Stability for the mailing problem

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ABSTRACT. We prove that optimal traffic plans for the mailing problem in $\mathbb{R}^d$ are stable with respect to variations of the given coupling, above the critical exponent $\alpha = 1 - 1/d$, thus solving an open problem stated in the book *Optimal transportation networks*, by Bernot, Caselles and Morel. We apply our novel result to study some regularity properties of the minimizers of the mailing problem, showing that only finitely many connected components of an optimal traffic plan meet together at any branching point.

ABSTRACT. Nous prouvons que les plans de trafic optimaux pour le problème d'acheminement dans $\mathbb{R}^d$ sont stables par rapport aux variations du couplage donné, au-dessus de l'exposant critique $\alpha = 1 - 1/d$, résolvant ainsi un problème ouvert posé dans le livre *Optimal transportation networks* de Bernot, Caselles et Morel. Nous appliquons notre nouveau résultat à l'étude de quelques propriétés de régularité des minimiseurs du problème d'acheminement, en montrant que seul un nombre fini de composantes connexes d'un plan de trafic optimal se rencontrent en n'importe quel point de branchement.

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1. INTRODUCTION

The optimal branched transportation is a variant of the Monge-Kantorovich optimal transportation of mass, in which the mass particles are assumed to interact (rather than travelling independently) while moving from a source to a target distribution. In particular there is a gain in the cost of the transportation whenever some mass is transported in a grouped way. A consequence of this assumption is that the particles' paths form a one-dimensional network which develops branched structures.

The interest in branched transportation arises from the observation of common structures in many natural supply-demand systems, such as the nerves of a leaf, river basins, or the nervous, the bronchial, and the cardiovascular system. It has been also used to model several human-designed systems, including power supply, urban planning, and irrigation.

In order to translate in mathematical terms the convenience of grouping mass during the transportation, one considers a cost functional obtained integrating along the network created by the particles' trajectories a subadditive function of the intensity of the flow. The first *discrete* model was introduced by Gilbert in [Gil67]. Then it was extended by Xia to a *continuous* framework adopting an Eulerian formulation (describing the particles' flow) which uses Radon vector-valued measures, or, equivalently, 1-dimensional currents, called in this framework *traffic paths*, (see [Xia03]). At about the same time a Lagrangian formulation (describing the particles' trajectories) of the continuous version of the problem was proposed by Maddalena, Morel, and Solimini [MSM03], using the notion of *traffic plans*, i.e. measures on the set of Lipschitz curves.

A great interest has been devoted to branched transportation problems in the last years, with results concerning existence [Xia03, MSM03, BCM05, BCM08, BBS11, Peg17, CDRMS17], regularity [Xia04, BBS06, DS07b, DS07a, MS10, Xia11, BS14, BW18] and strategies to compute minimizers [OS11, CMF16, BLS15, MOV16, BOO16, MM16b, MM16a, Gol17, CDRM18, MMT17].

In the present paper we restrict our attention to the so called *mailing problem* (also known as *who goes where problem*): a version of the branched transportation problem in which not

only the initial and target mass distributions are given, but also a coupling between the two measures: in other words one knows a priori where each mass particle should be moved, and the only unknown is an optimal transportation network realizing such coupling.

More precisely we address an open problem on the stability of optimal traffic plans for the mailing problem, with respect to variations of the given mass distribution, raised in [BCM09, Remark 6.13]. Let us introduce some preliminary notation in order to state the problem and our main result.

Let $X := \overline{B(0,R)} \subset \mathbb{R}^d$ and fix a finite measure $\pi \in \mathscr{M}_+(X \times X)$. A traffic plan is a finite measure $\mathbf{P} \in \mathscr{M}_+(\mathrm{Lip}_1)$ on the space of 1-Lipschitz curves $\gamma : [0, \infty) \to X$, supported on curves which are eventually constant. The topology on Lip_1 is induced by the uniform convergence on compact subsets of $[0, \infty)$. Given a sequence $\{\mathbf{P}_n\}_{n \in \mathbb{N}} \subset \mathscr{M}_+(\mathrm{Lip}_1)$, we will say that

$$\mathbf{P}_n \overset{*}{\rightharpoonup} \mathbf{P}, \qquad \text{if and only if} \qquad \int_{\mathrm{Lip}_1} f(\gamma) d\mathbf{P}_n(\gamma) \to \int_{\mathrm{Lip}_1} f(\gamma) d\mathbf{P}(\gamma)$$

for every $f : \mathrm{Lip}_1 \mapsto \mathbb{R}$ continuous and compactly supported with respect to the considered topology in Lip_1. We say that $\mathbf{P}$ is associated to the coupling π if it satisfies

$$(e_0, e_\infty)_\sharp \mathbf{P} = \pi, \quad \text{where} \qquad e_0(\gamma) := \gamma(0) \quad \text{and} \qquad e_\infty(\gamma) := \lim_{t \to \infty} \gamma(t).$$

For given $\alpha \in (0, 1)$, the α-energy of $\mathbf{P}$ is defined by

$$\mathcal{E}^\alpha(\mathbf{P}) := \int_{\mathrm{Lip}_1} \int_{\mathbb{R}^+} |\gamma(t)|_{\mathbf{P}}^{\alpha-1} |\dot{\gamma}(t)| dt \, d\mathbf{P}(\gamma),$$

where the multiplicity at a point x is given by

$$|x|_{\mathbf{P}} := \mathbf{P}(\{\gamma \in \mathrm{Lip}_1 : \gamma(t) = x, \quad \text{for some } t\}).$$

We say that a traffic plan $\mathbf{P}$, with a coupling π, is optimal, and we write $\mathbf{P} \in \mathbf{OTP}(\pi)$, if

$$\mathcal{E}^\alpha(\mathbf{P}) \le \mathcal{E}^\alpha(\mathbf{P}'), \qquad \text{for every } \mathbf{P}' \text{ such that } (e_0, e_\infty)_\sharp \mathbf{P}' = \pi.$$

On optimal traffic plans, the α-energy (when it is finite) is proved to coincide with the α-mass

$$\mathbb{M}^\alpha(\mathbf{P}) := \int_{\mathbb{R}^d} |x|_{\mathbf{P}}^\alpha d\mathscr{H}^1,$$

see Proposition 2.7. Notice that this integral is well defined because the integrand is non-zero just on a 1-rectifiable set, see Theorem 2.6. We consequently recover the desired notion of cost, i.e. a quantity which is computed integrating a subadditive function of the mass flow.

In the class of traffic plans, we consider the usual notion of weak* convergence of Radon measures. To get continuity of the map $(e_\infty)_\sharp$ on traffic plans, with respect to the weak* convergence, we should require a technical assumption on the class of traffic plans that we consider, namely that, denoting $T(\gamma)$ the stopping time of a curve γ, there exists a constant C such that each traffic plan $\mathbf{P}$ in the class satisfies

$$\int_{\mathrm{Lip}_1} T(\gamma) d\mathbf{P} \le C.$$

In this case we say that $\mathbf{P} \in \mathbf{TP}_C$. Notice that this is a tightness condition on a class of probability measures.

The main result of the paper is the following theorem.

1.1. Theorem. *Let $\alpha > 1 - \frac{1}{d}$ and $C > 0$. Let $\pi \in \mathscr{P}(X \times X)$ be a probability measure and $\{\pi_n\}_{n \in \mathbb{N}} \subseteq \mathscr{P}(X \times X)$ be such that*

$$\pi_n \overset{*}{\rightharpoonup} \pi, \qquad \text{weakly* in the sense of measures.} \tag{1.1}$$

For every $n \in \mathbb{N}$, let $\mathbf{P}_n \in \boldsymbol{OTP}(\pi_n)$ be an optimal traffic plan for the mailing problem and assume that $\mathbf{P}_n \in \boldsymbol{TP}_C$ and that there exists a traffic plan $\mathbf{P} \in \mathscr{P}(\mathrm{Lip}_1)$ satisfying

$$\mathbf{P}_n \stackrel{*}{\rightharpoonup} \mathbf{P}, \qquad as \; n \to \infty.$$

Then $\mathbf{P}$ is optimal, namely $\mathbf{P} \in \boldsymbol{OTP}(\pi)$ and moreover $\mathcal{E}^\alpha(\mathbf{P}_n) \to \mathcal{E}^\alpha(\mathbf{P})$.

In the regime $\alpha > 1 - \frac{1}{d}$, we prove a quantitative upper bound on the energy of an optimal traffic plan $\mathbf{P} \in \boldsymbol{OTP}(\pi)$ depending only on the mass of π and on the diameter of its support, see Corollary 3.8. Such bound is crucial in our strategy and the bound fails in general for $\alpha \leq 1 - \frac{1}{d}$: this is the reason why we can prove Theorem 1.1 only in the case $\alpha > 1 - \frac{1}{d}$.

