On some notions of tangent space to a measure

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Abstract. We consider some definitions of tangent space to a Radon measure μ on $\mathbb{R}^n$ which have been given in the literature. In particular we focus our attention on a recent distributional notion of tangent vector field to a measure and we compare it to other definitions coming from Geometric Measure Theory, based on the idea of blow–up. After showing some classes of examples, we prove an estimate from above for the dimension of the tangent spaces and a rectifiability theorem which also includes the case of measures supported on sets of variable dimension.

1. Introduction

We study some different notions of tangent space to a measure μ on $\mathbb{R}^n$, with special attention to the one given by Bouchitté, Buttazzo and Seppecher in [1].
We compare this definition with others usually employed in Geometric Measure Theory, based on the fundamental idea of blow–up of the measure μ. One is the definition of approximate tangent space to a measure [8] which generalizes the notion of approximate tangent space to a rectifiable subset of $\mathbb{R}^n$; another, given by Preiss, identifies the tangent space to a measure with a set of "tangent measures" (see [7], [4]).
We will denote respectively by T_μ, P_μ and $\mathrm{Tan}(\mu)$ the tangent spaces to a measure defined in [1], [8], [7]. While the last two are very useful to study geometric properties of a measure (in particular they allow rectifiability properties of μ to be deduced from its behaviour on the balls of $\mathbb{R}^n$), the main interest of the definition proposed in [1] is in applications to variational problems. For instance, in the same paper the authors give a model for the elastic energy of low dimensional structures, involving this notion of tangent space to a measure and a related definition of Sobolev–type spaces.
Another field where these tools turn out to be useful is shape optimization. It is quite natural to work in the class of measures, since in many interesting cases the minimizing sequences do not converge to a set. See [2] for an example of this approach.
An outline of the paper is as follows.
In Section 2 we recall the different definitions and state some comparison results between them. In particular we prove the inclusion $T_\mu \subseteq P_\mu$ (see Lemma 2.4), which in some cases may be strict, and a useful decomposition lemma for the measures of $\mathrm{Tan}(\mu)$ showing that all such measures are the product of a Hausdorff measure on T_μ with an arbitrary measure on the orthogonal complement of T_μ (see Lemma 2.6). Both these results are proved by means of blow–up techniques.
In Section 3 we consider some interesting classes of examples, some of them "regular", in which T_μ coincides with P_μ (see Examples 1-3), others "pathological", like self–similar sets for which T_μ reduces to zero (see Examples 4 and 5). In this contest some relations with normal currents and rectifiable varifolds are proved.
In Section 4 we first prove an estimate from above for the dimension of T_μ, then we establish a rectifiability theorem based on the behaviour of tangent spaces T_μ. Our estimate for the dimension of T_μ says, roughly speaking, that it cannot be greater than the Hausdorff dimension of the support of μ, while the rectifiability theorem, which is deduced from a criterion of Preiss, also includes the case of a sum of Hausdorff measures of variable dimension, each one concentrated on a different rectifiable set. This is possible by a nice property of the tangent space T_μ which we want finally to point out: its dimension may depend on the point of the support of μ, differently from what happens for the approximate tangent space P_μ.
Throughout the paper, all the measures considered will be positive Radon measures on $\mathbb{R}^n$, unless otherwise stated.

2. Comparison results

Different notions of tangent space to a Radon measure have been proposed in the literature; we focus our attention mainly on the one which has been recently introduced by Bouchitté, Buttazzo and Seppecher in [1]. We slightly modify their definition by considering a larger set of admissible tangent fields: we consider the space of vector fields on $\mathbb{R}^n$ given by

$$X_\mu = \{\Phi : \mathbb{R}^n \to \mathbb{R}^n \ : \ |\Phi| \in L^1_{\mathrm{loc}}(\mu) \ , \mathrm{div}(\Phi\mu) \in \mathcal{M}\} \ ,$$

where the divergence operator is in the distributional sense and $\mathcal{M}$ denotes the set of Radon measures on $\mathbb{R}^n$. We make the following

Definition 2.1. For μ–a.e. $x \in \mathbb{R}^n$, the tangent space to μ is given by

$$T_\mu(x) = \mu - \mathrm{ess}\bigcup\{\Phi(x) : \Phi \in X_\mu\} \ .$$

For the existence and properties of the μ–essential union, we refer to [9]. We recall that, by definition, T_μ is the only μ–measurable, closed valued multifunction on $\mathbb{R}^n$ such that

i) $\Phi \in X_\mu \Longrightarrow \Phi(x) \in T_\mu(x)$ for μ–a.e. $x \in \mathbb{R}^n$;

ii) for any other μ–measurable, closed valued multifunction Σ on $\mathbb{R}^n$ satisfying i), i.e. $\Phi \in \Sigma \Longrightarrow \Phi(x) \in \Sigma$ for μ–a.e. $x \in \mathbb{R}^n$, we have $T_\mu(x) \subseteq \Sigma(x)$ for μ–a.e. $x \in \mathbb{R}^n$.

