

**NON-AUTONOMOUS FUNCTIONALS, BORDERLINE CASES
AND RELATED FUNCTION CLASSES**

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*To Nina Nikolaevna Ural'tseva, with gratitude and admiration
for all her beautiful mathematics*

ABSTRACT. We consider a class of non-autonomous functionals characterised by the fact that the energy density changes its ellipticity and growth properties according to the point, and prove some regularity results for related minimisers. These results are the borderline counterpart of analogous ones previously derived for non-autonomous functionals with (p, q) -growth. We also discuss similar functionals related to Musielak-Orlicz spaces in which basic properties like density of smooth functions, boundedness of maximal and integral operators, and validity of Sobolev type inequalities naturally relate to the assumptions needed to prove regularity of minima.

1. ALMOST FIFTY YEARS OF DEGENERATE OPERATORS IN RUSSIA

In 1967 a seminal paper [60] of Ural'tseva appeared, featuring the proof of the $C^{1,\beta}$ -nature of energy solutions to the degenerate equation

$$(1.1) \quad -\operatorname{div}(|Du|^{p-2}Du) = 0 .$$

The one on the left hand side is nowadays very well known as the p -Laplacean operator. This operator is relevant in a large number of situations as for instance in the Calculus of Variations, in Geometric Analysis, in the theory of quasiconformal mappings, in the modelling of non-Newtonian fluids. Ural'tseva herself, by exhibiting a counterexample, showed that the regularity of solutions does not go beyond Hölder continuity of the gradient, for some exponent $\beta \in (0, 1)$. The proof of the Hölder gradient continuity result also appears in the second, yet untranslated, edition of the classical book [41]. Ural'tseva's fundamental result is at the origin of a huge literature, up to the point that it is nowadays hard to find another single nonlinear operator that has attracted so much attention as long as the elliptic regularity theory is concerned. We quote here the important paper of Uhlenbeck [59], where Ural'tseva's result has been extended to the vectorial case, and the papers [23, 29, 42, 47], where different proofs and extensions to equations with coefficients have been given.

The equation appearing in (1.1) is the Euler-Lagrange equation of the functional

$$(1.2) \quad w \mapsto \int |Dw|^p dx, \quad p > 1$$

and, in fact, several of the results and techniques coming from the analysis of (1.1) have been eventually found to be useful in the Calculus of Variations. Starting from the eighties, new models and functionals related to the one in (1.2) were developed by Zhikov [62, 63, 64, 65, 66], together with a group of Russian mathematicians, in order to describe the behaviour of strongly anisotropic materials in the context of homogenisation and nonlinear elasticity. These functionals revealed to be important also in the study of duality theory and in the context of the Lavrentiev phenomenon.

They are non-autonomous functionals of the form

$$(1.3) \quad w \mapsto \int_{\Omega} F(x, Dw) dx, \quad \Omega \subset \mathbb{R}^n,$$

where the integrand $F(x, z)$ is characterised by the fact that the growth with respect to the second variable z , i.e. the gradient variable, depends on the space variable x . This for instance allows to give better models for those composites made by different basic materials. In particular, in the paper [65], Zhikov considered in relation to the Lavrentiev phenomenon three different model functionals for this situation. These are

$$(1.4) \quad \begin{aligned} \mathcal{M}(w, \Omega) &:= \int_{\Omega} c(x) |Dw|^2 dx, & 0 < 1/c(\cdot) \in L^t(\Omega), \quad t > 1; \\ \mathcal{V}(w, \Omega) &:= \int_{\Omega} |Dw|^{p(x)} dx, & 1 < p(x) < \infty; \\ \mathcal{P}_{p,q}(w, \Omega) &:= \int_{\Omega} (|Dw|^p + a(x)|Dw|^q) dx, & 0 \leq a(x) \leq L, \quad 1 < p < q. \end{aligned}$$

The first functional is actually a well-known one. There is a loss of ellipticity on the set $\{c(x) = 0\}$ and it has been studied at length in the context of equations involving Muckenhoupt weights; see for instance [30]. New, interesting conditions for basic properties of function spaces related to this functional have been recently given in [6, 58, 67]. The functional \mathcal{V} has also been the object of intensive interest nowadays and a huge literature has been developed on it. This energy has been used to build models for strongly anisotropic materials: in a material made of different components, the exponent $p(x)$ dictates the geometry of the composite, that changes its hardening exponent according to the point. The same mechanism can be used to give models for non-newtonian fluids that change their viscosity in presence of an electro-magnetic field, that in this case influences the size of the exponent $p(x)$, see for instance [57]. Similar models appear in image segmentation problems [14]. At the same time, the functional \mathcal{V} originated several studies in function spaces theory, see for instance [18, 26]. The third functional $\mathcal{P}_{p,q}$ is probably the hardest to treat, since the change of the exponent, and therefore of the ellipticity, which appear around the zero set $\{a(x) = 0\}$ is the most dramatic one. In this respect, the functional $\mathcal{P}_{p,q}$ appears as an upgraded version of \mathcal{V} . Again, in this case, the modulating coefficient $a(x)$ dictates the geometry of the composite made by two differential materials, with hardening exponents p and q , respectively. As a matter of fact, the two functionals \mathcal{V} and $\mathcal{P}_{p,q}$ share several different properties, and the aim of this paper is also to emphasize the strong similarities between the two in a borderline situation. This will be described in the next section. Both functionals give a relevant example of energy underlying a so-called Musielak-Orlicz space [24, 46, 56] and this point is briefly described in Section 2.2 below.

The functionals displayed in (1.4) fall in the realm of the so-called functionals with non-standard growth conditions of (p, q) -type, according to Marcellini's terminology [49, 50, 51]. These are functionals of the type in (1.3), where the energy density satisfies

$$(1.5) \quad |z|^p \lesssim F(x, z) \lesssim |z|^q + 1, \quad 1 \leq p \leq q.$$

They have been the object of an intensive investigation during the last years, see for instance [8, 9, 11, 12, 13, 34, 43, 44, 54, 55]. Here we like to mention again the pioneering contribution of Ural'tseva & Urdaletova [61], who proved regularity

results for minimisers of functional of the type

$$w \mapsto \int_{\Omega} \sum_{i=1}^n |D_i w|^{p_i} dx, \quad 1 < p_1 \leq \dots \leq p_n.$$

Another significant model example of functional with (p, q) -growth is given by

$$(1.6) \quad w \mapsto \int_{\Omega} |Dw|^p \log(1 + |Dw|) dx, \quad p \geq 1,$$

which is a logarithmic perturbation of the functional in (1.2); see [51, 53]. Functionals with (p, q) occur in several problems from elasticity and mathematical materials science; they have been studied at length in the setting of relaxation and lower semicontinuity; see for instance [1, 32, 39].

Before going on, let us recall the definition of local minimiser of a general functional of the type

$$(1.7) \quad w \mapsto \int_{\Omega} F(x, w, Dw) dx, \quad \Omega \subset \mathbb{R}^n,$$

where the integrand $F: \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow [0, \infty)$ is a Carathéodory function satisfying a bound of the type

$$(1.8) \quad |z|^p \lesssim F(x, v, z)$$

whenever $(x, v, z) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, as for instance considered in (1.5).

Definition 1. A function $u \in W_{\text{loc}}^{1,1}(\Omega)$ is a local minimiser of the functional in (1.7), if and only if $F(x, u, Du) \in L_{\text{loc}}^1(\Omega)$ and the minimality condition

$$\int_{\text{supp}(u-v)} F(x, u, Du) dx \leq \int_{\text{supp}(u-v)} F(x, v, Dv) dx$$

is satisfied whenever $v \in W_{\text{loc}}^{1,1}(\Omega)$ is such that $\text{supp}(u - v) \subset \Omega$.

In this paper we shall never consider the one-dimensional case and therefore it will always be

$$n \geq 2.$$

Definition 1 and the lower bound in display (1.8) imply that $u \in W_{\text{loc}}^{1,p}(\Omega)$. In all the cases considered in this paper, all $W_{\text{loc}}^{1,1}$ -minimisers will automatically be in $W_{\text{loc}}^{1,p}$ since the lower bound in (1.8) will always be in force. Moreover, since all the forthcoming results are local in nature, we shall assume with no loss of generality global integrability of local minimisers and for this reason we shall several times assume that local minimisers are directly in $u \in W^{1,p}(\Omega)$. In the following, with u being a minimiser, we shall typically denote $\|Du\|_{L^p} \equiv \|Du\|_{L^p(\Omega)}$.

2. INTRODUCTION, RESULTS AND THE FUNCTIONAL SETTING

2.1. Regularity results. In the recent papers [15, 16] the last two named authors have investigated the regularity properties of local minimisers of the functional $\mathcal{P}_{p,q}$ in (1.4) under sharp regularity assumptions on the modulating coefficient $a(\cdot)$ (see [8, 28] for previous contributions on this functional). The main outcome is that there is a sharp interaction between the regularity properties of $a(\cdot)$ and the gap q/p . An example of this is the following

Theorem 2.1 ([15]). *Let $u \in W^{1,p}(\Omega)$ be a local minimiser of the functional $\mathcal{P}_{p,q}$ and assume that the conditions*

$$(2.1) \quad 0 \leq a(\cdot) \in C^{0,\alpha}(\Omega) \quad \text{and} \quad \frac{q}{p} < 1 + \frac{\alpha}{n}$$

hold. Then Du is locally Hölder continuous in Ω .

The bound on q/p appearing in (2.1) is sharp in the sense that the counterexamples constructed in [28, 33] show that as soon the condition (2.1) is violated there exists a coefficient function $a(\cdot) \in C^{0,\alpha}$ and a local minimiser u whose set of discontinuity points is arbitrary close to $n - p$; this means that minimisers can be almost as bad as any other $W^{1,p}$ -functions. As described in [15], Theorem 2.1 actually extends to the vectorial case, and, as long as the scalar case is concerned, to a larger class of more general functionals. The bound in (2.1) reflects in a sharp way the subtle interaction between the different kinds of ellipticity properties of the operator - given by the numbers p and q - and the regularity of the coefficient $a(\cdot)$ that dictates the phase transition. More precisely, this in turn relates to the kind of non-uniform ellipticity of the Euler-Lagrange equation of the functional, which is

$$(2.2) \quad -\operatorname{div} A(x, Du) = 0$$

where in this case it is

$$A(x, z) = |z|^{p-2}z + (q/p)a(x)|z|^{q-2}z.$$

The non-uniform ellipticity of equation in display (2.2), when evaluated on the specific solution u , is measured by the potential blow-up of the ratio

$$(2.3) \quad \frac{\text{highest eigenvalue of } \partial_z A(x, Du)}{\text{lowest eigenvalue of } \partial_z A(x, Du)} \approx 1 + a(x)|Du|^{q-p}.$$

Around the phase transition, zero set $\{a(x) = 0\}$ the ratio in (2.3) exhibits a potential blow-up with respect to the gradient of rate $q - p$; to compensate, $a(x)$ is required to be suitably small. This means that, since we are close to $\{a(x) = 0\}$, then α must be large enough as prescribed in (2.1). This heuristic reasoning is confirmed when looking at the functional \mathcal{V} : in this case, when $p(\cdot)$ is continuous, the variability of x produces a small change in the growth exponent and therefore the non-uniform ellipticity of the related Euler-Lagrange equation is modest. Therefore any Hölder continuity exponent of the variable exponent $p(\cdot)$ is sufficient to get regularity. It indeed follows the next

Theorem 2.2 ([17]). *Let $u \in W^{1,1}(\Omega)$ be a local minimiser of the functional \mathcal{V} and assume that $p(\cdot)$ is locally Hölder continuous in Ω . Then Du is locally Hölder continuous in Ω .*

This fact is even more visible when looking at the Hölder continuity of solutions rather than at the Hölder continuity of their gradients. Indeed, we have

Theorem 2.3. *Let $u \in W^{1,1}(\Omega)$ be a local minimiser of the functional \mathcal{V} and assume that the continuous function $p(\cdot)$ is such that $1 < p_1 \leq p(x) \leq p_2 < \infty$ holds whenever $x \in \Omega$. Let $\omega(\cdot)$ be its modulus of continuity in the sense that*

$$|p(x) - p(y)| \leq \omega(|x - y|) \quad \text{holds for every } x, y \in \Omega$$

and denote

$$(2.4) \quad \limsup_{r \rightarrow 0} \omega(r) \log \left(\frac{1}{r} \right) =: l.$$

Then

- [31] if $l < \infty$, then $u \in C_{\text{loc}}^{0,\beta}(\Omega)$ for some $\beta \in (0, 1)$
- [3] if $l = 0$, then $u \in C_{\text{loc}}^{0,\beta}(\Omega)$ for every $\beta \in (0, 1)$.

Exactly as in the case of conditions (2.1) for $a(\cdot)$, the logarithmic modulus of continuity in (2.4) is sharp in the sense that if $l = \infty$, then minimisers can be discontinuous. The example showing the sharpness of (2.4) is due to Zhikov [65], and is of the nature of the ones found for the functional $\mathcal{P}_{p,q}$ in [28, 33] (which have

been in fact inspired by the constructions in [65]). In general, when the variable exponent $p(\cdot)$ is discontinuous, regularity of minima depends on the topological properties of the graph of $p(\cdot)$: saddle points allow for counterexamples [65, 66], while certain geometries of the level sets with Lipschitz interphases still allow to prove Hölder continuity of minima [2, 5].

In this paper we give yet another example of this general regularity phenomenon, showing a limiting case of the functional $\mathcal{P}_{p,q}$ for which one can reproduce exactly the situation of Theorems 2.2-2.3 as a borderline case of Theorem 2.1. We shall indeed consider the functional defined by

$$(2.5) \quad \mathcal{P}_{\log}(w, \Omega) := \int_{\Omega} [|Dw|^p + a(x)|Dw|^p \log(e + |Dw|)] dx ,$$

where the non-negative function $a(\cdot)$ is supposed to be bounded. We have indeed the following:

Theorem 2.4. *Let $u \in W^{1,1}(\Omega)$ be a local minimiser of the functional \mathcal{P}_{\log} defined in (2.5) and assume that the function $a(\cdot)$ is non-negative and bounded. Let $\omega(\cdot)$ be its modulus of continuity in the sense that*

$$(2.6) \quad |a(x) - a(y)| \leq \omega(|x - y|) \quad \text{holds for every } x, y \in \Omega$$

and denote

$$(2.7) \quad \limsup_{r \rightarrow 0} \omega(r) \log \left(\frac{1}{r} \right) =: l .$$

Then

- if $l < \infty$, then $u \in C_{\text{loc}}^{0,\beta}(\Omega)$ for some $\beta \in (0, 1)$
- if $l = 0$, then $u \in C_{\text{loc}}^{0,\beta}(\Omega)$ for every $\beta \in (0, 1)$
- if $\omega(r) \lesssim r^\sigma$ with $\sigma \in (0, 1)$, then Du is locally Hölder continuous in Ω .

The above theorem tells that, when looking at \mathcal{P}_{\log} , since the phase transition is between $|Du|^p$ to $|Du|^p \log(e + |Du|)$, we do not need to require the Hölder continuity of the coefficient $a(\cdot)$ to avoid discontinuity of minima, but a modulus of continuity which is exactly dually calibrated to the size of the transition suffices. The logarithmic modulus of continuity therefore naturally emerges. At the same time, the conditions required on the modulus of continuity of $a(\cdot)$ are exactly the same as those required on the variable exponent $p(\cdot)$ in Theorem 2.3. This is linked to the fact that a variable exponent produces a logarithmic perturbation in the gradient when x varies, which is exactly the one that is incorporated in \mathcal{P}_{\log} . The results above are linked to a general principle that prescribes that the oscillations with respect to the coefficients are linked to the rate of the non-uniform ellipticity with respect to the gradient variable as described in (2.3); in this case on the right-hand side of (2.3) we have $1 + a(x) \log(e + |Du|)$. This balance is not only linked to the regularity properties of minima, but is rather a general principle characterizing the structure in question. In fact, in the next section we shall briefly describe how the same condition ruling regularity of minima allows to use the various energies in question to define function spaces enjoining basic properties like, for instance, boundedness of maximal operators, density of smooth functions, Sobolev inequalities. These properties can be in turn used to prove regularity of minima, as shown in [15, 16]. Concluding the discussion about Theorem 2.4, it is then interesting to determine similar conditions for more general non-autonomous functionals, allowing for regularity of minima. See also Remark 2.1 below.

