \in f_m(B(x,\delta)) + B(0,\delta)$ for every $m \geq \bar{m}$, thus

$$y \in \bigcap_{m=\bar{m}}^{\infty} f_m(B(x,\delta)) + B(0,\delta).$$

It follows that $\bigcup_{m=k}^{\infty} f_m(B(x,\delta)) + B(0,\delta)$ is a connected set for every $k \geq \bar{m}$ as a union of connected sets with nonempty intersection.

Clearly, for $k_2 > k_1$ one has

$$\overline{\bigcup_{m=k_2}^{\infty} f_m(B(x,\delta)) + B(0,\delta)} \subseteq \overline{\bigcup_{m=k_1}^{\infty} f_m(B(x,\delta)) + B(0,\delta)}$$

and hence Theorem [10, Theorem 6.1.18] ensures that also

$$\bigcap_{k=\bar{m}}^{\infty} \bigcup_{m=k}^{\infty} f_m(B(x,\delta)) + B(0,\delta)$$

is connected. On the other hand, from Lemma 3.1 it is easy to check that

$$\bigcap_{k=\bar{m}}^{\infty} \bigcup_{m=k}^{\infty} f_m(B(x,\delta)) + B(0,\delta) = \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} f_m(B(x,\delta)) + B(0,\delta)$$

thus the proof is concluded. $\square$

Lemma 3.3. *Let $f \in \mathcal{S}$. Then $\tilde{f}(x) = f(x)$ for a.e. $x \in [0,1]^2$.*

Proof. This property is a direct consequence of the fact that for almost all $x \in [0,1]^2$ one has $f_k(x) \to f(x)$ and (2.7). Let $x \in [0,1]^2$ be one of such points, then for every $\delta > 0$ there exists $\bar{k} = \bar{k}(\delta)$ such that for all $k \geq \bar{k}$ one has $|f_k(x) - f(x)| < \delta$ and, in particular, $f(x) \in \tilde{f}(x)$. Suppose now, for sake of contradiction, that $\tilde{f}(x)$ contains another element $a \neq f(x)$, then $|f(x) - a| \geq t > 0$ for some t. Then, by definition of $\tilde{f}$, there has to be a sequence $x_n \to x$ and subsequence k_n such that $f_{k_n}(x_n) \to a$. Fix now $\delta < t/100$, then there exist $\bar{n}$ and $k_{\bar{n}}$ big enough so that

$$|x_{\bar{n}} - x| < \delta, \quad |f_{k_{\bar{n}}}(x) - f(x)| < \delta, \quad |f_{k_{\bar{n}}}(x_{\bar{n}}) - a| < \delta,$$

which imply, in turn,

$$|f_{k_{\bar{n}}}(x_{\bar{n}}) - f_{k_{\bar{n}}}(x)| > \frac{t}{2}.$$

In other words, $f_{k_{\bar{n}}}(x_{\bar{n}}), f_{k_{\bar{n}}}(x) \in f_{k_{\bar{n}}}(B(x,\delta))$ and $|f_{k_{\bar{n}}}(x_{\bar{n}}) - f_{k_{\bar{n}}}(x)| > \frac{t}{2}$. By Corollary 2.7 we obtain

$$\frac{t}{2} \leq \frac{C}{\delta}|Df_{k_{\bar{n}}}|\big(B(x,2\delta)\big).$$

By Proposition 2.2 we obtain that for a.e. $\delta > 0$ (where $|Df|(\partial B(x,2\delta)) = 0$) we can take $k_{\bar{n}} \to \infty$ to get

$$\frac{t}{2} \leq \frac{C}{\delta}|Df|\big(B(x,2\delta)\big).$$

Finally, taking the limit as $\delta \to 0^+$ we obtain a contradiction with (2.7). $\square$

Lemma 3.4. *Let $f \in \mathcal{S}$. Then $\tilde{f}(x) = \tilde{h}^{-1}(x)$, where $\tilde{h}^{-1}(x) = \{y \in [0,1]^2 : \tilde{h}(y) \ni x\}$ is meant as the preimage of a point by multifunction and not as the usual preimage set.*

Proof. "$\supset$" Let $y \in \tilde{h}^{-1}(x)$, then $x \in \tilde{h}(y)$ and by (3.2) we can find $y_n \to y$ such that $f_{k_n}^{-1}(y_n) \to x$. We set $x_n = f_{k_n}^{-1}(y_n)$ and we obtain $x_n \to x$ such that $f_{k_n}(x_n) = y_n \to y$. It follows that $y \in \tilde{f}(x)$.
"$\subset$" Let $y \in \tilde{f}(x)$, then we can find $x_n \to x$ such that $f_{k_n}(x_n) \to y$. It follows that for $y_n = f_{k_n}(x_n)$ we have $y_n \to y$ and $f_{k_n}^{-1}(y_n) \to x$. Thus $x \in \tilde{h}(y)$ and $y \in \tilde{h}^{-1}(x)$. $\square$

Lemma 3.5. *Let $f \in \mathcal{S}$. Then f is differentiable a.e. in $[0,1]^2$.*

Proof. We claim that

$$\operatorname{osc}_{B(x,r)} f \leq \liminf_k \operatorname{osc}_{B(x,r)} f_k \tag{3.3}$$

holds for every x and $r < \operatorname{dist}(x, \partial(0,1)^2)$. For every $\varepsilon > 0$ we can find $a_\varepsilon, b_\varepsilon \in B(x,r)$ for which $f_k(a_\varepsilon) \to f(a_\varepsilon)$ and $f_k(b_\varepsilon) \to f(b_\varepsilon)$ in the image and

$$\operatorname{osc}_{B(x,r)} f < |f(a_\varepsilon) - f(b_\varepsilon)| + \varepsilon. \tag{3.4}$$

Then, for a fixed $\varepsilon > 0$, it follows that

$$|f(a_\varepsilon) - f_k(a_\varepsilon)| < \varepsilon, \quad \text{and} \quad |f(b_\varepsilon) - f_k(b_\varepsilon)| < \varepsilon$$

if k is big enough. As a consequence, from (3.4) we obtain

$$\operatorname{osc}_{B(x,r)} f < \liminf_k |f_k(a_\varepsilon) - f_k(b_\varepsilon)| + 3\varepsilon \leq \liminf_k \operatorname{osc}_{B(x,r)} f_k + 3\varepsilon$$

and (3.3) follows by sending ε to 0.

For $x \in (0,1)^2$ we choose $0 < r < \frac{1}{2}\operatorname{dist}(x, \partial(0,1)^2)$ so that $|Df|(\partial B(x,2r)) = 0$. By (3.3) and (2.8) we obtain

$$\frac{\operatorname{osc}_{B(x,r)} f}{r} \leq \frac{\liminf_k \operatorname{osc}_{B(x,r)} f_k}{r} \leq C\frac{\liminf_k |Df_k|(B(x,2r))}{r^2}.$$

By Proposition 2.2 we obtain

$$\frac{\operatorname{osc}_{B(x,r)} f}{r} \leq C\frac{|Df|(B(x,2r))}{r^2}$$

and by Radon–Nikodym theorem we know that $\lim_{r\to 0+}$ of the right-hand side is finite a.e. The a.e. differentiability of f follows by Stepanov Theorem 2.1. $\square$

4. Images of disjoint sets by $\tilde{f}$ are essentially disjoint

The aim of this section is to show that the multifunction representative of our $f \in \mathcal{S}$ defined in (3.1) does not depend on the approximating sequence f_k. Further we show that images of disjoint balls by $\tilde{f}$ intersect in a set of measure zero.

For almost every $t \in (0,1)$ we know that the restriction of a BV mapping is a one-dimensional BV on the line $\{y = t\}$ (see e.g. [1, Section 3.11]). As such the restriction to the line has a left and a right-continuous representative which we will call $\hat{f}$ and $\check{f}$ respectively. Since $|D_1 f|(\{y = t\}) < \infty$, f has at most countably many jumps on $\{y = t\}$, and the sum of the size of these jumps is finite. We denote the jump points as x_j and we define the line segment $A_j(t) = [\hat{f}(x_j)\check{f}(x_j)]$ and call $A(t) = \bigcup_j A_j(t)$. We use this notation in the following formulation.

Lemma 4.1. *Let $f_m \in BV(Q, \mathbb{R}^2)$ be a sequence of homeomorphisms of the unit square Q with $f_m(x) = x$ for $x \in \partial Q$ converging strictly to f in BV such that for almost every t the restriction of f_m on to the line $(0,1) \times \{t\}$ converges strictly to the restriction of f onto $(0,1) \times \{t\}$. Then for every $\varepsilon > 0$ there exists an $M(\varepsilon)$ such that for all $m \geq M$ we have*

$$f_m(\{y = t\}) \subset \Big(\hat{f}(\{y = t\}) \cup A(t)\Big) + B(0, \varepsilon). \tag{4.1}$$

and

$$\Big(\hat{f}(\{y = t\}) \cup A(t)\Big) \subset f_m(\{y = t\}) + B(0, \varepsilon). \tag{4.2}$$

Proof. We start by separating the jumps into two categories, small and big. A big jump is a jump of size $\varepsilon/100$ or larger, the other jumps are small. Obviously there are only a finite number J of large jumps, call them $\{x_1, x_2, \ldots, x_J\}$. Now for the jump at x_j, $1 \leq j \leq J$ we find a left $I_j^- = (x_j^-, x_j)$ and right $I_j^+ = (x_j, x_j^+)$ neighborhood. These intervals are chosen such that $|D_1 f|(I_j^{\pm}) < \frac{\varepsilon}{100}$ and that $|D_1 f|(\{x_j^{\pm}\}) = 0$. From Proposition 2.2 we thus get that f_m converges strictly on (x_j^-, x_j^+). We cover the rest of $\{y = t\}$ with finitely many intervals $I_j, j = J+1, \ldots, K$ so that f_m converges strictly in BV on each I_j and $|D_1 f|(I_j) < \varepsilon/50$. We have in total K intervals, so there must be some positive minimum length of the intervals, which we call $d > 0$.

We have that $\|f_m - f\|_{L^1(I_j)} \to 0$ and so we have an m_0 such that if $m \geq m_0$ the set $\{x : |f(x) - f_m(x)| \leq \varepsilon/100\} \cap I_j$ has positive measure for all $1 \leq j \leq K$. This is because $\mathcal{L}^1(I_j) \geq d$ and we may assume that $\|f_m - f\|_{L^1(I_j)} < d\varepsilon/100$. Also, by strict convergence, we may assume that the variation of f_m on I_j, $J < j \leq K$, is less than $\varepsilon/25$ as soon as $m \geq m_0$. So for an interval I_j, $J < j \leq K$, that does not contain a large jump we have an $s_m \in I_j$ such that $|f(s_m) - f_m(s_m)| \leq \varepsilon/100$ and

$$f_m(I_j) \subset B(f_m(s_m), \varepsilon/25) \subset B(f(s_m), 5\varepsilon/100) \subset f(I_j) + B(0, \varepsilon) \tag{4.3}$$

and by similar arguments and the lower semicontinuity of total variation

$$f(I_j) \subset B(f(s_m), \varepsilon/25) \subset B(f_m(s_m), 5\varepsilon/100) \subset f_m(I_j) + B(0, \varepsilon) \tag{4.4}$$

which will be helpful to show (4.1) and (4.2) for these intervals.