We also point out that, for every measure μ, the vector space $T_\mu(x)$ is well–defined up to a μ–negligible set, and its dimension as a linear subspace of $\mathbb{R}^n$ may depend on the point x.

We now aim to relate Definition 2.1 to the notions of tangent space to a measure given by Simon [8] and by Preiss [7], both based on the crucial idea of blow–up.

For any fixed $x^0 \in \mathbb{R}^n$ and any positive real number ρ, let us denote by $\mu_{x^0,\rho}$ the Borel measure on $\mathbb{R}^n$ defined by $\mu_{x^0,\rho}(B) = \mu(x^0 + \rho B)$ for every Borel subset B of $\mathbb{R}^n$.

Definition 2.2. Let $P_\mu = P_\mu(x^0)$ be a k–dimensional subspace of $\mathbb{R}^n$, with $k \leq n$. Then P_μ is said to be the approximate tangent space for μ at x if there exists a positive constant $\theta = \theta(x^0)$, which is called the multiplicity at x^0, such that, when ρ converges to zero

$$\frac{1}{\rho^k}\mu_{x^0,\rho} \ \rightharpoonup \ \theta(x^0)\mathcal{H}^k \llcorner P_\mu(x^0) \tag{2.1}$$

in the vague topology on measures.

For a general Radon measure μ, the existence μ–almost everywhere of $P_\mu = P_\mu(x^0)$ is not guaranteed. We have the following fundamental characterization of the measures admitting a k–dimensional approximate tangent space μ–almost everywhere (see [8], Theorem 11.8).

Theorem 2.3. *The measure μ has a k–dimensional approximate tangent space $P_\mu(x)$ at μ–a.e. $x \in \mathbb{R}^n$ if and only if $\mu = \theta\mathcal{H}^k \llcorner M$, where M is a countably k–rectifiable subset of $\mathbb{R}^n$ and θ is a nonnegative $\mathcal{H}^k$–measurable function on $\mathbb{R}^n$.*

A more general rectifiability criterion based on tangent spaces T_μ will be proved in Section 4. For the moment we state a comparison result between definitions 2.1 and 2.2, which is proved analogously to Lemma 5.2 of [1].

Lemma 2.4. *Suppose that $P_\mu = P_\mu(x)$ associates to μ–a.e. point x the k–dimensional approximate tangent space $P_\mu(x)$ to μ at x. Then*

$$T_\mu(x) \subseteq P_\mu(x) \qquad \text{for } \mu\text{–a.e. } x \in \mathbb{R}^n \ . \tag{2.2}$$

Proof. For the minimality property ii) of T_μ, it is sufficient to prove that, for every tangent field Φ, we have $\Phi(x) \in P_\mu(x)$ for μ–a.e. $x \in \mathbb{R}^n$.
Let $\Phi \in X_\mu$; we observe that $P_\mu(x)^\perp$ is spanned by

$$\left\{ \int_{P_\mu(x)} \nabla\psi(y) \, d\mathcal{H}^k(y) \quad : \quad \psi \in \mathcal{D}(B) \right\} ,$$

where B is the unit ball of $\mathbb{R}^n$. Thus, we only need to prove that, for μ–a.e. $x \in \mathbb{R}^n$,

$$\Phi(x) \cdot \int_{P_\mu(x)} \nabla\psi(y) \, d\mathcal{H}^k(y) = 0 \qquad \forall \psi \in \mathcal{D}(B) \ .$$

We can suppose that x is a Lebesgue point for Φ with respect to μ and that both μ and the measure $m = |\operatorname{div}(\Phi\mu)|$ have finite k–dimensional density at x (by Theorem 2.3, $\mu = \theta\mathcal{H}^k \llcorner M$). Let us set $\psi_\rho(y) = \psi\left(\frac{y-x}{\rho}\right)$ and $M = \max_B |\psi|$; then

$$\begin{aligned}
\theta(x) \left| \Phi(x) \cdot \int_{P_\mu(x)} \nabla\psi(y) \, d\mathcal{H}^k(y) \right| &= \lim_{\rho\to 0} \frac{1}{\rho^k} \left| \Phi(x) \cdot \int_{B_\rho(x)} \nabla\psi\left(\frac{y-x}{\rho}\right) \, d\mu(y) \right| \\
&= \lim_{\rho\to 0} \frac{\rho}{\rho^k} \left| \int_{B_\rho(x)} \Phi(y) \cdot \nabla\psi_\rho(y) \, d\mu(y) \right| \\
&= \lim_{\rho\to 0} \frac{\rho}{\rho^k} \left| \int_{B_\rho(x)} \psi_\rho(y) \, d(\operatorname{div}(\Phi\mu))(y) \right| \\
&\leq \lim_{\rho\to 0} \rho M \frac{m(B_\rho(x))}{\rho^k}
\end{aligned}$$

and the last limit is zero because the k–dimensional density of m at x is finite. ■

Next we want to show the strict relation between the linear space T_μ and the quite general concept of tangent measure for μ at a point x^0, studied by Preiss in [7] (see also [4]).
He considers all the possible limits of sequences given by blowing up μ around x^0. More precisely, he calls a measure ν on $\mathbb{R}^n$ a tangent measure for μ at x^0 if ν is nonzero and if there exist two sequences of positive real numbers λ_i and ρ_i (with ρ_i converging to zero) such that $\frac{\mu_{x^0,\rho_i}}{\lambda_i}$ converge to ν in the vague topology on measures.

