SOME ELEMENTARY QUESTIONS IN THE CALCULUS OF VARIATIONS

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ABSTRACT. We discuss some elementary questions in the calculus of variations related to the Lavrentiev phenomenon and the uniqueness of the minimizers.

1. THE LAVRENTIEV PHENOMENON

Suppose we have a *standard* functional in the calculus of variations

$$\mathcal{F}(y) = \int_a^b f(t, y(t), y'(t)) \, dt$$

with a Lagrangian function $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. If we consider the minimization of $\mathcal{F}$ on the set of functions such that $y(a) = y_a$ and $y(b) = y_b$, belonging to different classes as $AC([a, b])$ (absolutely continuous functions or equivalently functions in $W^{1,1}(a, b)$), $Lip([a, b])$ (Lipschitz functions) or $C^1([a, b])$, the so called *Lavrentiev phenomenon* [2] could happen, that is, for instance

$$\inf_{\substack{y \in AC([a,b]) \\ y(a) = y_a \\ y(b) = y_b}} \mathcal{F}(y) \quad < \quad \inf_{\substack{y \in Lip([a,b]) \\ y(a) = y_a \\ y(b) = y_b}} \mathcal{F}(y)$$

in this case between $AC([a, b])$ and its subset $Lip([a, b])$.

An easy example due to Manià [3] is the functional $\mathcal{F}$ associated to $f(t, y, \xi) = (y^3 - t)^2 \xi^6$ defined on $\mathcal{A} = \{y \in AC([0, 1]) \; : \; y(0) = 0 \text{ and } y(1) = 1\}$. The function $\overline{y}(t) = \sqrt[3]{t}$ belongs to $AC([0, 1])$ but not to $Lip([a, b])$ and trivially achieves the null minimum of the functional $\mathcal{F}$, while it can be shown that there exists a positive constant m such that for every Lipschitz function $y : [0, 1] \to \mathbb{R}$ with $y(0) = 0$ and $y(1) = 1$ there holds $\mathcal{F}(y) \geq m > 0$. A nice survey on the Lavrentiev phenomenon is the paper [1].

We are going to exhibit some examples of the Lavrentiev phenomenon between AC/Lip and also between Lip/C^1 where both the two *infima* above are actually *minima* achieved by some function in the relative class, with particular attention to the "regularity" properties of the Lagrangians. Up to our knowledge this kind of examples are not present in literature.

1.1. **Lavrentiev phenomenon between Lip and C^1.**
Let us define the nonnegative function $g : [-1, 1] \times \mathbb{R} \to \mathbb{R}$,

$$g(t, y) = \frac{\left(y^2 - t^2\right)^2}{\left(y^2 + t^2/\sqrt{2}\,\right)^2} \tag{1.1}$$

when $y, t \neq 0$ and $f(0, 0) = 0$ for every $t \in [-1, 1]$. Then, the behavior of g is depicted in the following figure, with particular attention to the zones where g is less (in white) or greater than one (in gray):

---

y

1

$g < 1$

0 1 0 1 1 $\dfrac{\sqrt{2-\sqrt{2}}}{2}$

2 2 $g > 1$ t

-1 1

1 1 $-\dfrac{\sqrt{2-\sqrt{2}}}{2}$

0 0 -1

1
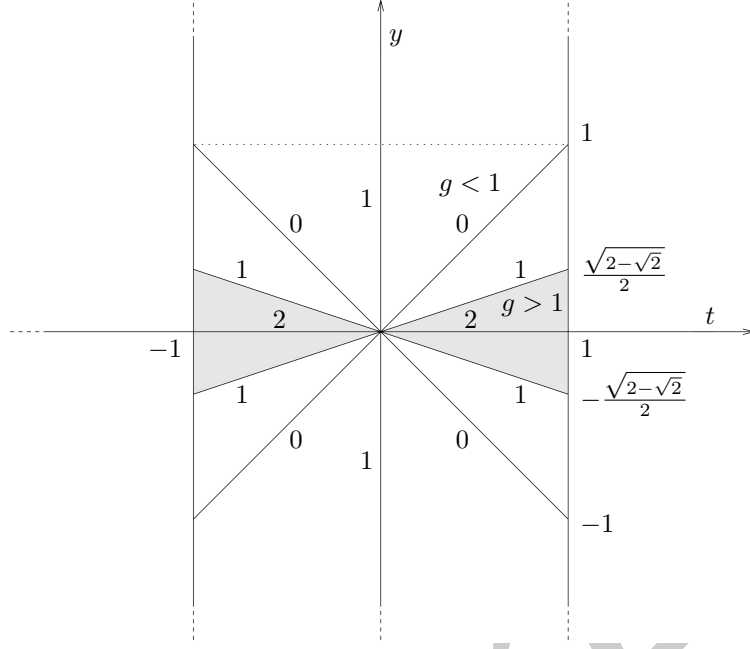
Then we consider the nonnegative Lagrangian f given by

