

Second order differentiation formula on $\text{RCD}^*(K, N)$ spaces

Nicola Gigli * Luca Tamanini †

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Abstract

Aim of this paper is to prove the second order differentiation formula for $H^{2,2}$ functions along geodesics in $\text{RCD}^*(K, N)$ spaces with $N < \infty$. This formula is new even in the context of Alexandrov spaces, where second order differentiation is typically related to semiconvexity.

We establish this result by showing that W_2 -geodesics can be approximated up to second order, in a sense which we shall make precise, by entropic interpolation. In turn this is achieved by proving new, even in the smooth setting, estimates concerning entropic interpolations which we believe are interesting on their own. In particular we obtain:

- equiboundedness of the densities along the entropic interpolations,
- local equi-Lipschitz continuity of the Schrödinger potentials,
- a uniform weighted L^2 control of the Hessian of such potentials.

Finally, the techniques adopted in this paper can be used to show that in the RCD setting the viscous solution of the Hamilton-Jacobi equation can be obtained via a vanishing viscosity method, in accordance with the smooth case.

With respect to a previous version, where the space was assumed to be compact, in this paper the second order differentiation formula is proved in full generality.

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*SISSA, Trieste. email: ngigli@sissa.it

†Institut für Angewandte Mathematik, Universität Bonn. email: tamanini@iam.uni-bonn.de

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1 Introduction

In the last ten years there has been a great interest in the study of metric measure spaces with Ricci curvature bounded from below, see for instance [44], [55], [56], [29], [5], [6], [27], [7], [48], [49], [30], [24], [26], [38], [9], [47], [13], [12]. The starting points of this research line have been the seminal papers [44] and [55], [56] which linked lower Ricci bounds on metric measure spaces to properties of entropy-like functionals in connection with W_2 -geometry. Later ([5]) it emerged that also Sobolev calculus is linked to W_2 -geometry and building on top of this the original definition of CD spaces by Lott-Sturm-Villani has evolved into that of RCD spaces ([6], [27]).

An example of link between Sobolev calculus and W_2 -geometry is the following result (a minor variant of a statement in [24]). It says that we can safely take one derivative of a $W^{1,2}(X)$ function along an optimal geodesic test plan π , i.e. a test plan satisfying

$$\iint_0^1 |\dot{\gamma}_t|^2 dt d\pi(\gamma) = W_2^2((e_0)_*\pi, (e_1)_*\pi).$$

Theorem 1.1 (First order differentiation formula). *Let (X, d, m) be an $RCD(K, \infty)$ space, π an optimal geodesic test plan with bounded support (equivalently: such that $\{\gamma_t : t \in [0, 1], \gamma \in \text{supp}(\pi)\} \subset X$ is bounded) and $f \in W^{1,2}(X)$.*

Then the map $[0, 1] \ni t \mapsto f \circ e_t \in L^2(\pi)$ is in $C^1([0, 1], L^2(\pi))$ and we have

$$\frac{d}{dt}(f \circ e_t) = \langle \nabla f, \nabla \phi_t \rangle \circ e_t,$$

for every $t \in [0, 1]$, where $e_t : C([0, 1], X) \rightarrow X, \gamma \mapsto \gamma_t$ is the evaluation map and ϕ_t is any function such that for some $s \neq t, s \in [0, 1]$, the function $-(s-t)\phi_t$ is a Kantorovich potential from $(e_t)_\pi$ to $(e_s)_*\pi$.*

Recall that on $RCD(K, \infty)$ spaces every W_2 -geodesic (μ_t) between measures with bounded density and support is such that $\mu_t \leq C m$ for every $t \in [0, 1]$ and some $C > 0$ ([49]), so that between two such measures there always exists a (unique) optimal geodesic test plan with bounded support. Thus the theorem also says that we can find ‘many’ C^1 functions on RCD spaces. We remark that such C^1 regularity - which was crucial in [24] - is non-trivial even if the function f is assumed to be Lipschitz and that statements about C^1 smoothness are quite rare in metric geometry.

Furthermore, projecting from π to $\mu_t := (e_t)_*\pi$ one can see that Theorem 1.1 immediately implies

$$\frac{d}{dt} \int f d\mu_t = \int \langle \nabla f, \nabla \phi_t \rangle d\mu_t \tag{1.1}$$

and one might think of this identity as an ‘integrated’ version of the basic formula

$$\frac{d}{dt}f(\gamma_t) = df(\gamma'_t)$$

valid in the smooth framework; at the technical level the proof of the claim has to do with the fact that the geodesic (μ_t) solves the continuity equation

$$\frac{d}{dt}\mu_t + \operatorname{div}(\nabla\varphi_t\mu_t) = 0, \quad (1.2)$$

where the φ_t ’s are appropriate choices of Kantorovich potentials (see also [28] in this direction), and with the fact that $\nabla\varphi_t = \nabla\phi_t$ (see Lemma A.7 below).

In [25], the first author developed a second-order calculus on RCD spaces, in particular defining the space $H^{2,2}(X)$ and for $f \in H^{2,2}(X)$ the Hessian $\operatorname{Hess}(f)$, see [25] and the Appendix. It is then natural to ask whether an ‘integrated’ version of the second order differentiation formula

$$\frac{d^2}{dt^2}f(\gamma_t) = \operatorname{Hess}(f)(\gamma'_t, \gamma'_t) \quad \text{for } \gamma \text{ geodesic}$$

holds in this framework. In this paper we provide affirmative answer to this question, our main result being:

Theorem 1.2 (Second order differentiation formula). *Let (X, d, \mathbf{m}) be an $\operatorname{RCD}^*(K, N)$ space, $N < \infty$, π an optimal geodesic test plan with bounded support and $f \in H^{2,2}(X)$.*

Then the map $[0, 1] \ni t \mapsto f \circ e_t \in L^2(\pi)$ is in $C^2([0, 1], L^2(\pi))$ and we have

$$\frac{d^2}{dt^2}(f \circ e_t) = \operatorname{Hess}(f)(\nabla\phi_t, \nabla\phi_t) \circ e_t, \quad (1.3)$$

for every $t \in [0, 1]$, where ϕ_t is as in Theorem 1.1.

An equivalent formulation, which is the one we shall actually prove (see Theorem 5.12) and is more in the spirit of (1.1), is the following:

Theorem 1.3 (Second order differentiation formula (2nd form)). *Let (X, d, \mathbf{m}) be an $\operatorname{RCD}^*(K, N)$ space, $N < \infty$, $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ be such that $\mu_0, \mu_1 \leq C\mathbf{m}$ for some $C > 0$, with compact supports and let (μ_t) be the unique W_2 -geodesic connecting μ_0 to μ_1 . Also, let $f \in H^{2,2}(X)$.*

Then the map

$$[0, 1] \ni t \mapsto \int f d\mu_t \in \mathbb{R}$$

belongs to $C^2([0, 1])$ and it holds

$$\frac{d^2}{dt^2} \int f d\mu_t = \int \operatorname{Hess}(f)(\nabla\phi_t, \nabla\phi_t) d\mu_t, \quad (1.4)$$

for every $t \in [0, 1]$, where ϕ_t is any function such that for some $s \neq t$, $s \in [0, 1]$, the function $-(s-t)\phi_t$ is a Kantorovich potential from μ_t to μ_s .

Let us comment about the assumptions in Theorem 1.2 and Theorem 1.3:

- The first order differentiation formula is valid on general $\text{RCD}(K, \infty)$ spaces, while for the second order one we need to assume finite dimensionality. This is due to the strategy of our proof, which among other things uses the Li-Yau inequality.
- There exist optimal geodesic test plans without bounded support (if $K = 0$ or the densities of the initial and final marginals decay sufficiently fast) but in this case the functions ϕ_t appearing in the statement(s) are not Lipschitz. As such it seems hard to have $\text{Hess}(h)(\nabla\phi_t, \nabla\phi_t) \circ e_t \in L^1(\pi)$ and thus we cannot really hope for anything like (1.3), (1.4) to hold: this explains the need of the assumption on bounded supports.

Having at disposal such second order differentiation formula is interesting not only at the theoretical level, but also for applications to the study of the geometry of RCD spaces. For instance, the proofs of both the splitting theorem [24] and of the ‘volume cone implies metric cone’ [19] in this setting can be greatly simplified by using such formula (in this direction, see [58] for comments about the splitting). Also, one aspect of the theory of RCD spaces which is not yet clear is whether they have constant dimension: for Ricci-limit spaces this is known to be true by a result of Colding-Naber [18] which uses second order derivatives along geodesics in a crucial way. Thus our result is necessary to replicate Colding-Naber argument in the non-smooth setting (but not sufficient: they also use a calculus with Jacobi fields which as of today does not have a non-smooth counterpart).

Let us discuss the strategy of the proof. Our starting point is a related second order differentiation formula obtained in [25], available under proper regularity assumptions:

Theorem 1.4. *Let (μ_t) be a W_2 -absolutely continuous curve solving the continuity equation*

$$\frac{d}{dt}\mu_t + \text{div}(X_t\mu_t) = 0,$$

for some vector fields $(X_t) \subset L^2(TX)$ in the following sense: for every $f \in W^{1,2}(X)$ the map $t \mapsto \int f d\mu_t$ is absolutely continuous and it holds

$$\frac{d}{dt} \int f d\mu_t = \int \langle \nabla f, X_t \rangle d\mu_t.$$

Assume that

- (i) $t \mapsto X_t \in L^2(TX)$ is absolutely continuous,
- (ii) $\sup_t \{ \|X_t\|_{L^2} + \|X_t\|_{L^\infty} + \|\nabla X_t\|_{L^2} \} < +\infty$.

Then for $f \in H^{2,2}(X)$ the map $t \mapsto \int f d\mu_t$ is $C^{1,1}$ and the formula

$$\frac{d^2}{dt^2} \int f d\mu_t = \int \text{Hess}(f)(X_t, X_t) + \langle \nabla f, \frac{d}{dt}X_t + \nabla_{X_t}X_t \rangle d\mu_t \quad (1.5)$$

holds for a.e. $t \in [0, 1]$.

If the vector fields X_t are of gradient type, so that $X_t = \nabla\phi_t$ for every t and the ‘acceleration’ a_t is defined as

$$\frac{d}{dt}\phi_t + \frac{|\nabla\phi_t|^2}{2} =: a_t$$

then (1.5) reads as

$$\frac{d^2}{dt^2} \int f d\mu_t = \int \text{Hess}(f)(\nabla\phi_t, \nabla\phi_t) d\mu_t + \int \langle \nabla f, \nabla a_t \rangle d\mu_t. \quad (1.6)$$

In the case of geodesics, the functions φ_t appearing in (1.2) solve (in a sense which we will not make precise here) the Hamilton-Jacobi equation

$$\frac{d}{dt}\varphi_t + \frac{|\nabla\varphi_t|^2}{2} = 0, \quad (1.7)$$

thus in this case the acceleration a_t is identically 0. Hence if the vector fields $(\nabla\varphi_t)$ satisfy the regularity requirements (i), (ii) in the last theorem we would easily be able to establish Theorem 1.2. However in general this is not the case; informally speaking this has to do with the fact that for solutions of the Hamilton-Jacobi equations we do not have sufficiently strong second order estimates.

In order to establish Theorem 1.2 it is therefore natural to look for suitable ‘smooth’ approximation of geodesics for which we can apply Theorem 1.4 above and then pass to the limit in formula (1.5). Given that the lack of smoothness of W_2 -geodesic is related to the lack of smoothness of solutions of (1.7), also in line with the classical theory of viscous approximation for the Hamilton-Jacobi equation there is a quite natural thing to try: solve, for $\varepsilon > 0$, the equation

$$\frac{d}{dt}\varphi_t^\varepsilon = \frac{|\nabla\varphi_t^\varepsilon|^2}{2} + \frac{\varepsilon}{2}\Delta\varphi_t^\varepsilon, \quad \varphi_0^\varepsilon := \varphi,$$

where φ is a given, fixed, Kantorovich potential for the geodesic (μ_t) , and then solve

$$\frac{d}{dt}\mu_t^\varepsilon - \text{div}(\nabla\varphi_t^\varepsilon\mu_t^\varepsilon) = 0, \quad \mu_0^\varepsilon := \mu_0.$$

This plan can actually be pursued and following the ideas in this paper one can show that if the space (X, d, \mathfrak{m}) is $\text{RCD}^*(K, N)$ and the geodesic (μ_t) is made of measures with equibounded densities, then as $\varepsilon \downarrow 0$:

- i) the curves (μ_t^ε) W_2 -uniformly converge to the geodesic (μ_t) and the measures μ_t^ε have equibounded densities.
- ii) the functions φ_t^ε are equi-Lipschitz and converge both uniformly and in the $W^{1,2}$ -topology to the only viscous solution (φ_t) of (1.7) with φ as initial datum; in particular the continuity equation (1.2) for the limit curve holds.

These convergence results are based on Hamilton’s gradient estimates and the Li-Yau inequality and are sufficient to pass to the limit in the term with the Hessian in (1.6). For these curves the acceleration is given by $a_t^\varepsilon = -\frac{\varepsilon}{2}\Delta\varphi_t^\varepsilon$ and thus we are left to prove that the quantity

$$\varepsilon \int \langle \nabla f, \nabla\Delta\varphi_t^\varepsilon \rangle d\mu_t^\varepsilon$$

goes to 0 in some sense. However, there appears to be *no hope of obtaining this by PDE estimates*. The problem is that this kind of viscous approximation can produce in the limit a curve which is not a geodesic if φ is not c -concave: shortly said, this happens as soon as

a shock appears in Hamilton-Jacobi. Since there is no hope for formula (1.4) to be true for non-geodesics, we see that there is little chance of obtaining it via such viscous approximation.

We therefore use another way of approximating geodesics: the slowing down of entropic interpolations. Let us briefly describe what this is in the familiar Euclidean setting.

Fix two probability measures $\mu_0 = \rho_0 \mathcal{L}^d$, $\mu_1 = \rho_1 \mathcal{L}^d$ on \mathbb{R}^d . The Schrödinger functional equations are

$$\rho_0 = f \mathbf{h}_1 g \qquad \rho_1 = g \mathbf{h}_1 f, \qquad (1.8)$$

the unknown being the Borel functions $f, g : \mathbb{R}^d \rightarrow [0, \infty)$, where $\mathbf{h}_t f$ is the heat flow starting at f evaluated at time t . It turns out that in great generality these equations admit a solution which is unique up to the trivial transformation $(f, g) \mapsto (cf, g/c)$ for some constant $c > 0$. Such solution can be found in the following way: let \mathbf{R} be the measure on $(\mathbb{R}^d)^2$ whose density w.r.t. \mathcal{L}^{2d} is given by the heat kernel $r_t(x, y)$ at time $t = 1$ and minimize the Boltzmann-Shannon entropy $H(\gamma | \mathbf{R})$ among all transport plans γ from μ_0 to μ_1 . The Euler equation for the minimizer forces it to be of the form $f \otimes g \mathbf{R}$ for some Borel functions $f, g : \mathbb{R}^d \rightarrow [0, \infty)$, where $f \otimes g(x, y) := f(x)g(y)$ (we shall reprove this known result in Proposition 2.1). Then the fact that $f \otimes g \mathbf{R}$ is a transport plan from μ_0 to μ_1 is equivalent to (f, g) solving (1.8).

Once we have found the solution of (1.8) we can use it in conjunction with the heat flow to interpolate from ρ_0 to ρ_1 by defining

$$\rho_t := \mathbf{h}_t f \mathbf{h}_{1-t} g.$$

This is called entropic interpolation. Now we slow down the heat flow: fix $\varepsilon > 0$ and by mimicking the above find $f^\varepsilon, g^\varepsilon$ such that

$$\rho_0 = f^\varepsilon \mathbf{h}_{\varepsilon/2} g^\varepsilon \qquad \rho_1 = g^\varepsilon \mathbf{h}_{\varepsilon/2} f^\varepsilon,$$

(the factor 1/2 plays no special role, but is convenient in computations). Then define

$$\rho_t^\varepsilon := \mathbf{h}_{t\varepsilon/2} f^\varepsilon \mathbf{h}_{(1-t)\varepsilon/2} g^\varepsilon.$$

The remarkable and non-trivial fact here is that as $\varepsilon \downarrow 0$ the curves of measures $(\rho_t^\varepsilon \mathcal{L}^d)$ converge to the W_2 -geodesic from μ_0 to μ_1 .

The first connections between Schrödinger equations and optimal transport have been obtained by Mikami in [45] for the quadratic cost on \mathbb{R}^d ; later Mikami-Thieullen [46] showed that a link persists even for more general cost functions. The statement we have just made about convergence of entropic interpolations to displacement ones has been proved by Léonard in [41]. Actually, Léonard worked in much higher generality: as it is perhaps clear from the presentation, the construction of entropic interpolation can be done in great generality, as only a heat kernel is needed. He also provided a basic intuition about why such convergence is in place: the basic idea is that if the heat kernel admits the asymptotic expansion $\varepsilon \log r_\varepsilon(x, y) \sim -\frac{d^2(x, y)}{2}$ (in the sense of Large Deviations), then the rescaled entropy functionals $\varepsilon H(\cdot | \mathbf{R}_\varepsilon)$ converge to $\frac{1}{2} \int d^2(x, y) d\cdot$ (in the sense of Γ -convergence). We refer to [43] for a deeper discussion of this topic, historical remarks and much more.

Starting from these intuitions and results, working in the setting of $\text{RCD}^*(K, N)$ spaces we gain new information about the convergence of entropic interpolations to displacement

ones. In order to state our results, it is convenient to introduce the Schrödinger potentials $\varphi_t^\varepsilon, \psi_t^\varepsilon$ as

$$\varphi_t^\varepsilon := \varepsilon \log \mathbf{h}_{t\varepsilon/2} f^\varepsilon \qquad \psi_t^\varepsilon := \varepsilon \log \mathbf{h}_{(1-t)\varepsilon/2} g^\varepsilon.$$

In the limit $\varepsilon \downarrow 0$ these will converge to forward and backward Kantorovich potentials along the limit geodesic (μ_t) (see below). In this direction, it is worth to notice that while for $\varepsilon > 0$ there is a tight link between potentials and densities, as we trivially have

$$\varphi_t^\varepsilon + \psi_t^\varepsilon = \varepsilon \log \rho_t^\varepsilon,$$

in the limit this becomes the well known (weaker) relation that is in place between forward/backward Kantorovich potentials and measures (μ_t) :

$$\begin{aligned} \varphi_t + \psi_t &= 0 && \text{on } \text{supp}(\mu_t), \\ \varphi_t + \psi_t &\leq 0 && \text{on } X, \end{aligned}$$

see e.g. Remark 7.37 in [59] (paying attention to the different sign convention). By direct computation one can verify that $(\varphi_t^\varepsilon), (\psi_t^\varepsilon)$ solve the Hamilton-Jacobi-Bellman equations

$$\frac{d}{dt} \varphi_t^\varepsilon = \frac{1}{2} |\nabla \varphi_t^\varepsilon|^2 + \frac{\varepsilon}{2} \Delta \varphi_t^\varepsilon \qquad - \frac{d}{dt} \psi_t^\varepsilon = \frac{1}{2} |\nabla \psi_t^\varepsilon|^2 + \frac{\varepsilon}{2} \Delta \psi_t^\varepsilon, \quad (1.9)$$

thus introducing the functions

$$\vartheta_t^\varepsilon := \frac{\psi_t^\varepsilon - \varphi_t^\varepsilon}{2}$$

it is not hard to check that it holds

$$\frac{d}{dt} \rho_t^\varepsilon + \text{div}(\nabla \vartheta_t^\varepsilon \rho_t^\varepsilon) = 0 \quad (1.10)$$

and

$$\frac{d}{dt} \vartheta_t^\varepsilon + \frac{|\nabla \vartheta_t^\varepsilon|^2}{2} = a_t^\varepsilon, \qquad \text{where} \quad a_t^\varepsilon := -\frac{\varepsilon^2}{8} \left(2\Delta \log \rho_t^\varepsilon + |\nabla \log \rho_t^\varepsilon|^2 \right).$$

With this said, our main results about entropic interpolations can be summarized as follows. Under the assumptions that the metric measure space (X, d, \mathbf{m}) is $\text{RCD}^*(K, N)$, $N < \infty$, and that ρ_0, ρ_1 belong to $L^\infty(X)$ with bounded supports it holds:

- Zeroth order

- *bound* For some $C > 0$ we have $\rho_t^\varepsilon \leq C$ for every $\varepsilon \in (0, 1)$ and $t \in [0, 1]$.
- *convergence* The curves $(\rho_t^\varepsilon \mathbf{m})$ W_2 -uniformly converge to the unique W_2 -geodesic (μ_t) from μ_0 to μ_1 and setting $\rho_t := \frac{d\mu_t}{d\mathbf{m}}$ it holds $\rho_t^\varepsilon \xrightarrow{*} \rho_t$ in $L^\infty(X)$ for all $t \in [0, 1]$.

- First order

- *bound* For any $t \in (0, 1]$ the functions $\{\varphi_t^\varepsilon\}_{\varepsilon \in (0, 1)}$ are locally equi-Lipschitz. Similarly for the ψ 's.
- *convergence* For every sequence $\varepsilon_n \downarrow 0$ there is a subsequence - not relabeled - such that for any $t \in (0, 1]$ the functions φ_t^ε converge both locally uniformly and in $W_{loc}^{1,2}(X)$ to a function φ_t such that $-\varphi_t$ is a Kantorovich potential from μ_t to μ_0 . Similarly for the ψ 's.

- Second order For every $\delta \in (0, 1/2)$ we have

- *bound*

$$\begin{aligned} \sup_{\varepsilon \in (0,1)} \iint_{\delta}^{1-\delta} (|\text{Hess}(\vartheta_t^\varepsilon)|_{\text{HS}}^2 + \varepsilon^2 |\text{Hess}(\log \rho_t^\varepsilon)|_{\text{HS}}^2) \rho_t^\varepsilon dt d\mathbf{m} < \infty, \\ \sup_{\varepsilon \in (0,1)} \iint_{\delta}^{1-\delta} (|\Delta \vartheta_t^\varepsilon|^2 + \varepsilon^2 |\Delta \log \rho_t^\varepsilon|^2) \rho_t^\varepsilon dt d\mathbf{m} < \infty. \end{aligned} \quad (1.11)$$

Notice that since in general the Laplacian is not the trace of the Hessian, there is no direct link between these two bounds.

- *convergence* For every function $h \in W^{1,2}(X)$ with $\Delta h \in L^\infty(X)$ it holds

$$\lim_{\varepsilon \downarrow 0} \iint_{\delta}^{1-\delta} \langle \nabla h, \nabla a_t^\varepsilon \rangle \rho_t^\varepsilon dt d\mathbf{m} = 0. \quad (1.12)$$

With the exception of the convergence $\rho_t^\varepsilon \mathbf{m} \rightarrow \mu_t$, all these results are new even on smooth manifolds (in fact, even on \mathbb{R}^d).

The zeroth and first order bounds are both consequences of the Hamilton-Jacobi-Bellman equations (1.9) satisfied by the φ 's and ψ 's and can be obtained from Hamilton's gradient estimate and the Li-Yau inequality. The facts that the limit curve is the W_2 -geodesic and that the limit potentials are Kantorovich potentials are consequence of the fact that we can pass to the limit in the continuity equation (1.10) and that the limit potentials satisfy the Hamilton-Jacobi equation. In this regard it is key that we approximate at the same time both the 'forward' potentials ψ and the 'backward' one φ : see the proof of Proposition 5.4 and recall that the simple viscous approximation may converge to curves which are not W_2 -geodesics.

Notice that these zeroth and first order convergences are sufficient to pass to the limit in the term with the Hessian in (1.6). As said, also the viscous approximation could produce the same kind of convergence.

The crucial advantage of dealing with entropic interpolations (which has no counterpart in viscous approximation) is thus in the second order bounds and convergence results which show that the term with the acceleration in (1.6) vanishes in the limit and thus eventually allows us to prove our main result Theorem 1.2. In this direction, we informally point out that being the geodesic equation a second order one, in searching for an approximation procedure it is natural to look for one producing some sort of second order convergence.

The limiting property (1.12) is mostly a consequence - although perhaps non-trivial - of the bound (1.11) (see in particular Lemma 4.10 and the proof of Theorem 5.12), thus let us focus on how to get (1.11). The starting point here is a formula due to Léonard [39], who realized that there is a connection between entropic interpolation and lower Ricci bounds: he computed the second order derivative of the entropy along entropic interpolations and in this direction our contribution has been the rigorous proof in the RCD framework of his formal computations, thus getting

$$\frac{d^2}{dt^2} H(\mu_t^\varepsilon | \mathbf{m}) = \int \rho_t^\varepsilon d(\Gamma_2(\vartheta_t^\varepsilon) + \frac{\varepsilon^2}{4} \Gamma_2(\log(\rho_t^\varepsilon))) = \frac{1}{2} \int \rho_t^\varepsilon d(\Gamma_2(\varphi_t^\varepsilon) + \Gamma_2(\psi_t^\varepsilon)), \quad (1.13)$$

where Γ_2 is the 'iterated carré du champ' operator defined as

$$\Gamma_2(f) := \Delta \frac{|\nabla f|^2}{2} - \langle \nabla f, \nabla \Delta f \rangle$$

(in the setting of RCD spaces some care is needed when handling this object, but let us neglect this issue here).

Observe that if $h : [0, 1] \rightarrow \mathbb{R}^+$ is a convex function, then $-\frac{h(0)}{t} \leq h'(t) \leq \frac{h(1)}{1-t}$ for any $t \in (0, 1)$ and thus

$$\int_{\delta}^{1-\delta} h''(t) dt = h'(1-\delta) - h'(\delta) \leq \frac{h(1)}{1-\delta} + \frac{h(0)}{\delta}. \quad (1.14)$$

If we assume for simplicity that $K = 0$ we have $\mathbf{\Gamma}_2 \geq 0$, so that (1.13) tells in particular that $t \mapsto H(\mu_t^\varepsilon | \mathbf{m})$ is convex for any $\varepsilon > 0$, and if we also assume that $\mathbf{m}(X) = 1$ such function is non-negative. Therefore (1.14) gives that for any $\delta \in (0, 1/2)$ it holds

$$\sup_{\varepsilon \in (0,1)} \int_{\delta}^{1-\delta} \rho_t^\varepsilon d(\mathbf{\Gamma}_2(\vartheta_t^\varepsilon) + \frac{\varepsilon^2}{4} \mathbf{\Gamma}_2(\log(\rho_t^\varepsilon))) dt \leq \frac{H(\mu_1 | \mathbf{m})}{1-\delta} + \frac{H(\mu_0 | \mathbf{m})}{\delta} < \infty. \quad (1.15)$$

Recalling the Bochner inequalities ([20], [9], [25])

$$\mathbf{\Gamma}_2(\eta) \geq |\text{Hess}(\eta)|_{\text{HS}}^2 \mathbf{m}, \quad \mathbf{\Gamma}_2(\eta) \geq \frac{(\Delta \eta)^2}{N} \mathbf{m},$$

we see that (1.11) follows from (1.15). Then with some work (see Lemma 4.10 and Theorem 5.12 for the details) starting from (1.11) we can deduce (1.12) which in turn ensures that the term with the acceleration in (1.6) vanishes in the limit $\varepsilon \downarrow 0$, thus leading to our main result Theorem 1.2.

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2 The Schrödinger problem

Let (X, τ) be a Polish space, $\mu_0, \mu_1 \in \mathcal{P}(X)$ and \mathbf{R} be a non-negative Radon measure on X^2 . Recall that $\gamma \in \mathcal{P}(X^2)$ is called transport plan for μ_0, μ_1 provided $\pi_*^0 \gamma = \mu_0$ and $\pi_*^1 \gamma = \mu_1$, where $\pi^0, \pi^1 : X^2 \rightarrow X$ are the canonical projections. We are interested in finding a transport plan of the form

$$\gamma = f \otimes g \mathbf{R}$$

for certain Borel functions $f, g : X \rightarrow [0, \infty)$, where $f \otimes g(x, y) := f(x)g(y)$. As we shall see in this short section, in great generality this problem can be solved in a unique way and the plan γ can be found as the minimum of

$$\gamma' \mapsto H(\gamma' | \mathbf{R})$$

among all transport plans from μ_0 to μ_1 , where $H(\cdot | \cdot)$ is the Boltzmann-Shannon entropy. For appropriate choice of the reference measure \mathbf{R} (which will also be our choice in the following), this minimization problem is called Schrödinger problem, we refer to [43] for a survey on the topic.

Let us first recall the definition of the relative entropy functional in the case of a reference measure with possibly infinite mass (see [42] for more details). Given a σ -finite measure ν on a Polish space (Y, τ') , there exists a measurable function $W : Y \rightarrow [0, \infty)$ such that

$$z_W := \int e^{-W} d\nu < +\infty.$$

Introducing the probability measure $\nu_W := z_W^{-1} e^{-W} \nu$, for any $\sigma \in \mathcal{P}(Y)$ such that $\int W d\sigma < +\infty$ the Boltzmann-Shannon entropy is defined as

$$H(\sigma | \nu) := H(\sigma | \nu_W) - \int W d\sigma - \log z_W \quad (2.1)$$

where $H(\sigma | \nu_W)$ is in turn defined as

$$H(\sigma | \tilde{\nu}) := \begin{cases} \int \rho \log(\rho) d\tilde{\nu} & \text{if } \sigma = \rho \tilde{\nu} \\ +\infty & \text{if } \sigma \not\ll \tilde{\nu} \end{cases}$$

for all $\tilde{\nu} \in \mathcal{P}(Y)$; notice that Jensen's inequality and the fact that $\tilde{\nu} \in \mathcal{P}(Y)$ grant that $\int \rho \log(\rho) d\tilde{\nu}$ is well defined and non-negative, in particular the definition makes sense. The definition is meaningful, because if $\int W' d\sigma < +\infty$ for another function W' such that $z_{W'} < +\infty$, then

$$H(\sigma | \nu_W) - \int W d\sigma - \log z_W = H(\sigma | \nu_{W'}) - \int W' d\sigma - \log z_{W'}.$$

Hence $H(\cdot | \nu)$ is well defined for all $\sigma \in \mathcal{P}(Y)$ such that $\int W d\sigma < +\infty$ for some non-negative measurable function W with $z_W < +\infty$.

The following proposition collects the basic properties of the minimizer of the Schrödinger problem; we emphasize that point (i) of the statement is already known in the literature on the subject (see in particular [40], [11] and [50]) and there are similarities between point (ii) and some results in [11]. A complete proof has already been presented in [33] for the compact case; here we adapt the arguments to our more general case. Notice that Radon measures on Polish spaces are always σ -finite, hence the above discussion about the Boltzmann-Shannon entropy applies.

Proposition 2.1. *Let (X, τ, \mathbf{m}) be a Polish space equipped with a non-negative Radon measure \mathbf{m} and let \mathbf{R} be a non-negative Radon measure on X^2 such that $\pi_*^0 \mathbf{R} = \pi_*^1 \mathbf{R} = \mathbf{m}$ and*

$$\mathbf{m} \otimes \mathbf{m} \ll \mathbf{R} \ll \mathbf{m} \otimes \mathbf{m}.$$

Let $\mu_0 = \rho_0 \mathbf{m}$ and $\mu_1 = \rho_1 \mathbf{m}$ be Borel probability measures and assume that there exists a Borel function $B : X \rightarrow [0, \infty)$ such that

$$\int_{X^2} e^{-B(x)-B(y)} d\mathbf{R}(x, y) < \infty \quad \int B d\mu_0 < \infty \quad \int B d\mu_1 < \infty. \quad (2.2)$$

Then the following holds.

i) Assume that

$$H(\mu_0 \otimes \mu_1 | \mathbf{R}) < +\infty. \quad (2.3)$$

Then:

i-a) There exists a unique minimizer γ of $H(\cdot | \mathbf{R})$ among all transport plans from μ_0 to μ_1 .

i-b) $\gamma = f \otimes g \mathbf{R}$ for appropriate Borel functions $f, g : X \rightarrow [0, \infty)$ which are \mathbf{m} -a.e. unique up to the trivial transformation $(f, g) \rightarrow (cf, g/c)$ for some $c > 0$.

ii) Assume that $\rho_0, \rho_1 \in L^\infty(X, \mathbf{m})$ and that for some $c > 0$ it holds

$$\mathbf{R} \geq c \mathbf{m} \otimes \mathbf{m} \quad \text{in } P_0 \times P_1, \quad (2.4)$$

where $P_0 := \{\rho_0 > 0\}$ and $P_1 := \{\rho_1 > 0\}$. Then:

ii-a) The bound (2.3) holds.

ii-b) The functions f, g given by point (i-b) above are in $L^1 \cap L^\infty(\mathbf{m})$ with

$$\|f\|_{L^\infty(\mathbf{m})} \|g\|_{L^1(\mathbf{m})} \leq \frac{\|\rho_0\|_{L^\infty(\mathbf{m})}}{c} \quad \text{and} \quad \|f\|_{L^1(\mathbf{m})} \|g\|_{L^\infty(\mathbf{m})} \leq \frac{\|\rho_1\|_{L^\infty(\mathbf{m})}}{c} \quad (2.5)$$

and γ is the only transport plan which can be written as $f' \otimes g' \mathbf{R}$ for $f', g' : X \rightarrow [0, \infty)$ Borel.

proof

(i-a) Existence follows by the direct method of calculus of variations: the class of transport plans is not empty, narrowly compact (see e.g. [4]) and $H(\cdot | \mathbf{R})$ is well defined therein; indeed by assumption $\int W d\sigma < +\infty$ with $W(x, y) := B(x) + B(y)$ for all transport plan σ . Moreover by (2.1) we have that

$$H(\sigma | \mathbf{R}) = H(\sigma | \mathbf{R}_W) - \int B d\mu_0 - \int B d\mu_1 - \log z_W,$$

so that $H(\cdot | \mathbf{R})$ is narrowly lower semicontinuous on the class of transport plans.

Since $H(\cdot | \mathbf{R})$ is strictly convex, uniqueness is equivalent to the existence of a $\gamma \in \text{Adm}(\mu_0, \mu_1)$ with finite entropy w.r.t. \mathbf{R} and by (2.3) we conclude.