Adopting this definition, the set of tangent measures for μ at x is shown to be non–empty for μ–a.e. x (see [7], Theorem 2.5). Therefore, we prefer to take as normalization constants $\lambda_i = \mu(B_{\rho_i}(x^0))$, instead of arbitrary positive numbers. In this way the mass on the unit ball B of the measure $\frac{\mu_{x^0,\rho_i}}{\mu(B_{\rho_i}(x^0))}$ is equal to one for every i, assuring the existence of a weakly convergent subsequence.

Definition 2.5. We say that a measure ν on the unit ball B of $\mathbb{R}^n$ belongs to the set $\mathrm{Tan}(\mu, x)$ of tangent measures for μ at the point x^0, if there exists a sequence of positive real numbers ρ_i converging to zero such that, when $i \to \infty$,

$$\frac{\mu_{x^0,\rho_i}}{\mu(B_{\rho_i}(x^0))} \rightharpoonup \nu \tag{2.3}$$

as measures.

We can prove a structure result for measures belonging to $\mathrm{Tan}(\mu, x)$, showing that all such measures are of product type.

Lemma 2.6. *For μ–a.e. $x \in \mathbb{R}^n$, any measure $\nu \in \mathrm{Tan}(\mu, x)$ is of the form*

$$\left(\mathcal{H}^{k(x)}_{T_\mu(x)} \times \sigma\right) \llcorner B \ , \tag{2.4}$$

where $k(x)$ is the dimension of $T_\mu(x)$, where $\mathcal{H}^{k(x)}_{T_\mu(x)}$ is the $k(x)$–dimensional Hausdorff measure on $T_\mu(x)$, and σ is a measure of locally finite mass on $T_\mu(x)^\perp$.

Proof. Let $L_\mu(x)$ be the set of vectors $v \in \mathbb{R}^n$ such that every measure $\nu \in \mathrm{Tan}(\mu, x)$ can be decomposed as a product $\left(\mathcal{H}^1_{\langle v\rangle} \times \sigma\right) \llcorner B$, where $\mathcal{H}^1_{\langle v\rangle}$ is the one–dimensional Hausdorff measure on the line $\langle v\rangle$ and σ is a measure on $v^\perp$.
The conclusion of the lemma is equivalent to $T_\mu(x) \subseteq L_\mu(x)$ for μ–a.e. x. So, by using the minimality property ii) of T_μ, it is enough to show that every tangent field Φ satisfies μ–almost everywhere $\Phi(x) \in L_\mu(x)$.
Let $\Phi \in X_\mu$, and let us suppose that $D_{\Phi(x)}\nu = 0$ for every measure $\nu \in \mathrm{Tan}(\mu, x)$, where $D_{\Phi(x)}\nu$ is intended in the distributional sense, i.e.

$$\langle D_{\Phi(x)}\nu, \psi\rangle = \int_B \frac{\partial\psi(y)}{\partial\Phi(x)}\, d\nu(y) \qquad \forall \psi \in \mathcal{D}(B) \ ;$$

then, by standard regularization it can be easily checked that $\Phi(x) \in L_\mu(x)$.
It remains to prove that for $\Phi \in X_\mu$ and $\nu \in \mathrm{Tan}(\mu, x)$, at μ–a.e. x we have $D_{\Phi(x)}\nu = 0$, that is,

$$\Phi(x) \cdot \int_B \nabla\psi(y)\, d\nu(y) = 0 \qquad \forall \psi \in \mathcal{D}(B) \ .$$

Such equality can be performed by the same argument used in the proof of Lemma 2.4. ■

3. Examples

Example 1 (*Lipschitz manifolds*).

We improve a result of [1], by showing that if μ is equal to $\mathcal{H}^k$ on a k–dimensional manifold M, then T_μ coincides with the classical tangent space, even if M is Lipschitz rather than of class C^2. We denote by $T_M(x)$ the classical tangent space to M at a point $x \in M$, which exists $\mathcal{H}^k$–almost everywhere on M by Rademacher's theorem.