Theorem 2.4 summarises some of the more general results obtained in Theorems 4.1 as long as the Hölder continuity of minima is obtained, for some potentially small exponent. This means that the assumption $l < \infty$, with l being defined in (2.7), is

considered. The remaining higher regularity results, including the gradient Hölder continuity, are then proved for the model case \mathcal{P}_{\log} in Section 5. More precisely, the second assertion in Theorem 2.4 will be proved in Paragraph 5.2 while the last one will be proved in Paragraph 5.3. More results, including extensions to the case where minima are vector valued, are finally described in Section 6.

2.2. Conditions for regularity and Musielak-Orlicz spaces. Musielak-Orlicz spaces are, roughly speaking, Orlicz type spaces where the underlying Young functions is allowed to depend on space variable x too. Nowadays the most popular example of these spaces is given by the variable exponent Lebesgue spaces $L^{p(x)}$. For the definition we recall that if $H: \Omega \times [0, \infty) \rightarrow [0, \infty)$ is a Caratheodory function which is convex with respect to the second variable and such that $H(x, 0) = 0$, the space $L^H(\Omega; \mathbb{R}^k)$ is then defined as the set of measurable maps $f: \Omega \rightarrow \mathbb{R}^k$ such that

$$\int_{\Omega} H(x, |f(x)|) dx < \infty .$$

The natural Luxemburg type norm, exactly as in the case of classical Orlicz spaces (see Section 3.3 below and in particular (3.13)), is then naturally defined by

$$(2.8) \quad \|f\|_{L^H} := \inf \left\{ \lambda > 0 : \int_{\Omega} H \left(x, \frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\} .$$

Related Sobolev spaces can be then defined by prescribing that a function has distributional derivatives in L^H ; these are usually denoted as $W^{1,H}(\Omega)$. We will not expand on Musielak-Orlicz spaces any further here, but we just point out a few connections with regularity of minima of functionals of the type in (1.3) where the following bounds are satisfied:

$$H(x, |z|) \lesssim F(x, z) \lesssim H(x, |z|) + 1 .$$

In the case of the variable exponent spaces $L^{p(x)}$ it is $H(x, t) \equiv t^{p(x)}$. The case we are mostly interested in here is the one given by

$$(2.9) \quad H(x, t) := t^p + a(x)t^q, \quad t \geq 0 .$$

The phenomenon we want to emphasize is that, exactly as in the case of the variable exponent spaces $L^{p(x)}$, the same conditions ensuring regularity of minima of functionals of the type in (1.3) also allow for proving basic properties of the related function spaces. These include: boundedness of maximal operators, Sobolev inequalities, density of smooth functions. We shall present a few facts from [15, 28, 52] that show this interplay in the case (2.9). We start with maximal inequalities. Let $\Omega \subset \mathbb{R}^n$; we recall that the restricted maximal operator, for maps $f \in L^1(\Omega; \mathbb{R}^k)$ is defined as

$$M(f)(x) := M_{\Omega}(f)(x) := \sup_{B_{\varrho}(x) \subset \Omega, \varrho \leq 1} \int_{B_{\varrho}(x)} |f(y)| dy .$$

Theorem 2.5 ([15, 52, 65]). *Let $1 < p \leq q$ and $\alpha \in (0, 1]$ verify (2.1). For every $t \geq 1$ the maximal inequality*

$$\int_{\Omega} [H(x, |M(f)|)]^t dx \leq c \left(1 + [a]_{0,\alpha}^t \|f\|_{L^p(\Omega)}^{t(q-p)} \right) \int_{\Omega} [H(x, |f|)]^t dx$$

holds whenever $f \in L^p(\Omega)$, for a constant c depending only on the quantities n, p, q, α and t .

A first corollary of the previous result is given by the following

Theorem 2.6 ([15]). *Let $1 < p \leq q$ and $\alpha \in (0, 1]$ verify (2.1). Then there exist a constant c depending only on $n, p, q, [a]_{0, \alpha}$ and $\|Df\|_{L^p(B_R)}$, and exponents $d_1 > 1 > d_2$, depending only on n, p, q, α , such that*

$$\left(\int_{B_R} \left[H \left(x, \left| \frac{f - (f)_{B_R}}{R} \right| \right) \right]^{d_1} dx \right)^{1/d_1} \leq c \left(\int_{B_R} [H(x, |Df|)]^{d_2} dx \right)^{1/d_2}$$

holds whenever $f \in W^{1,p}(B_R)$, and whenever $B_R \subset \Omega$ is such that $R \leq 1$.

Another corollary of Theorem 2.5 asserts the density of smooth functions, or, equivalently, the absence of Lavrentiev phenomenon for the functional $\mathcal{P}_{p,q}$. This is contained in the following

Theorem 2.7 ([28, 65]). *Let $1 < p \leq q$ and $\alpha \in (0, 1]$ verify (2.1) and let $f: \Omega \rightarrow \mathbb{R}^k$ be such that*

$$\int_{\Omega} H(x, |f|) dx < \infty .$$

Then for every open subset $\Omega' \Subset \Omega$ there exists a sequence of smooth maps $\{f_k\} \subset C^\infty(\Omega'; \mathbb{R}^k)$ such that $f_k \rightarrow f$ strongly in $W^{1,p}$ and

$$\int_{\Omega'} H(x, |f_k|) dx \rightarrow \int_{\Omega'} H(x, |f|) dx .$$

In the papers [15, 16, 28] there is an approach that exhibits an interplay between the function spaces properties of Theorems 2.5-2.7 and regularity of minima. The above results can be used to obtain regularity of minimisers of the functional $\mathcal{P}_{p,q}$. On the other hand, the examples of irregular minima (when condition (2.1) fails) are constructed exhibiting the occurrence of the Lavrentiev phenomenon and therefore maximal estimates as those in Theorem 2.5 fail (since they would imply the absence of Lavrentiev phenomenon, as described [15, section 4]). For interesting results about basic harmonic analysis properties in Musielak-Orlicz spaces and further Sobolev type inequalities we refer for instance to [46].

Remark 2.1 (Future directions). We have reported Theorems 2.5-2.7 referring to the function $H(\cdot)$ defined in (2.9), under conditions (2.1). The point is that by letting $H(x, t) = t^{p(x)}$ similar results hold in the case the function $p(x)$ satisfies condition (2.4) with $l < \infty$; see for instance [18, 26]. The same phenomenon again holds for the functional \mathcal{M} from (1.4), where we therefore consider $H(x, t) := c(x)t^2$, as it is known from the theory of Muckenhoupt weights. It is therefore natural to conceive that there are more general conditions, ruling simultaneously basic properties of Musielak-Orlicz spaces and regularity of minima of related integral functionals.

3. PRELIMINARIES

3.1. Notation. In what follows we denote by c a general positive constant possibly varying from line to line; special occurrences will be denoted by c_1, c_*, \bar{c} or the like. All such constants will always be *larger or equal than one*; moreover relevant dependencies on parameters will be emphasised using parentheses, i.e., $c_1 \equiv c_1(n, p, q)$ means that c_1 depends on n, p, q . We denote by

$$B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$$

the open ball with center x_0 and radius $r > 0$; when not important, or clear from the context, we shall omit denoting the centre just denoting as follows: $B_r \equiv B_r(x_0)$. Unless otherwise stated, different balls in the same context will have the same

centre. With $\mathcal{B} \subset \mathbb{R}^n$ being a measurable set with positive, finite measure $|\mathcal{B}| > 0$, and with $g: \mathcal{B} \rightarrow \mathbb{R}^k$, $k \geq 1$, being a measurable map, we shall denote by

$$(g)_{\mathcal{B}} \equiv \int_{\mathcal{B}} g(x) dx := \frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} g(x) dx$$

its integral average. The Sobolev exponent p^* is $np/(n-p)$ if $p < n$ or every number larger than p , in the case $p \geq n$. For any $x > 0$, $\log x$ is the natural logarithm of x and for $\gamma > 0$, we shall denote by $\log^\gamma(e+x)$ the quantity $[\log(e+x)]^\gamma$.

3.2. N -functions setting. In the following we shall need some results which apply in a more general setting, the one of N -functions.

Let us consider a convex function $\varphi: [0, \infty) \rightarrow [0, \infty)$, such that

$$(3.1) \quad \varphi \in C^1([0, \infty)) \cap C^2((0, \infty)), \quad \varphi(0) = \varphi'(0) = 0, \\ \varphi'(t) \text{ increasing and } \lim_{t \rightarrow \infty} \varphi'(t) = \infty.$$

We define the auxiliary vector field $V: \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$(3.2) \quad V_\varphi(z) := \left(\frac{\varphi'(|z|)}{|z|} \right)^{1/2} z$$

whenever $z \in \mathbb{R}^n$; by (3.1) it turns out to be a bijection of \mathbb{R}^n . Under the assumption

$$(3.3) \quad \frac{1}{c_\varphi} \leq \frac{\varphi''(t)t}{\varphi'(t)} \leq c_\varphi, \quad \text{for all } t > 0 \text{ and some } c_\varphi \geq 1,$$

V_φ describes the monotonicity properties of the map $[\varphi'(|z|)/|z|]z$; indeed, for $z_1, z_2 \in \mathbb{R}^n$ it holds that

$$(3.4) \quad \frac{1}{c} |V_\varphi(z_1) - V_\varphi(z_2)|^2 \leq \left\langle \frac{\varphi'(|z_1|)}{|z_1|} z_1 - \frac{\varphi'(|z_2|)}{|z_2|} z_2, z_1 - z_2 \right\rangle \leq c |V_\varphi(z_1) - V_\varphi(z_2)|^2,$$

with a constant $c \geq 1$ depending on n and c_φ . The previous relation will be used also to prove the following algebraic fact, which is a consequence of [27, Lemma 2.4]. This asserts that there exists a constant c , depending only on n and the constant c_φ appearing in (3.3), such that

$$(3.5) \quad \varphi''(|z_1| + |z_2|) |z_1 - z_2|^2 \leq c |V_\varphi(z_1) - V_\varphi(z_2)|^2.$$

Moreover, we have that $|V_\varphi(z)|^2$ is comparable to $\varphi(z)$, namely for every $z \in \mathbb{R}^n$

$$(3.6) \quad \frac{1}{c} |V_\varphi(z)|^2 \leq \varphi(|z|) \leq c |V_\varphi(z)|^2,$$

$c \equiv c(c_\varphi)$; this follows from the fact that from (3.3), by integration by parts, we also have $\varphi(t)/c(c_\varphi) \leq \varphi'(t)t \leq c(c_\varphi)\varphi(t)$ for every $t \geq 0$. We shall denote by $V_p(\cdot) \equiv V_\phi$ for the choice $\varphi(t) = t^p/p$, namely

$$(3.7) \quad V_p(z) := |z|^{(p-2)/2} z,$$

and we denote $V_{\log} \equiv V_\phi$ for the choice $\varphi(t) = t^p \log(e+t)/p$, that is

$$(3.8) \quad V_{\log}(z) := \left(|z|^{p-2} \log(e+|z|) + \frac{|z|^{p-1}}{p(e+|z|)} \right)^{1/2} z.$$

In the first case, the next elementary inequality, which holds whenever $z_1, z_2 \in \mathbb{R}^n$, is classic:

$$(3.9) \quad \frac{1}{c} (|z_1| + |z_2|)^{(p-2)/2} |z_1 - z_2| \leq |V_p(z_1) - V_p(z_2)| \leq c (|z_1| + |z_2|)^{(p-2)/2} |z_1 - z_2|$$

valid for every $z_1, z_2 \in \mathbb{R}^n$, where c still depends on n, p (see [37]). In particular, when $p \geq 2$,

$$(3.10) \quad |z_1 - z_2|^p \leq c |V_p(z_1) - V_p(z_2)|^2$$

holds, again for a constant c depending on n, p .

Remark 3.1. Note that if φ_1, φ_2 are as above and moreover they satisfy (3.3) with constants $c_{\varphi_1}, c_{\varphi_2}$, their sum $\varphi := \varphi_1 + \varphi_2$ satisfies (3.3) with constant $c_\varphi := 2 \max\{c_{\varphi_1}, c_{\varphi_2}\}$. Indeed,

$$\frac{\varphi_1''(t)t}{\varphi_1'(t) + \varphi_2'(t)} + \frac{\varphi_2''(t)t}{\varphi_1'(t) + \varphi_2'(t)} \leq \frac{\varphi_1''(t)t}{\varphi_1'(t)} + \frac{\varphi_2''(t)t}{\varphi_2'(t)} \leq c_{\varphi_1} + c_{\varphi_2}$$

and

$$\frac{\varphi_1''(t)t}{\varphi_1'(t) + \varphi_2'(t)} + \frac{\varphi_2''(t)t}{\varphi_1'(t) + \varphi_2'(t)} \geq \frac{[\varphi_1''(t) + \varphi_2''(t)]t}{2 \max\{\varphi_1'(t), \varphi_2'(t)\}} \geq \frac{1}{2} \min\left\{\frac{1}{c_{\varphi_1}}, \frac{1}{c_{\varphi_2}}\right\}.$$

Remark 3.2. We will use the previous results in particular for the choice

$$(3.11) \quad \varphi(t) := t^p + a_0 t^p \log(e + t), \quad p > 1, \quad a_0 \geq 0.$$

It is easy to see that it satisfies all the properties mentioned above and in particular satisfies (3.3) with $c_\varphi \equiv c_\varphi(p)$ being *independent of a_0* , in view of the previous Remark 3.1.

An estimate which will be useful later is the following one: its proof can be found in [25, Theorem 7].

Lemma 3.1. *Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ as above, in particular satisfying (3.1) and (3.3), and let $B_R \equiv B_R(x_0) \subset \Omega$. Then there exist an exponent $d_1 \in (0, 1)$ and a constant c , both depending on n and c_φ , such that*

$$(3.12) \quad \int_{B_R} \varphi\left(\frac{|f - (f)_{B_R}|}{R}\right) dx \leq c \left(\int_{B_R} [\varphi(|Df|)]^{d_1} dx \right)^{1/d_1}$$

holds for every function $f \in W^{1,1}(\Omega)$ such that $\int_\Omega \varphi(|Df|) dx < \infty$.

The following Sobolev-type inequality is on the other hand taken from [7, Proposition 3.5]

Lemma 3.2. *Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ as above, in particular satisfying (3.1) and (3.3), and let $B_R \equiv B_R(x_0) \subset \Omega$. There exists a constant c depending on n and c_φ such that*

$$\int_{B_R} \left[\varphi\left(\frac{|f|}{R}\right) \right]^{\frac{n}{n-1}} dx \leq c \left(\int_{B_R} \varphi(|Df|) dx \right)^{\frac{n}{n-1}}$$

holds for every weakly differentiable function $f \in W_0^{1,1}(B_R)$ such that $\varphi(|Df|) \in L^1(B_R)$.