(i-b) The uniqueness part of the claim is trivial, so we concentrate on existence. Finiteness of the entropy in particular grants that $\gamma \ll \mathbf{R}$. Put $p := \frac{d\gamma}{d\mathbf{R}}$ and let $P_0 := \{\rho_0 > 0\}$, $P_1 := \{\rho_1 > 0\}$. We start claiming that

$$p > 0 \quad \mathbf{m} \otimes \mathbf{m}\text{-a.e. on } P_0 \times P_1. \quad (2.6)$$

Notice that since $\mathbf{m} \otimes \mathbf{m}$ and \mathbf{R} are mutually absolutely continuous, the claim makes sense and arguing by contradiction we shall assume that $\mathbf{R}(Z) > 0$, where $Z := (P_0 \times P_1) \cap \{p = 0\}$.

Let $s := \frac{d(\mu_0 \otimes \mu_1)}{d\mathbf{R}}$ and for $\lambda \in (0, 1)$ let us define $\Phi(\lambda) : X^2 \rightarrow \mathbb{R}$ by

$$\Phi(\lambda) := \frac{u(p + \lambda(s - p)) - u(p)}{\lambda}, \quad \text{where } u(z) := z \log(z).$$

The convexity of u grants that $\Phi(\lambda) \leq u(s) - u(p) \in L^1(X^2, \mathbf{R})$ (recall (2.3)) and that $\Phi(\lambda)$ is monotone decreasing as $\lambda \downarrow 0$. Moreover, on Z we have $\Phi(\lambda) \downarrow -\infty$ \mathbf{R} -a.e. as $\lambda \downarrow 0$, thus the monotone convergence theorem ensures that

$$\lim_{\lambda \downarrow 0} \frac{H(\gamma + \lambda(\mu_0 \otimes \mu_1 - \gamma) | \mathbf{R}) - H(\gamma | \mathbf{R})}{\lambda} = -\infty.$$

Since $\gamma + \lambda(\mu_0 \otimes \mu_1 - \gamma)$ is a transport plan from μ_0 to μ_1 for $\lambda \in (0, 1)$, this is in contradiction with the minimality of γ which grants that the left-hand side is non-negative, hence Z is \mathbb{R} -negligible, as desired.

Let us now pick $h \in L^\infty(X^2, \gamma)$ such that $\pi_*^0(h\gamma) = \pi_*^1(h\gamma) = 0$ and $\varepsilon \in (0, \|h\|_{L^\infty(\gamma)}^{-1})$. Then $(1 + \varepsilon h)\gamma$ is a transport plan from μ_0 to μ_1 and noticing that hp is well defined \mathbb{R} -a.e. we have

$$\begin{aligned} \|u((1 + \varepsilon h)p)\|_{L^1(\mathbb{R})} &= \int |(1 + \varepsilon h)p \log((1 + \varepsilon h)p)| \, d\mathbb{R} \\ &\leq \int (1 + \varepsilon h)p |\log p| \, d\mathbb{R} + \int (1 + \varepsilon h) |\log(1 + \varepsilon h)| \, d\gamma \\ &\leq \|1 + \varepsilon h\|_{L^\infty(\gamma)} \|p \log p\|_{L^1(\mathbb{R})} + \|(1 + \varepsilon h) \log(1 + \varepsilon h)\|_{L^\infty(\gamma)}, \end{aligned}$$

so that $u((1 + \varepsilon h)p) \in L^1(\mathbb{R})$. Then again by the monotone convergence theorem we get

$$\lim_{\varepsilon \downarrow 0} \frac{H((1 + \varepsilon h)\gamma | \mathbb{R}) - H(\gamma | \mathbb{R})}{\varepsilon} = \int \lim_{\varepsilon \downarrow 0} \frac{u((1 + \varepsilon h)p) - u(p)}{\varepsilon} \, d\mathbb{R} = \int hp(\log p + 1) \, d\mathbb{R}.$$

By the minimality of γ we know that the left-hand side in this last identity is non-negative, thus after running the same computation with $-h$ in place of h and noticing that the choice of h grants that $\int hp \, d\mathbb{R} = \int h \, d\gamma = 0$ we obtain

$$\int hp \log(p) \, d\mathbb{R} = 0 \quad \forall h \in L^\infty(\gamma) \text{ such that } \pi_*^0(h\gamma) = \pi_*^1(h\gamma) = 0. \quad (2.7)$$

The rest of the argument is better understood by introducing the spaces $V, {}^\perp W \subset L^1(\gamma)$ and $V^\perp, W \subset L^\infty(\gamma)$ as follows

$$\begin{aligned} V &:= \{f \in L^1(\gamma) : f = \varphi \oplus \psi \text{ for some } \varphi \in L^0(\mathfrak{m}|_{P_0}), \psi \in L^0(\mathfrak{m}|_{P_1})\}, \\ W &:= \{h \in L^\infty(\gamma) : \pi_*^0(h\gamma) = \pi_*^1(h\gamma) = 0\}, \\ V^\perp &:= \{h \in L^\infty(\gamma) : \int fh \, d\gamma = 0 \, \forall f \in V\}, \\ {}^\perp W &:= \{f \in L^1(\gamma) : \int fh \, d\gamma = 0 \, \forall h \in W\}, \end{aligned}$$

where here and in the following the function $\varphi \oplus \psi$ is defined as $\varphi \oplus \psi(x, y) := \varphi(x) + \psi(y)$. Notice that the Euler equation (2.7) reads as $\log(p) \in {}^\perp W$ and our thesis as $\log(p) \in V$; hence to conclude it is sufficient to show that ${}^\perp W \subset V$.

Claim 1: V is a closed subspace of $L^1(\gamma)$.

We start claiming that $f \in V$ if and only if $f \in L^1(\gamma)$ and

$$f(x, y) + f(x', y') = f(x, y') + f(x', y) \quad \mathfrak{m} \otimes \mathfrak{m} \otimes \mathfrak{m} \otimes \mathfrak{m}\text{-a.e. } (x, x', y, y') \in P_0^2 \times P_1^2. \quad (2.8)$$

Indeed the ‘only if’ follows trivially from $\gamma \ll \mathfrak{m} \otimes \mathfrak{m}$ and the definition of V . For the ‘if’ we apply Fubini’s theorem to get the existence of $x' \in P_0$ and $y' \in P_1$ such that

$$f(x, y) + f(x', y') = f(x, y') + f(x', y) \quad \mathfrak{m} \otimes \mathfrak{m}\text{-a.e. } x, y \in P_0 \times P_1.$$

Thus $f = f(\cdot, y') \oplus (f(x', \cdot) - f(x', y'))$, as desired.

Now notice that since (2.6) grants that $(\mathbf{m} \otimes \mathbf{m})|_{P_0 \times P_1} \ll \gamma$, we see that the condition (2.8) is closed w.r.t. $L^1(\gamma)$ -convergence.

Claim 2: $V^\perp \subset W$.

Let $h \in L^\infty(\gamma) \setminus W$, so that either the first or second marginal of $h\gamma$ is non-zero. Say the first. Thus since $\pi_*^0 \gamma = \mu_0$ we have $\pi_*^0(h\gamma) = f_0 \mu_0$ for some $f_0 \in L^\infty(\mu_0) \setminus \{0\}$. Then the function $f := f_0 \oplus 0 = f_0 \circ \pi^0$ belongs to V and we have

$$\int h f d\gamma = \int f_0 \circ \pi^0 d(h\gamma) = \int f_0 d\pi_*^0(h\gamma) = \int f_0^2 d\mu_0 > 0,$$

so that $h \notin V^\perp$.

Claim 3: ${}^\perp W \subset V$.

Let $f \in L^1(\gamma) \setminus V$, use the fact that V is closed and the Hahn-Banach theorem to find $h \in L^\infty(\gamma) \sim L^1(\gamma)^*$ such that $\int f h d\gamma \neq 0$ and $\int \tilde{f} h d\gamma = 0$ for every $\tilde{f} \in V$. Thus $h \in V^\perp$ and hence by the previous step $h \in W$. The fact that $\int f h d\gamma \neq 0$ shows that $f \notin {}^\perp W$, as desired.

(ii-a) The bounds (2.2) and (2.4) give that $\int e^{-B(x)-B(y)} d(\mathbf{m} \otimes \mathbf{m})|_{P_0 \times P_1} < \infty$, which together with (2.2) again grants that $H(\mu_0 \otimes \mu_1 | (\mathbf{m} \otimes \mathbf{m})|_{P_0 \times P_1})$ is well defined. The assumption that $\rho_0, \rho_1 \in L^\infty(\mathbf{m})$ then ensures that $H(\mu_0 \otimes \mu_1 | (\mathbf{m} \otimes \mathbf{m})|_{P_0 \times P_1})$ is finite, hence the claim follows by direct computations:

$$\begin{aligned} H(\mu_0 \otimes \mu_1 | \mathbf{R}) &= H(\mu_0 \otimes \mu_1 | (\mathbf{m} \otimes \mathbf{m})|_{P_0 \times P_1}) + \int \log \left(\frac{d((\mathbf{m} \otimes \mathbf{m})|_{P_0 \times P_1})}{d\mathbf{R}} \right) \rho_0 \otimes \rho_1 d(\mathbf{m} \otimes \mathbf{m}) \\ &\leq H(\mu_0 \otimes \mu_1 | (\mathbf{m} \otimes \mathbf{m})|_{P_0 \times P_1}) - \log(c) < \infty. \end{aligned}$$

(ii-b) Then let σ be a transport plan from μ_0 to μ_1 such that $\sigma = f' \otimes g' \mathbf{R}$ for suitable non-negative Borel functions f', g' . We claim that in this case it holds $f', g' \in L^\infty(\mathbf{m})$, leading in particular to the claim in the statement about γ .

By disintegrating \mathbf{R} w.r.t. π^0 , from $\pi_*^0(f' \otimes g' \mathbf{R}) = \rho_0 \mathbf{m}$ and $\mathbf{R}_0 = \mathbf{m}$ we get that

$$f'(x) \int g'(y) d\mathbf{R}_x(y) = \rho_0(x) < +\infty, \quad \text{for } \mathbf{m}\text{-a.e. } x \quad (2.9)$$

whence $g' \in L^1(\mathbf{R}_x)$ for \mathbf{m} -a.e. x . Notice then that the sets where f' and g' are positive must coincide with P_0 and P_1 respectively, up to \mathbf{m} -negligible sets, so that nothing changes in (2.9) if we restrict the integral to P_1 . Moreover, since from (2.4) we have that $\mathbf{R}_x \geq c \mathbf{m}$ in P_1 for \mathbf{m} -a.e. $x \in P_0$, we see that $g' \in L^1(\mathbf{m})$ with

$$c \|g'\|_{L^1(\mathbf{m})} \leq \int g'(y) d\mathbf{R}_x(y) \quad \text{for } \mathbf{m}\text{-a.e. } x \in P_0$$

and thus (2.9) yields

$$f' \leq \frac{\|\rho_0\|_{L^\infty(\mathbf{m})}}{c \|g'\|_{L^1(\mathbf{m})}}, \quad \mathbf{m}\text{-a.e. in } P_0,$$

which is the first in (2.5), because in $X \setminus P_0$ we already know that f' vanishes \mathbf{m} -a.e. By interchanging the roles of f' and g' , the same conclusion follows for g' .

For the uniqueness of γ , put $\varphi := \log f'$, $\psi := \log g'$ and notice that, by what we have just proved, they are bounded from above. On the other hand

$$\int \varphi \oplus \psi \, d\sigma = H(\sigma | \mathbb{R}) > -\infty$$

because, as already remarked in the proof of (i), (2.2) implies that $H(\cdot | \mathbb{R})$ is well defined on $\text{Adm}(\mu_0, \mu_1)$. From these two facts we infer that

$$\varphi \circ \pi^0, \psi \circ \pi^1 \in L^1(\sigma). \quad (2.10)$$

Putting for brevity $p' := f' \otimes g'$ and arguing as before to justify the passage to the limit inside the integral we get

$$\begin{aligned} \frac{d}{d\lambda} H((1-\lambda)\sigma + \lambda\gamma | \mathbb{R})|_{\lambda=0+} &= \int (p - p') \log(p') \, d\mathbb{R} \\ &= \int \varphi \oplus \psi \, d(\gamma - \sigma) \\ \text{(by (2.10))} &= \int \varphi \, d\pi_*^0(\gamma - \sigma) + \int \psi \, d\pi_*^1(\gamma - \sigma) \\ \text{(because } \sigma \text{ and } \gamma \text{ have the same marginals)} &= 0. \end{aligned}$$

This equality and the convexity of $H(\cdot | \mathbb{R})$ yield $H(\sigma | \mathbb{R}) \leq H(\gamma | \mathbb{R})$ and being γ the unique minimum of $H(\cdot | \mathbb{R})$ among transport plans from μ_0 to μ_1 , we conclude that $\sigma = \gamma$. \square

The above result is valid in the very general framework of Polish spaces. We shall now restate it in the form we shall need in the context of RCD spaces.

Recall that on a finite-dimensional $\text{RCD}^*(K, N)$ space (X, d, \mathbf{m}) , \mathbf{m} satisfies the volume growth condition (A.20), so that we can choose $W = d^2(\cdot, \bar{x})$ for any $\bar{x} \in X$ in (2.1). Setting $z := \int e^{-d^2(\cdot, \bar{x})} \, d\mathbf{m}$ and

$$\tilde{\mathbf{m}} := z^{-1} e^{-d^2(\cdot, \bar{x})} \mathbf{m},$$

the definition (2.1) becomes

$$H(\mu | \mathbf{m}) = H(\mu | \tilde{\mathbf{m}}) - \int d^2(\cdot, \bar{x}) \, d\mu - \log z \quad (2.11)$$

and this shows that $H(\cdot | \mathbf{m})$ is well defined on $\mathcal{P}_2(X)$ and W_2 -lower semicontinuous. Let us also remind that on RCD spaces there is a well defined heat kernel $r_\varepsilon[x](y)$ (see (A.3) and (A.4)). The choice of working with $r_{\varepsilon/2}$ in the following statement is convenient for the computations we will do later on.

Theorem 2.2. *Let (X, d, \mathbf{m}) be an $\text{RCD}^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in [1, \infty)$ endowed with a non-negative Radon measure \mathbf{m} . For $\varepsilon > 0$ define $\mathbb{R}^{\varepsilon/2} \in \mathcal{P}(X^2)$ as*

$$d\mathbb{R}^{\varepsilon/2}(x, y) := r_{\varepsilon/2}[x](y) \, d\mathbf{m}(x) \, d\mathbf{m}(y).$$

Also, let $\mu_0, \mu_1 \in \mathcal{P}(X)$ be Borel probability measures with bounded densities and supports.

Then there exist and are uniquely \mathbf{m} -a.e. determined (up to multiplicative constants) two Borel non-negative functions $f^\varepsilon, g^\varepsilon : X \rightarrow [0, \infty)$ such that $f^\varepsilon \otimes g^\varepsilon \mathbb{R}^{\varepsilon/2}$ is a transport plan from μ_0 to μ_1 . In addition, $f^\varepsilon, g^\varepsilon$ belong to $L^\infty(X)$ and their supports are included in $\text{supp}(\mu_0)$ and $\text{supp}(\mu_1)$ respectively.

proof Start observing that $\mathbf{R}_0^{\varepsilon/2} = \mathbf{R}_1^{\varepsilon/2} = \mathbf{m}$ and if we set $B := \mathbf{d}^2(\cdot, \bar{x})$ with $\bar{x} \in X$ arbitrarily chosen, then the second and third in (2.2) are automatically satisfied; for the first one notice that

$$\int_{X^2} e^{-B \oplus B} d\mathbf{R}^{\varepsilon/2} = \int \left(e^{-\mathbf{d}^2(y, \bar{x})} d\mathbf{R}_x^{\varepsilon/2}(y) \right) e^{-\mathbf{d}^2(x, \bar{x})} d\mathbf{m}(x),$$

$e^{-\mathbf{d}^2(y, \bar{x})} \leq 1$, $\mathbf{R}_x^{\varepsilon/2}$ is a probability measure and recall (A.20). Hence Proposition 2.1, the fact that the Gaussian estimates (A.5) on the heat kernel grant that there are constants $0 < c_\varepsilon \leq C_\varepsilon < +\infty$ such that

$$c_\varepsilon \mathbf{m} \otimes \mathbf{m} \leq \mathbf{R}^{\varepsilon/2} \leq C_\varepsilon \mathbf{m} \otimes \mathbf{m}$$

in $P_0 \times P_1$ and the fact that $f^\varepsilon \otimes g^\varepsilon \mathbf{R}^{\varepsilon/2}$ is a transport plan from μ_0 to μ_1 provide us with the conclusion. \square

3 Hamilton's and Li-Yau's estimates

Here we recall Hamilton's gradient estimate and Li-Yau Laplacian estimates for $\log h_t u$, where u is a non-negative function.

Let us start with the following result, which we shall frequently use later on without explicit mention:

Proposition 3.1. *Let $(X, \mathbf{d}, \mathbf{m})$ be an $\text{RCD}^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in [1, \infty)$, $t > 0$ and $u_0 \in L^2 \cap L^\infty(X)$ be non-negative and not identically zero. Put $u_t := h_t u_0$.*

Then $\log u_t \in \text{Test}_{loc}^\infty(X)$.

proof By (A.7) $u_t \in \text{Test}^\infty(X)$ and by (A.5) u_t is locally bounded away from 0. Taking into account the fact that \log is smooth on $(0, \infty)$, the conclusion easily follows from (A.6). \square

We now recall Hamilton's gradient estimate on $\text{RCD}(K, \infty)$ spaces, which is known to be true from [37]:

Theorem 3.2 (Hamilton's gradient estimate). *Let $(X, \mathbf{d}, \mathbf{m})$ be an $\text{RCD}^*(K, \infty)$ space with $K \in \mathbb{R}$ and let $u_0 \in L^p \cap L^\infty(X)$ be positive with $p \in [1, \infty)$. Put $u_t := h_t u_0$ for all $t > 0$. Then*

$$t |\nabla \log u_t|^2 \leq (1 + 2K^- t) \log \left(\frac{\|u_0\|_{L^\infty(X)}}{u_t} \right), \quad \mathbf{m}\text{-a.e.}$$

for all $t > 0$, where $K^- := \max\{0, -K\}$.

proof In [37] this result has been stated for proper $\text{RCD}(K, \infty)$ spaces; still, the assumption that bounded sets are relatively compact is never used so that the proof works in general RCD spaces. We remark that in [37] the authors refer to [27], [3], [6] and [51] for the various calculus rules and that in these latter references no properness assumption is made. \square

In the finite-dimensional case, thanks to the Gaussian estimates for the heat kernel we can easily obtain a bound independent of the L^∞ norm of the initial datum:

Theorem 3.3. *Let $(X, \mathbf{d}, \mathbf{m})$ be an $\text{RCD}^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in [1, \infty)$. Then there is a constant C depending on K, N only such that for any $u_0 \in L^1(X)$ non-negative, not identically 0 and with bounded support the inequality*

$$|\nabla \log(u_t)|^2 \leq C \left(1 + \frac{1}{t} \right) \left(1 + t + \frac{D_0^2(x)}{t} \right), \quad \mathbf{m}\text{-a.e.} \quad (3.1)$$

holds for all $t > 0$, where $u_t := \mathbf{h}_t u$ and

$$D_0(x) := \sup_{y \in \text{supp}(u_0)} \mathbf{d}(x, y).$$

In particular, for every $0 < \delta \leq T < \infty$ and $\bar{x} \in \mathbf{X}$ there is a constant $C_{\delta, T} > 0$ depending on K, N, δ, T, \bar{x} and the diameter of $\text{supp}(u_0)$ such that for every $\varepsilon \in (0, 1)$ it holds

$$\varepsilon |\nabla \log(u_{\varepsilon t})| \leq C_{\delta, T} (1 + \mathbf{d}(\cdot, \bar{x})) \quad \forall t \in [\delta, T]. \quad (3.2)$$

proof Recall the representation formula (A.4)

$$u_t(x) = \int u_0(y) r_t[y](x) \, \mathbf{d}\mathbf{m}(y) = \int_{\text{supp}(u_0)} u_0(y) r_t[y](x) \, \mathbf{d}\mathbf{m}(y) \quad \forall x \in \mathbf{X}$$

and that for the transition probability densities $r_t[y](x)$ we have the Gaussian estimates (A.5), which can be simplified as

$$\frac{C_0}{\mathbf{m}(B_{\sqrt{t}}(y))} \exp\left(-\frac{\mathbf{d}^2(x, y)}{3t} - C_2 t\right) \leq r_t[x](y) \leq \frac{C_1}{\mathbf{m}(B_{\sqrt{t}}(y))} e^{C_2 t} \quad \forall x, y \in \mathbf{X},$$

for appropriate constants C_0, C_1, C_2 depending only on K, N . Therefore, we have

$$\begin{aligned} \|u_t\|_{L^\infty} &= \sup_x u_t(x) \leq C_1 e^{C_2 t} \int_{\text{supp}(u_0)} \frac{u(y)}{\mathbf{m}(B_{\sqrt{t}}(y))} \, \mathbf{d}\mathbf{m}(y), \\ \inf_x u_{2t}(x) &\geq C_0 e^{-2C_2 t} e^{-\frac{D_0^2(x)}{t}} \int_{\text{supp}(u_0)} \frac{u(y)}{\mathbf{m}(B_{\sqrt{2t}}(y))} \, \mathbf{d}\mathbf{m}(y) > 0. \end{aligned}$$

By the fact that \mathbf{m} is uniformly locally doubling we know that it holds

$$\mathbf{m}(B_{\sqrt{2t}}(y)) \leq \mathbf{m}(B_{\sqrt{t}}(y)) C_3 e^{C_4 \sqrt{t}} \quad \forall y \in \mathbf{X}, \, t > 0,$$

where C_3, C_4 only depend on K, N . As a consequence, the above yields

$$\frac{\|u_t\|_{L^\infty}}{u_{2t}(x)} \leq C_5 e^{3C_2 t + C_4 \sqrt{t} + \frac{D_0^2(x)}{t}} \quad \forall x \in \mathbf{X}, \, t > 0.$$

We now apply Proposition 3.2 with u_t in place of u_0 (notice that the assumptions are fulfilled) to get

$$t |\nabla \log(u_{2t})|^2 \leq (1 + 2K^- t) \log\left(\frac{\|u_t\|_{L^\infty}}{u_{2t}}\right) \leq (1 + 2K^- t) \left(\log C_5 + 3C_2 t + C_4 \sqrt{t} + \frac{D_0^2(x)}{t}\right)$$

\mathbf{m} -a.e., which is (equivalent to) the bound (3.1). The last statement is now obvious, noticing that

$$D_0(x) \leq D_0(\bar{x}) + \mathbf{d}(x, \bar{x})$$

for any $\bar{x} \in \text{supp}(u_0)$. □

A further result that we shall need soon is the Li-Yau inequality in the form proved by Baudoin and Garofalo (see [22] for the case of finite mass and [35] for the general one).

Theorem 3.4 (Li-Yau inequality). *Let (X, d, \mathbf{m}) be an $\text{RCD}^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in [1, \infty)$ and let $u_0 \in L^p(X)$ for some $p \in [1, \infty)$ be non-negative. Put $u_t := h_t u_0$ for all $t > 0$. Then*

$$|\nabla \log u_t|^2 \leq e^{-2Kt/3} \frac{\Delta u_t}{u_t} + \frac{NK}{3} \frac{e^{-4Kt/3}}{1 - e^{-2Kt/3}}, \quad \mathbf{m}\text{-a.e.} \quad (3.3)$$

for all $t > 0$, where $\frac{NK}{3} \frac{e^{-4Kt/3}}{1 - e^{-2Kt/3}}$ is understood as $\frac{N}{2t}$ when $K = 0$.

We restate such inequality in the form that we shall use:

Theorem 3.5. *Let (X, d, \mathbf{m}) be an $\text{RCD}^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in [1, \infty)$. Then for every $0 < \delta \leq T < \infty$ and $\bar{x} \in X$ there exists a constant $C_{\delta, T} > 0$ depending on K, N, δ, T, \bar{x} and the diameter of $\text{supp}(u_0)$ such that the following holds.*

For any $u_0 \in L^1(X)$ non-negative, not identically zero and with bounded support and for any $\varepsilon \in (0, 1)$ it holds

$$\varepsilon \Delta \log(h_{\varepsilon t}(u_0)) \geq -C_{\delta, T}(1 + d^2(\cdot, \bar{x})) \quad \forall t \in [\delta, T]. \quad (3.4)$$

proof Rewrite the Li-Yau inequality (3.3) as

$$e^{-2Kt/3} \underbrace{\left(\frac{\Delta u_t}{u_t} - |\nabla \log u_t|^2 \right)}_{=\Delta \log u_t} \geq (1 - e^{-2Kt/3}) |\nabla \log u_t|^2 - \frac{NK}{3} \frac{e^{-4Kt/3}}{1 - e^{-2Kt/3}}$$

and use Hamilton's gradient estimate (3.2) to control $|\nabla \log u_t|^2$ in the right-hand side. \square

4 The Schrödinger problem: properties of the solutions

4.1 The setting

Let us fix once for all the assumptions and notations which we shall use from now on.

Setting 4.1. (X, d, \mathbf{m}) is an $\text{RCD}^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in [1, \infty)$ and $\mu_0 = \rho_0 \mathbf{m}$, $\mu_1 = \rho_1 \mathbf{m}$ are two absolutely continuous Borel probability measures with bounded densities and supports.

For any $\varepsilon > 0$ we consider the couple $(f^\varepsilon, g^\varepsilon)$ given by Theorem 2.2 normalized in such a way that

$$\int \log(h_{\frac{\varepsilon}{2}} f^\varepsilon) \rho_1 \, d\mathbf{m} = 0, \quad (4.1)$$

then we set $\rho_0^\varepsilon := \rho_0$, $\rho_1^\varepsilon := \rho_1$, $\mu_0^\varepsilon := \mu_0$, $\mu_1^\varepsilon := \mu_1$ and

$$\begin{cases} f_t^\varepsilon := h_{\varepsilon t/2} f^\varepsilon \\ \varphi_t^\varepsilon := \varepsilon \log f_t^\varepsilon \\ \text{for } t \in (0, 1] \end{cases} \quad \begin{cases} g_t^\varepsilon := h_{\varepsilon(1-t)/2} g^\varepsilon \\ \psi_t^\varepsilon := \varepsilon \log g_t^\varepsilon \\ \text{for } t \in [0, 1) \end{cases} \quad \begin{cases} \rho_t^\varepsilon := f_t^\varepsilon g_t^\varepsilon \\ \mu_t^\varepsilon := \rho_t^\varepsilon \mathbf{m} \\ \vartheta_t^\varepsilon := \frac{1}{2}(\psi_t^\varepsilon - \varphi_t^\varepsilon) \\ \text{for } t \in (0, 1) \end{cases}$$

In order to investigate the time behaviour of the functions just defined, let us introduce the weighted L^2 and $W^{1,2}$ spaces. The weight we will always consider is e^{-V} with $V = Md^2(\cdot, \bar{x})$; because of (A.20), $e^{-V} \mathbf{m}$ has finite mass for every $M > 0$. For $L^2(X, e^{-V} \mathbf{m})$ no comments are required. The weighted Sobolev space is denoted and defined as

$$W^{1,2}(X, e^{-V} \mathbf{m}) := \{f \in W_{loc}^{1,2}(X) : f, |Df| \in L^2(X, e^{-V} \mathbf{m})\}$$

where $|Df|$ is the local minimal weak upper gradient already introduced. Since V is locally bounded, $W^{1,2}(X, e^{-V} \mathbf{m})$ turns out to coincide with the Sobolev space built over the metric measure space $(X, d, e^{-V} \mathbf{m})$, thus motivating the choice of the notation. The advantage of dealing with $L^2(X, e^{-V} \mathbf{m})$ and $W^{1,2}(X, e^{-V} \mathbf{m})$ is the fact that they are Hilbert spaces, unlike $L_{loc}^2(X)$ and $W_{loc}^{1,2}(X)$.

After this premise, let us begin with a couple of quantitative estimates for f_t^ε , g_t^ε and ρ_t^ε .

Lemma 4.2. *With the same assumptions and notation as in Setting 4.1 and defining*

$$v_s := \inf_{y \in \text{supp}(\rho_0) \cup \text{supp}(\rho_1)} \mathbf{m}(B_{\sqrt{s}}(y)) \quad V_s := \sup_{y \in \text{supp}(\rho_0) \cup \text{supp}(\rho_1)} \mathbf{m}(B_{\sqrt{s}}(y)), \quad (4.2)$$

for any $\bar{x} \in X$ there exist positive constants C_1, \dots, C_9 depending on $K, N, \rho_0, \rho_1, \bar{x}$ only such that the following bounds hold:

i) For any $\varepsilon > 0$ and $t \in (0, 1]$ we have

$$\frac{C_1}{V_{\varepsilon t/2}} \|f^\varepsilon\|_{L^1(\mathbf{m})} \exp\left(-\frac{C_2 d^2(\cdot, \bar{x})}{\varepsilon t} - \frac{C_3}{\varepsilon t}\right) \leq f_t^\varepsilon \leq \frac{C_4}{v_{\varepsilon t/2}} \|f^\varepsilon\|_{L^1(\mathbf{m})} \exp\left(-\frac{C_5 d^2(\cdot, \bar{x})}{\varepsilon t} + \frac{C_6}{\varepsilon t}\right) \quad (4.3)$$

and analogously for g_t^ε and $t \in [0, 1)$.

ii) For any $\varepsilon \in (0, 1)$ and $t \in [0, 1]$ we have

$$\rho_t^\varepsilon \leq \frac{C_7}{\varepsilon^{N/2}} \exp\left(\frac{C_8 - C_9 d^2(\cdot, \bar{x})}{\varepsilon}\right). \quad (4.4)$$

proof

(i) Direct consequence of the representation formula (A.4), the Gaussian estimates (A.5) and the fact that ρ_0 and f^ε have the same support

(ii) We shall indicate by C a constant depending only on $K, N, \rho_0, \rho_1, \bar{x}$ whose value might change in the various instances it appears. Start from $\rho_t^\varepsilon = f_t^\varepsilon g_t^\varepsilon \leq f_t^\varepsilon \|g_t^\varepsilon\|_{L^\infty} \leq f_t^\varepsilon \|g^\varepsilon\|_{L^\infty}$, then use the bounds (4.3), (2.5) and notice that the constant c appearing in (2.5) is $\geq \frac{C}{V_{\varepsilon/2}} e^{-\frac{C}{\varepsilon}}$ to obtain

$$\rho_t^\varepsilon \leq \frac{C V_{\varepsilon/2}}{v_{\varepsilon t/2}} \exp\left(\frac{C - C d^2(\cdot, \bar{x})}{\varepsilon t}\right).$$

Noticing that the the Bishop-Gromov inequality (A.18) ensures that for every $s \in [0, 1]$ it holds $V_s \leq C \mathbf{m}(B_1(\bar{x}))$ and $v_s \geq C \mathbf{m}(B_1(\bar{x})) s^{N/2}$, we obtain the claim for $t \in [1/2, 1]$. The case $t \in [0, 1/2]$ follows by a symmetric argument. \square

The following proposition collects the basic properties of the functions defined in Setting 4.1 and the respective ‘PDEs’ solved:

Proposition 4.3. *With the same assumptions and notation as in Setting 4.1, the following holds.*

All the functions are well defined and for any $\varepsilon > 0$:

a) $f_t^\varepsilon, g_t^\varepsilon, \rho_t^\varepsilon$ belong to $\text{Test}^\infty(\mathbf{X})$ for all $t \in \mathcal{J}$, where \mathcal{J} is the respective domain of definition (for (ρ_t^ε) we pick $\mathcal{J} = (0, 1)$);

b) $\varphi_t^\varepsilon, \psi_t^\varepsilon, \vartheta_t^\varepsilon$ belong to $\text{Test}_{loc}^\infty(\mathbf{X})$ for all $t \in \mathcal{J}$, where \mathcal{J} is the respective domain of definition.

