# DAMAGE MODEL FOR PLASTIC MATERIALS AT FINITE STRAINS

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Abstract. We consider a model for nonlinear elastoplasticity coupled with incomplete damage. The internal energy of the deformed elastoplastic body depends on the deformation $y$, on the plastic strain $P$, and on an internal variable $z$ describing the damage level of the medium. We consider a dissipation distance $D$ between internal states accounting for coupled plastic deformation and damage. Moving from time-discretization we prove the existence of a rate-independent quasistatic evolution of the system.

## 1. Introduction

Failure in ductile materials, such as metals or polymers, proceeds from the initiation of micro-defects, followed by their diffuse growth accompanied by large irreversible deformations, up to the formation of localized macroscopic cracks. Altogether, these phenomena constitute *ductile fracture*, e.g., [16] or [21, Section 1.1.3], and are of primary concern in predictive modeling of forming processes in industrial practice.

Continuum-based models for ductile fracture must involve two dissipative mechanisms: *damage* and *plasticity*; see, e.g., [5] for an overview. Damage accounts for the stiffness reduction due to initiation, growth, and coalescence of defects, e.g., [21, Chapter 7], whereas plasticity quantifies the development of permanent strains within the material, e.g., [21, Chapter 7]. Moreover, the two mechanisms interact, resulting in the need for *coupled* damage-plasticity models, e.g., [21, Section 7.4.1].

In what follows, we adopt the format of *generalized standard materials* [19] and assume that the material behavior is governed by a stored energy density and a dissipation potential. Under small strains, the first local energy-based model for coupled damage-plasticity was introduced by Ju [20]. Later on, Alessi et al. [2, 3] developed its non-local extension by including gradients of a damage variable into the stored energy, in the spirit of regularized variational models for brittle fracture by Bourdin et al. [7]. Such enrichment introduces an additional length scale into the energy functional to characterize the regions to which damage localizes, rendering the model objective with respect to spatial discretization; see also [1] for an overview and comparison of available formulations. Very recently, this class of models has been extended to a finite-strain regime independently by Ambati et al. [4], Borden et al. [6], and Miehe et al. [23]. The last formulation involves additional regularization with gradients of plastic strains to control the localization of permanent strains, too. We invite an interested Reader to [4, 6, 23] for illustration of predictive power of these models, including their experimental validation.

Apart from providing a convenient approach to constitutive modeling, the framework of generalized standard materials naturally leads to the notion of *energetic solutions* – a solution concept for rate-independent problems developed by Mielke and co-workers [26, 29] that characterizes the evolution of state variables by conditions of global stability and energy conservation. The existence of an energetic solution for small-strain damage-plastic models has recently been shown by Crismale [10], who further extended his result to gradient plasticity coupled with damage [11]. However, existence results for finite-strain models are currently lacking, although finite-strain damage [28] and gradient plasticity [22, 24, 25, 27, 30] were successfully addressed within the energetic solution concept.

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In the current work, we prove the existence of an energetic solution to models of incomplete damage coupled with gradient plasticity at finite strains, under structural assumptions that comply with contemporary engineering models [1].

More specifically, we consider an elastoplastic body $\Omega \subset \mathbb{R}^d$ subjected to a deformation $y : \Omega \to \mathbb{R}^d$. In nonlinear plasticity it is commonly assumed that the deformation gradient $\nabla y$ complies for the multiplicative decomposition $\nabla y = F_e P$ where $F_e : \Omega \to \mathbb{R}^{d\times d}$ and $P : \Omega \to SL(d)$ stand for the elastic and plastic strains. Moreover we introduce an internal scalar variable $z : \Omega \to [0, 1]$ describing the damage of the medium, where the value $z(x) = 1$ corresponds to an undamaged status of $\Omega$ at $x$, while values close to zero mean that the body is highly damaged. The *internal stored energy* of a material state $(y, P, z)$ is given by

$$\mathcal{W}(y, P, z) = \int_\Omega W_{\mathrm{el}}(x, F_e, z) + W_{\mathrm{h}}(x, P, z) + \frac{\nu}{r_1}|\nabla P|^{r_1} + \frac{\mu}{r_2}|\nabla z|^{r_2}\, dx,$$

see Section 2.5. In this expression, the first term is the elastic energy, the second represents the energy related to hardening effects, the third and fourth are regularization terms which from a physical point of view can be viewed as surface energies penalizing spatial variations of the internal variables $P$ and $z$. More precisely, we can expect $P$ and $z$, to change values on length scales of order $\mu^{1/(r_1-1)}$ and $\nu^{1/(r_2-1)}$, respectively. This would schematically correspond to the observation of the emergence of lower dimensional substructures in plasticity and damage, namely plastic shear bands and cracks. The correlation between the variables $P$ and $z$ partly relies on the behavior of the internal energy, which will be monotone in $z$ at fixed $P$ (see Section 2.6). The evolution is driven by a time-dependent external loading $\ell$ which completes the *total energy* of the system given by

$$\mathcal{E}(t, y, P, z) = \mathcal{W}(y, P, z) - \langle \ell(t), y\rangle,$$

where the dual product $\langle \ell(t), y\rangle$ is defined in (2.24). We consider rate-independent evolution of the energy $\mathcal{E}$ coupled with a dissipation distance between internal states given by

$$\mathcal{D}(P, z, \hat{P}, \hat{z}) = \int_\Omega D(P(x), z(x), \hat{P}(x), \hat{z}(x))dx.$$

The latter depends on the joint behavior of plastic strain and damage. This coupling is implemented in the non-symmetric distance

$$D(P, z, \hat{P}, \hat{z}) = \begin{cases} \kappa|z - \hat{z}| + \rho(\hat{z})D_{\mathrm{p}}(P, \hat{P}), & \text{if } z \geq \hat{z}, \\ \infty, & \text{else,} \end{cases}$$

which we comment in Section 2.3. Here, $\rho$ is a positive, monotone increasing function and $D_{\mathrm{p}}$ is the classical plastic dissipation distance introduced by Mielke [24, 25]. The function $\rho$ models the fact that the material plasticizes more easily once it is damaged. We rely on the concept of *energetic solutions*, and consider a quasistatic evolution, namely a trajectory $[0, T] \ni t \mapsto (y(t), P(t), z(t))$ on a time interval $[0, T]$ satisfying at every time $t$ a stability condition and an energy balance, see Definition 3.1. Our main result Theorem 3.2 in Section 3 asserts the existence of an energetic solution for any compatible (stable) initial datum (see Section 3). To prove this result we apply a standard time-discretization scheme introduced by Mielke and co-workers [26, 29].

The paper is organized as follows. Section 2 introduces our model, emphasizing the treatment of the dissipation potential that accounts for damage and plastic processes. The existence proof, based on incremental energy minimization, is presented in Section 3. We note in passing that our analysis rests on the conditions of global stability; alternative solution concepts like viscous approximation, employed in a similar context by Crismale and Lazzaroni [12], or semistability, used by Roubíček and Valdman [32, 33], are excluded from consideration.

## 2. The model

2.1. **Preliminaries.** We first describe the setting of our model and then introduce some basic concepts of linear algebra and geodesic calculus which help to understand the model.

*Reference configuration.* In the sequel we will work on a bounded connected open set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary. This represents the reference configuration of an elastoplastic body. We assume that the boundary of $\Omega$ is the union of a Dirichlet and Neumann part, namely $\partial\Omega := \Gamma_D \cup \Gamma_N$, and suppose $\Gamma_D$ has strictly positive $(d-1)$-Hausdorff measure. Once we have fixed a Dirichlet boundary condition for the deformation $y : \Omega \to \mathbb{R}^3$, we will often make use of the Poincaré inequality

$$\|y\|_{W^{1,p}} \leq C\|\nabla y\|_{L^p},$$

which holds true for this domain since $\mathcal{H}^{d-1}(\Gamma_D) > 0$. Throughout the paper we will use the letter $C$ to denote a generic positive constant that may change from line to line.

*Matrices and groups.* We denote by $\mathbb{R}^{d\times d}$ the vector space of $d \times d$ matrices with real entries. The standard Euclidean inner product is denoted by double dots, namely $A : B = A_{ij}B_{ij}$ (summation convention). The symbols $\mathbb{R}^{d\times d}_{\text{sym}}$ and $\mathbb{R}^{d\times d}_{\text{anti}}$ denote the subspaces of $\mathbb{R}^{d\times d}$ consisting of symmetric and anti-symmetric matrices, respectively. The symbol $\mathbb{R}^{d\times d}_{\text{dev}}$ stands for deviatoric matrices, where deviatoric means tracefree. We will also employ the following notation for common matrix groups

$$\begin{aligned}
GL(d) &:= \{A \in \mathbb{R}^{d\times d} : \det A \neq 0\},\\
GL^+(d) &:= \{A \in \mathbb{R}^{d\times d} : \det A > 0\},\\
SL(d) &:= \{A \in \mathbb{R}^{d\times d} : \det A = 1\},\\
SO(d) &:= \{A \in SL(d) : A^T A = AA^T = I\}.
\end{aligned}$$

*Norms.* We will consistently use the notation $|\cdot|$ for norms of tensors and scalars, e.g. $|A| = (A : A)^{1/2}$. This notation will be used in general for $k$-tensors of every order. On the other hand we use the double-bar notation $\|\cdot\|$ for norms on function spaces, e.g. $\|f\|_{L^1} = \int_\Omega |f(x)|dx$.

*Polar decomposition.* For all $A \in GL(d)$ there exists a unique decomposition

$$A = RT, \tag{2.1}$$

with $R \in SO(d)$ and $T \in \mathbb{R}^{d\times d}_{\text{sym}}$ positive definite. If moreover $A \in SL(d)$ then it is easy to see that both $T$ and $R$ must have determinant equal to 1. Furthermore, as $T$ is symmetric, there exists an orthogonal matrix $Q$ and a diagonal matrix $\Lambda$ such that

$$T = Q\Lambda Q^T. \tag{2.2}$$

The diagonal matrix $\Lambda$ has the positive eigenvalues $\lambda_i$ of $T$ on the diagonal. The matrix $\xi = \text{diag}(\log\lambda_1, \dots, \log\lambda_d)$ then satisfies

$$\Lambda = e^\xi \qquad \text{and} \qquad T = Qe^\xi Q^T = e^{Q\xi Q^T}, \tag{2.3}$$

where the last equality follows from the fact that $Q$ is invertible and $Q^{-1}e^A Q = e^{Q^{-1}AQ}$ for all $A \in \mathbb{R}^{d\times d}$.

*Geodesic exponential map vs. matrix exponential.* Let $G$ be a (matrix) Lie group, e.g. $SO(d)$ or $SL(d)$. The (geodesic) exponential map is defined by

$$\begin{aligned}
\text{Exp} : T_e G &\to G\\
v &\mapsto \gamma_v(1)
\end{aligned}$$

where $\gamma_v$ is the unique geodesic starting from the identity $e \in G$ with initial velocity $v$ lying in the tangent space to $G$ at the identity. It is easy to show that the tangent space of $SL(d)$ (resp. $SO(d)$) at the identity is $\mathbb{R}^{d\times d}_{\text{dev}}$ (resp. $\mathbb{R}^{d\times d}_{\text{anti}}$), see [8, Example I.9.4., Exercise I.17(b)]. It is important to remark that in general the geodesic exponential map defined above differs from the algebraic exponential of a matrix $\xi$ we used before and denoted by $e^\xi$. In fact it was shown in [24, Theorem 6.1] that for the left-invariant metric induced by the standard Euclidean scalar product $A : B$ the geodesics on $SL(d)$ starting from $P(0)$ in direction of $\xi \in \mathbb{R}^{d\times d}_{\text{dev}}$ are given by

$$P(t) = P(0)e^{t\xi^T}e^{t(\xi-\xi^T)}.$$

Notice that for tracefree matrices in general $\xi^T\xi \neq \xi\xi^T$, that is why $\text{Exp}(\xi) \neq e^\xi$. For antisymmetric matrices however, the product commutes. This implies that on $SO(d)$ the geodesics are exactly given by $P(t) = P(0)e^{t\xi}$ for $\xi \in \mathbb{R}^{d\times d}_{\text{anti}}$.