Now we consider $1 \leq j \leq J$. Notice that since $|D_1 f|(I_j^{\pm}) < \varepsilon/100$ we have

$$\begin{aligned} &|\hat{f}(x) - \hat{f}(x_j)| < \varepsilon/100 \text{ for } x \in I_j^- \text{ and} \\ &|\check{f}(x) - \check{f}(x_j)| < \varepsilon/100 \text{ for } x \in I_j^+. \end{aligned} \tag{4.5}$$

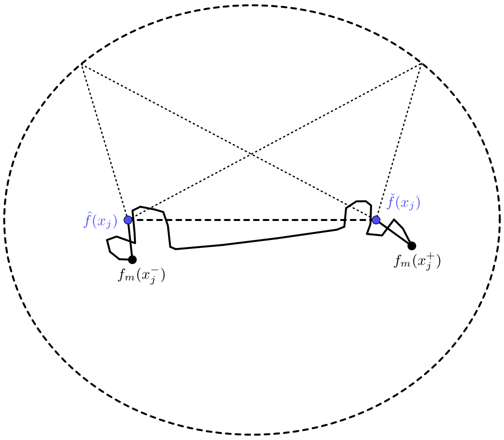


Fig. 1. An ellipse with foci at the ends of a jump and a path from $f_m(x_j^-)$ to $f_m(x_j^+)$ that must stay inside the ellipse.

There exists a m_0 such that for each $1 \le j \le J$ and for each $m \ge m_0$ there is a pair of points $s_m^+ \in I_j^+$ and $s_m^- \in I_j^-$ such that $|f_m(s_m^\pm) - f(s_m^\pm)| < \varepsilon/100$. By strict convergence we can also assume for all $m \ge m_0$ that

$$|D_1 f_m|(I_j) < |\hat{f}(x_j) - \check{f}(x_j)| + \varepsilon/25. \tag{4.6}$$

Using (4.5) we have that $|f(s_m^\pm) - f(x_j^\pm)| < \varepsilon/100$ and so

$$\begin{aligned} |f_m(s_m^-) - \hat{f}(x_j)| &< |f_m(s_m^-) - f(s_m^-)| + |f(s_m^-) - f(x_j^-)| + |f(x_j^-) - \hat{f}(x_j)| \\ &< \varepsilon/100 + \varepsilon/100 + \varepsilon/100 < \varepsilon/25 \end{aligned}$$

and similarly for $\check{f}(x_j)$ and s_m^+. Thus (see Fig. 1) we are able to connect any point in $f_m(I_j)$ with a single curve with endpoints $\hat{f}(x_j)$ and $\check{f}(x_j)$ and of length at most $|\hat{f}(x_j) - \check{f}(x_j)| + 3\varepsilon/25$. Thus $f_m(I_j)$ lies in the ellipse whose foci are the points $\hat{f}(x_j)$ and $\check{f}(x_j)$ and the sum of the distances to the foci is $|\hat{f}(x_j) - \check{f}(x_j)| + 3\varepsilon/25$. Now it is not hard to check that this ellipse lies inside the set $[\hat{f}(x_j), \check{f}(x_j)] + B(0, \sqrt{\varepsilon})$. Together with (4.3) we obtain (4.1). To finish the proof of (4.2) we notice that the curve constructed above joining the foci and having length $|\hat{f}(x_j) - \check{f}(x_j)| + 3\varepsilon/25$ cannot go too far from the major semiaxis of the ellipse. In fact, distance is at most the length of the minor semiaxis, whose length is bounded from above by $\sqrt{\varepsilon}$. Thus $f(I_j) \subset f_m(I_j) + B(0, \sqrt{\varepsilon})$. To get ε instead of $\sqrt{\varepsilon}$ one replaces the ε in the proof by ε^2. $\square$

Remark 4.2. Lemma 4.1 holds also for more general situations. Since the proof splits the line into small segments it is evident that it holds for rectangles $K \subset Q$ for which we have strict convergence on each side and the limit function is not discontinuous at the corners.

We define

$$\tilde{f}(A) := \bigcup_{x \in A} \tilde{f}(x).$$

The following (4.7) says that the 'limsup' of the definition (3.2) is in fact just a limit. It follows that the definition of $\tilde{f}$ (see (3.1)) does not depend on the approximating sequence.

Proposition 4.3. *Let $f_k \in BV(Q, \mathbb{R}^2)$ be a sequence of homeomorphisms of the unit square Q with $f_k(x) = x$ for $x \in \partial Q$ converging strictly to f in BV. For every $x \in Q$ we have*

$$\tilde{f}(x) = \bigcap_{\delta > 0} \bigcup_{k=1}^{\infty} \bigcap_{m=k}^{\infty} \Big[f_m(B(x, \delta)) + B(0, \delta) \Big]. \tag{4.7}$$

Moreover, let $g_k \in BV(Q, \mathbb{R}^2)$ be a sequence of homeomorphisms of the unit square Q with $g_k(x) = x$ for $x \in \partial Q$ converging strictly to f in BV and define

$$\hat{f}(x) := \left\{ y \in [0,1]^2 : \text{ there is } x_n \to x \text{ and } k_n \to \infty \text{ such that } g_{k_n}(x_n) \to y \right\}.$$

Then $\hat{f}(x) = \tilde{f}(x)$ for every x.

Proof. From (3.2) and elementary inclusion we have

$$\begin{aligned}\tilde{f}(x) &= \bigcap_{\delta>0} \bigcap_{k=1}^{\infty} \overline{\bigcup_{m=k}^{\infty} \left[f_m(B(x,\delta)) + B(0,\delta) \right]} \\ &\supset \bigcap_{\delta>0} \bigcup_{k=1}^{\infty} \bigcap_{m=k}^{\infty} \left[f_m(B(x,\delta)) + B(0,\delta) \right].\end{aligned}$$

It remains is to show the other inclusion. Take any sequence (x_l) which is converging to x and any subsequence f_{k_l} of f_k such that $f_{k_l}(x_l)$ converges to some point, call it a. We want to show that for every δ there exists a K_δ such that for all $k \geq K_\delta$ we have

$$a \in f_k(B(x,\delta)) + B(0,\delta).$$

For contradiction assume that we have a $\delta > 0$ and $k_n \nearrow \infty$ such that $a \notin f_{k_n}\big(B(x,\delta)\big) + B(0,\delta)$. Since we are taking a subsequence anyway, we may as well assume that it converges strictly on almost every line in BV (by Proposition 2.3) and similarly for f_{k_l}.

By Proposition 2.3 we can easily find an $\delta/4 < \eta < \delta/2$ such that f_{k_l} and f_{k_n} both converge strictly on $\{(y_1,y_2) : \ y_2 = x_2 \pm \eta\}$ and $\{(y_1,y_2) : \ y_1 = x_1 \pm \eta\}$ and hence also on $\partial Q(x,\eta)$. Now by Lemma 4.1 we obtain

$$f_{k_l}(\partial Q(x,\eta)) \subset \left[\hat{f}(\partial Q(x,\eta)) \cup A(\partial Q(x,\eta))\right] + B(0,\delta/4)$$

and

$$\left[\hat{f}(\partial Q(x,\eta)) \cup A(\partial Q(x,\eta))\right] \subset f_{k_n}(\partial Q(x,\eta)) + B(0,\delta/4)$$

where $A(\partial Q(x,\eta))$ denotes the corresponding parts of A on four sides of $Q(x,\eta)$. As f_{k_l} and f_{k_n} are homeomorphisms it follows that

$$\begin{aligned}f_{k_l}(Q(x,\eta)) &\subset f_{k_n}(Q(x,\eta)) + B(0,\delta/2) \\ &\subset f_{k_n}(B(x,\delta)) + B(0,\delta/2).\end{aligned}$$

For sufficiently large l we have $|f_{k_l}(x_l) - a| < \delta/2$ but this is in contradiction with $a \notin f_{k_n}(B(x,\delta)) + B(0,\delta)$.

To prove $\hat{f}(x) = \tilde{f}(x)$ assume for contrary that there is $a \in \hat{f}(x) \setminus \tilde{f}(x)$. We can find a sequence $x_l \to x$ and subsequence g_{k_l} such that $g_{k_l}(x_l) \to a$. By $a \notin \tilde{f}(x)$ we can find $\delta > 0$ and $k_n \to \infty$ such that

$$a \notin f_{k_n}(B(x,\delta)) + B(0,\delta).$$

Since we are taking a subsequence anyway, we may as well assume that it converges strictly on almost every line in BV and similarly for g_{k_l}. Now similarly to the argument above we obtain a contradiction. □

As a corollary we obtain that for homeomorphisms f our definition of $\tilde{f}(x)$ is the correct one.

Corollary 4.4. *Let f be a homeomorphism. Then $\tilde{f}(x) = \{f(x)\}$ for every $x \in Q$.*

Indeed, it is enough to apply Proposition 4.3 to $g_k = f$.

Conclusion (4.7) also holds for more general sets. By the proof of Proposition 4.3 and Remark 4.2 we have the same conclusion for rectangles where we have strict convergence on every side.

Corollary 4.5. *With the assumptions of Proposition 4.3 and Remark 4.2 we have*

$$\tilde{f}([0,1]\times[0,t]) = \bigcap_{\delta>0}\bigcup_{k=1}^{\infty}\bigcap_{m=k}^{\infty}\Big[f_m([0,1]\times[0,t+\delta]) + B(0,\delta)\Big]$$

and

$$\tilde{f}(Q(x,r)) = \bigcap_{\delta>0}\bigcup_{k=1}^{\infty}\bigcap_{m=k}^{\infty}\Big[f_m(Q(x,r+\delta)) + B(0,\delta)\Big].$$

It follows easily that $\tilde{f}(A)$ is measurable for each open and each closed set $A\subset[0,1]^2$.

Finally we can show that images of disjoint open sets are essentially disjoint.

Proposition 4.6. *Let $f_k\in BV(Q,\mathbb{R}^2)$ be a sequence of homeomorphisms of the unit square Q with $f_k(x)=x$ for $x\in\partial Q$ converging strictly to f in BV. Then the image of disjoint open strips are essentially disjoint, i.e. let $0<a<b\leq c<d<1$ then*

$$\mathcal{L}^2\big(\tilde{f}(\{a<x_2<b\})\cap\tilde{f}(\{c<x_2<d\})\big)=0.$$

Proof. Firstly assume that $b<c$. By Proposition 2.3 we find a subsequence of f_k, call it f_m, which converges strictly on almost every line and therefore on the line $\{x_2=e\}$ with $b<e<c$. By Proposition 4.3 and Corollary 4.5 we know that we can define $\tilde{f}(x)$ using only f_m.