Theorem 3.1. *Let $\mu = \mathcal{H}^k \llcorner M$, where $k \leq n$ and M is a k–dimensional Lipschitz manifold in $\mathbb{R}^n$. Then $T_\mu(x) = T_M(x)$ for μ–a.e. $x \in M$.*

Proof. Since in this case the approximate tangent space to μ coincides with T_M, by Lemma 2.4 we have immediately $T_\mu(x) \subseteq T_M(x)$ for μ–a.e. $x \in M$.
In order to prove the opposite inclusion we observe that T_μ is local on open sets of $\mathbb{R}^n$, in other words, if $A \subseteq \mathbb{R}^n$ is open and we have measures μ_1 and μ_2 with $\mu_1 \llcorner A = \mu_2 \llcorner A$, then $T_{\mu_1}(x) = T_{\mu_2}(x)$ μ–a.e. in A. Therefore, we can suppose without loss of generality that M is the graph of a Lipschitz function u from a regular open set $\Omega \subseteq \mathbb{R}^k$ to $\mathbb{R}^{n-k}$. Then, it is enough to prove that for any fixed $v \in \mathbb{R}^k$, the vector field $\Phi(x) = \frac{(v, \partial_v u(x))}{J(x)}$ belongs to X_μ, where $J(x)$ denotes the Jacobian of the map $x \mapsto (x, u(x))$.
We show that $\mathrm{div}(\Phi\mu)$ is a Radon measure. For any test function $\psi \in \mathcal{D}(\mathbb{R}^n)$ we have

$$\begin{aligned}
\int \Phi \cdot \nabla\psi \, d\mu &= \int_M \Phi \cdot \nabla\psi \, d\mathcal{H}^k \\
&= \int_\Omega (v, \partial_v u(x)) \cdot \nabla\psi(x, u(x)) \, dx \\
&= \int_\Omega \partial_v \big[\psi(x, u(x))\big] \, dx \\
&= \int_{\partial\Omega} \psi(x, u(x))\, v \cdot n \, d\mathcal{H}^{k-1}
\end{aligned}$$

where n is the external unit normal vector to $\partial\Omega$. ■

Example 2 (*relations with currents*).

We first show that there exists a natural one–to–one correspondence between the set of normal 1–currents in $\mathbb{R}^n$ (see for instance [8, Chapter 6], [5]) and the set of pairs (μ, Φ), where μ is a Radon measure on $\mathbb{R}^n$ and Φ is a unit tangent vector field to μ.

Theorem 3.2. *If T is a normal 1–current with total variation μ_T and orientation τ, the vector field τ is tangent to the measure μ_T, that is*

$$\tau \in X_{\mu_T} \ .$$

Conversely, for any given Radon measure μ on $\mathbb{R}^n$ which admits a tangent field $\Phi \in X_\mu$ with $|\Phi| = 1$ μ–a.e., we obtain a normal 1–current by setting

$$T(\omega) = \int_{\mathbb{R}^n} \langle \omega(x), \Phi(x) \rangle \, d\mu_T(x) \qquad \forall \omega \in \mathcal{D}_1(\mathbb{R}^n) \ . \tag{3.1}$$

PROOF. Since ∂T has locally finite mass, we have

$$\int_{\mathbb{R}^n} \tau \cdot \nabla\psi \; d\mu_T = \int_{\mathbb{R}^n} \langle d\psi, \tau \rangle \; d\mu_T = T(d\psi) = \partial T(\psi) \; ,$$

which shows that $\tau \in X_{\mu_T}$.
Conversely, as Φ belongs to X_μ, by the same argument the 1–current T defined by (3.1) is immediately seen to be normal. ■

In the case of k–dimensional currents there exists a natural way to produce, starting from a normal and *simple* k–current T, a pair $(\mu, \{\Phi_1, \ldots, \Phi_{\binom{n}{k-1}}\})$, where μ is a Radon measure (which, as one expects, will be the total variation of T) and $\{\Phi_1, \ldots, \Phi_{\binom{n}{k-1}}\}$ is a set of tangent fields to μ.
Let c be the contraction operator from $\bigwedge^{k-1}(\mathbb{R}^n) \times \bigwedge_k(\mathbb{R}^n)$ into $\mathbb{R}^n$ defined by

$$c(\omega, v_1 \wedge \ldots \wedge v_k) = \sum_{i=1}^{k} (-1)^{i-1} \langle \omega, v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_k \rangle v_i \; .$$

and denote by $dx^{i_1 \cdots i_{k-1}}$ the $(k-1)$–form $dx^{i_1} \wedge \ldots \wedge dx^{i_{k-1}}$.

Theorem 3.3. *If T is a normal and simple k–current with total variation μ_T and orientation $\tau_1 \wedge \ldots \wedge \tau_k$, we have*

$$c(dx^{i_1 \cdots i_{k-1}}, \tau_1 \wedge \ldots \wedge \tau_k) \in X_{\mu_T}$$

for every multi–index $(i_1, \ldots, i_{k-1})$ with $1 \le i_1 < \ldots < i_{k-1} \le n$.