*Rotations.* In two dimensions $SO(2)$ consists of all rotations of the form

$$R_\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

with $\alpha \in [-\pi, \pi)$. In such a case the rotation $R_\alpha$ can be expressed as an exponential matrix

$$R_\alpha = e^{\alpha L} \qquad \text{with} \quad L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.4}$$

For general dimension $d$, since $SO(d)$ is a compact connected Lie group, the exponential map

$$\begin{aligned} \mathrm{Exp} : \mathbb{R}^{d\times d}_{\text{anti}} &\to SO(d) \\ \xi &\mapsto e^{\xi} \end{aligned}$$

is surjective [18, Corollary 11.10.]. Therefore, for every $R \in SO(d)$ there exists $\xi \in \mathbb{R}^{d\times d}_{\text{anti}}$ such that $R = e^{\xi}$. We can use the spectral theory for real skew-symmetric matrices to bring $\xi$ to a block diagonal form. Namely, there exists an orthogonal matrix $Q$ such that $\xi = Q\Sigma Q^T$ with

$$\Sigma = \begin{pmatrix} \alpha_1 L_1 & & & \\ & \alpha_2 L_2 & & \\ & & \ddots & \\ & & & \alpha_p L_p \end{pmatrix}$$

where either $L_j = 0 \in \mathbb{R}$ or $L_j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{R}^{2\times 2}_{\text{anti}}$ and $\alpha_j \in \mathbb{R}$. Since $\Sigma$ is block diagonal its exponential is easily computed as

$$e^{\Sigma} = \begin{pmatrix} e^{\alpha_1 L_1} & & & \\ & e^{\alpha_2 L_2} & & \\ & & \ddots & \\ & & & e^{\alpha_p L_p} \end{pmatrix}, \quad e^{\alpha_j L_j} = \begin{pmatrix} \cos\alpha_j & \sin\alpha_j \\ -\sin\alpha_j & \cos\alpha_j \end{pmatrix} \text{ or } e^0 = 1.$$

Using the periodicity of sin and cos we have that the rotation $R$ can be written as

$$R = e^{Q\Sigma Q^T} = e^{Q\tilde{\Sigma} Q^T} \tag{2.5}$$

where $\tilde{\Sigma}$ is defined as $\Sigma$ but with $\tilde{\alpha}_j = \alpha_j \mod 2\pi$.

2.2. **Plastic dissipation distance.** The *(plastic) dissipation potential* is a mapping

$$R : \Omega \times SL(d) \times \mathbb{R}^{d\times d} \to [0, +\infty],$$

which is measurable in $x \in \Omega$ and convex and positively 1-homogeneous in the rate, i.e.,

$$R(x, P, \lambda \dot{P}) = \lambda R(x, P, \dot{P}) \qquad \text{for all } \lambda \geq 0.$$

In the following, not to overburden notation, we will drop the explicit dependence on $x \in \Omega$. We further assume plastic indifference which corresponds to requiring that

$$R(PQ, \dot{P}Q) = R(P, \dot{P}) \qquad \text{for all } Q \in SL(d). \tag{2.6}$$

This property implies that there exists a 1-homogeneous function $\hat{R} : \Omega \times \mathbb{R}^{d\times d} \to [0, +\infty]$ such that

$$R(P, \dot{P}) = \hat{R}(\dot{P}P^{-1}),$$

see [25] or [29, Section 4.2.1.1]. We assume there exist constants $c_0, c_1 > 0$ independent of $x \in \Omega$ such that

$$c_0|Q| \leq \hat{R}(Q) \leq c_1|Q| \qquad \text{for every } Q \in SL(d). \tag{2.7}$$

With the potential at disposal, we define the induced *plastic dissipation distance* on $SL(d)$ for any pair $P_1, P_2 \in SL(d)$ by

$$D_{\mathrm{p}}(P_1, P_2) = \inf \left\{ \int_0^1 R(P(s), \dot{P}(s))ds : P \in W^{1,\infty}([0,1]; SL(d)),\ P(0) = P_1,\ P(1) = P_2 \right\}. \tag{2.8}$$

Notice that due to plastic indifference we have that $D_{\rm p}(P_1,P_2)=\hat D_{\rm p}(P_2P_1^{-1})$ with

$$\hat D_{\rm p}(P)=\inf\left\{\int_0^1 \hat R(\dot P(s)P(s)^{-1})ds : P\in W^{1,\infty}([0,1];SL(d)),\ P(0)=I,\ P(1)=P\right\}.$$

Due to (2.7), the dissipation distance $\hat D_{\rm p}$ is equivalent to the standard Riemannian distance induced by the Euclidean scalar product on the Lie algebra $\mathbb{R}^{d\times d}_{\rm dev}$. In particular, we have that

$$c_0\hat d_{\rm SL}(P)\le \hat D_{\rm p}(P)\le c_1\hat d_{\rm SL}(P) \tag{2.9}$$

where

$$\hat d_{\rm SL}(P)=\inf\left\{\int_0^1 |\dot P(s)P^{-1}(s)|ds : P\in W^{1,\infty}([0,1];SL(d)),\ P(0)=I,\ P(1)=P\right\}.$$

As it was pointed out in [24], the geodesics with respect to $\hat R$ in direction $\xi$ are in general not known and even in the specific Riemannian case geodesics of $\hat d_{\rm SL}$ connecting the identity to $\xi$ are not given by $t\mapsto e^{t\xi}$. In particular it might happen that $\hat d_{\rm SL}(e^\xi)<|\xi|$. However from standard theory of Riemannian manifolds it is known that

$$d_{\rm SL}(P_0,P_1):=\hat d_{\rm SL}(P_1P_0^{-1})$$

is a metric on $SL(d)$, see pp. 19-20 of [8]. We conclude this discussion with the following result:

**Lemma 2.1** ($D_{\rm p}$ is a quasi-distance)**.** *The following properties hold true. For every $P_1,P_2,P_3,Q\in SL(d)$:*

(i) $D_{\rm p}(P_1,P_2)=0$ *if and only if* $P_1=P_2$,
(ii) $D_{\rm p}(P_1,P_3)\le D_{\rm p}(P_1,P_2)+D_{\rm p}(P_2,P_3)$
(iii) $D_{\rm p}(P_1Q,P_2Q)=D_{\rm p}(P_1,P_2)$.

*Proof.* The implication (i) follows from the previous remark that $d_{\rm SL}$ is a metric on $SL(d)$ which by (2.9) is equivalent to $D_{\rm p}$. Condition (ii) is easily checked, while (iii) follows from (2.6). □

Notice that $D_{\rm p}$ might be *not* symmetric. We now show that the quasi-distance $\hat D_{\rm p}$ has sublinear growth. To prove this upper bound the most important observation is that if $P\in SL(d)$ is such that $P=e^\xi$ for some $\xi$ then we may test the definition of $\hat D_{\rm p}(P)$ with the path $s\mapsto e^{s\xi}, s\in[0,1]$ and get

$$\hat D_{\rm p}(P)\le\int_0^1\hat R(e^{s\xi}\xi e^{-s\xi})ds=\int_0^1\hat R(\xi)ds=\hat R(\xi). \tag{2.10}$$

**Proposition 2.2.** *Assume that (2.6) and (2.7) hold. Then in any dimension $d\in\mathbb{N}, d\ge1$, there is a positive constant $C=C(d)>0$, independent of $x\in\Omega$, such that, for every $P_1,P_2\in SL(d)$*

$$D_{\rm p}(x,P_1,P_2)\le C(1+|P_1|+|P_2|). \tag{2.11}$$

*Proof.* For the Reader's convenience the proof is split into several steps. In steps 1-3 we show that for every $P\in SL(d)$

$$\hat D_{\rm p}(P)\le C(1+|P|) \tag{2.12}$$

for some constant $C=C(c_1,d)>0$ ($c_1$ being the constant in (2.7)). In step 4 we deduce the general statement of the proposition.

*Step 1.* Let $P\in SL(d)$ be arbitrary. We use the decompositions (2.1) and (2.2), namely

$$P=RT=RQ\Lambda Q^T, \tag{2.13}$$

where $R\in SO(d)$, $Q$ is orthogonal and $\Lambda$ is diagonal and can be written as the exponential of $\xi=\log\Lambda$ as in (2.3). We estimate the dissipation relative to the positive definite symmetric matrix $T=Q\Lambda Q^T$ using (2.10), and writing $T=Qe^\xi Q^T=e^{Q\xi Q^T}$,

$$\hat D_{\rm p}(T)\le\hat R(Q\xi Q^T)\le c_1|Q\xi Q^T|=c_1|\xi|, \tag{2.14}$$

where the second inequality follows from (2.7).

*Step 2.* We now claim that $|\xi|$ can be estimated in terms of $|T|$. First assume $d=2$ and remember that $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2)$ and $\xi = \operatorname{diag}(\log\lambda_1, \log\lambda_2)$ with $\lambda_1$, $\lambda_2$ being the eigenvalues of $T$, and $\lambda_1 = 1/\lambda_2$. Assume without loss of generality $\lambda_1 > 1$. We have

$$|\xi|^2 = |\log\lambda_1|^2 + |\log\lambda_2|^2 = |\log\lambda_1|^2 + |\log\lambda_1|^2 \leq 2|\lambda_1|^2 \leq 2|T|^2.$$

Let us now focus on the case $d = 3$. Assume first that $\xi = \operatorname{diag}(\log\lambda_1, \log\lambda_2, \log\lambda_3)$ with $\lambda_1$, $\lambda_2 > 1$ and $\lambda_3 = (\lambda_1\lambda_2)^{-1} < 1$. We have

$$|\xi|^2 = |\log\lambda_1|^2 + |\log\lambda_2|^2 + |\log(\lambda_1\lambda_2)|^2 \leq 3|\log\lambda_1|^2 + 3|\log\lambda_2|^2 \leq 3|T|^2.$$

In the case $\lambda_1, \lambda_2 < 1$ and $\lambda_3 > 1$ we write $\eta_i := \lambda_i^{-1}$ and

$$\begin{aligned}|\xi|^2 &= |\log\eta_1|^2 + |\log\eta_2|^2 + |\log(\eta_1\eta_2)|^2 \leq 3|\log\eta_1|^2 + 3|\log\eta_2|^2 \\ &\leq 3|\log(\eta_1\eta_2)|^2 = 3|\log(\lambda_3)|^2 \leq 3|T|^2.\end{aligned}$$

Therefore estimate (2.14) leads, in both cases $d = 2, 3$, to

$$\hat{D}_{\mathrm{p}}(T) \leq c_1\sqrt{d}\,|T|. \tag{2.15}$$

In the general case $d > 3$ we need to argue in a different way and will obtain an estimate that is slightly weaker than (2.15). We have $\prod_{i=1}^d \lambda_i = 1$, and thus $\sum_{i=1}^d \log(\lambda_i) = 0$. Assume $\lambda_1, \dots, \lambda_m > 1$ for some $m$ and $\lambda_{m+1}, \dots, \lambda_d \leq 1$. Let $\ell = \sum_{i=1}^m \log\lambda_i$, so that $\sum_{i=m+1}^d \log\lambda_i = -\ell$. Since $\log\lambda_i > 0$ for all $i = 1, \dots, m$ and $\log\lambda_i \leq 0$ for all $i = m+1, \dots, d$ we can write

$$\sum_{i=1}^m (\log\lambda_i)^2 \leq \ell^2, \qquad \sum_{i=m+1}^d (\log\lambda_i)^2 \leq \ell^2.$$

Then, Jensen's inequality implies that

$$\begin{aligned}|\xi|^2 = \sum_{i=1}^d (\log\lambda_i)^2 \leq 2\ell^2 = 2\left(\sum_{i=1}^m \log\lambda_i\right)^2 &\leq 2m\sum_{i=1}^m (\log\lambda_i)^2 \\ &\leq 2m\sum_{i=1}^m (\lambda_i)^2 \leq 2(d-1)\sum_{i=1}^d \lambda_i^2 = 2(d-1)|T|^2.\end{aligned}$$

In particular, estimate (2.14) leads to

$$\hat{D}_{\mathrm{p}}(T) \leq C_1|T|, \tag{2.16}$$

where $C_1 := c_1\sqrt{2(d-1)}$.

*Step 3.* Let us now give an estimate for the rotation $R$ in the decomposition (2.13). In case $d = 2$, thanks to (2.4), we have

$$\hat{D}_{\mathrm{p}}(R) \leq \hat{R}(\alpha L) \leq c_1\alpha|L| \leq \pi\sqrt{2}c_1.$$

For general dimensions, we use (2.5) to estimate

$$\hat{D}_{\mathrm{p}}(R) \leq \hat{R}(Q\tilde{\Sigma}Q^T) \leq c_1|Q\tilde{\Sigma}Q^T| = c_1|\tilde{\Sigma}| \leq C_2, \tag{2.17}$$

where $C_2 := c_1\sqrt{d}\pi$.