1.2. Remark. A comment about the $\mathbf{TP}_C$ assumption in Theorem 1.1 is necessary. In [BCM09, Proposition 6.12] the assumption $\mathbf{P}_n \in \mathbf{TP}_C$ is not explicitly stated, but it is tacitly used. Moreover, without such assumption the stability as well as the convergence of $(e_\infty)_\sharp \mathbf{P}_n$ to $(e_\infty)_\sharp \mathbf{P}$ could fail, both for the mailing problem and for the "standard" branched transportation problem. One can prove that, whenever a traffic plan $\mathbf{P} \in \mathbf{OTP}(\pi)$ is supported on curves which are parametrized by arc length, the quantity $\int_{\mathrm{Lip}_1} T(\gamma) d\mathbf{P}$ is bounded from above by the energy $\mathcal{E}^\alpha(\mathbf{P})$, up to a constant which depends only on the mass of π. Hence, by the discussion above, in Theorem 1.1 one could replace the assumption $\mathbf{P}_n \in \mathbf{TP}_C$ with the (stronger) assumption that every $\mathbf{P}_n$ is supported on curves parametrized by arc length.

Besides the obvious relevance for numerical applications, Theorem 1.1 can be used to carry on part of the regularity program for optimal traffic plans for the mailing problem, which is another open question in [BCM09, Section 15.3]. In Theorem 6.2 we prove that any branching point of an optimal traffic plan, supported on curves whose length is uniformly bounded from below, splits the support in only finitely many connected components.

2. NOTATION AND PRELIMINARIES

We denote by $\mathbb{R}^d$ the d-dimensional Euclidean space and by $B(x,r)$ the open ball $\{y \in \mathbb{R}^d : |x - y| < r\}$. For a set $A \subset \mathbb{R}^d$, we denote by $\overline{A}$ its closure and by $A^c := \mathbb{R}^d \setminus A$ its complementary set.

2.1. **Measures and rectifiable sets.** For a locally compact, separable metric space Y, we denote by $\mathscr{M}_+(Y)$ the set of positive finite Radon measures in Y, namely the set of positive measures on the σ-algebra of Borel sets of Y that are finite and inner regular. The set of probability measures, i.e. those measures μ satisfying $\mu(Y) = 1$, is denoted $\mathscr{P}(Y)$.

For $\mu, \nu \in \mathscr{M}_+(Y)$, we write $\mu \leq \nu$ if $\mu(A) \leq \nu(A)$ for every Borel set A. For a measure μ we denote by

$$\mathrm{supp}(\mu) := \bigcap \{C \subset Y : C \text{ is closed and } \mu(Y \setminus C) = 0\}$$

its *support*. We say that μ is *supported* on a Borel set E if $\mu(Y \setminus E) = 0$. For a Borel set E, $\mu \llcorner E$ is the restriction of μ to E, i.e. the measure defined by

$$[\mu \llcorner E](A) := \mu(E \cap A), \qquad \text{for every Borel set } A.$$

We denote by $|\mu|$ the total variation (or mass) of μ, i.e. $|\mu| = \mu(Y)$.

For a measure $\mu \in \mathscr{M}_+(Y)$, a metric space Z, and a Borel map $\eta : Y \to Z$, we let $\eta_\sharp \mu \in \mathscr{M}_+(Z)$ be the push-forward measure, namely

$$\eta_\sharp \mu(A) := \mu(\eta^{-1}(A)), \qquad \text{for every Borel set } A \subset Z.$$

We use $\mathscr{L}^d$ and $\mathscr{H}^k$ to denote respectively the d-dimensional Lebesgue measure on $\mathbb{R}^d$ and the k-dimensional Hausdorff measure (see [Sim83]).

A set $K \subset \mathbb{R}^d$ is said *countably k-rectifiable* (or simply *k-rectifiable*) if it can be covered, up to an $\mathscr{H}^k$-negligible set, by countably many k-dimensional submanifolds of class C^1.

2.2. Lipschitz curves. Let $X \subseteq \mathbb{R}^d$ be a compact, convex set. We denote by Lip_1 the space of 1-Lipschitz curves $\gamma : \mathbb{R}^+ \to X$, endowed with the metric of uniform convergence on compact subsets of $\mathbb{R}^+ := \{t \in \mathbb{R} : t \geq 0\}$.

For every $\gamma \in \mathrm{Lip}_1$, we denote its (possibly infinite) *stopping time* by

$$T(\gamma) := \inf\{t \in \mathbb{R}^+ : \dot{\gamma}(s) = 0, \quad \text{for a.e.} \quad s \geq t\}.$$

On the set $\{\gamma \in \mathrm{Lip}_1(X) : T(\gamma) < \infty\}$, we define the evaluations at time 0 and "at time ∞"

$$e_0(\gamma) := \gamma(0), \qquad e_\infty(\gamma) := \gamma(T(\gamma)).$$

2.3. Traffic plans. In this section, we recall the basic definitions concerning traffic plans (see [BCM09] for an exhaustive description).

We denote with $\mathbf{TP}(X)$ the space of *traffic plans* with values in X, i.e.

$$\mathbf{TP}(X) := \{\mathbf{P} \in \mathscr{M}_+(\mathrm{Lip}_1(X)) : \mathbf{P} \text{ is supported on } \{\gamma : T(\gamma) < \infty\}\}.$$

Every traffic plan $\mathbf{P}$ naturally identifies a *coupling* between an "irrigating" and an "irrigated" probability measure on $\mathbb{R}^d$. We consider the measure in $\mathscr{M}_+(X \times X)$:

$$\pi_{\mathbf{P}} := (e_0, e_\infty)_\sharp \mathbf{P},$$

called the *coupling* of $\mathbf{P}$ and, denoting e_1, e_2 the projections from $\mathbb{R}^d \times \mathbb{R}^d$ onto the first and the second components, we call the *irrigating* and *irrigated measure* of $\mathbf{P} \in \mathbf{TP}(X)$ respectively

$$\mu_{\mathbf{P}}^- := (e_1)_\sharp \pi_{\mathbf{P}} = (e_0)_\sharp \mathbf{P}, \qquad \mu_{\mathbf{P}}^+ := (e_2)_\sharp \pi_{\mathbf{P}} = (e_\infty)_\sharp \mathbf{P}. \tag{2.1}$$

For every $\pi \in \mathscr{M}_+(X \times X)$, we denote the set of traffic plans with coupling π by

$$\mathbf{TP}(\pi) := \{\mathbf{P} \in \mathbf{TP}(X) : \pi_{\mathbf{P}} = \pi\}.$$

Sometimes we will not fix a coupling of a traffic plan, but just the pair of the initial and the final measure of the transportation process. Given two measures $\mu^-, \mu^+ \in \mathscr{M}_+(X)$, we denote the set of traffic plans with marginals μ^-, μ^+ by

$$\mathbf{TP}(\mu^-, \mu^+) := \{\mathbf{P} \in \mathbf{TP}(X) : \mu_{\mathbf{P}}^- = \mu^- \text{ and } \mu_{\mathbf{P}}^+ = \mu^+\}.$$

2.4. Energies of traffic plans. For a traffic plan $\mathbf{P}$ and a point $x \in \mathbb{R}^d$, we define the *multiplicity* of $\mathbf{P} \in \mathbf{TP}(X)$ at x by

$$|x|_{\mathbf{P}} := \mathbf{P}(\{\gamma \in \mathrm{Lip}_1(X) : \gamma(t) = x, \quad \text{for some } t \in \mathbb{R}^+\}).$$

Roughly speaking, the multiplicity measures the amount of paths passing through a given point x, without counting repetitions or orientations. For any $\alpha \in [0, 1)$, we define a functional $\mathcal{E}^\alpha : \mathbf{TP}(X) \to [0, \infty]$, that we call *$\alpha$-energy*, defined by

$$\mathcal{E}^\alpha(\mathbf{P}) := \int_{\mathrm{Lip}_1} \int_{\mathbb{R}^+} |\gamma(t)|_{\mathbf{P}}^{\alpha-1} |\dot{\gamma}(t)| dt \, d\mathbf{P}(\gamma), \tag{2.2}$$

where we used the convention that $0^{\alpha-1} = +\infty$ for $\alpha \in [0, 1)$.