$$f(t, y, \xi) = \big|\, g_t(t,y) + g_y(t,y)\xi \,\big| = \frac{2(2+\sqrt{2})\big|ty\big(y^2 - t^2\big)\big(t\xi - y\big)\big|}{\big(y^2 + t^2/\sqrt{2}\big)^3}$$

when $t, y \neq 0$ and $f(0,0,\xi) = 0$ for every $\xi \in \mathbb{R}$.

Notice that the function f is locally Lipschitz outside the origin and $f(0, y, \xi) = 0$ for $y \neq 0$, hence it is a Caratheodory function and actually slightly nicer than that being continuous for *every* fixed $t \in [-1, 1]$, while usually the continuity in the second and third variables is asked only for *almost every* fixed $t \in [-1, 1]$.

The associated functional $\mathcal{F}(y) = \int_{-1}^{1} f(t, y(t), y'(t))\, dt$ defined on the space of Lipschitz functions $y : [-1, 1] \to \mathbb{R}$ satisfying $y(-1) = y(1) = 1$, has zero as minimum, realized by the function $\overline{y}(t) = |t|$. Indeed, for $t \neq 0$ we have

$$f(t, \overline{y}(t), \overline{y}'(t)) = \big|g_t(t, \overline{y}(t)) + g_y(t, \overline{y}(t))\overline{y}'(t)\big| = \left|\frac{d}{dt} g(t, \overline{y}(t))\right| = \left|\frac{d}{dt} g(t, |t|)\right| = 0$$

as $g(t, |t|) = 0$ for every $t \in [-1, 1]$, by the equation 1.1.

Let now $y \in C^1([-1, 1])$ with $y(-1) = y(1) = 1$, if its minimum of y is zero or negative, let $t_0 \leq t_1$ be the first and last point where y is equal to zero. If $t_0 \neq 0$, it is then easy to see that

$$\mathcal{F}(y) \geq \int_{-1}^{t_0} \big|g_t(t, y(t)) + g_y(t, y(t))y'(t)\big|\, dt \geq \left|\int_{-1}^{t_0} \frac{d}{dt} g(t, y(t))\, dt\right| = g(t_0, y(t_0)) = g(t_0, 0) = 2.$$

The same argument applied to t_1 shows that either $t_0 = t_1 = 0$, otherwise $\mathcal{F}(y) \geq 2$. If $t_0 = t_1 = 0$ the function y has a null minimum at $t = 0$ which implies (see the figure) that there exists a value $\overline{t} \in (0, 1)$ such that $g(-\overline{t}, y(-\overline{t})), g(\overline{t}, y(\overline{t})) > 1$. Then, arguing as before

$$\mathcal{F}(y) \geq \int_{-1}^{-\overline{t}} \left|\frac{d}{dt} g(t, y(t))\right| dt + \int_{\overline{t}}^{1} \left|\frac{d}{dt} g(t, y(t))\right| dt$$

$$\geq \left|\int_{-1}^{-\overline{t}} \frac{d}{dt} g(t, y(t))\, dt\right| + \left|\int_{\overline{t}}^{1} \frac{d}{dt} g(t, y(t))\, dt\right|$$

$$= g(-\overline{t}, y(-\overline{t})) + g(\overline{t}, y(\overline{t}))$$

$$> 2.$$

If the minimum of the function y is positive, then $y(0) > 0$ and $g(0, y(0)) = 1$, hence we have

$$\mathcal{F}(y) \geq \int_{-1}^{0} \left| \frac{d}{dt} g(t, y(t)) \right| dt + \int_{0}^{1} \left| \frac{d}{dt} g(t, y(t)) \right| dt$$

$$\geq \left| \int_{-1}^{0} \frac{d}{dt} g(t, y(t)) \, dt \right| + \left| \int_{0}^{1} \frac{d}{dt} g(t, y(t)) \, dt \right|$$

$$= 2g(0, y(0))$$

$$= 2.$$

In conclusion, for every function $y \in C^1([-1, 1])$ with $y(-1) = y(1) = 1$, we have $\mathcal{F}(y) \geq 2$ and since, by direct computation, we have $\mathcal{F}(\widetilde{y}) = 2$ for the constant function $\widetilde{y} \equiv 1$, this is a (nonunique) minimizer of $\mathcal{F}$ among such functions.