Let us denote $E:=f(\{x_2=e\})\cup A(e)$ and note that clearly $\mathcal{L}^2(E)=0$. By Lemma 4.1 we see that for every δ there exists an $M(\delta)$ such that for all $m\geq M$ we have

$$f_m(\{x_2=e\})\subset\Big(\hat{f}(\{x_2=e\})\cup A(e)\Big)+B(0,\delta)=E+B(0,\delta). \tag{4.8}$$

Let us fix δ_0 such that $e-b>\delta_0$ and $c-e>\delta_0$. Using Proposition 4.3 and (4.8) we obtain

$$\begin{aligned}\tilde{f}(\{a<x_2<b\}) &= \bigcap_{0<\delta<\delta_0}\bigcup_{k=1}^{\infty}\bigcap_{m=k}^{\infty}\Big[f_m([0,1]\times[a-\delta,b+\delta])+B(0,\delta)\Big]\\ &\subset \bigcap_{0<\delta<\delta_0}\bigcup_{k=1}^{\infty}\bigcap_{m=k}^{\infty}\Big[f_m([0,1]\times[0,e])+B(0,\delta)\Big]\end{aligned}$$

and similarly

$$\tilde{f}(\{c<x_2<d\})\subset\bigcap_{0<\delta<\delta_0}\bigcup_{k=1}^{\infty}\bigcap_{m=k}^{\infty}\Big[f_m([0,1]\times[e,1])+B(0,\delta)\Big].$$

Using these facts, (4.8) and the fact that f_m converges strictly to f on $\{x_2=e\}$ we obtain

$$\tilde{f}(\{a<x_2<b\})\cap\tilde{f}(\{c<x_2<d\})\subset\bigcap_{0<\delta<\delta_0}E+B(0,2\delta)=\overline{E}=E.$$

As $\mathcal{L}^2(E)=0$ we obtain our conclusion.

Now if $b=c$ then apply the above to the strips $a<y<b-\delta_n$ and $c+\delta_n<y<d$ for $\delta_n\searrow 0$ and the intersection of the image of $a<y<b$ and $b<y<d$ is the countable union of null sets. $\square$

Corollary 4.7. *Let $f_k \in BV(Q,\mathbb{R}^2)$ be a sequence of homeomorphisms of the unit square Q with $f_k(x) = x$ for $x \in \partial Q$ converging strictly to f in BV. Then there is at most a countable number of $t \in [0,1]$ such that $\tilde{f}(\{x_2 = t\})$ has positive measure.*

Proof. The image $\tilde{f}(\{x_2 = t\})$ is a subset of Q for all t. Now for a pair of distinct numbers $0 \leq s < t \leq 1$ we know that their images (by Proposition 4.3) have null intersection (by Proposition 4.6). $\square$

Remark 4.8. Similarly one can prove that the image of two disjoint balls must have null intersection and therefore also that the image of any two distinct points has null intersection.

5. The Jacobians and their equality

In [22] the authors proved that the distributional Jacobian of a mapping $u \in W^{1,p}(\Omega,\mathbb{R}^n)$, $p > n-1$, satisfying the (INV) condition and $\mathrm{Per}(\mathrm{im}(u,\Omega)) < \infty$ is a non-negative Radon measure. As mentioned in the introduction the (INV) condition means that interior (resp. exterior) of a disk is mapped inside (outside) the image of the boundary of the ball. For a precise definition see [22]. Moreover, the singular part of this measure is a countable set of points and the restriction of the distributional Jacobian to any point in the countable set is a multiple of the Dirac measure. The map forms a cavity there whose Lebesgue measure in the image equals the Jacobian measure of the point.

We would like to recover a similar result for our setting. In our setting one cannot necessarily define the distributional Jacobian (see e.g. [16] or [22]), i.e. the distribution

$$\mathcal{J}_f(\varphi) := -\int_\Omega f_1(x) J(\varphi(x), f_2(x))\, dx \text{ for } \varphi \in C_0^\infty(\mathbb{R}^2).$$

The natural analogy for BV mapping is

$$\mathcal{J}_f(\varphi) := -\int_\Omega f_1(x)\frac{\partial \varphi(x)}{\partial x_1} d\frac{\partial f^2(x)}{\partial x_2} + \int_\Omega f_1(x)\frac{\partial \varphi(x)}{\partial x_2} d\frac{\partial f^2(x)}{\partial x_1} \tag{5.1}$$

which is well-defined for $f \in BV \cap C$ but this need not be defined for us as essentially $f^1 \in L^1 \cap L^\infty$ may be undefined on a set of positive $|Df^2|$-measure.

Instead we define the non-negative Borel measure such that

$$\mathbb{J}_f(A) := \mathcal{L}^2(\tilde{f}(A)) \text{ for open sets } A,$$

where $\tilde{f}$ is the multivalued representative of f defined in (3.2).

Further we define the distributional Jacobian of f as

$$\mathcal{J}_f(\varphi) := \lim_{k\to\infty} \mathcal{J}_{f_k}(\varphi),$$

i.e. we show that this limit exists for every φ. It is clearly a nonnegative distribution ($\varphi \geq 0 \to \mathcal{J}_f(\varphi) \geq 0$) and hence can be represented by a Radon measure (see [21, Theorem 1.16]). We show that these two notions of Jacobians coincide $\mathcal{J}_f = \mathbb{J}_f$. As a corollary we get that we have at most countably many cavities of f, they correspond to a Dirac measure of $\mathbb{J}_f$ and $\mathcal{J}_f(x) = \mathbb{J}_f(x)$ is the measure of the cavity opened at x.

Theorem 5.1. *Let $f_k \in BV(Q,\mathbb{R}^2)$ be a sequence of homeomorphisms of the unit square Q with $f_k(x) = x$ for $x \in \partial Q$ converging strictly to f in BV. Then $\mathbb{J}_f, \mathcal{J}_f, \mathbb{J}_{f_k}$ and $\mathcal{J}_{f_k}$ are Radon measures and*

$$\mathcal{J}_f = \mathbb{J}_f.$$

Moreover, $\mathbb{J}_{f_k} = \mathcal{J}_{f_k}$*, this sequence converges weak* to* $\mathbb{J}_f$ *in measures and there exists a countable set* N_x *such that*

$$\lim_{k\to\infty} \mathbb{J}_{f_k}(B(x,r)) = \mathbb{J}_f(B(x,r)) \tag{5.2}$$

for every $x \in Q$ *and every* $r \in [0,1] \setminus N_x$ *with* $r < \operatorname{dist}(x, \partial(0,1)^2)$.

Lemma 5.2. *Let* $f_k \in BV(Q,\mathbb{R}^2)$ *be a sequence of homeomorphisms of the unit square* Q *with* $f_k(x) = x$ *for* $x \in \partial Q$ *converging strictly to* f *in* BV*. Then there is a countable set* N *such that for every* $t \in [0,1] \setminus N$ *we have for* $\tilde{f}$*, defined in (3.2), the following*

$$\mathbb{J}_f\big([0,1]\times[0,t]\big) := \mathcal{L}^2\big(\tilde{f}([0,1]\times[0,t])\big) = \lim_{k\to\infty} \mathcal{L}^2\big(f_k([0,1]\times[0,t])\big). \tag{5.3}$$

Remark 5.3. An immediate result of Lemma 5.2 is that for any $a \in Q$ and for almost every $r > 0$ such that $B(a,r) \subset\subset Q$ we have for $\tilde{f}$ defined in (3.2) the following

$$\mathbb{J}_f\big(B(a,r)\big) := \mathcal{L}^2\big(\tilde{f}(B(a,r))\big) = \lim_{k\to\infty} \mathcal{L}^2\big(f_k(B(a,r))\big).$$

To prove this it suffices to notice that the polar-coordinates mapping, Ψ_a, centered at a is locally bi-Lipschitz on $\mathbb{R}^2 \setminus \{a\}$, thus there exists a representative of $f \circ \Psi_a^{-1}$, whose restriction to almost all lines is a function of bounded variation and then continue with the proof below.

Proof of Lemma 5.2. We prove that for every subsequence of $\lim_{k\to\infty}\mathcal{L}^2\big(f_k([0,1]\times[0,t))\big)$ we find a converging subsequence and that the value of this limit is in fact the same as in the claim. So assume that f_{k_l} is an arbitrary subsequence of f_k from which we choose a further subsequence $f_m = f_{k_{l_m}}$ which converges strictly on almost every line (see Proposition 2.3).

Let us fix $t \in (0,1)$ such that f_m converges to f strictly on $\{x_2 = t\}$ and

$$\mathcal{L}^2\big(\tilde{f}([0,1]\times\{t\})\big) = 0.$$

By Proposition 4.3 and Corollary 4.5 we know that we can define $\tilde{f}(x)$ using only f_m. Let $\varepsilon > 0$. As

$$\lim_{s\to t-} \mathcal{L}^2\big(\tilde{f}([0,1]\times[0,s])\big) = \mathcal{L}^2\big(\tilde{f}([0,1]\times[0,t])\big)$$

we can fix $s < t$ such that

$$0 \le \mathcal{L}^2\big(\tilde{f}([0,1]\times[0,t])\big) - \mathcal{L}^2\big(\tilde{f}([0,1]\times[0,s])\big) < \varepsilon. \tag{5.4}$$

Let us denote $E := \hat{f}(\{x_2 = t\}) \cup A(t)$ and note that clearly $\mathcal{L}^2(E) = 0$. By Lemma 4.1 we see that for every δ there exists an $M(\delta)$ such that for all $m \ge M$ we have

$$f_m(\{x_2 = t\}) \subset \Big(\hat{f}(\{x_2 = t\}) \cup A(t)\Big) + B(0,\delta) = E + B(0,\delta). \tag{5.5}$$

Using Proposition 4.3 and (5.5) we obtain

$$\begin{aligned}
\tilde{f}(\{x_2 \le s\}) &= \bigcap_{0<\delta<t-s} \bigcup_{k=1}^{\infty} \bigcap_{m=k}^{\infty} \Big[f_m([0,1]\times[0,s+\delta]) + B(0,\delta)\Big] \\
&\subset \bigcap_{0<\delta<t-s} \bigcup_{k=1}^{\infty} \bigcap_{m=k}^{\infty} \Big[f_m([0,1]\times[0,t]) + B(0,\delta)\Big] \\
&\subset \bigcap_{0<\delta<t-s} \bigcup_{k=1}^{\infty} \bigcap_{m=k}^{\infty} \Big[f_m([0,1]\times[0,t]) \cup \big[E + B(0,2\delta)\big]\Big] \\
&= \bigcup_{k=1}^{\infty} \bigcap_{m=k}^{\infty} \Big[f_m([0,1]\times[0,t]) \cup \overline{E}\Big].
\end{aligned}$$

Since $\mathcal{L}^2(\overline{E}) = 0$ and f_m converges to f strictly on $\{x_2 = t\}$ we obtain with the help of (5.4) and choice of a nice subsequence

$$\begin{aligned}\mathcal{L}^2(\tilde{f}(\{x_2 \le t\})) &\le \varepsilon + \mathcal{L}^2(\bigcup_{k=1}^{\infty}\bigcap_{m=k}^{\infty} f_m([0,1]\times[0,t])) \\ &= \varepsilon + \lim_{m\to\infty} \mathcal{L}^2(f_m(\{x_2 \le t\})).\end{aligned}$$

By passing $\varepsilon \to 0$ we obtain one inequality and analogously

$$\mathcal{L}^2(\tilde{f}(\{y \ge t\})) \le \lim_{m\to\infty} \mathcal{L}^2(f_m(\{y \ge t\})).$$

Since we have chosen t so that $\mathcal{L}^2(f_m(\{y = t\})) = 0 = \mathcal{L}^2(\tilde{f}(\{y = t\}))$, we obtain

$$\begin{aligned}\mathcal{L}^2(\tilde{f}(\{y \le t\})) &= 1 - \mathcal{L}^2(\tilde{f}(\{y \ge t\})) \\ &\ge \lim_m \big(1 - \mathcal{L}^2(f_m(\{y \ge t\}))\big) \\ &= \lim_m \mathcal{L}^2(f_m(\{y \le t\})).\end{aligned}$$

Thus for every f_{k_l} we find a subsequence whose limit is as in the claim and our proof is finished. $\square$

Now we are in a position to prove the main result of this section.