For any $\varepsilon > 0$, $\mathcal{C} \subset \mathcal{J}$ compact and $\bar{x} \in \mathbf{X}$ there exists $M = M(K, N, \rho_0, \rho_1, \mathcal{C}, \bar{x}) > 0$ such that all the curves $(f_t^\varepsilon), (g_t^\varepsilon), (\rho_t^\varepsilon)$ belong to $AC(\mathcal{C}, W^{1,2}(\mathbf{X}))$ and $(\varphi_t^\varepsilon), (\psi_t^\varepsilon), (\vartheta_t^\varepsilon)$ to $AC(\mathcal{C}, W^{1,2}(\mathbf{X}, e^{-V}\mathbf{m}))$, where \mathcal{J} is the respective domain of definition (for (ρ_t^ε) we pick $\mathcal{J} = (0, 1)$) and $V = Md^2(\cdot, \bar{x})$; their time derivatives are given by the following expressions for a.e. $t \in [0, 1]$:

$$\begin{aligned} \frac{d}{dt} f_t^\varepsilon &= \frac{\varepsilon}{2} \Delta f_t^\varepsilon & \frac{d}{dt} g_t^\varepsilon &= -\frac{\varepsilon}{2} \Delta g_t^\varepsilon \\ \frac{d}{dt} \varphi_t^\varepsilon &= \frac{1}{2} |\nabla \varphi_t^\varepsilon|^2 + \frac{\varepsilon}{2} \Delta \varphi_t^\varepsilon & -\frac{d}{dt} \psi_t^\varepsilon &= \frac{1}{2} |\nabla \psi_t^\varepsilon|^2 + \frac{\varepsilon}{2} \Delta \psi_t^\varepsilon \\ \frac{d}{dt} \rho_t^\varepsilon + \text{div}(\rho_t^\varepsilon \nabla \vartheta_t^\varepsilon) &= 0 & \frac{d}{dt} \vartheta_t^\varepsilon + \frac{|\nabla \vartheta_t^\varepsilon|^2}{2} &= -\frac{\varepsilon^2}{8} \left(2\Delta \log \rho_t^\varepsilon + |\nabla \log \rho_t^\varepsilon|^2 \right). \end{aligned}$$

Moreover, for every $\varepsilon > 0$ we have:

i)

$$\sup_{t \in \mathcal{C}} \|h_t^\varepsilon\|_{L^\infty(\mathbf{X})} + \text{Lip}(h_t^\varepsilon) + \|\Delta h_t^\varepsilon\|_{W^{1,2}(\mathbf{X})} < \infty \quad (4.5)$$

if (h_t^ε) is equal to any of $(f_t^\varepsilon), (g_t^\varepsilon), (\rho_t^\varepsilon)$ and

$$\sup_{t \in \mathcal{C}} \|e^{-V} h_t^\varepsilon\|_{L^\infty(\mathbf{X})} + \|e^{-V} \text{lip}(h_t^\varepsilon)\|_{L^\infty(\mathbf{X})} + \|\Delta h_t^\varepsilon\|_{W^{1,2}(\mathbf{X}, e^{-V}\mathbf{m})} < \infty \quad (4.6)$$

if (h_t^ε) is equal to any of $(\varphi_t^\varepsilon), (\psi_t^\varepsilon), (\vartheta_t^\varepsilon)$; in both cases, \mathcal{C} is a compact subset of the respective domain of definition \mathcal{J} (for (ρ_t^ε) we pick $\mathcal{J} = (0, 1)$),

ii) $\mu_t^\varepsilon \in \mathcal{P}_2(\mathbf{X})$ for every $t \in [0, 1]$ and $(\rho_t^\varepsilon) \in C([0, 1], L^2(\mathbf{X}))$,

iii) we have $f_t^\varepsilon \rightarrow f^\varepsilon$ and $g_t^\varepsilon \rightarrow g^\varepsilon$ in $L^2(\mathbf{X})$ as $t \downarrow 0$ and $t \uparrow 1$ respectively.

proof

Properties of $(f_t^\varepsilon), (g_t^\varepsilon)$. Recalling (A.7) we see that $f_{t_0}^\varepsilon \in \text{Test}^\infty(\mathbf{X})$ for any $t_0 > 0$. Then the maximum principle for the heat flow, the fact that it is a contraction in $W^{1,2}(\mathbf{X})$ and the Bakry-Émery gradient estimates (A.9) together with the Sobolev-to-Lipschitz property grant that (4.5) holds for (f_t^ε) . The fact that $(f_t^\varepsilon) \in AC(\mathcal{C}, W^{1,2}(\mathbf{X}))$ and that it solves the stated scaled heat equation is trivial. The fact that $f_t^\varepsilon \rightarrow f^\varepsilon$ in $L^2(\mathbf{X})$ as $t \downarrow 0$ follows from the L^2 -continuity of the heat flow.

Properties of $(\varphi_t^\varepsilon), (\psi_t^\varepsilon)$. By Proposition 3.1 we know that $\varphi_t^\varepsilon \in \text{Test}_{loc}^\infty(\mathbf{X})$ and from the chain and Leibniz rules we see that

$$\begin{aligned} \nabla \varphi_t^\varepsilon &= \varepsilon \frac{\nabla f_t^\varepsilon}{f_t^\varepsilon}, \\ \Delta \varphi_t^\varepsilon &= \varepsilon \left(\frac{\Delta f_t^\varepsilon}{f_t^\varepsilon} - \frac{|\nabla f_t^\varepsilon|^2}{(f_t^\varepsilon)^2} \right), \\ \nabla \Delta \varphi_t^\varepsilon &= \varepsilon \left(\frac{\nabla \Delta f_t^\varepsilon}{f_t^\varepsilon} - \frac{\Delta f_t^\varepsilon \nabla f_t^\varepsilon}{(f_t^\varepsilon)^2} - \frac{\nabla |\nabla f_t^\varepsilon|^2}{(f_t^\varepsilon)^2} + \frac{2|\nabla f_t^\varepsilon|^2 \nabla f_t^\varepsilon}{(f_t^\varepsilon)^3} \right). \end{aligned}$$

These identities, (4.5) for (f_t^ε) , estimate (A.8) and (4.3) imply that for any \bar{x} there is $M > 0$ such that for $V := Md^2(\cdot, \bar{x})$ the bound (4.6) for (φ_t^ε) holds, as claimed. Similarly, we see that $|\nabla\varphi_t^\varepsilon|^2 \in L^2_{loc}((0, 1], W^{1,2}(X, e^{-V}\mathbf{m}))$.

The expressions for $\nabla\varphi_t^\varepsilon, \Delta\varphi_t^\varepsilon$ and the equation for (f_t^ε) also grant that \mathbf{m} -a.e. it holds

$$\frac{d}{dt}\varphi_t^\varepsilon = \frac{1}{2}|\nabla\varphi_t^\varepsilon|^2 + \frac{\varepsilon}{2}\Delta\varphi_t^\varepsilon \quad (4.7)$$

for a.e. t and since we have seen that the right-hand side belongs to $L^2_{loc}((0, 1], W^{1,2}(X, e^{-V}\mathbf{m}))$, this shows at once that $(\varphi_t^\varepsilon) \in AC_{loc}((0, 1], W^{1,2}(X, e^{-V}\mathbf{m}))$ and that (4.7) holds when the left-hand side is intended as limit of the difference quotients in $W^{1,2}(X, e^{-V}\mathbf{m})$, as claimed.

The same arguments apply to ψ_t^ε .

Properties of $(\rho_t^\varepsilon), (\vartheta_t^\varepsilon)$. The bound (4.5) for $(f_t^\varepsilon), (g_t^\varepsilon)$ and the Leibniz rules for the gradient and Laplacian give the bound (4.5) for (ρ_t^ε) and also grant that $(\rho_t^\varepsilon) \in AC_{loc}((0, 1], L^2(X))$. To see that this curve is absolutely continuous with values in $W^{1,2}(X)$ notice that

$$\frac{d}{dt}\rho_t^\varepsilon = \frac{\varepsilon}{2}\left(g_t^\varepsilon\Delta f_t^\varepsilon - f_t^\varepsilon\Delta g_t^\varepsilon\right)$$

and recall (4.5) for $f_t^\varepsilon, g_t^\varepsilon$. The stated equation for (ρ_t^ε) is now a matter of direct computation:

$$\begin{aligned} \frac{\varepsilon}{2}\left(g_t^\varepsilon\Delta f_t^\varepsilon - f_t^\varepsilon\Delta g_t^\varepsilon\right) &= \frac{\varepsilon}{2}\rho_t^\varepsilon\left(\Delta\log f_t^\varepsilon + |\nabla\log f_t^\varepsilon|^2 - \Delta\log g_t^\varepsilon - |\nabla\log g_t^\varepsilon|^2\right) \\ &= \rho_t^\varepsilon\frac{1}{\varepsilon}\left(\frac{|\nabla\varphi_t^\varepsilon|^2}{2} - \frac{|\nabla\psi_t^\varepsilon|^2}{2} + \frac{\varepsilon}{2}\Delta\varphi_t^\varepsilon - \frac{\varepsilon}{2}\Delta\psi_t^\varepsilon\right) \\ &= \rho_t^\varepsilon\left(-\langle\nabla\vartheta_t^\varepsilon, \nabla\log\rho_t^\varepsilon\rangle - \Delta\vartheta_t^\varepsilon\right) \\ &= -\langle\nabla\vartheta_t^\varepsilon, \nabla\rho_t^\varepsilon\rangle - \rho_t^\varepsilon\Delta\vartheta_t^\varepsilon = -\operatorname{div}(\rho_t^\varepsilon\nabla\vartheta_t^\varepsilon). \end{aligned}$$

It is clear that $\rho_t^\varepsilon \geq 0$ for every ε, t , hence the identity

$$\int \rho_t^\varepsilon \, d\mathbf{m} = \int h_{\varepsilon t/2} f^\varepsilon h_{\varepsilon(1-t)/2} g^\varepsilon \, d\mathbf{m} = \int f^\varepsilon h_{\varepsilon/2} g^\varepsilon \, d\mathbf{m} = \int \rho_0^\varepsilon \, d\mathbf{m} = 1$$

shows that $\mu_t^\varepsilon \in \mathcal{P}(X)$. The fact that μ_t^ε has finite second moment is a direct consequence of the Gaussian bound (4.4) and the volume growth estimate (A.20).

For the L^2 -continuity of ρ_t^ε in $t = 0, 1$, by the L^2 -continuity of the heat flow and the fact that $f^\varepsilon, g^\varepsilon \in L^\infty$ (Theorem 2.2) we see that $\rho_t^\varepsilon \rightarrow f^\varepsilon h_{\varepsilon/2} g^\varepsilon$ and $\rho_t^\varepsilon \rightarrow h_{\varepsilon/2} f^\varepsilon g^\varepsilon$ as $t \rightarrow 0, 1$ respectively. Hence all we have to check is that

$$\rho_0 = f^\varepsilon h_{\varepsilon/2} g^\varepsilon \quad \rho_1 = g^\varepsilon h_{\varepsilon/2} f^\varepsilon, \quad (4.8)$$

but as already noticed in the proof of Theorem 2.2, these are equivalent to the fact that $f^\varepsilon \otimes g^\varepsilon \mathbf{R}^{\varepsilon/2}$ is a transport plan from μ_0 to μ_1 ; hence, (4.8) holds by the very choice of $(f^\varepsilon, g^\varepsilon)$ made.

Finally, the fact that $(\vartheta_t^\varepsilon)$ belongs to $AC_{loc}((0, 1], W^{1,2}(X, e^{-V}\mathbf{m}))$ and satisfies the bound (4.6) is a direct consequence of the analogous property for $(\varphi_t^\varepsilon), (\psi_t^\varepsilon)$. The equation for its time derivative comes by direct computation:

$$\begin{aligned} \frac{d}{dt}\vartheta_t^\varepsilon + \frac{|\nabla\vartheta_t^\varepsilon|^2}{2} &= -\frac{|\nabla\psi_t^\varepsilon|^2}{4} - \frac{\varepsilon}{4}\Delta\psi_t^\varepsilon - \frac{|\nabla\varphi_t^\varepsilon|^2}{4} - \frac{\varepsilon}{4}\Delta\varphi_t^\varepsilon + \frac{|\nabla\psi_t^\varepsilon|^2}{8} + \frac{|\nabla\varphi_t^\varepsilon|^2}{8} - \frac{\langle\nabla\psi_t^\varepsilon, \nabla\varphi_t^\varepsilon\rangle}{4} \\ &= -\frac{\varepsilon^2}{4}\Delta\log\rho_t^\varepsilon - \frac{1}{8}\left(|\nabla\psi_t^\varepsilon|^2 + |\nabla\varphi_t^\varepsilon|^2 + 2\langle\nabla\varphi_t^\varepsilon, \nabla\psi_t^\varepsilon\rangle\right) \\ &= -\frac{\varepsilon^2}{8}\left(2\Delta\log\rho_t^\varepsilon + |\nabla\log\rho_t^\varepsilon|^2\right), \end{aligned}$$

hence the proof is complete. \square

Using the terminology adopted in the literature (see [43]) we shall refer to:

- φ_t^ε and ψ_t^ε as Schrödinger potentials, in connection with Kantorovich ones;
- $(\mu_t^\varepsilon)_{t \in [0,1]}$ as entropic interpolation, in analogy with displacement one.

4.2 Uniform estimates for the densities and the potentials

We start collecting information about quantities which remain bounded as $\varepsilon \downarrow 0$.

Proposition 4.4 (Locally uniform Lipschitz and Laplacian controls for the potentials). *With the same assumptions and notations as in Setting 4.1 the following holds.*

For all $\delta \in (0, 1)$ and $\bar{x} \in X$ there exists $C > 0$ which only depends on K, N, δ, \bar{x} such that

$$\text{lip}(\varphi_t^\varepsilon) \leq C(1 + d(\cdot, \bar{x})), \quad \mathbf{m}\text{-a.e.} \quad (4.9a)$$

$$\Delta\varphi_t^\varepsilon \geq -C(1 + d^2(\cdot, \bar{x})), \quad \mathbf{m}\text{-a.e.} \quad (4.9b)$$

for every $t \in [\delta, 1]$ and $\varepsilon \in (0, 1)$.

Furthermore, for all $M > 0$ there exists $C' > 0$ which only depends on K, N, δ, \bar{x}, M such that

$$\int |\Delta\varphi_t^\varepsilon| e^{-Md^2(\cdot, \bar{x})} d\mathbf{m} \leq C' \quad (4.10)$$

for every $t \in [\delta, 1]$ and $\varepsilon \in (0, 1)$. Analogous bounds hold for the ψ_t^ε 's in the time interval $[0, 1 - \delta]$.

proof Fix $\delta \in (0, 1)$ and $\bar{x} \in X$ as in the statement and notice that the bound (3.2) yields

$$|\nabla\varphi_t^\varepsilon| = \varepsilon |\nabla \log h_{\frac{\varepsilon t}{2}} f^\varepsilon| \leq C(1 + d(\cdot, \bar{x})) \quad \forall t \in [\delta, 1], \varepsilon \in (0, 1).$$

Thus recalling the Sobolev-to-Lipschitz property (A.10) we obtain the bound (4.9a). The bound (4.9b) is a restatement of (3.4). Finally, let $M > 0$ and χ a 1-Lipschitz cut-off function with bounded support; notice that $|h| = h + 2h^-$ whence

$$\int \chi e^{-Md^2(\cdot, \bar{x})} |\Delta\varphi_t^\varepsilon| d\mathbf{m} = \int \chi e^{-Md^2(\cdot, \bar{x})} \Delta\varphi_t^\varepsilon d\mathbf{m} + 2 \int \chi e^{-Md^2(\cdot, \bar{x})} (\Delta\varphi_t^\varepsilon)^- d\mathbf{m}.$$

Integration by parts, the fact that $|\nabla d^2(\cdot, \bar{x})| = 2d(\cdot, \bar{x})$, $|\nabla\chi| \leq 1$ and $0 \leq \chi \leq 1$ then imply that

$$\begin{aligned} \int \chi e^{-Md^2(\cdot, \bar{x})} |\Delta\varphi_t^\varepsilon| d\mathbf{m} &\leq \int e^{-Md^2(\cdot, \bar{x})} |\nabla\varphi_t^\varepsilon| d\mathbf{m} + 2M \int d(\cdot, \bar{x}) e^{-Md^2(\cdot, \bar{x})} |\nabla\varphi_t^\varepsilon| d\mathbf{m} \\ &\quad + 2 \int e^{-Md^2(\cdot, \bar{x})} (\Delta\varphi_t^\varepsilon)^- d\mathbf{m} \end{aligned}$$

and taking into account (4.9a) and (4.9b), the bound (4.10) follows.

For ψ_t^ε the argument is the same. \square

The gradient estimates that we just obtained together with the Gaussian bounds on $f_t^\varepsilon, g_t^\varepsilon, \rho_t^\varepsilon$ that we previously proved have the following direct implication, which we shall frequently use later on to justify our computations:

Lemma 4.5. *With the same assumptions and notation as in Setting 4.1, the following holds.*

For any $\varepsilon \in (0, 1)$ and $t \in (0, 1)$ let h_t^ε denote any of $\varphi_t^\varepsilon, \psi_t^\varepsilon, \vartheta_t^\varepsilon, \log \rho_t^\varepsilon$ and, for any $n \in \mathbb{N}$, let H_t^ε denote any of the functions.

$$\rho_t^\varepsilon |\nabla h_t^\varepsilon|^n, \quad \rho_t^\varepsilon \log \rho_t^\varepsilon |\nabla h_t^\varepsilon|^n, \quad |\nabla \rho_t^\varepsilon| |\nabla h_t^\varepsilon|^n, \quad \Delta \rho_t^\varepsilon |\nabla h_t^\varepsilon|^n \quad \rho_t^\varepsilon \langle \nabla h_t^\varepsilon, \nabla \Delta h_t^\varepsilon \rangle. \quad (4.11)$$

Then $H_t^\varepsilon \in L^1(X)$ for every $\varepsilon, t \in (0, 1)$. Moreover for every $\delta \in (0, 1/2)$ we have

$$\lim_{R \rightarrow \infty} \sup_{t \in [\delta, 1-\delta]} \int_{X \setminus B_R(\bar{x})} |H_t^\varepsilon| \, d\mathbf{m} = 0 \quad \forall \varepsilon \in (0, 1), \quad \bar{x} \in X. \quad (4.12)$$

Finally, $(0, 1) \ni t \mapsto \int H_t^\varepsilon \, d\mathbf{m}$ is continuous.

proof

General considerations We shall repeatedly use the fact that if h_1 has Gaussian decay and h_2 polynomial growth, i.e.

$$h_1 \leq c_1 \exp(-c_2 \mathbf{d}^2(\cdot, \bar{x})), \quad h_2 \leq c_3 (1 + \mathbf{d}^{c_4}(\cdot, \bar{x}))$$

for some $c_1, \dots, c_4 > 0$, $\bar{x} \in X$, then their product $h_1 h_2$ belongs to $L^1 \cap L^\infty(X)$: the L^∞ bound is obvious, the one for the L^1 -norm is a direct consequence of the volume growth (A.20) and explicit computations.

For what concerns the continuity of $(0, 1) \ni t \mapsto \int H_t^\varepsilon \, d\mathbf{m}$, notice that Proposition 4.3 yields that all the maps $(0, 1) \ni t \mapsto |\nabla h_t^\varepsilon| \in L^2(X, e^{-V} \mathbf{m})$ and $(0, 1) \ni t \mapsto \rho_t^\varepsilon, |\nabla \rho_t^\varepsilon|, \Delta \rho_t^\varepsilon \in L^2(X)$ are continuous (for $\Delta \rho_t^\varepsilon$ use the fact that $\Delta \rho_t^\varepsilon = g_t^\varepsilon \Delta f_t^\varepsilon + f_t^\varepsilon \Delta g_t^\varepsilon + 2 \langle \nabla f_t^\varepsilon, \nabla g_t^\varepsilon \rangle$ and the continuity of $(0, 1) \ni t \mapsto \Delta f_t^\varepsilon, \Delta g_t^\varepsilon \in L^2(X)$). Hence all the functions in (4.11), with the possible exception of the last one, are continuous from $(0, 1)$ to $L^0(X)$ equipped with the topology of convergence in measure on bounded sets. Therefore the continuity of $(0, 1) \ni t \mapsto \int H_t^\varepsilon \, d\mathbf{m}$ for these maps will follow as soon as we show that they are, locally in $t \in (0, 1)$, uniformly dominated by an $L^1(X)$ function. Given that such domination also gives (4.12), we shall focus on proving it.

Finally, we shall consider only the case $h_t^\varepsilon = \varphi_t^\varepsilon$, as the estimates for ψ_t^ε can be obtained by symmetric arguments and the ones for $\vartheta_t^\varepsilon, \log \rho_t^\varepsilon$ follow from the identities $\vartheta_t^\varepsilon = \frac{\psi_t^\varepsilon - \varphi_t^\varepsilon}{2}$, $\varepsilon \log \rho_t^\varepsilon = \varphi_t^\varepsilon + \psi_t^\varepsilon$.

Study of $\rho_t^\varepsilon |\nabla h_t^\varepsilon|^n$. By (4.9a) we know that $|\nabla \varphi_t^\varepsilon|$ has linear growth locally uniform in $t \in (0, 1)$; hence $|\nabla \varphi_t^\varepsilon|^n$ has polynomial growth locally uniform in $t \in (0, 1)$. Since ρ_t^ε has Gaussian bounds by (4.4), we deduce that $\rho_t^\varepsilon |\nabla h_t^\varepsilon|^n$ is, locally in $t \in (0, 1)$, uniformly dominated.

Study of $\rho_t^\varepsilon \log \rho_t^\varepsilon |\nabla h_t^\varepsilon|^n$. Writing $\log \rho_t^\varepsilon = \log f_t^\varepsilon + \log g_t^\varepsilon$ and using (4.3) we see that $|\log \rho_t^\varepsilon|$ has quadratic growth locally uniform in $t \in (0, 1)$. Thus the claim follows as before.

Study of $|\nabla \rho_t^\varepsilon| |\nabla h_t^\varepsilon|^n$. Notice that $|\nabla \rho_t^\varepsilon| = \rho_t^\varepsilon |\nabla \log \rho_t^\varepsilon|$ and observe that from $\varepsilon \log \rho_t^\varepsilon = \varphi_t^\varepsilon + \psi_t^\varepsilon$ and (4.9a) we have that $|\nabla \log \rho_t^\varepsilon|$ has linear growth locally uniform in $t \in (0, 1)$.

Study of $|\Delta \rho_t^\varepsilon| |\nabla h_t^\varepsilon|^n$. Write

$$|\Delta \rho_t^\varepsilon| \leq f_t^\varepsilon |\Delta g_t^\varepsilon| + g_t^\varepsilon |\Delta f_t^\varepsilon| + 2\varepsilon^{-2} \rho_t^\varepsilon |\nabla \varphi_t^\varepsilon| |\nabla \psi_t^\varepsilon|$$

and notice that the term $\rho_t^\varepsilon |\nabla \varphi_t^\varepsilon| |\nabla \psi_t^\varepsilon|$ can be handled as before and that by (A.7) and the maximum principle for the heat flow we have that

$$\Delta f_t^\varepsilon, \Delta g_t^\varepsilon \text{ are bounded in } L^\infty(X) \text{ locally uniformly in } t \in (0, 1). \quad (4.13)$$

Hence the conclusion follows from the Gaussian bounds (4.3).

Study of $\rho_t^\varepsilon \langle \nabla h_t^\varepsilon, \nabla \Delta h_t^\varepsilon \rangle$. Notice that

$$\begin{aligned} |\nabla \Delta \varphi_t^\varepsilon| &\leq \varepsilon \left(\frac{|\nabla \Delta f_t^\varepsilon|}{f_t^\varepsilon} + \frac{|\Delta f_t^\varepsilon| |\nabla f_t^\varepsilon|}{(f_t^\varepsilon)^2} + \frac{|\nabla |\nabla f_t^\varepsilon|^2|}{(f_t^\varepsilon)^2} + \frac{2|\nabla f_t^\varepsilon|^3}{(f_t^\varepsilon)^3} \right) \\ &\leq \frac{1}{f_t^\varepsilon} \left(\varepsilon |\nabla \Delta f_t^\varepsilon| + |\Delta f_t^\varepsilon| |\nabla \varphi_t^\varepsilon| + 2|\nabla \varphi_t^\varepsilon| |\text{Hess}(f_t^\varepsilon)|_{\text{HS}} + 2\varepsilon^{-2} f_t^\varepsilon |\nabla \varphi_t^\varepsilon|^3 \right) \end{aligned}$$

and therefore, using also $2g_t^\varepsilon |\nabla \varphi_t^\varepsilon|^2 |\text{Hess}(f_t^\varepsilon)|_{\text{HS}} \leq g_t^\varepsilon |\nabla \varphi_t^\varepsilon|^4 + g_t^\varepsilon |\text{Hess}(f_t^\varepsilon)|_{\text{HS}}^2$ we get

$$\begin{aligned} \rho_t^\varepsilon |\nabla \varphi_t^\varepsilon| |\nabla \Delta \varphi_t^\varepsilon| \\ \leq \varepsilon g_t^\varepsilon |\nabla \varphi_t^\varepsilon| |\nabla \Delta f_t^\varepsilon| + g_t^\varepsilon |\Delta f_t^\varepsilon| |\nabla \varphi_t^\varepsilon|^2 + g_t^\varepsilon |\nabla \varphi_t^\varepsilon|^4 + g_t^\varepsilon |\text{Hess}(f_t^\varepsilon)|_{\text{HS}}^2 + 2\varepsilon^{-2} \rho_t^\varepsilon |\nabla \varphi_t^\varepsilon|^4. \end{aligned}$$

The last term in the right-hand side is dominated locally uniformly in $t \in (0, 1)$ by what already proved. Similarly, the term $g_t^\varepsilon |\nabla \varphi_t^\varepsilon|^4$ is, locally in t , dominated thanks to the Gaussian bounds on g_t^ε , a domination for $g_t^\varepsilon |\Delta f_t^\varepsilon| |\nabla \varphi_t^\varepsilon|^2$ then follows using (4.13). Writing $\nabla \Delta f_t^\varepsilon = \nabla h_{t-\delta} \Delta f_\delta^\varepsilon$ for any $t \geq \delta > 0$ and using (A.7) and the Bakry-Émery estimates (A.9) we see that

$$|\nabla \Delta f_t^\varepsilon| \text{ is, locally in } t, \text{ uniformly bounded in } L^\infty(X), \quad (4.14)$$

thus a local uniform domination for $\varepsilon g_t^\varepsilon |\nabla \varphi_t^\varepsilon| |\nabla \Delta f_t^\varepsilon|$ follows.

It remains to consider the term $g_t^\varepsilon |\text{Hess}(f_t^\varepsilon)|_{\text{HS}}^2$: we know from (A.13) that $|\text{Hess}(f_t^\varepsilon)|_{\text{HS}} \in L^2(X)$ and from (4.3) that $g_t^\varepsilon \in L^\infty(X)$. This is sufficient to conclude that $\rho_t^\varepsilon \langle \nabla h_t^\varepsilon, \nabla \Delta h_t^\varepsilon \rangle \in L^1(X)$. To prove (4.12), thanks to the dominations previously obtained, it is enough to prove that

$$\lim_{R \rightarrow \infty} \int (1 - \chi_R) g_t^\varepsilon |\text{Hess}(f_t^\varepsilon)|_{\text{HS}}^2 \, \text{d}\mathbf{m} = 0 \quad \text{locally uniformly in } t \in (0, 1), \quad (4.15)$$

where for any $R > 0$ the function χ_R is a cut-off given by Lemma A.2. From (A.16) we have

$$\begin{aligned} \int (1 - \chi_R) g_t^\varepsilon |\text{Hess}(f_t^\varepsilon)|_{\text{HS}}^2 \, \text{d}\mathbf{m} &\leq \int \Delta((1 - \chi_R) g_t^\varepsilon) \frac{|\nabla f_t^\varepsilon|^2}{2} \\ &\quad + (1 - \chi_R) g_t^\varepsilon (\langle \nabla f_t^\varepsilon, \nabla \Delta f_t^\varepsilon \rangle - K |\nabla f_t^\varepsilon|^2) \, \text{d}\mathbf{m}. \end{aligned}$$

By (4.14), the already noticed fact that $|\nabla f_t^\varepsilon|$ is also uniformly bounded in $L^\infty(X)$ locally in $t \in (0, 1)$ and the Gaussian bounds (4.3) on g_t^ε we see that the second addend in the last integral is, locally in $t \in (0, 1)$, dominated by an $L^1(X)$ function.

For the first addend we write

$$\Delta((1 - \chi_R) g_t^\varepsilon) = -g_t^\varepsilon \Delta \chi_R - 2 \langle \nabla \chi_R, \nabla g_t^\varepsilon \rangle + (1 - \chi_R) \Delta g_t^\varepsilon \quad |\nabla f_t^\varepsilon| = \varepsilon^{-1} f_t^\varepsilon |\nabla \varphi_t^\varepsilon|$$

and use the properties of χ_R given by Lemma A.2 and those of $g_t^\varepsilon, f_t^\varepsilon, |\nabla \varphi_t^\varepsilon|$ that we already mentioned to deduce that $\Delta((1 - \chi_R) g_t^\varepsilon)$ is bounded in $L^\infty(X)$ and $|\nabla f_t^\varepsilon|^2$ dominated in

$L^1(X)$, both locally uniformly in $t \in (0, 1)$. Hence (4.15) follows from the fact that $(1 - \chi_R), |\nabla \chi_R|, \Delta \chi_R$ are identically 0 on $B_R(\bar{x})$.

It remains to prove that $t \mapsto \int \rho_t^\varepsilon \langle \nabla \varphi_t^\varepsilon, \nabla \Delta \varphi_t^\varepsilon \rangle \, d\mathbf{m}$ is continuous and thanks to (4.12) to this aim it is sufficient to show that for any $R > 0$ the map $t \mapsto \int \chi_R \rho_t^\varepsilon \langle \nabla \varphi_t^\varepsilon, \nabla \Delta \varphi_t^\varepsilon \rangle \, d\mathbf{m}$ is continuous. To see this, notice that

$$\int \chi_R \rho_t^\varepsilon \langle \nabla \varphi_t^\varepsilon, \nabla \Delta \varphi_t^\varepsilon \rangle \, d\mathbf{m} = - \int (\chi_R \langle \nabla \rho_t^\varepsilon, \nabla \varphi_t^\varepsilon \rangle + \rho_t^\varepsilon \langle \nabla \chi_R, \nabla \varphi_t^\varepsilon \rangle) \Delta \varphi_t^\varepsilon + \chi_R \rho_t^\varepsilon |\Delta \varphi_t^\varepsilon|^2 \, d\mathbf{m}, \quad (4.16)$$

and that the maps $t \mapsto \rho_t^\varepsilon, \varphi_t^\varepsilon$ are continuous with values in $W^{1,2}(X), W^{1,2}(X, e^{-V} \mathbf{m})$ respectively. Also, writing $\Delta \varphi_t^\varepsilon = \varepsilon \frac{\Delta f_t^\varepsilon}{f_t^\varepsilon} - \varepsilon |\nabla \varphi_t^\varepsilon|^2$, using the continuity of $t \mapsto f_t^\varepsilon, \Delta f_t^\varepsilon \in L^2(X)$, the bound (4.13), the fact that f_t^ε is bounded from below on $\text{supp}(\chi_R)$ by a positive constant depending continuously on t also taking into account what previously proved we see that the integrand in the right-hand side of (4.16) is continuous as map with values in $L^0(X, \mathbf{m}|_{\text{supp}(\chi_R)})$ and, locally in t , uniformly dominated by an $L^1(X)$ -function. This is sufficient to conclude. \square

Proposition 4.6 (Uniform L^∞ bound on the densities). *With the same assumptions and notations as in Setting 4.1 the following holds.*

For every $\bar{x} \in X$ there exist constants $C, C' > 0$ which depends on $K, N, \bar{x}, \rho_0, \rho_1$ such that

$$\rho_t^\varepsilon \leq C e^{-C' d^2(\cdot, \bar{x})} \quad \mathbf{m}\text{-a.e.} \quad (4.17)$$

for every $t \in [0, 1]$ and for every $\varepsilon \in (0, 1)$.

proof From (4.4) and direct manipulation we see that there are constants $c, c', r > 0$ depending on $K, N, \bar{x}, \rho_0, \rho_1$ only such that

$$\rho_t^\varepsilon(x) \leq c e^{-c' d^2(x, \bar{x})} \quad \forall x \notin B_r(\bar{x}), \quad \varepsilon \in (0, 1), \quad t \in [0, 1], \quad (4.18)$$

hence to conclude it is sufficient to show that there exists a constant $M > 0$ depending on $K, N, \bar{x}, \rho_0, \rho_1$ only such that

$$\|\rho_t^\varepsilon\|_{L^\infty(X)} \leq M \quad \forall \varepsilon \in (0, 1), \quad t \in [0, 1]. \quad (4.19)$$

For later purposes it will be useful to observe that from (4.18) and the volume growth estimate (A.20) it follows that there is $R > r$ such that

$$\int_{X \setminus B_R(\bar{x})} (\rho_t^\varepsilon)^p d^2(\cdot, \bar{x}) \, d\mathbf{m} \leq 1 \quad \forall \varepsilon \in (0, 1), \quad t \in [0, 1], \quad p \geq 2. \quad (4.20)$$

Now fix $\varepsilon > 0$. We know from Proposition 4.3 that $(\rho_t^\varepsilon) \in C([0, 1], L^2(X)) \cap AC_{loc}((0, 1), L^2(X))$ and by the maximum principle for the heat equation $\rho_t^\varepsilon \leq C_\varepsilon$ for all $t \in [0, 1]$, thus for any $p > 2$ the function $E_p : [0, 1] \rightarrow [0, \infty)$ defined by

$$E_p(t) := \int (\rho_t^\varepsilon)^p \, d\mathbf{m},$$

belongs to $C([0, 1]) \cap AC_{loc}((0, 1))$. An application of the dominated convergence theorem grants that its derivative can be computed passing the limit inside the integral, obtaining

$$\frac{d}{dt} E_p(t) = p \int (\rho_t^\varepsilon)^{p-1} \frac{d}{dt} \rho_t^\varepsilon \, d\mathbf{m} = -p \int (\rho_t^\varepsilon)^{p-1} \text{div}(\rho_t^\varepsilon \nabla \vartheta_t^\varepsilon) \, d\mathbf{m}.$$

Then the definition of ϑ_t^ε , (4.9a), (4.9b) and (4.4) allow to justify the integration by parts, whence

$$\begin{aligned} \frac{d}{dt} E_p(t) &= p(p-1) \int (\rho_t^\varepsilon)^{p-1} \langle \nabla \rho_t^\varepsilon, \nabla \vartheta_t^\varepsilon \rangle \, d\mathbf{m} \\ &= (p-1) \int \langle \nabla (\rho_t^\varepsilon)^p, \nabla \vartheta_t^\varepsilon \rangle \, d\mathbf{m} = -(p-1) \int (\rho_t^\varepsilon)^p \Delta \vartheta_t^\varepsilon \, d\mathbf{m} \end{aligned}$$

and recalling that $\vartheta_t^\varepsilon = \psi_t^\varepsilon - \frac{\varepsilon}{2} \log \rho_t^\varepsilon$ we obtain (for the same reasons as above, the integrals are well defined) that

$$\frac{d}{dt} E_p(t) = -(p-1) \int (\rho_t^\varepsilon)^p \Delta \psi_t^\varepsilon \, d\mathbf{m} + \frac{\varepsilon}{2} (p-1) \int (\rho_t^\varepsilon)^p \Delta \log \rho_t^\varepsilon \, d\mathbf{m}. \quad (4.21)$$

Now notice (the same arguments as above justify integration by parts) that

$$\int (\rho_t^\varepsilon)^p \Delta \log \rho_t^\varepsilon \, d\mathbf{m} = -p \int (\rho_t^\varepsilon)^{p-1} \langle \nabla \rho_t^\varepsilon, \nabla \log \rho_t^\varepsilon \rangle \, d\mathbf{m} = -p \int (\rho_t^\varepsilon)^{p-2} |\nabla \rho_t^\varepsilon|^2 \, d\mathbf{m} \leq 0$$

and choose $\delta := \frac{1}{2}$ and $T := 1$ in (3.4) to get the existence of a constant $c'' > 0$ depending on K, N, \bar{x} and the diameters of the supports of ρ_0, ρ_1 such that $\Delta \psi_t^\varepsilon \geq -c''(1 + \mathbf{d}^2(\cdot, \bar{x}))$ for any $t \in [0, 1/2]$. Thus from (4.21) we have

$$\frac{d}{dt} E_p(t) \leq c''(p-1)E_p(t) + c''(p-1) \int (\rho_t^\varepsilon)^p \mathbf{d}^2(\cdot, \bar{x}) \, d\mathbf{m} \quad a.e. \, t \in [0, 1/2]$$

and recalling (4.20) we get

$$\frac{d}{dt} E_p(t) \leq c''(p-1)E_p(t) + c''(p-1) \int_{B_R(\bar{x})} (\rho_t^\varepsilon)^p \mathbf{d}^2(\cdot, \bar{x}) \, d\mathbf{m} + 1 \leq c'''(p-1)E_p(t) + 1$$

for a.e. $t \in [0, 1/2]$. Then Gronwall's lemma gives

$$E_p(t) \leq \left(E_p(0) + \frac{1}{c'''(p-1)} \right) e^{c'''(p-1)t} \quad \forall t \in [0, 1/2].$$

Passing to the p -th roots, writing $E_p(t) = \|\rho_t^\varepsilon\|_{L^{p-1}(\mu_t^\varepsilon)}^{p-1}$ and observing that, being μ_t^ε a probability measure, we have $\|h\|_{L^p(\mu_t^\varepsilon)} \uparrow \|h\|_{L^\infty(\mu_t^\varepsilon)}$ as $p \rightarrow \infty$, we obtain

$$\|\rho_t^\varepsilon\|_{L^\infty} \leq e^{c'''} \|\rho_0\|_{L^\infty}, \quad \forall t \in [0, 1/2].$$

Switching the roles of ρ_0 and ρ_1 we get the analogous control for $t \in [\frac{1}{2}, 1]$, whence the claim (4.19) with $M := e^{c'''} \max\{\|\rho_0\|_{L^\infty}, \|\rho_1\|_{L^\infty}\}$. \square

4.3 The entropy along entropic interpolations

In [39] Léonard computed the first and second derivatives of the relative entropy along entropic interpolations: here we are going to show that his computations are fully justifiable in our setting. As we shall see later on, these formulas will be the crucial tool for showing that the acceleration of the entropic interpolation goes to 0 in the suitable weak sense.

