# LOSS OF REGULARITY FOR THE CONTINUITY EQUATION WITH NON-LIPSCHITZ VELOCITY FIELD

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Abstract. We consider the Cauchy problem for the continuity equation in space dimension $d \geq 2$. We construct a divergence-free velocity field uniformly bounded in all Sobolev spaces $W^{1,p}$, for $1 \leq p < \infty$, and a smooth compactly supported initial datum such that the unique solution to the continuity equation with this initial datum and advecting field does not belong to any Sobolev space of positive fractional order at any positive time. We also construct velocity fields in $W^{r,p}$, with $r > 1$, and solutions of the continuity equation with these velocities that exhibit some loss of regularity, as long as the Sobolev space $W^{r,p}$ does not embed in the space of Lipschitz functions. Our constructions are based on examples of optimal mixers from the companion paper *Exponential self-similar mixing by incompressible flows* (Preprint arXiv:1605.02090), and have been announced in *Exponential self-similar mixing and loss of regularity for continuity equations* (C. R. Math. Acad. Sci. Paris, **352** (2014), no. 11).

## 1. Introduction

We study the issue of *propagation of regularity* of the initial datum $\bar{\rho}$ for solutions $\rho$ to the Cauchy problem for the continuity equation in $d$ space dimensions:

$$\begin{cases} \partial_t \rho + \operatorname{div}(u\rho) = 0 \\ \rho(0, \cdot) = \bar{\rho}\,, \end{cases} \tag{1}$$

where the velocity field $u = u(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and the datum $\bar{\rho} = \bar{\rho}(x) : \mathbb{R}^d \to \mathbb{R}$ are given, and $\rho = \rho(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$. We restrict to velocity fields for which the (distributional) spatial divergence $\operatorname{div} u$ vanishes, in which case the continuity equation coincides with the transport equation $\partial_t \rho + u \cdot \nabla \rho = 0$. Specific regularity assumptions on the velocity field will be introduced later in the paper.

By *propagation of regularity* we mean the following statement: if $\bar{\rho}$ belongs to some regularity space, then the solution $\rho(t, \cdot)$ at time $t > 0$ belongs to the same regularity space with an estimate on the norm.

The question whether propagation of regularity holds for solutions of equation (1) has a classical and simple answer when the velocity field is regular enough, namely Lipschitz continuous with respect to space uniformly with respect to time, i.e.,

$$|u(t,x) - u(t,y)| \leq L|x - y|, \tag{2}$$

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for any $t \geq 0$ and $x$, $y \in \mathbb{R}^d$, for some constant $L > 0$ depending on $u$, but not on $t$, $x$, $y$. The infimum over all $L$ satisfying the above estimate is called the Lipschitz constant of $u$ and denoted by $\mathrm{Lip}(u)$.

With this regularity on the velocity field, the classical Cauchy-Lipschitz theory applies, and the Cauchy problem for the continuity equation (1) has a unique solution, which is moreover transported by the unique flow of the velocity field $u$, that is,

$$\rho(t,x) = \bar{\rho}\big(X(t,\cdot)^{-1}(x)\big)\,, \tag{3}$$

where the flow $X = X(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is the solution of

$$\begin{cases} \dot{X}(t,x) = u(t, X(t,x)) \\ X(0,x) = x\,. \end{cases} \tag{4}$$

Grönwall's estimate ensures that the flow and its inverse are Lipschitz continuous with respect to space with a bound on their Lipschitz constant that is exponential in time and in $L$, where $L$ is as the constant appearing in (2):

$$\mathrm{Lip}\,\big(X(t,\cdot)\big) \leq \exp(tL)\,, \qquad \mathrm{Lip}\,\big(X(t,\cdot)^{-1}\big) \leq \exp(tL). \tag{5}$$

In particular, from (3) it follows that

$$\mathrm{Lip}\,\big(\rho(t,\cdot)\big) \leq \mathrm{Lip}\,(\bar{\rho})\,\exp(tL)\,, \tag{6}$$

and an analogous result holds for the Hölder regularity of the solution. Alternatively, the exponential bound in (6) can be proved by performing simple energy estimates directly on the continuity equation (1).

When the velocity field is not Lipschitz continuous, but still retains some weak regularity, a theory of well posedness for the continuity equation (1) and the ordinary differential equation (4) has been developed, starting with the seminal works by DiPerna-Lions [10] and Ambrosio [3]. In these two papers, the velocity field is assumed to have, respectively, Sobolev or $BV$ regularity with respect to the spatial variables, with an integral dependence on time. (By $BV$ regularity we mean that the velocity is of bounded variation.) In this setting, one can prove existence, uniqueness, and stability of weak solutions of the partial differential equation (PDE) (1), and of suitable flows (the so-called *regular Lagrangian flows*), solutions of the ordinary differential equation (ODE) (4). (See [4] for a recent survey on research in this area.)

The well-posedness of the continuity equation with only Sobolev or $BV$ velocities makes it rather natural to ask whether any propagation of regularity for solutions of (1) holds when the advecting field has only the weak regularity in [3, 10]. A positive answer to this question would be very relevant in particular for applications to non-linear problems, as propagation of regularity would imply strong compactness of solutions, allowing the use of approximation schemes for non-linear problems that include continuity equations.

To the best of our knowledge, the literature addressing this question is rather limited. Several results of classical flavor for "almost Lipschitz" velocity fields (i.e., with a modulus of continuity satisfying the Osgood condition) are collected for instance in [5]. Whether or not any propagation of regularity holds seems a subtle question that, in particular, will depend on dimension. As a matter of fact, while in one space dimension it is not difficult to see that $BV$ regularity of the initial datum is propagated in time for solutions of the transport equation with a Sobolev velocity field, without additional conditions on its divergence, in [7] the authors show that, in two space dimensions, neither $BV$ regularity nor continuity of the data are preserved under the flow, if the velocity field lies in the intersection of a Sobolev and a Hölder space. Their construction is based on the lack of pointwise uniqueness for solutions of (4), which can be used to generate trajectories that collapse quickly by time reversion. See also [11] for a related analysis concerning $H^{3/2}$ velocity fields on the circle.