We denote the set of optimal traffic plans with coupling $\pi \in \mathscr{M}_+(X \times X)$ as

$$\mathbf{OTP}(\pi) := \{\mathbf{P} \in \mathbf{TP}(\pi) : \mathcal{E}^\alpha(\mathbf{P}) \leq \mathcal{E}^\alpha(\mathbf{P}') \quad \forall \mathbf{P}' \in \mathbf{TP}(\pi)\}.$$

The existence of an optimal traffic plan follows from the lower semicontinuity of the α-energy, which can be found in [BCM09] (see also [Peg17]). The lower semicontinuity holds only in a certain subclass of $\mathbf{TP}(X)$. For every $C > 0$ we define the class $\mathbf{TP}_C(X)$ as the traffic plans $\mathbf{P} \in \mathbf{TP}(X)$ such that

$$\int_{\mathrm{Lip}_1} T(\gamma) \, d\mathbf{P} \leq C. \tag{2.3}$$

2.5. Lemma. *Let $C > 0$ and let $\{\mathbf{P}_n\}_{n \in \mathbb{N}}$ be a sequence of traffic plans in $\boldsymbol{TP}_C(\mathbb{R}^d)$ weakly* converging to a traffic plan $\mathbf{P}$. Then*

$$\liminf_{n \to \infty} \mathcal{E}^\alpha(\mathbf{P}_n) \geq \mathcal{E}^\alpha(\mathbf{P}).$$

Let us denote $\mathcal{E}^\alpha(\pi) := \mathcal{E}^\alpha(\mathbf{P})$ for some $\mathbf{P} \in \mathbf{OTP}(\pi)$. We recall the main result on the structure of traffic plans with finite α-energy:

2.6. Theorem. (see [BCM09, Theorem 4.10]). *Let $\alpha \in [0, 1)$ and $\mathbf{P}$ be a traffic plan with $\mathcal{E}^\alpha(\mathbf{P}) < \infty$. Then there exists a 1-rectifiable set E such that*

$$\mathscr{H}^1(\mathrm{Im}(\gamma) \setminus E) = 0, \qquad \text{for } \mathbf{P}\text{-a.e. } \gamma.$$

In the case $\mathcal{E}^\alpha(\mathbf{P}) < \infty$, we can define an alternative notion of "energy", called α-*mass*. Let E be the set associated to $\mathbf{P}$ as in Theorem 2.6, we denote

$$\mathbb{M}^\alpha(\mathbf{P}) := \int_E |x|_{\mathbf{P}}^\alpha d\mathscr{H}^1(x). \tag{2.4}$$

Moreover we recall the relationship between the functionals $\mathcal{E}^\alpha$ and $\mathbb{M}^\alpha$.

2.7. Proposition. (see [BCM09, Proposition 4.8, Proposition 4.9]). *For every $\mathbf{P} \in \boldsymbol{TP}(\pi_{\mathbf{P}})$ it holds*

$$\mathcal{E}^\alpha(\mathbf{P}) \geq \mathbb{M}^\alpha(\mathbf{P}). \tag{2.5}$$

If moreover $\mathbf{P} \in \boldsymbol{OTP}(\pi_{\mathbf{P}})$, then

$$\mathcal{E}^\alpha(\mathbf{P}) = \mathbb{M}^\alpha(\mathbf{P}). \tag{2.6}$$

Furthermore, for every $\pi \in \mathscr{M}_+(X \times X)$, we have that

$$\inf\{\mathcal{E}^\alpha(\mathbf{P}) : \mathbf{P} \in \boldsymbol{TP}(\pi)\} = \inf\{\mathbb{M}^\alpha(\mathbf{P}) : \mathbf{P} \in \boldsymbol{TP}(\pi)\}. \tag{2.7}$$

Optimal traffic plans satisfy a structural condition, usually called *simple path property*, see [BCM09, Definition 4.7]:

2.8. Definition. *For each $\gamma \in \mathrm{Lip}_1$, we define*

$$D(\gamma) = \{x \in \mathbb{R}^d | \exists s, t \in [0, +\infty) : s \neq t, \gamma(s) = \gamma(t) = x, \gamma([s,t]) \neq \{x\}\}.$$

We say that a traffic plan $\mathbf{P}$ satisfies the simple path property if

$$\int_{\mathrm{Lip}_1} \mathscr{H}^1(D(\gamma)) \, d\mathbf{P}(\gamma) = 0.$$

2.9. Remark. In [BCM09, Proposition 4.8] it is proved that the conclusion (2.6) holds even if $\mathbf{P}$ satisfies the simple path property (but it is not necessarily an optimum).

Optimal traffic plans enjoy an important regularity property, usually called *single path property*. Roughly speaking all fibres between two given points coincide (but they may have opposite orientations). See [BCM09, Proposition 7.4] and the discussion before, for a formal definition.

The last prerequisite, crucial for our proof, is the fact that, when $\alpha > 1 - \frac{1}{d}$, the minimal transportation cost between two probability measures supported on a compact set is bounded by a constant which depends only on the diameter of the set.

2.10. Theorem. (see [BCM09, Corollary 6.9]). *Let $\alpha > 1 - \frac{1}{d}$, and let $\mu^-, \mu^+ \in \mathscr{M}_+(X)$. Then there exists $\mathbf{P} \in \boldsymbol{TP}(\mu^-, \mu^+)$ satisfying*

$$\mathcal{E}^\alpha(\mathbf{P}) \leq C_{\alpha,d} \, \mathrm{diam}(\mathrm{X}) |\mu^-|^\alpha,$$

where $C_{\alpha,d}$ is a constant depending only on α and d.

3. Concatenation and compatibility with the mailing problem

3.1. **Disintegration.** Let $\mathbf{P}$ be any traffic plan, with coupling π. By the disintegration theorem for Radon measures, see [AFP00, Theorem 2.28], applied to $\mathbf{P}$, with respect to the map $(e_0, e_\infty) : \mathrm{Lip}_1 \to X \times X$ (which evaluates each curve at its initial and final point), we deduce that $\mathbf{P}$ can be written as

$$\mathbf{P} = \pi \otimes \mathbf{P}^{x,y},$$

for a suitable family of probability measures $\mathbf{P}^{x,y}$ supported on the set of curves which begin at x and end at y. The latter means that

$$\mathbf{P}(A) := \int_{X \times X} \mathbf{P}^{x,y}(\{\gamma \in A : (e_0, e_\infty)(\gamma) = (x, y)\}) d\pi(x, y), \quad \text{for every Borel set } A \subset \mathrm{Lip}_1.$$

Moreover, given two measures $\mu, \nu \in \mathscr{M}_+(X)$, we will denote $\mu \times \nu \in \mathscr{M}_+(X \times X)$ the standard product measure satisfying $\mu \times \nu(A \times B) = \mu(A)\nu(B)$ for every couple of Borel subsets $A, B \subset X$. This is useful to prove the following lemma:

3.2. Lemma. *Let $N \in \mathbb{N}$ and for every $i = 1, ..., N$ let $\mu_i^-, \mu_i^+ \in \mathscr{M}_+(X)$ such that $\mu_i^-(X) = \mu_i^+(X)$. Let $\mathbf{P} \in \mathbf{TP}\big(\sum_{i=1}^N \mu_i^- \times \mu_i^+\big)$. Then for every $i = 1, ..., N$ there exists $\mathbf{P}^i \in \mathbf{TP}\big(\mu_i^- \times \mu_i^+\big)$ such that*

$$\mathbf{P} = \sum_{i=1}^N \mathbf{P}^i. \tag{3.1}$$

Proof. We disintegrate the traffic plan $\mathbf{P}$ with respect to (e_0, e_∞) to get a family of probability measures $\{\mathbf{P}^{x,y}\}_{(x,y) \in X \times X}$ such that

$$\mathbf{P} = \Big(\sum_{i=1}^N \mu_i^- \times \mu_i^+\Big) \otimes \mathbf{P}^{x,y},$$

namely for every test function $\varphi : \mathrm{Lip}_1(X) \to \mathbb{R}$

$$\int \varphi(\gamma) \, d\mathbf{P}(\gamma) = \int \varphi(\gamma) \, d\mathbf{P}^{x,y}(\gamma) \, d\Big(\sum_{i=1}^N \mu_i^- \times \mu_i^+\Big)(x, y) = \sum_{i=1}^N \int \varphi(\gamma) \, d\mathbf{P}^{x,y}(\gamma) \, d(\mu_i^- \times \mu_i^+)(x, y). \tag{3.2}$$

For every $i = 1, ..., N$ we define

$$\mathbf{P}^i := \big(\mu_i^- \times \mu_i^+\big) \otimes \mathbf{P}^{x,y}.$$

Then (3.1) is directly implied by (3.2). Recalling that for every $(x, y) \in X \times X$ and for $\mathbf{P}^{x,y}$-a.e γ, it holds $(e_0(\gamma), e_\infty(\gamma)) = (x, y)$, for every test function $\psi : X \times X \to \mathbb{R}$ we compute

$$\int \psi(x, y) \, d(e_0, e_\infty)_\# \mathbf{P}^i(x, y) = \int \psi(e_0(\gamma), e_\infty(\gamma)) \, d\mathbf{P}^i(\gamma)$$

$$= \int \psi(e_0(\gamma), e_\infty(\gamma)) \, d\mathbf{P}^{x,y}(\gamma) d\big(\mu_i^- \times \mu_i^+\big)(x, y) \tag{3.3}$$

$$= \int \psi(x, y) \, d\big(\mu_i^- \times \mu_i^+\big)(x, y),$$

which reads $(e_0, e_\infty)_\# \mathbf{P}^i = \mu_i^- \times \mu_i^+$, i.e. $\mathbf{P}^i \in \mathbf{TP}\big(\mu_i^- \times \mu_i^+\big)$. $\qquad \square$

3.3. Concatenation. In this paper we will often need to perform the following operation: given two traffic plans $\mathbf{P}_1 \in \mathbf{TP}(\mu_1^-, \mu_1^+)$ and $\mathbf{P}_2 \in \mathbf{TP}(\mu_2^-, \mu_2^+)$, with $\mu_1^+ = \mu_2^-$, we want to define a traffic plan $\mathbf{P} \in \mathbf{TP}(\mu_1^-, \mu_2^+)$, obtained as a "concatenation" of the curves in $\mathbf{P}_1$ and those in $\mathbf{P}_2$. We begin by defining the concatenation of two curves.