Hence, this is an example of a functional with Lavrentiev phenomenon between Lip and C^1 where both the two *minima* are achieved, moreover, the Lagrangian is (slightly better than) a Caratheodory function.

Remark 1.1. By standard approximation techniques, it is easy to see that if the Lagrangian is continuous or it is Caratheodory and bounded on bounded subsets, the infima of $\mathcal{F}$ on Lip or C^1 are equal, hence we cannot further improve its regularity.

A natural question is whether it is possible to have an analogous example where the functional is finite on every Lipschitz function.

1.2. **Lavrentiev phenomenon between AC and Lip.**

Let us define the nonnegative function $g : [-1, 1] \times \mathbb{R} \to \mathbb{R}$,

$$g(t, y) = \frac{\left(y^6 - t^2 \right)^2}{\left(y^6 + t^2 / \sqrt{2} \right)^2} \tag{1.2}$$

when $t, y \neq 0$ and $f(0, 0) = 0$ for every $t \in [-1, 1]$. Then, the behavior of g is depicted in the following figure, with particular attention to the zones where g is less (in white) or greater than one (in gray):
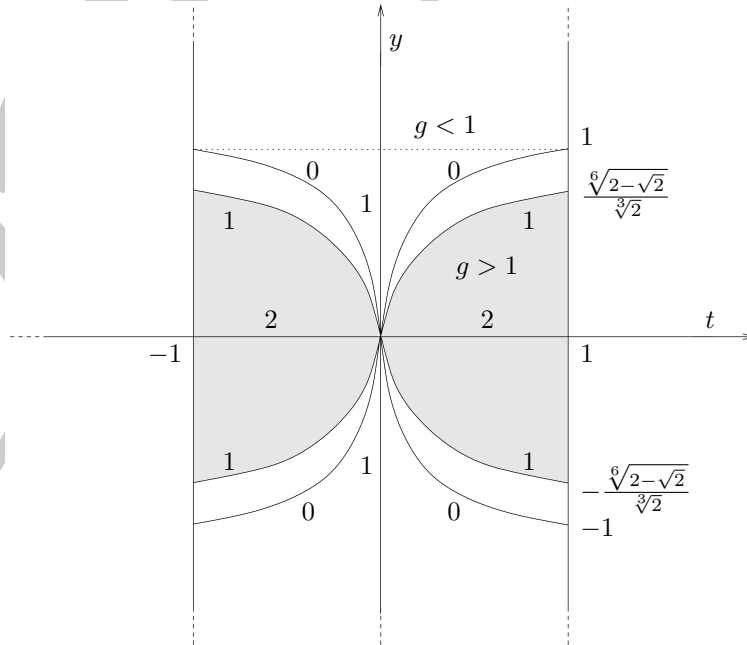
We consider the nonnegative Lagrangian f given by

$$f(t, y, \xi) = \left| g_t(t, y) + g_y(t, y)\xi \right| = \frac{2(2 + \sqrt{2}) \left| ty \left(y^6 - t^2 \right) \left(3t\xi - y \right) \right|}{\left(y^6 + t^2 / \sqrt{2} \right)^3}$$

when $y, t \neq 0$ and $f(0, 0, \xi) = 0$ for every $\xi \in \mathbb{R}$. Again, the function f is locally Lipschitz outside the origin and $f(0, y, \xi) = 0$ for $y \neq 0$, hence it is a Caratheodory function continuous for *every* fixed $t \in [-1, 1]$. Then, the argument goes as in the previous section: the associated functional $\mathcal{F}(y) = \int_{-1}^{1} f(t, y(t), y'(t)) \, dt$ defined on the space of absolutely continuous functions $y : [-1, 1] \to \mathbb{R}$ satisfying $y(-1) = y(1) = 1$, has zero as minimum, realized by the function $\overline{y}(t) = \sqrt[3]{|t|}$.