On the other hand, a few positive results on propagation of some (low) regularity have become available recently in the literature. The quantitative estimates for regular Lagrangian flows in [8] show that any regular Lagrangian flow can be approximated (in the Lusin sense) with Lipschitz maps, with a quantitative control on the Lipschitz constant of the approximation that depends exponentially on the size of the neglected set. This approximation result implies the propagation of a suitable "Lipexp regularity" of the initial datum (see [8]). More recently, in [6] the authors prove preservation under the flow of "a logarithm of a derivative" for the solution of the continuity equation, without even assuming boundedness of the divergence of the velocity field, and use it for a new theory of existence of solutions to the compressible Navier-Stokes equations. This result is obtained by means of a delicate argument, which utilizes the propagation of a weighted norm of the solution, with a suitable choice of weights, via an adjoint problem. See also [15] for a related approach in the context of incompressible velocity fields, described in terms of Fourier multipliers and proved with the use of multilinear harmonic analysis [16].

In view of the above results, it is interesting to try and understand whether any regularity measured in terms of weak derivatives in standard ways, e.g. Sobolev regularity, is propagated when the advecting velocity has only Sobolev or $BV$ regularity. This is again a rather delicate question because the flow map may be discontinuous.

In this article, we work directly with the continuity equation, rather than with the flow induced by the advecting velocity, that is, we employ directly the PDE and we do not explicitly refer to the underlying ODE for the flow. We exploit results on optimal mixing, which give a sharp rate of decay to zero of negative norms, and interpolation to show blow up on positive norms, whenever these are available.

Our main result shows that, for a Sobolev but not Lipschitz velocity field, the (fractional) Sobolev regularity of the data is not maintained for positive time. The loss of regularity is particularly striking when the velocity field has only one derivative in $L^p$, for $1 \leq p < \infty$. In this case, the initial data looses any positive fractional Sobolev regularity instantaneously, even when the datum is smooth and compactly supported.

**Theorem 1.** *Assume $d \geq 2$. There exists a bounded velocity field $v$ and a bounded solution $\theta$ of the continuity equation (1) with velocity $v$, both defined for $t \geq 0$, such that:*

*(i) for any $1 \leq p < \infty$ the velocity field $v$ is bounded in $\dot{W}^{1,p}(\mathbb{R}^d)$ uniformly in time;*
*(ii) the initial datum $\bar{\theta} := \theta(0, \cdot)$ belongs to $C_c^\infty(\mathbb{R}^d)$;*
*(iii) the solution $\theta(t, \cdot)$ does not belong to $\dot{H}^s(\mathbb{R}^d)$ for any $s > 0$ and $t > 0$.*

*Moreover both $v$ and $\theta$ are compactly supported in space, and can be taken to be smooth on the complement of a point in $\mathbb{R}^d$.*

*Remark* 2. This dramatic loss of regularity indicates a severe lack of continuity of the solution map in Sobolev spaces. Indeed, the inverse flow map $X^{-1}(t,\cdot)$ cannot be in any $\dot{H}^s(\mathbb{R}^d)$, with $s > 0$, for $t > 0$, otherwise the solution would retain Sobolev regularity from formula (3) as the initial data is smooth. The representation formula (3) is still valid, though not pointwise, for the unique weak and renormalized solution to (1). In a recent paper [14], Jabin gives in fact an example of a velocity field in the Sobolev space $W^{1,p}(\mathbb{R}^d)$, for $1 \le p < \infty$, with respect to the space variables, for which the associated regular Lagrangian flow is not in $W^{1,\tilde{p}}(\mathbb{R}^d)$ for any $\tilde{p}$. The proof exploits a randomization argument in the choice of the rotations of certain basic blocks. In fact, Jabin observed that his construct gives the same lack of regularity for the flow in case of velocity fields in $W^{r,p}(\mathbb{R}^d)$, provided $p < \frac{d}{r-1}$, where $r$ is even allow to be larger than 1 and $p$ to be smaller than 1. Compare this with our Theorem 3 below.

In the above Theorem 1, we employ homogeneous Sobolev spaces (defined below in Section 2), since our construction is based on a suitable rescaling argument of certain building blocks. Homogeneous Sobolev norm have the sharpest behavior under rescaling as opposed to non-homogeneous spaces. In our context, working with homogeneous spaces is not a restriction, since in the example we construct both the velocity field and the solution are uniformly bounded, hence the inhomogeneous norm will be bounded if and only if the homogeneous norm is bounded. Additionally, we state the loss of regularity for the solution only in terms of $L^2$-based Sobolev spaces $H^s$, $s \in \mathbb{R}_+$. While these norms have physical meaning, using $L^2$-based Sobolev norms is a technical restriction, which allows us to use equivalent Gagliardo-type norms for the fractional spaces $H^s$, $0 < s < 1$. These norms in turn imply an almost orthogonality property (see Lemma 7) in these spaces. We expect that a statement analogous to that of Theorem 1 can be made concerning blow-up of $L^p$-Sobolev norms, $1 < p < \infty$.

Our proof is based on a previous construction [2] (see also [1]) of optimal mixers in the context of two-dimensional incompressible flows under Sobolev bounds on the velocity. Informally, mixing of the passive scalar $\rho$ amounts to creation of stretching and filamentation in the scalar field that accounts for a small negative Sobolev norm of $\rho$. Since the $L^2$ norm of the solution is preserved under the flow, by interpolation positive Sobolev norms have to be large. The strategy of proof has two main steps:

**Step 1.** Construction of a basic element that saturates the Grönwall estimate (5)–(6) in integral sense: positive Sobolev norms of the solution grow exponentially in time. This construction is essentially taken from our previous paper [2].

**Step 2.** Iteration and scaling, which turn the exponential growth into an instantaneous blow up for the positive Sobolev norms of the solution, while still keeping a control on the regularity of the velocity field and the initial datum.

We notice that it is straightforward to saturate the Grönwall estimate for Step 1 above in pointwise sense. The relevance of our example is, instead, to saturate it in integral sense.

As a matter of fact, one can prove a complementary result to that in Theorem 1 in the setting of velocity fields with higher regularity, showing loss of some regularity. By higher regularity, we mean here in higher Sobolev space $W^{r,p}$ with $r > 1$, as long as they do not embed in the Lipschitz class.