3.3. The $L^p \log^\gamma L$ spaces. For every bounded open set $\Omega \subset \mathbb{R}^n$ and every convex, strictly increasing function φ with $\lim_{t \rightarrow 0} \varphi(t)/t = 0$ and $\lim_{t \rightarrow \infty} \varphi(t)/t = \infty$, we consider the Orlicz space $L^\varphi(\Omega)$ with the Luxemburg norm

$$(3.13) \quad \|f\|_\varphi = \inf \left\{ \lambda > 0 : \int_\Omega \varphi\left(\frac{|f|}{\lambda}\right) dx \leq 1 \right\}.$$

When $\varphi(t) = t^p/p$, the previous quantity defines an averaged L^p norm; when $\varphi(t) = t^p \log^\gamma(e + t)$, for $p \geq 1, \gamma > 0$, the Orlicz space $L^\varphi(\Omega)$ is denoted by $L^p \log^\gamma L$ and it consists of the measurable functions such that

$$\int_\Omega |f|^p \log^\gamma(e + |f|) dx < \infty.$$

It is known that the $L \log^\gamma L$ norm, denoted in the following by $\|f\|_{L \log^\gamma L}$, is estimated by an averaged L^q norm for every $q > 1$

$$\|f\|_{L \log^\gamma L(B_R)} \leq c(q) \left(\int_{B_R} |f|^q dx \right)^{1/q}.$$

Moreover, Iwaniec & Verde [38] showed that $\|f\|_{L \log^\gamma L(B_R)}$ is equivalent to the quantity

$$\int_{B_R} |f| \log^\gamma \left(e + \frac{|f|}{(|f|)_{B_R}} \right) dx$$

via a constant depending on n, γ . Overall, for every $q > 1$ and $f \in L^q(B_R)$ we have

$$(3.14) \quad \int_{B_R} |f| \log^\gamma \left(e + \frac{|f|}{(|f|)_{B_R}} \right) dx \leq c(n, \gamma, q) \left(\int_{B_R} |f|^q dx \right)^{1/q}.$$

3.4. Estimates for frozen functionals. Here we collect some regularity results for minimisers of frozen functionals obtained by \mathcal{P}_{\log} , defined in (2.5), by considering the case in which the coefficient $a(\cdot)$ is constant. Several of such estimates find their root in the seminal work of Lieberman [43] as long as the scalar case is concerned. The results in the vectorial can be inferred from [27]. We start by higher regularity results, thereby considering functionals of the type

$$(3.15) \quad \mathcal{P}_0(w, \Omega) := \int_{\Omega} [|Dw|^p + a_0 |Dw|^p \log(e + |Dw|)] dx$$

where $a_0 \geq 0$ is a constant.

Theorem 3.1 ([27, 43]). *Let $v \in W^{1,p}(\Omega)$ be a local minimiser of the functional \mathcal{P}_0 defined in (3.15). There exists $\tilde{\alpha} \in (0, 1)$, depending only on n, p , but otherwise independent of $a_0 \geq 0$ and of the minimiser v , such that $Dv \in C_{\text{loc}}^{0, \tilde{\alpha}}(\Omega; \mathbb{R}^n)$. Moreover, whenever $B_R \subset \Omega$, the following inequalities hold:*

$$(3.16) \quad \sup_{B_{R/2}} \left(|Dv|^p + a_0 |Dv|^p \log(e + |Dv|) \right) \leq c \int_{B_R} \left(|Dv|^p + a_0 |Dv|^p \log(e + |Dv|) \right) dx$$

and, for every $0 < \varrho \leq R$

$$(3.17) \quad \int_{B_\varrho} |Dv - (Dv)_{B_\varrho}|^p dx \leq c \left(\frac{\varrho}{R} \right)^{\tilde{\alpha} p} \int_{B_R} \left(|Dv|^p + a_0 |Dv|^p \log(e + |Dv|) \right) dx,$$

where again c depends only on n, p .

Proof. The proof is a consequence of the results in [27], that we apply with the choice $\varphi(t) := t^p + a_0 t^p \log(e + t)$ with the vector field $V_\varphi(\cdot)$ that has been defined in (3.2); see also [7, Lemma 4.1]. Estimate (3.16) is actually nothing but the content of [27, Lemma 5.8], and we therefore concentrate on the proof of (3.17). The next thing we are using from [27] is the following key excess estimate:

$$(3.18) \quad \int_{B_\varrho} |V_\varphi(Dv) - (V_\varphi(Dv))_{B_\varrho}|^2 dx \leq c \left(\frac{\varrho}{R} \right)^{\alpha_0} \int_{B_R} |V_\varphi(Dv) - (V_\varphi(Dv))_{B_R}|^2 dx$$

holds whenever $\varrho \in (0, R)$ for constants $c \geq 1$ and $\alpha_0 \in (0, 1)$ both depending only on n, p . We now proceed with the proof of (3.17), and we preliminarily remark that it is sufficient to prove the inequality when $\varrho \leq R/2$. Since V_φ is a bijection of \mathbb{R}^n in itself, we then find $A \in \mathbb{R}^n$ such that

$$(3.19) \quad V_\varphi(A) = (V_\varphi(Dv))_{B_\varrho}.$$

We then separate two cases; the first is when $p \geq 2$. In this case, using (3.5) and (3.19), we have

$$\begin{aligned}
& \int_{B_\rho} |Dv - (Dv)_{B_\rho}|^p dx \\
& \leq 2^p \int_{B_\rho} |Dv - A|^p dx \\
& \leq c \int_{B_\rho} (|Dv| + |A|)^{p-2} |Dv - A|^2 dx \\
& \leq c \int_{B_\rho} \varphi''(|Dv| + |A|) |Dv - A|^2 dx \\
& \leq c \int_{B_\rho} |V_\varphi(Dv) - V_\varphi(A)|^2 dx \\
& = c \int_{B_\rho} |V_\varphi(Dv) - (V_\varphi(Dv))_{B_\rho}|^2 dx \\
& \leq c \left(\frac{\rho}{R}\right)^{\alpha_0} \int_{B_R} |V_\varphi(Dv) - (V_\varphi(Dv))_{B_R}|^2 dx \\
& \leq c \left(\frac{\rho}{R}\right)^{\alpha_0} \int_{B_R} (|V_\varphi(Dv)|^2 + |(V_\varphi(Dv))_{B_R}|^2) dx \\
& \leq c \left(\frac{\rho}{R}\right)^{\alpha_0} \int_{B_R} (|Dv|^p + a_0 |Dv|^p \log(e + |Dv|)) dx .
\end{aligned}$$

Therefore (3.17) obviously follows with $\tilde{\alpha} := \alpha_0/p$. Notice that in the last line we have used the very definition of the vector field $V_\varphi(\cdot)$ plus a few elementary computations. In the case $1 < p < 2$ we argue somehow differently and we also need to use estimate (3.16). Indeed, also using (3.6), we have

$$\begin{aligned}
& \int_{B_\rho} |Dv - (Dv)_{B_\rho}|^p dx \\
& \leq 2^p \int_{B_\rho} (|Dv| + |A|)^{p(p-2)/2} |Dv - A|^p (|Dv| + |A|)^{p(2-p)/2} dx \\
& \leq c \left(\int_{B_\rho} (|Dv| + |A|)^{p-2} |Dv - A|^2 dx \right)^{p/2} \left(\int_{B_\rho} (|Dv| + |A|)^p dx \right)^{(2-p)/2} \\
& \leq c \left(\int_{B_\rho} \varphi''(|Dv| + |A|) |Dv - A|^2 dx \right)^{p/2} \\
& \quad \cdot \left(\int_{B_\rho} (|V_\varphi(Dv)|^2 + |V_\varphi(A)|^2) dx \right)^{(2-p)/2} \\
& \leq c \left(\int_{B_\rho} |V_\varphi(Dv) - V_\varphi(A)|^2 dx \right)^{p/2} \\
& \quad \cdot \left(\int_{B_\rho} (|V_\varphi(Dv)|^2 + |(V_\varphi(Dv))_{B_\rho}|^2) dx \right)^{(2-p)/2} \\
& \leq c \left(\int_{B_\rho} |V_\varphi(Dv) - (V_\varphi(Dv))_{B_\rho}|^2 dx \right)^{(2-p)/2} \sup_{B_\rho} |V_\varphi(Dv)|^{2-p}
\end{aligned}$$

$$\begin{aligned}
&\leq c \left(\frac{\varrho}{R}\right)^{\alpha_0 p/2} \left(\int_{B_R} |V_\varphi(Dv) - (V_\varphi(Dv))_{B_R}|^2 dx \right)^{p/2} \\
&\quad \cdot \left(\sup_{B_\varrho} \varphi(|Dv|) \right)^{(2-p)/2} \\
&\leq c \left(\frac{\varrho}{R}\right)^{\alpha_0 p/2} \left(\int_{B_R} (|V_\varphi(Dv)|^2 + |(V_\varphi(Dv))_{B_R}|^2) dx \right)^{p/2} \\
&\quad \cdot \left(\int_{B_R} (|Dv|^p + a_0 |Dv|^p \log(e + |Dv|)) dx \right)^{(2-p)/2} \\
&\leq c \left(\frac{\varrho}{R}\right)^{\alpha_0 p/2} \int_{B_R} (|Dv|^p + a_0 |Dv|^p \log(e + |Dv|)) dx .
\end{aligned}$$

In the case $1 < p < 2$ we therefore have that (3.17) holds with $\tilde{\alpha} := \alpha_0/2$. The proof is complete. \square

The next result deals with the Hölder continuity of so-called quasi-minima (see [37]) of the functional \mathcal{P}_0 defined in (3.15). In this setting, a function $v \in W^{1,1}(\Omega)$ is a Q -minimiser of \mathcal{P}_0 for $Q \geq 1$, iff

$$|Dv|^p + a_0 |Dv|^p \log(e + |Dv|) \in L^1(\Omega)$$

and the quasi-minimality condition

$$(3.20) \quad \mathcal{P}_0(v, K) \leq Q \mathcal{P}_0(w, K)$$

holds for every $w \in W^{1,1}(\Omega)$ and compact set $K \subset \Omega$, such that $\text{supp}(v - w) \subset K$ and $|Dw|^p + a_0 |Dw|^p \log(e + |Dw|) \in L^1(\Omega)$. We then have the next theorem, which follows from [43, Section 6], for the choice $G(t) = t^p + a_0 t^p \log(e + t)$:

Theorem 3.2 ([43]). *Let $v \in W^{1,p}(\Omega)$ be a Q -minimiser of the functional \mathcal{P}_0 defined in (3.15). Then there exists a positive constant c and an exponent $\beta_0 \in (0, 1)$, both depending on n, p, Q , but otherwise independent of a_0 and of the Q -minimiser v , such that*

$$\text{osc}(v, B_\varrho) \leq c \left(\frac{\varrho}{R}\right)^{\beta_0} \text{osc}(v, B_R)$$

holds whenever $B_\varrho \subset B_R \subset \Omega$ are concentric balls.

For minimisers of the classic functional in (1.2), we have the following Calderón-Zygmund result up to the boundary; for the proof, see [40, Theorem 7.7] and [10].

Theorem 3.3. *Let $v \in u + W_0^{1,p}(B_{R/2})$ a minimiser to the functional (1.2) in $B_{R/2}$, where the function u belongs to $W^{1,q}(B_{R/2})$, for some $q \geq p$. Then $v \in W^{1,q}(B_{R/2})$ and the estimate*

$$\int_{B_{R/2}} |Dv|^q dx \leq c \int_{B_{R/2}} |Du|^q dx$$

holds for a constant $c \equiv c(n, p, q)$.

3.5. Miscellanea. We collect here some useful lemmata and inequalities often used in the rest of the paper. To begin with, we report two classical iteration results; see [37, Lemma 7.3] for the following one.

Lemma 3.3. *Let $\phi: [0, \tilde{R}] \rightarrow [0, \infty)$ be a non-decreasing function, such that the following inequality holds for some $\varepsilon \geq 0$ and whenever $0 < \varrho \leq R \leq \tilde{R}$:*

$$\phi(\varrho) \leq \tilde{c} \left[\left(\frac{\varrho}{R}\right)^n + \varepsilon \right] \phi(R) .$$

Then for every $\delta \in (0, n)$ there exists $\bar{\varepsilon} \equiv \bar{\varepsilon}(n, \delta, \bar{c}) > 0$ such that if $\varepsilon \leq \bar{\varepsilon}$, then

$$\phi(\varrho) \leq \bar{c} \left(\frac{\varrho}{R} \right)^{n-\delta} \phi(R)$$

holds whenever $0 < \varrho \leq R \leq \tilde{R}$ and for a constant $\bar{c} \equiv \bar{c}(n, \delta, \bar{c})$.

The next one is [37, Lemma 6.1].

Lemma 3.4. *Let $h: [r_1, r_2] \rightarrow [0, \infty)$ be a function such that*

$$h(r) \leq \theta h(s) + \frac{A}{(s-r)^\ell} \quad \text{for every } r_1 \leq r < s \leq r_2,$$

where $\theta \in (0, 1)$, $A \geq 0$ and $\ell > 0$. Then

$$h(r) \leq \frac{c(\theta, \ell) A}{(r_2 - r_1)^\ell}.$$

The following result encodes the self-improving character of reverse Hölder's inequalities, and a version can be found in [37, Section 6.4]. The original, seminal version is due to Gehring [35].

Lemma 3.5 (Gehring's Lemma). *Let $f \in L^1(\Omega)$ be such that*

$$\int_{B_{R/2}} |f| dx \leq \bar{c} \left(\int_{B_R} |f|^d dx \right)^{1/d}$$

holds for some exponent $d \in (0, 1)$, some constant $\bar{c} \geq 1$ and for every ball $B_R \subset \Omega$ with radius $R \leq R_0$. Then there exists an exponent δ_g , depending on d, \bar{c} , such that $f \in L_{\text{loc}}^{1+\delta_g}(\Omega)$ and the estimate

$$\int_{B_{R/2}} |f|^{1+\delta_g} dx \leq c(d, \bar{c}, R_0) \left(\int_{B_R} |f| dx \right)^{1+\delta_g}$$

holds for every ball $B_R \subset \Omega$ with radius R , such that $R \leq R_0$.

Useful properties of the logarithm function of later frequent use are the following:

$$(3.21) \quad \log(e + xy) \leq \log(e + x) + \log(e + y), \quad \log(e + Ax) \leq A \log(e + x)$$

for every $x, y \geq 0$ and $A \geq 1$. Another useful property is

$$(3.22) \quad \begin{aligned} (x + y)^p \log(e + x + y) &\leq (2x)^p \log(e + 2x) + (2y)^p \log(e + 2y) \\ &\leq 2^{p+1} x^p \log(e + x) + 2^{p+1} y^p \log(e + y), \end{aligned}$$

for all $x, y \geq 0$ and all $p \geq 1$.

4. DE GIORGI'S AND GEHRING'S THEORIES

Here we present low order regularity results. In particular, we prove the first assertion of Theorem 2.4, that will follow as a corollary of Theorem 4.1 below. The assumptions of this last theorem will be more general than those of Theorem 2.4 and will cover a larger family of functionals, also allowing for measurable dependence on the variable x of the integrand, while the relation with the model functional \mathcal{P}_{\log} defined in (2.5) will be retained only in terms of growth conditions of the integrand. Specifically, we shall consider functionals of the type

$$(4.1) \quad W^{1,1}(\Omega) \ni w \mapsto \mathcal{F}(w, \Omega) := \int_{\Omega} F(x, w, Dw) dx,$$

where the energy density $F: \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is initially only assumed to be a Carathéodory function satisfying the following growth conditions:

$$(4.2) \quad \nu H(x, z) \leq F(x, v, z) \leq LH(x, z)$$

whenever $x \in \Omega$, $v \in \mathbb{R}$ and $z \in \mathbb{R}^n$, where $0 < \nu \leq 1 \leq L$. We recall here the notation introduced in (2.9), and we present a slight modification of it aimed at simplifying the writing: namely, we shall denote

$$(4.3) \quad H(x, z) := |z|^p + a(x)|z|^p \log(e + |z|).$$

The function $H(x, z)$, with some ambiguity, will be considered both in the case $z \in \mathbb{R}$ and $z \in \mathbb{R}^n$; the meaning will anyway be always clear from the context.