3.4. Definition (Concatenation of two curves). *Given two curves $\gamma_1, \gamma_2 \in \mathrm{Lip}_1$ such that $T(\gamma_1) < \infty$ and $e_\infty(\gamma_1) = e_0(\gamma_2)$, we define their* concatenation *$c(\gamma_1, \gamma_2) \in \mathrm{Lip}_1$ as*

$$c(\gamma_1, \gamma_2)(t) := \begin{cases} \gamma_1(t) & \text{if } t \le T(\gamma_1) \\ \gamma_2(t - T(\gamma_1)) & \text{if } t > T(\gamma_1). \end{cases}$$

We now extend the map c to any pair (γ_1, γ_2) of Lipschitz curves defining the concatenation map $conc : \mathrm{Lip}_1 \times \mathrm{Lip}_1 \to \mathrm{Lip}_1$ as follows:

$$conc(\gamma_1, \gamma_2) := \begin{cases} c(\gamma_1, \gamma_2) & \text{if } T(\gamma_1) < \infty \text{ and } e_\infty(\gamma_1) = \gamma_2(0) \\ 0 & \text{otherwise,} \end{cases}$$

where 0 denotes the constant curve at the point 0.

The concatenation induces an analogous operation on traffic plans, which is in general a multi-map.

3.5. Definition (Set of concatenations of two given traffic plans). *Let $\mu^-, \nu, \mu^+ \in \mathscr{M}_+(X)$ with the same total mass. Let $\mathbf{P}_1 \in \mathbf{TP}(\mu^-, \nu)$ and $\mathbf{P}_2 \in \mathbf{TP}(\nu, \mu^+)$. A concatenation of $\mathbf{P}_1$ and $\mathbf{P}_2$ is any traffic plan of the form $\mathbf{P} = conc_{\#}P$ which belongs to $\mathbf{TP}(\mu^-, \mu^+)$ and where $P \in \mathscr{M}_+(\mathrm{Lip}_1 \times \mathrm{Lip}_1)$ satisfies*

(1) *for P-a.e. (γ_1, γ_2), we have that γ_1 is eventually constant and $e_\infty(\gamma_1) = \gamma_2(0)$.*
(2) *$(\rho_i)_{\#}P = \mathbf{P}_i$ for $i = 1, 2$, where $\rho_1, \rho_2 : \mathrm{Lip}_1 \times \mathrm{Lip}_1 \to \mathrm{Lip}_1$ denote the projections on the first and second component, respectively.*

We notice that the set of P's that generate a concatenation of $\mathbf{P}_1$ and $\mathbf{P}_2$ as in Definition 3.5 is convex, since (1) and (2) are stable under convex combinations.

One may wonder why we give the definition of the set of concatenations instead of choosing a more canonical representative in this set (for instance, the one described in Lemma 3.6 (4) below). The reason is hidden in Lemma 3.6 (5) below: the concatenation, seen as a multi-map is stable by addition. This is not the case, in general, if one fixes a selection of the multi-map.

The existence of a concatenation satisfying Definition 3.5 is trivial in some simple cases: for instance, if we disintegrate $\mathbf{P}_1$ and $\mathbf{P}_2$ as

$$\mathbf{P}_i = \nu \otimes \mathbf{P}_i^x \qquad \text{for } i = 1, 2,$$

with respect to e_∞ and e_0 respectively, and if we assume that, for ν-a.e. x, $\mathbf{P}_1^x$ (resp $\mathbf{P}_2^x$) is supported on a single curve ending in (resp. starting at) the point x, then the (unique) concatenation of $\mathbf{P}_1$ and $\mathbf{P}_2$ is obtained by concatenating for ν-a.e. x the unique curve in the support of $\mathbf{P}_1^x$ (that ends in x) with the unique curve in the support of $\mathbf{P}_2^x$ (that starts at x).

When we drop the assumption that $\mathbf{P}_1^x$ and $\mathbf{P}_2^x$ are Dirac deltas for ν-a.e. x, the operation is more involved, and the set of all concatenations has infinitely many elements. In the following lemma, we summarize the properties of the set of concatenations.

3.6. Lemma (Properties of the set of concatenations). *Let $\mu^-, \nu, \mu^+ \in \mathscr{P}(X)$. Let $\mathbf{P}_1 \in \mathbf{TP}(\mu^-, \nu)$, $\mathbf{P}_2 \in \mathbf{TP}(\nu, \mu^+)$ and let $\mathbf{P}$ be any concatenation of $\mathbf{P}_1$ and $\mathbf{P}_2$. Then*

(1) *we have the following inequalities regarding the multiplicities*

$$\max\{|z|_{\mathbf{P}_1}, |z|_{\mathbf{P}_2}\} \le |z|_{\mathbf{P}} \le |z|_{\mathbf{P}_1} + |z|_{\mathbf{P}_2} \qquad \text{for every } z \in \mathbb{R}^d; \tag{3.4}$$

(2) *for every $\phi : \mathbb{R}^d \mapsto [0, \infty)$ Borel function, setting $A_\phi(\gamma) := \int_0^\infty \phi(\gamma(t)) |\dot\gamma(t)| \, dt$, we have*

$$\int_{\mathrm{Lip}_1} A_\phi(\gamma) \, d\mathbf{P}(\gamma) = \int_{\mathrm{Lip}_1} A_\phi(\gamma) \, d\mathbf{P}_1(\gamma) + \int_{\mathrm{Lip}_1} A_\phi(\gamma) \, d\mathbf{P}_2(\gamma);$$

(3) *we have the energy bounds*

$$\mathcal{E}^\alpha(\mathbf{P}) \leq \mathcal{E}^\alpha(\mathbf{P}_1) + \mathcal{E}^\alpha(\mathbf{P}_2), \qquad \mathbb{M}^\alpha(\mathbf{P}) \leq \mathbb{M}^\alpha(\mathbf{P}_1) + \mathbb{M}^\alpha(\mathbf{P}_2); \tag{3.5}$$

(4) *the set of concatenations of $\mathbf{P}_1$ and $\mathbf{P}_2$ is nonempty;*

(5) *the set of concatenations is stable under addition: if $\tilde{\mathbf{P}}_1 \in \mathbf{TP}(\tilde\mu^-, \tilde\nu)$, $\tilde{\mathbf{P}}_2 \in \mathbf{TP}(\tilde\nu, \tilde\mu^+)$ and $\tilde{\mathbf{P}}$ is a concatenation of $\tilde{\mathbf{P}}_1$ and $\tilde{\mathbf{P}}_2$, then $\mathbf{P} + \tilde{\mathbf{P}}$ is a concatenation of $\mathbf{P}_1 + \tilde{\mathbf{P}}_1$ and $\mathbf{P}_2 + \tilde{\mathbf{P}}_2$.*

Proof. By Definition 3.5, there exists $P \in \mathscr{P}(\mathrm{Lip}_1 \times \mathrm{Lip}_1)$ such that

$$\int \varphi(\gamma) \, d\mathbf{P}(\gamma) = \int \varphi(conc(\gamma_1, \gamma_2)) \, dP(\gamma_1, \gamma_2). \tag{3.6}$$

for every Borel function $\varphi : \mathrm{Lip}_1 \to \mathbb{R}$.

We claim that $\mathbf{P}$ satisfies (3.4). Indeed, given $z \in \mathbb{R}^d$ we apply (3.6) with $\varphi(\gamma) = 1_{z \in Im(\gamma)}$ and we notice that $1_{z \in Im(conc(\gamma_1, \gamma_2))} = \max\{1_{z \in Im(\gamma_1)}, 1_{z \in Im(\gamma_2)}\}$ to get

$$|z|_\mathbf{P} = \int 1_{z \in Im(\gamma)} \, d\mathbf{P}(\gamma) = \int \max\{1_{z \in Im(\gamma_1)}, 1_{z \in Im(\gamma_2)}\} \, dP(\gamma_1, \gamma_2).$$

We estimate

$$|z|_\mathbf{P} \geq \int 1_{z \in Im(\gamma_1)} \, dP(\gamma_1, \gamma_2) = \int 1_{z \in Im(\gamma_1)} \, d(\rho_1)_\# P(\gamma_1) = \int 1_{z \in Im(\gamma_1)} \, d\mathbf{P}_1(\gamma_1) = |z|_{\mathbf{P}_1} \tag{3.7}$$

The same argument shows that $|z|_\mathbf{P} \geq |z|_{\mathbf{P}_2}$. In order to prove the second inequality in (3.4) we employ the fact that $\max\{1_{z \in Im(\gamma_1)}, 1_{z \in Im(\gamma_2)}\} \leq 1_{z \in Im(\gamma_1)} + 1_{z \in Im(\gamma_2)}$ and the same computations as above. This concludes the proof of (1).