**Theorem 3.** *Assume $d \geq 2$. Let $r > 1$ and $1 \leq p < \infty$ be such that $p < \frac{d}{r-1}$. Let $T > 0$ and $\sigma > 0$ be fixed. Then there exists a bounded velocity field $v$ and a solution $\theta$ of the continuity equation (1) with velocity $v$, both defined for $0 \leq t \leq T$, such that:*

*(i) the velocity field $v$ is bounded in $\dot{W}^{r,p}(\mathbb{R}^d)$ uniformly in time;*
*(ii) the initial datum $\bar{\theta} := \theta(0, \cdot)$ belongs to $\dot{H}^{\sigma}(\mathbb{R}^d)$;*
*(iii) the solution $\theta(t, \cdot)$ does not belong to $\dot{H}^{\sigma}(\mathbb{R}^d)$ for any $t > 0$.*

*Moreover, there is $\bar{\mu} = \bar{\mu}(r, p, d) < 1$ such that*

*(iv) for any $\mu > \bar{\mu}$ the solution $\theta(T, \cdot)$ does not belong to $\dot{H}^{\mu\sigma}(\mathbb{R}^d)$ at the final time $T$.*

*Moreover both $v$ and $\theta$ are compactly supported in space, and can be taken smooth on the complement of a point in $\mathbb{R}^d$.*

Some comments are in order.

(1) The loss of regularity in Theorem 3 is much weaker than that in Theorem 1. This is due to the fact that the examples of optimal mixers in [2] allow for exponential mixing only when the velocity field has a uniform in time bound on the first derivative in $L^p$. If the bound is on higher derivatives, then the mixing is only polynomial in time, which is not sufficient to show instantaneous blow up of the positive norms by rescaling. If we could construct an example of more than polynomial mixing with equibounds on higher derivatives, then we could prove a stronger version of Theorem 3. (See Theorem 8 for more details on the examples of optimal mixers from [2].)

(2) If $r < 1$ and we assume the velocity field is in $W^{r,p}$, then in general solutions are not unique. However, we can prove the existence of one solution as in Theorem 1.

(3) The solution in Theorem 3 is unique in $L^2$ if $p \geq 2$. If $\sigma > d/2$, the solution can be taken to be bounded, and is the unique one for any $p \geq 1$.

(4) In the proof of both theorems we do not discuss the case $p = 1$ for the velocity. This choice is by convenience, since we define Sobolev spaces using Fourier multipliers and this definition is not equivalent to the definition using weak derivatives and interpolation for $p = 1$ and $p = \infty$. Clearly, the result holds also for $p = 1$ as stated by embedding, as $W^{r,p} \subset W^{r,1}$ for any $p > 1$.

*Remark* 4. Further variants of Theorem 1 are possible. For example, by modifying the proof of Theorem 3, one can prove instantaneous loss of some regularity, but with further constraints on the regularity of the velocity field and of the initial data. Informally, the regularity index $r > 1$ must be sufficiently close to 1, $\sigma$ must be sufficiently large, and $s$ sufficiently close to $\sigma$. Furthermore, these conditions are coupled together. For sake of clarity and to keep the article contained, we opt not to include a precise statement and a proof of these variants.

*Remark* 5. Quoc-Hung Nguyen pointed out to us that the same strategy of proof of Theorems 1 and 3 can be used to construct a divergence-free velocity field in $W^{1,p}(\mathbb{R}^d)$, for a given $1 \leq p < \infty$, and an initial datum in $\dot{H}^{\sigma}(\mathbb{R}^d)$ for some $\sigma > 0$, such that some logarithmic regularity of the solution is instantaneously lost, that is, for $\alpha > 0$ sufficiently large $\int_{\mathbb{R}^d} \big|\log|\xi|\big|^{\alpha} |\hat{\theta}(t, \xi)|^2 \, d\xi = \infty$ for $t > 0$. Compare with the positive results in [8, 6, 15], which guarantees propagation of such a logarithmic regularity with $\alpha \leq 1$ if $1 < p \leq \infty$, and with $\alpha \leq 2$ if $2 \leq p \leq \infty$.

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## 2. Preliminaries: fractional Sobolev spaces

We begin by recalling some basic definitions and properties of homogeneous Sobolev spaces of real order that will be extensively used in this paper. For a complete exposition we refer the reader for instance to [5, 9, 12, 18].

Let $s \geq 0$ and $1 < p < \infty$. We say that a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ belongs to the homogeneous Sobolev space $\dot{W}^{s,p}(\mathbb{R}^d)$ if the Fourier transform $\hat{f}$ of $f$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and

$$\mathcal{F}^{-1}\left(|\xi|^s \hat{f}(\xi)\right) \in L^p(\mathbb{R}^d)\,. \tag{7}$$

If this is the case, we define $\|f\|_{\dot{W}^{s,p}(\mathbb{R}^d)}$ to be the $L^p(\mathbb{R}^d)$ norm of the function in (7).

If $s = k \in \mathbb{N}$, then $\dot{W}^{k,p}(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ coincides with the usual Sobolev space $W^{k,p}(\mathbb{R}^d)$ consisting of those $L^p(\mathbb{R}^d)$ functions possessing weak derivatives of order less or equal than $k$ in $L^p(\mathbb{R}^d)$. This equivalence however does not hold in the borderline cases $p = 1$ and $p = \infty$. We also recall the well-known fact that $\dot{W}^{r,p}(\mathbb{R}^d)$ continuously embeds into the space of Lipschitz continuous functions provided $p > \frac{d}{r-1}$.