If $y \in Lip([-1, 1])$ with $y(-1) = y(1) = 1$, either it "crosses" the vertical axis with $y(0) > 0$, or $\mathcal{F}(y) \geq 2$, exactly by the same reasonings and computations of the case Lip/C^1. If $y(0) > 0$, hence $g(0, y(0)) = 1$, we have as before

$$
\begin{aligned}
\mathcal{F}(y) &\geq \int_{-1}^{0} \left| \frac{d}{dt} g(t, y(t)) \right| dt + \int_{0}^{1} \left| \frac{d}{dt} g(t, y(t)) \right| dt \\
&\geq \left| \int_{-1}^{0} \frac{d}{dt} g(t, y(t)) \, dt \right| + \left| \int_{0}^{1} \frac{d}{dt} g(t, y(t)) \, dt \right| \\
&= 2g(0, y(0)) \\
&= 2.
\end{aligned}
$$

In conclusion, for every function $y \in Lip([-1, 1])$ with $y(-1) = y(1) = 1$, we have $\mathcal{F}(y) \geq 2$ and since, by direct computation, we have $\mathcal{F}(\widetilde{y}) = 2$ for the constant function $\widetilde{y} \equiv 1$, this is a (nonunique) minimizer of $\mathcal{F}$ among such functions. Hence, also in this case we have the Lavrentiev phenomenon between Lip and C^1 and both *minima* are achieved, with a Lagrangian (slightly better than) a Caratheodory function.

Remark 1.2. In this case, as the "polynomial" example of Manià shows, the regularity or the local boundedness of the Lagrangian do not play a role (anyway, if the Lagrangian is globally bounded it is easy to see that the Lavrentiev phenomenon cannot appear, by standard approximation). It would be interesting to find an example with a continuous or even polynomial Lagrangian, in particular we do not know if in the one of Manià the minimum on the subset of the Lipschitz functions is actually achieved.

Remark 1.3. We can "glue together" the previous examples considering the Borel Lagrangian f given by

$$f(t, y, \xi) = f_1(t + 1, y, \xi)$$

for $t \in [-2, 0]$ and

$$f(t, y, \xi) = f_2(t - 1, y, \xi)$$

when $t \in [0, 2]$, where f_1 is the Lagrangian defined in the Section 1.1 and f_2 is the one in the Section 1.2. Then, considering the functional $\mathcal{F}(y) = \int_{-2}^{2} f(t, y(t), y'(t)) \, dt$ on the functions $y : [-2, 2] \to \mathbb{R}$ satisfying $y(-2) = y(2) = 1$ and being AC, Lipschitz or C^1, we see that

- the minimum on $AC([-2, 2])$ is zero, realized by the function $\overline{y}(t) = |t+1|$ for $t \in [-2, 0]$ and $\overline{y}(t) = \sqrt[3]{|t - 1|}$ for $t \in [0, 2]$,
- the minimum on $Lip([-2, 2])$ is 2, realized by the function $\widehat{y}(t) = |t + 1|$ for $t \in [-2, 0]$ and $\widehat{y}(t) = 1$ for $t \in [0, 2]$,
- the minimum on $C^1([-2, 2])$ is 4, realized by the constant function $\widetilde{y} \equiv 1$.

Hence, we have a "double" Lavrentiev phenomenon with all the three minima achieved.

2. Uniqueness of minimizers in presence of "partial" strict convexity of the Lagrangian

It is almost trivial to show that if the Lagrangian $f = f(t, y, \xi)$ is strictly convex in the pair (y, ξ) for every $t \in [a, b]$ the minimizer of the functional $\mathcal{F}(y) = \int_{a}^{b} f(t, y(t), y'(t)) \, dt$ is unique. We want to discuss when the same conclusion holds weakening such convexity hypothesis, that is, we only assume that the map $(y, \xi) \mapsto f(t, y, \xi)$ is convex for every $t \in [a, b]$ and strictly convex in only one of the variables y or ξ. This is easily seen to be true if the Lagrangian is

"separable", that is $f(t, y, \xi) = g(t, y) + h(t, \xi)$.

We will see that for smooth Lagrangians the conclusion is (almost) true for onedimensional scalar problems with fixed values at the borders.