PROOF. Let ψ be a test function in $\mathcal{D}(\mathbb{R}^n)$. Then

$$\begin{aligned}\int_{\mathbb{R}^n} c(dx^{i_1 \cdots i_{k-1}}, \tau_1 \wedge \ldots \wedge \tau_k) \cdot \nabla\psi \; d\mu_T &= \int_{\mathbb{R}^n} \langle d\psi \wedge dx^{i_1 \cdots i_{k-1}}, \tau_1 \wedge \ldots \wedge \tau_k \rangle \; d\mu_T \\ &= \int_{\mathbb{R}^n} \langle d(\psi dx^{i_1 \cdots i_{k-1}}), \tau_1 \wedge \ldots \wedge \tau_k \rangle \; d\mu_T \\ &= \partial T(\psi dx^{i_1 \cdots i_{k-1}})\end{aligned}$$

which means, since T has locally finite mass, that the field $c(dx^{i_1 \cdots i_{k-1}}, \tau_1 \wedge \ldots \wedge \tau_k)$ belongs to X_{μ_T}. ■

Remark 3.4. For $k > 1$ we cannot repeat the same argument as the case $k = 1$ to produce a normal k–current, starting from a Radon measure μ and a set of k linearly independent vector fields $\Phi_1, \ldots, \Phi_k$ in X_μ. Indeed, because of the possible non–smoothness of the fields Φ_i, we cannot integrate by parts in order to show that the boundary of the k–current

$$T(\omega) = \int_{\mathbb{R}^n} \langle \omega, \Phi_1 \wedge \ldots \wedge \Phi_k \rangle \; d\mu_T \qquad \forall \omega \in \mathcal{D}_k(\mathbb{R}^n)$$

has locally finite mass.

Example 3 (*rectifiable k–varifolds*).

For an appropriate class of rectifiable k–varifolds V (see [8]) the approximate tangent space to the weight measure μ_V coincides with T_{μ_V}.

Theorem 3.5. *Let $\mu_V = \theta \mathcal{H}^k \llcorner M$ be the weight measure of a rectifiable k–varifold $V = V(M, \theta)$, and suppose that the first variation δV is a Radon measure.*
Then $T_{\mu_V} = P_{\mu_V}$ μ_V–a.e.

PROOF. By (2.2) we only need to prove the inclusion $P_{\mu_V} \subseteq T_{\mu_V}$ for μ_V–a.e. $x \in M$. If e_i are the coordinate fields, it is sufficient to show that $e_i^M \in X_{\mu_V}$ for $i = 1, \dots, n$, where e_i^M denotes the projection of e_i on P_{μ_V}. For every test function $\psi \in \mathcal{D}(\mathbb{R}^n)$, if $\nabla^M \psi$ is the projection of $\nabla \psi$ on P_{μ_V}, we have

$$\int_{\mathbb{R}^n} e_i^M \cdot \nabla \psi \, d\mu_V = \int_{\mathbb{R}^n} \nabla_i^M \psi \, d\mu_V = \langle (\delta V)_i, \psi \rangle \ .$$

Then, since the measure δV is locally finite, we get $e_i^M \in X_{\mu_V}$ for every $i = 1, \dots, n$. ■

Example 4 (*Cantor–like sets*).

We consider now the case of measures μ concentrated on Cantor–like sets. The interest in such examples arises from the fact that the study of the corresponding T_μ throws some light on possible pathological behaviours of tangent spaces.

i) For $n = 1$, we take $\mu = \mathcal{H}^\alpha \llcorner C$, where $0 < \alpha \leq 1$ and C is an α–dimensional Cantor set, that is, a closed subset of $[0, 1]$ with a dense complement and $\mathcal{H}^\alpha(C) \in (0, +\infty)$.
Then $T_\mu(x) = \{0\}$ for μ–a.e. x.
Indeed, one can see that every vector field $\Phi \in X_\mu$ must be identically zero. In fact, since by definition the distributional derivative of $\Phi\mu$ is a Radon measure, we have $\Phi\mu = f\mathcal{H}^1$, where f is a function with bounded variation. Thus, in the case $\alpha < 1$ we infer that $\Phi = 0$ μ–a.e. from the fact that μ is singular with respect to $\mathcal{H}^1$. In the case $\alpha = 1$ we get the same conclusion by observing that f must be absolutely continuous and it vanishes on the dense set $[0, 1] \setminus C$, being equal to $\Phi\chi_C$.
As an immediate consequence we get the following corollaries.