We will be frequently interested in the behavior of homogeneous Sobolev norms under rescaling of a given function $f$. If we set

$$f_\lambda(x) = f\left(\frac{x}{\lambda}\right)\,,$$

then it holds

$$\|f_\lambda\|_{\dot{W}^{s,p}(\mathbb{R}^d)} = \lambda^{\frac{d}{p}-s} \|f\|_{\dot{W}^{s,p}(\mathbb{R}^d)}\,. \tag{8}$$

In the particular case $p = 2$ we use the notation $\dot{H}^s(\mathbb{R}^d)$. For $s \in \mathbb{R}$, a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ belongs to the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$ if the Fourier transform $\hat{f}$ of $f$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and

$$\|f\|^2_{\dot{H}^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{f}(\xi)|^2 \, d\xi < \infty\,. \tag{9}$$

As before, it holds

$$\|f_\lambda\|_{\dot{H}^s(\mathbb{R}^d)} = \lambda^{\frac{d}{2}-s} \|f\|_{\dot{H}^s(\mathbb{R}^d)}\,. \tag{10}$$

In fact, more general definitions can be given removing the requirement that $\hat{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, setting the entire theory in the context of tempered distributions modulo polynomials (as done for instance in [12]). In this paper we always deal with bounded functions with compact support, for which the local summability of the Fourier transform is guaranteed. Therefore we prefer to follow the somehow simplified approach above.

In our work, homogeneous spaces will be used only to measure the "size" of given functions and velocity fields, which in fact will be typically regular but with large norm. In particular, the issue of completeness (which holds only for specific values of $s$ given our definition) and seminorms will not arise. With abuse of language, we will then refer to seminorms as norms.

The following interpolation property holds. If $s_1 < s < s_2$ and $s = \vartheta s_1 + (1-\vartheta)s_2$, then

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} \leq \|f\|^{\vartheta}_{\dot{H}^{s_1}(\mathbb{R}^d)} \|f\|^{1-\vartheta}_{\dot{H}^{s_2}(\mathbb{R}^d)} . \tag{11}$$

In particular $\dot{H}^{s_1}(\mathbb{R}^d) \cap \dot{H}^{s_2}(\mathbb{R}^d) \subset \dot{H}^s(\mathbb{R}^d)$. Analogous inequalities hold for $\dot{W}^{s,p}(\mathbb{R}^d)$, but we will not need them in this paper.

We recall that, for every $0 < s < 1$, the $\dot{H}^s(\mathbb{R}^d)$ (semi) norm is equivalent (up to a dimensional factor) to the Gagliardo (semi) norm, given by :

$$\left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^2}{|x-y|^{d+2s}} \, dx \, dy \right)^{1/2} . \tag{12}$$

We recall that one can equivalently introduce Gagliardo (semi) norms that are $L^p$ based for $1 \leq p < \infty$:

$$\left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}} \, dx \, dy \right)^{1/p} .$$

However, the space of $L^p$ functions for which this norm is finite is a Besov space, namely the space $\dot{B}^s_{p,p}$, and $\dot{W}^{s,p} = \dot{B}^s_{p,p}$ if and only if $p = 2$ (see e.g. [18, Chapter 2]). This is one of the reasons why we state the loss of regularity in $L^2$-Sobolev spaces only.

Using (12), we prove a localization property in $\dot{H}^s$ in the following sense. If a function has compact support, then we exhibit a quantitative relation between the Sobolev norm of this function on the whole space and the Sobolev norm of the function localized on a set containing its support. This lemma is essentially Lemma 5.1 in [9], and we include a proof here for the reader's sake.

**Lemma 6.** *Let $0 < s < 1$. Let $K \subset \Omega \subset \mathbb{R}^d$ and assume* $\operatorname{dist}(K, \Omega^c) = \lambda > 0$. *Then, for every function $f \in \dot{H}^s(\mathbb{R}^d)$ with support contained in $K$, it holds*

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^2}{|x-y|^{d+2s}} \, dx \, dy \leq \int_{\Omega} \int_{\Omega} \frac{|f(x)-f(y)|^2}{|x-y|^{d+2s}} \, dx \, dy + \frac{C_d}{s} \frac{1}{\lambda^{2s}} \|f\|^2_{L^2(\mathbb{R}^d)} ,$$

*where $C_d$ is a dimensional constant.*

*Proof.* Since $f$ vanishes in the complement of $\Omega$, we have

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^2}{|x-y|^{d+2s}} \, dx \, dy = \int_{\Omega} \int_{\Omega} \frac{|f(x)-f(y)|^2}{|x-y|^{d+2s}} \, dx \, dy + 2 \int_{\Omega^c} \int_{\Omega} \frac{|f(x)|^2}{|x-y|^{d+2s}} \, dx \, dy .$$

Therefore we need to estimate the last integral. We can compute

$$\begin{aligned}\int_{\Omega^c}\int_{\Omega}\frac{|f(x)|^2}{|x-y|^{d+2s}}\,dx\,dy &= \int_{\Omega}|f(x)|^2\int_{\Omega^c}\frac{\mathbf{1}_K(x)}{|x-y|^{d+2s}}\,dy\,dx\\ &\le \int_{\Omega}|f(x)|^2\int_{B(x,\lambda)^c}\frac{\mathbf{1}_K(x)}{|x-y|^{d+2s}}\,dy\,dx\\ &= C_d\int_{\Omega}|f(x)|^2\int_{\lambda}^{\infty}\frac{r^{d-1}}{r^{d+2s}}\,dr = \frac{C_d}{2s}\frac{1}{\lambda^{2s}}\|f\|^2_{L^2(\mathbb{R}^d)}\,.\end{aligned}$$

This concludes the proof. □

We use the previous lemma to show a sort of "orthogonality property" for Sobolev functions with disjoint supports.

**Lemma 7.** *Let* $0 < s < 1$. *For* $i = 1, 2$, *let* $K_i \subset \Omega_i \subset \mathbb{R}^d$ *and assume* $\operatorname{dist}(K_i, \Omega_i^c) = \lambda_i > 0$. *Moreover, assume* $\Omega_1 \cap \Omega_2 = \emptyset$. *Then, given* $f_i \in \dot{H}^s(\mathbb{R}^d)$ *with support contained in* $K_i$, *there exists a dimensional constant* $C_d$ *such that:*

$$\|f_1+f_2\|^2_{\dot{H}^s(\mathbb{R}^d)} \ge \|f_1\|^2_{\dot{H}^s(\mathbb{R}^d)} + \|f_2\|^2_{\dot{H}^s(\mathbb{R}^d)} - \frac{C_d}{s}\left[\frac{1}{\lambda_1^{2s}}\|f_1\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{\lambda_2^{2s}}\|f_2\|^2_{L^2(\mathbb{R}^d)}\right].$$