*Step 4.* We first observe that by Lemma 2.1(ii),(iii) we have that

$$\hat{D}_{\mathrm{p}}(PQ) \leq \hat{D}_{\mathrm{p}}(Q) + \hat{D}_{\mathrm{p}}(P), \quad \text{for all } P, Q \in SL(d). \tag{2.18}$$

Now let $P_1, P_2 \in SL(d)$. We use the polar decomposition

$$P_i = R_iT_i = R_iQ_i\Lambda_iQ_i^T, \quad i = 1, 2.$$

Using (2.16), (2.17) and (2.18) we obtain

$$\begin{aligned}D_{\mathrm{p}}(P_1, P_2) = \hat{D}_{\mathrm{p}}(P_2P_1^{-1}) &= \hat{D}_{\mathrm{p}}(R_2T_2T_1^{-1}R_1^{-1}) \\ &\leq \hat{D}_{\mathrm{p}}(R_1^{-1}) + \hat{D}_{\mathrm{p}}(T_1^{-1}) + \hat{D}_{\mathrm{p}}(T_2) + \hat{D}_{\mathrm{p}}(R_2) \\ &\leq 2C_2 + C_1|T_2| + \hat{D}_{\mathrm{p}}(T_1^{-1}).\end{aligned}$$

Now $T_1^{-1} = Q_1\Lambda_1^{-1}Q_1^T$ and

$$\hat{D}_{\mathrm{p}}(T_1^{-1}) = \hat{R}(Q_1\log\Lambda_1^{-1}Q_1^T) \leq c_1|\log\Lambda_1^{-1}| = c_1|\log\Lambda_1|.$$

As in step 2 we deduce that $c_1|\log\Lambda_1| \le C_1|T_1|$. Altogether we have shown that

$$D_{\mathrm{p}}(P_1,P_2) \le 2C_2 + C_1(|T_1|+|T_2|) \le 2C_2 + C_1C_3(|P_1|+|P_2|) \tag{2.19}$$

where $C_1 = c_1\sqrt{2(d-1)}$, $C_2 = c_1\sqrt{d}\pi$ and $C_3 = \sup\{|R| : R \in SO(d)\}$. □

We will also need the following statement on continuity.

**Lemma 2.3.** *The dissipation distance $D_{\mathrm{p}} : \Omega \times SL(d) \times SL(d) \to [0,\infty)$ is a Carathéodory function (i.e. measurable in the first variable and continuous in the other variables for a.e. fixed $x \in \Omega$).*

*Proof.* The measurability of $D_{\mathrm{p}}(\cdot,P_0,P_1)$ follows from measurability of $\hat{R}$. To show continuity let $P^*,P \in SL(d)$ be fixed and let $P_k \to P$ in $SL(d)$. Then (dropping the $x$-variable dependence) we use the triangle inequality to estimate

$$|D_{\mathrm{p}}(P^*,P_k) - D_{\mathrm{p}}(P^*,P)| \le D_{\mathrm{p}}(P,P_k) = \hat{D}_{\mathrm{p}}(P_kP^{-1})$$

Therefore it suffices to show that $\hat{D}_{\mathrm{p}}(\hat{P}_k) \to 0$ for any sequence $\hat{P}_k \to I$. Since $\hat{P}_k \in SL(d)$ we can use the decompositions (2.1) and (2.3) as well as the spectral theory (2.5) to write

$$\hat{P}_k = e^{Q_k\Sigma_kQ_k^T}e^{\tilde{Q}_k\xi_k\tilde{Q}_k^T}$$

where $Q_k,\tilde{Q}_k$ are orthogonal and $\Sigma_k,\xi_k \to 0$ since $\hat{P}_k \to I$. We again use the triangle inequality and (2.10) to estimate

$$\hat{D}_{\mathrm{p}}(\hat{P}_k) \le \hat{R}(Q_k\Sigma_kQ_k^T) + \hat{R}(\tilde{Q}_k\xi_k\tilde{Q}_k^T) \le c_1\left(|\Sigma_k|+|\xi_k|\right) \to 0 \quad \text{as } k\to\infty.$$

This proves the claimed continuity. □

2.3. **Dissipation with damage.** Let $\rho : \Omega\times\mathbb{R} \to \mathbb{R}^+$ be a Carathéodory function (measurable in $x \in \Omega$ for every $t \in \mathbb{R}$, continuous in $t$ for a.e. $x \in \Omega$). We will make the following assumption

(H) $\rho$ is non-decreasing in the second variable and constant on the intervals $(-\infty,0]$ and $[1,+\infty)$.

Let $\kappa \in L^\infty(\Omega;\mathbb{R}^+)$ be such that $\kappa(x) \ge \kappa_0 > 0$ for a.e. $x \in \Omega$. Given $x \in \Omega$, $z_1,z_2 \in [0,1]$ and $P_1,P_2 \in SL(d)$ we define the (coupled) dissipation between $(P_1,z_1)$ and $(P_2,z_2)$ at $x$ as

$$\begin{aligned} D(x,P_1,z_1,P_2,z_2) = \inf\Bigg\{ \int_0^1 & S(x,\dot{z}(s)) + \rho(x,z(s))R(x,P(s),\dot{P}(s))ds : \\ & (P,z) \in W^{1,\infty}([0,1];SL(d)\times[0,1]), \\ & P(0)=P_1,\ P(1)=P_2,\ z(0)=z_1,\ z(1)=z_2\Bigg\}, \end{aligned}$$

where

$$S(x,\dot{z}) := \begin{cases} \kappa(x)|\dot{z}| & \text{if } \dot{z} \le 0, \\ \infty & \text{else.} \end{cases}$$

Thanks to the monotonicity of $\rho(x,\cdot)$ we can prove the following.

**Proposition 2.4.** *Let $x \in \Omega$, $z_1,z_2 \in [0,1]$ and $P_1,P_2 \in SL(d)$. Then*

$$D(x,P_1,z_1,P_2,z_2) = S(x,z_2-z_1) + \rho(x,z_2)D_{\mathrm{p}}(x,P_1,P_2). \tag{2.20}$$

*Proof.* We consider the following two cases.

*Case* $z_1 < z_2$: In this case the right-hand side of (2.20) is infinite. So we need to show that $D$ is infinite too. This follows as, for every path $z \in W^{1,\infty}([0,1];[0,1])$ connecting $z_1$ to $z_2$, the measure $\mathcal{L}^1(\{\dot{z} > 0\})$ is strictly positive. By definition of $S$ the path has infinite dissipation length.

*Case* $z_1 \geq z_2$: Since $\rho$ is non-decreasing, by definition of $S$ every path of finite dissipation satisfies $\dot{z} \leq 0$ a.e. on $[0,1]$. We know that

$$\begin{aligned} D(x, P_1, z_1, P_2, z_2) &= \inf \left\{ \int_0^1 S(x, \dot{z}(s))ds + \int_0^1 \rho(x, z(s))R(x, P(s), \dot{P}(s))ds : \dot{z} \leq 0 \right\} \\ &\geq S(x, z_2 - z_1) + \inf \left\{ \int_0^1 \rho(x, z(s))R(x, P(s), \dot{P}(s))ds : z_2 \leq z \leq z_1 \right\} \\ &\geq S(x, z_2 - z_1) + \rho(x, z_2)D_{\mathrm{p}}(x, P_1, P_2). \end{aligned}$$

To show the opposite inequality let $P_k \in W^{1,\infty}([0,1]; SL(d))$ be a sequence with $P_k(0) = P_1,\ P_k(1) = P_2$ such that, for any $k$,

$$\int_0^1 R(x, P_k(s), \dot{P}_k(s))ds \leq D_{\mathrm{p}}(x, P_1, P_2) + \frac{1}{k}. \tag{2.21}$$

Let $z_k \in W^{1,\infty}([0,1]; [z_2, z_1])$ be the function

$$z_k(s) = \begin{cases} k(z_2 - z_1)(s - \frac{1}{k}) + z_2, & \text{if } 0 \leq s \leq \frac{1}{k}, \\ z_2, & \text{else.} \end{cases}$$

Moreover let $\zeta : [\frac{1}{k}, 1] \to [0,1]$ be the unique affine function such that $\zeta(\frac{1}{k}) = 0$, $\zeta(1) = 1$, and let

$$\tilde{P}_k(t) = \begin{cases} P_1 & \text{for } t \in [0, \frac{1}{k}], \\ P_k(\zeta(t)) & \text{for } t \in [\frac{1}{k}, 1]. \end{cases}$$

Notice that $\tilde{P}_k$ is Lipschitz continuous as well. Since $\tilde{P}_k$ is constant on $[0, 1/k]$ it follows that $R(x, \cdot, \dot{P}_k) = 0$ on $[0, 1/k]$, and by 1-homogeneity (i.e. $R(x, \cdot, \alpha\dot{P}) = \alpha R(x, \cdot, \dot{P})$ for $\alpha \geq 0$) we have

$$\begin{aligned} D(x, P_1, z_1, P_2, z_2) &\leq S(x, z_2 - z_1) + \int_{1/k}^1 \rho(x, z_k(s))R(x, \tilde{P}_k(s), \dot{\tilde{P}}_k(s))ds \\ &= S(x, z_2 - z_1) + \rho(x, z_2) \int_{1/k}^1 R(x, P_k(\zeta(s)), \dot{P}_k(\zeta(s)))\dot{\zeta}(s)ds \\ &= S(x, z_2 - z_1) + \rho(x, z_2) \int_0^1 R(x, P_k(t), \dot{P}_k(t))dt \\ &\overset{(2.21)}{\leq} S(x, z_2 - z_1) + \rho(x, z_2) \left( D_{\mathrm{p}}(x, P_1, P_2) + \frac{1}{k} \right) \end{aligned}$$

where we used the change of variables $t = \zeta(s)$. Now taking the limit $k \to \infty$ on the right-hand side we conclude. □

We can now define the dissipation between two internal states $(P_1, z_1), (P_2, z_2) : \Omega \to SL(d) \times [0,1]$, namely

$$\mathcal{D}(P_1, z_1, P_2, z_2) = \int_\Omega D(x, P_1(x), z_1(x), P_2(x), z_2(x))dx$$

and we are in position to introduce the *total dissipation* of a damage-plastic process. Let $(P, z) : [s,t] \to L^1(\Omega; SL(d)) \times L^1(\Omega; [0,1])$, we define

$$\mathrm{Diss}(P, z; s, t) := \sup \sum_{i=1}^N \mathcal{D}(P(r_{i-1}), z(r_{i-1}), P(r_i), z(r_i)), \tag{2.22}$$

where the supremum is computed over all partitions $s = r_0 < r_1 < \cdots < r_{N-1} < r_N = t$, and all $N \in \mathbb{N}$.

2.4. **State spaces.** The space of admissible states, denoted by $\mathcal{Q}$, is the triple $\mathcal{Y}\times\mathcal{P}\times\mathcal{Z}$ where

$$\begin{aligned}
&\mathcal{Y} := W^{1,q_Y}_D(\Omega;\mathbb{R}^d) := \{y\in W^{1,q_Y}(\Omega;\mathbb{R}^d) : y = \mathrm{id} \text{ on } \Gamma_D\},\\
&\mathcal{P} := W^{1,r_1}(\Omega; SL(d)),\\
&\mathcal{Z} := W^{1,r_2}(\Omega;[0,1]).
\end{aligned}$$

for some coefficients $q_Y > d$ and $r_1, r_2 > 1$. We will also consider the case $r_1, r_2 = 1$, which calls for a $BV$-setting. The space $\mathcal{Q}$ is endowed with the weak topologies of the Sobolev spaces, namely we write, for instance

$$P_k \rightharpoonup P \text{ in } \mathcal{P} \text{ if and only if } P_k \rightharpoonup P \text{ weakly in } W^{1,r_1}(\Omega;\mathbb{R}^{d\times d}).$$

Because of the fixed Dirichlet boundary condition, by Poincaré inequality, weak convergence in $\mathcal{Y} = W^{1,q_Y}_D(\Omega;\mathbb{R}^d)$ is equivalent to weak convergence of gradients, i.e.,

$$y_k \rightharpoonup y \text{ in } \mathcal{Y} \text{ if and only if } \nabla y_k \rightharpoonup \nabla y \text{ weakly in } L^{q_Y}(\Omega;\mathbb{R}^{d\times d}).$$

Notice that the space $\mathcal{P}$ is not a linear subspace of $W^{1,r_1}(\Omega;\mathbb{R}^{d\times d})$ because the target space is the manifold $SL(d)$. Nevertheless weak limits of sequences $(P_k)_{k\in\mathbb{N}}\subset\mathcal{P}$ are again in $\mathcal{P}$, and we might assume to take values in $SL(d)$ almost everywhere. This follows since weak convergence in $\mathcal{P}$ implies strong convergence of $P_k \to P$ in $L^1(\Omega)$.

Sometimes we use the short notation $q = (y, P, z)$ for elements in $\mathcal{Q}$. It is also convenient to occasionally use the variable $q$ in the dissipation distance $\mathcal{D}$ although it depends only on the internal variables and is therefore independent of $y$.