Proof of Theorem 5.1. Firstly let us consider the distributional Jacobians $\mathcal{J}_{f_k}$ and the corresponding measure-theoretic Jacobians $\mathbb{J}_{f_k}$. We show firstly that $\mathcal{J}_{f_k}$ is a non-negative Radon measure and then prove the same for the former and that both are the limit of the same smooth approximations and therefore must be equal.

By [24, Theorem 1.4] we can approximate any homeomorphism $f_k \in BV$ by diffeomorphisms $g_{k,l}$ such that $g_{k,l} \to f_k$ uniformly and weak* in BV as $l \to \infty$. For the smooth maps $g_{k,l}$ the equality between $J_{g_{k,l}}$, $\mathcal{J}_{g_{k,l}}$ and $\mathbb{J}_{g_{k,l}}$, when understood as measures, is standard (see e.g. [16, Section 2.2]) and hence $\mathcal{J}_{g_{k,l}} = \mathbb{J}_{g_{k,l}}$ as measures. Since $g_{k,l} \to f_k$ uniformly and weak* in BV there is no obstacle in passing to the limit in the definition of the distributional Jacobian (5.1) and we obtain

$$\lim_{l\to\infty} \mathcal{J}_{g_{k,l}}(\varphi) = \mathcal{J}_{f_k}(\varphi) \text{ for every } \varphi \in C_0^\infty(Q).$$

The non-negativity of the limit comes from the non-negativity of the sequence and a non-negative distribution is a Radon measure (see [21, Theorem 1.16]).

We claim that $\mathbb{J}_{g_{k,l}} \to \mathbb{J}_{f_k}$ weak* in Radon measures. We already know that the sequence converges but have to prove that the limit can be represented by $\mathbb{J}_{f_k}$. Let us fix a rectangle $R \subset Q$ such that $\mathcal{L}^2(\tilde{f}_k(\partial R)) = 0$; since f_k are in BV it is clear that almost every rectangle satisfies this condition. As $g_{k,l}$ and f_k are homeomorphisms and $g_{k,l} \to f_k$ uniformly we obtain that

$$\lim_{l\to\infty} \mathbb{J}_{g_{k,l}}(R) = \lim_{l\to\infty} \mathcal{L}^2(g_{k,l}(R)) = \mathcal{L}^2(f_k(R)) = \mathbb{J}_{f_k}(R). \tag{5.6}$$

The proof of the weak* convergence is as follows. We take a $\varphi \in \mathcal{C}(\overline{Q})$, $\varepsilon > 0$ and find δ such that

$$|\varphi(a) - \varphi(b)| < \varepsilon \text{ for every } |a - b| < 2\delta.$$

Almost every line parallel to the x or y-axis has the property that its image in f_k has zero two-dimensional measure. We divide the square Q by lines parallel to the x and y-axis each of which is a $\mathbb{J}_{f_k}$-zero-measure set and each at a distance of less than δ from his neighbor. Thus we have divided Q into rectangles $\{R\}_{R\in\mathcal{R}}$ and so

$$\int_Q \varphi\, d\mathbb{J}_{f_k} = \sum_{R\in\mathcal{R}} \int_R \varphi\, d\mathbb{J}_{f_k}. \tag{5.7}$$

By our choice of δ we can find a constant k_R for every rectangle R in our grid such that $|\varphi(x) - k_R| < \varepsilon$ for all $x \in R$. Now we calculate

$$\langle \mathbb{J}_{g_{k,l}}, \varphi\rangle = \int_Q \varphi\, d\mathbb{J}_{g_{k,l}} = \sum_{R\in\mathcal{R}} \int_R \varphi\, d\mathbb{J}_{g_{k,l}}$$

and because $|\varphi(x) - k_R| < \varepsilon$ we have

$$\Big| \int_R \varphi\, d\mathbb{J}_{g_{k,l}} - k_R \mathbb{J}_{g_{k,l}}(R)\Big| < \varepsilon \mathbb{J}_{g_{k,l}}(R).$$

Since $\sum_R \mathbb{J}_{g_{k,l}}(R) = \mathcal{L}^2(Q)$ we have

$$\Big|\langle \mathbb{J}_{g_{k,l}}, \varphi\rangle - \sum_{R\in\mathcal{R}} k_R \mathbb{J}_{g_{k,l}}(R)\Big| < \varepsilon \mathcal{L}^2(Q).$$

Similarly we obtain for $\mathbb{J}_{f_k}$ the corresponding estimate by using (5.7)

$$\Big| \int_Q \varphi\, d\mathbb{J}_{f_k} - \sum_{R\in\mathcal{R}} k_R \mathbb{J}_{f_k}(R)\Big| < \varepsilon \mathcal{L}^2(Q).$$

Thus we have

$$\begin{aligned}\Big|\langle \mathbb{J}_{g_{k,l}}, \varphi\rangle - \int_Q \varphi\, d\mathbb{J}_{f_k}\Big| &\le \Big| \int_Q \varphi\, d\mathbb{J}_{g_{k,l}} - \sum_R k_R \mathbb{J}_{g_{k,l}}(R)\Big| \\ &\quad + \Big|\sum_R k_R \mathbb{J}_{g_{k,l}}(R) - \sum_R k_R \mathbb{J}_{f_k}(R)\Big| \\ &\quad + \Big|\int_Q \varphi\, d\mathbb{J}_{f_k} - \sum_{R\in\mathcal{R}} k_R \mathbb{J}_{f_k}(R)\Big| \\ &< 2\varepsilon + \Big|\sum_R k_R \mathbb{J}_{g_{k,l}}(R) - \sum_{R\in\mathcal{R}} k_R \mathbb{J}_{f_k}(R)\Big|\end{aligned}$$

and

$$\sum_{R\in\mathcal{R}} k_R \mathbb{J}_{g_{k,l}}(R) - \sum_{R\in\mathcal{R}} k_R \mathbb{J}_{f_k}(R) \le \|\varphi\|_\infty \sum_{R\in\mathcal{R}} |\mathbb{J}_{g_{k,l}}(R) - \mathbb{J}_{f_k}(R)|$$

but the sum above is finite and, by (5.6), $\mathbb{J}_{g_{k,l}}(R)$ each converges to $\mathbb{J}_{f_k}(R)$ and so one easily sees that in the limit this tends to 0. Thus we have proven that, for every $\varepsilon > 0$ there is an $L(\varepsilon)$ such that if $l > L$ then

$$\Big|\langle \mathbb{J}_{g_{k,l}}, \varphi\rangle - \int_Q \varphi\, d\mathbb{J}_{f_k}\Big| < 3\varepsilon.$$

Thus $\mathbb{J}_{g_{k,l}}$ converges weak* in Radon measures to $\mathbb{J}_{f_k}$. But this limit must also be $\mathcal{J}_{f_k}$ and so the two are equal.

Clearly $\mathbb{J}_{f_k}$ is a bounded sequence and therefore has a converging subsequence. We now show that in fact that, $\mathbb{J}_f$ is a Radon measure and is the limit of $\mathbb{J}_{f_k}$, i.e. we want to show that the left hand side of (5.8) makes sense and the limit holds

$$\int_Q \varphi\, d\mathbb{J}_f = \lim_{k\to\infty} \int_Q \varphi\, d\mathbb{J}_{f_k} \text{ for every } \varphi \in C(\overline{Q}). \tag{5.8}$$

As before we take a $\varphi \in \mathcal{C}(\overline{Q})$, $\varepsilon > 0$ and find δ such that

$$|\varphi(a) - \varphi(b)| < \varepsilon \text{ for every } |a-b| < 2\delta.$$

By Lemma 5.2 there is a bad countable set $N \subset [0,1]$. We divide the square Q by lines parallel to the x and y-axis with their corresponding coordinates not in the set N and each at a distance of less than δ from his neighbor. Thus we have divided Q into rectangles $\{R\}_{R\in\mathcal{R}}$ on each of which we have (5.3). We have chosen these lines so that their image has two dimensional measure zero (see Corollary 4.7) and so we may assume that the lines are $\mathbb{J}_f$ and $\mathbb{J}_{f_k}$-zero-measure sets. Thus in the following we can use the fact that

$$\int_Q \varphi \, d\mathbb{J}_{f_k} = \sum_{R\in\mathcal{R}} \int_R \varphi \, d\mathbb{J}_{f_k}$$

and again as $\mathbb{J}_f$ is an outer Borel measure, the sides of R are not in the set N from Lemma 5.2 and by Proposition 4.6 we have

$$\int_Q \varphi \, d\mathbb{J}_f = \sum_{R\in\mathcal{R}} \int_R \varphi \, d\mathbb{J}_f.$$

From here our calculations are identical to the smooth-to-homeomorphic case and so we get $\mathbb{J}_f = \lim_k \mathbb{J}_{f_k}$ weak* in measures. It follows that the distributional Jacobian equals $\mathbb{J}_f$ as

$$\mathcal{J}_f(\varphi) := \lim_{k\to\infty} \mathcal{J}_{f_k}(\varphi) = \lim_{k\to\infty} \mathbb{J}_{f_k}(\varphi) = \mathbb{J}_f(\varphi).$$

The claim that

$$\lim_{k\to\infty} \mathbb{J}_{f_k}(B(x,r)) = \mathbb{J}_f(B(x,r))$$

for every $x \in Q$ and every $r \in [0,1] \setminus N_x$ is straightforward application of the polar coordinates (as in Remark 5.3) and Lemma 5.2. $\square$

6. Characterization of fractures

We define the set of fractures of f to be

$$Frac := \{x : \ \mathcal{H}^1(\tilde{f}(x)) > 0\}. \tag{6.1}$$

The aim of this section is to estimate the total size of fractures with the total variation of the derivative of f.