In order to prove (2), we observe that for every γ_1 and γ_2 with $T(\gamma_1) < \infty$ and $\gamma_1(\infty) = \gamma_2(0)$, we can write

$$A_\phi(conc(\gamma_1, \gamma_2)) = \int_0^\infty \phi(conc(\gamma_1, \gamma_2)(t)) \left| \frac{d}{dt} conc(\gamma_1, \gamma_2)(t) \right| dt$$

$$= \int_0^{T(\gamma_1)} \phi(\gamma_1(t)) |\dot\gamma_1(t)| dt + \int_{T(\gamma_1)}^\infty \phi(\gamma_2(t - T(\gamma_1))) |\dot\gamma_2(t - T(\gamma_1))| dt \tag{3.8}$$

$$= \int_0^\infty \phi(\gamma_1(t)) |\dot\gamma_1(t)| \, dt + \int_0^\infty \phi(\gamma_2(t)) |\dot\gamma_2(t)| \, dt = A_\phi(\gamma_1) + A_\phi(\gamma_2).$$

We plug the map $\varphi := A_\phi$ in equation (3.6) to deduce

$$\int A_\phi(\gamma) \, d\mathbf{P}(\gamma) = \int A_\phi(conc(\gamma_1, \gamma_2)) \, dP(\gamma_1, \gamma_2) \overset{(3.8)}{=} \int A_\phi(\gamma_1) + A_\phi(\gamma_2) \, dP(\gamma_1, \gamma_2)$$

$$= \int A_\phi(\gamma_1) \, dP(\gamma_1, \gamma_2) + \int A_\phi(\gamma_2) \, dP(\gamma_1, \gamma_2) = \int A_\phi(\gamma_1) \, d\mathbf{P}_1(\gamma_1) + \int A_\phi(\gamma_2) \, d\mathbf{P}_2(\gamma_2),$$

which gives (2).

Integrating the second inequality in (3.4) with respect to $\mathscr{H}^1$ along the set $\{|z|_{\mathbf{P}_1} > 0\} \cup \{|z|_{\mathbf{P}_2} > 0\}$ (which is 1-rectifiable as long as $\mathbf{P}_1$ and $\mathbf{P}_2$ have finite α-mass), we deduce the

second inequality in (3.5). For the first inequality, for every γ_1 and γ_2 with $\gamma_1(\infty) = \gamma_2(0)$, we use (3.8) with $\phi := |\cdot|_{\mathbf{P}}^{\alpha-1}$ and by (3.4) we deduce

$$\int_0^\infty |conc(\gamma_1,\gamma_2)(t)|_{\mathbf{P}}^{\alpha-1}\left|\frac{d}{dt}conc(\gamma_1,\gamma_2)(t)\right|dt = \int_0^\infty |\gamma_1(t)|_{\mathbf{P}}^{\alpha-1}|\dot\gamma_1(t)|\,dt + \int_0^\infty |\gamma_2(t)|_{\mathbf{P}}^{\alpha-1}|\dot\gamma_2(t)|\,dt$$
$$\leq \int_0^\infty |\gamma_1(t)|_{\mathbf{P}_1}^{\alpha-1}|\dot\gamma_1(t)|\,dt + \int_0^\infty |\gamma_2(t)|_{\mathbf{P}_2}^{\alpha-1}|\dot\gamma_2(t)|\,dt.$$

Integrating the previous inequality with respect to P, we get that

$$\mathcal{E}^\alpha(\mathbf{P}) = \int\int_0^\infty |\gamma(t)|_{\mathbf{P}}^{\alpha-1}|\dot\gamma(t)|\,dt\,d\mathbf{P}(\gamma) = \int\int_0^\infty |conc(\gamma_1,\gamma_2)|_{\mathbf{P}}^{\alpha-1}\left|\frac{d}{dt}conc(\gamma_1,\gamma_2)\right|dt\,dP(\gamma_1,\gamma_2)$$
$$\leq \int\int_0^\infty |\gamma_1(t)|_{\mathbf{P}_1}^{\alpha-1}|\dot\gamma_1(t)|\,dt\,dP(\gamma_1,\gamma_2) + \int\int_0^\infty |\gamma_2(t)|_{\mathbf{P}_2}^{\alpha-1}|\dot\gamma_2(t)|\,dt\,dP(\gamma_1,\gamma_2)$$
$$= \int\int_0^\infty |\gamma_1(t)|_{\mathbf{P}_1}^{\alpha-1}|\dot\gamma_1(t)|\,dt\,d\mathbf{P}_1(\gamma_1) + \int\int_0^\infty |\gamma_2(t)|_{\mathbf{P}_2}^{\alpha-1}|\dot\gamma_2(t)|\,dt\,d\mathbf{P}_2(\gamma_2)$$
$$= \mathcal{E}^\alpha(\mathbf{P}_1) + \mathcal{E}^\alpha(\mathbf{P}_2),$$

which yields (3).

To show (4), we consider the disintegration of $\mathbf{P}_1$ and $\mathbf{P}_2$ with respect to the common marginal ν. Let $\{\mathbf{P}_1^x\}_{x\in X}$ and $\{\mathbf{P}_2^x\}_{x\in X}$ be families of probability measures representing the disintegration of $\mathbf{P}_1$ and $\mathbf{P}_2$ with respect to ν. In other words, for every $x \in \mathbb{R}^d$, $\mathbf{P}_1^x \in \mathscr{P}(\mathrm{Lip}_1)$ is supported on curves that end at x and $\mathbf{P}_2^x$ is supported on curves that begin at x, and

$$\mathbf{P}_1 = \nu \otimes \mathbf{P}_1^x, \qquad \mathbf{P}_2 = \nu \otimes \mathbf{P}_2^x.$$

We define

$$P := \nu \otimes (\mathbf{P}_1^x \times \mathbf{P}_2^x),$$

namely for every C^0 test function $\varphi : \mathrm{Lip}_1 \times \mathrm{Lip}_1 \to \mathbb{R}$

$$\int \varphi(\gamma_1,\gamma_2)\,dP(\gamma_1,\gamma_2) = \int \varphi(\gamma_1,\gamma_2)\,d\mathbf{P}_1^x(\gamma_1)\,d\mathbf{P}_2^x(\gamma_2)\,d\nu(x). \tag{3.9}$$

The measure P satisfies all the properties in Definition 3.5; indeed $(\rho_i)_\# P = \nu \otimes \mathbf{P}_i^x = \mathbf{P}_i$ and for ν-a.e. x, for $\mathbf{P}_1^x \times \mathbf{P}_2^x$-a.e. (γ_1,γ_2) we have that $\gamma_1(\infty) = x = \gamma_2(0)$. We define

$$\mathbf{P} := conc_\# P.$$

For every test function $\psi : X \to \mathbb{R}$ we can compute

$$\int \psi(x)\,d(e_0)_\#\mathbf{P}(x) = \int \psi(e_0(\gamma))\,d\mathbf{P}(\gamma) = \int \psi(e_0(conc(\gamma_1,\gamma_2)))\,dP(\gamma_1,\gamma_2)$$
$$\stackrel{(3.9)}{=} \int \psi(e_0(conc(\gamma_1,\gamma_2)))\,d\mathbf{P}_1^x(\gamma_1)\,d\mathbf{P}_2^x(\gamma_2)\,d\nu(x)$$
$$= \int \psi(e_0(\gamma_1))\,d\mathbf{P}_1^x(\gamma_1)\,d\nu(x) = \int \psi(e_0(\gamma_1))\,d\mathbf{P}_1(\gamma_1) \tag{3.10}$$
$$= \int \psi(x)\,d(e_0)_\#\mathbf{P}_1(x) = \int \psi(x)\,d\mu^-(x)$$

and analogously

$$\int \psi(x)\,d(e_\infty)_\#\mathbf{P}(x) \stackrel{(3.9)}{=} \int \psi(e_\infty(conc(\gamma_1,\gamma_2)))\,d\mathbf{P}_1^x(\gamma_1)\,d\mathbf{P}_2^x(\gamma_2)\,d\nu(x)$$
$$= \int \psi(e_\infty(\gamma_2))\,d\mathbf{P}_2(\gamma_2) = \int \psi(x)\,d(e_\infty)_\#\mathbf{P}_2(x) = \int \psi(x)\,d\mu^+(x). \tag{3.11}$$

We deduce that $\mathbf{P} \in \mathbf{TP}(\mu^-,\mu^+)$ and we conclude that $\mathbf{P}$ is a concatenation of $\mathbf{P}_1$ and $\mathbf{P}_2$.