2.5. **Energy.** We consider the following energy

$$\begin{aligned}
\mathcal{E}(t,y,P,z) = &\int_\Omega W_{\mathrm{el}}(x,\nabla y P^{-1}, z) + W_{\mathrm{h}}(x,P,z)dx\\
&+\frac{\nu}{r_1}\int_\Omega |\nabla P|^{r_1}dx + \frac{\mu}{r_2}\int_\Omega |\nabla z|^{r_2}dx - \langle \ell(t), y\rangle,
\end{aligned}\tag{2.23}$$

for some material parameters $\nu,\mu > 0$. The symbol $\langle\cdot,\cdot\rangle$ denotes the duality product between $\mathcal{Y}^*$ and $\mathcal{Y}$, and the mapping $t\mapsto \ell(t)\in\mathcal{Y}^*$ represents external loading of the mechanical system. This load can be split as

$$\langle \ell(t), y\rangle = \int_\Omega f(t)\cdot y\,dx + \int_{\Gamma_N}\tau(t)\cdot y\,d\mathcal{H}^{d-1},\tag{2.24}$$

where $f(t)$ is a prescribed bulk force and $\tau(t)$ is a prescribed traction on the Neumann boundary $\Gamma_N$. Assumptions on the regularity of $t\mapsto \ell(t)$ will be specified in Section 2.6. The quantity

$$\mathcal{W}_{\mathrm{el}}(y,P,z) := \int_\Omega W_{\mathrm{el}}(x,\nabla y P^{-1}, z)dx,$$

is the elastic energy of the system and the term

$$\mathcal{W}_{\mathrm{h}}(y,P,z) := \int_\Omega W_{\mathrm{h}}(x,P,z)dx,$$

represents the energy related to hardening instead. The terms in (2.23) involving $\nabla P$ and $\nabla z$ are higher order energetic terms which have the role of regularizations introducing internal length scales. Notice that the elastic energy depends solely on the elastic strain $F_{\mathrm{e}} := \nabla y P^{-1}$ whereas the hardening depends solely on the plastic strain $P$.

2.6. **Assumptions on the energy.** It is convenient to denote the total bulk energy density (without regularization) by

$$W(x,y,P,z) = W_{\mathrm{el}}(x,\nabla y P^{-1}, z) + W_{\mathrm{h}}(x,P,z).$$

We complete our hypotheses on the energy with the following conditions:

- *Objectivity:*
  We want the energy density $W$ to be frame-indifferent. To guarantee this we suppose that

$$W_{\mathrm{el}}(x, QF_{\mathrm{e}}, z) = W_{\mathrm{el}}(x,F_{\mathrm{e}},z) \quad \forall Q\in SO(d), F_{\mathrm{e}}\in GL^+(d).\tag{2.25}$$

- *Non-interpenetrability:*
  We assume that $W_{\rm el}(x, F_{\rm e}, z) = +\infty$ if $\det F_{\rm e} \leq 0$, and

$$W_{\rm el}(x, F_{\rm e}, z) \to \infty \text{ as } \det F_{\rm e} \to 0^+. \tag{2.26}$$

- *Lower semicontinuity and coercivity:*
  We assume that $W_{\rm el}, W_{\rm h}$ are normal integrands, meaning that $W_{\rm el}(\cdot, F, z)$ is measurable for every $F, z$ and $W_{\rm el}(x, \cdot, \cdot)$ is lower-semicontinuous for a.e. $x \in \Omega$; the analogue for $W_{\rm h}$ holds. Moreover, we assume that $W_{\rm el}(x, \cdot, z)$ is *polyconvex*, i.e.

$$W_{\rm el}(x, F_{\rm e}, z) = \mathbb{W}_{\rm conv}(x, \mathbb{M}(F_{\rm e}), z), \tag{2.27}$$

  where $\mathbb{W}_{\rm conv}(x, \cdot, z)$ is convex for a.e. $x \in \Omega$ and every $z \in [0, 1]$, and $\mathbb{M}(F_{\rm e})$ denotes the vector of all minors of the elastic strain $F_{\rm e}$. For instance in dimension $d = 3$:

$$\mathbb{M}(F_{\rm e}) = (F_{\rm e}, \text{cof } F_{\rm e}, \det F_{\rm e}).$$

  Furthermore, we assume *coercivity* bounds

$$W_{\rm el}(x, F, z) \geq C_1 |F|^{q_F} - C_2, \tag{2.28a}$$

$$W_{\rm h}(x, P, z) \geq C_1 |P|^{q_P} - C_2, \tag{2.28b}$$

  for some constants $C_1, C_2 > 0$ and exponents satisfying

$$\frac{1}{q_F} + \frac{1}{q_P} \leq \frac{1}{q_Y} < \frac{1}{d}, \tag{2.29}$$

  see [29, Section 4.1.3].
- *Monotonicity and continuity of damage:*
  We further assume continuity and monotonicity in $z$, i.e. an increase of damage leads to a release of stored energy. More precisely,

$$z \leq \hat{z} \Rightarrow W(x, y, P, z) \leq W(x, y, P, \hat{z}). \tag{2.30}$$

- *Regularity of the loading:*
  We require $\ell$ to be absolutely continuous in time, i.e.

$$\ell \in W^{1,1}(0, T; \mathcal{Y}^*), \tag{2.31}$$

  see [15, Section 4] for a similar assumption on the loading.

**Remark 2.5** (Ciarlet-Nečas condition)**.** By assumption (2.26) it is clear that a finite energy solution satisfies the *local* non-interpenetration $\det \nabla y > 0$ a.e. in $\Omega$. It would be possible to guarantee *global* non-self-interpenetration involving the so called Ciarlet-Nečas condition [9], which reads

$$\int_\Omega \det (\nabla y) dx \leq \mathcal{L}^d(y(\Omega)).$$

In order to achieve this we would simply change the state space $\mathcal{Y}$ to

$$\mathcal{Y}_{CN} := \left\{ y \in \mathcal{Y} : \int_\Omega \det (\nabla y) dx \leq \mathcal{L}^d(y(\Omega)) \right\}$$

and remark that, due to the condition $q_Y > d$ in (2.29), convergence of $\nabla y_k \rightharpoonup \nabla y$ in $L^{q_Y}(\Omega)$ implies convergence of $\det (\nabla y_k) \rightharpoonup \det (\nabla y)$ in $L^1(\Omega)$. This shows that $\mathcal{Y}_{CN}$ is weakly closed in $\mathcal{Y}$.

## 3. Quasistatic evolution

We follow the concept of energetic solutions, which is solely based on the energy functional $\mathcal{E}$, the dissipation distance $\mathcal{D}$ and the state space $\mathcal{Q}$ introduced above. Given an external load $\ell : [0, T] \to \mathcal{Y}^*$ and suitable initial conditions $(y_0, P_0, z_0) \in \mathcal{Q}$ we look for an energetic solution $(y, P, z) : [0, T] \to \mathcal{Q}$. We first introduce the concept of stable states at a given time $t \in [0, T]$: this is defined via the subset $\mathcal{S}(t)$ of admissible states defined as

$$\mathcal{S}(t) = \left\{ (y, P, z) \in \mathcal{Q} : \mathcal{E}(t, y, P, z) \leq \mathcal{E}(t, \hat{y}, \hat{P}, \hat{z}) + \mathcal{D}(P, z, \hat{P}, \hat{z}) \;\; \forall (\hat{y}, \hat{P}, \hat{z}) \in \mathcal{Q} \right\}.$$

An energetic solution is asked to satisfy the following energy balance (E) and global stability condition (S).

**Definition 3.1.** We say that $(y, P, z) : [0, T] \to \mathcal{Q}$ is an energetic solution for the initial conditions $(y_0, P_0, z_0) \in \mathcal{Q}$ if $(y(0), P(0), z(0)) = (y_0, P_0, z_0)$, the map $s \mapsto \partial_t \mathcal{E}(s, y(s), P(s), z(s))$ belongs to $L^1(0, T)$, $\mathcal{E}(t, y(t), P(t), z(t)) < \infty$ for all $t \in [0, T]$, and the two following conditions are satisfied:

(S) $(y(t), P(t), z(t)) \in \mathcal{S}(t)$,

(E) $\mathcal{E}(t, y(t), P(t), z(t)) + \mathrm{Diss}(P, z; 0, t) = \mathcal{E}(0, y_0, P_0, z_0) + \int_0^t \partial_t \mathcal{E}(s, y(s), P(s), z(s)) ds$,

where $\mathrm{Diss}(P, z; 0, t)$ is defined in (2.22).

In order to show existence of energetic solutions we resort in applying the existence theory introduced and developed by Mielke and coauthors in a series of papers and books (see [26] or more recently [29] and references therein). Along the existence proof we are called to check that, under the assumptions stated in the previous section, the following conditions are satisfied:

(C1) The dissipation $\mathcal{D}$ satisfies the following two properties:
(i) $\forall (P_1, z_1), (P_2, z_2) \in \mathcal{P} \times \mathcal{Z}$ :
$$\mathcal{D}(P_1, z_1, P_2, z_2) = 0 \;\Leftrightarrow\; P_1 = P_2,\; z_1 = z_2.$$
(ii) $\forall (P_i, z_i) \in \mathcal{P} \times \mathcal{Z}, i = 0, 1, 2$ :
$$\mathcal{D}(P_0, z_0, P_2, z_2) \le \mathcal{D}(P_0, z_0, P_1, z_1) + \mathcal{D}(P_1, z_1, P_2, z_2).$$
(C2) $\mathcal{D} : \mathcal{P} \times \mathcal{Z} \times \mathcal{P} \times \mathcal{Z} \to [0, +\infty]$ is lower-semicontinuous.
(C3) There exists a function $\lambda \in L^1(0, T)$ such that for all $q \in \mathcal{Q}$ the following implication holds true:
$$\mathcal{E}(t, q) < \infty \Rightarrow \partial_t \mathcal{E}(\cdot, q) : [0, T] \to \mathbb{R} \text{ is integrable and}$$
$$|\partial_t \mathcal{E}(t, q)| \le \lambda(t)(1 + \mathcal{E}(t, q)).$$
(C4) For all $t \in [0, T]$, the map $q \mapsto \mathcal{E}(t, q)$ has compact sublevels.
(C5) The set of stable states is closed on $[0, T] \times \mathcal{Q}$: namely, for every sequence $(t_k, q_k)$ such that $q_k \in \mathcal{S}(t_k)$ for every $k$, $t_n \to t$, and $q_k \rightharpoonup q$ in $\mathcal{Q}$, we have $q \in \mathcal{S}(t)$.

Let us observe that in our case

$$\partial_t \mathcal{E}(t, y(t), P(t), z(t)) = -\langle \dot{\ell}(t), y(t) \rangle \tag{3.1}$$

is linear in $y$ and thus the map $q \to \partial_t \mathcal{E}(t, q)$ is weakly continuous for almost every fixed $t \in [0, T]$. Notice also that (C1) and (C2) imply that for any bounded sequence $(P_k, z_k)_{k \in \mathbb{N}} \in \mathcal{P} \times \mathcal{Z}$ we have

$$\min\{\mathcal{D}(P, z, P_k, z_k), \mathcal{D}(P_k, z_k, P, z)\} \to 0 \quad \Rightarrow \quad (P_k, z_k) \rightharpoonup (P, z) \text{ in } \mathcal{P} \times \mathcal{Z}$$

as was observed in [22, Lemma 4.1], because bounded sets in $\mathcal{P} \times \mathcal{Z}$ are precompact (with respect to the weak topologies). As a consequence we may use the generalized version of *Helly's selection principle* stated in [29, Theorem 2.1.24]. We now formulate the central result of the paper.