Proposition 6.1. *Let $f \in BV(Q,\mathbb{R}^2)$ be a strict limit of BV homeomorphisms $f_k : Q \to Q$, with $f_k|_{\partial Q} = id$. Given $x \in Q$ then*

$$C|Df|(B(x,4r)) \geq r \operatorname{diam}(\tilde{f}(B(x,r))) \geq r \operatorname{diam}(\tilde{f}(x)) \tag{6.2}$$

is satisfied by all $r \in (0, \frac{1}{4}\operatorname{dist}(x,\partial Q))$.

Proof. As the second inequality is trivial, it is enough to prove only the first one.

By Proposition 4.3 and Corollary 4.5 we know that

$$\tilde{f}(B(x,r)) = \bigcap_{\delta>0} \bigcup_{k=1}^{\infty} \bigcap_{m=k}^{\infty} \Big[f_m(B(x,r+\delta)) + B(0,\delta) \Big].$$

It follows that we find $z_m, w_m \in B(x, \frac{3}{2}r)$ and a subsequence k_m such that

$$|f_{k_m}(z_m) - f_{k_m}(w_m)| \geq \frac{1}{4}\operatorname{diam}(\tilde{f}(B(x,r))).$$

By Corollary 2.7 we obtain

$$C|Df_{k_m}|(B(x,3r)) \geq C\operatorname{osc}_{B(x,\frac{3}{2}r)} f_{k_m} \geq r\operatorname{diam}(\tilde{f}(B(x,r))).$$

This is true for every r such that $B(x,3r) \subset Q$. If we consider an $r > 0$ such that $|Df|(S(x,3r)) = 0$ we obtain

$$C|Df|(B(x,3r)) \geq r\operatorname{diam}(\tilde{f}(B(x,r)))$$

by the strict convergence of the total variation (see Proposition 2.2).

The condition $|Df|(S(x,3r)) = 0$ is true for all but countably many radii. Thus this estimate holds for almost every radius ρ, $3r \leq \rho \leq 4r$ and we obtain our claim. $\square$

The following corollary, on $Frac$ from (6.1), follows immediately from Remark 2.5 and Proposition 6.1.

Corollary 6.2.

(1) *If $x \in Frac$ then $x \in \Theta_f$. Moreover, $\mathcal{H}^1(Frac \setminus Jump_f) = 0$.*
(2) *$Frac$ is 1-rectifiable and thus has σ-finite $\mathcal{H}^1$-measure.*

Now, we show that at $\mathcal{H}^1$-a.e. point of $Frac$ the $\tilde{f}(x)$ is actually a line segment.

Proposition 6.3. *Let $f \in BV(Q,\mathbb{R}^2)$ be a strict limit of BV homeomorphisms $f_k : Q \to Q$, with $f_k|_{\partial Q} = id$. We have*

$$\mathcal{H}^1_\infty(\tilde{f}(X)) = \mathcal{H}^1(\tilde{f}(X)) \tag{6.3}$$

at $\mathcal{H}^1$-a.e. $X \in Frac$.

Proof. Recall that $Frac$ is rectifiable and, therefore, contained (up to zero $\mathcal{H}^1$-measure set) in a countable union of (possibly rotated and translated) Lipschitz graphs. Take any such graph. Without loss of generality we may assume that the graph is given by $\varphi : (0,1) \to \mathbb{R}$.

Since $Frac$ has σ-finite $\mathcal{H}^1$-measure, there exists a set $E \subset (0,1)$ with full $\mathcal{H}^1$-measure such that for every $x \in E$ we have $\mathcal{H}^1(Jump_f \cap (\{x\} \times \mathbb{R})) = 0$.

On almost every line parallel to the coordinate axis f is a one dimensional BV function. We denote the restriction of f to the line parallel to x-axis with y-coordinate y_0 by $f^x(\cdot, y_0)$. Analogously we define $f^y(x_0, \cdot)$. Also, on almost every line the set of discontinuities of f^x and f^y is the intersection of the jump set and the line in question. For proofs of these facts see [1, Section 3.11] for example. Notice that to use these properties we must choose the so called precise representative of f, but the choice of representative does not affect the sets $\tilde{f}(X)$ and we will assume that f is given by the precise representative.

In the following we will use notation

$$(x,[a,b]) := \{(x,y) : y \in [a,b]\}$$

for vertical intervals and analogous notation for horizontal intervals.

Let us consider a point $X = (x, \varphi(x)) \in Frac$ such that all the above properties hold on vertical line through X. By Lemma 3.3 and Fubini's theorem we may choose X so that $\tilde{f}(X) = \{f(X)\}$ $\mathcal{H}^1$-almost everywhere on almost every line parallel to y-axis. Moreover, we need x, the first coordinate of X, to satisfy following Lebesgue point property. Let $0 \le q_- < q_+ \le 1$ be two rational numbers, then the function

$$t \mapsto |Df^y|(t, [q_-, q_+]) \tag{6.4}$$

is integrable and, thus, almost every t satisfies

$$\frac{1}{2r} \int_{[x-r,x+r]} \big||Df^y|(t, [q_-, q_+]) - |Df^y|(x, [q_-, q_+])\big| dt = o(r). \tag{6.5}$$

Since there is only countable number of rational intervals $[q_-, q_+]$ we infer that for almost every t the function (6.4) satisfies (6.5) for *every* interval $[q_-, q_+]$. We consider only x that satisfy this condition.

As f is BV on the line, we know that following limits exist

$$\lim_{t \to 0^+} f(x, \varphi(x) + t) = A \tag{6.6}$$

and

$$\lim_{t \to 0^+} f(x, \varphi(x) - t) = B. \tag{6.7}$$

We will show that $\tilde{f}(X) = \langle A, B\rangle$, where $\langle A, B\rangle$ is the line segment between A and B. From the choice of x (see (6.6) and (6.7)) it is easy to see that $A, B \in \tilde{f}(x, \varphi(x))$. Especially, $A, B \in \tilde{f}(K)$, for any rectangle K containing $(x, \varphi(x))$. By Lemma 3.2 $\tilde{f}(x, \varphi(x))$ is connected. To show that actually $\tilde{f}(x, \varphi(x)) = \langle A, B\rangle$ we show that for any $\varepsilon > 0$ there exists a rectangle K, containing $(x, \varphi(x))$, such that f_k converges strictly on ∂K and the curve $f_k(\partial K)$ has length smaller than $2|A - B| + \varepsilon$. This implies that no point outside $\langle A, B\rangle$ is in $\tilde{f}(x, \varphi(x))$.

We start by selecting a subsequence, which is also denoted with f_k, which converges strictly on almost every line parallel to the coordinate axes (see Proposition 2.3). Now choose a rational interval $[q_-, q_+]$, depending on our choice of X, such that $\varphi(x) \in [q_-, q_+]$ and

$$|Df^y|(x, (\varphi(x), q_+)) + |Df^y|(x, (q_-, \varphi(x))) < \varepsilon. \tag{6.8}$$

Now for almost every $0 < t'$ such that

$$[\varphi(x) - t', \varphi(x) + t'] \subset [q_-, q_+] \tag{6.9}$$

there exists $\eta(t') > 0$, also depending on X, such that

$$|Df^x|([x - \eta,\, x + \eta], \varphi(x) + t') + |Df^x|([x - \eta,\, x + \eta], \varphi(x) - t') < \varepsilon, \tag{6.10}$$

This is true since we assumed in the beginning that $\mathcal{H}^1(Jump_f \cap (\{x\} \times \mathbb{R})) = 0$.

We know that for almost every t' with (6.9) and $\eta' < \eta(t')$ the subsequence f_k converges strictly on the boundary of

$$K = [x - \eta', x + \eta'] \times [\varphi(x) - t', \varphi(x) + t'].$$

Moreover, by (6.5) we may take η' such that

$$\begin{aligned} \big||Df^y|(x - \eta', [q_-, q_+]) - |Df^y|(x, [q_-, q_+])\big| &< \varepsilon \\ \big||Df^y|(x + \eta', [q_-, q_+]) - |Df^y|(x, [q_-, q_+])\big| &< \varepsilon. \end{aligned} \tag{6.11}$$

Due to the strict convergence on ∂K we have for large enough k

$$\left||Df^y|(s,[\varphi(x)-t',\varphi(x)+t'])-|Df^y_k|([(s,[\varphi(x)-t',\varphi(x)+t'])])\right|<\varepsilon \tag{6.12}$$

for $s=x-\eta'$, $x+\eta'$ and

$$\left||Df^y|([x-\eta',x+\eta'],\varphi(x)+r)-|Df^y_k|([x-\eta',x+\eta'],\varphi(x)+r)\right|<\varepsilon \tag{6.13}$$

for $r=-t'$, t'.

We may now estimate the length of the curve $f_k(\partial K)$ for large k as follows

$$\begin{aligned}\mathcal{H}^1(f_k(\partial K))&=\sum_{s\in\{-\eta',\eta\}}\int_{\{x+s\}\times[\varphi(x)-t',\varphi(x)+t']}|Df^y_k|+\sum_{r\in\{-t',t'\}}\int_{[x-\eta',x+\eta']\times\{r\}}|Df^x_k|\\&\underbrace{\leq}_{(6.12),(6.13)}\sum_{s\in\{-\eta',\eta\}}\int_{\{x+s\}\times[\varphi(x)-t',\varphi(x)+t']}|Df^y|\\&\quad+\sum_{r\in\{-t',t'\}}\int_{[x-\eta',x+\eta']\times\{r\}}|Df^x|+4\varepsilon\\&\underbrace{\leq}_{(6.10),(6.11)}2\int_{\{x\}\times[q_-,q_+]}|Df^y|+6\varepsilon\underbrace{\leq}_{(6.8)}2|A-B|+10\varepsilon.\end{aligned} \tag{6.14}$$

By Proposition 4.3 we see that for any given $\delta>0$

$$\tilde{f}(x,\varphi(x))\subset f_k(K)+B(0,\delta) \tag{6.15}$$

for $k\geq C(\delta)$. Notice that here f_k is actually a subsequence of the original f_k, but Proposition 4.3 also implies that $\tilde{f}(K)$ does not change when we take the subsequence.

As we mentioned earlier $A,B\in\tilde{f}(x,\varphi(x))$ and by (6.15) we find points A' and B' in $f_k(K)$ with distance δ to points A and B respectively. Thus the line segment $\langle A',B'\rangle$ is contained in the δ-neighborhood of $\langle A,B\rangle$ since $|A-B|-2\delta\leq|A'-B'|$. By (6.14) $f_k(\partial K)$ is a curve of length at most $2|A-B|+10\varepsilon$ going around points A' and B'. Similarly to the ellipse arguments of Lemma 4.1 we see that all points in $f_k(K)$ are in $\sqrt{10\varepsilon|A'-B'|}$-neighborhood of $\langle A',B'\rangle$. Thus, $f_k(K)$ is contained in the $(\sqrt{10\varepsilon|A'-B'|}+\delta)$-neighborhood of $\langle A,B\rangle$ and, finally by (6.15), $\tilde{f}(x,\varphi(x))$ is in $(\sqrt{10\varepsilon|A'-B'|}+2\delta)$-neighborhood of $\langle A,B\rangle$. This does not depend on k anymore and δ and ε are arbitrary. Thus $\tilde{f}(x,\varphi(x))=\langle A,B\rangle$.