*The above formula generalizes to:*

$$\left\|\sum_n f_n\right\|^2_{\dot{H}^s(\mathbb{R}^d)} \ge \limsup_{N\to\infty}\sum_{n=1}^{N}\left[\|f_n\|^2_{\dot{H}^s(\mathbb{R}^d)} - \frac{C_d}{s}\frac{1}{\lambda_n^{2s}}\|f_n\|^2_{L^2(\mathbb{R}^d)}\right]. \tag{13}$$

We note that, for $s = 0$, $s = 1$, we have true orthogonality:

$$\|f_1+f_2\|^2_{\dot{H}^s(\mathbb{R}^d)} = \|f_1\|^2_{\dot{H}^s(\mathbb{R}^d)} + \|f_2\|^2_{\dot{H}^s(\mathbb{R}^d)}\,,$$

owing to the local nature of the corresponding norms.

*Proof of Lemma 7.* It is enough to observe that

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|f_1(x)+f_2(x)-f_1(y)-f_2(y)|^2}{|x-y|^{d+2s}}\,dx\,dy\\ &\qquad\ge \int_{\Omega_1}\int_{\Omega_1}\frac{|f_1(x)-f_1(y)|^2}{|x-y|^{d+2s}}\,dx\,dy + \int_{\Omega_2}\int_{\Omega_2}\frac{|f_2(x)-f_2(y)|^2}{|x-y|^{d+2s}}\,dx\,dy\end{aligned}$$

and to apply the previous lemma. □

## 3. An example of exponential mixing

As outlined in the Introduction, the first step in proving our main result, Theorem 1, is the construction of a regular velocity field $u$ and of a regular solution $\rho$ to the continuity equation with velocity $u$ such that $u$ is Lipschitz uniformly in time and the positive Sobolev norms of $\rho$ grow exponentially in time.

Given that all Lebesgue norms of $\rho$ are constant in time, one way to achieve the exponential growth in time of the positive Sobolev norms is to ensure an exponential decay of the negative norms in view of the interpolation inequality (11). Negative Sobolev norms will decay in time if the flow of $u$ is sufficiently "mixing", which informally means that the action of the flow generates small scales in the scalar field. It was recently shown [13, 17] that, under a $W^{1,p}$ bound on $u$ uniform in time, where $1 < p \le \infty$, the rate of decay of negative Sobolev norms for the advected scalar is indeed at most exponential in time.

In [2], we constructed examples of velocity fields satisfying the required $W^{1,p}$ bound and passive scalar fields advected by $u$ that saturate the exponential rate of decay of the negative norms (see [19] for a related example). We will refer to these examples as "optimal mixers". We also consider velocity fields with higher regularity, see (14) and (15) below. Our examples are based on a quasi self-similar geometric construction.

**Theorem 8** ([2])**.** *Assume $d \ge 2$ and let $\mathcal{Q} \subset \mathbb{R}^d$ be the open cube with unit side centered at the origin of $\mathbb{R}^d$. There exist a velocity field $u \in C^\infty([0,+\infty[\times\mathbb{R}^d)$ and a corresponding (non trivial) solution $\rho \in C^\infty([0,+\infty[\times\mathbb{R}^d)$ of the continuity equation (1) such that:*

*(i) $u(t,\cdot)$ is bounded, divergence-free, and compactly supported in $\mathcal{Q}$ for any $t$;*
*(ii) $\rho(t,\cdot)$ has zero average and is bounded and compactly supported in $\mathcal{Q}$ for any $t$;*
*(iii) $u(t,\cdot)$ belongs to $\mathrm{Lip}\,(\mathbb{R}^d)$ uniformly in time;*
*(iv) for any $r \ge 0$ and $1 \le p \le \infty$, there exist constants $b > 0$ and $B_r > 0$ such that*

$$\|u(t,\cdot)\|_{\dot W^{r,p}(\mathbb{R}^d)} \le B_r \exp\big((r-1)\,bt\big)\,; \tag{14}$$

*(v) for any $0 < s < 2$, there exist constants $c > 0$ and $\hat C_s > 0$ such that*

$$\|\rho(t,\cdot)\|_{\dot H^{-s}(\mathbb{R}^d)} \le \hat C_s \exp(-sct) \qquad t \ge 0\,. \tag{15}$$

*Remark* 9. By the interpolation formula (11) with $s_1 = -s$, $s_2 = s$, and $\vartheta = 1/2$, which reads

$$\|\rho(t,\cdot)\|_{L^2(\mathbb{R}^d)} \le \|\rho(t,\cdot)\|^{1/2}_{\dot H^{-s}(\mathbb{R}^d)}\|\rho(t,\cdot)\|^{1/2}_{\dot H^{s}(\mathbb{R}^d)}\,,$$

together with the fact that the $L^2$ norm of the solution $\rho$ is conserved in time, estimate (15) implies the following lower bound:

$$\begin{aligned}\|\rho(t,\cdot)\|_{\dot H^{s}(\mathbb{R}^d)} &\ge \|\rho(t,\cdot)\|^2_{L^2(\mathbb{R}^d)}\|\rho(t,\cdot)\|^{-1}_{\dot H^{-s}(\mathbb{R}^d)} \ge \|\rho(0,\cdot)\|^2_{L^2(\mathbb{R}^d)}\hat C_s^{-1}\exp(sct)\\ &= C_s\exp(sct) \quad \text{ for every } 0 < s < 2.\end{aligned} \tag{16}$$

This bound can be seen as saturating Grönwall inequality (5)–(6) and will be used in the proof of Theorems 1 and 3.