Let us assume that $f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a continuous Lagrangian, convex in the pair (y, ξ) for every $t \in [a, b]$ and let $\mathcal{F}(y) = \int_a^b f(t, y(t), y'(t))\, dt$ the associated functional. If y, z are a pair of minimizers for $\mathcal{F}$ on the set $\mathcal{A} = \{y \in C^1([a, b]) : y(a) = y_a \text{ e } y(b) = y_b\}$, by the convexity of f we have

$$f\big(t, \lambda y(t) + (1 - \lambda)z(t), \lambda y'(t) + (1 - \lambda)z'(t)\big) \le \lambda f(t, y(t), y'(t)) + (1 - \lambda)f(t, z(t), z'(t))$$

for every $t \in [a, b]$ and $\lambda \in [0, 1]$. If for some pair (t_0, λ) this inequality is strict, then this would also hold for t in a neighborhood of t_0 (and λ fixed) and after integration on $[a, b]$, we would have the contradiction

$$\mathcal{F}(\lambda y + (1 - \lambda)z) < \lambda \mathcal{F}(y) + (1 - \lambda)\mathcal{F}(z) = m,$$

where $m = \mathcal{F}(y) = \mathcal{F}(z)$ is the minimum of $\mathcal{F}$. It follows that for every $t \in [a, b]$ and $\lambda \in [0, 1]$ we have

$$f\big(t, \lambda y(t) + (1 - \lambda)z(t), \lambda y'(t) + (1 - \lambda)z'(t)\big) = \lambda f(t, y(t), y'(t)) + (1 - \lambda)f(t, z(t), z'(t)). \quad (2.1)$$

2.1. Strict convexity in the second variable.

Let us assume that we are dealing with a scalar problem, that is, $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and we further assume that $f = f(t, y, \xi)$ is strictly convex in the variable y, for every $t \in [a, b]$ and $\xi \in \mathbb{R}$. Then, if the associated functional $\mathcal{F}(y)$ has a minimizer on the set $\mathcal{A} = \{y \in C^1([a, b]) : y(a) = y_a \text{ e } y(b) = y_b\}$, such a minimizer is unique.

Indeed, having any two minimizers $y, z \in \mathcal{A}$ the same values at the borders of $[a, b]$, either $y(t) \le z(t)$ for every $t \in [a, b]$, or there exists $c \in (a, b)$ point of positive maximum for the function $y - z$ where clearly holds $y'(c) = z'(c)$, then, by equality (2.1) with $t = c$ and $\lambda = 1/2$ we get

$$
\begin{aligned}
f\Big(c, \frac{y(c) + z(c)}{2}, y'(c)\Big) &= f\Big(c, \frac{y(c) + z(c)}{2}, \frac{y'(c) + z'(c)}{2}\Big) \\
&= \frac{f(c, y(c), y'(c)) + f(c, z(c), z'(c))}{2} \\
&= \frac{f(c, y(c), y'(c)) + f(c, z(c), y'(c))}{2}.
\end{aligned}
$$

Hence, by the strict convexity of f in the variable y, we then have that $y(c) = z(c)$ which is a contradiction, thus $y(t) \le z(t)$ for every $t \in [a, b]$. By the symmetry of y and z in this argument, we conclude that $y = z$.

- The same argument works also if f is only Caratheodory and extends to multidimensional *scalar* functionals
- We do not know if the conclusion holds also in the *vectorial case*, even onedimensional.

2.2. Strict convexity in the third variable.

Let us assume that $f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is of class C^3, convex in the pair (y, ξ) for every $t \in [a, b]$ and with the matrix $f_{\xi_i \xi_j}$ everywhere positive definite (hence, f is strictly convex in the variable ξ). Then, if the associated functional $\mathcal{F}(y) = \int_a^b f(t, y(t), y'(t))\, dt$ has a minimizer on the set $\mathcal{A} = \{y \in C^1([a, b]) : y(a) = y_a \text{ e } y(b) = y_b\}$, such a minimizer is unique.

Indeed, if y, z are two minimizers, they are of class C^3 and satisfy the Euler–Lagrange equation of $\mathcal{F}$ which is a second order ODE that can be put in normal form $y''(t) = F(t, y(t), y'(t))$ with the function F of class C^1 (hence locally Lipschitz). Then, by equality (2.1) with $t = a$ and $\lambda = 1/2$ we get

$$f\Big(a, y_a, \frac{y'(a) + z'(a)}{2}\Big) = \frac{f(a, y_a, y'(a)) + f(a, y_a, z'(a))}{2}.$$

Hence, by the strict convexity of f in the variable ξ we conclude $y'(a) = z'(a)$, the uniqueness theorem of ODEs thus implies that $y = z$.

- We do not know if the same conclusion holds when f is less regular and simply strictly convex in the variable ξ.
- We also do not know if the conclusion holds in the multidimensional case.

Putting together the two cases, we conclude that if the convex Lagrangian f of a scalar, onedimensional functional is smooth and $f_{yy} > 0$ or $f_{\xi\xi} > 0$ everywhere, then the minimization problem with fixed boundary values has a unique minimizer (if it exists).

References

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