**Theorem 3.2** (Existence of energetic solutions)**.** *Let* $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ *be the triple introduced in Section 2. Let* $q_0 = (y_0, P_0, z_0) \in \mathcal{S}(0)$ *be a stable initial state. Then there exists an energetic solution* $q = (y, P, z) : [0, T] \to \mathcal{Q}$ *for the initial conditions* $q_0$*.*

Given an interval $[0, T]$ and a positive natural number $n$, we denote by $\Pi_n$ the family of partitions of $[0, T]$ into $n$ intervals, namely the family of $n$-tuples of real numbers satisfying $0 = t_0 < t_1 < \cdots < t_n = T$. We define the family of partitions of arbitrary length as

$$\Pi = \bigcup_{n=1}^{\infty} \Pi_n.$$

Given $\sigma \in \Pi$ the symbol $\Delta(\sigma)$ will denote the fineness of the partition $\sigma$, namely

$$\Delta(\sigma) := \max_k |t_k - t_{k-1}|.$$

**Theorem 3.3** (Existence via incremental minimization)**.** *For every stable initial data $q_0 \in \mathcal{S}(0)$ and every sequence of partitions $\sigma_n \in \Pi$ of $[0,T]$ with fineness $\Delta_n := \Delta(\sigma_n)$ tending to zero as $n \to \infty$, we can find a trajectory $q_n : [0,T] \to \mathcal{Q}$ with $q(0) = q_0$ which is piecewise constant on the partition, right-continuous and satisfies*

$$q_n(t) \in \mathcal{S}(t), \tag{3.2}$$

$$\mathcal{E}(t, q_n(t)) + \mathrm{Diss}(P_n, z_n; s, t) - \mathcal{E}(s, q_n(s)) \le -\int_s^t \langle \dot{\ell}(r), y_n(r) \rangle dr \tag{3.3}$$

*for every $s, t \in \sigma_n$. Moreover, there exists a subsequence and an energetic solution $q = (y, P, z) : [0,T] \to \mathcal{Q}$ for the initial conditions $q_0$ with the following properties:*

$$\begin{aligned} \forall t \in [0,T]: &\quad P_{n_k}(t) \rightharpoonup P(t) \ \text{in } \mathcal{P}, \\ \forall t \in [0,T]: &\quad z_{n_k}(t) \rightharpoonup z(t) \ \text{in } \mathcal{Z}, \\ \forall s,t \in [0,T]: &\quad \mathrm{Diss}(P_{n_k}, z_{n_k}; s, t) \to \mathrm{Diss}(P, z; s, t), \\ \forall t \in [0,T]: &\quad \mathcal{E}(t, q_{n_k}(t)) \to \mathcal{E}(t, q(t)), \end{aligned}$$

*and*

$$\langle \dot{\ell}(\cdot), y_{n_k}(\cdot) \rangle \to \langle \dot{\ell}(\cdot), y(\cdot) \rangle \quad \text{in } L^1(0,T). \tag{3.4}$$

Notice that the statement of Theorem 3.3 is actually stronger than that of Theorem 3.2 because it additionally provides a way to construct energetic solutions using incremental minimization.

In order to prove Theorem 3.3 we start by checking conditions (C1)-(C5).

*Proof of (C1):*
The dissipation $\mathcal{D}$ is defined as an integral over $\Omega$ of the non-negative function $D$. By Proposition 2.4 for almost every $x \in \Omega$ we have

$$D(x, P_0(x), z_0(x), P_1(x), z_1(x)) = \kappa(x)(z_0(x) - z_1(x)) + \rho(z_1(x)) D_{\mathrm{p}}(x, P_0(x), P_1(x)),$$

with $\rho(z_1(x)) D(x, P_0(x), P_1(x)) \ge 0$, and $\kappa(x) \ge \kappa_0 > 0$. It is thus easily seen that if $D(\cdot, P_0, z_0, P_1, z_1) = 0$ a.e. in $\Omega$ it must be $z_0 = z_1$ a.e. on $\Omega$. Now, since $\rho$ is strictly positive $D_{\mathrm{p}}(x, P_0(x), P_1(x)) = 0$ for a.e. $x \in \Omega$, which in turn implies $P_0 = P_1$ a.e. by Lemma 2.1(i). This proves point (i) of (C1).

We now prove the triangle inequality (ii). Let $(P_i, z_i) \in \mathcal{P} \times \mathcal{Z}$ for $i = 1, 2, 3$. We can assume without loss of generality that $z_1 \ge z_2 \ge z_3$ a.e. on $\Omega$. Otherwise the right-hand side of the triangle inequality is $+\infty$. Fix $x \in \Omega$ and for the sake of simplicity do not write the $x$-dependence in the next formulas. We use Lemma 2.1(ii), Proposition 2.4, and the monotonicity of $\rho$ to estimate

$$\begin{aligned} D(P_1, z_1, P_3, z_3) &= \kappa(z_1 - z_3) + \rho(z_3) D_{\mathrm{p}}(P_1, P_3) \\ &= \kappa(z_1 - z_2) + \kappa(z_2 - z_3) + \rho(z_3) D_{\mathrm{p}}(P_1, P_3) \\ &\le \kappa(z_1 - z_2) + \kappa(z_2 - z_3) + \rho(z_3)(D_{\mathrm{p}}(P_1, P_2) + D_{\mathrm{p}}(P_2, P_3)) \\ &\le \kappa(z_1 - z_2) + \rho(z_2) D_{\mathrm{p}}(P_1, P_2) + \kappa(z_2 - z_3) + \rho(z_3) D_{\mathrm{p}}(P_2, P_3) \\ &= D(P_1, z_1, P_2, z_2) + D(P_2, z_2, P_3, z_3). \end{aligned}$$

We then conclude by integrating over $\Omega$.

*Proof of (C2):*
We have to show that whenever $(P_k, z_k, \hat{P}_k, \hat{z}_k) \rightharpoonup (P, z, \hat{P}, \hat{z})$ in $(\mathcal{P} \times \mathcal{Z})^2$ then

$$\mathcal{D}(P, z, \hat{P}, \hat{z}) \le \liminf_{k \to \infty} \mathcal{D}(P_k, z_k, \hat{P}_k, \hat{z}_k).$$

By compactness the convergence of $(P_k, z_k, \hat{P}_k, \hat{z}_k)$ to $(P, z, \hat{P}, \hat{z})$ above is strong in $L^1(\Omega)$. By Proposition 2.4 it suffices to show that

$$\int_\Omega S(x, z(x) - \hat{z}(x)) dx \le \liminf_{k \to \infty} \int_\Omega S(x, z_k(x) - \hat{z}_k(x)) dx \tag{3.5}$$

and

$$\int_\Omega \rho(x,\hat z)D_{\mathrm p}(x,P,\hat P)dx = \lim_{k\to\infty}\int_\Omega \rho(x,\hat z_k)D_{\mathrm p}(x,P_k,\hat P_k)dx \tag{3.6}$$

The implication (3.5) simply follows from Fatou's Lemma since $S$ is non-negative and lower semicontinuous in the second component. Indeed we can choose a subsequence that realizes the lim inf in (3.5). By taking another subsequence, we can further assume that $z_{k_l}\to z$ and $\hat z_{k_l}\to\hat z$ a.e. on $\Omega$. By Fatou's Lemma

$$\begin{aligned}\int_\Omega S(x,z(x)-\hat z(x))dx &\le \liminf_{l\to\infty}\int_\Omega S(x,z_{k_l}(x)-\hat z_{k_l}(x))dx\\ &=\liminf_{k\to\infty}\int_\Omega S(x,z_k(x)-\hat z_k(x))dx.\end{aligned}$$

In order to prove (3.6) we use that $\rho(x,\cdot)$ is continuous. As shown in Lemma 2.3, $D_{\mathrm p}(x,\cdot,\cdot)$ is continuous as well and using the sublinear growth (2.11) we conclude by Dominated Convergence Theorem.

*Proof of (C3):*
From the very definition of the energy we have

$$\partial_t\mathcal E(t,P,z) = -\langle\dot\ell(t),y\rangle.$$

Since $\nabla y = F_{\mathrm e}P$, we infer by Hölder and Young inequalities

$$\|\nabla y\|_{L^{q_Y}} \le C\|F_{\mathrm e}\|_{L^{q_F}}\|P\|_{L^{q_P}} \le C\frac{\|F_{\mathrm e}\|^{q_F}_{L^{q_F}}}{q_F} + C\frac{\|P\|^{q_P}_{L^{q_P}}}{q_P}, \tag{3.7}$$

where we have used

$$\frac{1}{q_Y}\ge\frac{1}{q_F}+\frac{1}{q_P}.$$

Hence, if $\ell\in W^{1,1}(0,T;\mathcal Y^*)$ we infer

$$\begin{aligned}\partial_t\mathcal E(t,y,P,z) &\le C\|\dot\ell(t)\|_{\mathcal Y^*}\|\nabla y\|_{L^{q_Y}}\\ &\overset{(3.7)}{\le} C\lambda(t)(\|F\|^{q_F}_{L^{q_F}}+\|P\|^{q_P}_{L^{q_P}})\\ &\overset{(2.28)}{\le} C\lambda(t)(1+\mathcal E(t,y,P,z))\end{aligned}$$

where $\lambda(t) := \|\dot\ell(t)\|_{\mathcal Y^*}\in L^1(0,T)$.

*Proof of (C4):*
To assume that all sublevels of the energy are compact is equivalent to saying that sublevels are precompact and closed. We start by showing (sequential) precompactness. Let $t\in[0,T]$ and assume that we have a sequence $q_k=(y_k,P_k,z_k)\in\mathcal Q$ which satisfies $\mathcal E(t,q_k)\le C$. Using coercivity (2.28) we see that

$$\begin{aligned}\mathcal E(t,q_k) \ge& c\left(\|\nabla y_kP_k^{-1}\|^{q_F}_{L^{q_F}}+\|P_k\|^{q_P}_{L^{q_P}}+\|\nabla P_k\|^{r_1}_{L^{r_1}}+\|\nabla z_k\|^{r_2}_{L^{r_2}}\right)\\ &-C_1\|\nabla y_k\|_{L^{q_Y}}-C_2,\end{aligned} \tag{3.8}$$

where $C_1 = C\|\ell\|_{L^\infty([0,T];\mathcal Y^*)}$ by assumption (2.31). By Young's inequality we deduce that, for any $\mu>0$,

$$C_1\|\nabla y_k\|_{L^{q_Y}} \le \mu^{-1}C+\frac{\mu\|\nabla y_k\|^{q_Y}_{L^{q_Y}}}{q_Y}. \tag{3.9}$$

Additionally, in view of (2.29), we have

$$\|\nabla y_k\|^{q_Y}_{L^{q_Y}} \le \|\nabla y_kP_k^{-1}\|^{q_Y}_{L^{q_F}}\|P_k\|^{q_Y}_{L^{q_P}} \le C\left(\|\nabla y_kP_k^{-1}\|^{q_F}_{L^{q_F}}+\|P_k\|^{q_P}_{L^{q_P}}\right). \tag{3.10}$$

Combining (3.8), (3.9), and (3.10) and choosing $\mu>0$ suitably small, we readily see that

$$\|\nabla y_k\|^{q_Y}_{L^{q_Y}}+\|P_k\|^{q_P}_{L^{q_P}}+\|\nabla P_k\|^{r_1}_{L^{r_1}}+\|\nabla z_k\|^{r_2}_{L^{r_2}}\le C. \tag{3.11}$$

Then, there exists a subsequence such that

$$\begin{aligned} z_{k_l} &\to z^* \text{ in } L^1(\Omega) \text{ and pointwise a.e.} \\ z_{k_l} &\rightharpoonup z^* \text{ weakly in } W^{1,r_2}(\Omega). \end{aligned}$$

Notice that $z_k \in [0,1]$ a.e. on $\Omega$, thus $z_k$ stays uniformly bounded in $L^\infty(\Omega)$, so that by Vitali's Convergence Theorem we infer

$$z_{k_l} \to z^* \text{ in } L^\sigma(\Omega)$$

for all $\sigma \geq 1$. Similarly we argue for $P_k$, which is uniformly bounded in $L^{\bar{q}_P}(\Omega)$ with

$$\bar{q}_P := \max\{q_P, r_1^*\},$$

($r_1^*$ being the Sobolev exponent associated to $r_1$) and we extract a subsequence such that

$$\begin{aligned} P_{k_l} &\rightharpoonup P^* \text{ weakly in } W^{1,r_1}(\Omega), \\ P_{k_l} &\to P^* \text{ in } L^s(\Omega) \end{aligned}$$

for every $s \in [1, \bar{q}_P)$. Furthermore, we infer that

$$\begin{aligned} \nabla y_{k_l} &\rightharpoonup \nabla y^* \text{ weakly in } L^{q_Y}(\Omega), \\ y_{k_l} &\to y^* \text{ in } L^s(\Omega) \end{aligned}$$

for every $s \in [1, q_Y^*]$ thanks to the Dirichlet boundary condition on $y$. In particular, we have checked that

$$q_{k_l} \rightharpoonup q^* \text{ in } \mathcal{Q},$$

which is nothing but sequential *pre*compactness.