To prove (5), let $P, \tilde{P} \in \mathscr{M}_+(\mathrm{Lip}_1 \times \mathrm{Lip}_1)$ be associated to $\mathbf{P}, \tilde{\mathbf{P}}$ as in Definition 3.5. Then $P + \tilde{P}$ verifies the properties of Definition 3.5 (where in property (2) we replace $\mathbf{P}_i$ with $\mathbf{P}_i + \tilde{\mathbf{P}}_i$) and therefore $\mathbf{P} + \tilde{\mathbf{P}} = conc_\#(P + \tilde{P}) \in \mathbf{TP}(\mu^- + \tilde{\mu}^-, \mu^+ + \tilde{\mu}^+)$ is a concatenation of $\mathbf{P}_1 + \tilde{\mathbf{P}}_1$ and $\mathbf{P}_2 + \tilde{\mathbf{P}}_2$. □

When dealing with the mailing problem, the concatenation of two traffic plans needs to take into account the coupling between the initial and the final measures. In general it is not true that, given two traffic plans $\mathbf{P}_1$ and $\mathbf{P}_2$ with a common marginal and given a coupling π between the initial measure of $\mathbf{P}_1$ and the final measure of $\mathbf{P}_2$, there exists a concatenation of $\mathbf{P}_1$ and $\mathbf{P}_2$ with coupling π. On the other hand, in the special case when the common marginal is a Dirac delta, we are allowed to prescribe any coupling π, as shown in the next lemma.

3.7. Lemma (Concatenation with prescribed coupling through a Dirac delta). *Let* $\mathbf{P}_1 \in \mathbf{TP}(\mu^-, \delta_{x_0})$, $\mathbf{P}_2 \in \mathbf{TP}(\delta_{x_0}, \mu^+)$, *and let* $\pi \in \mathscr{P}(X \times X)$ *with* $(e_1)_\#\pi = \mu^-$ *and* $(e_2)_\#\pi = \mu^+$. *Then there exists a concatenation* $\mathbf{P}$ *of* $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *such that*

$$\mathbf{P} \in \mathbf{TP}(\pi).$$

Proof. Let $\{\mathbf{P}_1^x\}_{x \in X}$ be a 1-parameter family of probability measures representing the disintegration of $\mathbf{P}_1$ with respect to the first marginal. In particular, for μ^--a.e. x, $\mathbf{P}_1^x$ is supported on curves that begin at x. Similarly, let $\{\mathbf{P}_2(x)\}_{x \in X}$ be a 1-parameter family of probability measures representing the disintegration of $\mathbf{P}_2$ with respect to the second marginal

$$\mathbf{P}_1 = \mu^-(x) \otimes \mathbf{P}_1^x \qquad \mathbf{P}_2 = \mu^+(x) \otimes \mathbf{P}_2^x.$$

We define the traffic plan $\mathbf{P}$ through the disintegration

$$P := \pi(x, y) \otimes (\mathbf{P}_1^x \times \mathbf{P}_2^y), \qquad \mathbf{P} := conc_\# P,$$

recalling that this means that the measure $\mathbf{P}$ is the unique measure that satisfies for every bounded Borel test function $\varphi : \mathrm{Lip}_1 \to \mathbb{R}$

$$\int \varphi(\gamma) \, d\mathbf{P}(\gamma) = \int \varphi(conc(\gamma_1, \gamma_2)) \, d\mathbf{P}_1^x(\gamma_1) \, d\mathbf{P}_2^y(\gamma_2) \, d\pi(x, y). \qquad (3.12)$$

We notice that P satisfies the properties in Definition 3.5.

We observe that $\mathbf{P} \in \mathbf{TP}(\pi)$. To verify this, we test (3.12) with $\varphi \circ (e_0, e_\infty) : \mathrm{Lip}_1 \to \mathbb{R}$

$$\int \varphi(x, y) \, d(e_0, e_\infty)_\# \mathbf{P}(x, y) = \int \varphi(e_0(\gamma), e_\infty(\gamma)) \, d\mathbf{P}(\gamma)$$

$$= \int \varphi(\gamma_1(0), e_\infty(\gamma_2)) \, d\mathbf{P}_1^x(\gamma_1) \, d\, \mathbf{P}_2^y(\gamma_2) \, d\pi(x, y)$$

$$= \int \varphi(x, y) \, d\mathbf{P}_1^x(\gamma_1) \, d\, \mathbf{P}_2^y(\gamma_2) \, d\pi(x, y) = \int \varphi(x, y) \, d\pi(x, y).$$

This shows that $\mathbf{P}$ is a concatenation of $\mathbf{P}_1$ and $\mathbf{P}_2$ with $\mathbf{P} \in \mathbf{TP}(\pi)$. □

With the previous lemma we prove the equivalent of Theorem 2.10 for the mailing problem.

3.8. Corollary (Upper bound on the energy for the mailing problem). *Let* $\alpha > 1 - \frac{1}{d}$. *Then there exists constant* $c(d, \alpha)$ *such that if* $\pi \in \mathscr{M}_+(X \times X)$ *is supported on a set of diameter less than or equal than* L, *then*

$$\mathcal{E}^\alpha(\pi) \le c(d, \alpha)\pi(X \times X)^\alpha L.$$

Proof. Let $x_0 \in X$ be any point in the support of $(e_1)_\#\pi$. By Theorem 2.10 there exist two traffic plans $\mathbf{P}_1 \in \mathbf{TP}((e_1)_\#\pi, \delta_{x_0})$ and $\mathbf{P}_2 \in \mathbf{TP}(\delta_{x_0}, (e_2)_\#\pi)$ satisfying

$$\mathcal{E}^\alpha(\mathbf{P}_i) \le C_{\alpha,d}[(e_1)_\#\pi](X)^\alpha L = C_{\alpha,d}\pi(X \times X)^\alpha L.$$

Applying Lemma 3.7 to $\mathbf{P}_1$ and $\mathbf{P}_2$ and Lemma 3.6 (3), we find that

$$\mathcal{E}^\alpha(\pi) \leq C_{\alpha,d}\big(\mu^-(X)^\alpha + \mu^+(X)^\alpha\big)L = 2C_{\alpha,d}\pi(X \times X)^\alpha L.$$

$\square$

3.9. Remark. A consequence of Corollary 3.8 is that, above the critical threshold, there exists a constant $C = C(X, d, \alpha)$ such that if an optimal traffic plan $\mathbf{P} \in \mathbf{OTP}(\pi_{\mathbf{P}})$ of unit mass is supported on curves which are parametrized by arc-length, then $\mathbf{P} \in \mathbf{TP}_C$. Indeed we have

$$c(d, \alpha)\mathrm{diam}(X) \geq \mathcal{E}^\alpha(\mathbf{P}) = \int_{\mathrm{Lip}_1} \int_{\mathbb{R}^+} |\gamma(t)|_{\mathbf{P}}^{\alpha-1}|\dot\gamma(t)|dt\, d\mathbf{P}(\gamma) \geq \int_{\mathrm{Lip}_1} T(\gamma)\, d\mathbf{P}(\gamma).$$

In particular, the energy $\mathcal{E}^\alpha$ is always lower semicontinuous along a converging sequence of such optimal traffic plans (compare with Lemma 2.5).

We conclude this section on concatenations with a generalization of Lemma 3.7. Roughly speaking, it says that, if we have three traffic plans $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$, which are "admissible" for a concatenation, i.e. $(e_\infty)_\#\mathbf{P}_i = (e_0)_\#\mathbf{P}_{i+1}$, for $i = 1, 2$, and moreover $(e_\infty)_\sharp\mathbf{P}_2$ is a Dirac delta at a point x_0, then we can realize any coupling between the initial measure $(e_0)_\#\mathbf{P}_1$ and the final measure $(e_\infty)_\#\mathbf{P}_3$, via a suitable concatenation of the three traffic plans. Generalizing this construction, one could prove that whenever two or more traffic plan which can be concatenated have a Dirac delta in one of the intermediate marginals, then every coupling between the first and the last marginal can be realized by a concatenation.

3.10. Lemma. *Let $x_0 \in \mathbb{R}^d$, μ_1, μ_2, μ_3 be measures on $\mathbb{R}^d$ with the same mass. Let $\mathbf{P}_0$ be a traffic plan between μ_1 and μ_3, $\mathbf{P}_1$ between μ_1 and μ_2, $\mathbf{P}_2$ between μ_2 and δ_{x_0}, $\mathbf{P}_3$ between δ_{x_0} and μ_3.*

Then there exists a concatenation $\mathbf{P}_4$ between $\mathbf{P}_2$ and $\mathbf{P}_3$ and a concatenation $\mathbf{P}_5$ between $\mathbf{P}_1$ and $\mathbf{P}_4$ such that $\mathbf{P}_5$ has the same coupling as $\mathbf{P}_0$.

Only in the proof of this lemma, we will use the following notation. For $1 \leq j \leq i$, we use

$$e_j : \underbrace{\mathbb{R}^d \times \cdots \times \mathbb{R}^d}_{i \text{ times}} \to \mathbb{R}^d$$

to denote the projection on the j-th copy of $\mathbb{R}^d$.

The proof will rely on the following lemma, known as gluing lemma, which can be found in [San15, Lemma 5.5]. This time e_1, e_2, e_3 are the projections on X, Y, Z respectively.