*Remark* 10. In fact, the example in [2] was constructed in two space dimensions. However, it is not difficult to modify such construction to make it $d$ dimensional. One can argue as follows. Call $\tilde u(t,x_1,x_2)$ and $\tilde\rho(t,x_1,x_2)$ the velocity field and solution constructed in [2]. Fix $\eta = \eta(x_3,\dots,x_d)$ smooth, with compact support in $B(0,1/4) \subset \mathbb{R}^{d-2}$ and $\eta = 1$ on $B(0,1/8)$. Moreover, fix $\bar\eta = \bar\eta(x_3,\dots,x_d)$ smooth, with compact support in $B(0,1/2) \subset \mathbb{R}^{d-2}$ and $\bar\eta = 1$ on $B(0,1/4)$. Then, setting

$$u(t,x_1,\dots,x_d) = \tilde u(t,x_1,x_2)\bar\eta(x_3,\dots,x_d)\,, \quad \rho(t,x_1,\dots,x_d) = \tilde\rho(t,x_1,x_2)\eta(x_3,\dots,x_d),$$

yields a velocity field $u$ and solution $\rho$ defined on $\mathbb{R}^d$ that satisfy the conclusions of Theorem 8.

## 4. Proof of Theorems 1 and 3

The main idea in the proof of both Theorems 1 and 3 consists in constructing the velocity field $v$ and the associated solution $\theta$ by patching together a countable number of velocity fields $u_n$ and solutions $\rho_n$ saturating Grönwall's inequality, the existence of which follows from Theorem 8 (see also Remark 9), after rescaling in both space and time, and in size as well for the solution, in a suitable manner. We then translate them so as to be supported in cubes that accumulate towards a point in space. The pairs $u_n$, $\rho_n$ serve as building blocks in our construction.

The choice of the rescaling parameters is the key point of the proof and is done in such a way that the regularity of the velocity field and that of the initial datum for the solution can be controlled, while at the same time any regularity of the solution at later times is destroyed.

In the proof of Theorem 1, we choose the parameters for the rescaling to ensure that the velocity field has Sobolev regularity uniformly in time, while we exploit that the rescaling in time can be chosen so that the rate of exponential growth of the Sobolev norms of the solution grows enough at each step to become in the limit an instantaneous blow up.

When $r > 1$, in view of estimate (14), the Sobolev norm of order $r$ of the velocity field is not uniformly bounded in time, rather it grows exponentially in time. One then needs to balance this exponential growth with the exponential growth for the Sobolev norms of the solution, which accounts for the weaker result in Theorem 3.

4.1. **A geometric construction.** Fix a sequence $\{\lambda_n\}$, with $\lambda_n > 0$ and $\lambda_n \downarrow 0$, to be determined later. For every $n$ consider an open cube $Q_n$ in $\mathbb{R}^d$ with side of length $3\lambda_n$. The first condition we impose on $\{\lambda_n\}$ is that

$$\left.\begin{array}{c}\{Q_n\} \text{ can be chosen to be contained} \\ \text{in a compact set and convergent to a point}\end{array}\right\} \qquad \Longleftarrow \qquad \sum_n \lambda_n < \infty\,. \tag{A}$$

If (A) is satisfied, we can choose the centers of the cubes so that all cubes are contained in a compact set in $\mathbb{R}^d$ and accumulate to a point, i.e., every open ball centered at such a point contains all the cubes but finitely many. By saying that condition (A) is satisfied, we mean that the condition on the right hand side, which implies the one on the left, is satisfied. The same will be true for conditions (B)–(D) (and variants) below.

Given $u$ and $\rho$ as in Theorem 8, we fix two more sequences $\{\tau_n\}$ and $\{\gamma_n\}$ with $\tau_n > 0$ and $\gamma_n > 0$ to be also determined later, where $\tau_n \downarrow 0$. Up to a translation, in each cube $Q_n$ we set

$$u_n(t,x) = \frac{\lambda_n}{\tau_n}\, u\left(\frac{t}{\tau_n}, \frac{x}{\lambda_n}\right)\,, \qquad \rho_n(t,x) = \gamma_n\, \rho\left(\frac{t}{\tau_n}, \frac{x}{\lambda_n}\right)\,. \tag{17}$$

It is immediate to check that $\rho_n$ is a solution of the continuity equation with velocity field $u_n$, that is

$$\partial_t \rho_n + \operatorname{div}(u_n \rho_n) = 0 \qquad \text{on } \mathbb{R}^d. \tag{18}$$

Moreover, we observe that for every $n$ we have

$$\operatorname{dist}(\operatorname{Supp} \rho_n\,,\, Q_n^c) \geq \lambda_n\,,$$

since the support of $\rho_n$ is contained in a square of side $\lambda_n$ with the same center as $Q_n$, the side of which has length $3\lambda_n$.

We set

$$v := \sum_n u_n\,, \qquad \theta := \sum_n \rho_n\,, \tag{19}$$

where convergence of both series is taken pointwise almost everywhere. We will prove later that, for a suitable choice of the three sequences of parameters, $\lambda_n$, $\tau_n$, $\gamma_n$, the function $\theta$ is a (weak) solution of the continuity equation with velocity field $v$:

$$\partial_t \theta + \operatorname{div}(v\theta) = 0\,. \tag{20}$$

4.2. **Regularity of the velocity field.** We estimate the $\dot{W}^{r,p}(\mathbb{R}^d)$ norm of the velocity field $v$, at a fixed time $t$:

$$\begin{aligned}
\|v(t,\cdot)\|_{\dot{W}^{r,p}(\mathbb{R}^d)} &\overset{(19)}{\le} \sum_n \|u_n(t,\cdot)\|_{\dot{W}^{r,p}(\mathbb{R}^d)} \overset{(17)}{=} \sum_n \frac{\lambda_n}{\tau_n}\left\| u\left(\frac{t}{\tau_n},\frac{\cdot}{\lambda_n}\right)\right\|_{\dot{W}^{r,p}(\mathbb{R}^d)} \\
&\overset{(8)}{=} \sum_n \frac{\lambda_n}{\tau_n}\lambda_n^{\frac{d}{p}-r}\left\| u\left(\frac{t}{\tau_n},\cdot\right)\right\|_{\dot{W}^{r,p}(\mathbb{R}^d)} \\
&\overset{(14)}{\le} \sum_n \frac{\lambda_n^{1-r+\frac{d}{p}}}{\tau_n} B_r \exp\left(\frac{(r-1)bt}{\tau_n}\right)\,,
\end{aligned}$$

where we have explicitly referenced which property we are using for each inequality. Therefore, the fact that $v(t,\cdot)$ belongs to $\dot{W}^{r,p}(\mathbb{R}^d)$ is implied by the following condition:

$$v(t,\cdot) \in \dot{W}^{r,p}(\mathbb{R}^d) \qquad \Longleftarrow \qquad \sum_n \frac{\lambda_n^{1-r+\frac{d}{p}}}{\tau_n} \exp\left(\frac{(r-1)bt}{\tau_n}\right) < \infty\,. \tag{B}$$