It remains to show the lower semicontinuity of $\mathcal{E}$, which is equivalent to closedness of sublevels. Take a sequence $q_k \rightharpoonup q$ in $\mathcal{Q}$ where $q_k = (y_k, P_k, z_k)$ and assume without loss of generality that $\sup_k \mathcal{E}(t, q_k) \leq C$. We can use (3.11) and choose a subsequence such that

$$\lim_{l\to\infty} \mathcal{E}(t, q_{k_l}) = \liminf_{k\to\infty} \mathcal{E}(t, q_k)$$

and

$$\begin{aligned} \nabla y_{k_l} &\rightharpoonup \nabla y \text{ weakly in } L^{q_Y}(\Omega), \\ \nabla P_{k_l} &\rightharpoonup \nabla P \text{ weakly in } L^{r_1}(\Omega), \\ \nabla z_{k_l} &\rightharpoonup \nabla z \text{ weakly in } L^{r_2}(\Omega), \\ P_{k_l} &\to P \text{ in } L^s(\Omega), \\ z_{k_l} &\to z \text{ in } L^\sigma(\Omega) \end{aligned}$$

for every $s \in [1, \bar{q}_P), \sigma \in [1, \infty)$. Now in order to use (2.27) we need to show that

$$\mathbb{M}(\nabla y_k P_k^{-1}) \rightharpoonup \mathbb{M}(\nabla y P^{-1}) \quad \text{ in } L^1(\Omega).$$

This result was established in [27] and can be found in [22, Proposition 5.1] or [29, Lemma 4.1.3] in a slightly more general framework. The convergence is proven under the assumption that

$$\frac{1}{q_Y} + \frac{d-1}{s} \leq 1$$

which is indeed satisfied here since $\bar{q}_P > q_Y > d$ and therefore $s$ can be chosen larger than $d$. The lower semicontinuity of

$$(y, P, z) \mapsto \int_\Omega W_{\text{el}}(x, \nabla y P^{-1}, z) + W_{\text{h}}(x, P, z) dx$$

now follows from classical theory due to polyconvexity of the integrand. It was pointed out in [29] that the classical assumption of $W$ being a Carathéodory function can be relaxed to the one of a normal integrand using a Yosida-Moreau regularization.

Let us emphasize that the assumption that $r_1, r_2$ are strictly greater than 1 is not needed and one could consider $r_1 = r_2 = 1$ as well, at the expense of rewriting the argument in BV-spaces.

The proof of (C5) is typically the hardest part and we will establish it in the next section by arguing as in Thomas [34–36]. The process in finding mutual recovery sequences used therein is almost directly applicable to our setting.

3.1. **Closedness of stable states (C5).** This closedness relies on finding a suitable recovery sequence. In [28] this was achieved for $r_2 > d$, in which case damage is continuous in space. In the papers [34, 35], this was generalized to $1 < r_2 < d$ first and in [36] to the case where damage is of bounded variation ($r_2 = 1$). We will adapt the arguments contained in these references to our model.

We want to prove that if $(t_k, q_k)$ is a sequence such that $q_k \in \mathcal{S}(t_k)$, $t_k \to t$, and $q_k \rightharpoonup q$ in $\mathcal{Q}$, then $q \in \mathcal{S}(t)$. Thus we need to ensure that for every $\hat{q} \in \mathcal{Q}$

$$0 \le \mathcal{E}(t, \hat{q}) + \mathcal{D}(q, \hat{q}) - \mathcal{E}(t, q).$$

In order to show this we will provide a so-called *mutual recovery sequence* (see [28, 31]) $\hat{q}_k \rightharpoonup \hat{q}$ such that

$$\limsup_{k\to\infty} \Big(\mathcal{E}(t_k, \hat{q}_k) + \mathcal{D}(q_k, \hat{q}_k) - \mathcal{E}(t_k, q_k)\Big) \le \mathcal{E}(t, \hat{q}) + \mathcal{D}(q, \hat{q}) - \mathcal{E}(t, q). \tag{3.12}$$

Indeed, since by stability of $q_k$ we have for every $\hat{q}_k \in \mathcal{Q}$

$$0 \le \mathcal{E}(t_k, \hat{q}_k) + \mathcal{D}(q_k, \hat{q}_k) - \mathcal{E}(t_k, q_k), \tag{3.13}$$

the lim sup bound (3.12) together with (3.13) implies the claim $q \in \mathcal{S}(t)$.

Notice that if the dissipation $\mathcal{D}$ was continuous (not only lower semicontinuous) then (3.12) would hold true even for the constant recovery sequence $\hat{q}_k = \hat{q}$ because $\mathcal{E}$ is lower semicontinuous and $\mathcal{E}(\cdot, \hat{q})$ is continuous. In the present case however, the dissipation $\mathcal{D}$ is not continuous, since $D(P_0, z_0, P_1, z_1)$ is only continuous on its domain $\{z_0 \ge z_1\}$ (compare to the assumptions on $D$ in [22, Conditions (3.5) and Remark 3.2]). In the next lemma we show that nonetheless $\mathcal{D}$ is continuous on its domain.

**Lemma 3.4.** *Let us define the domain of $\mathcal{D}$ as*

$$\mathbb{D} = \Big\{(P, z, \hat{P}, \hat{z}) \in (\mathcal{P} \times \mathcal{Z})^2 : \mathcal{D}(P, z, \hat{P}, \hat{z}) < \infty\Big\}.$$

*Then $\mathcal{D} : \mathbb{D} \to [0, \infty)$ is continuous.*

*Proof.* By Proposition 2.4,

$$\mathcal{D}(P, z, \hat{P}, \hat{z}) = \int_\Omega S(x, \hat{z} - z) + \rho(x, \hat{z}) D_{\mathrm{p}}(x, P, \hat{P}) dx.$$

Now take a sequence $(P_k, z_k, \hat{P}_k, \hat{z}_k) \in \mathbb{D}$ such that $(P_k, z_k, \hat{P}_k, \hat{z}_k) \rightharpoonup (P, z, \hat{P}, \hat{z})$ in $(\mathcal{P} \times \mathcal{Z})^2$. Then the convergence is strong in $L^1(\Omega)$ and for every subsequence $(k_l)_{l\in\mathbb{N}}$ we can find a further subsequence $(k_{l_j})_{j\in\mathbb{N}}$ such that $(z_{k_{l_j}}, P_{k_{l_j}}, \hat{z}_{k_{l_j}}, \hat{P}_{k_{l_j}}) \to (z, P, \hat{z}, \hat{P})$ a.e. in $\Omega$. Observe that

$$\mathbb{D} = \Big\{(P, z, \hat{P}, \hat{z}) \in (\mathcal{P} \times \mathcal{Z})^2 : z(x) \ge \hat{z}(x) \text{ for a.e. } x \in \Omega\Big\}.$$

Thus,

$$\mathcal{D}(P_k, z_k, \hat{P}_k, \hat{z}_k) = \int_\Omega \kappa(x)(z_k(x) - \hat{z}_k(x)) + \rho(x, \hat{z}_k) D_{\mathrm{p}}(x, P_k, \hat{P}_k) dx$$

for every $k \in \mathbb{N}$. By Lemma 2.3 the integrand converges pointwise a.e. on $\Omega$ for the sequence $(k_{l_j})_j$ to the corresponding limit. We can further estimate the integrand, using (2.11), by

$$2\|\kappa\|_{L^\infty} + C\|\rho\|_{L^\infty}(1 + |P_k(x)| + |\hat{P}_k(x)|).$$

This bound allows us to use the Dominated Convergence Theorem. Hence,

$$\lim_{j\to\infty} \int_\Omega \kappa(z_{k_{l_j}} - \hat{z}_{k_{l_j}}) + \rho(\hat{z}_{k_{l_j}}) D_{\mathrm{p}}(P_{k_{l_j}}, \hat{P}_{k_{l_j}}) dx = \int_\Omega \kappa(z - \hat{z}) + \rho(\hat{z}) D_{\mathrm{p}}(P, \hat{P}) dx, \tag{3.14}$$

where, for simplicity, we have again omitted the $x$-dependence. Noticing that $\kappa(x)(z(x) - \hat{z}(x)) = S(x, z(x) - \hat{z}(x))$ for a.e. $x \in \Omega$, the right-hand side of (3.14) is nothing but $\mathcal{D}(P, z, \hat{P}, \hat{z})$ and the statement follows. □

**Lemma 3.5.** *Let $q_k \in \mathcal{S}(t_k)$ such that $t_k \to t$ and $q_k \rightharpoonup q$ in $\mathcal{Q}$. Then for every $\hat{q} \in \mathcal{Q}$ we can find a sequence $\hat{q}_k$ such that (3.12) holds true.*

*Proof.* We proceed in several steps following the proof in [35, Theorem 3.14]:

*Step 1.* Let $(t_k, q_k)$ be as in the lemma and $\hat{q} = (\hat{y}, \hat{P}, \hat{z}) \in \mathcal{Q}$ be arbitrary. We first set $\hat{y}_k := \hat{y}$ for all $k$. From this choice it is possible to reduce to the case $t_k = t$ for every $k$. Indeed, we claim that the limsup in (3.12) coincides with

$$\limsup_{k\to\infty} \Big(\mathcal{E}(t, \hat{q}_k) + \mathcal{D}(q_k, \hat{q}_k) - \mathcal{E}(t, q_k)\Big).$$

Consider the difference

$$\begin{aligned}|\mathcal{E}(t_k, \hat{q}_k) - \mathcal{E}(t, \hat{q}_k) + \mathcal{E}(t, q_k) - \mathcal{E}(t_k, q_k)| &= |\langle \ell(t_k) - \ell(t), \hat{y}_k\rangle + \langle \ell(t_k) - \ell(t), y_k\rangle| \\ &\le 2C\|\ell(t_k) - \ell(t)\|_{\mathcal{Y}^*} \to 0\end{aligned}$$

as $k \to \infty$ since by (2.31) $\ell \in W^{1,1}(0, T; \mathcal{Y}^*) \subset C^0(0, T; \mathcal{Y}^*)$. This shows the claim.

*Step 2.* If $\mathcal{E}(t, \hat{q}) + \mathcal{D}(q, \hat{q}) = +\infty$ then (3.12) holds trivially. Let us therefore assume $\mathcal{E}(t, \hat{q}) < \infty$ and $\mathcal{D}(q, \hat{q}) < \infty$. This in particular implies

$$\hat{z} \le z, \tag{3.15}$$

a.e. on $\Omega$. We define the recovery sequence as $\hat{q}_k := (\hat{y}, \hat{P}, \hat{z}_k)$ where

$$\hat{z}_k := \min\{(\hat{z} - \delta_k)^+, z_k\}$$

and $\delta_k > 0$ is a sequence that will be chosen later (tending to zero as $k \to +\infty$).
We now claim that $\hat{z}_k \rightharpoonup \hat{z}$ weakly in $W^{1,r_2}(\Omega)$. Indeed, by construction $\hat{z}_k$ is bounded in $W^{1,r_2}(\Omega)$. So for every subsequence $\hat{z}_{k_l}$ there exists a further subsequence $\hat{z}_{k_{l_j}}$ and a limit $z^*$ (a priori depending on the subsequence we choose) such that

$$\begin{aligned}&\hat{z}_{k_{l_j}} \rightharpoonup z^* \text{ weakly in } W^{1,r_2}(\Omega),\\ &\hat{z}_{k_{l_j}} \to z^* \text{ in } L^{r_2}(\Omega),\\ &\hat{z}_{k_{l_j}} \to z^* \text{ a.e. on } \Omega.\end{aligned}$$

But by definition of $\hat{z}_k$, it follows that it converges to $\hat{z}$ a.e. on $\Omega$. Thus, $z^* = \hat{z}$ independently of the subsequence and we have shown

$$\hat{z}_k \rightharpoonup \hat{z} \text{ weakly in } W^{1,r_2}(\Omega).$$

Notice that $(q_k, \hat{q}_k) \in \mathbb{D}$ because $\hat{z}_k \le z_k$. Therefore, by Lemma 3.4

$$\limsup_{k\to\infty} \mathcal{D}(q_k, \hat{q}_k) = \lim_{k\to\infty} \mathcal{D}(q_k, \hat{q}_k) = \mathcal{D}(q, \hat{q}).$$

*Step 3.* It remains to show that

$$\limsup_{k\to\infty} \Big(\mathcal{E}(t, \hat{q}_k) - \mathcal{E}(t, q_k)\Big) \le \mathcal{E}(t, \hat{q}) - \mathcal{E}(t, q). \tag{3.16}$$

To achieve this we need to choose the sequence $\delta_k$ in such a way that $\mathcal{L}^d(\{z_k < (\hat{z} - \delta_k)^+\})$ goes to zero as $k \to \infty$. This particularly implies $\hat{z}_k \to z$ in $L^\sigma$ for all $\sigma \ge 1$. Recall that

$$\mathcal{E}(t, y, P, z) = \int_\Omega W(x, y, P, z)dx + \frac{\nu}{r_1}\int_\Omega |\nabla P|^{r_1}dx + \frac{\mu}{r_2}\int_\Omega |\nabla z|^{r_2} - \langle \ell(t), y\rangle,$$

where $W(x, y, P, z) = W_{\mathrm{el}}(x, \nabla y P^{-1}, z) + W_{\mathrm{h}}(x, P, z)$. So the difference $\mathcal{E}(t, \hat{q}_k) - \mathcal{E}(t, q_k)$ on the left hand side of (3.16) consists of two parts. Namely,

$$\int_\Omega \Big(W(x, \hat{y}, \hat{P}, \hat{z}_k) - W(x, y, P, z_k)\Big)dx \tag{3.17}$$

and

$$\frac{\mu}{r_2}\int_\Omega \Big(|\nabla \hat{z}_k|^{r_2} - |\nabla z_k|^{r_2}\Big)dx. \tag{3.18}$$

Taking the lim sup as $k \to \infty$ in (3.17) and using lower semicontinuity of $W$, it suffices to show that

$$\lim_{k\to\infty} \int_\Omega W(x, \hat{y}, \hat{P}, \hat{z}_k)dx = \int_\Omega W(x, \hat{y}, \hat{P}, \hat{z})dx. \tag{3.19}$$

Now, for every subsequence $(k_l)_{l\in\mathbb{N}}$ we can choose a further subsequence $(k_{l_j})_{j\in\mathbb{N}}$ such that $\hat{z}_{k_{l_j}} \to \hat{z}$ a.e. on $\Omega$. Since $W$ is continuous in $z$, this implies

$$W(x, \hat{y}, \hat{P}, \hat{z}_{k_{l_j}}) \to W(x, \hat{y}, \hat{P}, \hat{z}) \text{ a.e. on } \Omega.$$

By using $\hat{z}_{k_{l_j}} \leq \hat{z}$, monotonicity (2.30), and coercivity (2.28) we get the uniform bound

$$-2C_2 \leq W(x, y, P, \hat{z}_{k_{l_j}}) \leq W(x, y, P, \hat{z}) \in L^1(\Omega).$$

Therefore, (3.19) follows from the Dominated Convergence Theorem.