Let $\omega(\cdot)$ be the modulus of continuity of $a(\cdot)$, as in (2.6). Without loss of generality this can be assumed to be concave and we shall assume this for the rest of the paper. The natural assumption to develop the basic regularity is

$$(4.4) \quad \limsup_{r \rightarrow 0} \omega(r) \log\left(\frac{1}{r}\right) < \infty,$$

or, in other words

$$(4.5) \quad \omega(r) \log\left(\frac{1}{r}\right) \leq \tilde{L} \quad \text{for every } r \leq 1.$$

Theorem 4.1 (Basic regularity). *Let $u \in W^{1,p}(\Omega)$ be a local minimiser of the functional \mathcal{F} defined in (4.1), under the assumption (4.5) on the modulus of continuity $\omega(\cdot)$ of the function $a(\cdot)$. Then:*

- (Gehring's theory) *There exists a positive integrability exponent $\delta_g > 0$, depending only on $n, p, \nu, L, \|Du\|_{L^p}, \tilde{L}$, such that*

$$(4.6) \quad H(x, Du) \in L_{\text{loc}}^{1+\delta_g}(\Omega).$$

More precisely, the local reverse Hölder's inequality

$$(4.7) \quad \left(\int_{B_{R/2}} [H(x, Du)]^{1+\delta_g} dx \right)^{1/(1+\delta_g)} \leq c \int_{B_R} H(x, Du) dx$$

holds for every ball $B_R \subset \Omega$ and for a constant c , depending only on $n, p, \nu, L, \|Du\|_{L^p}, \tilde{L}$. In particular, if $p > n/(1 + \delta_g)$, then u is locally Hölder continuous.

- (Vectorial Gehring's theory) *(4.6) and (4.7) hold also in the case the minimiser $u : \Omega \rightarrow \mathbb{R}^N$ is vector valued, i.e. $N > 1$; the constant c and δ_g depend also on N .*
- (De Giorgi's theory) *u is locally bounded. Moreover, when $p \leq n/(1 + \delta_g)$, for every open subset $\Omega' \Subset \Omega$ there exists $\beta \in (0, 1)$, depending only on $n, p, \|a\|_{L^\infty}, \|Du\|_{L^p}, \nu, L$ and $\|u\|_{L^\infty(\Omega')}$, such that*

$$u \in C_{\text{loc}}^{0,\beta}(\Omega').$$

Remark 4.1. The reader will notice that all the constants in the above and forthcoming a priori estimates depend also on the specific minimisers in question via the quantity $\|Du\|_{L^p}$. On the contrary, in the standard case of functionals with p -polynomial growth, the constants do not depend on the solution, but they just depend on the ellipticity ratio L/ν , and of course on n, p . The dependence on $\|Du\|_{L^p}$ is natural and typically occurs in non-uniformly elliptic problems of the type in (2.2); it stems from the fact that the ratio appearing on the left hand since of (2.3) depends on the gradient of the solution u .

We start by stating the following intrinsic Caccioppoli inequality; the proof is very similar to that in [15, Section 9]. Note that here it is not necessary any regularity assumption on $a(\cdot)$, except for its boundedness.

Lemma 4.1 (Caccioppoli type inequality). *Let $u \in W^{1,p}(\Omega)$ be a local minimiser of the functional defined in (4.1), with $a(\cdot)$ being bounded and non-negative; let $B_R \subset \Omega$ be a ball. The inequality*

$$(4.8) \quad \int_{B_{r_1}} H(x, D(u-k)_\pm) dx \leq c \int_{B_{r_2}} H\left(x, \frac{(u-k)_\pm}{r_2-r_1}\right) dx$$

holds for every $k \in \mathbb{R}$, $0 < r_1 < r_2 \leq R$, with a constant c depending only on n, p, ν, L . In particular,

$$(4.9) \quad \int_{B_{R/2}} H(x, Du) dx \leq c \int_{B_R} H\left(x, \frac{u-k}{R}\right) dx$$

holds.

Proof. The proof adapts the arguments developed in [36]. We restrict ourselves to the case “+”, since for the version with “-”, it is sufficient to note that $-u$ is still a local minimiser of a functional as \mathcal{F} with integrand $F(x, v, z)$ replaced by $F(x, -v, -z)$, that still satisfies (4.2). Let us fix r_1, r_2 as in the statement, and also choose r, s such that $r_1 \leq r < s \leq r_2$. We use as competitor the function $w = u - \eta(u-k)_+$, with $\eta \in C_c^\infty(B_s)$, $0 \leq \eta \leq 1$, $\eta \equiv 1$ on B_r and $|D\eta| \leq c/(s-r)$; we have, using the minimality of u and (4.2), that

$$\int_{B_s \cap \{u > k\}} H(x, Du) dx \leq \frac{L}{\nu} \int_{B_s \cap \{u > k\}} H(x, Dw) dx .$$

Note that, since $u \in W^{1,p}(B_R)$ and in particular $(u-k)_+^p \log(e + (u-k)_+) \in L^1(B_R)$ by Sobolev embedding theorem, all the integrals here are finite. Performing simple algebraic manipulations and using (3.22), yields

$$\begin{aligned} \int_{B_r} H(x, D(u-k)_+) dx &\leq c_* \int_{B_s \setminus B_r} H(x, D(u-k)_+) dx \\ &\quad + c_* \int_{B_s} \left| \frac{(u-k)_+}{s-r} \right|^p \left(1 + a(x) \log \left(e + \left| \frac{(u-k)_+}{s-r} \right| \right) \right) dx \end{aligned}$$

for some $c_* \equiv c_*(n, p, \nu, L)$. By “filling the hole”, that is adding to both sides the quantity

$$c_* \int_{B_r} H(x, D(u-k)_+) dx$$

we get, using (3.21)₂

$$\begin{aligned} \int_{B_r} H(x, D(u-k)_+) dx &\leq \theta \int_{B_s} H(x, D(u-k)_+) dx \\ &\quad + c \int_{B_s} \left| \frac{(u-k)_+}{s-r} \right|^p \left(1 + a(x) \log \left(e + \left| \frac{(u-k)_+}{s-r} \right| \right) \right) dx \\ &\leq \theta \int_{B_s} H(x, D(u-k)_+) dx \\ &\quad + \frac{c(r_2-r_1)}{(s-r)^{p+1}} \int_{B_s} (u-k)_+^p \left(1 + a(x) \log \left(e + \left| \frac{(u-k)_+}{r_2-r_1} \right| \right) \right) dx , \end{aligned}$$

with $\theta \equiv \theta(n, p, \nu, L) = c_*/(c_* + 1) \in (0, 1)$ and $c \equiv c(n, p, \nu, L)$. At this point Lemma 3.4 applied with the choice

$$h(t) \equiv \int_{B_t} H(x, D(u-k)_+) dx$$

yields (4.8). \square

Note that the previous Caccioppoli's inequality holds in particular for minimisers of the functional \mathcal{P}_{\log} in (2.5).

Remark 4.2. Taking $k = (u)_{B_R}$ in (4.9) we get

$$(4.10) \quad \int_{B_{R/2}} H(x, Du) dx \leq c \int_{B_R} H\left(x, \frac{u - (u)_{B_R}}{R}\right) dx$$

holds whenever $B_R \subset \Omega$ for a constant c depending only on n, p, ν, L .

4.1. Gehring's theory. To prove the first part of Theorem 4.1, in light of Lemma 3.5, it is enough to show that there exists an exponent $d \in (0, 1)$, depending on n, p, ν, L , such that

$$(4.11) \quad \int_{B_{R/2}} H(x, Du) dx \leq c \left(\int_{B_R} [H(x, Du)]^d dx \right)^{1/d}$$

for every ball $B_R \subset \Omega$ with $R \leq 1/e$ and for some constant c depending only on $n, p, \nu, L, \|Du\|_{L^p}, \tilde{L}$. Indeed, (4.7) immediately follows from Lemma 3.5, while (4.6) then follows from a standard covering argument. This will be done in the scalar case only, since the vectorial version of the result follows verbatim, with an additional dependence on N of the various constants. We consider two cases, according to the occurrence of the inequality

$$(4.12) \quad a_i(R) := \inf_{B_R} a \leq 2\omega(R).$$

We first consider the first case, i.e., we suppose that (4.12) is satisfied. Thus

$$a(x) = [a(x) - a_i(R)] + a_i(R) \leq \omega(2R) + a_i(R) \leq 2\omega(R) + a_i(R) \leq 5\omega(R)$$

holds for every $x \in B_R$. Notice that the concavity of $\omega(\cdot)$ has been used to estimate $\omega(2R) \leq 2\omega(R)$: we shall use a similar computation several times below. Recalling (3.21)₁, that $R \leq 1/e$ and using Poincaré inequality we have that

$$(4.13) \quad \begin{aligned} \log\left(e + \left|\frac{u - (u)_{B_R}}{R}\right|\right) &\leq c \left[1 + \log\left(e + \left|\frac{u - (u)_{B_R}}{R}\right|^p\right)\right] \\ &\leq c + c \log\left(e + \frac{|u - (u)_{B_R}|^p}{(|u - (u)_{B_R}|^p)_{B_R}}\right) + c \log\left(e + \frac{(|u - (u)_{B_R}|^p)_{B_R}}{R^p}\right) \\ &\leq c + c \log\left(e + \frac{|u - (u)_{B_R}|^p}{(|u - (u)_{B_R}|^p)_{B_R}}\right) + c \log\left(e + \frac{\|Du\|_{L^p}^p}{R^n}\right) \\ &\leq c \log\left(e + \frac{|u - (u)_{B_R}|^p}{(|u - (u)_{B_R}|^p)_{B_R}}\right) + c \log\left(\frac{1}{R}\right) \end{aligned}$$

for some $c \equiv c(n, p, \|Du\|_{L^p})$. Applying this inequality together with estimates (4.10) and (3.14) with $f \equiv |u - (u)_{B_R}|^p/R^p$, $\gamma = 1$, we deduce that for every $q > p$ and $R \leq 1/e$, the estimations

$$\begin{aligned} \int_{B_{R/2}} H(x, Du) dx &\leq c \int_{B_R} H\left(x, \frac{u - (u)_{B_R}}{R}\right) dx \\ &\leq c \int_{B_R} \left|\frac{u - (u)_{B_R}}{R}\right|^p + c\omega(R) \left|\frac{u - (u)_{B_R}}{R}\right|^p \log\left(e + \left|\frac{u - (u)_{B_R}}{R}\right|\right) dx \\ &\leq c \left(1 + \omega(R) \log\left(\frac{1}{R}\right)\right) \int_{B_R} \left|\frac{u - (u)_{B_R}}{R}\right|^p dx \\ &\quad + c\omega(R) \int_{B_R} \left|\frac{u - (u)_{B_R}}{R}\right|^p \log\left(e + \frac{|u - (u)_{B_R}|^p}{(|u - (u)_{B_R}|^p)_{B_R}}\right) dx \\ &\leq c \left(1 + \omega(R) \log\left(\frac{1}{R}\right)\right) \left[\int_{B_R} \left|\frac{u - (u)_{B_R}}{R}\right|^p dx + \left(\int_{B_R} \left|\frac{u - (u)_{B_R}}{R}\right|^q dx \right)^{p/q} \right] \end{aligned}$$

hold for some constant c which depends only on $n, p, \nu, L, \|Du\|_{L^p}$. Thanks to Sobolev embedding theorem, we find exponents

$$q_* < p < q$$

in such a way that the following inequality holds:

$$\left(\int_{B_R} \left| \frac{u - (u)_{B_R}}{R} \right|^q dx \right)^{p/q} \leq c \left(\int_{B_R} |Du|^{q_*} dx \right)^{p/q_*}.$$

Therefore, by (4.5) and Hölder's inequality

$$\begin{aligned} \int_{B_{R/2}} H(x, Du) dx &\leq c \int_{B_R} \left| \frac{u - (u)_{B_R}}{R} \right|^p dx + c \left(\int_{B_R} \left| \frac{u - (u)_{B_R}}{R} \right|^q dx \right)^{p/q} \\ &\leq c \left(\int_{B_R} \left| \frac{u - (u)_{B_R}}{R} \right|^q dx \right)^{p/q} \\ &\leq c \left(\int_{B_R} |Du|^{q_*} dx \right)^{p/q_*} \\ &\leq c \left(\int_{B_R} [H(x, Du)]^{q_*/p} dx \right)^{p/q_*}, \end{aligned}$$

hold for some constant c which depends only on $n, p, \nu, L, \|Du\|_{L^p}, \tilde{L}$.

When (4.12) is not satisfied, namely, when

$$a_i(R) \equiv \inf_{B_R} a > 2\omega(R) \quad \text{holds for every } x \in B_R$$

we have that

$$a(x) = [a(x) - a_i(R)] + a_i(R) \leq \omega(2R) + a_i(R) \leq 2\omega(R) + a_i(R) \leq 2a_i(R).$$

Therefore (4.10) gives

$$\int_{B_{R/2}} H(x, Du) dx \leq c \int_{B_R} \left| \frac{u - (u)_{B_R}}{R} \right|^p \left(1 + a_i(R) \log \left(e + \left| \frac{u - (u)_{B_R}}{R} \right| \right) \right) dx.$$

Finally, we apply the Sobolev-Poincaré type inequality (3.12) for the choice $\varphi(t) := t^p + a_0 t^p \log(e + t)$, see Remark 3.2. This yields

$$\begin{aligned} \int_{B_R} \left| \frac{u - (u)_{B_R}}{R} \right|^p \left(1 + a_0 \log \left(e + \left| \frac{u - (u)_{B_R}}{R} \right| \right) \right) dx \\ \leq c \left(\int_{B_R} |Du|^{pd_1} (1 + a_0 \log(e + |Du|))^{d_1} dx \right)^{1/d_1}, \end{aligned}$$

with $d_1 \equiv d_1(n, p) \in (0, 1)$ and c depending only on n and p . Applying this inequality with $a_0 = a_i(R)$ we deduce that

$$\begin{aligned} \int_{B_{R/2}} H(x, Du) dx &\leq c \left(\int_{B_R} |Du|^{pd_1} (1 + a_i(R) \log(e + |Du|))^{d_1} dx \right)^{1/d_1} \\ &\leq c \left(\int_{B_R} [H(x, Du)]^{d_1} dx \right)^{1/d_1}. \end{aligned}$$

In both cases, we have proved (4.11) with $d = \max\{d_1, q_*/p\} \in (0, 1)$.

Remark 4.3. In the previous proof, the only dependence of the constant c in (4.11) on $\|Du\|_{L^p}$ (and therefore the only dependence of δ_g and c in the statement of Theorem 4.1) comes from the estimate in the last line of (4.13). If the minimiser u is assumed a priori to be bounded, then the statement of Gehring's theory holds with constants depending only on n, p, ν, L, \tilde{L} , and $\|u\|_{L^\infty(\Omega)}$. Indeed, in (4.13)

one can estimate $(|u - (u)_{B_R}|^p)_{B_R}$ from above with $2^p \|u\|_{L^\infty(\Omega)}^p$ instead of using Poincaré inequality.

Remark 4.4. Using the approach based on the two alternatives described in the proof above it is also possible to prove the intrinsic Sobolev-type inequality contained in the following:

Theorem 4.2. *Let $1 < p$, $a(\cdot)$ and $\omega(\cdot)$ be as in (2.6) with (4.5) in force, and $H(\cdot)$ be as in (4.3). Then there exist an exponent $\bar{d} > 1$, depending only on n, p , such that*

$$(4.14) \quad \left(\int_{B_R} \left[H \left(x, \frac{f}{R} \right) \right]^{\bar{d}} dx \right)^{1/\bar{d}} \leq c \int_{B_R} H(x, Df) dx$$

holds whenever $B_R \equiv B_R(x_0)$ is a ball with radius $R \leq 1/e$ and $f \in W_0^{1,1}(B_R)$ is such that the right-hand side of (4.14) is finite. The constant c depends only on n, p, \tilde{L} and $\|Df\|_{L^p(B_R)}$ and has the form $c(n, p, \tilde{L})(1 + \|Df\|_{L^p(B_R)})$.