Moreover, the definition of $u_n$, which is given in (17), and the fact that the cubes $\{Q_n\}$ have been chosen to be pairwise disjoint, gives the following implication:

$$v(t,\cdot) \in L^\infty(\mathbb{R}^d) \text{ uniformly in } t \qquad \Longleftarrow \qquad \left\{\frac{\lambda_n}{\tau_n}\right\} \text{ bounded.} \tag{$\tilde{\mathrm{B}}$}$$

4.3. **Regularity of the initial datum.** For a given $\sigma \ge 0$, we estimate the $\dot{H}^\sigma(\mathbb{R}^d)$ norm of the initial datum $\bar\theta = \theta(0,\cdot)$ as follows:

$$\begin{aligned}
\|\bar\theta(\cdot)\|_{\dot{H}^\sigma(\mathbb{R}^d)} &\overset{(19)}{\le} \sum_n \|\rho_n(0,\cdot)\|_{\dot{H}^\sigma(\mathbb{R}^d)} \overset{(17)}{=} \sum_n \gamma_n \left\|\rho\left(0,\frac{\cdot}{\lambda_n}\right)\right\|_{\dot{H}^\sigma(\mathbb{R}^d)} \\
&\overset{(10)}{=} \sum_n \gamma_n \lambda_n^{\frac{d}{2}-\sigma}\|\rho\|_{\dot{H}^\sigma(\mathbb{R}^d)}\,.
\end{aligned}$$

Hence, we obtain the further condition

$$\bar\theta \in \dot{H}^\sigma(\mathbb{R}^d) \qquad \Longleftarrow \qquad \sum_n \gamma_n \lambda_n^{\frac{d}{2}-\sigma} < \infty\,. \tag{C}$$

In Theorem 1, we require that the solution $\theta$ is bounded, which follows if $\bar\theta$ is bounded. Using the definition of $\rho_n$ in (17) and again the fact that the cubes $\{Q_n\}$ have been chosen to be pairwise disjoint gives the following implication:

$$\theta(t,\cdot) \in L^\infty(\mathbb{R}^d) \text{ uniformly in } t \qquad \Longleftarrow \qquad \{\gamma_n\} \text{ bounded.} \tag{$\tilde{\mathrm{C}}$}$$

In Theorem 3, we only require that $\theta$ belongs to $L^2(\mathbb{R}^d)$ uniformly in time, which again follows if $\bar\theta \in L^2(\mathbb{R}^d)$. This requirement corresponds to condition (C) with $\sigma = 0$, that is,

$$\theta(t,\cdot) \in L^2(\mathbb{R}^d) \text{ uniformly in } t \qquad \Longleftarrow \qquad \sum_n \gamma_n \lambda_n^{\frac{d}{2}} < \infty\,. \tag{$\hat{\mathrm{C}}$}$$

4.4. **Loss of regularity for the solution.** To show loss of regularity for $\theta$ at any $t > 0$, it is enough to show that the solution does not belong to $\dot H^s(\mathbb{R}^d)$, for $0 < s < 1$, since $\theta$ is at least in $L^2$ (by condition ($\tilde{\mathrm{C}}$) or ($\hat{\mathrm{C}}$) and the compact support above) and $H^s(\mathbb{R}^d) = \dot H^s(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ forms a scale of spaces. We then fix an arbitrary $0 < s < 1$, and estimate the norm of $\theta$ in $\dot H^s(\mathbb{R}^d)$ from below as follows:

$$\begin{aligned}
\|\theta(t,\cdot)\|^2_{\dot H^s(\mathbb{R}^d)} &\overset{(13)}{\geq} \limsup_{N\to\infty} \sum_{n=1}^N \left( \|\rho_n(t,\cdot)\|^2_{\dot H^s(\mathbb{R}^d)} - \frac{C_d}{s}\frac{1}{\lambda_n^{2s}} \|\rho_n(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} \right) \\
&\overset{(17)}{=} \limsup_{N\to\infty} \sum_{n=1}^N \left( \gamma_n^2 \left\| \rho\left(\frac{t}{\tau_n}, \frac{\cdot}{\lambda_n}\right)\right\|^2_{\dot H^s(\mathbb{R}^d)} - \frac{C_d}{s}\frac{1}{\lambda_n^{2s}}\gamma_n^2 \left\| \rho\left(\frac{t}{\tau_n}, \frac{\cdot}{\lambda_n}\right)\right\|^2_{L^2(\mathbb{R}^d)} \right) \\
&\overset{(10)}{=} \limsup_{N\to\infty} \sum_{n=1}^N \gamma_n^2 \lambda_n^{d-2s} \left[ \left\| \rho\left(\frac{t}{\tau_n}, \cdot\right)\right\|^2_{\dot H^s(\mathbb{R}^d)} - \frac{C_d}{s} \left\| \rho\left(\frac{t}{\tau_n}, \cdot\right)\right\|^2_{L^2(\mathbb{R}^d)} \right] \\
&\overset{(16)}{\geq} \limsup_{N\to\infty} \sum_{n=1}^N \gamma_n^2 \lambda_n^{d-2s} \left[ C_s^2 \exp\left(\frac{2sct}{\tau_n}\right) - \frac{C_d \hat C_0}{s} \right],
\end{aligned}$$

where again we have referenced the property used at each step. In the last inequality, we have denoted the $L^2(\mathbb{R}^d)$ norm of $\rho$, which is conserved in time, by $\hat C_0$. Therefore, since we require $\tau_n \downarrow 0$, for any given $t > 0$ we have

$$\theta(t,\cdot) \notin \dot H^s(\mathbb{R}^d) \qquad \Longleftarrow \qquad \sum_n \gamma_n^2 \lambda_n^{d-2s} \exp\left(\frac{2sct}{\tau_n}\right) = \infty\,. \tag{D}$$

4.5. **Verify that $\theta$ is a solution of the continuity equation with velocity $v$.** We assume that the sequences $\{\lambda_n\}$, $\{\tau_n\}$, and $\{\gamma_n\}$ have been chosen so that conditions (B), ($\tilde{\mathrm{B}}$), and ($\tilde{\mathrm{C}}$) (or ($\hat{\mathrm{C}}$)) are satisfied. Then, the series in (19) converge strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ on any finite time interval. This fact, together with the validity of (18) and the fact that the cubes $\{Q_n\}$ have been chosen to be pairwise disjoint, implies the validity of (20).