We start by noticing that a form of Bochner inequality for the Schrödinger potentials can be deduced. Observe that in general the object $\Gamma_2(\varphi_t^\varepsilon)$ is not a well defined measure, because in some sense it can have both infinite positive mass and infinite negative mass, nevertheless, thanks to Lemma 4.5, its action on ρ_t^ε can still be defined: we will put

$$\langle \Gamma_2(h_t^\varepsilon), \rho_t^\varepsilon \rangle := \int \left(\frac{1}{2} \Delta \rho_t^\varepsilon |\nabla h_t^\varepsilon|^2 - \rho_t^\varepsilon \langle \nabla h_t^\varepsilon, \nabla \Delta h_t^\varepsilon \rangle \right) dm,$$

where h_t^ε is equal to any of $\varphi_t^\varepsilon, \psi_t^\varepsilon, \vartheta_t^\varepsilon, \log \rho_t^\varepsilon$, and notice that Lemma 4.5 ensures that the integral in the right-hand side is well defined and finite. We then have:

Lemma 4.7. *With the same assumptions and notations as in Setting 4.1, for any $\varepsilon > 0$ and $t \in (0, 1)$ we have*

$$\langle \Gamma_2(h_t^\varepsilon), \rho_t^\varepsilon \rangle \geq \int (|\text{Hess}(h_t^\varepsilon)|_{\text{HS}}^2 + K|\nabla h_t^\varepsilon|^2) \rho_t^\varepsilon dm \quad (4.22a)$$

$$\langle \Gamma_2(h_t^\varepsilon), \rho_t^\varepsilon \rangle \geq \int \left(\frac{(\Delta h_t^\varepsilon)^2}{N} + K|\nabla h_t^\varepsilon|^2 \right) \rho_t^\varepsilon dm \quad (4.22b)$$

where h_t^ε is equal to any of $\varphi_t^\varepsilon, \psi_t^\varepsilon, \vartheta_t^\varepsilon, \log \rho_t^\varepsilon$.

proof Fix $\varepsilon > 0$, $t \in (0, 1)$ and, for given $\bar{x} \in X$ and $R > 0$, let $\chi_R \in \text{Test}^\infty(X)$ be a cut-off function with support in $B_{R+1}(\bar{x})$ and such that $\chi_R \equiv 1$ in $B_R(\bar{x})$. Then we know that $\chi_{R+1}\varphi_t^\varepsilon \in \text{Test}^\infty(X)$ and thus (A.16) holds for it, namely

$$\Gamma_2(\chi_{R+1}\varphi_t^\varepsilon) \geq (|\text{Hess}(\chi_{R+1}\varphi_t^\varepsilon)|_{\text{HS}}^2 + K|\nabla(\chi_{R+1}\varphi_t^\varepsilon)|^2) m.$$

Multiplying both sides of the inequality by $\chi_R \rho_t^\varepsilon$, integrating over X and using the locality of the various differential operators appearing we obtain

$$\int \left(\frac{1}{2} \Delta(\chi_R \rho_t^\varepsilon) |\nabla \varphi_t^\varepsilon|^2 - \chi_R \rho_t^\varepsilon \langle \nabla \varphi_t^\varepsilon, \nabla \Delta \varphi_t^\varepsilon \rangle \right) dm \geq \int \chi_R \rho_t^\varepsilon (|\text{Hess}(\varphi_t^\varepsilon)|_{\text{HS}}^2 + K|\nabla \varphi_t^\varepsilon|^2) dm. \quad (4.23)$$

By monotone convergence we have that

$$\begin{aligned} \lim_{R \rightarrow \infty} \int \chi_R \rho_t^\varepsilon |\nabla \varphi_t^\varepsilon|^2 dm &= \int \rho_t^\varepsilon |\nabla \varphi_t^\varepsilon|^2 dm, \\ \lim_{R \rightarrow \infty} \int \chi_R \rho_t^\varepsilon |\text{Hess}(\varphi_t^\varepsilon)|_{\text{HS}}^2 dm &= \int \rho_t^\varepsilon |\text{Hess}(\varphi_t^\varepsilon)|_{\text{HS}}^2 dm. \end{aligned}$$

and thus the right-hand side of (4.23) goes to the right-hand side of (4.22a). Now notice that

$$\Delta(\chi_R \rho_t^\varepsilon) = \chi_R \Delta \rho_t^\varepsilon + 2 \langle \nabla \rho_t^\varepsilon, \nabla \chi_R \rangle + \rho_t^\varepsilon \Delta \chi_R$$

and that the choice of χ_R grants that $|\chi_R|, |\nabla \chi_R|, |\Delta \chi_R|$ are uniformly bounded and m -a.e. converge to 1, 0, 0 respectively as $R \rightarrow \infty$. Hence Lemma 4.5 and the dominated convergence theorem give that the left-hand side of (4.23) goes to $\langle \Gamma_2(h_t^\varepsilon), \rho_t^\varepsilon \rangle$, thus settling the proof of (4.22a) for $h_t^\varepsilon = \varphi_t^\varepsilon$. The other claims follow by similar means taking (A.17) into account. \square

Now we are in position for motivating Léonard's computations, thus getting the formulas for the first and second derivative of the entropy along entropic interpolations.

Proposition 4.8. *With the same assumptions and notations as in Setting 4.1 the following holds.*

For any $\varepsilon > 0$ the map $t \mapsto H(\mu_t^\varepsilon | \mathbf{m})$ belongs to $C([0, 1]) \cap C^2(0, 1)$ and for every $t \in (0, 1)$ it holds

$$\frac{d}{dt} H(\mu_t^\varepsilon | \mathbf{m}) = \int \langle \nabla \rho_t^\varepsilon, \nabla \vartheta_t^\varepsilon \rangle \, d\mathbf{m} = \frac{1}{2\varepsilon} \int (|\nabla \psi_t^\varepsilon|^2 - |\nabla \varphi_t^\varepsilon|^2) \rho_t^\varepsilon \, d\mathbf{m}, \quad (4.24a)$$

$$\frac{d^2}{dt^2} H(\mu_t^\varepsilon | \mathbf{m}) = \left\langle \Gamma_2(\vartheta_t^\varepsilon), \rho_t^\varepsilon \right\rangle + \frac{\varepsilon^2}{4} \left\langle \Gamma_2(\log \rho_t^\varepsilon), \rho_t^\varepsilon \right\rangle = \frac{1}{2} \left\langle \Gamma_2(\varphi_t^\varepsilon), \rho_t^\varepsilon \right\rangle + \frac{1}{2} \left\langle \Gamma_2(\psi_t^\varepsilon), \rho_t^\varepsilon \right\rangle. \quad (4.24b)$$

proof By Lemma 4.5 we know that the central and right-hand sides in (4.24a) and (4.24b) exist, are finite and continuously depend on $t \in (0, 1)$. Also, the equality between the central and right-hand sides follows trivially from the relations $\vartheta_t^\varepsilon = \frac{\psi_t^\varepsilon - \varphi_t^\varepsilon}{2}$ and $\varepsilon \log \rho_t^\varepsilon = \varphi_t^\varepsilon + \psi_t^\varepsilon$.

Thus to conclude it is sufficient to show that $t \mapsto H(\mu_t^\varepsilon | \mathbf{m})$ is in $C([0, 1])$ and that the identities (4.24a) and (4.24b) hold for a.e. $t \in (0, 1)$.

Lemma 4.5 ensures that $t \mapsto H(\mu_t^\varepsilon | \mathbf{m})$ is continuous in $(0, 1)$. To check continuity in $t = 0, 1$, thanks to the fact that $(\rho_t^\varepsilon) \in C([0, 1], L^2(X))$ by Proposition 4.3 and arguing as in the proof of Lemma 4.5, it is sufficient to show that $\rho_t^\varepsilon \log \rho_t^\varepsilon$ is dominated by an $L^1(X)$ function. To see this, write

$$\rho_t^\varepsilon \log \rho_t^\varepsilon = g_t^\varepsilon f_t^\varepsilon \log f_t^\varepsilon + f_t^\varepsilon g_t^\varepsilon \log g_t^\varepsilon$$

and notice that for $t \in [0, 1/2]$ the bound (4.3) ensures that the function g_t^ε is uniformly bounded above by a Gaussian and that $\log g_t^\varepsilon$ has a quadratic growth. On the other hand, we know by Theorem 2.2 that $f_0^\varepsilon = f^\varepsilon$ is in L^∞ , thus the maximum principle for the heat flow and the fact that $z \mapsto z \log z$ is bounded from below give that the L^∞ norms of $f_t^\varepsilon, f_t^\varepsilon \log f_t^\varepsilon$ are uniformly bounded in $t \in [0, 1/2]$. As discussed in the proof of Lemma 4.5, this is sufficient to conclude and a similar arguments yields the desired bound for $t \in [1/2, 1]$.

Now fix $\varepsilon > 0$ and for $R > 0$ let $\chi_R \in \text{Test}^\infty(X)$ be a cut-off function as given by Lemma A.2. Notice that Lemma 4.5 grants that

$$\int \chi_R \rho_t^\varepsilon \log \rho_t^\varepsilon \, d\mathbf{m} \quad \rightarrow \quad \int \rho_t^\varepsilon \log \rho_t^\varepsilon \, d\mathbf{m} \quad \text{as } R \rightarrow \infty \text{ for every } t \in (0, 1). \quad (4.25)$$

Also, Proposition 4.3 tells that $(\rho_t^\varepsilon) \in AC_{loc}((0, 1), L^2(X))$ and that it is, locally in $t \in (0, 1)$ and in space, uniformly bounded away from 0 and ∞ . Therefore, for $u(z) := z \log z$ we have that $(0, 1) \ni t \mapsto \chi_R u(\rho_t^\varepsilon) \in L^2(X)$ is absolutely continuous. In particular, so is $\int \chi_R u(\rho_t^\varepsilon) \, d\mathbf{m}$ and it is then clear that

$$\frac{d}{dt} \int \chi_R u(\rho_t^\varepsilon) \, d\mathbf{m} = \int \chi_R (\log(\rho_t^\varepsilon) + 1) \frac{d}{dt} \rho_t^\varepsilon \, d\mathbf{m}, \quad \text{a.e. } t.$$

Using the formula for $\frac{d}{dt} \rho_t^\varepsilon$ provided by Proposition 4.3 we then get

$$\begin{aligned} \frac{d}{dt} \int \chi_R u(\rho_t^\varepsilon) \, d\mathbf{m} &= - \int \chi_R (\log(\rho_t^\varepsilon) + 1) \text{div}(\rho_t^\varepsilon \nabla \vartheta_t^\varepsilon) \, d\mathbf{m} = \int \langle \nabla (\chi_R (\log(\rho_t^\varepsilon) + 1)), \nabla \vartheta_t^\varepsilon \rangle \rho_t^\varepsilon \, d\mathbf{m} \\ &= \int \chi_R \langle \nabla \rho_t^\varepsilon, \nabla \vartheta_t^\varepsilon \rangle \, d\mathbf{m} + \int \langle \nabla \chi_R, \nabla \vartheta_t^\varepsilon \rangle (\log \rho_t^\varepsilon + 1) \rho_t^\varepsilon \, d\mathbf{m}. \end{aligned}$$

Since $|\nabla\chi_R|$ is uniformly bounded and identically 0 on $B_R(\bar{x})$, Lemma 4.5 grants that the last expression in the above identity converges to $\int \langle \nabla\rho_t^\varepsilon, \nabla\vartheta_t^\varepsilon \rangle \, \mathrm{d}\mathbf{m}$ as $R \rightarrow \infty$ locally uniformly in $t \in (0, 1)$. This fact, (4.25) and the initial discussion give $C^1((0, 1))$ regularity for $t \mapsto H(\mu_t^\varepsilon | \mathbf{m})$ and (4.24a).

For (4.24b), notice that from Proposition 4.3 we know that $(\rho_t^\varepsilon) \in AC_{loc}((0, 1), W^{1,2}(\mathbf{X}))$ and $(\vartheta_t^\varepsilon) \in AC_{loc}((0, 1), W^{1,2}(\mathbf{X}, e^{-V}\mathbf{m}))$ with $V = M\mathbf{d}^2(\cdot, \bar{x})$, for some $\bar{x} \in \mathbf{X}$ and $M > 0$ sufficiently large. Hence $(0, 1) \ni t \mapsto \chi_R \langle \nabla\rho_t^\varepsilon, \nabla\vartheta_t^\varepsilon \rangle \in L^2(\mathbf{X})$ is absolutely continuous. In particular, so is $\int \chi_R \langle \nabla\rho_t^\varepsilon, \nabla\vartheta_t^\varepsilon \rangle \, \mathrm{d}\mathbf{m}$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \chi_R \langle \nabla\rho_t^\varepsilon, \nabla\vartheta_t^\varepsilon \rangle \, \mathrm{d}\mathbf{m} = \int \chi_R \left(\left\langle \nabla \frac{\mathrm{d}}{\mathrm{d}t} \rho_t^\varepsilon, \nabla\vartheta_t^\varepsilon \right\rangle + \left\langle \nabla\rho_t^\varepsilon, \nabla \frac{\mathrm{d}}{\mathrm{d}t} \vartheta_t^\varepsilon \right\rangle \right) \, \mathrm{d}\mathbf{m}, \quad \text{a.e. } t.$$

Thus from the formulas for $\frac{\mathrm{d}}{\mathrm{d}t}\rho_t^\varepsilon, \frac{\mathrm{d}}{\mathrm{d}t}\vartheta_t^\varepsilon$ provided in Proposition 4.3 we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int \chi_R \langle \nabla\rho_t^\varepsilon, \nabla\vartheta_t^\varepsilon \rangle \, \mathrm{d}\mathbf{m} &= \underbrace{\int -\chi_R \langle \nabla(\operatorname{div}(\rho_t^\varepsilon \nabla\vartheta_t^\varepsilon)), \nabla\vartheta_t^\varepsilon \rangle \, \mathrm{d}\mathbf{m}}_{A_t(R)} \\ &\quad + \underbrace{\int \chi_R \langle \nabla\rho_t^\varepsilon, \nabla \left(-\frac{1}{2} |\nabla\vartheta_t^\varepsilon|^2 - \frac{\varepsilon^2}{4} \Delta \log(\rho_t^\varepsilon) - \frac{\varepsilon^2}{8} |\nabla \log(\rho_t^\varepsilon)|^2 \right) \rangle \, \mathrm{d}\mathbf{m}}_{B_t(R)}. \end{aligned}$$

Now notice that a few integration by parts and the Leibniz rule give

$$\begin{aligned} A_t(R) &= \int \operatorname{div}(\rho_t^\varepsilon \nabla\vartheta_t^\varepsilon) \langle \nabla\chi_R, \nabla\vartheta_t^\varepsilon \rangle \, \mathrm{d}\mathbf{m} + \int \chi_R \operatorname{div}(\rho_t^\varepsilon \nabla\vartheta_t^\varepsilon) \Delta\vartheta_t^\varepsilon \, \mathrm{d}\mathbf{m} \\ &= \int \langle \nabla\rho_t^\varepsilon, \nabla\vartheta_t^\varepsilon \rangle \langle \nabla\chi_R, \nabla\vartheta_t^\varepsilon \rangle \, \mathrm{d}\mathbf{m} - \int \chi_R \rho_t^\varepsilon \langle \nabla\vartheta_t^\varepsilon, \nabla\Delta\vartheta_t^\varepsilon \rangle \, \mathrm{d}\mathbf{m} \end{aligned}$$

and

$$\begin{aligned} B_t(R) &= \int \frac{1}{2} |\nabla\vartheta_t^\varepsilon|^2 \operatorname{div}(\chi_R \nabla\rho_t^\varepsilon) - \frac{\varepsilon^2}{4} \chi_R \langle \nabla\rho_t^\varepsilon, \nabla\Delta \log \rho_t^\varepsilon \rangle + \frac{\varepsilon^2}{8} |\nabla \log \rho_t^\varepsilon|^2 \operatorname{div}(\chi_R \nabla\rho_t^\varepsilon) \, \mathrm{d}\mathbf{m} \\ &= \int \frac{1}{2} \chi_R \Delta\rho_t^\varepsilon |\nabla\vartheta_t^\varepsilon|^2 - \frac{\varepsilon^2}{4} \chi_R \rho_t^\varepsilon \langle \nabla \log \rho_t^\varepsilon, \nabla\Delta \log \rho_t^\varepsilon \rangle + \frac{\varepsilon^2}{8} \chi_R \Delta\rho_t^\varepsilon |\nabla \log \rho_t^\varepsilon|^2 \, \mathrm{d}\mathbf{m} \\ &\quad + \int \frac{1}{2} |\nabla\vartheta_t^\varepsilon|^2 \langle \nabla\chi_R, \nabla\rho_t^\varepsilon \rangle + \frac{\varepsilon^2}{8} |\nabla \log \rho_t^\varepsilon|^2 \langle \nabla\chi_R, \nabla\rho_t^\varepsilon \rangle \, \mathrm{d}\mathbf{m}. \end{aligned}$$

Since $|\nabla\chi_R|$ is uniformly bounded and identically 0 on $B_R(\bar{x})$, Lemma 4.5 gives that

$$A_t(R) + B_t(R) \quad \rightarrow \quad \left\langle \Gamma_2(\vartheta_t^\varepsilon), \rho_t^\varepsilon \right\rangle + \frac{\varepsilon^2}{4} \left\langle \Gamma_2(\log \rho_t^\varepsilon), \rho_t^\varepsilon \right\rangle \quad \text{as } R \rightarrow \infty$$

locally uniformly in $t \in (0, 1)$.

This fact, the convergence of $\int \chi_R \langle \nabla\rho_t^\varepsilon, \nabla\vartheta_t^\varepsilon \rangle \, \mathrm{d}\mathbf{m}$ to $\int \langle \nabla\rho_t^\varepsilon, \nabla\vartheta_t^\varepsilon \rangle \, \mathrm{d}\mathbf{m}$ as $R \rightarrow \infty$ (which is also consequence of Lemma 4.5) and the initial discussion give the conclusion. \square

As a first consequence of the formulas just obtained, we show that some quantities remain bounded as $\varepsilon \downarrow 0$:

Lemma 4.9 (Bounded quantities). *With the same assumptions and notations of Setting 4.1, for any $\bar{x} \in X$ we have*

$$\sup_{\varepsilon \in (0,1), t \in [0,1]} \int d^2(\cdot, \bar{x}) \rho_t^\varepsilon dm < \infty, \quad (4.26a)$$

$$\sup_{\varepsilon \in (0,1), t \in [0,1]} |H(\mu_t^\varepsilon | \mathbf{m})| < \infty, \quad (4.26b)$$

$$\sup_{\varepsilon \in (0,1)} \iint_0^1 \left(|\nabla \vartheta_t^\varepsilon|^2 + \varepsilon^2 |\nabla \log \rho_t^\varepsilon|^2 \right) \rho_t^\varepsilon dt dm < \infty, \quad (4.26c)$$

and for any $\delta \in (0, \frac{1}{2})$

$$\sup_{\varepsilon \in (0,1)} \iint_\delta^{1-\delta} \left(|\text{Hess}(\vartheta_t^\varepsilon)|_{\text{HS}}^2 + \varepsilon^2 |\text{Hess}(\log \rho_t^\varepsilon)|_{\text{HS}}^2 \right) \rho_t^\varepsilon dt dm < \infty, \quad (4.27a)$$

$$\sup_{\varepsilon \in (0,1)} \iint_\delta^{1-\delta} \left(|\Delta \vartheta_t^\varepsilon|^2 + \varepsilon^2 |\Delta \log \rho_t^\varepsilon|^2 \right) \rho_t^\varepsilon dt dm < \infty. \quad (4.27b)$$

proof (4.26a) follows from (4.17) and the volume growth (A.20). As regards (4.26b), notice that (4.26a) and (2.11) give a uniform lower bound on $H(\mu_t^\varepsilon | \mathbf{m})$; for the upper bound notice that (4.17) implies a uniform quadratic growth on $\log \rho_t^\varepsilon$.

Let us now pass to (4.26c) and observe that Proposition 4.4 together with (4.26a) grants

$$\sup_{\varepsilon \in (0,1)} \iint_{\frac{1}{2}}^1 |\nabla \varphi_t^\varepsilon|^2 \rho_t^\varepsilon dt dm + \iint_0^{\frac{1}{2}} |\nabla \psi_t^\varepsilon|^2 \rho_t^\varepsilon dt dm < \infty. \quad (4.28)$$

As a second step, notice that (4.24a) gives

$$\begin{aligned} \iint_0^{\frac{1}{2}} |\nabla \varphi_t^\varepsilon|^2 \rho_t^\varepsilon dt dm &= \iint_0^{\frac{1}{2}} |\nabla \psi_t^\varepsilon|^2 \rho_t^\varepsilon dt dm - 2\varepsilon \int_0^{\frac{1}{2}} \frac{d}{dt} H(\mu_t^\varepsilon | \mathbf{m}) dt \\ &= \iint_0^{\frac{1}{2}} |\nabla \psi_t^\varepsilon|^2 \rho_t^\varepsilon dt dm + 2\varepsilon (H(\mu_0 | \mathbf{m}) - H(\mu_{\frac{1}{2}}^\varepsilon | \mathbf{m})) \end{aligned}$$

so that taking into account (4.26b) and (4.28) we see that the right-hand side is uniformly bounded for $\varepsilon \in (0, 1)$. Using again (4.28) we deduce that

$$\sup_{\varepsilon \in (0,1)} \iint_0^1 |\nabla \varphi_t^\varepsilon|^2 \rho_t^\varepsilon dt dm < \infty.$$

A symmetric argument provides the analogous bound for (ψ_t^ε) and thus recalling that $\vartheta_t^\varepsilon = \frac{1}{2}(\psi_t^\varepsilon - \varphi_t^\varepsilon)$ and $\varepsilon \log \rho_t^\varepsilon = \psi_t^\varepsilon + \varphi_t^\varepsilon$ we obtain (4.26c).

Now use the fact that $\vartheta_t^\varepsilon = -\varphi_t^\varepsilon + \frac{\varepsilon}{2} \log \rho_t^\varepsilon$ in conjunction with (4.24a) to get

$$\begin{aligned} \frac{d}{dt} H(\mu_t^\varepsilon | \mathbf{m})|_{t=\delta} &= - \int \langle \nabla \rho_\delta^\varepsilon, \nabla \varphi_\delta^\varepsilon \rangle dm + \frac{\varepsilon}{2} \int \langle \nabla \rho_\delta^\varepsilon, \nabla \log \rho_\delta^\varepsilon \rangle dm \\ &= \int \rho_\delta^\varepsilon \Delta \varphi_\delta^\varepsilon dm + \frac{\varepsilon}{2} \int \frac{|\nabla \rho_\delta^\varepsilon|^2}{\rho_\delta^\varepsilon} dm \geq \int \rho_\delta^\varepsilon \Delta \varphi_\delta^\varepsilon dm. \end{aligned}$$

Recalling the lower bound (4.9b) and (4.26a), we get that for some constant C_δ independent on ε it holds

$$\frac{d}{dt}H(\mu_t^\varepsilon | \mathbf{m})|_{t=\delta} \geq -C_\delta \quad \forall \varepsilon \in (0, 1)$$

and an analogous argument starting from $\vartheta_t^\varepsilon = \psi_t^\varepsilon - \frac{\varepsilon}{2} \log \rho_t^\varepsilon$ yields $\frac{d}{dt}H(\mu_t^\varepsilon | \mathbf{m})|_{t=1-\delta} \leq C_\delta$ for every $\varepsilon \in (0, 1)$. Therefore

$$\sup_{\varepsilon \in (0,1)} \int_\delta^{1-\delta} \frac{d^2}{dt^2}H(\mu_t^\varepsilon | \mathbf{m}) = \sup_{\varepsilon \in (0,1)} \left(\frac{d}{dt}H(\mu_t^\varepsilon | \mathbf{m})|_{t=1-\delta} - \frac{d}{dt}H(\mu_t^\varepsilon | \mathbf{m})|_{t=\delta} \right) < \infty.$$

The bounds (4.27a) and (4.27b) then come from this last inequality used in conjunction with (4.24b), (4.26c) and the weighted Bochner inequalities (4.22a) and (4.22b) respectively. \square

With the help of the previous lemma we can now prove that some crucial quantities vanish in the limit $\varepsilon \downarrow 0$; as we shall see in the proof of our main Theorem 5.12, this is what we will need to prove that the acceleration of the entropic interpolations goes to 0 as ε goes to zero.

Lemma 4.10 (Vanishing quantities). *With the same assumptions and notations of Setting 4.1, for any $\delta \in (0, \frac{1}{2})$ we have*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \iint_\delta^{1-\delta} \rho_t^\varepsilon |\Delta \log \rho_t^\varepsilon| dt d\mathbf{m} = 0, \quad (4.29a)$$

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \int_\delta^{1-\delta} \rho_t^\varepsilon |\nabla \log \rho_t^\varepsilon|^2 dt d\mathbf{m} = 0, \quad (4.29b)$$

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \iint_\delta^{1-\delta} \rho_t^\varepsilon |\Delta \log \rho_t^\varepsilon| |\nabla \log \rho_t^\varepsilon| dt d\mathbf{m} = 0, \quad (4.29c)$$

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \iint_\delta^{1-\delta} \rho_t^\varepsilon |\nabla \log \rho_t^\varepsilon|^3 dt d\mathbf{m} = 0. \quad (4.29d)$$

proof For (4.29a) we notice that

$$\varepsilon^2 \iint_\delta^{1-\delta} \rho_t^\varepsilon |\Delta \log \rho_t^\varepsilon| dt d\mathbf{m} \leq \varepsilon \sqrt{1-2\delta} \sqrt{\varepsilon^2 \iint_\delta^{1-\delta} \rho_t^\varepsilon |\Delta \log \rho_t^\varepsilon|^2 dt d\mathbf{m}}$$

and the fact that, by (4.27b), the last square root is uniformly bounded in $\varepsilon \in (0, 1)$.

For (4.29b) we start observing that Lemma 4.11 below applies to ρ_t^ε , because by Proposition 4.3 $\rho_t^\varepsilon \in \text{Test}^\infty(X) \cap L^1(X)$ and

$$\Delta \rho_t^\varepsilon = f_t^\varepsilon \Delta g_t^\varepsilon + g_t^\varepsilon \Delta f_t^\varepsilon + 2\langle \nabla f_t^\varepsilon, \nabla g_t^\varepsilon \rangle \in L^1(X).$$

Hence, from the identity $\rho_t^\varepsilon |\nabla \log \rho_t^\varepsilon|^2 = -\rho_t^\varepsilon \Delta \log \rho_t^\varepsilon + \Delta \rho_t^\varepsilon$ and the fact that $\int \Delta \rho_t^\varepsilon d\mathbf{m} = 0$ we get

$$\varepsilon^2 \iint_\delta^{1-\delta} \rho_t^\varepsilon |\nabla \log \rho_t^\varepsilon|^2 dt d\mathbf{m} = -\varepsilon^2 \iint_\delta^{1-\delta} \rho_t^\varepsilon \Delta \log \rho_t^\varepsilon dt d\mathbf{m} \leq \varepsilon^2 \iint_\delta^{1-\delta} \rho_t^\varepsilon |\Delta \log \rho_t^\varepsilon| dt d\mathbf{m}$$

and then conclude by (4.29a).

For (4.29c) we observe that

$$\begin{aligned} \varepsilon^2 \iint_{\delta}^{\varepsilon^{1-\delta}} \rho_t^\varepsilon |\Delta \log \rho_t^\varepsilon| |\nabla \log \rho_t^\varepsilon| dt dm \\ \leq \sqrt{\varepsilon^2 \iint_{\delta}^{\varepsilon^{1-\delta}} \rho_t^\varepsilon |\Delta \log \rho_t^\varepsilon|^2 dt dm} \sqrt{\varepsilon^2 \iint_{\delta}^{\varepsilon^{1-\delta}} \rho_t^\varepsilon |\nabla \log \rho_t^\varepsilon|^2 dt dm}, \end{aligned}$$

and use the fact that the first square root in the right-hand side is bounded (by (4.27b)) and the second one goes to 0 (by (4.29b)).

To prove (4.29d) we start again from the identity $\rho_t^\varepsilon |\nabla \log \rho_t^\varepsilon|^2 = -\rho_t^\varepsilon \Delta \log \rho_t^\varepsilon + \Delta \rho_t^\varepsilon$ to get

$$\iint_{\delta}^{\varepsilon^{1-\delta}} \rho_t^\varepsilon |\nabla \log \rho_t^\varepsilon|^3 dt dm = - \iint_{\delta}^{\varepsilon^{1-\delta}} \rho_t^\varepsilon \Delta \log \rho_t^\varepsilon |\nabla \log \rho_t^\varepsilon| dt dm + \iint_{\delta}^{\varepsilon^{1-\delta}} \Delta \rho_t^\varepsilon |\nabla \log \rho_t^\varepsilon| dt dm.$$

After a multiplication by ε^2 we see that the first integral on the right-hand side vanishes as $\varepsilon \downarrow 0$ thanks to (4.29c). For the second we start noticing that an application of the dominated convergence theorem ensures that

$$\iint_{\delta}^{\varepsilon^{1-\delta}} \Delta \rho_t^\varepsilon |\nabla \log \rho_t^\varepsilon| dt dm = \lim_{\eta \downarrow 0} \iint_{\delta}^{\varepsilon^{1-\delta}} \Delta \rho_t^\varepsilon \sqrt{\eta + |\nabla \log \rho_t^\varepsilon|^2} dt dm, \quad (4.30)$$

then we observe that for every $\eta > 0$ the map $z \mapsto \sqrt{\eta + z}$ is in $C^1 \cap \text{LIP}([0, \infty))$ and since $|\nabla \log \rho_t^\varepsilon|^2 \in W^{1,2}(\mathbf{X}, e^{-V})$ for $V = M d^2(\cdot, \bar{x})$ and suitable \bar{x}, M (recall Proposition 4.3) we deduce that $\sqrt{\eta + |\nabla \log \rho_t^\varepsilon|^2} \in W^{1,2}(\mathbf{X}, e^{-V})$ as well. Thus by the chain rule for gradients, the Leibniz rule (A.15) and also using a cut-off argument in conjunction with Lemma 4.5 to justify integration by parts, we see that it holds

$$\begin{aligned} \left| \iint_{\delta}^{\varepsilon^{1-\delta}} \Delta \rho_t^\varepsilon \sqrt{\eta + |\nabla \log \rho_t^\varepsilon|^2} dt dm \right| \\ = \left| \iint_{\delta}^{\varepsilon^{1-\delta}} \frac{\rho_t^\varepsilon}{2\sqrt{\eta + |\nabla \log \rho_t^\varepsilon|^2}} \langle \nabla \log \rho_t^\varepsilon, \nabla |\nabla \log \rho_t^\varepsilon|^2 \rangle dt dm \right| \\ = \left| \iint_{\delta}^{\varepsilon^{1-\delta}} \frac{\rho_t^\varepsilon}{\sqrt{\eta + |\nabla \log \rho_t^\varepsilon|^2}} \text{Hess}(\log \rho_t^\varepsilon) (\nabla \log \rho_t^\varepsilon, \nabla \log \rho_t^\varepsilon) dt dm \right| \\ \leq \iint_{\delta}^{\varepsilon^{1-\delta}} \rho_t^\varepsilon |\text{Hess}(\log \rho_t^\varepsilon)|_{\text{HS}} |\nabla \log \rho_t^\varepsilon| dt dm \end{aligned}$$

and being this true for any $\eta > 0$, from (4.30) we obtain

$$\begin{aligned} \varepsilon^2 \left| \iint_{\delta}^{\varepsilon^{1-\delta}} \Delta \rho_t^\varepsilon |\nabla \log \rho_t^\varepsilon| dt dm \right| &\leq \varepsilon^2 \iint_{\delta}^{\varepsilon^{1-\delta}} \rho_t^\varepsilon |\text{Hess}(\log \rho_t^\varepsilon)|_{\text{HS}} |\nabla \log \rho_t^\varepsilon| dt dm \\ &\leq \sqrt{\varepsilon^2 \iint_{\delta}^{\varepsilon^{1-\delta}} \rho_t^\varepsilon |\text{Hess}(\log \rho_t^\varepsilon)|_{\text{HS}}^2 dt dm} \\ &\quad \times \sqrt{\varepsilon^2 \iint_{\delta}^{\varepsilon^{1-\delta}} \rho_t^\varepsilon |\nabla \log \rho_t^\varepsilon|^2 dt dm}. \end{aligned}$$

In this last expression the first square root is uniformly bounded in $\varepsilon \in (0, 1)$ by (4.27a), while the second one vanishes as $\varepsilon \downarrow 0$ thanks to (4.29b).