The inequality in display (4.14) is the analog of a similar Sobolev type inequality proved in [15, Theorem 1.6] for minimisers of the functional $\mathcal{P}_{p,q}$ defined in (1.4). See also the proof proposed in [16].

4.2. De Giorgi's theory. Here we prove the remaining part of Theorem 4.1, namely the local Hölder continuity of minimisers, for some exponent $\beta \in (0, 1)$. The first step involves the proof of local boundedness.

4.2.1. Local boundedness of minimisers. In our case $p < n$, local boundedness of minimisers of \mathcal{F} , and therefore of the model case \mathcal{P}_{\log} , can be for instance inferred by the results of [21], dealing with the more general (p, q) -growth conditions.

Theorem 4.3. *Every local minimiser of the functional \mathcal{F} defined in (4.1), under the sole assumption (4.2), with $a \in L^\infty(\Omega)$, is locally bounded.*

The previous result follows from the ones in [21, Theorem 2.2] since a minimiser u of the functional \mathcal{F} defined in (4.1) is easily seen to be a Q -minimiser of the convex functional

$$w \mapsto \int_{\Omega} H(x, Dw) dx .$$

This means that, accordingly to the definition already given in (3.20), there exists $Q \geq 1$ such that $H(x, Du) \in L^1(\Omega)$ and

$$(4.15) \quad \int_K H(x, Du) dx \leq Q \int_K H(x, Dw) dx$$

holds for every $w \in W^{1,1}(\Omega)$ and compact subset $K \subset \Omega$ such that $\text{supp}(u - w) \subset K$. Indeed, consider such a w and observe that, thanks to (4.2), we can estimate

$$\begin{aligned} \int_K H(x, Du) dx &\leq \frac{1}{\nu} \int_K F(x, u, Du) dx \\ &\leq \frac{1}{\nu} \int_K F(x, w, Dw) dx \leq \frac{L}{\nu} \int_K H(x, Dw) dx \end{aligned}$$

so that (4.15) follows with $Q \equiv L/\nu$.

4.2.2. *Reducing the oscillation in the two phases setting.* We here start the proof of the Hölder continuity of minima. Due to Theorem 4.3 we know that, whenever $\Omega' \Subset \Omega$ is an open subset as in the last part of the statement of Theorem 4.1, we have that

$$\|u\|_{L^\infty(\Omega')} < \infty .$$

Once fixed a ball $B_r \equiv B_r(x_0) \subset \Omega'$, similarly to the previous Paragraph 4.1, the proof will split into two parts, according to the fact that the following condition holds or not:

$$(4.16) \quad a_s(r) \equiv a_s(x_0, r) := \sup_{B_r} a(\cdot) \leq 4\omega(r).$$

In the case the previous condition is satisfied, we have the following result about the reduction of the oscillation of u .

Proposition 4.1. *Let $u \in W^{1,p}(\Omega)$ be a local minimiser of the functional \mathcal{F} defined in (4.1), under the assumptions (4.2) and (4.5), with $p < n$. Let $B_r \equiv B_r(x_0) \subset \Omega'$, $r \leq 1/e$ and suppose that (4.16) holds. Then there exists a constant $\vartheta \in (0, 1)$, depending on $n, p, \nu, L, \tilde{L}, \|u\|_{L^\infty(\Omega')}$, such that*

$$(4.17) \quad \operatorname{osc}_{B_{r/4}} u \leq \vartheta \operatorname{osc}_{B_r} u.$$

We start with the following ‘‘almost standard Caccioppoli’s estimate’’: in the case (4.16) holds, then the intrinsic Caccioppoli’s estimate of Lemma 4.1 can be improved in order to get estimate (4.18) below. This last one is a Caccioppoli estimate of the type that holds for functionals with standard p -growth. This is the ultimate effect of an assumption as (4.16), that allows to treat terms as $a(x)|Du|^p \log(e + |Du|)$ as a perturbation of $|Du|^p$, when u is a minimiser of the functional in (4.1).

Lemma 4.2 (Almost standard Caccioppoli’s estimate). *Let $u \in W^{1,p}(\Omega)$ be a local minimiser of (4.1) under the assumptions (4.2) and (4.5) and let $B_r \equiv B_r(x_0) \subset \Omega'$ be a ball with radius $r \leq 1/e$; moreover, suppose that (4.16) holds. Then, for every $k \in \mathbb{R}$ such that $|k| < \|u\|_{L^\infty(\Omega')}$, the inequality*

$$(4.18) \quad \int_{B_{r_1}} |D(u - k)_\pm|^p dx \leq c \left(\frac{r}{r_2 - r_1} \right)^{p+1} \int_{B_{r_2}} \frac{(u - k)_\pm^p}{r^p} dx$$

holds for every $r/2 \leq r_1 < r_2 \leq r$ and a constant depending on n, p, ν, L, \tilde{L} and $\|u\|_{L^\infty(\Omega')}$.

Proof. From (4.8) and (4.16) we infer

$$\begin{aligned} & \int_{B_{r_1}} |D(u - k)_\pm|^p dx \\ & \leq c \int_{B_{r_2}} \left(1 + a_s(r) \log \left(e + \left| \frac{(u - k)_\pm}{r_2 - r_1} \right| \right) \right) \left| \frac{(u - k)_\pm}{r_2 - r_1} \right|^p dx \\ & \leq c \left(\frac{r}{r_2 - r_1} \right)^p \int_{B_{r_2}} \left(1 + \omega(r) \log \left(e + \frac{2\|u\|_{L^\infty}}{r_2 - r_1} \right) \right) \frac{(u - k)_\pm^p}{r^p} dx . \end{aligned}$$

Using (3.21)₂ and the fact that $r \leq 1/e$ we estimate

$$\begin{aligned} \log \left(e + \frac{2\|u\|_{L^\infty(\Omega')}}{r_2 - r_1} \right) & \leq \frac{r}{r_2 - r_1} \log \left(e + \frac{2\|u\|_{L^\infty(\Omega')}}{r} \right) \\ & \leq c(\|u\|_{L^\infty(\Omega')}) \frac{r}{r_2 - r_1} \log \left(\frac{1}{r} \right) . \end{aligned}$$

Recalling (4.5), this yields (4.18) since in our case $r_1 \approx r_2 \approx r$. \square

Having (4.18) at hand, the proof of the following lemma is completely analogous to the density lemmas valid in the usual case of functionals with standard p -growth.

Lemma 4.3. *Let the assumptions of Proposition 4.1 hold, and moreover assume that the density condition*

$$(4.19) \quad \frac{|B_r \cap \{u \geq \sup_{B_r} u - \text{osc}_{B_r} u/2\}|}{|B_r|} \leq \frac{1}{2}$$

holds. Then there exists a constant c depending on $n, p, \nu, L, \tilde{L}, \|u\|_{L^\infty}$ such that

$$\frac{|B_{r/2} \cap \{u \geq \sup_{B_r} u - \text{osc}_{B_r} u/2^j\}|}{|B_{r/2}|} \leq \frac{c}{j^{1^*/p'}}$$

holds for every $j \in \mathbb{N}$.

Proof. Once noticing that (4.18), for the choices $r_1 = r/2$ and $r_2 = r$, reduces to

$$\int_{B_{r/2}} |D(u - k)_\pm|^p dx \leq c \int_{B_r} \frac{(u - k)_\pm^p}{r^p} dx,$$

the proof is identical to that in the standard case. See, for instance, [37, Lemma 7.2]. \square

At this point, the argument needed to prove Proposition 4.1, due to De Giorgi [22], is classic. We give the proof for the convenience of the reader.

Proof of Proposition 4.1. We assume without loss of generality that (4.19) holds; if it would not be so, we could just consider $-u$, which locally minimise a functional with energy density $\tilde{F}(x, w, z) := F(x, -w, -z)$, which clearly satisfies (4.2); if u does not satisfy (4.19), then it does $-u$. For $j \in \mathbb{N}$ fixed, consider the radii and the levels

$$r_i := \frac{r}{4} + \frac{r}{2^{i+2}}, \quad \tilde{r}_i := \frac{r_{i+1} + r_i}{2}, \quad k_i := \sup_{B_r} u - \frac{1}{2^j} \left(\frac{1}{2} + \frac{1}{2^{i+1}} \right) \text{osc}_{B_r} u$$

for $i \in \mathbb{N}_0$; call $B_i := B_{r_i}$, $\tilde{B}_i := B_{\tilde{r}_i}$. Then we apply Caccioppoli's inequality (4.18)₊ with $r_1 = (r_i + r_{i+1})/2$, $r_2 = r_i$ and $k = k_i$; this yields

$$\int_{\tilde{B}_i} |D(u - k_i)_+|^p dx \leq c 2^{(i+3)(p+1)} \int_{B_i} \frac{(u - k_i)_+^p}{r_i^p} dx.$$

Now take a cut-off function $\eta \in C_c^\infty(\tilde{B}_i)$, $\eta \in [0, 1]$, such that $\eta \equiv 1$ on B_{i+1} and $|D\eta| \leq c 2^i/r$; using also Sobolev's inequality (recall that here $p < n$),

$$\begin{aligned} \left(\int_{B_{i+1}} (u - k_i)_+^{p^*} dx \right)^{p/p^*} &\leq c(n, p) \left(\int_{\tilde{B}_i} [(u - k_i)_+ \eta]^{p^*} dx \right)^{p/p^*} \\ &\leq c \tilde{r}_i^p \int_{\tilde{B}_i} |D[(u - k_i)_+ \eta]|^p dx \\ &\leq c 2^{i(p+1)} \int_{B_i} (u - k_i)_+^p dx. \end{aligned}$$

Since

$$2^{-(i+j+2)} \chi_{\{u \geq k_{i+1}\}} \text{osc}_{B_r} u \leq (u - k_i)_+ \leq 2^{-j} \chi_{\{u \geq k_i\}} \text{osc}_{B_r} u \quad \text{in } B_r$$

by the definition of the levels k_i , we end up with

$$A_{i+1} \leq c 2^{c(n,p)i} A_i^{1 + \frac{p}{n-p}}, \quad \text{with } A_i := \int_{B_i} \chi_{\{u \geq k_i\}} dx,$$

for a constant $c \equiv c(n, p, \nu, L, \tilde{L}, \|u\|_{L^\infty(\Omega')})$. A standard hypergeometric Lemma from [22] at this point ensures that $A_i \rightarrow 0$, provided that

$$A_0 = \frac{|B_{r/2} \cap \{u \geq \sup_{B_r} u - 2^{-j} \operatorname{osc}_{B_r} u\}|}{|B_{r/2}|} \leq \frac{1}{\tilde{c}},$$

with \tilde{c} depending on $n, p, \nu, L, \tilde{L}, \|u\|_{L^\infty(\Omega')}$ but *not on* j . This, in view of Lemma 4.3, can be guaranteed by choosing j , depending on $n, p, \nu, L, \tilde{L}, \|u\|_{L^\infty(\Omega')}$, large enough. We thus end up with

$$u \leq \sup_{B_r} u - 2^{-j-1} \operatorname{osc}_{B_r} u \quad \text{a.e. in } B_{r/4},$$

that is (4.17) with $\vartheta = 1 - 2^{-j-1}$, after subtracting $\inf_{B_r} u$. \square

4.2.3. Proof of the Hölder continuity. We show in this paragraph that, fixed a ball $B_R \equiv B_R(x_0) \subset \Omega'$ with radius smaller than one, the estimate

$$(4.20) \quad \operatorname{osc}_{B_\rho} u \leq c \left(\frac{\rho}{R} \right)^\beta \operatorname{osc}_{B_R} u$$

holds for every $0 < \rho \leq R$, with a constant c depending on $n, p, \nu, L, \tilde{L}, \|u\|_{L^\infty(\Omega')}$ and an exponent β as in Theorem 4.1. These constants are independent of the starting ball $B_R(x_0)$ and therefore this implies, up to a standard covering argument, the local Hölder continuity of the minimiser in question. We use an exit-time argument, i.e. we consider the set

$$\mathcal{L} := \{m \in \mathbb{N}_0 : (4.16) \text{ is not satisfied in } B_{4^{-m}R}\} \subset \mathbb{N}_0$$

and its minimum $\ell := \min \mathcal{L}$. If $\mathcal{L} = \emptyset$, we set $\ell = \infty$. Now, if $\ell \geq 1$, then (4.16) is satisfied in $B_{4^{-m}R}$ for $m = 0, \dots, \ell - 1$ and by induction we have, for the same indexes, using Proposition 4.1,

$$(4.21) \quad \operatorname{osc}_{B_{4^{-(m+1)}R}} u \leq \vartheta^m \operatorname{osc}_{B_R} u \quad \implies \quad \operatorname{osc}_{B_\rho} u \leq c \left(\frac{\rho}{R} \right)^{\beta_1} \operatorname{osc}_{B_R} u$$

for any $\rho \in (4^{-(\ell+1)}R, R]$, by a simple interpolative argument, with $\beta_1 = \log_{1/4} \vartheta$. If $\ell = \infty$, then we are finished; indeed (4.21) is valid whenever $\rho \in (0, R]$. If $\ell \in \mathbb{N}$, then the goal now is to show that u in $B_{4^{-\ell}R}$ is a Q -minimiser of a functional without x -dependence. Indeed, since we know that

$$\sup_{B_{4^{-\ell}R}} a(\cdot) > 4\omega(4^{-\ell}R),$$

then there exists $\bar{x} \in B_{4^{-\ell}R}$ such that $a(\bar{x}) > 4\omega(4^{-\ell}R)$. Then, for every $x \in B_{4^{-\ell}R}$, we have $a(\bar{x}) - a(x) \leq \omega(2 \cdot 4^{-\ell}R) \leq 2\omega(4^{-\ell}R)$ and therefore it holds that

$$(4.22) \quad \frac{1}{2}a(\bar{x}) \leq a(\bar{x}) - 2\omega(4^{-\ell}R) \leq a(x) \leq a(\bar{x}) + 2\omega(4^{-\ell}R) \leq 2a(\bar{x}).$$

Thus we have that

$$F(x, u, Du) \approx H(x, Du) \approx |Du|^p(1 + a(\bar{x}) \log(e + |Du|))$$

holds up to a constant depending only on ν, L , and therefore it follows that

$$|Du|^p(1 + a(\bar{x}) \log(e + |Du|)) \in L^1(B_{4^{-\ell}R}).$$

Moreover, for every $w \in W^{1,1}(B_{4^{-\ell}R})$ such that $|Dw|^p(1 + a(\bar{x}) \log(e + |Dw|)) \in L^1(B_{4^{-\ell}R})$ and compact subset K such that $\operatorname{supp}(u - w) \subset K \subset B_{4^{-\ell}R}$, we have

$$\begin{aligned} \int_K |Du|^p(1 + a(\bar{x}) \log(e + |Du|)) dx &\leq 2 \int_K H(x, Du) dx \\ &\leq \frac{2}{\nu} \int_K F(x, u, Du) dx \end{aligned}$$

$$\begin{aligned}
&\leq \frac{2}{\nu} \int_K F(x, w, Dw) dx \\
&\leq \frac{2L}{\nu} \int_K H(x, Dw) dx \\
&\leq \frac{4L}{\nu} \int_K |Dw|^p (1 + a(\bar{x}) \log(e + |Dw|)) dx .
\end{aligned}$$

Note that we have used both (4.2) and (4.22) twice, together with the minimality of u . Hence u is a Q -minimiser of the functional \mathcal{P}_0 defined in (3.15) with $a_0 := a(\bar{x})$ in $B_{4^{-\ell}R}$ in the sense of (3.20), with $Q = 4L/\nu$. At this point we are allowed to use Theorem 3.2 that ensures that the inequality

$$(4.23) \quad \operatorname{osc}_{B_\rho} u \leq c \left(\frac{\rho}{4^{-\ell}R} \right)^{\beta_0} \operatorname{osc}_{B_{4^{-\ell}R}} u$$

holds whenever $\rho \leq 4^{-\ell}R$, with $\beta_0 \in (0, 1)$ and c both depending on $n, p, L/\nu$. Merging (4.21) with the second inequality appearing in (4.23) yields (4.20) in the case $\ell \in \mathbb{N}$, and with $\beta := \min\{\beta_0, \beta_1\}$. Finally, if $\ell = 0$, then we can directly show that u is a $4L/\nu$ -minimiser in B_R and in this case (4.23) is (4.20) and we are finished.