3.11. Lemma. *Let X, Y, Z be locally compact, separable metric spaces. Let $\pi_1 \in \mathscr{P}(X \times Y)$ and $\pi_2 \in \mathscr{P}(Y \times Z)$. Then there exists $\pi \in \mathscr{P}(X \times Y \times Z)$ such that $(e_1, e_2)_\#\pi = \pi_1$ and $(e_2, e_3)_\#\pi = \pi_2$.*

Proof of Lemma 3.10. For every $i = 0, \ldots, 3$ we set $\pi_i = (e_0, e_\infty)_\#\mathbf{P}_i$. Applying Lemma 3.11, we build $\tilde\pi \in \mathscr{P}(\mathbb{R}_x^d \times \mathbb{R}_y^d \times \mathbb{R}_z^d)$ such that $(e_1, e_2)_\#\tilde\pi = \pi_1$ and $(e_2, e_3)_\#\tilde\pi = \pi_2$. Next, we apply again Lemma 3.11 between $\tilde\pi$ (commuting the spaces in the product in the following order: $(\mathbb{R}_y^d \times \mathbb{R}_z^d) \times \mathbb{R}_x^d$) and π_0 to obtain, after commuting the spaces back, $\pi \in \mathscr{P}(\mathbb{R}_x^d \times \mathbb{R}_y^d \times \mathbb{R}_z^d \times \mathbb{R}_w^d)$ such that

$$(e_1, e_2)_\#\pi = \pi_1, \qquad (e_1, e_4)_\#\pi = \pi_0.$$

We notice moreover that

$$(e_2, e_3)_\#\pi = \pi_2, \qquad (e_3, e_4)_\#\pi = \pi_3,$$

because $(e_3)_\#\pi = \delta_{x_0}$ and there exists a unique coupling π_2 with marginals μ_2 and δ_{x_0} and a unique coupling π_3 with marginals δ_{x_0} and μ_3.

Denote by $\mathbf{P}_4$ any concatenation of $\mathbf{P}_2$ and $\mathbf{P}_3$ which realizes the coupling $\pi_4 := (e_2, e_4)_\# \pi$ (it can be obtained by Lemma 3.7). To construct $\mathbf{P}_5$, we disintegrate $\mathbf{P}_1$ and $\mathbf{P}_4$ with respect to (e_0, e_∞) to get

$$\mathbf{P}_1 = \pi_1(x, y) \otimes \mathbf{P}_1^{x,y} = \pi(x, y, z, w) \otimes \mathbf{P}_1^{x,y}$$
$$\mathbf{P}_4 = \pi_4(y, w) \otimes \mathbf{P}_4^{y,w} = \pi(x, y, z, w) \otimes \mathbf{P}_4^{y,w}.$$

We can now define

$$\mathbf{P}_5 := \mathrm{conc}_\# \big(\pi(x, y, z, w) \otimes (\mathbf{P}_1^{x,y} \times \mathbf{P}_4^{y,w}) \big).$$

Since for every x, y, z, w the measures $\mathbf{P}_1^{x,y}$ and $\mathbf{P}_4^{y,w}$ are supported on two sets of curves with a common extreme in y, their concatenation is well defined. Moreover, we need to verify property (2) of Definition (3.5)

$$(\rho_1)_\# \big(\pi(x, y, z, w) \otimes (\mathbf{P}_1^{x,y} \times \mathbf{P}_4^{y,w}) \big) = \pi(x, y, z, w) \otimes (\rho_1)_\# (\mathbf{P}_1^{x,y} \times \mathbf{P}_4^{y,w}) = \pi(x, y, z, w) \otimes \mathbf{P}_1^{x,y} = \mathbf{P}_1$$

and similarly $(\rho_2)_\# \big(\pi(x, y, z, w) \otimes (\mathbf{P}_2^{x,y} \times \mathbf{P}_4^{y,w}) \big) = \mathbf{P}_2$, so that $\mathbf{P}_5$ is a well defined concatenation between $\mathbf{P}_1$ and $\mathbf{P}_4$. Finally we check that $\mathbf{P}_0$ and $\mathbf{P}_5$ have the same coupling

$$(e_0, e_\infty)_\# \mathbf{P}_5 = \big((e_0, e_\infty) \circ \mathrm{conc} \big)_\# \big(\pi(x, y, z, w) \otimes (\mathbf{P}_1^{x,y} \times \mathbf{P}_4^{y,w}) \big)$$
$$= \pi(x, y, z, w) \otimes \big(\big((e_0, e_\infty) \circ \mathrm{conc} \big)_\# (\mathbf{P}_1^{x,y} \times \mathbf{P}_4^{y,w}) \big)$$
$$= \pi(x, y, z, w) \otimes \delta_{x,w} = (e_1, e_4)_\# \pi = \pi_0 = (e_0, e_\infty)_\# \mathbf{P}_0. \qquad \square$$

4. Stability for the mailing problem

In this section we prove a slightly more refined version of the stability Theorem 1.1, which is more suited to the application presented in Section 6. We deduce Theorem 1.1 as a corollary.

4.1. Theorem. *Let $\alpha > 1 - \frac{1}{d}$. Let $\pi_\infty \in \mathscr{P}(X \times X)$ with marginals μ^- and μ^+ and $\{\pi_n\}_{n \in \mathbb{N}} \subseteq \mathscr{P}(X \times X)$ be a sequence with marginals μ_n^-, μ_n^+ such that*

$$\pi_n \overset{*}{\rightharpoonup} \pi_\infty \qquad \text{weakly}^* \text{ in the sense of measures.} \tag{4.1}$$

Then for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for every sequence $\{\mathbf{P}_i \in \mathbf{TP}(\pi_i)\}_{i \in \mathbb{N} \cup \{\infty\}}$ and for every $n, m \geq n_0$ (with $m = \infty$ possibly), there exist two traffic plans $\mathbf{P}_{n,m}^1 \in \mathbf{TP}(\mu_n^-, \mu_m^-)$, $\mathbf{P}_{n,m}^2 \in \mathbf{TP}(\mu_m^+, \mu_n^+)$ such that

$$\mathcal{E}^\alpha(\mathbf{P}_{n,m}^1) \leq \varepsilon, \qquad \mathcal{E}^\alpha(\mathbf{P}_{n,m}^2) \leq \varepsilon$$

and there exists a traffic plan obtained as a concatenation between $\mathbf{P}_m$ and $\mathbf{P}_{n,m}^2$, and a further concatenation $\tilde{\mathbf{P}}_{n,m}$ between $\mathbf{P}_{n,m}^1$ and the latter concatenation such that $\tilde{\mathbf{P}}_{n,m} \in \mathbf{TP}(\pi_n)$.

Proof of Theorem 1.1. We apply Theorem 4.1 with $m = \infty$, being $\mathbf{P}_\infty \in \mathbf{OTP}(\pi_\infty)$, to find a competitor $\tilde{\mathbf{P}}_{n,\infty} \in \mathbf{TP}(\pi_n)$ given by a concatenation of $\mathbf{P}_\infty$ and $\mathbf{P}_{n,\infty}^2$ and a further concatenation of $\mathbf{P}_{n,\infty}^1$ and the latter concatenation; hence with the energy estimate

$$\mathcal{E}^\alpha(\tilde{\mathbf{P}}_{n,\infty}) \leq \mathcal{E}^\alpha(\mathbf{P}_{n,\infty}^1) + \mathcal{E}^\alpha(\mathbf{P}_\infty) + \mathcal{E}^\alpha(\mathbf{P}_{n,\infty}^2).$$

Moreover, for every $\varepsilon > 0$, for n large enough, the "connections" have small energy $\mathcal{E}^\alpha(\mathbf{P}_{n,\infty}^1), \mathcal{E}^\alpha(\mathbf{P}_{n,\infty}^2) \leq \varepsilon$. Letting $n \to \infty$, by the lower semicontinuity of the energy (see Lemma 2.5) and since $\mathbf{P}_n \in \mathbf{OTP}(\pi_n)$, we have

$$\mathcal{E}^\alpha(\mathbf{P}) \leq \liminf_{n \to \infty} \mathcal{E}^\alpha(\mathbf{P}_n) \leq \liminf_{n \to \infty} \mathcal{E}^\alpha(\tilde{\mathbf{P}}_{n,\infty}) \leq \mathcal{E}^\alpha(\mathbf{P}_\infty) + 2\varepsilon.$$

Since the previous formula holds for every $\varepsilon > 0$, we deduce that $\mathbf{P} \in \mathbf{OTP}(\pi)$.