Let us now show that the lim sup as $k \to \infty$ of the expression in (3.18) is less or equal to $\frac{\mu}{r_2}\int_\Omega(|\nabla\hat{z}|^{r_2} - |\nabla z|^{r_2})dx$. We define

$$B_k = \{z_k < (\hat{z} - \delta_k)^+\}$$
$$A_k = \Omega \setminus B_k$$

Since $B_k \subset \{|z - z_k| \geq \delta_k\}$ thanks to (3.15), we can use Markov's inequality to show that

$$\mathcal{L}^d(B_k) \leq \frac{1}{\delta_k^{r_2}} \int_\Omega |z - z_k|^{r_2}dx.$$

As we want this to go to 0 we impose that

$$\delta_k = \|z - z_k\|_{L^{r_2}}^{1/r_2}.$$

Now, we can write, by the definition of $\hat{z}_k$,

$$\begin{aligned}\frac{\mu}{r_2}\int_\Omega \Big(|\nabla\hat{z}_k|^{r_2} - |\nabla z_k|^{r_2}\Big)dx &= \frac{\mu}{r_2}\left(\int_{A_k}\Big(|\nabla\hat{z}|^{r_2} - |\nabla z_k|^{r_2}\Big)dx + \int_{B_k}\Big(\underbrace{|\nabla\hat{z}_k|^{r_2} - |\nabla z_k|^{r_2}}_{=0}\Big)dx\right)\\ &= \frac{\mu}{r_2}\int_{A_k}\Big(|\nabla\hat{z}|^{r_2} - |\nabla z_k|^{r_2}\Big)dx.\end{aligned}$$

We take the lim sup above as $k \to \infty$ and use that $I_{A_k}\nabla z_k \rightharpoonup \nabla z$ weakly in $L^{r_2}(\Omega)$ (here $I_{A_k}$, the characteristic function of $A_k$, converges to 1 strongly in $L^q(\Omega)$ for any $q \in [1, +\infty)$, while $\nabla z_k$ tends to $\nabla z$ weakly in $L^p(\Omega)$ for all $p < r_2$; the equiboundedness of $I_{A_k}\nabla z_k$ on $L^{r_2}$ implies the claim) to get

$$\begin{aligned}\limsup_{k\to\infty}\frac{\mu}{r_2}\int_{A_k}\Big(|\nabla\hat{z}|^{r_2} - |\nabla z_k|^{r_2}\Big)dx &= \frac{\mu}{r_2}\left(\int_\Omega |\nabla\hat{z}|^{r_2}dx - \liminf_k \int_{A_k}|\nabla z_k|^{r_2}dx\right)\\ &\leq \frac{\mu}{r_2}\left(\int_\Omega |\nabla\hat{z}|^{r_2}dx - \int_\Omega |\nabla z|^{r_2}dx\right)\end{aligned}$$

by weak lower semicontinuity of the norm. This shows (3.16) and finishes the proof. □

**Remark 3.6** (Case $r_1 = r_2 = 1$)**.** In this special case, instead of $\frac{\mu}{r_2}\int_\Omega |\nabla z|^{r_2}dx$ the damage variable $z$ is regularized by

$$\mu|Dz|(\Omega),$$

where $z$ belongs to $BV(\Omega; [0, 1])$ and $|Dz|(\Omega)$ denotes its total variation. To deal with this case we refer to [36] and sketch the main argument here. For the plastic variable $P$ we proceed analogously replacing $\frac{\nu}{r_1}\int_\Omega |\nabla P|^{r_1}dx$ by the total variation $\nu|DP|(\Omega)$. Then we define the mutual recovery sequence as

$$\hat{q}_k = (\hat{y}, \hat{P}, \hat{z}_k),$$

with $\hat{z}_k$ defined as

$$\hat{z}_k = \begin{cases}\hat{z} - \delta_k & \text{on } A_k := \{0 \leq \hat{z} - \delta_k \leq z_k\}\\ z_k & \text{on } B_k := \{0 \leq z_k < \hat{z} - \delta_k\}\\ 0 & \text{on } C_k := \Omega \setminus (A_k \cup B_k).\end{cases}$$

With this choice we have that $0 \le \hat{z}_k \le z_k$. We have to verify that we can choose $\delta_k$ so that the following three conditions are satisfied:

(1) $\hat{z}_k \in BV(\Omega;[0,1])$, $\mathcal{L}^d(B_k)+\mathcal{L}^d(C_k)\to 0$, and $\hat{z}_k \to \hat{z}$ strongly in $L^1(\Omega)$,
(2) $\lim_{k\to\infty}\mathcal{D}(q_k,\hat{q}_k)=\mathcal{D}(q,\hat{q})$,
(3) $\limsup_{k\to\infty}\mathcal{E}(t_k,\hat{q}_k)-\mathcal{E}(t_k,q_k)\le\mathcal{E}(t,\hat{q})-\mathcal{E}(t,q)$.

To check (1) one follows the lines of the proof of Lemma 2.13 in [36]: this procedure consists into choosing $\delta_k \in [m_k^{1/2}, m_k^{1/4}]$, with

$$m_k := \max\{k^{-1}, \|z-z_k\|_{L^1}\}.$$

To verify (2) we first redefine the domain of $\mathcal{D}$ as

$$\mathbb{D}=\left\{(P,z,\hat{P},\hat{z})\in\big(BV(\Omega;SL(d))\times BV(\Omega;[0,1])\big)^2:\mathcal{D}(P,z,\hat{P},\hat{z})<\infty\right\}.$$

Then, in the spirit of Lemma 3.4, we can prove that $\mathcal{D}:\mathbb{D}\to[0,\infty)$ is continuous. The proof of Lemma 3.4 can be adapted observing that weak convergence in $BV$ implies that $z_k\to z$ strongly in $L^1(\Omega)$, then the same arguments can be used.

Let us verify (3). The term $\int_\Omega W(x,y,P,\hat{z}_k)-W(x,y,P,z_k)dx$ is treated as in the proof of Lemma 3.5. Finally the inequality

$$\limsup_{k\to\infty}|D\hat{z}_k|(\Omega)-|Dz_k|(\Omega)\le|D\hat{z}|(\Omega)-|Dz|(\Omega),$$

is achieved as in the proof of Lemma 2.13 in [36].

3.2. **Proof of Theorem 3.3.** We are now in position to prove the main theorem, see also [26]. We proceed in several steps.

**Step 1: Approximation via incremental minimization.** Let $\sigma_n=\{0=t_0^n<t_1^n<\cdots<t^n_{N(n)}=T\}\in\Pi, n\in\mathbb{N}$, be a sequence of partitions such that the fineness $\Delta(\sigma_n)$ tends to zero as $n$ tends to $\infty$. For fixed $n$ we iteratively solve for

$$(y_j,P_j,z_j)\in\underset{(\hat{y},\hat{P},\hat{z})\in\mathcal{Q}}{\operatorname{argmin}}\left\{\mathcal{E}(t_j,\hat{y},\hat{P},\hat{z})+\mathcal{D}(P_{j-1},z_{j-1},\hat{P},\hat{z})\right\},\quad j\in\{1,\dots,N(n)\}. \tag{3.20}$$

Note that (C2) and (C4) guarantee the existence of minimizers. This selection satisfies $q_j=(y_j,P_j,z_j)\in\mathcal{S}(t_j)$. This can be seen by using the minimum property in (3.20) and the triangle inequality (C1ii). Testing the minimum in (3.20) by $q_{j-1}$ we infer

$$\mathcal{E}(t_j,q_j)-\mathcal{E}(t_{j-1},q_{j-1})+\mathcal{D}(q_{j-1},q_j)\le\int_{t_{j-1}}^{t_j}\partial_t\mathcal{E}(s,q_{j-1})ds=-\int_{t_{j-1}}^{t_j}\langle\dot{\ell}(s),y_{j-1}\rangle ds.$$

Summing up over $j$ from $k+1$ to $l$ we get

$$\mathcal{E}(t_l,q_l)-\mathcal{E}(t_k,q_k)+\sum_{j=k+1}^{l}\mathcal{D}(q_{j-1},q_j)\le-\sum_{j=k+1}^{l}\int_{t_{j-1}}^{t_j}\langle\dot{\ell}(s),y_{j-1}\rangle ds, \tag{3.21}$$

for every $k,l\in\{0,\dots,N(n)\}$ with $k\le l$. We define the right-continuous piecewise constant approximation

$$q_n(t):=q_{j-1},\qquad\text{for } t\in[t_{j-1},t_j),$$

which turns (3.21) into

$$\mathcal{E}(t,q_n(t))+\mathrm{Diss}(P_n,z_n;s,t)\le\mathcal{E}(s,q(s))-\int_s^t\langle\dot{\ell}(r),y_n(r)\rangle dr,$$

for every $s,t\in\sigma_n$. We have just established (3.2) and (3.3). The next goal will be to pass this inequality to the limit.

**Step 2: A priori estimates.** We use (3.3) together with (C3) so that for $t\in\sigma_n$

$$\mathcal{E}(t,q_n(t))+\mathrm{Diss}(P_n,z_n;0,t)\le\mathcal{E}(0,q_0)+\int_0^t\lambda(s)(1+\mathcal{E}(s,q_n(s)))ds.$$

Using Gronwall's inequality and uniform continuity of $\mathcal{E}(\cdot, q)$ guaranteed by (2.31) we can establish the bound

$$\mathcal{E}(t, q_n(t)) \le (1 + \mathcal{E}(0, q_0)) \exp\left(\int_0^t \lambda(s) ds\right) \le C$$

for every $t \in [0, T]$. This leads to

$$\sup_{t\in[0,T]} \mathcal{E}(t, q_n(t)) + \mathrm{Diss}(P_n, z_n; 0, T) \le C. \tag{3.22}$$

**Step 3: Selection of subsequences.** The dissipation distance satisfies (C1) and (C2) and due to (C4) the sequence $(P_n, z_n)$ takes values in a compact subset of $\mathcal{P} \times \mathcal{Z}$. Moreover, its dissipation is bounded uniformly in $n$. Therefore we can use Helly's selection principle [29, Theorem 2.1.24] and find a subsequence and functions $P, z : [0, T] \to \mathcal{P} \times \mathcal{Z}$, $\delta : [0, T] \to [0, C]$ such that the following hold:

$$\forall t \in [0, T] : \qquad P_{n_k}(t) \rightharpoonup P(t) \text{ in } \mathcal{P}, \tag{3.23a}$$

$$\forall t \in [0, T] : \qquad z_{n_k}(t) \rightharpoonup z(t) \text{ in } \mathcal{Z}, \tag{3.23b}$$

$$\forall t \in [0, T] : \qquad \delta_{n_k}(t) := \mathrm{Diss}(P_{n_k}, z_{n_k}; 0, t) \to \delta(t), \tag{3.23c}$$

$$\forall s, t \in [0, T] : \qquad \mathrm{Diss}(P, z; s, t) \le \delta(t) - \delta(s). \tag{3.23d}$$

Let us define the sequence

$$\theta_n(t) := -\langle \dot{\ell}(t), y_n(t)\rangle.$$

It is easy to check that $\theta_n$ is bounded in $L^1(0, T)$ and equi integrable. Indeed, for every interval $I \subset [0, T]$ :

$$\int_I |\theta_n(s)| ds \le \int_I \|\dot{\ell}(s)\|_{\mathcal{Y}^*} \|y_n(s)\|_{\mathcal{Y}} ds \le C \int_I \|\dot{\ell}(s)\|_{\mathcal{Y}^*} ds$$

and, since $\dot{\ell} \in L^1(0, T; \mathcal{Y}^*)$, for every $\varepsilon > 0$ there exists a $\eta > 0$ such that if $|I| < \eta$ we have

$$\int_I \|\dot{\ell}(s)\|_{\mathcal{Y}^*} ds < \varepsilon/C.$$

Thus, we can use Dunford-Pettis Theorem [14] or [29, Theorem B.3.8] and find a further (not relabeled) subsequence such that

$$\theta_{n_k} \rightharpoonup \theta \quad \text{ weakly in } L^1(0, T). \tag{3.24}$$

Notice that we did not construct a limit for the deformation yet because we are only able to use Helly's selection principle on the dissipative variables. We can still use the fact that $y_n$ is controlled by the energy for every fixed time $t$.