Finally we have $\mathcal{H}^1_\infty(\tilde{f}(x,\varphi(x)))=\mathcal{H}^1(\tilde{f}(x,\varphi(x)))$ for $\mathcal{H}^1$-almost every point in the set $Frac$. $\square$

Our next result gives an estimate for the size of the fractures.

Theorem 6.4. *Let $f\in BV(Q,\mathbb{R}^2)$ be a strict limit of BV homeomorphisms*

$$f_k:Q\to Q,$$

with $f_k|_{\partial Q}=id$. Then we have

$$\int_{Frac}\mathcal{H}^1(\tilde{f}(x))\,d\mathcal{H}^1(x)\leq C|Df|(Q). \tag{6.16}$$

Proof. As $Frac$ is a (countably) 1-rectifiable set, it is contained in a countable union of rotated graphs of Lipschitz functions. That is, we have Lipschitz functions $\varphi_i:(0,1)\to\mathbb{R}$ and denote by G_i a certain rotation of graph of φ_i. Then

$$Frac\subset\bigcup G_i.$$

We define sequences of sets used to estimate the integral over $Frac$. Let $E_1 = Frac \cap G_1$. Inductively, we set

$$E_i = \overline{Frac} \cap (G_i \setminus \bigcup_{j=1}^{i-1} G_j)$$

and for each E_i we define

$$E_i^k = E_i \setminus \Big(\bigcup_{j=1}^{i-1} E_j + B\big(0, \tfrac{1}{k}\big)\Big).$$

That is, E_i^k to be the subset of E_i which is at distance $\frac{1}{k}$ from all G_j for $j < i$.

From these definitions we have that $\bigcup_{j=1}^{i} E_j$ are closed sets and following inclusions are true

$$Frac \subset \bigcup E_i, \tag{6.17}$$

for all i

$$E_i \setminus (\bigcup_{j=1}^{i-1} E_j) = \bigcup_{k=1}^{\infty} \overline{E_i^k} = \bigcup_{k=1}^{\infty} E_i^k, \tag{6.18}$$

and

$$\bigcup_{i=1}^{j} \bigcup_{k=1}^{\infty} E_i^k = \bigcup_{i=1}^{j} E_i. \tag{6.19}$$

The first two follow directly from the definitions. The third one follows from (6.18).

Consider the set E_i^k, for some k and i. We claim that for every $\eta > 0$ we have

$$\int_{E_i^k} \mathcal{H}^1(\tilde{f}(x))\, d\mathcal{H}^1(x) \le C|Df|(E_i^k + B(0,\eta)). \tag{6.20}$$

Before the proof of (6.20) we show how our claim follows.

By taking η to zero we get from (6.20)

$$\int_{E_i^k} \mathcal{H}^1(\tilde{f}(x))\, d\mathcal{H}^1(x) \le C|Df|(\overline{E_i^k}).$$

Now taking k to infinity gives (recall (6.18))

$$\int_{E_i \setminus \cup_{j=1}^{i-1} E_j} \mathcal{H}^1(\tilde{f}(x))\, d\mathcal{H}^1(x) \le C|Df|(E_i \setminus \bigcup_{j=1}^{i-1} E_j).$$

Finally summing over all i we obtain

$$\begin{aligned}\int_{Frac} \mathcal{H}^1(\tilde{f}(x))\, d\mathcal{H}^1(x) &\le \sum_i \int_{E_i} \mathcal{H}^1(\tilde{f}(x))\, d\mathcal{H}^1(x)\\ &\le C\sum_i |Df|(E_i \setminus \bigcup_{j=1}^{i-1} E_j) \le |Df|(\overline{Frac}).\end{aligned}$$

Now we prove (6.20). Our set E_i^k lies on a graph of a Lipschitz function $\varphi : (0,1) \to \mathbb{R}$, thus E_i^k is a 1-rectifiable set with finite one dimensional measure. This implies (see for example [21, Theorem 16.2]) that $\mathcal{H}^1$-a.e. point in E_i^k has density 1, i.e.

$$\lim_{r\to 0} \frac{\mathcal{H}^1(B(x,r) \cap E_i^k)}{2r} = 1. \tag{6.21}$$

In $Frac$ we have the following doubling property. For any $x \in Frac$ there exists arbitrarily small r with

$$\operatorname{diam}(\tilde{f}(B(x,r))) \leq 2\operatorname{diam}(\tilde{f}(B(x,\frac{r}{20}))). \tag{6.22}$$

If this were not true then we would have for all small r

$$\operatorname{diam}(\tilde{f}(B(x,r))) > 2\operatorname{diam}(\tilde{f}(B(x,\frac{r}{20}))).$$

and by iteration for all k

$$2^{-k}\operatorname{diam}(\tilde{f}(B(x,r))) > \operatorname{diam}(\tilde{f}(B(x,\frac{r}{20^k}))) \geq \operatorname{diam}(\tilde{f}(x)) > 0$$

which is impossible.

We use a covering argument to estimate the integral over E_i^k. Choose for each $x \in E_i^k$ $r_x < \eta$ such that (6.22) is satisfied and we have for all $r \leq r_x$

$$\mathcal{H}^1(B(x,r) \cap E_i^k) \leq 3r.$$

The latter is made possible by (6.21). Then the collection $\{B(x,\frac{r_x}{5}) : x \in E_i^k\}$ is covering and using Vitali $5r$-covering theorem we obtain a sequence of disjoint balls $\{B(x_i,\frac{r_i}{5})\}_{i=1}^{\infty}$ such that $\{B(x_i,r_i)\}_{i=1}^{\infty}$ covers E_i^k.

Then, using Propositions 6.1 and 6.3 we may compute

$$\begin{aligned}\int_{Frac}\mathcal{H}^1_{\infty}(\tilde{f}(x))\,d\mathcal{H}^1(x) &\leq \sum_{i=1}^{\infty}\int_{B(x_i,r_i)}\mathcal{H}^1_{\infty}(\tilde{f}(x))\,d\mathcal{H}^1(x) \leq C\sum_{i=1}^{\infty}\operatorname{diam}(\tilde{f}(B(x_i,r_i)))r_i\\ &\leq C\sum_{i=1}^{\infty}\operatorname{diam}\big(\tilde{f}(B(x_i,\frac{r_i}{20}))\big)\frac{r_i}{20} \leq C\sum_{i=1}^{\infty}|Df|(B(x_i,\frac{r_i}{5})) \leq C|Df|(E_i^k + B(0,\eta)).\end{aligned}$$

This finishes the proof of (6.20) and thus also of our claim. $\square$

7. Examples

Example 7.1. Let us denote for $k \in \mathbb{N}$

$$f_k(x) = \begin{cases} x\Big(\dfrac{1+k}{1+k|x|^2}\Big)^{\frac{1}{2}} & \text{for } |x| \leq 1,\\ x & \text{for } |x| \geq 1.\end{cases}$$

These functions are homeomorphisms, they map circles around 0 onto circles and rays from the origin into the same ray. It is not difficult (see e.g. [16, Lemma 2.1]) to show that

$$f_k(x) \to f_c(x) := \frac{x}{|x|} \text{ strongly in } W^{1,p}(B(0,1),\mathbb{R}^2) \text{ for every } 1 \leq p < 2$$

and hence also strictly in BV.

The limit $f_c(x)$ maps circles centered at the origin onto similar circles and it squeezes $B(0,1)$ onto $\partial B(0,1)$. This is the example of so called *cavitation* as f is discontinuous at 0 and maps the origin in some sense to the whole created cavity $B(0,1)$ (see Fig. 2). The multivalued representative of this function $\tilde{f}_c$ (see 3.1) satisfies $\tilde{f}_c(0) = B(0,1)$.

Example 7.2. Let $k \in \mathbb{N}$ and $[x,y] \in [-2,2]^2$. We consider a piecewise linear function $l_{k,x}(y)$ such that

$$l_{k,x}(\pm 2) = \pm 2 \text{ and } l_{k,x}(\pm\tfrac{1}{k}) = \pm((1-|x|)_+ + \tfrac{1}{k}),$$

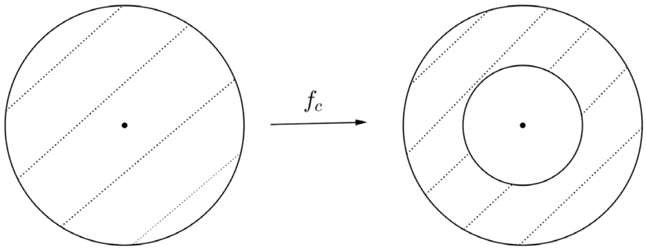


Fig. 2. A cavitation.

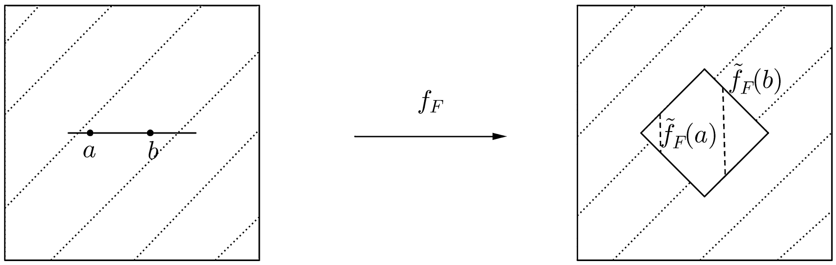


Fig. 3. Fracture opens segment onto square. Images of points a, b on the segment are straight lines $\tilde{f}_F(a)$ and $\tilde{f}_F(b)$.

i.e.

$$l_{k,x}(y) = \begin{cases} \Big(1 - \dfrac{(1-|x|)_+}{2-\frac{1}{k}}\Big)y - 2\dfrac{(1-|x|)_+}{2-\frac{1}{k}} & \text{for } y \in [-2, -\dfrac{1}{k}], \\ \big(1 + k(1-|x|)_+\big)y & \text{for } y \in [-\dfrac{1}{k}, \dfrac{1}{k}], \\ \Big(1 - \dfrac{(1-|x|)_+}{2-\frac{1}{k}}\Big)y + 2\dfrac{(1-|x|)_+}{2-\frac{1}{k}} & \text{for } y \in [\dfrac{1}{k}, 2]. \end{cases}$$

It is easy to see that $l_{k,x} : [-2, 2] \to [-2, 2]$ is a homeomorphism and that

$$l_{k,x} \underset{k\to\infty}{\to} l_x(y) := \Big(1 - \frac{(1-|x|)_+}{2}\Big)y + \operatorname{sgn} x(1-|x|)_+$$

strictly in BV, i.e. the limit function is increasing and has a jump of size $2(1-|x|)_+$ at 0.