Corollary 3.6. *The tangent space T_μ is not local on Borel subsets of $\mathbb{R}^n$.*

Corollary 3.7. *The inclusion (2.2) may be strict.*

In fact, the measures $\mathcal{H}^1$ and $\mathcal{H}^1 \llcorner C$ coincide on the Borel set C but their tangent spaces are different almost everywhere on C.

ii) For $n = 2$, let us consider a product measure $\mu = \mu_1 \times \mu_2$ on $\mathbb{R}^2$, where μ_1 is Lebesgue measure on $(0, 1)$ and μ_2 is a measure as in the previous example, i.e. μ_2 is $\mathcal{H}^1 \llcorner C$, with C a 1–dimensional Cantor set. One can easily check that $T_\mu(x) = T_{\mu_1}(x) \times T_{\mu_2}(x)$ whenever $\mu = \mu_1 \times \mu_2$. Therefore in this case $T_\mu(x) = \mathbb{R}$ for μ–a.e. x.

Corollary 3.8. *If μ is concentrated on a set S and if the dimension of $T_\mu(x)$ is a positive constant k for μ–a.e. $x \in S$, the Hausdorff dimension of S may be strictly larger than k, so that in general we have no relation between the measures μ and $\mathcal{H}^k \llcorner S$.*

Example 5 (*the Koch curve*).
Let $K \subset \mathbb{R}^2$ be the Koch curve with Hausdorff dimension $\alpha = \log 4/\log 3$.

Theorem 3.9. *If $\mu = \mathcal{H}^\alpha \llcorner K$, then $T_\mu(x) = \{0\}$ for μ–a.e. x.*

Proof. For a fixed $x^0 \in K$, by the self–similarity of the Koch curve (see [3] for further details), it is easily checked that there is a measure ν in $\mathrm{Tan}(\mu, x^0)$, which, up to a multiplicative factor, coincides with $\mathcal{H}^\alpha$ restricted to the intersection of the unit ball with at most two copies of K. If the tangent space $T_\mu(x^0)$ was one–dimensional, then, by Lemma 2.6, the measure ν would be a product $\mathcal{H}^1 \times \sigma$ and this is impossible. On the other hand, the dimension of $T_\mu(x^0)$ cannot be two, by an upper estimate for the dimension of the tangent space that we are going to prove in the next section, Theorem 4.2. ∎

4. An estimate for the dimension and a rectifiability theorem

In the first part of the section we show some estimates from above for the dimension of T_μ. Then we deduce, from a criterion proved by Preiss in [7], a rectifiability theorem based on tangent spaces T_μ. Our result is in some sense more general than Theorem 2.3, since it also includes the case of a measure μ given by a sum of Hausdorff measures of different dimensions, each one concentrated on a rectifiable set.
We adopt the usual notation $\theta^*_\alpha(\mu, x)$ and $\theta_{*\alpha}(\mu, x)$ for the α–dimensional upper and lower densities of the measure μ at the point x, whose definition can be found for instance in [8, pag.10].

Lemma 4.1. *Let α be a real number, $0 \le \alpha \le n$, and let $E \subseteq \mathbb{R}^n$ be a set where*

$$\theta^*_\alpha(\mu, x) = +\infty \ .$$

Then, for μ–a.e. $x \in E$, the dimension of $T_\mu(x)$ cannot be larger than the integer part of α.

Proof. We claim that the following condition is satisfied when $x \in E$ and $t \in (0, 1)$:

$$\limsup_{\rho \to 0^+} \frac{\mu(B_{t\rho}(x))}{\mu(B_\rho(x))} \ge t^\alpha \ . \tag{4.1}$$

Suppose by contradiction that there exist $x \in E$, $t \in (0, 1)$ and $\overline{\rho} > 0$ such that

$$\mu(B_{t\rho}(x)) \le t^\alpha \mu(B_\rho(x)) \qquad \forall \rho \in (0, \overline{\rho}] \ .$$

Then in particular

$$\mu(B_{t^n\overline{\rho}}(x)) \le t^{n\alpha} \mu(B_{\overline{\rho}}(x)) \qquad \forall n \in \mathbb{N} \ .$$

If for every $\rho \in (0, \overline{\rho}]$ we choose an integer n such that ρ belongs to the interval $(t^{n+1}\overline{\rho}, t^n\overline{\rho}]$, we get

$$\mu(B_\rho(x)) \leq \mu(B_{t^n\overline{\rho}}(x)) \leq t^{n\alpha}\mu(B_{\overline{\rho}}(x)) \leq [(t\overline{\rho})^{-\alpha}\mu(B_{\overline{\rho}}(x))]\rho^\alpha \ .$$

But, since $[(t\overline{\rho})^{-\alpha}\mu(B_{\overline{\rho}}(x))]$ does not depend on the choice of $\rho \in (0, \overline{\rho}]$, the above inequality contradicts the hypothesis on $\theta^*_\alpha(\mu, x)$, and therefore (4.1) holds.

Let us denote by Q^n_ρ an open cube of $\mathbb{R}^n$ centered at the origin with sides of semi–length ρ, either orthogonal or parallel to $T_\mu(x)$. Then there exists a constant $c(n) < 1$ such that $\overline{Q^n_{c(n)}} \subseteq B$.