The last claim follows from the fact that under the stated additional regularity properties of ρ_0, ρ_1 we can take $\delta = 0$ in (4.27a), (4.27b). Then we argue as before. \square

Lemma 4.11. *Let (X, d, \mathbf{m}) be an $\text{RCD}^*(K, \infty)$ space with $K \in \mathbb{R}$ endowed with a Borel non-negative measure \mathbf{m} which is finite on bounded sets and $h \in D(\Delta) \cap L^1(X)$ with $\Delta h \in L^1(X)$. Then*

$$\int \Delta h d\mathbf{m} = 0$$

proof Let $\bar{x} \in X$, $R > 0$ and $\chi_R \in \text{Test}^\infty(X)$ be a cut-off function as given by Lemma A.2. Then

$$\left| \int \chi_R \Delta h d\mathbf{m} \right| = \left| \int \Delta \chi_R h d\mathbf{m} \right| = \left| \int_{X \setminus B_{R+1}(\bar{x})} \Delta \chi_R h d\mathbf{m} \right| \leq \|\Delta \chi_R\|_{L^\infty(X)} \int_{X \setminus B_{R+1}(\bar{x})} h d\mathbf{m}.$$

Since Lemma A.2 ensures that $\|\Delta \chi_R\|_{L^\infty(X)}$ is uniformly bounded in R , the conclusion follows letting $R \rightarrow \infty$ in the above. \square

5 From entropic to displacement interpolations

5.1 Compactness

Starting from the uniform estimates discussed in Section 4, let us first prove that when we pass to the limit as $\varepsilon \downarrow 0$, up to subsequences Schrödinger potentials and entropic interpolations converge in a suitable sense to limit potentials and interpolations.

To formulate the result we need to introduce the Banach space $(C(X, e^{-V}), \|\cdot\|_{C(X, e^{-V})})$, where $V = M d^2(\cdot, \bar{x})$ for some $\bar{x} \in X$ and $M > 0$: the norm $\|\cdot\|_{C(X, e^{-V})}$ is defined as

$$\|f\|_{C(X, e^{-V})} := \sup_{x \in X} |f(x)| e^{-V(x)}$$

and $C(X, e^{-V}) := \{f \in C(X) : \|f\|_{C(X, e^{-V})} < \infty\}$.

Proposition 5.1 (Compactness for measures). *With the same assumptions and notations as in Setting 4.1 the following holds.*

For any sequence $\varepsilon_n \downarrow 0$ there exists a subsequence, not relabeled, such that the curves $(\mu_t^{\varepsilon_n})$ uniformly converge in $(\mathcal{P}_2(X), W_2)$ to a limit curve (μ_t) belonging to $AC([0, 1], (\mathcal{P}_2(X), W_2))$. Moreover, there is $M > 0$ so that

$$\mu_t \leq M \mathbf{m} \quad \forall t \in [0, 1] \tag{5.1}$$

and setting $\rho_t := \frac{d\mu_t}{d\mathbf{m}}$ it holds

$$\rho_t^{\varepsilon_n} \xrightarrow{*} \rho_t \quad \text{in } L^\infty(X) \quad \forall t \in [0, 1]. \tag{5.2}$$

proof Fix $\varepsilon \in (0, 1)$; we want to apply Theorem A.5 to (μ_t^ε) and $(\nabla \vartheta_t^\varepsilon)$. The continuity of $t \mapsto \rho_t^\varepsilon \in L^2(X)$ granted by Proposition 4.3 yields weak continuity of (μ_t) and (A.23a) is a consequence of (4.17). From the bound (4.26c) it follows (A.23b) and from the formula for $\frac{d}{dt} \rho_t^\varepsilon$ given in Proposition 4.3 and again the L^2 -continuity of (ρ_t^ε) on $[0, 1]$ it easily follows that (μ_t) and $(\vartheta_t^\varepsilon)$ solve the continuity equation in the sense of Theorem A.5. The conclusion of such theorem ensures that (μ_t^ε) is W_2 -absolutely continuous with

$$\int_0^1 |\dot{\mu}_t^\varepsilon|^2 dt = \iint_0^1 |\nabla \vartheta_t^\varepsilon|^2 \rho_t^\varepsilon dt d\mathbf{m}.$$

The bound (4.26c) grants that the right-hand side is uniformly bounded in $\varepsilon \in (0, 1)$ and since $\{(\mu_t^\varepsilon)\}_\varepsilon$ is tight and 2-uniformly integrable by (4.17) (hence W_2 -compact), this is sufficient to ensure the compactness of the family $\{(\mu_t^\varepsilon)\}_\varepsilon$ in $C([0, 1], (\mathcal{P}_2(\mathbf{X}), W_2))$ and, by the lower semicontinuity of the kinetic energy, the fact that any limit curve (μ_t) is absolutely continuous. The bound (5.1) is then a direct consequence of the uniform bound (4.19) and the convergence property (5.2) comes from the weak convergence of the measures and the uniform bound on the densities. \square

Proposition 5.2 (Compactness for potentials). *With the same assumptions and notations as in Setting 4.1 the following holds.*

For any sequence $\varepsilon_n \downarrow 0$ there exists a subsequence, not relabeled, such that for all $\bar{x} \in \mathbf{X}$ and $M > 0$, putting $V := Md^2(\cdot, \bar{x})$ we have:

i) *For every $\delta \in (0, 1)$ there exists $C > 0$ which only depends on K, N, δ, \bar{x} such that*

$$|\varphi_t^\varepsilon|, |\psi_{1-t}^\varepsilon| \leq C(1 + d^2(\cdot, \bar{x})) \quad \forall t \in [\delta, 1], \varepsilon \in (0, 1) \quad (5.3)$$

ii) *The curves $(\varphi_t^{\varepsilon_n}), (\psi_t^{\varepsilon_n})$ converge locally uniformly on \mathcal{J} with values in $L^1(\mathbf{X}, e^{-V} \mathbf{m})$ to limit curves $(\varphi_t), (\psi_t) \in AC_{loc}(\mathcal{J}, L^1(\mathbf{X}, e^{-V} \mathbf{m}))$ respectively, where $\mathcal{J} := (0, 1]$ for the φ 's, $\mathcal{J} := [0, 1)$ for the ψ 's;*

iii) *For all $t \in I$, the functions $\varphi_t^{\varepsilon_n}, \psi_t^{\varepsilon_n}$ also converge in $C(\mathbf{X}, e^{-V})$ to φ_t, ψ_t .*

iv) *For any $\delta \in (0, 1)$ there exists $C > 0$ which only depends on K, N, δ, \bar{x} such that*

$$\sup_{t \in [\delta, 1]} \text{lip}(\varphi_t) + \sup_{t \in [0, 1-\delta]} \text{lip}(\psi_t) \leq C(1 + d(\cdot, \bar{x})), \quad \mathbf{m}\text{-a.e.}; \quad (5.4)$$

v) *Finally, up to pass to a suitable subsequence to obtain the existence of limit measures μ_t as in Proposition 5.1 above, for every $t \in (0, 1)$ it holds*

$$\begin{aligned} \varphi_t + \psi_t &\leq 0 && \text{on } \mathbf{X}, \\ \varphi_t + \psi_t &= 0 && \text{on } \text{supp}(\mu_t). \end{aligned} \quad (5.5)$$

Similarly, the curves $(\vartheta_t^{\varepsilon_n})$ and functions $\vartheta_t^{\varepsilon_n}$ converge in $(0, 1)$ to the limit curve $t \mapsto \vartheta_t := \frac{1}{2}(\psi_t - \varphi_t)$ and functions ϑ_t in the same sense as above.

proof

(i) We start claiming that for all $\varepsilon > 0$ and $t, s \in (0, 1]$ with $t < s$ it holds

$$\|\varphi_t^\varepsilon - \varphi_s^\varepsilon\|_{L^1(\mathbf{X}, e^{-V} \mathbf{m})} \leq \iint_t^s e^{-V} \left(\frac{|\nabla \varphi_r^\varepsilon|^2}{2} + \frac{\varepsilon}{2} |\Delta \varphi_r^\varepsilon| \right) dr \, d\mathbf{m}. \quad (5.6)$$

Indeed, by Proposition 4.3 we know that $(\varphi_t^\varepsilon) \in AC([\delta, 1], W^{1,2}(\mathbf{X}, e^{-V'} \mathbf{m}))$ with $V' := M'd^2(\cdot, \bar{x})$ and $M' = M'(\delta)$ sufficiently large, for any $\delta \in (0, 1)$. Thus for any cut-off function $\chi_R \in \text{Test}^\infty(\mathbf{X})$ with $\chi_R \equiv 1$ on $B_R(\bar{x})$ and support in $B_{R+1}(\bar{x})$, we have $(\chi_R \varphi_t^\varepsilon) \in AC_{loc}((0, 1], W^{1,2}(\mathbf{X}, e^{-V} \mathbf{m}))$ and since $W^{1,2}(\mathbf{X}, e^{-V} \mathbf{m}) \subset L^1(\mathbf{X}, \mathbf{m})$ (because $e^{-V} \mathbf{m}$ is a finite measure) a fortiori this is true for $(\chi_R e^{-V} \varphi_t^\varepsilon)$. From the formula for $\frac{d}{dt} \varphi_t^\varepsilon$ (Proposition 4.3) this implies

$$\|\chi_R(\varphi_t^\varepsilon - \varphi_s^\varepsilon)\|_{L^1(\mathbf{X}, e^{-V} \mathbf{m})} \leq \iint_t^s \chi_R e^{-V} \left(\frac{|\nabla \varphi_r^\varepsilon|^2}{2} + \frac{\varepsilon}{2} |\Delta \varphi_r^\varepsilon| \right) dr \, d\mathbf{m},$$

so that the claim (5.6) follows by letting $R \rightarrow \infty$ and using the monotone convergence theorem. Denoting by C_δ a constant depending on $K, N, \rho_0, \rho_1, \bar{x}, \delta$, but independent of ε, t , whose value might change on the various occurrences it appears, estimates (4.9a) and (4.10) give

$$\|\varphi_t^\varepsilon - \varphi_s^\varepsilon\|_{L^1(X, e^{-V}\mathbf{m})} \leq C_\delta |s - t| \quad \forall \varepsilon \in (0, 1), \forall t, s \in [\delta, 1]. \quad (5.7)$$

Now we observe that from (4.9a) and the fact that X is a geodesic space it follows that

$$|\varphi_t^\varepsilon(x) - \varphi_t^\varepsilon(\bar{x})| \leq C_\delta d(x, \bar{x})(1 + d(x, \bar{x})) \leq C_\delta(1 + d^2(x, \bar{x})) \quad \forall \varepsilon \in (0, 1), t \in [\delta, 1]. \quad (5.8)$$

Which already tells that φ_t^ε has quadratic growth (with constants possibly depending on t, ε). For μ with finite second moment, integrate (5.8) w.r.t. μ in the x variable to get

$$|\varphi_t^\varepsilon(\bar{x}) - \int \varphi_t^\varepsilon d\mu| \leq \int |\varphi_t^\varepsilon(x) - \varphi_t^\varepsilon(\bar{x})| d\mu(x) \leq C_\delta \int (1 + d^2(x, \bar{x})) d\mu(x) \quad (5.9)$$

then pick $t := 1$, $\mu := \mu_1$ and recall that the normalization chosen for $(f^\varepsilon, g^\varepsilon)$ in Setting 4.1 reads as $\int \varphi_1^\varepsilon d\mu_1 = 0$ to deduce that $\sup_{\varepsilon \in (0, 1)} |\varphi_1^\varepsilon(\bar{x})| < \infty$ and thus (5.8) gives

$$|\varphi_1^\varepsilon| \leq C_\delta(1 + d^2(x, \bar{x})) \quad \forall \varepsilon \in (0, 1)$$

which in turn implies $\sup_{\varepsilon \in (0, 1)} \|\varphi_1^\varepsilon\|_{L^1(X, e^{-V}\mathbf{m})} < \infty$. This bound in conjunction with (5.7) gives

$$\|\varphi_t^\varepsilon\|_{L^1(X, e^{-V}\mathbf{m})} \leq C_\delta \quad \forall \varepsilon \in (0, 1), \forall t \in [\delta, 1],$$

so that picking $\mu := e^{-V}\mathbf{m}$ in (5.9) we see that $|\varphi_t^\varepsilon(\bar{x})| \leq C_\delta$ for every $\varepsilon \in (0, 1)$, $t \in [\delta, 1]$ and in conclusion (5.8) gives (5.3) for the φ_t^ε 's.

Following the same lines of thought, the bound (5.3) for ψ_t^ε will follow provided we are able to show that for some measures μ^ε with uniformly bounded second moment it holds

$$\sup_{t \in [0, 1 - \delta]} \sup_{\varepsilon \in (0, 1)} \left| \int \psi_t^\varepsilon d\mu^\varepsilon \right| < \infty$$

for any $\delta \in (0, 1)$. We pick $\mu^\varepsilon := \mu_{1/2}^\varepsilon$: (4.26a) gives the uniform bound on the second moment, while multiplying by $\rho_{1/2}^\varepsilon$ the identity $\varphi_{1/2}^\varepsilon + \psi_{1/2}^\varepsilon = \varepsilon \log \rho_{1/2}^\varepsilon$ and integrating we get

$$\int \varphi_{1/2}^\varepsilon d\mu_{1/2}^\varepsilon + \int \psi_{1/2}^\varepsilon d\mu_{1/2}^\varepsilon = H(\mu_{1/2}^\varepsilon | \mathbf{m})$$

and the conclusion follows from (4.26b) and (5.3) in conjunction with (4.26a).

(ii) By Ascoli-Arzelà's theorem, for given $\bar{x} \in X$ and $C > 0$ the set of functions φ on X such that

$$|\varphi| \leq C(1 + d^2(\cdot, \bar{x})) \quad \text{lip}(\varphi) \leq C(1 + d(\cdot, \bar{x}))$$

is a compact subset of $C(X, e^{-V}\mathbf{m})$. Thus for any $\delta \in (0, 1)$ the estimates (5.3) and (4.9a) give that $\{\varphi_t^\varepsilon : \varepsilon \in (0, 1), t \in [\delta, 1]\}$ is compact in $C(X, e^{-V}\mathbf{m})$ and thus a fortiori also compact in $L^1(X, e^{-V}\mathbf{m})$. This fact, (5.7), the arbitrariness of $\delta \in (0, 1)$ and again Ascoli-Arzelà's theorem give the claim. Similarly for the ψ 's.

(iii) We know that for any $t \in (0, 1)$ we have $\varphi_t^{\varepsilon_n} \rightarrow \varphi_t$ in $L^1(X, e^{-V}\mathbf{m})$. We also noticed that for any $t \in (0, 1)$ the family $\{\varphi_t^{\varepsilon_n}\}_n$ is compact in $C(X, e^{-V}\mathbf{m})$, thus the claim follows. Similarly for the ψ 's.

(iv) We know that for any $x \in X$ it holds $\text{lip } \varphi_t(x) \leq \underline{\lim}_{r \downarrow 0} \text{Lip}(\varphi_t|_{B_r(x)})$ and, since X is geodesic, that $\text{Lip}(\varphi_t^\varepsilon|_{B_r(x)}) = \sup_{B_r(x)} \text{lip} \varphi_t^\varepsilon$. Thus the claim follows from the bound (4.9a) and the fact that $\text{Lip}(\varphi_t|_{B_r(x)}) \leq \underline{\lim}_{n \rightarrow \infty} \text{Lip}(\varphi_t^{\varepsilon_n}|_{B_r(x)})$, which in turn is a trivial consequence of the local uniform convergence we already proved. Similarly for the ψ 's.

(v) For the first in (5.5) we pass to the limit in the identity

$$\varphi_t^\varepsilon + \psi_t^\varepsilon = \varepsilon \log \rho_t^\varepsilon \quad (5.10)$$

recalling the uniform bound (4.19). To get the second in (5.5) we multiply both sides of (5.10) by ρ_t^ε and integrate to obtain

$$\int (\varphi_t^\varepsilon + \psi_t^\varepsilon) \rho_t^\varepsilon \, d\mathbf{m} = \varepsilon H(\mu_t^\varepsilon | \mathbf{m}).$$

Letting $\varepsilon = \varepsilon_n \downarrow 0$ we see that the right-hand side goes to 0 by (4.26b); then we use the fact that $W_2(\mu_t^{\varepsilon_n}, \mu_t) \rightarrow 0$, that the functions $\varphi_t^\varepsilon, \psi_t^\varepsilon$ have uniform quadratic growth and converge locally uniformly to φ_t, ψ_t respectively to obtain that the left-hand side goes to $\int \varphi_t + \psi_t \, d\mu_t$. This is sufficient to conclude. \square

5.2 Identification of the limit curve and potentials

We now show that the limit interpolation is the geodesic from μ_0 to μ_1 and the limit potentials are Kantorovich potentials. We shall make use of the following simple lemma valid on general metric measure spaces:

Lemma 5.3. *Let (Y, d_Y, \mathbf{m}_Y) be a complete separable metric measure space endowed with a non-negative measure \mathbf{m}_Y which is finite on bounded sets and assume that $W^{1,2}(Y)$ is separable. Let π be a test plan and $f \in W^{1,2}(Y)$. Then $t \mapsto \int f \circ e_t \, d\pi$ is absolutely continuous and*

$$\left| \frac{d}{dt} \int f \circ e_t \, d\pi \right| \leq \int |df|(\gamma_t) |\dot{\gamma}_t| \, d\pi(\gamma) \quad \text{a.e. } t \in [0, 1], \quad (5.11)$$

where the exceptional set can be chosen to be independent on f .

Moreover, if $(f_t) \in AC([0, 1], L^2(Y)) \cap L^\infty([0, 1], W^{1,2}(Y))$, then the map $t \mapsto \int f_t \circ e_t \, d\pi$ is also absolutely continuous and

$$\frac{d}{ds} \left(\int f_s \circ e_s \, d\pi \right) \Big|_{s=t} = \int \left(\frac{d}{ds} f_s \Big|_{s=t} \right) \circ e_t \, d\pi + \frac{d}{ds} \left(\int f_t \circ e_s \, d\pi \right) \Big|_{s=t} \quad \text{a.e. } t \in [0, 1].$$

proof The absolute continuity of $t \mapsto \int f \circ e_t \, d\pi$ and the bound (5.11) are trivial consequences of the definitions of test plans and Sobolev functions. The fact that the exceptional set can be chosen independently on f follows from the separability of $W^{1,2}(Y)$ and standard approximation procedures, carried out, for instance, in [25].

For the second part, we start noticing that the second derivative in the right-hand side exists for a.e. t thanks to what we have just proved, so that the claim makes sense. The absolute continuity follows from the fact that for any $t_0, t_1 \in [0, 1]$, $t_0 < t_1$ it holds

$$\begin{aligned} \left| \int f_{t_1} \circ e_{t_1} - f_{t_0} \circ e_{t_0} \, d\pi \right| &\leq \left| \int f_{t_1} \circ e_{t_1} - f_{t_1} \circ e_{t_0} \, d\pi \right| + \left| \int f_{t_1} - f_{t_0} \, d(e_{t_0})_* \pi \right| \\ &\leq \iint_{t_0}^{t_1} |df_{t_1}|(\gamma_t) |\dot{\gamma}_t| \, dt \, d\pi(\gamma) + \iint_{t_0}^{t_1} \left| \frac{d}{dt} f_t \right| \, dt \, d(e_{t_0})_* \pi \end{aligned}$$

and our assumptions on (f_t) and π . Now fix a point t of differentiability for (f_t) and observe that the fact that $\frac{f_{t+h}-f_t}{h}$ strongly converges in $L^2(Y)$ to $\frac{d}{dt}f_t$ and $(e_{t+h})_*\pi$ weakly converges to $(e_t)_*\pi$ as $h \rightarrow 0$ and the densities are equibounded is sufficient to get

$$\lim_{h \rightarrow 0} \int \frac{f_{t+h}-f_t}{h} \circ e_{t+h} d\pi = \int \frac{d}{dt}f_t \circ e_t d\pi = \lim_{h \rightarrow 0} \int \frac{f_{t+h}-f_t}{h} \circ e_t d\pi.$$

Hence the conclusion comes dividing by h the trivial identity

$$\begin{aligned} \int f_{t+h} \circ e_{t+h} - f_t \circ e_t d\pi &= \int f_t \circ e_{t+h} - f_t \circ e_t d\pi + \int f_{t+h} \circ e_t - f_t \circ e_t d\pi + \\ &\quad + \int (f_{t+h} - f_t) \circ e_{t+h} - (f_{t+h} - f_t) \circ e_t d\pi \end{aligned}$$

and letting $h \rightarrow 0$. □

We now prove that in the limit the potentials evolve according to the Hopf-Lax semigroup (recall formula (A.25)).

Proposition 5.4 (Limit curve and potentials). *With the same assumptions and notations as in Setting 4.1 the following holds.*

The limit curve (μ_t) given by Proposition 5.1 is unique (i.e. independent on the sequence $\varepsilon_n \downarrow 0$) and is the only W_2 -geodesic connecting μ_0 to μ_1 .

For any $\bar{x} \in X$, $M > 0$ and any limit curve (φ_t) given by Proposition 5.2, (φ_t) is in $AC_{loc}((0, 1], C(X, e^{-V})) \cap L_{loc}^\infty((0, 1], W^{1,2}(X, e^{-V}\mathbf{m}))$, where $V := Md^2(\cdot, \bar{x})$, and for any $t_0, t_1 \in (0, 1]$, $t_0 < t_1$ we have

$$-\varphi_{t_1} = Q_{t_1-t_0}(-\varphi_{t_0}) \tag{5.12a}$$

$$\int \varphi_{t_0} d\mu_{t_0} - \int \varphi_{t_1} d\mu_{t_1} = \frac{1}{2(t_1 - t_0)} W_2^2(\mu_{t_0}, \mu_{t_1}) \tag{5.12b}$$

and $-(t_1 - t_0)\varphi_{t_1}$ is a Kantorovich potential from μ_{t_1} to μ_{t_0} . Similarly, for V as above and any limit curve (ψ_t) given by Proposition 5.2, (ψ_t) belongs to $AC_{loc}([0, 1], C(X, e^{-V})) \cap L_{loc}^\infty([0, 1], W^{1,2}(X, e^{-V}\mathbf{m}))$ and for every $t_0, t_1 \in [0, 1]$, $t_0 < t_1$ we have

$$-\psi_{t_0} = Q_{t_1-t_0}(-\psi_{t_1}) \tag{5.13a}$$

$$\int \psi_{t_1} d\mu_{t_1} - \int \psi_{t_0} d\mu_{t_0} = \frac{1}{2(t_1 - t_0)} W_2^2(\mu_{t_0}, \mu_{t_1}) \tag{5.13b}$$

and $-(t_1 - t_0)\psi_{t_0}$ is a Kantorovich potential from μ_{t_0} to μ_{t_1} .

proof

Inequality \leq in (5.12a). Pick $x, y \in X$, $r > 0$, define

$$\nu_x^r := \frac{1}{\mathbf{m}(B_r(x))} \mathbf{m}|_{B_r(x)} \quad \nu_y^r := \frac{1}{\mathbf{m}(B_r(y))} \mathbf{m}|_{B_r(y)}$$

and π^r as the only lifting of the only W_2 -geodesic from ν_x^r to ν_y^r (recall point (i) of Theorem A.6). Since ν_x^r, ν_y^r have compact support and $\pi^r \in \text{OptGeo}(\nu_x^r, \nu_y^r)$, there exist $\bar{x} \in X$ and $R > 0$ sufficiently large such that

$$\text{supp}((e_t)_*\pi^r) \subset B_R(\bar{x}), \quad \forall t \in [0, 1]. \tag{5.14}$$

Let χ be a Lipschitz cut-off function with bounded support such that $\chi \equiv 1$ in $B_R(\bar{x})$. Then, let $\varepsilon \in (0, 1)$ and $0 < t_0 < t_1 \leq 1$, put $\tilde{\varphi}_t^\varepsilon := \chi \varphi_t^\varepsilon$ and observe that $(\tilde{\varphi}_t^\varepsilon) \in AC_{loc}((0, 1], L^2(X)) \cap L_{loc}^\infty((0, 1], W^{1,2}(X))$ by Proposition 4.3 and the compactness of the support of χ ; thus, by Lemma 5.3 applied to π^r and $t \mapsto \tilde{\varphi}_{(1-t)t_0+tt_1}^\varepsilon$, we get

$$\frac{d}{dt} \int \tilde{\varphi}_{(1-t)t_0+tt_1}^\varepsilon \circ e_t d\pi^r \geq \int (t_1 - t_0) \frac{d}{ds} \tilde{\varphi}_s^\varepsilon|_{s=(1-t)t_0+tt_1}(\gamma_t) - |d\tilde{\varphi}_{(1-t)t_0+tt_1}^\varepsilon|(\gamma_t)|\dot{\gamma}_t| d\pi^r(\gamma).$$

As (5.14) implies that $\chi(\gamma_t) = 1$ for all $t \in [0, 1]$ for π^r -a.e. γ , $\tilde{\varphi}^\varepsilon$ can be replaced by φ^ε in the inequality above and, recalling the expression for $\frac{d}{dt} \varphi_t^\varepsilon$ and using Young's inequality, we obtain

$$\frac{d}{dt} \int \varphi_{(1-t)t_0+tt_1}^\varepsilon \circ e_t d\pi^r \geq \int \varepsilon \frac{t_1 - t_0}{2} \Delta \varphi_{(1-t)t_0+tt_1}^\varepsilon(\gamma_t) - \frac{1}{2(t_1 - t_0)} |\dot{\gamma}_t|^2 d\pi^r(\gamma).$$

Integrating in time and recalling that π^r is optimal we get

$$\int \varphi_{t_1}^\varepsilon d\nu_y^r - \int \varphi_{t_0}^\varepsilon d\nu_x^r \geq -\frac{1}{2(t_1 - t_0)} W_2^2(\nu_y^r, \nu_x^r) + \iint_0^1 \varepsilon \frac{t_1 - t_0}{2} \Delta \varphi_{(1-t)t_0+tt_1}^\varepsilon \circ e_t dt d\pi^r.$$

Let $\varepsilon \downarrow 0$ along the sequence (ε_n) for which $(\varphi_t^{\varepsilon_n})$ converges to our given (φ_t) in the sense of Proposition 5.2 and use the uniform bound (4.10) and the fact that π^r has bounded compression to deduce that

$$\int \varphi_{t_1} d\nu_y^r - \int \varphi_{t_0} d\nu_x^r \geq -\frac{1}{2(t_1 - t_0)} W_2^2(\nu_y^r, \nu_x^r)$$

and finally letting $r \downarrow 0$ we conclude from the arbitrariness of $x \in X$ that

$$-\varphi_{t_1}(y) \leq Q_{t_1-t_0}(-\varphi_{t_0})(y) \quad \forall y \in X. \quad (5.15)$$

Inequality \geq in (5.12a). To prove the opposite inequality we fix $\bar{x} \in X$, $r > 1$, again $0 < t_0 < t_1 \leq 1$ and let $\tilde{R} > r$ to be fixed later. Let $\chi_{\tilde{R}} \in \text{Test}^\infty(X)$ be given by Lemma A.2, define the vector field $X_t^\varepsilon := \chi_{\tilde{R}} \nabla \varphi_t^\varepsilon$ and apply Theorem A.4 to $((t_1 - t_0)X_{(1-t)t_1+tt_0}^\varepsilon)$: the inequality

$$\text{div} X_t^\varepsilon \geq \chi_{\tilde{R}} \Delta \varphi_t^\varepsilon - |\nabla \chi_{\tilde{R}}| |\nabla \varphi_t^\varepsilon|$$

and the bounds (4.9a), (4.9b) on $\nabla \varphi_t^\varepsilon, \Delta \varphi_t^\varepsilon$ ensure that the theorem is applicable and we obtain existence of the regular Lagrangian flow F^ε . Notice that from (4.9a) we know that $|X_t^\varepsilon| \leq C'(1 + d(\cdot, \bar{x}))$ for all $t \in [t_0, 1]$ for some $C' < \infty$ independent of \tilde{R}, ε , therefore for \mathfrak{m} -a.e. x we have

$$\frac{d}{dt} d(F_t^\varepsilon(x), \bar{x}) \leq \text{ms}_t(F^\varepsilon(x)) \stackrel{(A.22)}{=} (t_1 - t_0) |X_{(1-t)t_1+tt_0}^\varepsilon|(F_t^\varepsilon(x)) \leq C'(1 + d(F_t^\varepsilon(x), \bar{x}))$$

for a.e. t and thus Gronwall's Lemma implies the existence of R independent of \tilde{R}, ε such that for \mathfrak{m} -a.e. x it holds

$$x \in B_r(\bar{x}) \quad \Rightarrow \quad F_t(x) \in B_R(\bar{x}) \quad \forall t \in [0, 1]. \quad (5.16)$$

We now fix $\tilde{R} := R$ and put $\pi^\varepsilon := \mathfrak{m}(B_r(\bar{x}))^{-1}(F^\varepsilon)_* \mathfrak{m}|_{B_r(\bar{x})}$, where $F^\varepsilon : X \rightarrow C([0, 1], X)$ is the \mathfrak{m} -a.e. defined map which sends x to $t \mapsto F_t^\varepsilon(x)$, and observe that the bound (A.21) and

the identity (A.22) provided by Theorem A.4 coupled with the estimates (4.9a), (4.9b) on $\nabla\varphi_t^\varepsilon, \Delta\varphi_t^\varepsilon$ and the fact that $\chi_R \in \text{Test}^\infty(X)$ ensure that π^ε is a test plan with

$$\sup_{\varepsilon \in (0,1)} \iint_0^1 |\dot{\gamma}_t|^2 dt d\pi^\varepsilon(\gamma) < \infty \quad \text{and} \quad (e_t)_* \pi^\varepsilon \leq C\mathbf{m} \quad \forall t \in [0,1], \varepsilon \in (0,1), \quad (5.17)$$

for some $C < \infty$. Now put $\tilde{\varphi}_t^\varepsilon := \chi_R \varphi_t^\varepsilon$ and notice that the definition of π^ε and (5.16) ensures that for every $t \in [0,1]$ we have $\tilde{\varphi}_t^\varepsilon = \varphi_t^\varepsilon (e_t)_* \pi^\varepsilon$ -a.e. Moreover we have $(\tilde{\varphi}_t^\varepsilon) \in AC_{loc}((0,1], L^2(X)) \cap L_{loc}^\infty((0,1], W^{1,2}(X))$, thus by Lemma 5.3 applied to π^ε and $t \mapsto \tilde{\varphi}_{(1-t)t_1+tt_0}^\varepsilon$ we obtain

$$\begin{aligned} \frac{d}{dt} \int \varphi_{(1-t)t_1+tt_0}^\varepsilon \circ e_t d\pi^\varepsilon &= \frac{d}{dt} \int \tilde{\varphi}_{(1-t)t_1+tt_0}^\varepsilon \circ e_t d\pi^\varepsilon \\ &= \int (t_0 - t_1) \frac{d}{ds} \tilde{\varphi}_s^\varepsilon|_{s=(1-t)t_1+tt_0} \circ e_t d\pi^\varepsilon + \frac{d}{ds} \int \tilde{\varphi}_{(1-t)t_1+tt_0}^\varepsilon \circ e_s d\pi^\varepsilon|_{s=t} \\ &= \int (t_0 - t_1) \frac{d}{ds} \varphi_s^\varepsilon|_{s=(1-t)t_1+tt_0} \circ e_t d\pi^\varepsilon + (t_1 - t_0) \int d\tilde{\varphi}_{(1-t)t_1+tt_0}^\varepsilon(X_t^\varepsilon) \circ e_t d\pi^\varepsilon \\ &= \int \left(\frac{t_0 - t_1}{2} |d\varphi_{(1-t)t_1+tt_0}^\varepsilon|^2 + \varepsilon \frac{t_0 - t_1}{2} \Delta\varphi_{(1-t)t_1+tt_0}^\varepsilon + (t_1 - t_0) |d\varphi_{(1-t)t_1+tt_0}^\varepsilon|^2 \right) \circ e_t d\pi^\varepsilon \\ &= \int \left(\frac{t_1 - t_0}{2} |d\varphi_{(1-t)t_1+tt_0}^\varepsilon|^2 + \varepsilon \frac{t_0 - t_1}{2} \Delta\varphi_{(1-t)t_1+tt_0}^\varepsilon \right) \circ e_t d\pi^\varepsilon. \end{aligned}$$

Integrating in time and recalling (A.22) we deduce

$$\int \varphi_{t_0}^\varepsilon \circ e_1 - \varphi_{t_1}^\varepsilon \circ e_0 d\pi^\varepsilon = \iint_0^1 \frac{1}{2(t_1 - t_0)} |\dot{\gamma}_t|^2 + \varepsilon \frac{t_0 - t_1}{2} \Delta\varphi_{(1-t)t_1+tt_0}^\varepsilon(\gamma_t) dt d\pi^\varepsilon(\gamma). \quad (5.18)$$

Now, as before, we let $\varepsilon \downarrow 0$ along the sequence (ε_n) for which $(\varphi_t^{\varepsilon_n})$ converges to our given (φ_t) in the sense of Proposition 5.2: the first in (5.17) grants that (π^ε) is tight in $\mathcal{P}(C([0,1], X))$ (because $\gamma \mapsto \int_0^1 |\dot{\gamma}_t|^2 dt$ has locally compact sublevels and $(e_0)_* \pi^\varepsilon = \mathbf{m}(B_r(\bar{x}))^{-1} \mathbf{m}|_{B_r(\bar{x})}$) and thus up to pass to a subsequence, not relabeled, we can assume that (π^{ε_n}) weakly converges to some $\pi \in \mathcal{P}(C([0,1], X))$. The second in (5.17) and the bound (4.10) grant that the term with the Laplacian in (5.18) vanishes in the limit and thus taking into account the lower semicontinuity of the 2-energy we deduce that

$$\int \varphi_{t_0} \circ e_1 - \varphi_{t_1} \circ e_0 d\pi \geq \frac{1}{2(t_1 - t_0)} \iint_0^1 |\dot{\gamma}_t|^2 dt d\pi \geq \frac{1}{2(t_1 - t_0)} \int d^2(\gamma_0, \gamma_1) d\pi(\gamma).$$

Now notice that (5.15) implies that

$$\frac{d^2(\gamma_0, \gamma_1)}{2(t_1 - t_0)} \geq \varphi_{t_0}(\gamma_1) - \varphi_{t_1}(\gamma_0) \quad (5.19)$$

for any curve γ , hence the above gives

$$\int \varphi_{t_0} \circ e_1 - \varphi_{t_1} \circ e_0 d\pi \geq \frac{1}{2(t_1 - t_0)} \int d^2(\gamma_0, \gamma_1) d\pi(\gamma) \geq \int \varphi_{t_0} \circ e_1 - \varphi_{t_1} \circ e_0 d\pi$$

thus forcing the inequalities to be equalities. In particular, equality in (5.19) holds for π -a.e. γ and since $(e_0)_* \pi = \mathbf{m}|_{B_r(\bar{x})}$, this is the same as to say that for \mathbf{m} -a.e. $y \in B_r(\bar{x})$ equality

holds in (5.15). Since both sides of (5.15) are continuous in y , we deduce that equality holds for any $y \in B_r(\bar{x})$ and the arbitrariness of r allows to conclude that equality actually holds for any $y \in X$.