5. GRADIENT HÖLDER REGULARITY

In this section we propose the rest of the proof of Theorem 2.4, i.e. the last two assertions are considered. This means that we are first proving, in Paragraph 5.2 below, that $u \in C_{\text{loc}}^{0, \beta}(\Omega)$ for every $\beta \in (0, 1)$ assuming that $l = 0$ (this is defined in (2.7)). Then, in Paragraph 5.3, we shall finally prove the Hölder gradient continuity of minima assuming that $a(\cdot)$ is itself Hölder continuous. In the following \tilde{L} denotes a finite constant such that (4.5) holds.

5.1. Comparison lemma and decay estimate. In the following lemma, we estimate the difference between the energy of a minimiser of \mathcal{P}_{\log} on a ball $B_{R/2} \equiv B_{R/2}(x_0)$ such that $B_{R/2} \subset \Omega$, that is $\mathcal{P}_{\log}(u, B_{R/2})$, and the energy of the same minimiser computed when considering a related energy, obtained from the first one by freezing the coefficient $a(\cdot)$, that is

$$(5.1) \quad \int_{B_{R/2}} (|Du|^p + a_i(R)|Du|^p \log(e + |Du|)) dx ,$$

where

$$(5.2) \quad a_i(R) \equiv a_i(x_0, R) = \inf_{B_R} a(\cdot) .$$

Under the assumption (4.4), and therefore under (4.5), the energy with frozen functional is comparable with the whole energy of a minimiser; under the assumption (2.7) with $l = 0$, the former is a small perturbation of the latter. The proof is based on the reverse Hölder inequality (4.7), which is in fact the only point where we are going to use the minimality of u , and on the inequality in $L \log L$ spaces contained in display (3.14).

Lemma 5.1 (Energy comparison). *Let $u \in W^{1, p}(\Omega)$ be a local minimiser of \mathcal{P}_{\log} , let $B_R \equiv B_R(x_0) \subset \Omega$ be a ball with radius $R \leq 1/e$, and let $a_i(R)$ be as in (5.2). Moreover, let $\omega(\cdot)$ be as in (2.6) and assume that (4.5) holds. Then, for every $\gamma > 0$, there exists a constant $c \equiv c(n, p, \|Du\|_{L^p}, \tilde{L}, \gamma)$ such that the following inequality holds:*

$$(5.3) \quad \int_{B_{R/2}} [a(x) - a_i(R)]^\gamma |Du|^p \log^\gamma(e + |Du|) dx$$

$$\leq c \left[\omega(R) \log \left(\frac{1}{R} \right) \right]^\gamma \int_{B_R} H(x, Du) dx.$$

Proof. Assumption (4.5) allows to apply the higher integrability result (4.6) of Theorem 4.1, that gives that $H(x, Du) \in L_{\text{loc}}^{1+\delta_g}(\Omega)$, for a positive exponent $\delta_g > 0$ depending only on $n, p, \|Du\|_{L^p}, \tilde{L}$. Moreover, there exists a constant c , still depending on the same parameters, such that the following reverse inequality holds for every ball $B_R \subset \Omega$ with radius $R \leq 1/e$:

$$(5.4) \quad \begin{aligned} \int_{B_{R/2}} |Du|^{p+p\delta_g} dx &\leq \int_{B_{R/2}} [H(x, Du)]^{1+\delta_g} dx \\ &\leq c \left(\int_{B_R} H(x, Du) dx \right)^{1+\delta_g}. \end{aligned}$$

We estimate the left-hand side of (5.3) by mean of (2.6)

$$\begin{aligned} \int_{B_{R/2}} [a(x) - a_i(R)]^\gamma |Du|^p \log^\gamma(e + |Du|) dx \\ \leq c [\omega(R)]^\gamma \int_{B_{R/2}} |Du|^p \log^\gamma(e + |Du|^p) dx. \end{aligned}$$

To estimate the right-hand side in the above display, we use (3.21)₁ and (3.14) to deduce that

$$\begin{aligned} &\int_{B_{R/2}} |Du|^p \log^\gamma(e + |Du|^p) dx \\ &\leq c(\gamma) \int_{B_{R/2}} |Du|^p \log^\gamma \left(e + \frac{|Du|^p}{(|Du|^p)_{B_R}} \right) dx \\ &\quad + c(\gamma) \log^\gamma(e + (|Du|^p)_{B_R}) \int_{B_{R/2}} |Du|^p dx \\ &\leq cR^n \left(\int_{B_{R/2}} |Du|^{p(1+\delta_g)} dx \right)^{1/(1+\delta_g)} + c \log^\gamma \left(\frac{1}{R} \right) \int_{B_{R/2}} |Du|^p dx \end{aligned}$$

for a constant c depending on $n, p, \|Du\|_{L^p}$ and γ . By the reverse Hölder inequality (5.4) and the fact that $R \leq 1/e$, we then obtain

$$\begin{aligned} &\int_{B_{R/2}} |Du|^p \log^\gamma(e + |Du|) dx \\ &\leq cR^n \int_{B_R} H(x, Du) dx + c \log^\gamma \left(\frac{1}{R} \right) \int_{B_{R/2}} H(x, Du) dx \\ &\leq c \log^\gamma \left(\frac{1}{R} \right) \int_{B_R} H(x, Du) dx. \end{aligned}$$

From the last three displays we deduce (5.3). \square

Before going on, we premise a few basic facts on the validity of the weak formulation of the Euler-Lagrange equation of the functional \mathcal{P}_{\log} . In particular, we clarify the class of admissible test functions.

Remark 5.1. For the rest of the paper, we shall adopt the notation

$$(5.5) \quad f(z) := |z|^p \quad \text{and} \quad g(z) := |z|^p \log(e + |z|).$$

We show that the Euler-Lagrange equation

$$\int_{\Omega} \langle \partial f(Du) + a(x) \partial g(Du), D\varphi \rangle dx = 0$$

is valid for every $\varphi \in W^{1,p}(\Omega)$ with compact support, with $D\varphi \in L^p \log L(\Omega)$. An analogous fact holds in particular for the Euler-Lagrange equation of the functional in (5.1). This is a consequence of the following remarks (together with the fact that $a(\cdot)$ is bounded). First, note that we can concentrate on the part of the equation (5.1) involving $\partial g(\cdot)$ since the remaining part has a lower order growth. Given the N -function $g(\cdot)$ defined in (5.5), it is possible to define its Young conjugate $g^*(\cdot)$ by

$$g^*(s) := \sup_{r>0} (rs - g(r))$$

- note that by the properties of $g(\cdot)$, $g^*(s)$ is well defined for every $s \geq 0$. The Young conjugate enjoys the following general property:

$$(5.6) \quad g^*\left(\frac{g(s)}{s}\right) \approx g(s)$$

up to numerical constants, for all $s > 0$, while the following Hölder type inequality

$$\int_{\Omega} |FG| dx \leq 2 \|F\|_{L^g(\Omega)} \|G\|_{L^{g^*}(\Omega)}$$

holds for all measurable functions $F, G : \Omega \rightarrow \mathbb{R}^n$. Here the Luxemburg norm $\|\cdot\|_{L^g(\Omega)}$ is defined as in (3.13) with $\varphi \equiv g$. Hence we can estimate

$$\int_{\Omega} \langle a(\cdot) \partial g(Du), D\varphi \rangle dx \leq 2 \|a\|_{L^\infty} \|\partial g(Du)\|_{L^{g^*}(\Omega)} \|D\varphi\|_{L^g(\Omega)}.$$

We conclude by noting that $|\partial g(Du)| \approx |Du|^{p-1} \log(e + |Du|) = g(Du)/|Du|$ up to a constant depending on p , using (5.6), noting that

$$\lim_{n \rightarrow \infty} \|D\phi_n\|_{L^g(\Omega)} = 0 \iff \lim_{n \rightarrow \infty} \int_{\Omega} g(D\phi_n) dx = 0$$

and that smooth functions are dense in $L^p \log L(\Omega)$. A reference for the foregoing facts is for instance [43].

Next we prove a comparison lemma, where we estimate the distance between a minimiser of \mathcal{P}_{\log} and a minimiser of a frozen functional; note that in the following lines it could be $a_i(R) = 0$. In this last case the comparison functional is the usual p -Dirichlet energy and we shall use the vector field $V_p(\cdot)$, which has been defined in (3.7).

Lemma 5.2 (Comparison). *Let $u \in W^{1,p}(\Omega)$ be a local minimiser of \mathcal{P}_{\log} , and let $B_R \equiv B_R(x_0) \Subset \Omega$ be a ball with radius $R \leq 1/e$. Let $a_i(R)$ be as in (5.2). Moreover, let $\omega(\cdot)$ be as in (2.6) and assume that (4.5) holds. Finally, let $v \in W^{1,p}(B_{R/2})$ be the solution to the following Dirichlet problem:*

$$(5.7) \quad \begin{cases} v \mapsto \min_w \int_{B_{R/2}} (|Dw|^p + a_i(R)|Dw|^p \log(e + |Dw|)) dx \\ w \in u + W_0^{1,p}(B_{R/2}). \end{cases}$$

Then the inequality

$$(5.8) \quad \begin{aligned} & \int_{B_{R/2}} (|V_p(Du) - V_p(Dv)|^2 + a_i(R)|V_{\log}(Du) - V_{\log}(Dv)|^2) dx \\ & \leq c\omega(R) \log\left(\frac{1}{R}\right) \int_{B_R} H(x, Du) dx \end{aligned}$$

holds for a constant c depending only on $n, p, \|Du\|_{L^p}, \tilde{L}$.

Proof. Note that the problem (5.7) has a minimiser, as

$$(5.9) \quad Du \in L^{p(1+\delta_g)}(B_R) \subset L^p \log L(B_R)$$

by (4.6) and since $B_R \Subset \Omega$; therefore the class of competitors with finite energy in the Dirichlet class $u + W_0^{1,p}(B_{R/2})$ is non-empty. In particular, the minimality of v gives

$$(5.10) \quad \int_{B_{R/2}} (|Dv|^p + a_i(R)|Dv|^p \log(e + |Dv|)) dx \\ \leq \int_{B_{R/2}} (|Du|^p + a_i(R)|Du|^p \log(e + |Du|)) dx,$$

that in turns also implies

$$(5.11) \quad Dv \in L^p \log L(B_{R/2})$$

as the right hand side is finite by (5.9). This is immediate if $a_i(R) > 0$, while if $a_i(R) = 0$ this follows from Theorem 3.3 together with (5.9). Since both u and v are minimisers, using the corresponding Euler-Lagrange equations, recalling the content of Remark 5.1, the notation introduced in (5.5), and testing with $u-v \in W_0^{1,p}(B_{R/2})$ (notice that $Du - Dv \in L^p \log L(B_{R/2})$ by (5.9) and (5.11)), we compute

$$\begin{aligned} \mathcal{D}_1 &:= \int_{B_{R/2}} \langle \partial f(Du) - \partial f(Dv) + a_i(R)(\partial g(Du) - \partial g(Dv)), Du - Dv \rangle dx \\ &= \int_{B_{R/2}} \langle \partial f(Du) + a_i(R)\partial g(Du), Du - Dv \rangle dx \\ &= \int_{B_{R/2}} [a_i(R) - a(x)] \langle \partial g(Du), Du - Dv \rangle dx := \mathcal{D}_2. \end{aligned}$$

Then (3.4) applied to V_p and V_{\log} yields

$$(5.12) \quad \int_{B_{R/2}} (|V_p(Du) - V_p(Dv)|^2 + a_i(R)|V_{\log}(Du) - V_{\log}(Dv)|^2) dx \leq c|\mathcal{D}_2|,$$

where $c \equiv c(n, p)$. We estimate the integrand of \mathcal{D}_2 thanks to Young's inequality: indeed, for every $A > 0$, we have

$$\begin{aligned} &|[a_i(R) - a(x)] \langle \partial g(Du), Du - Dv \rangle| \\ &\leq c|a(x) - a_i(R)| |Du|^{p-1} (1 + \log(e + |Du|)) (|Du| + |Dv|) \\ &\leq \frac{c}{A^{\frac{1}{p-1}}} |a(x) - a_i(R)|^{p'} |Du|^p (1 + \log(e + |Du|))^{p'} + cA(|Du| + |Dv|)^p \\ &\leq \frac{c}{A^{\frac{1}{p-1}}} |a(x) - a_i(R)|^{p'} |Du|^p \left(1 + \log^{p'}(e + |Du|)\right) + cA(|Du|^p + |Dv|^p). \end{aligned}$$

Using Lemma 5.1 with $\gamma = p'$, the fact that $\log(1/R) \geq 1$, the minimality of v in (5.10), and finally the fact that $a_i(R) \leq a(x)$ in $B_{R/2}$, we deduce the following chain of inequalities:

$$(5.13) \quad |\mathcal{D}_2| \leq \frac{c}{A^{\frac{1}{p-1}}} \int_{B_{R/2}} |a(x) - a_i(R)|^{p'} |Du|^p \log^{p'}(e + |Du|) dx \\ + \frac{c}{A^{\frac{1}{p-1}}} [\omega(R)]^{p'} \int_{B_{R/2}} |Du|^p dx + cA \int_{B_{R/2}} (|Du| + |Dv|)^p dx \\ \leq \frac{c}{A^{\frac{1}{p-1}}} \left(\omega(R) \log\left(\frac{1}{R}\right) \right)^{p'} \int_{B_R} H(x, Du) dx \\ + cA \int_{B_{R/2}} (|Du|^p + a_i(R)|Du|^p \log(e + |Du|)) dx$$

$$\leq \left[cA + \frac{c}{A^{\frac{1}{p-1}}} \left(\omega(R) \log \left(\frac{1}{R} \right) \right)^{p'} \right] \int_{B_R} H(x, Du) dx$$

with c depending on $n, p, \|Du\|_{L^p}, \tilde{L}$. Choosing $A = \omega(R) \log(1/R)$, by (5.12) and (5.13) we deduce (5.8). \square

Thanks to the previous comparison lemma, we deduce a decay lemma for the minimiser of \mathcal{P}_{\log} .

Lemma 5.3 (Decay). *Let $u \in W^{1,p}(\Omega)$ be a local minimiser of \mathcal{P}_{\log} , let $B_R \equiv B_R(x_0) \Subset \Omega$ be a ball with radius $R \leq 1/e$, and let $a_i(R)$ be as in (5.2). Moreover, let $\omega(\cdot)$ be as in (2.6) and assume that (4.5) holds. Then the inequality*

$$(5.14) \quad \int_{B_\varrho} H(x, Du) dx \leq c_d \left[\left(\frac{\varrho}{R} \right)^n + \omega(R) \log \left(\frac{1}{R} \right) \right] \int_{B_R} H(x, Du) dx$$

holds whenever $0 < \varrho \leq R$ and $B_\varrho(x_0) \equiv B_\varrho \subset B_R \subset \Omega$, for a constant $c_d \equiv c_d(n, p, \|Du\|_{L^p}, \tilde{L})$.