It is not difficult to see that the following mappings are homeomorphisms

$$f_k([x, y]) := [x, l_{k,x}(y)] \text{ and that } f_k \underset{k\to\infty}{\to} f_F := [x, l_x(y)]$$

strictly in BV. Indeed, clearly $f_k \to f_F$ strongly in $W^{1,1}_{\text{loc}}$ on $[-2, 2]^2 \setminus ([-1, 1] \times \{0\})$, around the segment $[-1, 1] \times \{0\}$ the derivative $D_x f_F$ is bounded (as $k|y| \le 1$ for $y \in [-\frac{1}{k}, \frac{1}{k}]$) and the jump of f_F in y-direction corresponds to changes of $D_y f_k$ around the segment. The limiting map f_F gives us an example of the so called *fracture* around the segment $[-1, 1] \times \{0\}$ — see Fig. 3. Multifunction images of points $[x, 0]$, $x \in [-1, 1]$ are segments

$$\tilde{f}_F([x, 0]) = \{x\} \times [|x| - 1, 1 - |x|].$$

Example 7.3. Here we show that there is a sequence of homeomorphisms f_k such that $f_k(x, y) \to [x, y]$ weakly in BV but $\tilde{f}(0, 0) = [0, 1] \times \{0\}$ (or even $\tilde{f}(\{0, 0\}) = B(0, 1)$). This shows that it is reasonable to assume strict convergence to get a meaningful theory.

For $k \in \mathbb{N}$ let us define piecewise linear function

$$c_k(x) = \begin{cases} x & \text{for } x \le 0, \\ kx & \text{for } x \in [0, \dfrac{1}{k+1}], \\ \dfrac{1}{k}x + \dfrac{k-1}{k} & \text{for } x \in [\dfrac{1}{k+1}, 1], \\ x & \text{for } x \ge 1. \end{cases}$$

We set

$$f_k(x,y) = \Big[(1-k|y|)_+ c_k(x) + \big(1-(1-k|y|)_+\big)x, y\Big].$$

The first coordinate function is increasing as a function of x as it is convex combination of two increasing functions. Hence our f_k are homeomorphisms and it is easy to check that

$$f_k(x,y) = [x,y] \text{ for } [x,y] \notin [0,1] \times [-\tfrac{1}{k}, \tfrac{1}{k}]. \tag{7.1}$$

An elementary computation shows us that all partial derivatives are bounded by k on $[0,1] \times [-\frac{1}{k}, \frac{1}{k}]$ and hence

$$\int_{(-2,2)^2} |Df_k| < C \text{ for all } k.$$

It follows that there is weak* convergent subsequence in BV and in view of $f_k \to \mathrm{id}$ a.e. we have $f_k \to \mathrm{id}$ weakly in BV (possible for a subsequence). On the other hand we know that

$$f_k\big([0, \tfrac{1}{k+1}] \times \{0\}\big) = [0, \tfrac{k}{k+1}] \times \{0\}$$

and hence it is easy to see by (3.2) that $\tilde{f}(0,0) = [0,1] \times \{0\}$.

Let us now show in detail that the whole sequence f_k converges weakly and we do not need to select a subsequence. Let us first pick $\varphi \in C_c^1(\mathbb{R}^2)$. As f_k are Lipschitz we can use integration by parts to obtain

$$\int_{\mathbb{R}^2} Df_k(x)\varphi(x)\,dx = -\int_{\mathbb{R}^2} f_k(x) D\varphi(x)\,dx \text{ and } \int_{\mathbb{R}^2} I\varphi(x)\,dx = -\int_{\mathbb{R}^2} xD\varphi(x)\,dx,$$

where I is the identity matrix. By (7.1) we thus obtain

$$\Big|\int_{\mathbb{R}^2} (f_k(x) - x) D\varphi(x)\,dx\Big| \le \frac{2}{k}\|f_k(x) - x\|_\infty \|D\varphi\|_\infty \le \frac{4}{k}\|D\varphi\|_\infty \overset{k\to\infty}{\to} 0$$

and hence

$$\lim_{k\to\infty} \int_{\mathbb{R}^2} Df_k(x)\varphi(x)\,dx = \int_{\mathbb{R}^2} I\varphi(x)\,dx \text{ for every } \varphi \in C_c^1(\mathbb{R}^2). \tag{7.2}$$

Given $\varepsilon > 0$ and $\varphi_0 \in C_c(\mathbb{R}^2)$ we find $\varphi_0 \in C_c^1(\mathbb{R}^2)$ such that $\|\varphi - \varphi_0\|_\infty < \varepsilon$. Since $\sup_k \int_{[-2,2]^2} |Df_k| < C$ and $f = f_k$ outside $[-2,2]^2$ we obtain

$$\Big|\int_{\mathbb{R}^2} Df_k(x)\varphi_0(x)\,dx - \int_{\mathbb{R}^2} Df_k(x)\varphi(x)\,dx\Big| \le C\varepsilon$$

and hence it is easy to conclude that (7.2) actually holds for every $C_c(\mathbb{R}^2)$ test function.

Let us consider the mapping $R_k : \mathbb{R}^2 \to \mathbb{R}^2$ which rotates the plane around the origin by angle k. Let us define $\hat{f}_k := R_k \circ f_k \circ R_{-k}$. As before it is easy to see that $\hat{f}_k$ converges weakly in BV and since

$$|\{[x,y] : \ \hat{f}_k(x,y) \neq [x,y]\}| \to 0 \text{ and } |f_k(x,y) - [x,y]| \le 2$$

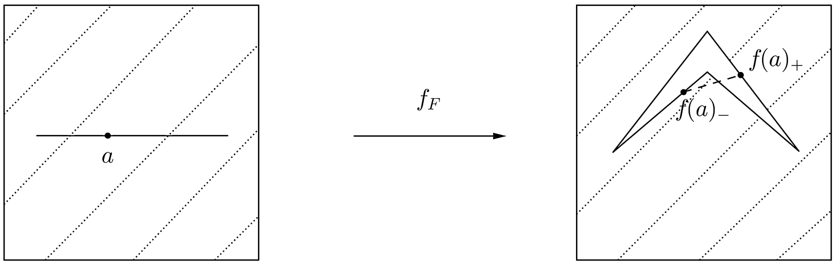


Fig. 4. Not all fractures can be created by strict limits.

we obtain that $f_k \to \mathrm{id}$ in L^1 and thus also weakly in BV. As in the previous paragraph we obtain that the whole sequence is weakly converging and we do not need to select a subsequence. Since (k mod 2π) is dense in $[0, 2\pi]$ we obtain that the limit multifunction mapping satisfies $\tilde{f}(0,0) = B(0,1)$. It follows that we cannot build a reasonable theory as e.g. the image of disjoint sets $\{0\}$ and $B([\frac{1}{2}, 0], \frac{1}{3})$ intersects in set of positive measure and thus it is not essentially disjoint. Let us note that we can even have $f_k(x,y) \to [x,y]$ a.e. (for a well-chosen subsequence). Indeed, choose k_l such that $\sum_l \frac{1}{k_l} < \infty$ and such that (k_l mod 2π) is dense in $[0, 2\pi]$. Then we still have $\tilde{f}(0,0) = B(0,1)$ and moreover $f_{k_l}(x,y) \to [x,y]$ a.e. outside of the set

$$\sum_{l=l_0}^{\infty} |\{f_{k_l}(x,y) \neq [x,y]\}| \leq \sum_{l=l_0}^{\infty} \frac{2}{k_l} \overset{l_0 \to \infty}{\to} 0.$$

Example 7.4. Let us point out that not all fractures can be realized as strict limits of BV homeomorphisms. In Fig. 4 we can see a fracture of a segment onto some nonconvex hole. Imagine that the two "images" of a point a on the fracture cannot be connected by a segment inside the nonconvex hole — see points $f(a)_+$ and $f(a)_-$ in Fig. 4. Then this cannot be realized as a strict limit on BV homeomorphisms as $\tilde{f}(a)$ has to lie inside the nonconvex hole and hence we lose some energy in the limiting process as the energy of the jump of the limit f is strictly smaller (see Proposition 6.3).

Example 7.5. There exists a sequence of finitely piecewise affine homeomorphisms f_k converging strongly in $W^{1,\infty}(Q, \mathbb{R}^2)$ with $f_k(x) = x$ on ∂Q but f_k^{-1} does not converge strictly in BV. Thus the formula

$$\int_Q |Df| = \int_{f(Q)} |Df^{-1}|$$

does not hold for f, limits of strictly converging homeomorphisms in BV.

We now show existence of such mapping. We create a mesh of triangles and rectangles in the preimage which will be the same for all f_k (see the left picture in Fig. 5). Each f_k is affine on each of the triangles and each of the rectangles. Since the mesh in the preimage is the same for all f_k and f_k converge uniformly to a nice piecewise affine mapping f we will obtain that f_k converge to f even in $W^{1,\infty}$. In Fig. 5 we denote the regions of the preimage on which f_1 is affine and we show where f_1 maps them to. Fig. 6 demonstrates how the sequence continues. We move the image of C4 closer and closer to the midpoint of the bottom side of the square and the image of C1 and C7 grow to fill the entire middle region of the square (their image in the limit is portrayed in Fig. 7). In each of the regions B2, B3, B4, C2, C3, C4, C5, C6, D3, D4 and D5 the map converges to a degenerate (or constant) affine map with vertices being mapped onto the midpoint of the bottom side of the square. It is elementary to notice that for sequence of affine maps A_n from a fixed triangle BCD such that each of the sequences $A_n(B)$ $A_n(C)$ $A_n(D)$, converges, that the sequence A_n converges strongly in $W^{1,\infty}(BCD^\circ)$. Therefore, since our map is affine on each region and converges uniformly over k on each region, we have Lipschitz convergence and thus also strict convergence in BV.