Other properties of φ_t . From Proposition 5.2 we already know that, for any $\bar{x} \in X$ and $M > 0$, $(\varphi_t) \in AC_{loc}((0, 1], L^1(X, e^{-V} \mathbf{m})) \cap L_{loc}^\infty((0, 1], W^{1,2}(X, e^{-V} \mathbf{m}))$, where $V := M d^2(\cdot, \bar{x})$. Since φ_t is a real-valued function for all $t \in (0, 1]$, (5.12a) tells us that for all $x \in X$, $t \mapsto \varphi_t(x)$ satisfies (A.26) for a.e. $t \in (0, 1]$; taking (5.4) into account, this yields that for all $\delta \in (0, 1)$ and $t_0, t_1 \in [\delta, 1]$ with $t_0 < t_1$

$$\|\varphi_{t_1} - \varphi_{t_0}\|_{C(X, e^{-V})} \leq \sup_{x \in X} \int_{t_0}^{t_1} \frac{(\text{lip}(\varphi_t)(x))^2}{2} dt \leq \left(\sup_{x \in X} C_\delta (1 + d(x, \bar{x})) e^{-V(x)} \right) |t_1 - t_0|$$

whence $(\varphi_t) \in AC_{loc}((0, 1], C(X, e^{-V})) \cap L_{loc}^\infty((0, 1], W^{1,2}(X, e^{-V} \mathbf{m}))$.

Up to extract a further subsequence - not relabeled - we can assume that the curves (μ_t^ε) converge to a limit curve (μ_t) as in Proposition 5.1. We claim that for any $t_0, t_1 \in (0, 1]$, $t_0 < t_1$ it holds

$$-\int \varphi_{t_1} d\mu_{t_1} + \int \varphi_{t_0} d\mu_{t_0} \geq \frac{1}{2(t_1 - t_0)} W_2^2(\mu_{t_0}, \mu_{t_1}) \quad (5.20)$$

and start observing that since $(\varphi_t) \in C((0, 1], C(X, e^{-V}))$ and $(\mu_t) \in AC([0, 1], (\mathcal{P}_2(X), W_2))$, by the uniform estimates (5.3) we see that both sides in (5.20) are continuous in t_0, t_1 , hence it is sufficient to prove (5.20) for $t_0, t_1 \in (0, 1)$.

Now fix $\bar{x} \in X$ and $R > 0$, let $\chi_R \in \text{Test}^\infty(X)$ be a cut-off function as in Lemma A.2 and observe that by Proposition 4.3 $t \mapsto \int \chi_R \varphi_t^\varepsilon \rho_t^\varepsilon d\mathbf{m}$ belongs to $C((0, 1]) \cap AC_{loc}((0, 1))$ with

$$\begin{aligned} -\frac{d}{dt} \int \chi_R \varphi_t^\varepsilon \rho_t^\varepsilon d\mathbf{m} &= \int \chi_R \left(-\frac{|\nabla \varphi_t^\varepsilon|^2}{2} - \frac{\varepsilon}{2} \Delta \varphi_t^\varepsilon - \langle \nabla \varphi_t^\varepsilon, \nabla \vartheta_t^\varepsilon \rangle \right) \rho_t^\varepsilon + \varphi_t^\varepsilon \langle \nabla \chi_R, \nabla \vartheta_t^\varepsilon \rangle \rho_t^\varepsilon d\mathbf{m} \\ &= \int \chi_R \left(\frac{|\nabla \vartheta_t^\varepsilon|^2}{2} - \frac{\varepsilon^2}{8} |\nabla \log \rho_t^\varepsilon|^2 - \frac{\varepsilon}{2} \Delta \varphi_t^\varepsilon \right) \rho_t^\varepsilon + \varphi_t^\varepsilon \langle \nabla \chi_R, \nabla \vartheta_t^\varepsilon \rangle \rho_t^\varepsilon d\mathbf{m} \end{aligned} \quad (5.21)$$

for a.e. $t \in (0, 1)$, having also used the identity $\varphi_t^\varepsilon = \frac{\varepsilon}{2} \log \rho_t^\varepsilon - \vartheta_t^\varepsilon$. By (5.3), (4.26a) and Lemma 4.5 it is readily verified that $\int \chi_R \varphi_t^\varepsilon \rho_t^\varepsilon d\mathbf{m} \rightarrow \int \varphi_t^\varepsilon \rho_t^\varepsilon d\mathbf{m}$ as $R \rightarrow \infty$ for any $t \in (0, 1)$ and that the rightmost side of (5.21) passes to the limit as $R \rightarrow \infty$ locally uniformly in $t \in (0, 1)$. Hence after an integration in t and letting $R \rightarrow \infty$ in (5.21) we obtain

$$-\int \varphi_{t_1}^\varepsilon d\mu_{t_1}^\varepsilon + \int \varphi_{t_0}^\varepsilon d\mu_{t_0}^\varepsilon = \iint_{t_0}^{t_1} \left(\frac{|\nabla \vartheta_t^\varepsilon|^2}{2} - \frac{\varepsilon^2}{8} |\nabla \log \rho_t^\varepsilon|^2 - \frac{\varepsilon}{2} \Delta \varphi_t^\varepsilon \right) \rho_t^\varepsilon dt d\mathbf{m}$$

As already noticed in the proof of Proposition 5.1, (μ_t^ε) and $(\nabla \vartheta_t^\varepsilon)$ satisfy the assumptions of Theorem A.5, thus from such theorem we deduce that

$$\iint_{t_0}^{t_1} \frac{|\nabla \vartheta_t^\varepsilon|^2}{2} \rho_t^\varepsilon dt d\mathbf{m} = \frac{1}{2} \int_{t_0}^{t_1} |\dot{\mu}_t^\varepsilon|^2 dt \geq \frac{1}{2(t_1 - t_0)} W_2^2(\mu_{t_0}^\varepsilon, \mu_{t_1}^\varepsilon).$$

Therefore

$$-\int \varphi_{t_1}^\varepsilon d\mu_{t_1}^\varepsilon + \int \varphi_{t_0}^\varepsilon d\mu_{t_0}^\varepsilon \geq \frac{1}{2(t_1 - t_0)} W_2^2(\mu_{t_0}^\varepsilon, \mu_{t_1}^\varepsilon) + \iint_{t_0}^{t_1} \left(-\frac{\varepsilon^2}{8} |\nabla \log \rho_t^\varepsilon|^2 - \frac{\varepsilon}{2} \Delta \varphi_t^\varepsilon \right) \rho_t^\varepsilon dt d\mathbf{m}.$$

We now pass to the limit in $\varepsilon = \varepsilon_n \downarrow 0$: we know from Proposition 5.1 that $W_2(\mu_t^{\varepsilon_n}, \mu_t) \rightarrow 0$ and together with (5.3) this also grants that the left-hand side trivially converges to the left-hand side of (5.20). The contribution of the term with $|\nabla \log \rho_t^\varepsilon|$ vanishes by (4.29b) and so does the one with $\Delta \varphi_t^\varepsilon$ by (4.10) and (4.17). Hence (5.20) is proved.

Now notice that (5.12a) can be rewritten as

$$-(t_1 - t_0)\varphi_{t_1} = ((t_1 - t_0)\varphi_{t_0})^c,$$

so that in particular $-(t_1 - t_0)\varphi_{t_1}$ is c -concave and $(-(t_1 - t_0)\varphi_{t_1})^c \geq (t_1 - t_0)\varphi_{t_0}$. Hence both (5.12b) and the fact that $-(t_1 - t_0)\varphi_{t_1}$ is a Kantorovich potential follow from

$$\begin{aligned} \frac{1}{2}W_2^2(\mu_{t_0}, \mu_{t_1}) &\geq \int -(t_1 - t_0)\varphi_{t_1} d\mu_{t_1} + \int (-(t_1 - t_0)\varphi_{t_1})^c d\mu_{t_0} \\ &\geq \int -(t_1 - t_0)\varphi_{t_1} d\mu_{t_1} + \int (t_1 - t_0)\varphi_{t_0} d\mu_{t_0} \stackrel{(5.20)}{\geq} \frac{1}{2}W_2^2(\mu_{t_0}, \mu_{t_1}) \end{aligned}$$

Then (5.13b) and the other claims about (ψ_t) are proved in the same way.

(μ_t) **is a geodesic.** Let $[t_0, t_1] \subset (0, 1)$, pick $t \in [0, 1]$ and put $t'_0 := (1 - t)t_1 + tt_0$. We know that $-(t_1 - t_0)\varphi_{t_1}$ and $-t(t_1 - t_0)\varphi_{t_1}$ are Kantorovich potentials from μ_{t_1} to μ_{t_0} and from μ_{t_1} to $\mu_{t'_0}$ respectively and thus by point (ii) of Theorem A.6 we deduce

$$W_2^2(\mu_{t_0}, \mu_{t_1}) = \int |d((t_1 - t_0)\varphi_{t_1})|^2 d\mu_{t_1} = \frac{1}{t^2} \int |d((t_1 - t'_0)\varphi_{t_1})|^2 d\mu_{t_1} = \frac{(t_1 - t_0)^2}{(t_1 - t'_0)^2} W_2^2(\mu_{t_1}, \mu_{t'_0}).$$

Swapping the roles of t_0, t_1 and using the ψ 's in place of the φ 's we then get

$$W_2(\mu_{t'_1}, \mu_{t'_0}) = \frac{t'_1 - t'_0}{t_1 - t_0} W_2(\mu_{t_1}, \mu_{t_0}) \quad \forall [t'_0, t'_1] \subset [t_0, t_1] \subset (0, 1).$$

This grants that the restriction of (μ_t) to any interval $[t_0, t_1] \subset (0, 1)$ is a constant speed geodesic. Since (μ_t) is continuous on the whole $[0, 1]$, this gives the conclusion. Since in this situation the W_2 -geodesic connecting μ_0 to μ_1 is unique (recall point (i) of Theorem A.6), by the arbitrariness of the subsequences chosen we also proved the uniqueness of the limit curve (μ_t) . \square

Remark 5.5 (The vanishing viscosity limit). The part of this last proposition concerning the properties of the φ_t^ε 's is valid in a context wider than the one provided by Schrödinger problem: we could restate the result by saying that if (φ_t^ε) solves

$$\frac{d}{dt}\varphi_t^\varepsilon = \frac{1}{2}|\nabla \varphi_t^\varepsilon|^2 + \frac{\varepsilon}{2}\Delta \varphi_t^\varepsilon \tag{5.22}$$

and φ_0^ε uniformly converges to some φ_0 , then φ_t^ε uniformly converges to $\varphi_t := -Q_t(-\varphi_0)$.

In this direction, it is worth recalling that in [2] and [21] it has been developed a theory of viscosity solutions for some first-order Hamilton-Jacobi equations on metric spaces. This theory applies in particular to the equation

$$\frac{d}{dt}\varphi_t = \frac{1}{2}\text{lip}(\varphi_t)^2 \tag{5.23}$$

whose only viscosity solution is given by the formula $\varphi_t := -Q_t(-\varphi_0)$.

Therefore, we have just proved that if one works not only on a metric space, but on a metric measure space which is an $\text{RCD}^*(K, N)$ space, then the solutions of the viscous approximation (5.22) converge to the unique viscosity solution of (5.23), in accordance with the classical case. \blacksquare

Remark 5.6. It is not clear whether the ‘full’ families $\varphi_t^\varepsilon, \psi_t^\varepsilon$ converge as $\varepsilon \downarrow 0$ to a unique limit. This is related to the non-uniqueness of the Kantorovich potentials in the classical optimal transport problem. \blacksquare

We shall now make use of the following lemma. It could be directly deduced from the results obtained by Cheeger in [14], however, the additional regularity assumptions on both the space and the function allow for a ‘softer’ argument based on the metric Brenier’s theorem, which we propose.

Lemma 5.7. *Let (Y, d_Y, \mathbf{m}_Y) be an $\text{RCD}^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in [1, \infty)$ and let $\phi : X \rightarrow \mathbb{R} \cup \{-\infty\}$ be a c -concave function not identically $-\infty$. Let Ω be the interior of the set $\{\phi > -\infty\}$. Then ϕ is locally Lipschitz on Ω and*

$$\text{lip } \phi = |\text{d}\phi|, \quad \mathbf{m}\text{-a.e. on } \Omega.$$

proof Lemma 3.3 in [32] grants that ϕ is locally Lipschitz on Ω and that $\partial^c \phi(x) \neq \emptyset$ for every $x \in \Omega$. The same lemma also grants that for $K \subset \Omega$ compact, the set $\cup_{x \in K} \partial^c \phi(x)$ is bounded. Recalling that $\partial^c \phi$ is the set of $(x, y) \in Y^2$ such that

$$\phi(x) + \phi^c(y) = \frac{1}{2} \mathbf{d}^2(x, y)$$

and that ϕ, ϕ^c are upper semicontinuous, we see that $\partial^c \phi$ is closed. Hence for $K \subset \Omega$ compact the set $\cup_{x \in K} \partial^c \phi(x)$ is compact and not empty and thus by the Kuratowski–Ryll–Nardzewski Borel selection theorem we deduce the existence of a Borel map $T : \Omega \rightarrow Y$ such that $T(x) \in \partial^c \phi(x)$ for every $x \in \Omega$.

Pick $\mu \in \mathcal{P}_2(Y)$ with $\text{supp}(\mu) \subset\subset \Omega$ and $\mu \leq C\mathbf{m}$ for some $C > 0$ and set $\nu := T_*\mu$. By construction, μ, ν have both bounded support, T is an optimal map and ϕ is a Kantorovich potential from μ to ν .

Hence point (iii) of Theorem A.6 applies and since $\text{lip } \phi = \max\{|D^+\phi|, |D^-\phi|\}$, by the arbitrariness of μ to conclude it is sufficient to show that $|D^+\phi| = |D^-\phi|$ \mathbf{m} -a.e. This easily follows from the fact that \mathbf{m} is doubling and ϕ Lipschitz, see Proposition 2.7 in [5]. \square

With this said, we can now show that the weighted energies of the Schrödinger potentials converge to the weighted energy of the limit ones:

Proposition 5.8. *With the same assumptions and notations as in Setting 4.1 the following holds.*

Let $\varepsilon_n \downarrow 0$ be a sequence such that $(\varphi_t^{\varepsilon_n}), (\psi_t^{\varepsilon_n})$ converge to limit curves $(\varphi_t), (\psi_t)$ as in Proposition 5.2 and let $V := M\mathbf{d}^2(\cdot, \bar{x})$ with $\bar{x} \in X$ and $M > 0$ arbitrary. Then for every $\delta \in (0, 1)$ we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \iint_{\delta}^1 e^{-V} |\text{d}\varphi_t^{\varepsilon_n}|^2 dt d\mathbf{m} &= \iint_{\delta}^1 e^{-V} |\text{d}\varphi_t|^2 dt d\mathbf{m}, \\ \lim_{n \rightarrow \infty} \iint_0^{1-\delta} e^{-V} |\text{d}\psi_t^{\varepsilon_n}|^2 dt d\mathbf{m} &= \iint_0^{1-\delta} e^{-V} |\text{d}\psi_t|^2 dt d\mathbf{m}. \end{aligned} \tag{5.24}$$

proof Fix $\delta \in (0, 1)$ and argue as in the proof of Proposition 5.2 to obtain that $t \mapsto \int e^{-V} \varphi_t^\varepsilon \, \mathbf{d}\mathbf{m}$ is absolutely continuous in $[\delta, 1]$ (see in particular (5.7)) and that

$$\int e^{-V} (\varphi_1^\varepsilon - \varphi_\delta^\varepsilon) \, \mathbf{d}\mathbf{m} = \frac{1}{2} \iint_\delta^1 e^{-V} (|\mathrm{d}\varphi_t^\varepsilon|^2 + \varepsilon \Delta \varphi_t^\varepsilon) \, \mathrm{d}t \, \mathbf{d}\mathbf{m}.$$

Choosing $\varepsilon := \varepsilon_n$, letting $n \rightarrow \infty$ and using the uniform bounds (4.10), (5.3) and the volume growth estimate (A.20) we obtain

$$\lim_{n \rightarrow \infty} \frac{1}{2} \iint_\delta^1 e^{-V} |\mathrm{d}\varphi_t^{\varepsilon_n}|^2 \, \mathrm{d}t \, \mathbf{d}\mathbf{m} = \lim_{n \rightarrow \infty} \int e^{-V} (\varphi_1^{\varepsilon_n} - \varphi_\delta^{\varepsilon_n}) \, \mathbf{d}\mathbf{m} = \int e^{-V} (\varphi_1 - \varphi_\delta) \, \mathbf{d}\mathbf{m}. \quad (5.25)$$

Combining (A.26) and (5.12a) we see that for any $x \in \mathbf{X}$ it holds

$$\frac{\mathrm{d}}{\mathrm{d}t} \varphi_t(x) = \frac{1}{2} ((\mathrm{lip} \varphi_t)(x))^2 \quad \text{a.e. } t \in [0, 1].$$

By Fubini's theorem, the same identity holds for $\mathcal{L}^1 \otimes \mathbf{m}$ -a.e. $(t, x) \in [\delta, 1] \times \mathbf{X}$. The identity (5.12a) also grants that φ_t is a multiple of a c -concave function, thus the thesis of Lemma 5.7 is valid for φ_t and recalling that $(\varphi_t) \in AC_{loc}((0, 1], L^1(\mathbf{X}, e^{-V} \mathbf{m}))$ by Proposition 5.2 we deduce that

$$\int e^{-V} (\varphi_1 - \varphi_\delta) \, \mathbf{d}\mathbf{m} = \int_\delta^1 \frac{\mathrm{d}}{\mathrm{d}t} \int e^{-V} \varphi_t \, \mathbf{d}\mathbf{m} \, \mathrm{d}t = \iint_\delta^1 e^{-V} \frac{|\mathrm{d}\varphi_t|^2}{2} \, \mathrm{d}t \, \mathbf{d}\mathbf{m},$$

which together with (5.25) gives the first in (5.24). The proof of the second is analogous. \square

As a direct consequence of the limit (5.24) and the local equi-Lipschitz bounds (4.9a) we obtain the following result. In order to state it, let us introduce the module $L^2(T^*\mathbf{X}, e^{-V} \mathbf{m})$ as $\{\omega \in L^0(T^*\mathbf{X}) : |\omega| \in L^2(\mathbf{X}, e^{-V} \mathbf{m})\}$; an analogous definition can be given for $L^2((T^*)^{\otimes 2} \mathbf{X})$.

Corollary 5.9. *With the same assumptions and notations as in Setting 4.1 the following holds.*

Let $\varepsilon_n \downarrow 0$ be a sequence such that $(\varphi_t^{\varepsilon_n}), (\psi_t^{\varepsilon_n})$ converge to limit curves $(\varphi_t), (\psi_t)$ as in Proposition 5.2. Then for every $\delta \in (0, 1)$, $\bar{x} \in \mathbf{X}$ and $M > 0$ we have

$$\begin{array}{llll} (\mathrm{d}\varphi_t^{\varepsilon_n}) & \rightarrow & (\mathrm{d}\varphi_t) & \text{in } L^2([\delta, 1], L^2(T^*\mathbf{X}, e^{-V} \mathbf{m})) \\ (\mathrm{d}\psi_t^{\varepsilon_n}) & \rightarrow & (\mathrm{d}\psi_t) & \text{in } L^2([0, 1 - \delta], L^2(T^*\mathbf{X}, e^{-V} \mathbf{m})) \\ (\mathrm{d}\varphi_t^{\varepsilon_n} \otimes \mathrm{d}\varphi_t^{\varepsilon_n}) & \rightarrow & (\mathrm{d}\varphi_t \otimes \mathrm{d}\varphi_t) & \text{in } L^2([\delta, 1], L^2((T^*)^{\otimes 2} \mathbf{X}, e^{-V} \mathbf{m})) \\ (\mathrm{d}\psi_t^{\varepsilon_n} \otimes \mathrm{d}\psi_t^{\varepsilon_n}) & \rightarrow & (\mathrm{d}\psi_t \otimes \mathrm{d}\psi_t) & \text{in } L^2([0, 1 - \delta], L^2((T^*)^{\otimes 2} \mathbf{X}, e^{-V} \mathbf{m})) \\ (\mathrm{d}\varphi_t^{\varepsilon_n} \otimes \mathrm{d}\psi_t^{\varepsilon_n}) & \rightarrow & (\mathrm{d}\varphi_t \otimes \mathrm{d}\psi_t) & \text{in } L^2([\delta, 1 - \delta], L^2((T^*)^{\otimes 2} \mathbf{X}, e^{-V} \mathbf{m})) \end{array} \quad (5.26)$$

where $V := M \mathrm{d}^2(\cdot, \bar{x})$.

proof Let V be as in the statement and start noticing that the closure of the differential grants that $\mathrm{d}\varphi_t^{\varepsilon_n} \rightharpoonup \mathrm{d}\varphi_t$ in $L^2(T^*\mathbf{X}, e^{-V} \mathbf{m})$ for all $t \in (0, 1]$. This and the fact that $(\mathrm{d}\varphi_t^{\varepsilon_n})$ is equibounded in $L^2([\delta, 1], L^2(T^*\mathbf{X}, e^{-V} \mathbf{m}))$, as a direct consequence of (4.9a), are sufficient to ensure that $(\mathrm{d}\varphi_t^{\varepsilon_n}) \rightharpoonup (\mathrm{d}\varphi_t)$ in $L^2([\delta, 1], L^2(T^*\mathbf{X}, e^{-V} \mathbf{m}))$. Given that the first in (5.24) grants convergence of the $L^2([\delta, 1], L^2(T^*\mathbf{X}, e^{-V} \mathbf{m}))$ -norms, we deduce strong convergence. This establishes the first limit.

Now observe that for every $\omega \in L^2([\delta, 1], L^2(T^*X, e^{-V} \mathbf{m}))$ the fact that $e^{-V} |d\varphi_t^{\varepsilon_n}|$ is uniformly bounded in $L^\infty([\delta, 1] \times X)$ for every $M > 0$ in the definition of V and the strong L^2 -convergence just proved ensure that $\langle d\varphi_t^{\varepsilon_n}, \omega_t \rangle \rightarrow \langle d\varphi_t, \omega_t \rangle$ in $L^2([\delta, 1] \times X, dt \otimes e^{-V} \mathbf{m})$. It follows that for any $\omega_1, \omega_2 \in L^2([\delta, 1], L^2(T^*X, e^{-V} \mathbf{m}))$ we have

$$\iint_{\delta}^1 e^{-V} \langle d\varphi_t^{\varepsilon_n}, \omega_{1,t} \rangle \langle d\varphi_t^{\varepsilon_n}, \omega_{2,t} \rangle dt d\mathbf{m} \rightarrow \iint_{\delta}^1 e^{-V} \langle d\varphi_t, \omega_{1,t} \rangle \langle d\varphi_t, \omega_{2,t} \rangle dt d\mathbf{m}$$

and thus to conclude it remains to prove that

$$\iint_{\delta}^1 e^{-V} |d\varphi_t^{\varepsilon_n} \otimes d\varphi_t^{\varepsilon_n}|_{\text{HS}}^2 dt d\mathbf{m} \rightarrow \iint_{\delta}^1 e^{-V} |d\varphi_t \otimes d\varphi_t|_{\text{HS}}^2 dt d\mathbf{m}.$$

Since $|v \otimes v|_{\text{HS}}^2 = |v|^4$ this is a direct consequence of the fact that $|d\varphi_t^{\varepsilon_n}|$ is uniformly bounded and converge to $|d\varphi_t|$ in $L^2([\delta, 1] \times X, dt \otimes e^{-V} \mathbf{m})$. Hence also the third limit is established.

The other claims follow by analogous arguments. \square

The estimates that we have for the functions φ 's tell nothing about their regularity as $t \downarrow 0$ and similarly little we know so far about the ψ 's for $t \uparrow 1$. We now see in which sense limit functions φ_0, ψ_1 exist. This is not needed for the proof of our main result, but we believe it is relevant on its own.

Thus let us fix $\varepsilon_n \downarrow 0$ so that $\varphi_t^{\varepsilon_n} \rightarrow \varphi_t$ for $t \in (0, 1]$ and $\psi_t^{\varepsilon_n} \rightarrow \psi_t$ for $t \in [0, 1)$ as in Proposition 5.2. Then define the functions $\varphi_0, \psi_1 : X \rightarrow \mathbb{R} \cup \{-\infty\}$ as

$$\begin{aligned} \varphi_0(x) &:= \inf_{t \in (0, 1]} \varphi_t(x) = \lim_{t \downarrow 0} \varphi_t(x), \\ \psi_1(x) &:= \inf_{t \in [0, 1)} \psi_t(x) = \lim_{t \uparrow 1} \psi_t(x). \end{aligned} \tag{5.27}$$

Notice that the fact that the inf are equal to the stated limits is a consequence of formulas (5.12a), (5.13a), which directly imply that for every $x \in X$ the maps $t \mapsto \varphi_t(x)$ and $t \mapsto \psi_{1-t}(x)$ are non-decreasing.

The main properties of φ_0, ψ_1 are collected in the following proposition:

Proposition 5.10. *With the same assumptions and notations as in Setting 4.1 and for φ_0, ψ_1 defined by (5.27) the following holds.*

- i) *The functions $-\varphi_t$ (resp. $-\psi_t$) Γ -converge to $-\varphi_0$ (resp. $-\psi_1$) as $t \downarrow 0$ (resp. $t \uparrow 1$).*
- ii) *For every $t \in (0, 1]$ we have*

$$Q_t(-\varphi_0) = -\varphi_t \quad Q_t(-\psi_1) = -\psi_{1-t}.$$

- iii) *It holds*

$$\varphi_0(x) = \begin{cases} -\psi_0(x) & \text{if } x \in \text{supp}(\rho_0) \\ -\infty & \text{otherwise} \end{cases} \quad \psi_1(x) = \begin{cases} -\varphi_1(x) & \text{if } x \in \text{supp}(\rho_1) \\ -\infty & \text{otherwise} \end{cases}$$

- iv) *We have*

$$\int \varphi_0 \rho_0 d\mathbf{m} + \int \psi_1 \rho_1 d\mathbf{m} = \frac{1}{2} W_2^2(\mu_0, \mu_1).$$

v) Define φ_0^ε on $\text{supp}(\rho_0)$ as $\varphi_0^\varepsilon := \varepsilon \log(f^\varepsilon)$ and let $\varepsilon_n \downarrow 0$ be such that $\varphi_t^{\varepsilon_n}, \psi_t^{\varepsilon_n}$ converge to φ_t, ψ_t as $n \rightarrow \infty$ as in Proposition 5.2.

Then the functions $\rho_0 \varphi_0^{\varepsilon_n}$, set to be 0 on $X \setminus \text{supp}(\rho_0)$, converge to $\rho_0 \varphi_0$ in $L^\infty(X)$ as $n \rightarrow \infty$.

With the analogous definition of $\rho_1 \psi_1^{\varepsilon_n}$ we have that these converge to $\rho_1 \psi_1$ in $L^\infty(X)$ as $n \rightarrow \infty$.

proof We shall prove the claims for φ_0 only, as those for ψ_1 follow along similar lines.

(i) For the Γ - $\overline{\lim}$ inequality we simply observe that by definition $-\varphi_0(x) = \lim_{t \downarrow 0} -\varphi_t(x)$. To prove the Γ - $\underline{\lim}$ inequality, use the fact that $-\varphi_t \geq -\varphi_s$ for $0 < t \leq s$ and the continuity of φ_s : for given (x_t) converging to x we have

$$\underline{\lim}_{t \downarrow 0} -\varphi_t(x_t) \geq \underline{\lim}_{t \downarrow 0} -\varphi_s(x_t) = -\varphi_s(x) \quad \forall s > 0.$$

The conclusion follows letting $s \downarrow 0$.

(ii) From $-\varphi_0 \geq -\varphi_s$ we deduce that

$$Q_t(-\varphi_0) \geq Q_t(-\varphi_s) \stackrel{(5.12a)}{=} -\varphi_{t+s} \quad \forall s \in (0, 1]$$

and thus letting $s \downarrow 0$ and using the continuity of $(0, 1] \ni t \mapsto \varphi_t(x)$ for all $x \in X$ we obtain $Q_t(-\varphi_0)(x) \geq -\varphi_t(x)$ for all $x \in X$. For the opposite inequality, notice that the second (4.3) gives

$$\varphi_t^\varepsilon \leq \varepsilon \log C_4 - \varepsilon \log v_{\varepsilon t/2} + \varepsilon \log \|f^\varepsilon\|_{L^1(X)} - \frac{C_5 d^2(\cdot, \bar{x})}{t} + \frac{C_6}{t} \quad (5.28)$$

for all $t \in (0, 1]$ with C_4, C_5, C_6 depending on $K, N, \rho_0, \rho_1, \bar{x}$ only and $v_{\varepsilon t/2}$ as in (4.2). We now claim that for every $\varepsilon \in (0, 1)$ we have

$$\varepsilon \log v_\varepsilon \geq -C, \quad \varepsilon \log \|f^\varepsilon\|_{L^1(X)} \leq C \quad (5.29)$$

for some constant C depending on $K, N, \rho_0, \rho_1, \bar{x}$ only. Indeed, from (A.19) we see that letting D be the diameter of $\text{supp}(\rho_0)$ and $c = c(D)$ a constant depending only on D we have

$$\mathbf{m}(B_{\sqrt{\varepsilon}}(x)) \geq c^{\log_2(D/\sqrt{\varepsilon})+1} \mathbf{m}(\text{supp}(\rho_0)) \quad \forall x \in \text{supp}(\rho_0).$$

Thus $v_\varepsilon \geq C^{\log_2(D/\sqrt{\varepsilon})+1} \mathbf{m}(\text{supp}(\rho_0))$ and thus the first in (5.29) follows. For the second we start noticing that the first inequality in (A.5), the identity $\int f^\varepsilon \otimes g^\varepsilon d\mathbf{R}^{\varepsilon/2} = 1$ and the fact that the supports of $f^\varepsilon, g^\varepsilon$ coincide with those of ρ_0, ρ_1 respectively give

$$\varepsilon \log \left(\|f^\varepsilon\|_{L^1(X)} \|g^\varepsilon\|_{L^1(X)} \right) = \varepsilon \log \int_{\text{supp}(\rho_0) \times \text{supp}(\rho_1)} f^\varepsilon \otimes g^\varepsilon d\mathbf{m} \leq \varepsilon \log(C_1 \mathbf{m}(B)) + \bar{D}^2 + C_2 \varepsilon^2, \quad (5.30)$$

for every $\varepsilon \in (0, 1)$, where \bar{D} is the diameter of $\text{supp}(\rho_0) \cup \text{supp}(\rho_1)$ and B is the 1-neighbourhood of $\text{supp}(\rho_0) \cup \text{supp}(\rho_1)$. Then recall the normalization (4.1), the identity $\log \rho_1 = \log g^\varepsilon + \log(\mathbf{h}_{\varepsilon/2} f^\varepsilon)$ and use Jensens' inequality for $-\log$ to obtain

$$H(\mu_1 | \mathbf{m}) = \int \rho_1 \log \rho_1 d\mathbf{m} = \int \log(g^\varepsilon) \rho_1 d\mathbf{m} \leq \log \int g^\varepsilon \rho_1 d\mathbf{m} \leq \log \left(\|g^\varepsilon\|_{L^1(X)} \|\rho_1\|_{L^\infty(X)} \right),$$

whence $\log \|g^\varepsilon\|_{L^1(X)} \geq H(\mu_1 | \mathbf{m}) - \log \|\rho_1\|_{L^\infty(X)}$ for all $\varepsilon \in (0, 1)$, which together with (5.30) gives the second in (5.29).

Therefore passing to the limit in (5.28) as $\varepsilon = \varepsilon_n \downarrow 0$ and recalling the local uniform convergence of $\varphi_t^{\varepsilon_n}$ to φ_t give $-\varphi_t \geq -\frac{\tilde{C}}{t} + \frac{C_5 d^2(\cdot, \bar{x})}{t}$ for every $t \in (0, 1]$, where $\tilde{C} \geq 0$ depends on $K, N, \rho_0, \rho_1, \bar{x}$ only. It follows that

$$-\varphi_t \geq \frac{C_5 d^2(\cdot, \bar{x})}{2t} \geq \frac{C_5}{2} d^2(\cdot, \bar{x}), \quad \forall t \in (0, 1], \quad x \notin B_{\sqrt{\frac{2C_5}{C}}}(\bar{x}). \quad (5.31)$$

Now fix $x \in X$ and a sequence $t_n \downarrow 0$: the bound (5.31) grants that there are $y_n \in X$ such that

$$Q_t(-\varphi_{t_n})(x) = \frac{d^2(x, y_n)}{2t} - \varphi_{t_n}(y_n)$$

and that these y_n range in a bounded set. Thus up to pass to a subsequence we can assume that $y_n \rightarrow y$ for some $y \in X$, so that taking into account the $\Gamma - \underline{\lim}$ inequality previously proved we get

$$\frac{d^2(x, y)}{2t} - \varphi_0(y) \leq \underline{\lim}_{n \rightarrow \infty} \frac{d^2(x, y_n)}{2t} - \varphi_{t_n}(y_n) = \underline{\lim}_{n \rightarrow \infty} Q_t(-\varphi_{t_n})(x) \stackrel{(5.12a)}{=} \underline{\lim}_{n \rightarrow \infty} -\varphi_{t_n+t}(x) = -\varphi_t(x)$$

which shows that $Q_t(-\varphi_0)(x) \leq -\varphi_t(x)$, as desired.