Proof. It is obviously sufficient to prove (5.14) for $\varrho \leq R/4$. Thanks to Lemma 5.1 with $\gamma = 1$ we have that

$$(5.15) \quad \begin{aligned} \int_{B_\varrho} H(x, Du) dx &\leq \int_{B_\varrho} (|Du|^p + a_i(R)|Du|^p \log(e + |Du|)) dx \\ &\quad + \int_{B_{R/2}} (a(x) - a_i(R)) |Du|^p \log(e + |Du|) dx \\ &\leq \int_{B_\varrho} (|Du|^p + a_i(R)|Du|^p \log(e + |Du|)) dx \\ &\quad + c \omega(R) \log \left(\frac{1}{R} \right) \int_{B_R} H(x, Du) dx, \end{aligned}$$

for a constant c depending only on $n, p, \|Du\|_{L^p}, \tilde{L}$. To estimate the first term appearing on the right hand side of (5.15), as in Lemma 5.2, we consider the function v defined in (5.7). By (3.6), applied with $\varphi(t) = t^p$ and $\varphi(t) = t^p \log(e + t)$, and therefore considering the vector field V_{\log} defined in (3.8), we obtain

$$(5.16) \quad \begin{aligned} &\int_{B_\varrho} (|Du|^p + a_i(R)|Du|^p \log(e + |Du|)) dx \\ &\leq c \int_{B_\varrho} (|V_p(Du)|^2 + a_i(R)|V_{\log}(Du)|^2) dx \\ &\leq c \int_{B_\varrho} (|V_p(Dv)|^2 + a_i(R)|V_{\log}(Dv)|^2) dx \\ &\quad + c \int_{B_{R/2}} (|V_p(Du) - V_p(Dv)|^2 + a_i(R)|V_{\log}(Du) - V_{\log}(Dv)|^2) dx \end{aligned}$$

for some constant c depending only on n and p . Using the sup estimate (3.16) for the minimiser of the frozen functional, we have that

$$\begin{aligned} &\int_{B_\varrho} (|V_p(Dv)|^2 + a_i(R)|V_{\log}(Dv)|^2) dx \\ &\leq c \sup_{B_\varrho} (|Dv|^p + a_i(R)|Dv|^p \log(e + |Dv|)) \\ &\leq c \int_{B_{R/2}} (|Dv|^p + a_i(R)|Dv|^p \log(e + |Dv|)) dx \end{aligned}$$

$$\begin{aligned} &\leq c \int_{B_{R/2}} (|Du|^p + a_i(R)|Du|^p \log(e + |Du|)) dx \\ &\leq c \int_{B_R} (|Du|^p + a_i(R)|Du|^p \log(e + |Du|)) dx \end{aligned}$$

for a constant c depending only on n, p . Estimate (5.14) follows using Lemma 5.2 to estimate the second integral on the right-hand side of (5.16), and connecting the resulting estimate to the content of the last display and eventually to (5.15). \square

5.2. Hölder continuity of minima. In this paragraph we prove the second assertion in Theorem 2.4. Therefore we shall consider local minimisers $u \in W^{1,p}(\Omega)$ of the functional \mathcal{P}_{\log} defined in (2.5), under the assumption (2.7) with $l = 0$, and we prove that $u \in C_{\text{loc}}^{0,\beta}(\Omega)$ holds for every $\beta \in (0, 1)$. In other words we are here assuming that, with $\omega(\cdot)$ being the modulus of continuity of the coefficient $a(\cdot)$ in the sense of (2.6), it holds that

$$(5.17) \quad \limsup_{r \rightarrow 0} \omega(r) \log\left(\frac{1}{r}\right) = 0.$$

We show the following: for every ball $B_R \Subset \Omega$ and exponent $\beta \in (0, 1)$, there exist a number $\varepsilon > 0$ and a radius $R_0 \leq 1/e$ such that, if $\omega(R) \log(1/R) \leq \varepsilon$ for every $R \leq R_0$, then the inequality

$$(5.18) \quad [u]_{\beta; B_{R/2}} \leq c(n, p, \|Du\|_{L^p}, \tilde{L}, R_0) \left(R^{p(1-\beta)} \int_{B_R} H(x, Du) dx \right)^{1/p}$$

holds. Here $[u]_{\beta; B_{R/2}}$ as usual denotes the Hölder seminorm of the function u in the ball $B_{R/2}$:

$$(5.19) \quad [u]_{\beta; B_{R/2}} := \sup_{x, y \in B_{R/2}, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\beta}.$$

To this end, we prove a Morrey type estimate. Namely, for every $\delta \in (0, n)$, there exist positive constants $c \equiv c(n, p, \|Du\|_{L^p}, \tilde{L}, \delta)$ and

$$(5.20) \quad R_0 \equiv R_0(n, p, \|Du\|_{L^p}, \tilde{L}, \delta) \leq 1/e,$$

such that the decay estimate

$$(5.21) \quad \int_{B_\varrho} H(x, Du) dx \leq c \left(\frac{\varrho}{R}\right)^{n-\delta} \int_{B_R} H(x, Du) dx$$

holds whenever $0 < \varrho \leq R \leq R_0$ and $B_\varrho \subset B_R \Subset \Omega$ are concentric balls. Since (5.21) implies that

$$\int_{B_\varrho} \left| \frac{u - (u)_{B_\varrho}}{\varrho} \right|^p dx \leq c \int_{B_\varrho} |Du|^p dx \leq c \left(\frac{\varrho}{R}\right)^{n-\delta} \int_{B_R} H(x, Du) dx$$

for every $0 < \varrho \leq R \leq R_0$ and $B_R \Subset \Omega$, the $C_{\text{loc}}^{0,\beta}$ -regularity of u , with $\beta = 1 - \delta/p$, follows using a standard covering argument and a well-known integral characterisation of Hölder continuity due to Campanato. The same arguments give the validity of the local estimate in (5.18). Since δ can be taken arbitrarily close to zero we can reach any $\beta \in (0, 1)$.

In order to prove (5.21), we apply Lemma 3.3 with the choice

$$\phi(\varrho) \equiv \int_{B_\varrho} H(x, Du) dx$$

and \tilde{c} being the constant c_d appearing in Lemma 5.3. More precisely, given any $\delta > 0$, we consider the number $\tilde{\varepsilon}$ given by Lemma 3.3 and depending on n, δ, c_d and

thus ultimately on $n, p, \|Du\|_{L^p}, \tilde{L}$ and δ . At this point, thanks to (5.17), we fix $R_0 > 0$ so that

$$\omega(R) \log\left(\frac{1}{R}\right) \leq \bar{\varepsilon} \quad \text{for every } R \leq R_0 .$$

Lemma 5.3 and the previous choice of R_0 imply that whenever $0 < \varrho \leq R \leq R_0$

$$\phi(\varrho) \leq c_d \left[\left(\frac{\varrho}{R}\right)^n + \varepsilon \right] \phi(R) ;$$

by Lemma 3.3, we deduce (5.21).

5.3. Hölder continuity of the gradient. Here we complete the proof of Theorem 2.4 demonstrating the validity of the third and last assertion, the one concerning the gradient Hölder continuity of minimisers. With the previous estimates in our hands we can then proceed modifying some of the arguments given in [47, 48] (see also [45] for more regularity on the standard p -Laplacean case). Let $u \in W^{1,p}(\Omega)$ be a local minimiser of the functional \mathcal{P}_{\log} defined in (2.5); we prove that, if

$$(5.22) \quad \omega(R) \leq \tilde{L}R^\sigma \quad \text{holds for every } R \leq 1$$

for some $\sigma \in (0, 1)$ and $\tilde{L} \geq 1$, then there exists $\beta \in (0, 1)$, depending only on n, p and σ , such that

$$(5.23) \quad Du \in C_{\text{loc}}^{0,\beta}(\Omega; \mathbb{R}^n) .$$

We start showing that for every ball $B_R \equiv B_R(x_0) \subset \Omega$ the inequality

$$(5.24) \quad \int_{B_\varrho} |Du - (Du)_{B_\varrho}|^p dx \leq c \left[\left(\frac{\varrho}{R}\right)^{\tilde{\alpha}p} + R^{\sigma/4} \left(\frac{R}{\varrho}\right)^n \right] \int_{B_R} H(x, Du) dx$$

holds whenever $0 < \varrho \leq R \leq 1/e$ and $B_\varrho \subset B_R \subset \Omega$ are concentric balls, for an exponent $\tilde{\alpha} \in (0, 1)$ depending only on n and p , and for a positive constant c depending on $n, p, \tilde{L}, \|Du\|_{L^p}$, and σ . We postpone the proof of (5.24) and show how it implies the local Hölder continuity of Du . Choosing $R = \varrho^{1-\varepsilon}$ in (5.24) and thereby taking

$$\varepsilon = \frac{\sigma}{4\tilde{\alpha}p + \sigma + 4n} \in (0, 1) ,$$

we deduce that for every ball $B_{\varrho^{1-\varepsilon}} \subset \Omega$ with $0 < \varrho^{1-\varepsilon} \leq 1/e$ the inequalities

$$(5.25) \quad \begin{aligned} \int_{B_\varrho} |Du - (Du)_{B_\varrho}|^p dx &\leq c \left[\varrho^{\varepsilon\tilde{\alpha}p} + \varrho^{(1-\varepsilon)\sigma/4-\varepsilon n} \right] \int_{B_{\varrho^{1-\varepsilon}}} H(x, Du) dx \\ &\leq c \varrho^{\varepsilon\tilde{\alpha}p} \int_{B_{\varrho^{1-\varepsilon}}} H(x, Du) dx \end{aligned}$$

hold for a constant c depending on $n, p, \|Du\|_{L^p}, \tilde{L}$, and σ . Next, with $\Omega' \Subset \Omega$ being an open subset, we define $R_1 = \min\{R_0, \text{dist}(\Omega', \partial\Omega)\}/4$ where R_0 has been determined in (5.20) and is such that (5.21) holds for $0 < \varrho < R \leq R_0$ (and therefore, in particular, R_1 depends only on the quantities $n, p, \|Du\|_{L^p}, \text{dist}(\Omega', \partial\Omega)$). For every $x_0 \in \Omega'$ and $0 < \rho < R_1^{1/(1-\varepsilon)}$, we apply (5.25) and (5.21) between the radii $\varrho^{1-\varepsilon}, R_1$, and with $\delta = \varepsilon\tilde{\alpha}p/[2(1-\varepsilon)]$ to deduce that

$$\begin{aligned} \int_{B_\varrho} |Du - (Du)_{B_\varrho}|^p dx &\leq c \varrho^{\varepsilon\tilde{\alpha}p - (1-\varepsilon)\delta} \int_{B_{R_1}} H(x, Du) dx \\ &= c \varrho^{\varepsilon\tilde{\alpha}p/2} \int_{B_{R_1}} H(x, Du) dx . \end{aligned}$$

By a well-known characterisation of Hölder continuity due to Campanato and Meyers, the previous inequality and the fact that $\Omega' \Subset \Omega$ is arbitrary, imply that $Du \in C_{\text{loc}}^{0,\beta}(\Omega; \mathbb{R}^n)$ for

$$\beta = \frac{\varepsilon \tilde{\alpha}}{2} = \frac{\tilde{\alpha} \sigma}{2(4\tilde{\alpha}p + \sigma + 4n)}$$

with the corresponding estimate

$$[Du]_{\beta;\Omega'} \leq c(n, p, \|Du\|_{L^p}, \tilde{L}, R_1) \left(\int_{\Omega} H(x, Du) dx \right)^{1/p},$$

see (5.19) for the notation. This completes the proof of (5.23).

In order to complete the proof we show the validity of (5.24). It is sufficient to do this for radii $0 < \varrho \leq R/2$ and with $R \leq 1/e$; in the case $R/2 < \varrho \leq R$ the inequality follows by trivial means. For this, we consider $a_i(R) = \inf_{B_R} a(\cdot)$ as in (5.2) and we define v to be the solution to the minimum problem in (5.7). Using the general property

$$\int_{B_\varrho} |Du - (Du)_{B_\varrho}|^p dx \leq 2^p \int_{B_\varrho} |Du - z|^p dx,$$

that holds for every $z \in \mathbb{R}^n$, we have

$$\begin{aligned} \int_{B_\varrho} |Du - (Du)_{B_\varrho}|^p dx &\leq 2^p \int_{B_\varrho} |Du - (Dv)_{B_\varrho}|^p dx \\ (5.26) \qquad \qquad \qquad &\leq 4^p \int_{B_\varrho} |Dv - (Dv)_{B_\varrho}|^p dx + 4^p \int_{B_\varrho} |Du - Dv|^p dx. \end{aligned}$$

To estimate the first term on the right-hand side, we apply (3.17) from Theorem 3.1 to v (with B_R there replaced by $B_{R/2}$ here and with $a_0 \equiv a_i(R)$) and we recall the minimality property of v . For some positive constants c and $\tilde{\alpha}$, depending only on n, p , we get

$$\begin{aligned} (5.27) \quad &\int_{B_\varrho} |Dv - (Dv)_{B_\varrho}|^p dx \\ &\leq c \left(\frac{\varrho}{R} \right)^{\tilde{\alpha}p} \int_{B_{R/2}} (|Dv|^p + a_i(R)|Dv|^p \log(e + |Dv|)) dx \\ &\leq c \left(\frac{\varrho}{R} \right)^{\tilde{\alpha}p} \int_{B_R} (|Du|^p + a_i(R)|Du|^p \log(e + |Du|)) dx \\ &\leq c \left(\frac{\varrho}{R} \right)^{\tilde{\alpha}p} \int_{B_R} H(x, Du) dx. \end{aligned}$$

As for the second term on the right-hand side of (5.26), we trivially have that

$$\int_{B_\varrho} |Du - Dv|^p dx \leq c \left(\frac{R}{\varrho} \right)^n \int_{B_{R/2}} |Du - Dv|^p dx.$$

If $p \geq 2$, by (3.10), Lemma 5.2, and the Hölder continuity of $a(\cdot)$ as in the assumption (5.22), we obtain

$$\begin{aligned} \int_{B_{R/2}} |Du - Dv|^p dx &\leq \int_{B_{R/2}} |V_p(Du) - V_p(Dv)|^2 dx \\ &\leq \omega(R) \log \left(\frac{1}{R} \right) \int_{B_R} H(x, Du) dx \\ &\leq \frac{cR^{\sigma/2}}{\sigma} \int_{B_R} H(x, Du) dx \end{aligned}$$

for some constant c depending only on $n, p, \tilde{L}, \|Du\|_{L^p}$. Instead, when $p < 2$, by (3.9), Hölder's inequality, applied with conjugate with exponents $2/p$ and $2/(2-p)$, Lemma 5.2, and again the minimality of v , we estimate

$$\begin{aligned}
& \int_{B_{R/2}} |Du - Dv|^p dx \\
& \leq c \int_{B_{R/2}} |V_p(Du) - V_p(Dv)|^p (|Du| + |Dv|)^{p(2-p)/2} dx \\
& \leq c \left(\int_{B_{R/2}} |V_p(Du) - V_p(Dv)|^2 dx \right)^{p/2} \left(\int_{B_{R/2}} (|Du| + |Dv|)^p dx \right)^{(2-p)/2} \\
& \leq c \left(\int_{B_{R/2}} |V_p(Du) - V_p(Dv)|^2 dx \right)^{p/2} \\
& \quad \cdot \left(\int_{B_{R/2}} (|Du| + a_i(R)|Du|^p \log(e + |Du|)) dx \right)^{(2-p)/2} \\
& \leq c R^{\sigma/4} \int_{B_R} H(x, Du) dx.
\end{aligned}$$

The above inequality again holds for some constant c depending only on n, p, \tilde{L}, σ and $\|Du\|_{L^p}$. In both cases we conclude that

$$(5.28) \quad \int_{B_\varrho} |Du - Dv|^p dx \leq c R^{\sigma/4} \left(\frac{R}{\varrho} \right)^n \int_{B_R} H(x, Du) dx$$

holds for a constant c depending on $n, p, \|Du\|_{L^p}, \tilde{L}$, and σ . By merging (5.27) and (5.28) to (5.26), we deduce that (5.24) holds. The proof of (5.24), and therefore the whole proof, is complete.