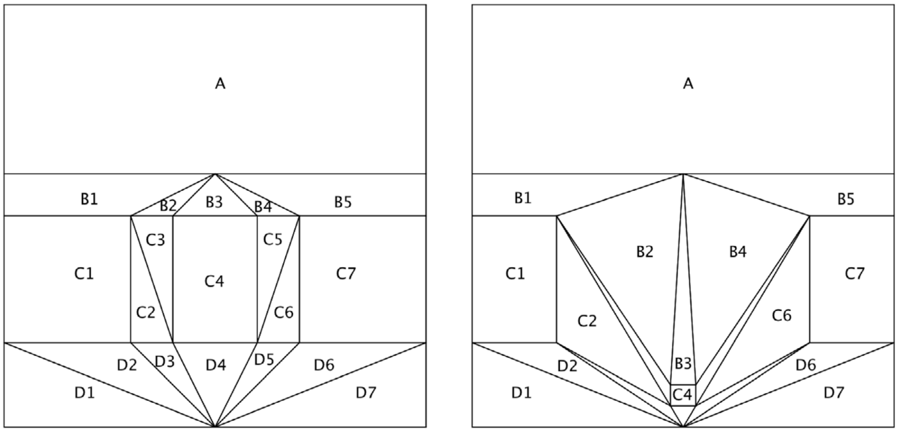
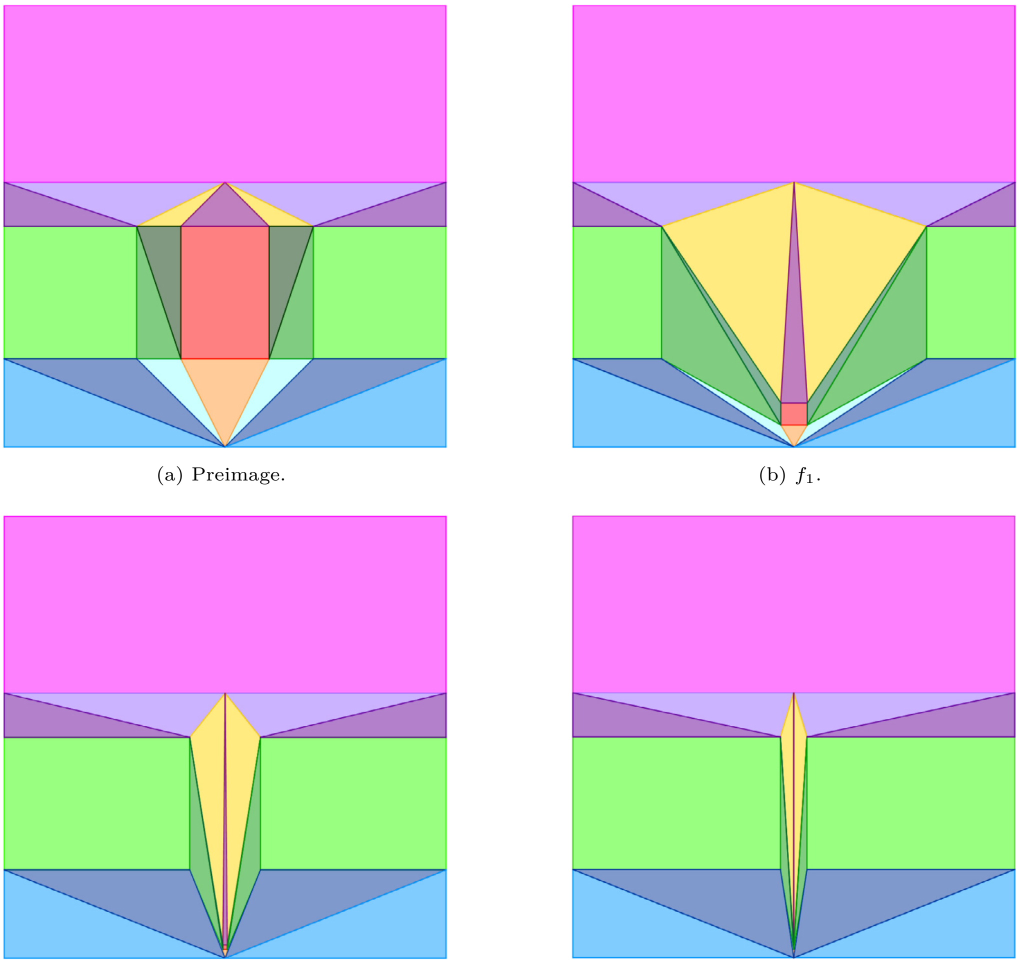


Fig. 5. Corresponding regions of the preimage and the image.



(a) Preimage.

(b) f_1.

(c) f_2.

(d) f_3.

Fig. 6. The sequence converging uniformly.

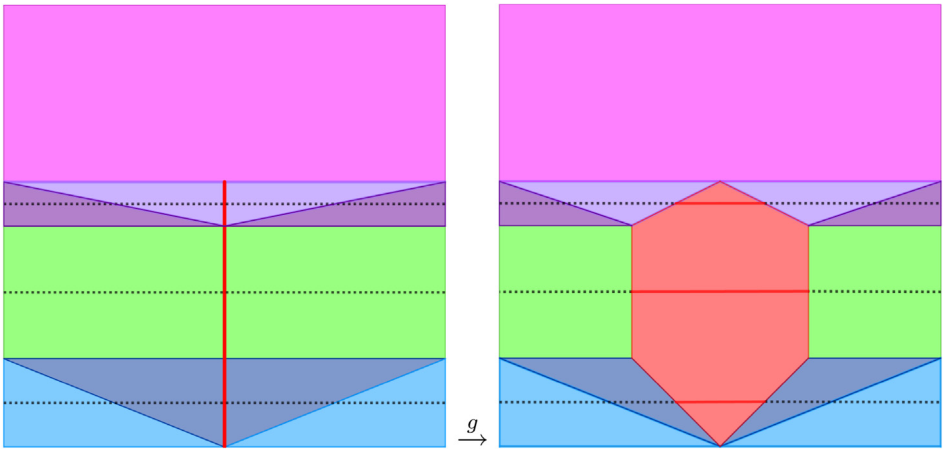


Fig. 7. The limit of the inverses sends horizontal lines onto horizontal lines.

Thus we have a sequence of homeomorphisms converging strongly in $W^{1,\infty}$. The inverse f_k^{-1} is in BV and $\int_Q |Df_k^{-1}| = \int_Q |Df_k|$ for every k and thus sequence $\int_Q |Df_k^{-1}|$ is bounded. Since f_k^{-1} converges in L^1 the sequence f_k^{-1} converges weakly in BV, call the limit g. We have the lower semi-continuity not just for the variation but also for the horizontal and vertical variations. We show that the horizontal variation of g is strictly less than that of the sequence and so f_k^{-1} has strictly more energy than g and f_k^{-1} does not converge strictly. The map g is depicted in Fig. 7.

The horizontal variation is easily calculated as horizontal segments are sent onto horizontal segments, except for the center point, where there is a jump between two points with the same y-coordinate. Thus the horizontal variation of g is 1. As we approach the limit however f_k maps B2 and B4 within an ε neighborhood of the bottom of the square (see Fig. 6). Therefore the image in f_k^{-1} of any horizontal line in the bottom quarter of the image which is ε or more above the bottom of the square must intersect the sets B2 and B4 in the preimage. Thus each such line must have horizontal variation of at least $1 + \delta$ for some fixed δ. Thus f_k^{-1} does not converge strictly.

Acknowledgments

The first author was supported by the grant DFG PR 1687/1-1, the second author was supported by the grant GAČR P201/18-07996S. The third author acknowledges financial support from the Spanish Ministry of Economy and Competitiveness, through the “María de Maeztu” Programme for Units of Excellence in R&D (MDM-2014-0445) and MTM-2016-77635-P (MICINN, Spain).

References

[1] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, in: Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.
[2] J. Ball, Convexity conditions and existence theorems in nonlinear elasticity, Arch. Ration. Mech. Anal. 63 (1978) 337–403.
[3] J. Ball, Global invertibility of Sobolev functions and the interpenetration of matter, Proc. Roy. Soc. Edinburgh Sect. A 88 (1981) 315–328.
[4] J.M. Ball, Some Open Problems in Elasticity, in: Geometry, Mechanics, and Dynamics, vol. 359, Springer, New York, 2002.
[5] J.M. Ball, Progress and puzzles in Nonlinear Elasticity, in: Proceedings of Course on Poly-, Quasi- and Rank-One Convexity in Applied Mechanics, in: CISM Courses and Lectures, Springer, 2010.
[6] D. Campbell, J. Onninen, A. Räbinä, V. Tengvall, Diffeomorphic Approximation of $W^{1,1}$ Planar Sobolev Monotone maps, in preparation 2017.

[7] S. Conti, C. De Lellis, Some remarks on the theory of elasticity for compressible Neohookean materials, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 2 (2003) 521–549.
[8] G. De Philippis, A. Pratelli, The closure of planar diffeomorphisms in Sobolev spaces, Preprint 2017.
[9] P. Di Gironimo, L. D'Onofrio, C. Sbordone, R. Schiattarella, Anisotropic Sobolev homeomorphisms, Ann. Acad. Sci. Fenn. Math. 36 (2011) 593–602.
[10] R. Engelking, General Topology, in: Mathematical Monographs, vol. 60, PWNPolish Scientific Publishers, Warsaw, 1977, p. 626.
[11] L. Evans, R. Gariepy, Measure Theory and Fine Properties of Functions, in: Studies in Advanced Mathematics, CRC Press, Boca Raton, 1992.
[12] D. Henao, C. Mora-Corral, Invertibility and weak continuity of the determinant for the modelling of cavitation and fracture in nonlinear elasticity, Arch. Ration. Mech. Anal. 197 (2010) 619–655.
[13] D. Henao, C. Mora-Corral, Fracture surfaces and the regularity of inverses for BV deformations, Arch. Ration. Mech. Anal. 201 (2011) 575–629.
[14] D. Henao, C. Mora-Corral, Lusin's condition and the distributional determinant for deformations with finite energy, Adv. Calc. Var. 5 (2012) 355–409.
[15] D. Henao, C. Mora-Corral, Regularity of inverses of Sobolev deformations with finite surface energy, J. Funct. Anal. 268 (2015) 2356–2378.
[16] S. Hencl, P. Koskela, Lectures on Mappings of Finite Distortion, in: Lecture Notes in Mathematics, vol. 2096, Springer, Cham, 2014, p. 176.
[17] S. Hencl, P. Koskela, J. Onninen, Homeomorphisms of bounded variation, Arch. Ration. Mech. Anal. 186 (2007) 351–360.
[18] T. Iwaniec, N.T. Koh, L.V. Kovalev, J. Onninen, Existence of energy-minimal diffeomorphisms between doubly connected domains, Invent. Math. 186 (3) (2011) 667–707.
[19] T. Iwaniec, J. Onninen, Monotone Sobolev mappings of planar domains and surfaces, Arch. Ration. Mech. Anal. 219 (1) (2016) 159–181.
[20] T. Iwaniec, J. Onninen, Limits of Sobolev homeomorphisms, J. Eur. Math. Soc. 19 (2) (2017) 473–505.
[21] P. Mattila, Geometry of Sets and Measures in Euclidean Spaces, in: Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1999.
[22] S. Müller, S.J. Spector, An existence theory for nonlinear elasticity that allows for cavitation, Arch. Ration. Mech. Anal. 131 (1) (1995) 1–66.
[23] S. Müller, Q. Tang, B.S. Yan, On a new class of elastic deformations not allowing for cavitation, Ann. Inst. H. Poincaré Anal. Non Linéaire 11 (1994) 217–243.
[24] A. Pratelli, E. Radici, On the planar minimal BV extension problem, Preprint 2017.
[25] J. Sivaloganathan, S. Spector, On the existence of minimizers with prescribed singular points in nonlinear elasticity, J. Elasticity 59 (2000) 83–113.
[26] V. Šverák, Regularity properties of deformations with finite energy, Arch. Ration. Mech. Anal. 100 (1988) 105–127.