(iii) For any $t \in (0, 1]$ we have

$$\varphi_0 \leq \varphi_t \stackrel{(5.5)}{\leq} -\psi_t$$

so that letting $t \downarrow 0$ and using the continuity of $[0, 1] \ni t \mapsto \psi_t(x)$ for all $x \in X$ we deduce that

$$\varphi_0 \leq -\psi_0 \quad \text{on } X.$$

Now notice that the fact that $-\varphi_0 \leq \Gamma - \underline{\lim}(-\varphi_t)$ implies that

$$\varphi_0(\gamma_0) \geq \overline{\lim}_{t \downarrow 0} \varphi_t(\gamma_t) \quad \forall \gamma \in C([0, 1], X). \quad (5.32)$$

Let π be the lifting of the W_2 -geodesic (μ_t) (recall point (i) of Theorem A.6); taking into account that the evaluation maps $e_t : C([0, 1], X) \rightarrow X$ are continuous and that $\text{supp}(\pi)$ is a compact subset of $C([0, 1], X)$, because given by constant speed geodesics running from the compact set $\text{supp}(\rho_0)$ to the compact $\text{supp}(\rho_1)$, it is easy to see that for every $\gamma \in \text{supp}(\pi)$ and $t \in [0, 1]$ we have $\gamma_t \in \text{supp}(\mu_t)$ and viceversa for every $x \in \text{supp}(\mu_t)$ there is $\gamma \in \text{supp}(\pi)$ with $\gamma_t = x$.

Thus let $x \in \text{supp}(\rho_0) = \text{supp}(\mu_0)$ and find $\gamma \in \text{supp}(\pi)$ with $\gamma_0 = x$: from the fact that $\gamma_t \in \text{supp}(\mu_t)$ and (5.5) we get

$$\varphi_0(x) \stackrel{(5.32)}{\geq} \overline{\lim}_{t \downarrow 0} \varphi_t(\gamma_t) = \overline{\lim}_{t \downarrow 0} -\psi_t(\gamma_t).$$

and since the continuity of $[0, 1] \ni t \mapsto \psi_t \in L^1(X, e^{-V})$ and the uniform local Lipschitz continuity of the ψ_t 's (both coming from Proposition 5.2) imply local uniform convergence of ψ_t to ψ_0 , we conclude $\varphi_0(x) \geq \psi_0(x)$.

Thus it remains to prove that $\varphi_0 = -\infty$ outside $\text{supp}(\rho_0)$. To this aim, we notice again that the supports of $f^\varepsilon, g^\varepsilon$ coincide with those of ρ_0, ρ_1 and use the second in (A.5) to get

$$\begin{aligned} f_t^\varepsilon(x) &= \mathfrak{h}_{\varepsilon t/2} f^\varepsilon(x) = \int f^\varepsilon(y) r_{\varepsilon t/2}(x, y) \, \mathfrak{d}\mathfrak{m}(y) \leq \frac{c_1}{v_{\varepsilon t/2}} e^{-c_2 \frac{d^2(x, \text{supp}(\rho_0))}{3\varepsilon t} + c_3 \varepsilon t} \int f^\varepsilon \, \mathfrak{d}\mathfrak{m}, \\ g_t^\varepsilon(x) &= \mathfrak{h}_{\varepsilon(1-t)/2} g^\varepsilon(x) = \int g^\varepsilon(y) r_{\varepsilon(1-t)/2}(x, y) \, \mathfrak{d}\mathfrak{m}(y) \leq \frac{c_4}{v_{\varepsilon(1-t)/2}} \int g^\varepsilon \, \mathfrak{d}\mathfrak{m}, \end{aligned}$$

for every $t \in (0, 1)$ and constants $c_i > 0$ depending on $K, N, \rho_0, \rho_1, \bar{x}$ only. From these bounds, the identity $\rho_t^\varepsilon = f_t^\varepsilon g_t^\varepsilon$ and the estimates (5.30) and (5.29) we deduce that

$$\overline{\lim}_{\varepsilon \downarrow 0} \varepsilon \log(\rho_t^\varepsilon(x)) \leq c_5 - c_6 \frac{d^2(x, \text{supp}(\rho_0))}{t} \quad \forall x \in X, t \in (0, 1). \quad (5.33)$$

Now let $\varepsilon_n \downarrow 0$ be the sequence such that $\varphi_t^{\varepsilon_n}, \psi_t^{\varepsilon_n}$ converge to φ_t, ψ_t as in Proposition 5.2 and put $S(x) := \sup_{\varepsilon \in (0, 1), t \in [0, 1/2]} |\psi_t^\varepsilon(x)| < \infty$ (recall (5.3)). The inequality

$$\varphi_t(x) = \lim_{n \rightarrow \infty} \varphi_t^{\varepsilon_n}(x) \leq \overline{\lim}_{n \rightarrow \infty} \varepsilon_n \log(\rho_t^{\varepsilon_n}(x)) - \lim_{n \rightarrow \infty} \psi_t^{\varepsilon_n}(x) \stackrel{(5.33)}{\leq} S(x) + c_5 - c_6 \frac{d^2(x, \text{supp}(\rho_0))}{t}$$

shows that if $x \notin \text{supp}(\rho_0)$ we have $\varphi_0(x) = \lim_{t \downarrow 0} \varphi_t(x) = -\infty$, as desired.

(iv) By the point (iii) just proven we have

$$\int \varphi_0 \rho_0 \, \mathfrak{d}\mathfrak{m} + \int \psi_1 \rho_1 \, \mathfrak{d}\mathfrak{m} = - \int \psi_0 \rho_0 \, \mathfrak{d}\mathfrak{m} - \int \varphi_1 \rho_1 \, \mathfrak{d}\mathfrak{m}$$

so that taking into account the weak continuity of $t \mapsto \mu_t$, the fact that the measures μ_t have equibounded supports and the continuity of $t \mapsto \varphi_t$ (resp. $t \mapsto \psi_t$) for t close to 1 (resp. close to 0) in the topology of local uniform convergence (direct consequence of the continuity in $L^1(X, e^{-V} \mathfrak{m})$ and the uniform local Lipschitz estimates provided by Proposition 5.2), we get

$$\begin{aligned} \int \varphi_0 \rho_0 \, \mathfrak{d}\mathfrak{m} + \int \psi_1 \rho_1 \, \mathfrak{d}\mathfrak{m} &= \lim_{t \downarrow 0} - \int \psi_t \rho_t \, \mathfrak{d}\mathfrak{m} - \int \varphi_{1-t} \rho_{1-t} \, \mathfrak{d}\mathfrak{m} \\ &\stackrel{(5.5)}{=} \lim_{t \downarrow 0} \int \varphi_t \rho_t \, \mathfrak{d}\mathfrak{m} - \int \varphi_{1-t} \rho_{1-t} \, \mathfrak{d}\mathfrak{m} \stackrel{(5.12b)}{=} \frac{1}{2} W_2^2(\mu_0, \mu_1). \end{aligned}$$

(v) Since $\rho_0 \in L^\infty(X)$, we also have $\rho_0 \log(\rho_0) \in L^\infty(X)$. The claim then follows from the identity $\rho_0 \varphi_0^\varepsilon = \varepsilon \rho_0 \log \rho_0 - \rho_0 \psi_0^\varepsilon$, the compactness of $\text{supp}(\rho_0)$, the local uniform convergence of $\psi_0^{\varepsilon_n}$ to ψ_0 as $n \rightarrow \infty$ and the fact that $\psi_0 = -\varphi_0$ on $\text{supp}(\rho_0)$. \square

5.3 Proof of the main theorem

We start with the following simple continuity statement:

Lemma 5.11. *With the same assumptions and notation as in Setting 4.1, let $t \mapsto \mu_t = \rho_t \mathfrak{m}$ be the W_2 -geodesic from μ_0 to μ_1 and $(\varphi_t)_{t \in (0, 1]}$ and $(\psi_t)_{t \in [0, 1)}$ any couple of limit functions given by Proposition 5.2.*

Then the maps

$$\begin{aligned} (0, 1] \ni t &\mapsto \rho_t \, \mathfrak{d}\varphi_t && \in L^2(T^*X) \\ [0, 1) \ni t &\mapsto \rho_t \, \mathfrak{d}\psi_t && \in L^2(T^*X) \\ (0, 1] \ni t &\mapsto \rho_t \, \mathfrak{d}\varphi_t \otimes \mathfrak{d}\varphi_t && \in L^2((T^*)^{\otimes 2}X) \\ [0, 1) \ni t &\mapsto \rho_t \, \mathfrak{d}\psi_t \otimes \mathfrak{d}\psi_t && \in L^2((T^*)^{\otimes 2}X) \end{aligned}$$

are all continuous w.r.t. the strong topologies.

proof By Lemma A.8 we know that for any $p < \infty$ we have $\rho_s \rightarrow \rho_t$ in $L^p(X)$ as $s \rightarrow t$ and thus in particular $\sqrt{\rho_s} \rightarrow \sqrt{\rho_t}$ as $s \rightarrow t$. Moreover, the compactness of the supports of ρ_0 and ρ_1 implies that there exist $\bar{x} \in X$ and $R > 0$ such that $\text{supp}(\rho_t) \subset B_R(\bar{x})$ for all $t \in [0, 1]$. Consider a Lipschitz cut-off function χ with support in $B_{R+1}(\bar{x})$ such that $\chi \equiv 1$ in $B_R(\bar{x})$. The closure of the differential and the fact that $\varphi_s \rightarrow \varphi_t$ weakly in $W^{1,2}(X, e^{-V} \mathbf{m})$ as $s \rightarrow t > 0$ (as a consequence of $(\varphi_t) \in C((0, 1], C(X, e^{-V})) \cap L_{loc}^\infty((0, 1), W^{1,2}(X, e^{-V} \mathbf{m}))$), see Proposition 5.4 and the notation therein) grant that $d\varphi_s \rightarrow d\varphi_t$ weakly in $L^2(T^*X, e^{-V} \mathbf{m})$ and thus $\chi d\varphi_s \rightarrow \chi d\varphi_t$ in $L^2(T^*X)$. Together with the previous claim about the densities, the fact that the latter are uniformly bounded in $L^\infty(X)$ and how χ is constructed, this is sufficient to conclude that $t \mapsto \sqrt{\rho_t} d\varphi_t \in L^2(T^*X)$ is weakly continuous.

We now claim that $t \mapsto \sqrt{\rho_t} d\varphi_t \in L^2(T^*X)$ is strongly continuous and to this aim we show that their $L^2(T^*X)$ -norms are constant. To see this, recall that by Proposition 5.4 we know that for $t \in (0, 1]$ the function $-(1-t)\psi_t$ is a Kantorovich potential from μ_t to μ_1 while from (5.5) and the locality of the differential we get that $|d\varphi_t| = |d\psi_t|$ μ_t -a.e., thus by point (iii) in Theorem A.6 we have that

$$\int |d\varphi_t|^2 \rho_t \, d\mathbf{m} = \frac{1}{(1-t)^2} \int |d(1-t)\psi_t|^2 \rho_t \, d\mathbf{m} = \frac{1}{(1-t)^2} W_2^2(\mu_t, \mu_1) = W_2^2(\mu_0, \mu_1).$$

Multiplying the $\sqrt{\rho_t} d\varphi_t$'s by $\sqrt{\rho_t}$ and using again the $L^2(X)$ -strong continuity of $\sqrt{\rho_t}$ and the uniform $L^\infty(X)$ -bound we conclude that $t \mapsto \rho_t d\varphi_t \in L^2(T^*X)$ is strongly continuous, as desired.

To prove the strong continuity of $t \mapsto \rho_t d\varphi_t \otimes d\varphi_t \in L^2((T^*)^{\otimes 2}X)$ we argue as in Corollary 5.9: the strong continuity of $t \mapsto \sqrt{\rho_t} d\varphi_t \in L^2(T^*X)$ and the fact that these are, locally in $t \in (0, 1]$, uniformly bounded (thanks again to $\text{supp}(\rho_t) \subset B_R(\bar{x})$ for all $t \in [0, 1]$), grant both that $t \mapsto \|\rho_t d\varphi_t \otimes d\varphi_t\|_{L^2((T^*)^{\otimes 2}X)}$ is continuous and that $t \mapsto \rho_t d\varphi_t \otimes d\varphi_t \in L^2((T^*)^{\otimes 2}X)$ is weakly continuous.

The claims about the ψ_t 's follow in the same way. \square

We now have all the tools needed to prove our main result. Notice that we shall not make explicit use of Theorem 1.4 but rather reprove it for (the restriction to $[\delta, 1 - \delta]$ of) entropic interpolations.

Theorem 5.12. *Let (X, d, \mathbf{m}) be an $\text{RCD}^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in [1, \infty)$. Let $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ be such that $\mu_0, \mu_1 \leq C\mathbf{m}$ for some $C > 0$, with compact supports and let (μ_t) be the unique W_2 -geodesic connecting μ_0 to μ_1 . Also, let $h \in H^{2,2}(X)$.*

Then the map

$$[0, 1] \ni t \mapsto \int h \, d\mu_t \in \mathbb{R}$$

belongs to $C^2([0, 1])$ and the following formulas hold for every $t \in [0, 1]$:

$$\begin{aligned} \frac{d}{dt} \int h \, d\mu_t &= \int \langle \nabla h, \nabla \phi_t \rangle \, d\mu_t, \\ \frac{d^2}{dt^2} \int h \, d\mu_t &= \int \text{Hess}(h)(\nabla \phi_t, \nabla \phi_t) \, d\mu_t, \end{aligned} \tag{5.34}$$

where ϕ_t is any function such that for some $s \neq t$, $s \in [0, 1]$, the function $-(s-t)\phi_t$ is a Kantorovich potential from μ_t to μ_s .

proof For the given μ_0, μ_1 introduce the notation of Setting 4.1 and then find $\varepsilon_n \downarrow 0$ such that $(\varphi_t^{\varepsilon_n}), (\psi_t^{\varepsilon_n})$ converge to limit curves $(\varphi_t), (\psi_t)$ as in Proposition 5.2.

By Lemma A.7 we know that the particular choice of the ϕ_t 's as in the statement does not affect the right-hand sides in (5.34), we shall therefore prove that such formulas hold for the choice $\phi_t := \psi_t$, which is admissible thanks to Proposition 5.4 whenever $t < 1$. The case $t = 1$ can be achieved swapping the roles of μ_0, μ_1 or, equivalently, with the choice $\phi_t = -\varphi_t$ which is admissible for $t > 0$.

Fix $h \in H^{2,2}(\mathbf{X})$ with compact support and for $t \in [0, 1]$ set

$$\mathbf{H}_n(t) := \int h \, d\mu_t^{\varepsilon_n} \quad \mathbf{H}(t) := \int h \, d\mu_t.$$

The bound (4.19) grants that the \mathbf{H}_n 's are uniformly bounded and the convergence in (5.2) that $\mathbf{H}_n(t) \rightarrow \mathbf{H}(t)$ for any $t \in [0, 1]$.

Since $(\rho_t^{\varepsilon_n}) \in AC_{loc}((0, 1), W^{1,2}(\mathbf{X}))$ we have that $\mathbf{H}_n \in AC_{loc}((0, 1))$ and, recalling the formula for $\frac{d}{dt}\rho_t^{\varepsilon}$ given by Proposition 4.3, that

$$\frac{d}{dt}\mathbf{H}_n(t) = \int h \frac{d}{dt}\rho_t^{\varepsilon_n} \, d\mathbf{m} = - \int h \operatorname{div}(\rho_t^{\varepsilon_n} \nabla \vartheta_t^{\varepsilon_n}) \, d\mathbf{m} = \int \langle \nabla h, \nabla \vartheta_t^{\varepsilon_n} \rangle \rho_t^{\varepsilon_n} \, d\mathbf{m}. \quad (5.35)$$

The fact that $\vartheta_t = \frac{\psi_t - \varphi_t}{2}$, the compactness of $\operatorname{supp}(h)$ and the bounds (4.19) and (4.9a) ensure that $|\frac{d}{dt}\mathbf{H}_n(t)|$ is uniformly bounded in n and $t \in [t_0, t_1] \subset (0, 1)$ and the compactness of $\operatorname{supp}(h)$ also allows us to use the convergence properties (5.26) and (5.2), which grant that

$$\iint_{t_0}^{t_1} \langle \nabla h, \nabla \vartheta_t^{\varepsilon_n} \rangle \rho_t^{\varepsilon_n} \, dt \, d\mathbf{m} \quad \rightarrow \quad \iint_{t_0}^{t_1} \langle \nabla h, \nabla \vartheta_t \rangle \rho_t \, dt \, d\mathbf{m}.$$

This is sufficient to pass to the limit in the distributional formulation of $\frac{d}{dt}\mathbf{H}_n(t)$ and taking into account that $\mathbf{H} \in C([0, 1])$ we have just proved that $\mathbf{H} \in AC_{loc}((0, 1))$ with

$$\frac{d}{dt}\mathbf{H}(t) = \int \langle \nabla h, \nabla \vartheta_t \rangle \rho_t \, d\mathbf{m} \quad (5.36)$$

for a.e. $t \in [0, 1]$. Recalling that $\vartheta_t = \frac{\psi_t - \varphi_t}{2}$, (5.5) and the locality of the differential we see that

$$\nabla \vartheta_t = \nabla \psi_t \quad \rho_t \mathbf{m}\text{-a.e.} \quad \forall t \in [0, 1], \quad (5.37)$$

and thus by Lemma 5.11 we see that the right-hand side of (5.36) is continuous in $t \in [0, 1]$, which then implies that $\mathbf{H} \in C^1([0, 1])$ and that the first in (5.34) holds for any $t \in [0, 1]$.

For the second derivative we assume in addition that $h \in \operatorname{Test}^\infty(\mathbf{X})$. Then we recall that $(\rho_t^{\varepsilon_n}) \in AC_{loc}((0, 1), W^{1,2}(\mathbf{X}))$ and $(\vartheta_t^{\varepsilon_n}) \in AC_{loc}((0, 1), W^{1,2}(\mathbf{X}, e^{-V} \mathbf{m}))$ with $V = M d^2(\cdot, \bar{x})$ for some $\bar{x} \in \mathbf{X}$ and $M > 0$ sufficiently large. Consider the rightmost side of (5.35) to get that $\frac{d}{dt}I_n(t) \in AC_{loc}((0, 1))$ and

$$\frac{d^2}{dt^2}\mathbf{H}_n(t) = \int \langle \nabla h, \nabla \frac{d}{dt}\vartheta_t^{\varepsilon_n} \rangle \rho_t^{\varepsilon_n} + \langle \nabla h, \nabla \vartheta_t^{\varepsilon_n} \rangle \frac{d}{dt}\rho_t^{\varepsilon_n} \, d\mathbf{m}$$

for a.e. t , so that defining the ‘acceleration’ a_t^{ε} as

$$a_t^{\varepsilon} := - \left(\frac{\varepsilon^2}{4} \Delta \log \rho_t^{\varepsilon} + \frac{\varepsilon^2}{8} |\nabla \log \rho_t^{\varepsilon}|^2 \right)$$

and recalling the formula for $\frac{d}{dt}\vartheta_t^\varepsilon$ given by Proposition 4.3 we have

$$\begin{aligned} \frac{d^2}{dt^2}H_n(t) &= \int \langle \nabla h, \nabla \left(-\frac{1}{2}|\nabla \vartheta_t^{\varepsilon_n}|^2 + a_t^{\varepsilon_n} \right) \rangle \rho_t^{\varepsilon_n} - \langle \nabla h, \nabla \vartheta_t^{\varepsilon_n} \rangle \operatorname{div}(\rho_t^{\varepsilon_n} \nabla \vartheta_t^{\varepsilon_n}) \, d\mathbf{m} \\ &= \int \left(-\frac{1}{2} \langle \nabla h, \nabla |\nabla \vartheta_t^{\varepsilon_n}|^2 \rangle + \langle \nabla (\langle \nabla h, \nabla \vartheta_t^{\varepsilon_n} \rangle), \nabla \vartheta_t^{\varepsilon_n} \rangle + \langle \nabla h, \nabla a_t^{\varepsilon_n} \rangle \right) \rho_t^{\varepsilon_n} \, d\mathbf{m} \\ \text{(by (A.15))} \quad &= \int \operatorname{Hess}(h)(\nabla \vartheta_t^{\varepsilon_n}, \nabla \vartheta_t^{\varepsilon_n}) \rho_t^{\varepsilon_n} \, d\mathbf{m} - \int (\Delta h + \langle \nabla h, \nabla \log \rho_t^{\varepsilon_n} \rangle) a_t^{\varepsilon_n} \rho_t^{\varepsilon_n} \, d\mathbf{m}. \end{aligned}$$

Since $\vartheta_t^\varepsilon = \frac{\psi_t^\varepsilon - \varphi_t^\varepsilon}{2}$ and $\operatorname{Hess}(h) \in L^2(T^{*\otimes 2}X)$ with compact support, by (5.26) and (5.2) we see that

$$\int \operatorname{Hess}(h)(\nabla \vartheta_t^{\varepsilon_n}, \nabla \vartheta_t^{\varepsilon_n}) \rho_t^{\varepsilon_n} \, d\mathbf{m} \xrightarrow{n \rightarrow \infty} \int \operatorname{Hess}(h)(\nabla \vartheta_t, \nabla \vartheta_t) \rho_t \, d\mathbf{m} \quad \text{in } L^1_{loc}(0, 1)$$

and since $|\nabla h|, \Delta h \in L^\infty(X)$, by Lemma 4.10 we deduce that

$$\int (\Delta h + \langle \nabla h, \nabla \log \rho_t^{\varepsilon_n} \rangle) a_t^{\varepsilon_n} \rho_t^{\varepsilon_n} \, d\mathbf{m} \rightarrow 0 \quad \text{in } L^1_{loc}(0, 1).$$

Hence we can pass to the limit in the distributional formulation of $\frac{d^2}{dt^2}H_n$ to obtain that $\frac{d}{dt}I \in AC_{loc}((0, 1))$ and

$$\frac{d^2}{dt^2}H(t) = \int \operatorname{Hess}(h)(\nabla \vartheta_t, \nabla \vartheta_t) \rho_t \, d\mathbf{m} \tag{5.38}$$

for a.e. t . Using again (5.37) and Lemma 5.11 we conclude that the right-hand side of (5.38) is continuous on $[0, 1)$, so that $H \in C^2([0, 1))$ and the second in (5.34) holds for every $t \in [0, 1)$.

It remains to remove the assumption that $h \in \operatorname{Test}^\infty(X)$ and has compact support. To this aim we claim that functions in $\operatorname{Test}^\infty(X)$ with compact support are dense in $H^{2,2}(X)$. To see this, let χ_R be as in Lemma A.2 and notice that the Leibniz rules for the gradient and the Laplacian easily give that $\chi_R h \rightarrow h$ in $W^{1,2}(X)$ and $\Delta(\chi_R h) \rightarrow \Delta h$ in $L^2(X)$ as $R \rightarrow \infty$ for every $h \in \operatorname{Test}^\infty(X)$. Hence by (A.13) we also have $\chi_R h \rightarrow h$ in $H^{2,2}(X)$. Taking into account that $\operatorname{Test}^\infty(X)$ is dense in $H^{2,2}(X)$ (recall (A.14)), our claim follows.

Now let $h \in H^{2,2}(X)$ be arbitrary and $(h_k) \subset \operatorname{Test}^\infty(X)$ with bounded support be $H^{2,2}$ -converging to h . Notice that we can choose the ϕ_t 's to be uniformly Lipschitz (e.g. by taking $\phi_t := \psi_t$ for $t \geq 1/2$, $\phi_t := -\varphi_t$ for $t < 1/2$ and recalling Proposition 4.4 and using a cut-off argument). The uniform L^∞ estimates (A.24), the fact that all the densities ρ_t are supported in a compact set B independent of $t \in [0, 1]$ and the L^2 -convergence of $h_k, \nabla h_k, \operatorname{Hess}(h_k)$ to $h, \nabla h, \operatorname{Hess}(h)$ respectively grant that as $k \rightarrow \infty$ we have that

$$\begin{aligned} \int h_k \, d\mu_t &\rightarrow \int h \, d\mu_t \\ \int \langle \nabla h_k, \nabla \phi_t \rangle \, d\mu_t &\rightarrow \int \langle \nabla h, \nabla \phi_t \rangle \, d\mu_t \\ \int \operatorname{Hess}(h_k)(\nabla \phi_t, \nabla \phi_t) \, d\mu_t &\rightarrow \int \operatorname{Hess}(h)(\nabla \phi_t, \nabla \phi_t) \, d\mu_t \end{aligned}$$

uniformly in $t \in [0, 1]$. This is sufficient to pass to the limit from the formulas (5.34) for the h_k 's to the one for h and also ensures the C^2 regularity of $t \mapsto \int h \, d\mu_t$. \square

5.4 Related differentiation formulas

In this last part we collect some direct consequences of Theorem 5.12. For the notion of covariant derivative and the related calculus rules we refer to [25].

Theorem 5.13. *Let (X, d, m) be an $\text{RCD}^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in [1, \infty)$. Then the following holds:*

- (i) *Let π be an optimal geodesic test plan with bounded support and $h \in H^{2,2}(X)$. Then the map $[0, 1] \ni t \mapsto h \circ e_t \in L^2(\pi)$ is in $C^2([0, 1], L^2(\pi))$ and we have*

$$\begin{aligned} \frac{d}{dt}(h \circ e_t) &= \langle \nabla h, \nabla \phi_t \rangle \circ e_t, \\ \frac{d^2}{dt^2}(h \circ e_t) &= \text{Hess}(h)(\nabla \phi_t, \nabla \phi_t) \circ e_t, \end{aligned}$$

for every $t \in [0, 1]$, where ϕ_t is any function such that for some $s \neq t$, $s \in [0, 1]$, the function $-(s - t)\phi_t$ is a Kantorovich potential from $(e_t)_*\pi$ to $(e_s)_*\pi$.

- (ii) *Let π be an optimal geodesic test plan with bounded support and $W \in H_C^{1,2}(X)$. Then the map $[0, 1] \ni t \mapsto \langle W, \nabla \phi_t \rangle \circ e_t \in L^2(\pi)$ is in $C^1([0, 1], L^2(\pi))$ and we have*

$$\frac{d}{dt}(\langle W, \nabla \phi_t \rangle \circ e_t) = (\nabla W : (\nabla \phi_t \otimes \nabla \phi_t)) \circ e_t,$$

for every $t \in [0, 1]$, where ϕ_t is as in (i).

- (iii) *Let μ_0, μ_1, ϕ_t be as in Theorem 5.12 and $W \in H_C^{1,2}(TX)$. Then the map*

$$[0, 1] \ni t \mapsto \int \langle W, \nabla \phi_t \rangle d\mu_t \in \mathbb{R}$$

belongs to $C^1([0, 1])$ and the following formula holds for every $t \in [0, 1]$

$$\frac{d}{dt} \int \langle W, \nabla \phi_t \rangle d\mu_t = \int \nabla W : (\nabla \phi_t \otimes \nabla \phi_t) d\mu_t.$$

proof

(i) Start observing that for π as in the assumptions and $\Gamma \subset C([0, 1], X)$ Borel with $\pi(\Gamma) > 0$, the curve $t \mapsto \pi(\Gamma)^{-1}(e_t)_*\pi|_\Gamma$ fulfills the assumptions of Theorem 5.12 with the same ϕ_t 's as in the current setting (and in particular, the ϕ_t 's can be chosen independently of Γ). Then, taking into account Lemma 5.11 it is easy to check that the maps $[0, 1] \ni t \mapsto h \circ e_t, \langle \nabla h, \nabla \phi_t \rangle \circ e_t, (\text{Hess}(h)(\nabla \phi_t, \nabla \phi_t) \circ e_t) \in L^2(\pi)$ are all continuous, and in particular with uniformly, in $t \in [0, 1]$, bounded $L^2(\pi)$ -norms.

Also, Theorem 5.12 applied to $t \mapsto \pi(\Gamma)^{-1}(e_t)_*\pi|_\Gamma$ gives, after integration and an application of Fubini's theorem, that for every $t, s \in [0, 1]$, $t < s$ we have

$$\begin{aligned} \int \eta(h \circ e_s - h \circ e_t) d\pi &= \int \eta \int_t^s \langle \nabla h, \nabla \phi_r \rangle \circ e_r dr d\pi \\ \int \eta(\langle \nabla h, \nabla \phi_s \rangle \circ e_s - \langle \nabla h, \nabla \phi_t \rangle \circ e_t) d\pi &= \int \eta \int_t^s \text{Hess}(h)(\nabla \phi_r, \nabla \phi_r) \circ e_r dr d\pi \end{aligned}$$

for every η of the form $\eta = \chi_\Gamma$ with $\Gamma \subset C([0, 1], X)$ Borel, where here and below the integral are intended in the Bochner sense. Then the fact that the linear span of such η 's is dense in $L^2(\boldsymbol{\pi})$ forces the equalities

$$\begin{aligned} h \circ e_s - h \circ e_t &= \int_t^s \langle \nabla h, \nabla \phi_r \rangle \circ e_r \, dr \\ \langle \nabla h, \nabla \phi_s \rangle \circ e_s - \langle \nabla h, \nabla \phi_t \rangle \circ e_t &= \int_t^s \text{Hess}(h)(\nabla \phi_r, \nabla \phi_r) \circ e_r \, dr \end{aligned}$$

which is the claim.

(ii) By (i) and the Leibniz rule for the covariant derivative (see [25]) we see that the claim holds for $W = \sum_{i=1}^n f_i \nabla g_i$, with $n \in \mathbb{N}$ and $(f_i), (g_i) \in \text{Test}^\infty(X)$. These vector fields are dense in $H_C^{1,2}(TX)$, hence the claim follows noticing that if $W_n \rightarrow W$ in $H_C^{1,2}(TX)$ and the ϕ_t 's are chosen uniformly Lipschitz (which as discussed in the proof of Theorem 5.12 is always admissible) then $\langle W_n, \nabla \phi_t \rangle \rightarrow \langle W, \nabla \phi_t \rangle$ and $\nabla W_n : (\nabla \phi_t \otimes \nabla \phi_t) \rightarrow \nabla W : (\nabla \phi_t \otimes \nabla \phi_t)$ in $L^2(X)$ as $n \rightarrow \infty$. Therefore, since $(e_t)_* \boldsymbol{\pi} \leq C \mathbf{m}$ for every $t \in [0, 1]$ and some $C > 0$, we have that

$$\begin{aligned} \langle W_n, \nabla \phi_t \rangle \circ e_t &\rightarrow \langle W, \nabla \phi_t \rangle \circ e_t \\ (\nabla W_n : (\nabla \phi_t \otimes \nabla \phi_t)) \circ e_t &\rightarrow (\nabla W : (\nabla \phi_t \otimes \nabla \phi_t)) \circ e_t \end{aligned}$$

in $L^2(\boldsymbol{\pi})$ uniformly in $t \in [0, 1]$. The conclusion follows.

(iii) Direct consequence of (ii) and an integration w.r.t. $\boldsymbol{\pi}$. □

A Reminders about analysis on RCD spaces

In this appendix we recall the basic definitions and properties of the various objects that we used in the body of the paper. We also provide detailed bibliographical references.

A.1 Sobolev calculus on RCD spaces

By $C([0, 1], (X, \mathbf{d}))$, or simply $C([0, 1], X)$, we denote the space of continuous curves with values on the metric space (X, \mathbf{d}) and for $t \in [0, 1]$ the **evaluation map** $e_t : C([0, 1], (X, \mathbf{d})) \rightarrow X$ is defined as $e_t(\gamma) := \gamma_t$. For the notion of **absolutely continuous curve** in a metric space and of **metric speed** see for instance Section 1.1 in [4]. The collection of absolutely continuous curves on $[0, 1]$ is denoted $AC([0, 1], (X, \mathbf{d}))$, or simply by $AC([0, 1], X)$.

By $\mathcal{P}(X)$ we denote the space of Borel probability measures on (X, \mathbf{d}) and by $\mathcal{P}_2(X) \subset \mathcal{P}(X)$ the subclass of those with finite second moment.

Let $(X, \mathbf{d}, \mathbf{m})$ be a complete and separable metric measure space endowed with a Borel non-negative measure which is finite on bounded sets.

For the definition of **test plans**, of the **Sobolev class** $S^2(X)$ and of **minimal weak upper gradient** $|Df|$ see [5] (and the previous works [14], [52] for alternative - but equivalent - definitions of Sobolev functions). The local counterpart of $S^2(X)$ is introduced as follows: $L_{loc}^2(X)$ is defined as the space of functions $f \in L^0(X)$ such that for all compact set $\Omega \subset X$ there exists a function $g \in L^2(X)$ such that $f = g$ \mathbf{m} -a.e. in Ω and the local Sobolev class $S_{loc}^2(X)$ is then defined as

$$S_{loc}^2(X) := \{f \in L^0(X) : \forall \Omega \subset\subset X \exists g \in S^2(X) \text{ s.t. } f = g \text{ } \mathbf{m}\text{-a.e. in } \Omega\}. \quad (\text{A.1})$$

The local minimal weak upper gradient of a function $f \in S_{loc}^2(X)$ is denoted by $|Df|$, omitting the locality feature, and defined for all $\Omega \subset\subset X$ as $|Df| := |Dg|$ \mathbf{m} -a.e. in Ω , where g is as in (A.1). The definition does not depend neither on Ω nor on the choice of g associated to it by locality of the minimal weak upper gradient.