By inequality (4.1), for μ–a.e. $x \in E$ and for every $t \in (0,1)$, we can choose a sequence of positive numbers ρ_i tending to zero such that the corresponding blown–up measures $\frac{\mu_{x,\rho_i}}{\mu(B_{\rho_i}(x))}$ converge to a measure $\nu \in \mathrm{Tan}(\mu, x)$, satisfying both the inequality

$$\nu\left(\overline{B_t}\right) \geq t^\alpha$$

and the product decomposition (2.4).

Let $d(x)$ be the dimension of $T_\mu(x)$. Since $\nu(Q^n_{c(n)}) \leq \nu(B) \leq 1$, if we take into account the representation of ν, we get

$$(4.2) \qquad [2c(n)]^{d(x)}\sigma(Q^{n-d(x)}_{c(n)}) \leq 1 \ .$$

On the other hand, since $\overline{B_t} \subseteq \overline{Q^n_t}$,

$$(4.3) \qquad (2t)^{d(x)}\sigma(Q^{n-d(x)}_t) = \nu\left(\overline{Q^n_t}\right) \geq \nu\left(\overline{B_t}\right) \geq t^\alpha \ .$$

Finally, if we choose $t < c(n)$, putting together (4.2) and (4.3), we get

$$t^{d(x)-\alpha} \geq [c(n)]^{d(x)} \ ,$$

which implies $d(x) \leq \alpha$, because t can be chosen arbitrarily small. Since $d(x)$ is an integer, we have obtained the result. ■

The following theorem is a straightforward consequence of Lemma 4.1.

Theorem 4.2. *Let E be any subset of $\mathbb{R}^n$. Then*

$$(4.4) \qquad \dim T_\mu(x) \leq \mathcal{H}\text{-}\dim(E) \qquad \text{for } \mu\text{–a.e. } x \in E \ .$$

Proof. If $\alpha > \mathcal{H}\text{-}\dim(E)$, then $\theta^*_\alpha(\mu, x) = +\infty$ for μ–a.e. $x \in E$. Therefore Lemma 4.1 immediately yields the estimate (4.4). ■

A nice property of tangent spaces in dimension one is given in the next lemma.

Lemma 4.3. *Let μ be a positive Radon measure on $\mathbb{R}$, and let μ^s be the singular part of μ with respect to the Lebesgue measure. Then $T_\mu(x) = \{0\}$ for μ^s–a.e. x.*

Proof. Let $\mu = \mu^a + \mu^s$, where μ^a is the absolutely continuous part of μ, and let g be the density of μ^a with respect to $\mathcal{H}^1$. Let Φ be a tangent field. Since the distributional derivative of $\Phi\mu$ is a Radon measure, there exists a function f with bounded variation such that $\Phi\mu = f\mathcal{H}^1$ (recall Example 4 of Section 3). Then Φ must be zero μ^s–almost everywhere, because $\Phi\mu^s = (f - \Phi g)\mathcal{H}^1$. ■

Remark 4.4. The analogous statement for a measure μ on $\mathbb{R}^n$ would be

$$\dim T_\mu(x) < n \qquad \text{for } \mu^s\text{–a.e. } x \ ,$$

where μ^s now denotes the singular part of μ with respect to $\mathcal{H}^n$. It is not clear whether this conjecture is true or not, because of the remarkable difference between the one–dimensional and the n–dimensional case (see also Example 2 of Section 3). We point out that this difference, essentially due to the presence of the divergence operator in Definition 2.1, rules out the use of slicing type techniques in treating the spaces T_μ.
Moreover, also the blow–up techniques, which are often useful, may fail, since the set $\mathrm{Tan}(\mu, x)$ can be sometimes very wild; O'Neil [6] has recently constructed a measure μ, for which, at μ–almost every point x, every measure belongs to $\mathrm{Tan}(\mu, x)$.

We now pass to the rectifiability result.
We denote by $\mathcal{R}_k$ the class of all measures of type $\theta\mathcal{H}^k \llcorner E$, with k an integer, E a countably $\mathcal{H}^k$–rectifiable subset of $\mathbb{R}^n$, and θ a positive function in $\mathcal{H}^k_{\mathrm{loc}}(E)$.

Theorem 4.5. *Let k be an integer, $k \le n$, and let E_k be the set of the points $x \in \mathbb{R}^n$ such that*

i) the dimension of $T_\mu(x)$ is equal to k;

ii) $\theta_{*k}(\mu, x)$ is positive and finite;

iii) the 'doubling condition' holds at x, i.e.

$$\limsup_{\rho \to 0^+} \frac{\mu(B_{2\rho}(x))}{\mu(B_\rho(x))} < +\infty \ .$$

Then the measure $\mu \llcorner E_k$ belongs to the class $\mathcal{R}_k$.

The proof of this theorem is based on the following rectifiability criterion of Preiss [7].