4.6. **Proof of Theorem 1.** We are now in the position to complete the proof of our main result.

In the case $r = 1$, condition (B) becomes

$$v(t, \cdot) \in \dot{W}^{1,p}(\mathbb{R}^d) \qquad \Longleftarrow \qquad \sum_n \frac{\lambda_n^{\frac{d}{p}}}{\tau_n} < \infty\,. \tag{$\hat{\mathrm{B}}$}$$

Our task is to find sequences $\{\lambda_n\}$, $\{\tau_n\}$, and $\{\gamma_n\}$ such that conditions (A), ($\hat{\mathrm{B}}$), ($\tilde{\mathrm{B}}$), (C), ($\tilde{\mathrm{C}}$), and (D) are satisfied, for any $1 < p < \infty$, $\sigma > 0$, $s > 0$, and $t > 0$. Choosing

$$\tau_n = \frac{1}{n^3} \qquad \text{and} \qquad \lambda_n = e^{-n}\,, \tag{21}$$

we immediately see that (A), ($\hat{\mathrm{B}}$), and ($\tilde{\mathrm{B}}$) are satisfied. If we take

$$\gamma_n = e^{-n^2}, \tag{22}$$

we see that (C) and ($\tilde{\mathrm{C}}$) are satisfied. We are left with having to check condition (D). With the choices in (21) and (22) for the parameters, the series appearing in condition (D) reads:

$$\sum_n \exp(-2n^2)\, \exp\big(-(d-2s)n\big)\, \exp(2sctn^3),$$

which clearly diverges since $sct > 0$. This concludes the proof of Theorem 1. $\square$

4.7. **Proof of Theorem 3.** The proof is a variation of the previous one. We fix $T > 0$ and choose

$$\tau_n = \frac{1}{n}\,.$$

We want to check (B) for $0 \le t \le T$. Such a condition now reads

$$\sum_n n \lambda_n^{1-r+\frac{d}{p}} \exp\big(n(r-1)bT\big) < \infty\,. \tag{23}$$

By the assumption $p < \frac{d}{r-1}$ (expressing the fact that we consider a Sobolev space that does not embed in the Lipschitz class), we have that

$$\beta = 1 - r + \frac{d}{p} > 0\,.$$

For $\alpha > 0$ to be determined, we choose

$$\lambda_n = \exp(-\alpha T n)\,.$$

Then, condition (23) becomes

$$\sum_n n \exp(-\alpha\beta T n) \exp\big(n(r-1)bT\big) < \infty\,.$$

This series indeed converges provided that

$$-\alpha\beta T + (r-1)bT < 0\,, \qquad \text{i.e.} \qquad \alpha > \frac{(r-1)b}{\beta}\,, \tag{24}$$

which is admissible. Under this condition, (B) holds and also (A) and ($\tilde{\mathrm{B}}$) follow.

With this choice of $\lambda_n$, condition (C) becomes

$$\sum_n \gamma_n \exp\left(\alpha\left(\sigma - \frac{d}{2}\right)Tn\right) < \infty\,.$$

We can guarantee that this condition holds if

$$\gamma_n = \frac{1}{n^2}\exp\left(\alpha\left(\frac{d}{2} - \sigma\right)Tn\right).$$

Moreover, condition ($\hat{\mathrm{C}}$) becomes

$$\sum_n \frac{1}{n^2}\exp\left(\alpha\left(\frac{d}{2} - \sigma\right)Tn\right)\Big[\exp(-\alpha Tn)\Big]^{\frac{d}{2}} = \sum_n \frac{1}{n^2}\exp\left(-\alpha\sigma Tn\right) < \infty\,,$$

and we can verify that it is satisfied. We note in passing that, in the case $\sigma > d/2$, the stronger condition ($\tilde{\mathrm{C}}$) in fact holds.

We now substitute all the above expressions into (D) and obtain that this condition is equivalent to having:

$$\begin{aligned}\sum_n \left(\gamma_n \lambda_n^{\frac{d}{2}-\sigma}\right)^2 \lambda_n^{2(\sigma-s)} \exp\left(\frac{2sct}{\tau_n}\right) &= \sum_n \frac{1}{n^4}\exp\big(-2(\sigma - s)\alpha Tn\big)\exp\big(2sctn\big)\\ &= \sum_n \frac{1}{n^4}\exp\Big(2n\big(-(\sigma-s)\alpha T + sct\big)\Big) = \infty\,.\end{aligned}$$

The above series diverges if

$$-(\sigma - s)\alpha T + sct > 0\,,$$

which is (trivially) the case for any $t \geq 0$ when $s > \sigma$. In the relevant case $0 < s \leq \sigma$, we see that the norm of the solution in $\dot{H}^s(\mathbb{R}^d)$ becomes infinity only after a time $(\sigma - s)\alpha T/sc$. Recalling that we control the Sobolev norm of the velocity field up to time $T$ only, only those regularity indices $s$ that satisfy

$$\frac{(\sigma - s)\alpha T}{sc} < T \qquad \Longleftrightarrow \qquad (\sigma - s)\alpha < sc\,, \tag{25}$$

lead to blow up of the norm. Recalling (24), we can rewrite (25) as

$$\frac{(\sigma - s)(r-1)b}{sc\beta} < 1 \qquad \Longleftrightarrow \qquad \frac{s}{\sigma} > 1 - \frac{c\beta}{c\beta + (r-1)b} = \bar{\mu}\,,$$

and $0 < \bar{\mu} < 1$. Therefore, the amount of regularity for the solution that can be lost is $\bar{\mu}\sigma$. Theorem 3 is now proved. □

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