6. RELATED FUNCTIONALS AND VECTORIAL CASES

In this section we point out a few extensions of the results contained in this paper. First we deal with the vectorial case, i.e. when $u: \Omega \rightarrow \mathbb{R}^N$ for $N > 1$. It is then easy to see that the last two assertions of Theorem 2.4 still hold in this case. Indeed, the whole proof remains the same, the only thing to do is to verify the validity of Theorem 3.1 in the vectorial case, which is in turn ensured by the results in [27] applied with the choice $\varphi(t) = t^p + a_0 t^p \log(e + t)$ as in (3.11). Indeed, the result in [27] allows to conclude with the Hölder continuity of the gradient of minima of functionals of the type

$$w \mapsto \int_{\Omega} \varphi(|Dw|) dx$$

where $\varphi(\cdot)$ is a function of the type described in Section 3.2 and satisfying (3.3) together with the Hölder type condition

$$(6.1) \quad |\varphi''(t+s) - \varphi''(t)| \leq \tilde{c}_\varphi \varphi''(t) \left(\frac{|s|}{t} \right)^\beta$$

for some $c > 0, \beta \in (0, 1]$ and for all $t > 0, s \in \mathbb{R}$ with $|s| < t/2$. Such assumptions are clearly verified by the function $\varphi(\cdot)$ defined in (3.11) and for a constant \tilde{c}_φ which is independent of a_0 ; this point is crucial in the above proofs. In particular, the results in [27] allow to conclude with the validity of (3.16) and (3.18); at this stage the proof of Theorem 3.1, and therefore of Theorem 2.4, follows as in the scalar case treated in the previous sections. We notice that assumption (6.1) is actually needed only in the vectorial case $N > 1$, while in the scalar case (3.3) suffices to

conclude with the result (and in particular with Theorem 3.1) by the analogous results in [43]. Summarising, we can state the following:

Theorem 6.1. *Let $u \in W^{1,1}(\Omega)$, $u: \Omega \rightarrow \mathbb{R}^N$, $N > 1$, be a vector valued local minimiser of the functional \mathcal{P}_{\log} defined in (2.5) and assume that the function $a(\cdot)$ is non-negative and bounded. Let $\omega(\cdot)$ be its modulus of continuity in the sense of (2.6) and let l be as in (2.7). Then*

- if $l = 0$, then $u \in C_{\text{loc}}^{0,\beta}(\Omega)$ for every $\beta \in (0, 1)$
- if $\omega(r) \lesssim r^\sigma$ with $\sigma \in (0, 1)$, then Du is locally Hölder continuous in Ω .

When looking at the functional \mathcal{P}_{\log} , it is clear that the main feature is the $L^p \log L$ -growth of the integrand with to the gradient-variable at those points x such that $a(x) > 0$. On the other hand, since the considered functional is degenerate on the zero set of the gradient, we might wonder what happens if we modify the integrand in such a way that also when the gradient approaches zero there is an unbalance between the two terms in the integrand. For instance, we might want to consider the related functional defined by

$$\tilde{\mathcal{P}}_{\log}(w, \Omega) := \int_{\Omega} [|Dw|^p + a(x)|Dw|^p \log(1 + |Dw|)] dx$$

which mixes-up the one in (1.2) and the one in (1.6) via the coefficient $a(\cdot)$. Indeed, by setting

$$(6.2) \quad \tilde{H}(x, z) := |z|^p + a(x)|z|^p \log(1 + |z|),$$

we see that when $|z|$ is small we then have

$$\tilde{H}(x, z) \approx \begin{cases} |z|^p + a(x)|z|^{p+1} & \text{if } a(x) > 0 \\ |z|^p & \text{if } a(x) = 0 \end{cases}$$

and therefore a different type of degeneracy occurs at the phase transition. Let us briefly show how the results proved in this paper extend to this case too. Indeed, Theorem 4.1 is just based on assumption (4.2). We can replace the function $H(\cdot)$ defined in (4.3) by $\tilde{H}(\cdot)$ which has been just defined in (6.2), and the proof follows exactly as before, with very minor variants. As far as Theorem 2.4 is concerned, we once again have to verify the applicability of Theorem 3.1, replacing $\log(e + |Dv|)$ by $\log(1 + |Dv|)$ in (3.16)-(3.17). To this aim we have to check that (3.3) and (6.1) are satisfied for the choice $\varphi(t) = t^p + a_0 t^p \log(1 + t)$; then the results of [27] again ensure the validity of (3.16) and (3.18) and Theorem 2.4 follows. All these arguments extend to the vectorial case as well: in turn, a direct computation leads to observe that (3.3) and (6.1) are again satisfied.

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REFERENCES

- [1] Acerbi E. & Bouchitté G. & Fonseca I.: Relaxation of convex functionals: the gap problem. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **20** (2003), 359–390.
- [2] Acerbi E. & Fusco N.: A transmission problem in the calculus of variations. *Calc. Var. Partial Differ. Equ.* **2** (1994), 1–16.
- [3] Acerbi E. & Mingione G.: Regularity results for a class of functionals with non-standard growth. *Arch. Rat. Mech. Anal.* **156** (2001), 121–140.

- [4] Acerbi E. & Mingione G.: Regularity results for electrorheological fluids: the stationary case. *C. R. Math. Acad. Sci. Paris* **334** (2002), 817–822.
- [5] Alkhutov Y.A.: On the Hölder continuity of $p(x)$ -harmonic functions. *Mat. Sb.* **196** (2) (2005), 3–28 (Russian); translation in *Sb. Math.* **196** (1-2) (2005), 147–171.
- [6] Ambrosio L. & Pinamonti A. & Speight G.: Weighted Sobolev spaces on metric measure spaces. Preprint 2014, <http://arxiv.org/abs/1406.3000>
- [7] Baroni P.: Riesz potential estimates for a general class of quasilinear equations. *Calc. Var. Partial Differ. Equ.*, doi: 10.1007/s00526-014-0768-z
- [8] Bildhauer M. & Fuchs M.: $C^{1,\alpha}$ -solutions to non-autonomous anisotropic variational problems. *Calc. Var. Partial Differ. Equ.* **24** (2005), 309–340.
- [9] Breit D.: New regularity theorems for non-autonomous variational integrals with (p, q) -growth. *Calc. Var. Partial Differ. Equ.* **44** (2012), 101–129.
- [10] Byun S. & Cho Y. & Wang L.: Calderón-Zygmund theory for nonlinear elliptic problems with irregular obstacles. *J. Funct. Anal.* **263** (2012), 3117–3143.
- [11] Carozza M. & Kristensen J. & Passarelli di Napoli A.: Higher differentiability of minimisers of convex variational integrals. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **28** (2011), 395–411.
- [12] Carozza M. & Kristensen J. & Passarelli di Napoli A.: Regularity of minimisers of autonomous convex variational integrals. *Ann. Scu. Sup. Pisa. Cl. Sci. (V)*, doi: 10.2422/2036-2145.201208_005.
- [13] Carozza M. & Kristensen J. & Passarelli di Napoli A.: On the validity of the Euler-Lagrange system. *Comm. Pure Appl. Anal.* **14** (2015), 51–62.
- [14] Chen Y. & Levine S. & Rao R.: Variable exponent, linear growth functionals in image restoration. *SIAM J. Appl. Math.* **66** (2006), 1383–1406.
- [15] Colombo M. & Mingione G.: Regularity for double phase variational problems. *Arch. Rat. Mech. Anal.*, doi: 10.1007/s00205-014-0785-2
- [16] Colombo M. & Mingione G.: Bounded minimisers of double phase variational integrals, to appear.
- [17] Coscia A. & Mingione G.: Hölder continuity of the gradient of $p(x)$ -harmonic mappings. *C. R. Acad. Sci. Paris Sér. I Math.* **328** (1999), 363–368.
- [18] Cruz-Uribe D. & Fiorenza A.: *Variable Lebesgue spaces. Foundations and harmonic analysis.* Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, Heidelberg, 2013.
- [19] Cupini G. & Marcellini P. & Mascolo E.: Local boundedness of solutions to quasilinear elliptic systems. *manuscripta math.* **137** (2012), 287–315.
- [20] Cupini G. & Marcellini P. & Mascolo E.: Existence and regularity for elliptic equations under p, q -growth. *Adv. Diff. Equ.* **19** (2014), 693–724.
- [21] Cupini G. & Marcellini P. & Mascolo E.: Local boundedness of minimisers with limit growth conditions, to appear.
- [22] De Giorgi E.: Sulla differenziabilità e l’analiticità delle estremali degli integrali multipli regolari, *Mem. Accad. Sci. Torino, Cl. Sci. Fis. Mat. Nat. (III)* **125** (3) (1957), 25–43.
- [23] E. DiBenedetto: $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, *Nonlinear Anal.* **7** (1983), 827–850.
- [24] Diening L.: Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces. *Bulletin des Sciences Mathématiques* **129** (2005), 657–700.
- [25] Diening L. & Ettwein F.: Fractional estimates for non-differentiable elliptic systems with general growth. *Forum Math.* **20** (2008), 523–556.
- [26] Diening L. & Harjulehto P. & Hästö P. & Růžička M.: *Lebesgue and Sobolev spaces with variable exponents.* Lecture Notes in Mathematics, 2017. Springer, Heidelberg, 2011.
- [27] Diening L. & Stroffolini B. & Verde A.: Everywhere regularity of functionals with φ -growth. *manuscripta math.* **129** (2009), 449–481.
- [28] Esposito L. & Leonetti F. & Mingione G.: Sharp regularity for functionals with (p, q) growth. *J. Diff. Equ.* **204** (2004), 5–55.
- [29] Evans L.C.: A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic p.d.e. *J. Diff. Equ.* **45** (1982), 356–373.
- [30] Fabes E.B. & Kenig C. E. & Serapioni R.: The local regularity of solutions of degenerate elliptic equations. *Commun. Partial Differ. Equ.* **7** (1982), 77–116.
- [31] Fan X. & Zhao D.: A class of De Giorgi type and Hölder continuity. *Nonlinear Anal. TMA* **36** (1999), 295–318.
- [32] Fonseca I. & Malý J.: Relaxation of multiple integrals below the growth exponent. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **14** (1997), 309–338.
- [33] Fonseca I. & Malý J. & Mingione G.: Scalar minimisers with fractal singular sets. *Arch. Rat. Mech. Anal.* **172** (2004), 295–307.
- [34] Fusco N. & Sbordone C.: Higher integrability of the gradient of minimizers of functionals with nonstandard growth conditions. *Comm. Pure Appl. Math.* **43** (1990), 673–683.

- [35] Gehring F. W.: The L^p -integrability of the partial derivatives of a quasiconformal mapping. *Acta Math.* **130** (1973), 265–277.
- [36] Giaquinta M. & Giusti E.: On the regularity of the minima of variational integrals. *Acta Math.* **148** (1982), 31–46.
- [37] Giusti E.: *Direct methods in the calculus of variations*. World Scientific Publishing Co., Inc., River Edge, NJ, 2003.
- [38] Iwaniec T. & Verde A.: On the operator $\mathcal{L}(f) = f \log |f|$. *J. Funct. Anal.* **169** (1999), 391–420.
- [39] Kristensen J.: Lower semicontinuity in Sobolev spaces below the growth exponent of the integrand. *Proc. Roy. Soc. Edinburgh Sect. A* **127** (1997), 797–817.
- [40] Kristensen J. & Mingione G.: The singular set of minima of integral functionals. *Arch. Rat. Mech. Anal.* **180** (2006), 331–398.
- [41] Ladyzhenskaya O.A. & Ural'tseva N.N.: *Linear and quasilinear elliptic equations. Second edition* (Russian) “Nauka”, Moscow 1973.
- [42] Lewis J. L.: Regularity of the derivatives of solutions to certain degenerate elliptic equations. *Indiana Univ. Math. J.* **32** (1983), 849–858.
- [43] Lieberman G.M.: The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations. *Commun. Partial Differ. Equ.* **16** (1991), 311–361.
- [44] Lieberman G.M.: Gradient estimates for a class of elliptic systems. *Ann. Mat. Pura Appl. (IV)* **164** (1993), 103–120.
- [45] Lindqvist P.: Notes on the p -Laplace equation. *Univ. Jyväskylä, Report* 102, (2006).
- [46] Maeda, F.Y. & Mizuta Y. & Ohno T. & Shimomura T.: Boundedness of maximal operators and Sobolev inequality on Musielak-Orlicz-Morrey spaces. *Bull. Math. Sci.* **137** (2013), 76–96.
- [47] Manfredi J.J.: Regularity for minima of functionals with p -growth. *J. Diff. Equ.* **76** (1988), 203–212.
- [48] Manfredi J.J.: Regularity of the gradient for a class of nonlinear possibly degenerate elliptic equations. *Ph.D. Thesis*. University of Washington, St. Louis, 1986.
- [49] Marcellini P.: On the definition and the lower semicontinuity of certain quasiconvex integrals. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **3** (1986), 391–409.
- [50] Marcellini P.: Regularity of minimisers of integrals of the calculus of variations with non standard growth conditions. *Arch. Rat. Mech. Anal.* **105** (1989), 267–284.
- [51] Marcellini P.: Regularity and existence of solutions of elliptic equations with p, q -growth conditions. *J. Diff. Equ.* **90** (1991), 1–30.
- [52] Mingione G. & Mucci D.: Integral functionals and the gap problem: sharp bounds for relaxation and energy concentration. *SIAM J. Math. Anal.* **36** (2005), 1540–1579.
- [53] Mingione G. & Siepe F.: Full $C^{1,\alpha}$ -regularity for minimisers of integral functionals with $L \log L$ -growth. *Z. Anal. Anwendungen* **18** (1999), 1083–1100.
- [54] Schmidt T.: Regularity of minimisers of $W^{1,p}$ -quasiconvex variational integrals with (p, q) -growth. *Calc. Var. Partial Differ. Equ.* **32** (2008), 1–24.
- [55] Schmidt T.: Regularity of relaxed minimisers of quasiconvex variational integrals with (p, q) -growth. *Arch. Rat. Mech. Anal.* **193** (2009), 311–337.
- [56] Musielak J.: *Orlicz spaces and Modular spaces*. Springer-Verlag, Berlin, 1983.
- [57] Růžička M.: *Electrorheological fluids: modeling and mathematical theory*. Lecture Notes in Mathematics, 1748. Springer-Verlag, Berlin, 2000.
- [58] Surnachev M. D.: Density of smooth functions in weighted Sobolev spaces with variable exponents. *Doklady Math.* **89** (2014), 146–150.
- [59] Uhlenbeck K.: Regularity for a class of non-linear elliptic systems. *Acta Math.* **138** (1977), 219–240.
- [60] Ural'tseva N.N.: Degenerate quasilinear elliptic systems. *Zap. Na. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **7** (1968), 184–222.
- [61] Ural'tseva N.N. & Urdaletova A.B.: The boundedness of the gradients of generalized solutions of degenerate quasilinear nonuniformly elliptic equations *Vestnik Leningrad Univ. Math.* **19** (1983) (russian), english. tran.: **16** (1984), 263–270.
- [62] Zhikov V.V.: Averaging of functionals of the calculus of variations and elasticity theory. *Izv. Akad. Nauk SSSR Ser. Mat.* **50** (1986), 675–710.
- [63] Zhikov V.V.: Lavrentiev phenomenon and homogenization for some variational problems. *C. R. Acad. Sci. Paris Sér. I Math.* **316** (1993), 435–439.
- [64] Zhikov V.V.: On Lavrentiev's Phenomenon. *Russian J. Math. Phys.* **3** (1995), 249–269.
- [65] Zhikov V.V.: On some variational problems. *Russian J. Math. Phys.* **5** (1997), 105–116.
- [66] Zhikov V.V. & Kozlov S. M. & Oleinik O. A.: *Homogenization of differential operators and integral functionals*. Springer-Verlag, Berlin, 1994.

- [67] Zhikov V.V.: Density of smooth functions in weighted Sobolev spaces. Doklady Math. **88** (2013), 669–673.

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