We define the limit deformation $y : [0, T] \to \mathcal{Y}$ as follows. Fix a time $t \in [0, T]$ and use (C4) and (3.22) to select a $t$-*dependent* subsequence $(N^t_k)_{k\in\mathbb{N}}$ of $(n_k)_{k\in\mathbb{N}}$ such that

$$\theta_{N^t_k}(t) \to \limsup_{k\to\infty} \theta_{n_k}(t) =: \theta_{\sup}(t)$$

and $y_{N^t_k}(t)$ converges weakly to some limit $\tilde{y}$ in $\mathcal{Y}$. We now define

$$y(t) := \tilde{y}$$

Notice that such $\tilde{y}$ may not be unique and may depend on the chosen subsequence. From definition we immediately see that

$$\theta_{\sup}(t) = \lim_{k\to\infty} \theta_{N^t_k}(t) = \lim_{k\to\infty} -\langle \dot{\ell}(t), y_{N^t_k}(t)\rangle = -\langle \dot{\ell}(t), y(t)\rangle \tag{3.25}$$

for every $t \in [0, T]$.

**Step 4: Stability.** We aim to show that the limit evolution defined in Step 3 is stable.

We define $\tau^t_n := \max\{\tau \in \sigma_n : \tau \le t\}$. Then by definition $\tau^t_n \to t$, $q_n(t) \rightharpoonup q(t)$ as $n = N^t_k \to \infty$ and $q_n(t) = q_n(\tau^t_n) \in \mathcal{S}(\tau^t_n)$ for every $n \in \mathbb{N}$ (see the definition of $q_n$ in step 1). Therefore $q(t) \in \mathcal{S}(t)$ by (C5).

In the next two steps we will show that the limit also satisfies the energy equality (E) so that $q(t) := (y(t), P(t), z(t))$ is an energetic solution.

**Step 5: Upper energy estimate.** Let $t \in [0, T]$ be fixed and $\tau_n^t$ as in Step 4. The goal here is to pass to the limit in (3.3) for $s = 0$ which reads

$$\mathcal{E}(\tau_n^t, q_n(\tau_n^t)) + \mathrm{Diss}(P_n, z_n; 0, \tau_n^t) \leq \mathcal{E}(0, q_0) - \int_0^{\tau_n^t} \langle \dot{\ell}(r), y_n(r) \rangle dr. \tag{3.26}$$

First we remark that $q_n(\tau_n^t) = q_n(t)$ and $\mathrm{Diss}(P_n, z_n; 0, \tau_n^t) = \mathrm{Diss}(P_n, z_n; 0, t)$ because $P_n, z_n$ are constant on $(\tau_n^t, t]$. Moreover, using (3.24) we get that

$$\lim_{k\to\infty} \int_0^{\tau_{n_k}^t} -\langle \dot{\ell}(r), y_{n_k}(r) \rangle dr = \int_0^t \theta(r) dr.$$

Let us remark that for a stable state $q = (y, P, z) \in \mathcal{S}(t)$ we have

$$\mathcal{E}(t, q) = \inf \{\mathcal{E}(t, \tilde{y}, P, z) : \tilde{y} \in \mathcal{Y}\}. \tag{3.27}$$

We claim that

$$\mathcal{E}(t, q(t)) \leq \liminf_{k\to\infty} \mathcal{E}(\tau_k^t, q_{n_k}(t)). \tag{3.28}$$

To see this we first notice that $\mathcal{E}$ is continuous in time and consider a subsequence such that

$$\liminf_{k\to\infty} \mathcal{E}(\tau_k^t, q_{n_k}(t)) = \lim_{l\to\infty} \mathcal{E}(\tau_{n_{k_l}}^t, q_{n_{k_l}}(t)),$$

with $q_{n_{k_l}}(t) \rightharpoonup \tilde{q} = (\tilde{y}, P(t), z(t))$. As $q_{n_{k_l}}(t) \in \mathcal{S}(\tau_{n_{k_l}}^t)$ we know that $\tilde{q} \in \mathcal{S}(t)$. By (3.27) we deduce now that $\mathcal{E}(t, \tilde{q}) = \mathcal{E}(t, q(t))$. This shows (3.28).

We now use (3.23c), (3.23d), (3.26) and (3.28) to get

$$\begin{aligned} \mathcal{E}(t, q(t)) + \mathrm{Diss}(P, z; 0, t) &\leq \liminf_{k\to\infty} \mathcal{E}(\tau_{n_k}^t, q_{n_k}(t)) + \lim_{k\to\infty} \mathrm{Diss}(P_{n_k}, z_{n_k}; 0, t) \\ &\leq \limsup_{k\to\infty} \mathcal{E}(\tau_{n_k}^t, q_{n_k}(t)) + \lim_{k\to\infty} \mathrm{Diss}(P_{n_k}, z_{n_k}; 0, t) \\ &\leq \mathcal{E}(0, q_0) + \int_0^t \theta(r) dr \\ &\leq \mathcal{E}(0, q_0) + \int_0^t \theta_{\sup}(r) dr \\ &= \mathcal{E}(0, q_0) - \int_0^t \langle \dot{\ell}(r), y(r) \rangle dr. \end{aligned} \tag{3.29}$$

**Step 6: Lower energy estimate.** Take any partition $\sigma = \{0 = r_0 < r_1 < \cdots < r_N = t\} \in \Pi$ of $[0, t]$. By the stability of the limit (step 4) one has that

$$\mathcal{E}(r_{i-1}, q(r_{i-1})) \leq \mathcal{E}(r_i, q(r_i)) - \mathcal{E}(r_i, q(r_i)) + \mathcal{E}(r_{i-1}, q(r_i)) + \mathcal{D}(q(r_{i-1}), q(r_i))$$

Summing this over $i = 1, \ldots, N$ we get

$$\mathcal{E}(t, q(t)) - \mathcal{E}(0, q(0)) + \sum_{i=1}^N \mathcal{D}(q(r_{i-1}), q(r_i)) \geq -\sum_{i=1}^N \int_{r_{i-1}}^{r_i} \langle \dot{\ell}(r), y(r_i) \rangle dr. \tag{3.30}$$

We can use

$$\sum_{i=1}^N \mathcal{D}(q(r_{i-1}), q(r_i)) \leq \mathrm{Diss}(P, z; 0, t)$$

to estimate the left hand side of (3.30) as desired. It remains to show that there exists a sequence of partitions

$$\sigma_n = \{0 = r_0^n < r_1^n < \cdots < r_{N(n)}^n = t\} \in \Pi, \quad n \in \mathbb{N}$$

such that

$$\int_0^t \langle \dot{\ell}(r), y(r) \rangle dr = \lim_{n\to\infty} \sum_{i=1}^{N(n)} \int_{r_{i-1}^n}^{r_i^n} \langle \dot{\ell}(r), y(r_i^n) \rangle dr. \tag{3.31}$$

The difficulty here is that we cannot assume $y$ to be measurable in time. On the other hand $\langle \dot{\ell}(\cdot), y(\cdot) \rangle$ is integrable and we can find a sequence of partitions such that the integral is approximated by its Riemann sums [17], i.e.

$$\int_0^t \langle \dot{\ell}(r), y(r) \rangle dr = \lim_{n\to\infty} \sum_{i=1}^{N(n)} \langle \dot{\ell}(r_i^n), y(r_i^n) \rangle (r_i^n - r_{i-1}^n). \tag{3.32}$$

Now, in order to get (3.31), we have to prove that

$$\lim_{n\to\infty} \left| \sum_{i=1}^{N(n)} \int_{r_{i-1}^n}^{r_i^n} \langle \dot{\ell}(r) - \dot{\ell}(r_i^n), y(r_i^n) \rangle dr \right| = 0. \tag{3.33}$$

Let us define $\dot{\ell}_n(r) := \dot{\ell}(r_i^n)$ for $r \in (r_{i-1}^n, r_i^n]$. Since

$$\sup\{\|y(r)\|_{\mathcal{Y}} : r \in [0,T]\} \le C,$$

to show (3.33), thanks to the uniform estimate of $y$, it suffices that

$$\lim_{n\to\infty} \int_0^t \|\dot{\ell}(r) - \dot{\ell}_n(r)\|_{\mathcal{Y}^*} dr = 0. \tag{3.34}$$

To show this we use a refined version of [13, Lemma 4.12] which states the approximation of Lebesgue integrals by Riemann sums.

**Lemma 3.7.** *Let $s < t$. Assume we have a countable family of Bochner integrable functions*

$$f_k : [s,t] \to X_k, \quad k \in \mathbb{N},$$

*where $X_k$ are Banach spaces. Then there exists a* ***k-independent*** *sequence of partitions*

$$\sigma_n = \{s = r_0^n < r_1^n < \cdots < r_{N(n)}^n = t\}, \quad n \in \mathbb{N},$$

*with fineness $\Delta(\sigma_n) \to 0$ such that*

$$\lim_{n\to\infty} \sum_{i=1}^{N(n)} \int_{r_{i-1}^n}^{r_i^n} \|f_k(r) - f_k(r_i^n)\| dr = 0$$

*for every $k \in \mathbb{N}$.*

The proof can be found in [13, Lemma 4.12] where the strategy of proof for this refined lemma is outlined in [13, Remark 4.13]. We use this lemma for the two functions $\langle \dot{\ell}(\cdot), y(\cdot) \rangle : [s,t] \to \mathbb{R}$ and $\dot{\ell} : [s,t] \to \mathcal{Y}^*$ to deduce (3.32) and (3.34).

**Step 7: Conclusion.** A combination of (3.29) with the lower estimate (step 6) gives the following chain of inequalities

$$\begin{aligned}
\mathcal{E}(t, q(t)) + \mathrm{Diss}(P, z; 0, t) &\le \liminf_{k\to\infty} \mathcal{E}(t, q_{n_k}(t)) + \lim_{k\to\infty} \mathrm{Diss}(P_{n_k}, z_{n_k}; 0, t) \\
&\le \limsup_{k\to\infty} \mathcal{E}(t, q_{n_k}(t)) + \delta(t) \\
&\le \mathcal{E}(0, q_0) + \int_0^t \theta(r) dr \\
&\le \mathcal{E}(0, q_0) - \int_0^t \langle \dot{\ell}(r), y(r) \rangle dr \\
&\le \mathcal{E}(t, q(t)) + \mathrm{Diss}(P, z; 0, t).
\end{aligned}$$

We hence deduce that equality holds everywhere, implying that

$$\theta(r) = \theta_{\sup}(r) = -\langle \dot{\ell}(r), y(r)\rangle \text{ for a.e. } r \in [0, T],$$

and

$$\begin{gathered} \mathrm{Diss}(P_{n_k}, z_{n_k}; 0, t) \to \mathrm{Diss}(P, z; 0, t), \\ \mathcal{E}(t, q_{n_k}(t)) \to \mathcal{E}(t, q(t)). \end{gathered}$$

It remains to show (3.4). We know that $\theta_{n_k} \rightharpoonup \theta$ in $L^1(0,T)$ and $\theta(t) = \limsup_{k\to\infty} \theta_{n_k}(t)$ for a.e. $t \in [0,T]$. Now,

$$\|\theta_{n_k} - \theta\|_{L^1} = \int_0^T (\theta - \theta_{n_k})dt + 2\int_0^T (\theta_{n_k} - \theta)^+ dt \tag{3.35}$$

where $f^+ := \max\{0, f\}$. The first integral converges to zero by weak convergence and the second integrand satisfies $0 \leq (\theta_{n_k} - \theta)^+ \leq \Theta_k := \sup_{l\geq k} \theta_{n_l} - \theta$. Due to equiboundedness of $\theta_{n_k}$ in $L^1$ we know that $\Theta_1 \in L^1(0,T)$. Therefore we can use Levi's Monotone Convergence Theorem for the monotone decreasing sequence $\Theta_k$ and conclude that also the second term in (3.35) converges to zero, entailing (3.4).

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