The Sobolev space $W^{1,2}(X)$ (resp. $W_{loc}^{1,2}(X)$) is defined as $L^2(X) \cap S^2(X)$ (resp. $L_{loc}^2 \cap S_{loc}^2(X)$). When endowed with the norm $\|f\|_{W^{1,2}}^2 := \|f\|_{L^2}^2 + \| |Df| \|_{L^2}^2$, $W^{1,2}(X)$ is a Banach space. The **Cheeger energy** is the convex and lower-semicontinuous functional $E : L^2(X) \rightarrow [0, \infty]$ given by

$$E(f) := \begin{cases} \frac{1}{2} \int |Df|^2 \, d\mathbf{m} & \text{for } f \in W^{1,2}(X) \\ +\infty & \text{otherwise} \end{cases}$$

(X, d, \mathbf{m}) is **infinitesimally Hilbertian** (see [27]) if $W^{1,2}(X)$ is Hilbert. In this case E is a Dirichlet form and its infinitesimal generator Δ , which is a closed self-adjoint operator on $L^2(X)$, is called **Laplacian** on (X, d, \mathbf{m}) and its domain denoted by $D(\Delta) \subset W^{1,2}(X)$. The flow (\mathbf{h}_t) associated to E is called **heat flow** (see [5]), and for any $f \in L^2(X)$ the curve $t \mapsto \mathbf{h}_t f \in L^2(X)$ is continuous on $[0, \infty)$, locally absolutely continuous on $(0, \infty)$ and the only solution of

$$\frac{d}{dt} \mathbf{h}_t f = \Delta \mathbf{h}_t f \quad \mathbf{h}_t f \rightarrow f \text{ as } t \downarrow 0.$$

If moreover (X, d, \mathbf{m}) is an $\text{RCD}(K, \infty)$ space (see [6]), the following a priori estimates hold true for every $f \in L^2(X)$ and $t > 0$:

$$E(\mathbf{h}_t f) \leq \frac{1}{4t} \|f\|_{L^2(X)}^2, \tag{A.2a}$$

$$\|\Delta \mathbf{h}_t f\|_{L^2(X)}^2 \leq \frac{1}{2t^2} \|f\|_{L^2(X)}^2. \tag{A.2b}$$

Still within the RCD framework, there exists the **heat kernel**, namely a function

$$(0, \infty) \times X^2 \ni (t, x, y) \mapsto r_t[x](y) = r_t[y](x) \in (0, \infty) \tag{A.3}$$

such that

$$\mathbf{h}_t f(x) = \int f(y) r_t[x](y) \, d\mathbf{m}(y) \quad \forall t > 0 \tag{A.4}$$

for every $f \in L^2(X)$. For every $x \in X$ and $t > 0$, $r_t[x]$ is a probability density and thus (A.4) can be used to extend the heat flow to $L^1(X)$ and shows that the flow is **mass preserving** and satisfies the **maximum principle**, i.e.

$$f \leq c \quad \mathbf{m} - a.e. \quad \Rightarrow \quad \mathbf{h}_t f \leq c \quad \mathbf{m} - a.e., \quad \forall t > 0.$$

For finite-dimensional $\text{RCD}^*(K, N)$ spaces (introduced in [27]) the heat kernel satisfies **Gaussian estimates**, i.e. for every $\delta > 0$ there are positive constants $C_1 = C_1(K, N, \delta)$ and $C_2 = C_2(K, N, \delta)$ such that for every $x, y \in X$ and $t > 0$ it holds

$$\frac{1}{C_1 \mathbf{m}(B_{\sqrt{t}}(y))} \exp\left(-\frac{d^2(x, y)}{(4 - \delta)t} - C_2 t\right) \leq r_t[x](y) \leq \frac{C_1}{\mathbf{m}(B_{\sqrt{t}}(y))} \exp\left(-\frac{d^2(x, y)}{(4 + \delta)t} + C_2 t\right), \tag{A.5}$$

see [36], which adapts the approach of [53], [54] to the RCD setting.

For general metric measure spaces, the **differential** is a well defined linear map d from $S^2(X)$ with values in the **cotangent module** $L^2(T^*X)$ (see [25]) which is a closed operator when seen as unbounded operator on $L^2(X)$. If (X, d, \mathbf{m}) is infinitesimally Hilbertian, which from now on we shall always assume, the cotangent module is canonically isomorphic to its dual, the **tangent module** $L^2(TX)$, and the isomorphism sends the differential df to the gradient ∇f . Replacing the language of L^2 -normed modules with the L^0 's one (see [25]), the differential can be extended to $d : S^2_{loc}(X) \rightarrow L^0(T^*X)$, where $L^0(T^*X)$ denotes the family of (measurable) 1-forms. The dual of $L^0(T^*X)$ as L^0 -normed module is denoted by $L^0(TX)$, it is canonically isomorphic to $L^0(T^*X)$ and its elements are called vector fields. With this said, $L^2_{loc}(T^*X) \subset L^0(T^*X)$ (resp. $L^2_{loc}(TX) \subset L^0(TX)$) is defined as the collection of the 1-forms ω such that $|\omega| \in L^2_{loc}(X)$ (resp. the vector fields v such that $|v| \in L^2_{loc}(X)$).

The **divergence** of a vector field is defined as (minus) the adjoint of the differential, i.e. we say that $v \in L^2(TX)$ (resp. $v \in L^2_{loc}(TX)$) has a divergence in $L^2(X)$ (resp. in $L^2_{loc}(X)$), and write $v \in D(\text{div})$ (resp. $v \in D(\text{div}_{loc})$), provided there is a function $g \in L^2(X)$ (resp. $g \in L^2_{loc}(X)$) such that

$$\int fg \, d\mathbf{m} = - \int df(v) \, d\mathbf{m} \quad \forall f \in W^{1,2}(X).$$

(resp. for all Lipschitz functions f with bounded support). In this case g is unique and is denoted $\text{div}(v)$. A function $f \in W^{1,2}_{loc}(X)$ has Laplacian in $L^2_{loc}(X)$, and we shall write $f \in D(\Delta_{loc})$, if there exists $g \in L^2_{loc}(X)$ such that

$$\int \phi g \, d\mathbf{m} = - \int \langle \nabla \phi, \nabla f \rangle \, d\mathbf{m}, \quad \forall \phi \text{ Lipschitz with bounded support}$$

and in this case, since g is unique, we set $\Delta f := g$. It can be verified that

$$f \in D(\Delta_{loc}) \text{ if and only if } \nabla f \in D(\text{div}_{loc}) \text{ and in this case } \Delta f = \text{div}(\nabla f),$$

in accordance with the smooth case.

As regards the properties of d, div, Δ , the differential satisfies the following calculus rules which we shall use extensively without further notice:

$$\begin{aligned} |df| &= |Df| & \mathbf{m}\text{-a.e.} & & \forall f \in S^2(X) \\ df &= dg & \mathbf{m}\text{-a.e. on } \{f = g\} & & \forall f, g \in S^2(X) \\ d(\varphi \circ f) &= \varphi' \circ f \, df & & & \forall f \in S^2(X), \varphi : \mathbb{R} \rightarrow \mathbb{R} \text{ Lipschitz} \\ d(fg) &= g \, df + f \, dg & & & \forall f, g \in L^\infty \cap S^2(X) \end{aligned}$$

where it is part of the properties the fact that $\varphi \circ f, fg \in S^2(X)$ for φ, f, g as above. For the divergence, the formula

$$\text{div}(fv) = df(v) + f \, \text{div}(v) \quad \forall f \in W^{1,2}(X), v \in D(\text{div}), \text{ such that } |f|, |v| \in L^\infty(X)$$

holds, where it is intended in particular that $fv \in D(\text{div})$ for f, v as above, and for the Laplacian

$$\begin{aligned} \Delta(\varphi \circ f) &= \varphi'' \circ f |df|^2 + \varphi' \circ f \Delta f \\ \Delta(fg) &= g \Delta f + f \Delta g + 2 \langle \nabla f, \nabla g \rangle \end{aligned}$$

where in the first equality we assume that $f \in D(\Delta)$, $\varphi \in C^2(\mathbb{R})$ are such that $f, |df| \in L^\infty(X)$ and $\varphi', \varphi'' \in L^\infty(\mathbb{R})$ and in the second that $f, g \in D(\Delta) \cap L^\infty(X)$ and $|df|, |dg| \in L^\infty(X)$ and it is part of the claims that $\varphi \circ f, fg$ are in $D(\Delta)$. On $S_{loc}^2(X)$ as well as on $D(\text{div}_{loc})$ and $D(\Delta_{loc})$ the same calculus rules hold with slight adaptations. For sake of information, we present the chain rule for differential and Laplacian, as they will be widely exploited without further mention.

Lemma A.1 (Calculus rules). *Let $(X, \mathbf{d}, \mathbf{m})$ be an $\text{RCD}^*(K, N)$ with $K \in \mathbb{R}$ and $N \in [1, \infty)$. Then:*

(i) *for all $f \in S_{loc}^2(X)$ and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ such that for all $K \subset\subset X$ there exists $I_K \subset\subset \mathbb{R}$ in such a way that $\mathcal{L}^1(f(K) \setminus I_K) = 0$ and $\varphi|_{I_K}$ is Lipschitz it holds*

$$d(\varphi \circ f) = \varphi' \circ f,$$

where it is part of the statement the fact that $\varphi \circ f \in S_{loc}^2(X)$ for φ, f as above; analogous statements hold for the gradient;

(ii) *for all $f \in D(\Delta_{loc})$ and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ such that $f, |df| \in L_{loc}^\infty(X)$ and $\varphi', \varphi'' \in L^\infty(\mathbb{R})$ it holds*

$$\Delta(\varphi \circ f) = \varphi'' \circ f |df|^2 + \varphi' \circ f \Delta f$$

where it is part of the claims that $\varphi \circ f \in D(\Delta_{loc})$.

The proof is based on the locality of such differentiation operators and the analogous properties of their global counterparts defined on $S^2(X), D(\Delta)$.

Beside this notion of L^2 -valued Laplacian, we shall also need that of measure-valued Laplacian ([27]). A function $f \in W^{1,2}(X)$ is said to have measure-valued Laplacian, and in this case we write $f \in D(\Delta)$, provided there exists a Borel (signed) measure μ whose total variation is finite on bounded sets and such that

$$\int g d\mu = - \int \langle \nabla g, \nabla f \rangle d\mathbf{m}, \quad \forall g \text{ Lipschitz with bounded support.}$$

In this case μ is unique and denoted Δf . This notion is compatible with the previous one in the sense that

$$f \in D(\Delta), \Delta f \ll \mathbf{m} \text{ and } \frac{d\Delta f}{d\mathbf{m}} \in L^2(X) \quad \Leftrightarrow \quad f \in D(\Delta) \text{ and in this case } \Delta f = \frac{d\Delta f}{d\mathbf{m}}.$$

On $\text{RCD}(K, \infty)$ spaces, the vector space of ‘test functions’ (see [51])

$$\text{Test}^\infty(X) := \left\{ f \in D(\Delta) \cap L^\infty(X) : |\nabla f| \in L^\infty(X), \Delta f \in L^\infty \cap W^{1,2}(X) \right\}$$

is an algebra dense in $W^{1,2}(X)$ for which it holds

$$f \in \text{Test}^\infty(X) \text{ and } \varphi \in C^\infty(\mathbb{R}) \quad \Rightarrow \quad \varphi \circ f \in \text{Test}^\infty(X). \quad (\text{A.6})$$

Combining the Gaussian estimates on $\text{RCD}^*(K, N)$ spaces, $N < \infty$, with the results in [51] we see that

$$f \in L^2 \cap L^\infty(X), t > 0 \quad \Rightarrow \quad \mathbf{h}_t(f) \in \text{Test}^\infty(X). \quad (\text{A.7})$$

The fact that $\text{Test}^\infty(\mathbf{X})$ is an algebra is based on the property

$$f \in \text{Test}^\infty(\mathbf{X}) \quad \Rightarrow \quad |df|^2 \in W^{1,2}(\mathbf{X}) \quad \text{with} \\ \int |d(|df|^2)|^2 \, \mathbf{m} \leq \| |df|^2 \|_{L^\infty}^2 \left(\| |df|^2 \|_{L^2} \| d\Delta f \|_{L^2} + |K| \| |df|^2 \|_{L^2}^2 \right) \quad (\text{A.8})$$

and actually a further regularity property of test functions is that

$$f \in \text{Test}^\infty(\mathbf{X}) \quad \Rightarrow \quad |df|^2 \in D(\Delta),$$

so that it is possible to introduce the **measure-valued Γ_2 operator** ([51]) as

$$\mathbf{\Gamma}_2(f) := \Delta \frac{|df|^2}{2} - \langle \nabla f, \nabla \Delta f \rangle \mathbf{m} \quad \forall f \in \text{Test}^\infty(\mathbf{X}).$$

By construction, the assignment $f \mapsto \mathbf{\Gamma}_2(f)$ is a quadratic form.

An important property of the heat flow on $\text{RCD}(K, \infty)$ spaces is the **Bakry-Émery contraction estimate** (see [6]):

$$|\text{dh}_t f|^2 \leq e^{-2Kt} \mathbf{h}_t(|df|^2) \quad \forall f \in W^{1,2}(\mathbf{X}), \quad t \geq 0. \quad (\text{A.9})$$

We also recall that $\text{RCD}(K, \infty)$ spaces have the **Sobolev-to-Lipschitz** property ([6], [24]), i.e.

$$f \in W^{1,2}(\mathbf{X}), \quad |df| \in L^\infty(\mathbf{X}) \quad \Rightarrow \quad \exists \tilde{f} = f \mathbf{m} - a.e. \quad \text{with} \quad \text{Lip}(\tilde{f}) \leq \| |df|^2 \|_{L^\infty}, \quad (\text{A.10})$$

and thus we shall typically identify Sobolev functions with bounded differentials with their Lipschitz representative; in particular this will be the case for functions in $\text{Test}^\infty(\mathbf{X})$.

A well-known consequence of lower Ricci curvature bounds (see e.g. [15], [16], [17]) is the existence of ‘**good cut-off functions**’, typically intended as cut-offs with bounded Laplacian; for our purposes the following result will be sufficient:

Lemma A.2. *Let (X, d, \mathbf{m}) be an $\text{RCD}^*(K, N)$ space with $K \in \mathbb{R}$ and $N \in [1, \infty)$. Then for all $R > 0$ and $x \in X$ there exists a function $\chi_R : X \rightarrow \mathbb{R}$ satisfying:*

- (i) $0 \leq \chi_R \leq 1$, $\chi_R \equiv 1$ on $B_R(x)$ and $\text{supp}(\chi_R) \subset B_{R+1}(x)$;
- (ii) $\chi_R \in \text{Test}^\infty(\mathbf{X})$.

Moreover, there exist constants $C, C' > 0$ depending on K, N only such that

$$\| |\nabla \chi_R| \|_{L^\infty(\mathbf{X})} \leq C, \quad \| \Delta \chi_R \|_{L^\infty(\mathbf{X})} \leq C'. \quad (\text{A.11})$$

The proof can be obtained following verbatim the arguments given in Lemma 3.1 of [47] (inspired by [8], see also [31] for an alternative approach): there the authors are interested in cut-off functions such that $\chi \equiv 1$ on $B_R(x)$ and $\text{supp}(\chi) \subset B_{2R}(x)$: for this reason they fix $R > 0$ and then claim that for all $x \in X$ and $0 < r < R$ there exists a cut-off function χ satisfying (i), (ii) and (A.11) with C, C' also depending on R . However, as far as one is concerned with cut-off functions χ where the distance between $\{\chi = 0\}$ and $\{\chi = 1\}$ is always equal to 1, the proof of [47] in the case $R = 1$ applies and does not affect (A.11).

A direct consequence of the existence of such cut-off functions is that

$$\begin{aligned} & \{f \in L^2_{loc}(X) : \forall \Omega \subset\subset X \exists g \in \text{Test}^\infty(X) \text{ s.t. } f = g \text{ m-a.e. in } \Omega\} \\ &= \left\{ f \in D(\Delta_{loc}) \cap L^\infty_{loc}(X) : |\nabla f| \in L^\infty_{loc}(X), \Delta f \in W^{1,2}_{loc}(X) \right\}. \end{aligned} \quad (\text{A.12})$$

Indeed the ‘ \subset ’ inclusion is obvious, while for the opposite one if f belongs to the second set and $\Omega \subset X$ is a bounded open set, consider a cut-off function $\chi \in \text{Test}^\infty(X)$ with compact support and $\chi \equiv 1$ on Ω : it is clear that $\chi f \in \text{Test}^\infty(X)$ and $\chi f \equiv f$ on Ω . We shall call the set in (A.12) the space of *local* test functions and denote it $\text{Test}^\infty_{loc}(X)$.

The existence of the space of test functions and the language of L^2 -normed L^∞ -modules allow to introduce the spaces $W^{2,2}(X)$ and $W^{1,2}_C(TX)$ as follows (see [25]). We first consider the tensor product $L^2((T^*)^{\otimes 2}X)$ of $L^2(T^*X)$ with itself. The pointwise norm on such module is denoted $|\cdot|_{\text{HS}}$ (as in the smooth case it coincides with the Hilbert-Schmidt one) and $\langle \cdot, \cdot \rangle$ is the scalar product inducing it. Then we say that a function $f \in W^{1,2}(X)$ belongs to $W^{2,2}(X)$ provided there exists $A \in L^2((T^*)^{\otimes 2}X)$ symmetric, i.e. such that $A(v_1, v_2) = A(v_2, v_1)$ m-a.e. for every $v_1, v_2 \in L^2(TX)$, for which it holds

$$\int hA(\nabla g, \nabla g) \, \text{d}\mathbf{m} = \int -\langle \nabla f, \nabla g \rangle \text{div}(h\nabla g) - h\langle \nabla f, \nabla \frac{|\nabla g|^2}{2} \rangle \, \text{d}\mathbf{m} \quad \forall g, h \in \text{Test}^\infty(X).$$

In this case A is unique, called **Hessian** of f and denoted by $\text{Hess}(f)$. The space $W^{2,2}(X)$ endowed with the norm

$$\|f\|_{W^{2,2}(X)}^2 := \|f\|_{L^2(X)}^2 + \|\text{d}f\|_{L^2(T^*X)}^2 + \|\text{Hess}(f)\|_{L^2((T^*)^{\otimes 2}X)}^2$$

is a separable Hilbert space which contains $\text{Test}^\infty(X)$ and in particular is dense in $W^{1,2}(X)$. It is proved in [25] that $D(\Delta) \subset W^{2,2}(X)$ with

$$\int |\text{Hess}(f)|_{\text{HS}}^2 \, \text{d}\mathbf{m} \leq \int (\Delta f)^2 - K|\nabla f|^2 \, \text{d}\mathbf{m} \quad \forall f \in D(\Delta). \quad (\text{A.13})$$

The space $H^{2,2}(X)$ is defined as the closure of $D(\Delta)$ in $W^{2,2}(X)$ and following the arguments in Proposition 4.3.18 in [25] it is not difficult to see that

$$\text{Test}^\infty(X) \text{ is dense in } H^{2,2}(X). \quad (\text{A.14})$$

It is unknown whether $H^{2,2}(X) = W^{2,2}(X)$ or not. We recall that

$$\text{d}\langle \nabla f, \nabla g \rangle = \text{Hess}(f)(\nabla g, \cdot) + \text{Hess}(g)(\nabla f, \cdot) \quad \forall f, g \in \text{Test}^\infty(X) \quad (\text{A.15})$$

and that the Hessian is a local operator, i.e. $\text{Hess}(f) = \text{Hess}(g)$ m-a.e. on $\{f = g\}$. Using this latter fact, for $f \in \text{Test}^\infty(X)$ we can define $\text{Hess}(f)$ as the element in the L^0 -completion of $L^2((T^*)^{\otimes 2}X)$ defined by

$$\text{Hess}(f) := \text{Hess}(g) \quad \text{m-a.e. on } \{f = g\} \quad \forall g \in \text{Test}^\infty(X).$$

The **Bochner inequality** on $\text{RCD}(K, \infty)$ spaces takes the form of an inequality between measures ([25] - see also the previous contributions [51], [57]):

$$\mathbf{\Gamma}_2(f) \geq (|\text{Hess}(f)|_{\text{HS}}^2 + K|\text{d}f|^2) \mathbf{m} \quad \forall f \in \text{Test}^\infty(X), \quad (\text{A.16})$$

and if the space is $\text{RCD}^*(K, N)$ for some finite N it also holds ([20], [9]):

$$\Gamma_2(f) \geq \left(\frac{(\Delta f)^2}{N} + K|df|^2 \right) \mathbf{m} \quad \forall f \in \text{Test}^\infty(\mathbf{X}). \quad (\text{A.17})$$

Notice that since the Laplacian is in general not the trace of the Hessian, the former does not trivially imply the latter (in connection to this, see [34]).

As regards the geometric features of finite-dimensional $\text{RCD}^*(K, N)$ spaces, we recall the **Bishop-Gromov inequality** in the form we shall need (see [55], [56]): for any $x \in \text{supp}(\mathbf{m})$ and for any $0 < r \leq R < \infty$ it holds

$$\frac{\mathbf{m}(B_r(x))}{\mathbf{m}(B_R(x))} \geq \frac{\int_0^r \sinh(t\sqrt{-K/(N-1)})^{N-1} dt}{\int_0^R \sinh(t\sqrt{-K/(N-1)})^{N-1} dt} \quad \frac{\mathfrak{s}_r(x)}{\mathfrak{s}_R(x)} \geq \left(\frac{\sinh(r\sqrt{-K/(N-1)})}{\sinh(R\sqrt{-K/(N-1)})} \right)^{N-1} \quad (\text{A.18})$$

(with standard adaptations and caveat if $K \geq 0$) where

$$\mathfrak{s}_r(x) := \limsup_{\delta \downarrow 0} \frac{1}{\delta} \mathbf{m}(\overline{B_{r+\delta}(x)} \setminus B_r(x)).$$

A couple of interesting consequences are the following: \mathbf{m} is uniformly locally doubling with an explicit expression for the local doubling constant, i.e. for all $x \in \mathbf{X}$ and $r > 0$ it holds

$$\mathbf{m}(B_{2r}(x)) \leq 2^N \cosh \left(2\sqrt{\frac{-K}{N-1}} r \right)^{N-1} \mathbf{m}(B_r(x)); \quad (\text{A.19})$$

and there exists a constant $C > 0$ such that the following volume growth condition is satisfied

$$\mathbf{m}(B_r(x)) \leq C e^{Cr}, \quad \forall x \in \mathbf{X}, r > 0. \quad (\text{A.20})$$

We conclude the section recalling the notion of Regular Lagrangian Flow, introduced by Ambrosio-Trevisan in [10] as the generalization to RCD spaces of the analogous concept existing on \mathbb{R}^d as proposed by Ambrosio in [1]:

Definition A.3 (Regular Lagrangian Flow). *Given $(v_t) \in L^1([0, 1], L^2(T\mathbf{X}))$, the function $F : [0, 1] \times \mathbf{X} \rightarrow \mathbf{X}$ is a Regular Lagrangian Flow for (v_t) provided:*

i) $[0, 1] \ni t \mapsto F_t(x)$ is continuous for every $x \in \mathbf{X}$

ii) for every $f \in \text{Test}^\infty(\mathbf{X})$ and \mathbf{m} -a.e. x the map $t \mapsto f(F_t(x))$ belongs to $W^{1,1}([0, 1])$ and

$$\frac{d}{dt} f(F_t(x)) = df(v_t)(F_t(x)) \quad \text{a.e. } t \in [0, 1].$$

iii) it holds

$$(F_t)_* \mathbf{m} \leq C \mathbf{m} \quad \forall t \in [0, 1]$$

for some constant $C > 0$.

In [10] the authors prove that under suitable assumptions on the v_t 's, Regular Lagrangian Flows exist and are unique. We shall use the following formulation of their result (weaker than the one provided in [10]):

Theorem A.4. *Let (X, d, \mathbf{m}) be an $\text{RCD}(K, \infty)$ space and $(v_t) \in L^1([0, 1], L^2(TX))$ be such that $v_t \in D(\text{div})$ for a.e. t and*

$$\text{div}(v_t) \in L^1([0, 1], L^2(X)) \quad (\text{div}(v_t))^- \in L^1([0, 1], L^\infty(X)).$$

Then there exists a unique, up to \mathbf{m} -a.e. equality, Regular Lagrangian Flow F for (v_t) . For such flow, the quantitative bound

$$(F_t)_* \mathbf{m} \leq \exp \left(\int_0^1 \|(\text{div}(v_t))^- \|_{L^\infty(X)} dt \right) \mathbf{m} \quad (\text{A.21})$$

holds for every $t \in [0, 1]$ and for \mathbf{m} -a.e. x the curve $t \mapsto F_t(x)$ is absolutely continuous and its metric speed $\text{ms}_t(F(x))$ at time t satisfies

$$\text{ms}_t(F(x)) = |v_t|(F_t(x)) \quad \text{a.e. } t \in [0, 1]. \quad (\text{A.22})$$

To be precise, (A.22) is not explicitly stated in [10]; its proof is anyway not hard and can be obtained, for instance, following the arguments in [25].

A.2 Optimal transport on RCD spaces

It is well known that on \mathbb{R}^d , curves of measures which are W_2 -absolutely continuous are in correspondence with appropriate solutions of the **continuity equation** ([4]). It has been proved in [28] that the same connection holds on arbitrary metric measure spaces (X, d, \mathbf{m}) , provided the measures are controlled by $C\mathbf{m}$ for some $C > 0$, the formulation of such result which we shall need is:

Theorem A.5 (Continuity equation and W_2 -AC curves). *Let (X, d, \mathbf{m}) be infinitesimally Hilbertian, $(\mu_t) \subset \mathcal{P}(X)$ be weakly continuous and $t \mapsto X_t \in L^0(TX)$ be a family of vector fields, possibly defined only for a.e. $t \in [0, 1]$. Assume that the map $t \mapsto \int |X_t|^2 d\mu_t$ is Borel and:*

$$\mu_t \leq C\mathbf{m} \quad \forall t \in [0, 1] \text{ for some } C > 0 \quad (\text{A.23a})$$

$$\int_0^1 \int |X_t|^2 d\mu_t dt < \infty \quad (\text{A.23b})$$

and that the continuity equation

$$\frac{d}{dt} \mu_t + \text{div}(X_t \mu_t) = 0,$$

is satisfied in the following sense: for any $f \in W^{1,2}(X)$ the map $[0, 1] \ni t \mapsto \int f d\mu_t$ is absolutely continuous and it holds

$$\frac{d}{dt} \int f d\mu_t = \int df(X_t) d\mu_t \quad \text{a.e. } t.$$

Then $(\mu_t) \in AC([0, 1], (\mathcal{P}(X), W_2))$ and

$$|\dot{\mu}_t|^2 = \int |X_t|^2 d\mu_t \quad \text{a.e. } t \in [0, 1].$$

Recall that given $f : X \rightarrow \mathbb{R}$ the upper and lower slopes $|D^+ f|, |D^- f| : X \rightarrow [0, \infty]$ are defined as 0 on isolated points and otherwise

$$|D^+ f|(x) := \overline{\lim}_{y \rightarrow x} \frac{(f(y) - f(x))^+}{d(x, y)} \quad |D^- f|(x) := \overline{\lim}_{y \rightarrow x} \frac{(f(y) - f(x))^-}{d(x, y)}.$$

Similarly, the **local Lipschitz constant** $\text{lip}(f) : X \rightarrow [0, \infty]$ is defined as 0 on isolated points and otherwise as

$$\text{lip} f(x) := \max\{|D^+ f|(x), |D^- f|(x)\} = \limsup_{y \rightarrow x} \frac{|f(x) - f(y)|}{d(x, y)}.$$

We also recall that the c -transform $\varphi^c : X \rightarrow \mathbb{R} \cup \{-\infty\}$ of a function $\varphi : X \rightarrow \mathbb{R} \cup \{-\infty\}$ is defined as

$$\varphi^c(x) := \inf_{y \in X} \frac{d^2(x, y)}{2} - \varphi(y)$$

and that φ is said to be **c -concave** provided $\varphi = \psi^c$ for some ψ . Also, given $\mu_0, \mu_1 \in \mathcal{P}_2(X)$, a function $\varphi : X \rightarrow \mathbb{R} \cup \{-\infty\}$ is called **Kantorovich potential** from μ_0 to μ_1 provided it is c -concave and

$$\int \varphi d\mu_0 + \int \varphi^c d\mu_1 = \frac{1}{2} W_2^2(\mu_0, \mu_1).$$

It is worth recalling that on general complete and separable metric spaces (X, d) we have that for $\mu_0, \mu_1 \in \mathcal{P}(X)$ with bounded support there exists a Kantorovich potential from μ_0 to μ_1 which is Lipschitz and bounded.

This can be obtained starting from an arbitrary Kantorovich potential ψ and then defining

$$\varphi(x) := \min \left\{ C, \inf_{y \in X} \frac{d^2(x, y)}{2} - \psi^c(y) \right\}$$

for C sufficiently big.

With this said, we recall the following version of Brenier-McCann theorem on RCD spaces ((i) comes from [23] and [49], (ii) from [6] and [27], (iii) from [5] and (iv) from [32]).

Theorem A.6. *Let (X, d, \mathbf{m}) be an $\text{RCD}(K, \infty)$ space and $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ with bounded support and such that $\mu_0, \mu_1 \leq C \mathbf{m}$ for some $C > 0$. Also, let φ be a Kantorovich potential for the couple (μ_0, μ_1) which is locally Lipschitz on a neighbourhood of $\text{supp}(\mu_0)$. Then:*

i) *There exists a unique geodesic (μ_t) from μ_0 to μ_1 , it satisfies*

$$\mu_t \leq C' \mathbf{m} \quad \forall t \in [0, 1] \text{ for some } C' > 0 \quad (\text{A.24})$$

and there is a unique lifting π of it, i.e. a unique measure $\pi \in \mathcal{P}(C([0, 1], X))$ such that $(e_t)_ \pi = \mu_t$ for every $t \in [0, 1]$ and $\iint_0^1 |\dot{\gamma}_t|^2 dt d\pi(\gamma) = W_2^2(\mu_0, \mu_1)$.*

ii) *For every $f \in W^{1,2}(X)$ the map $t \mapsto \int f d\mu_t$ is differentiable at $t = 0$ and*

$$\frac{d}{dt} \int f d\mu_t|_{t=0} = - \int df(\nabla \varphi) d\mu_0.$$

iii) *The identity*

$$|d\varphi|(\gamma_0) = |D^+ \varphi|(\gamma_0) = d(\gamma_0, \gamma_1)$$

holds for π -a.e. γ .

iv) If the space is $\text{RCD}^*(K, N)$ for some $N < \infty$, then (i), (ii), (iii) holds with μ_1 only assumed to be with bounded support, with the caveat that (A.24) holds in the form: for every $\delta \in (0, 1/2)$ there is $C_\delta > 0$ so that $\mu_t \leq C'_\delta \mathbf{m}$ for every $t \in [0, 1 - \delta]$.

A property related to the above is the fact that although the Kantorovich potentials are not uniquely determined by the initial and final measures, their gradients are. This is expressed by the following result, which also says that if we sit in the intermediate point of a geodesic and move to one extreme or the other, then the two corresponding velocities are one the opposite of the other (see Lemma 5.8 and Lemma 5.9 in [24] for the proof):

Lemma A.7. *Let (X, d, \mathbf{m}) be an $\text{RCD}(K, \infty)$ space with $K \in \mathbb{R}$ and $(\mu_t) \subset \mathcal{P}_2(X)$ a W_2 -geodesic such that $\mu_t \leq C\mathbf{m}$ for every $t \in [0, 1]$ for some $C > 0$. For $t \in [0, 1]$ let $\phi_t, \phi'_t : X \rightarrow \mathbb{R}$ be locally Lipschitz functions such that for some $s, s' \neq t$ the functions $-(s-t)\phi_t$ and $-(s'-t)\phi'_t$ are Kantorovich potentials from μ_t to μ_s and from μ_t to $\mu_{s'}$ respectively.*

Then

$$\nabla \phi_t = \nabla \phi_{t'} \quad \mu_t\text{-a.e.}$$

On RCD spaces, W_2 -geodesics made of measures with bounded density also have the weak continuity property of the densities expressed by the following lemma. The proof follows by a simple argument involving Young's measures and the continuity of the entropy along a geodesic (see Corollary 5.7 in [24]):

Lemma A.8. *Let (X, d, \mathbf{m}) be an $\text{RCD}(K, \infty)$ space with $K \in \mathbb{R}$ and $(\mu_t) \subset \mathcal{P}_2(X)$ a W_2 -geodesic such that $\mu_t \leq C\mathbf{m}$ for every $t \in [0, 1]$ for some $C > 0$. Let ρ_t be the density of μ_t .*

Then for any $t \in [0, 1]$ and any sequence $(t_n)_{n \in \mathbb{N}} \subset [0, 1]$ converging to t there exists a subsequence $(t_{n_k})_{k \in \mathbb{N}}$ such that

$$\rho_{t_{n_k}} \rightarrow \rho_t, \quad \mathbf{m}\text{-a.e.}$$

as $k \rightarrow \infty$.

We conclude recalling some properties of the **Hopf-Lax semigroup** in metric spaces, also in connection with optimal transport. For $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ and $t > 0$ we define the function $Q_t f : X \rightarrow \mathbb{R} \cup \{-\infty\}$ as

$$Q_t f(x) := \inf_{y \in X} \frac{d^2(x, y)}{2t} + f(y) \tag{A.25}$$

and set $t_* := \sup\{t > 0 : Q_t f(x) > -\infty \text{ for some } x \in X\}$; it is worth saying that t_* does not actually depend on x , since if $Q_t f(x) > -\infty$, then $Q_s f(y) > -\infty$ for all $s \in (0, t)$ and all $y \in X$. With this premise we have the following result (see [5]):

Proposition A.9. *Let (X, d) be a length space and $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$. Then for all $x \in X$ the map $(0, t_*) \ni t \mapsto Q_t f(x)$ is locally Lipschitz and*

$$\frac{d}{dt} Q_t f(x) + \frac{1}{2} \left(\text{lip} Q_t f(x) \right)^2 = 0 \quad \text{a.e. } t \in (0, t_*) \tag{A.26}$$

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