Theorem 4.6. *Let σ be a measure on $\mathbb{R}^n$ satisfying, for σ–a.e. x, these two conditions:*

A) if we set $\tau = 1 - 2^{-k-6}$ and

$$E_\rho(x) = \left\{ z \in B_\rho(x) \text{ s.t. } \exists\, s \in (0, \rho) : \frac{\sigma(B_s(z))}{s^k} \le \tau \frac{\sigma(B_\rho(x))}{\rho^k} \right\} \ ,$$

we have

$$\liminf_{\rho \to 0^+} \frac{\sigma(E_\rho(x))}{\sigma(B_\rho(x))} = 0 \ ;$$

B) if we set $\kappa = 8^{-k-9}k^{-4}$, we let $G_{n,k}$ denote the class of k–dimensional linear subspaces of $\mathbb{R}^n$, and

$$F_\rho(x) = \sup_{V \in G_{n,k}} \left\{ \inf_{z \in (x+V)\cap B_\rho(x)} \frac{\sigma(B_{\kappa\rho}(z))}{\sigma(B_\rho(x))} \right\} ,$$

we have

$$\liminf_{\rho \to 0^+} F_\rho(x) > 0 .$$

Then $\sigma \in \mathcal{R}_k$.

We are now in a position to prove Theorem 4.5.

Proof. We set $\sigma = \mu \llcorner E_k$, and we show that, under the hypotheses *i*), *ii*), and *iii*), both conditions A and B are satisfied σ–almost everywhere. It is enough to verify that such conditions hold for every density point of the set E_k with respect to μ. Since at such points the density of σ with respect to μ is equal to one, we can replace σ with μ.
Let us begin by checking condition A. We define, for positive constants ϵ and c, the sets

$$E(\epsilon, c) = \{z : \frac{\mu(B_r(z))}{r^k} \geq c \quad \forall r \in (0, \epsilon)\}$$

and

$$\widetilde{E}(\epsilon, c) = E(\epsilon, \tau c) \setminus \bigcup_{n=1}^{\infty} E\left(\frac{\epsilon}{n}, c\right) .$$

By *ii*), for μ–a.e. $x \in E_k$, we can choose $t > 0$ such that $\tau t < \theta_{*k}(\mu, x) < t$. Then, there exists $s > 0$ such that x belongs to $\widetilde{E}(s,t)$. We can also suppose that x is a density point of $\widetilde{E}(s,t)$, or, equivalently,

$$\lim_{\rho \to 0} \frac{\mu\left(\widetilde{E}^c(s,t) \cap B_\rho(x)\right)}{\mu(B_\rho(x))} = 0 .$$

It is easy to see that this property implies condition A, and consequently A is satisfied μ–almost everywhere.
We now verify condition B. First we observe that it is enough to check it at any point x where the decomposition property (2.4) holds. For every $\rho > 0$, let $z_\rho \in [x + T_\mu(x)] \cap B_\rho(x)$. We claim that

$$\liminf_{\rho \to 0^+} \frac{\mu(B_{\kappa\rho}(z_\rho))}{\mu(B_\rho(x))} > 0 . \tag{4.5}$$

Indeed, we can choose a sequence ρ_i of positive numbers tending to zero such that the following three conditions hold:

$$\lim_{i \to +\infty} \frac{\mu(B_{\kappa\rho_i}(z_{\rho_i}))}{\mu(B_{\rho_i}(x))} = \liminf_{\rho \to 0^+} \frac{\mu(B_{\kappa\rho}(z_\rho))}{\mu(B_\rho(x))} ;$$

$$\lim_{i\to+\infty} \frac{\mu_{x,\rho_i}}{\mu(B_{\rho_i}(x))} = \nu \in \mathrm{Tan}(\mu,x) \ ;$$

$$\lim_{i\to+\infty} \frac{z_{\rho_i} - x}{\rho_i} = z \in \overline{B} \cap T_\mu(x) \ .$$

Then by the above conditions and by the semicontinuity properties of weak convergence of measures, we get the estimate

$$\liminf_{\rho\to 0^+} \frac{\mu(B_{\kappa\rho}(z_\rho))}{\mu(B_\rho(x))} \geq \nu(B_\kappa(z) \cap B) \ . \tag{4.6}$$

Now, the doubling condition *iii*) implies that the origin of $\mathbb{R}^n$ belongs to the support of every measure $\nu \in \mathrm{Tan}(\mu,x)$. Then, if we choose a point $z' \in B \cap T_\mu(x)$ with $B_{\frac{\kappa}{2}}(z') \subseteq B_\kappa(z)$, we have

$$\nu(B_\kappa(z) \cap B) \geq \nu(B_{\frac{\kappa}{2}}(z') \cap B) = \nu((-z' + B_{\frac{\kappa}{2}}(z')) \cap B) > 0 \ . \tag{4.7}$$

Thus, combining (4.6) and (4.7), we get (4.5). Finally, observing that $T_\mu(x) \in G_{n,k}$ for μ–a.e. $x \in E_k$, we deduce that condition B is satisfied μ–almost everywhere. ■

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