To conclude the proof of Theorem 1.1, we need to show that $\mathcal{E}^\alpha(\mathbf{P}_n) \to \mathcal{E}^\alpha(\mathbf{P})$. To this aim, we apply Theorem 4.1 again with $m = \infty$, but $\mathbf{P}_\infty$ will be now chosen equal to $\mathbf{P}$ (that we already know is an optimal traffic plan). We deduce that for every $\varepsilon > 0$ and n large enough

there exists a competitor $\tilde{\mathbf{P}}_{n,\infty} \in \mathbf{TP}(\pi_n)$ given by a concatenation of $\mathbf{P}$ and $\mathbf{P}_{n,\infty}^2$ and a further concatenation between $\mathbf{P}_{n,\infty}^1$ and the latter concatenation; hence with the energy estimate

$$\mathcal{E}^\alpha(\tilde{\mathbf{P}}_{n,\infty}) \le \mathcal{E}^\alpha(\mathbf{P}_{n,\infty}^1) + \mathcal{E}^\alpha(\mathbf{P}) + \mathcal{E}^\alpha(\mathbf{P}_{n,\infty}^2).$$

Moreover, the connections have small energy $\mathcal{E}^\alpha(\mathbf{P}_{n,\infty}^1), \mathcal{E}^\alpha(\mathbf{P}_{n,\infty}^2) \le \varepsilon$. Letting $n \to \infty$, as above we have

$$\mathcal{E}^\alpha(\mathbf{P}) \le \liminf_{n\to\infty} \mathcal{E}^\alpha(\mathbf{P}_n) \le \limsup_{n\to\infty} \mathcal{E}^\alpha(\mathbf{P}_n) \le \limsup_{n\to\infty} \mathcal{E}^\alpha(\tilde{\mathbf{P}}_{n,\infty}) \le \mathcal{E}^\alpha(\mathbf{P}) + 2\varepsilon.$$

Since the previous inequality holds for every $\varepsilon > 0$, we deduce that $\mathcal{E}^\alpha(\mathbf{P}_n) \to \mathcal{E}^\alpha(\mathbf{P})$. $\qquad\square$

The proof of Theorem 4.1 relies on the following lemma, which employs in an essential way the condition $\alpha > 1 - \frac{1}{d}$. It builds a suitable covering of the support of a given measure, with balls satisfying the smallness condition in (4.4) below.

4.2. Lemma. Let $\alpha > 1 - \frac{1}{d}$, $\varepsilon > 0$ and let $\mu \in \mathscr{P}(X)$. Then there exists a countable family of balls $\{B_i = B(x_i, r_i)\}_{i \in \mathbb{N}}$ such that

$$\mu\left(\mathbb{R}^d \setminus \bigcup_{i=1}^\infty B_i\right) = 0, \tag{4.2}$$

$$\mu(\partial B_i) = 0 \qquad \text{for every } i \in \mathbb{N}, \tag{4.3}$$

$$\sum_{i=1}^\infty r_i \mu(B_i)^\alpha < \varepsilon. \tag{4.4}$$

4.3. Remark. When $\mu = f\mathscr{L}^d \llcorner X$ for some $f \in L^\infty(X)$, the previous lemma is trivial. Indeed, for any $r_0 > 0$, we can find a countable covering of X made by balls $\{B(x_i, r_i)\}_{i\in\mathbb{N}}$ with $r_i \le r_0$ and $\sum_{i=1}^\infty r_i^d \le \mathscr{L}^d(X) + 1$. We estimate

$$\sum_{i=1}^\infty r_i \mu(B_i)^\alpha \le \sum_{i=1}^\infty r_i^{1+\alpha d} \|f\|_{L^\infty}^\alpha \le r_0^{1+\alpha d - d} \sum_{i=1}^\infty r_i^d \|f\|_{L^\infty(\mathbb{R}^d)}^\alpha \le r_0^{1+\alpha d - d}(\mathscr{L}^d(X) + 1)\|f\|_{L^\infty}^\alpha.$$

By a suitable choice of r_0 and since $1 + \alpha d - d > 0$, this quantity can be made arbitrarily small.

Proof of Lemma 4.2. Let us consider

$$\delta := \frac{1}{1-\alpha} - d$$

and notice that $\delta > 0$ since $\alpha > 1 - \frac{1}{d}$. Let $D \subseteq X$ be the set of points such that the Radon-Nikodym density of μ with respect to $\mathscr{L}^d$ exists and is strictly positive (and possibly infinite)

$$D := \left\{x \in X : \text{ there exists } \frac{d\mu}{d\mathscr{L}^d}(x) := \lim_{r\to\infty} \frac{\mu(B_r(x))}{\omega_d r^d} \in (0, \infty]\right\}.$$

By the Besicovitch derivation theorem (see [AFP00, Theorem 2.22]) this set supports the measure μ; we aim to find a covering of μ-a.e. point of D made by balls that satisfy (4.3) and (4.4).

For any point $x \in D$, we claim that for every r small enough (depending on x)

$$\mu(B(x,r))^\alpha r \le \varepsilon\mu(B(x,r)). \tag{4.5}$$

Indeed, this condition (4.5) can be rewritten as

$$\left(\frac{1}{\varepsilon}\right)^{d+\delta} r^\delta \le \frac{\mu(B(x,r))}{r^d}. \tag{4.6}$$

Since the point x has positive Radon-Nikodym density $\frac{d\mu}{d\mathscr{L}^d}(x)$, the right-hand side in (4.6) is greater than $\frac{\omega_d}{2}\frac{d\mu}{d\mathscr{L}^d}(x) > 0$ for r small enough. On the other hand, the left-hand side of (4.6) converges to 0 as $r \to 0$. Hence the claim (4.5) holds.

For every $x \in D$, we also observe that for every r apart at most countably many

$$\mu(\partial B(x, r)) = 0. \tag{4.7}$$

The set D, because of the previous observation, has a fine covering made by all balls $B(x, r)$ centred at any point in D and satisfying (4.5) and (4.7). Hence, by the Vitali-Besicovitch covering lemma (see [AFP00, Theorem 2.19]), we can find a countable family of pairwise disjoint balls $\{B_i = B(x_i, r_i)\}_{i \in \mathbb{N}}$ satisfying the properties (4.5) and (4.7), that covers D up to a μ-null set. Hence, we estimate the left-hand side of (4.4) thanks to (4.5)

$$\sum_{i=1}^{\infty} r_i \mu(B_i)^\alpha \le \varepsilon \sum_{i=1}^{\infty} \mu(B_i) = \varepsilon. \qquad \square$$

Proof of Theorem 4.1. We develop the proof in several steps, briefly described here.

In the following, we consider two countable families of balls $B_i^\pm$ as in Lemma 4.2. We select two finite subfamilies, indexed by $1, ..., N^\pm$, which cover a big portion of the measures $\mu^\pm$ (Step 1 and 2).

In Step 3, we decompose the traffic plans $\mathbf{P}_n$, $\mathbf{P}_m$ in a sum of smaller traffic plans $\mathbf{P}_n^{i,j}$ and $\mathbf{P}_m^{i,j}$, according to the fact that each curve begins in a certain $B_i^- \setminus (\cup_{k=1}^{i-1} B_k^-)$, for some $i = 1, ..., N^-$ and ends in $B_j^+ \setminus (\cup_{k=1}^{j-1} B_k^+)$ for some $j = 1, ..., N^+$.

We define the residuum as the part of the traffic plans $\mathbf{P}_n$ and $\mathbf{P}_m$ that either begins outside $\cup_{k=1}^{N^-} B_k^-$ or ends outside $\cup_{k=1}^{N^+} B_k^+$; the construction which takes care of the residuum is carried out in Step 7 and is based on the smallness of the total mass carried by it.

In Step 4, we connect the initial points of each $\mathbf{P}_m^{i,j}$ with the initial points of $\mathbf{P}_n^{i,j}$ by means of a traffic plan; thanks to the choice of the balls given by Lemma 4.2, the sum of these traffic plans as i and j vary (denoted in the following by $\mathbf{P}^I$), can be built with arbitrarily small cost.

A similar construction works for the final points (Step 5) and we will denote the corresponding plan as $\mathbf{P}^{IV}$; in addition, doing this construction more carefully, one can ensure that the traffic plan connecting the final points of each $\mathbf{P}_n^{ij}$ with the final points of $\mathbf{P}_m^{ij}$ passes through the center of the ball B_j^+.

This last observation and Lemma 3.10 ensure that we can find a concatenation of the traffic plans $\mathbf{P}^I$, $\mathbf{P}_m - \mathbf{P}_m^{res}$ and $\mathbf{P}^{IV}$ which has the same marginals of $\mathbf{P}_n - \mathbf{P}_n^{res}$, as shown in Step 6.

Step 1: construction of a covering of $\operatorname{supp}(\mu^\pm)$. Let $\bar{\varepsilon} > 0$ to be chosen later (in terms only of $d, \alpha, \operatorname{diam}(X), \varepsilon$). Applying Lemma 4.2, we get that there exists a finite number of balls $\{B_i^\pm = B(x_i^\pm, r_i^\pm)\}_{i \in \mathbb{N}}$ covering $\operatorname{supp}(\mu^\pm)$ and satisfying the following properties:

$$\mu^\pm\Big(\mathbb{R}^d \setminus \bigcup_{i=1}^{\infty} B_i^\pm\Big) = 0, \tag{4.8}$$

$$\sum_{i=1}^{\infty} r_i^\pm \mu^\pm(B_i^\pm)^\alpha < \bar{\varepsilon}^\alpha, \tag{4.9}$$

$$\mu^\pm(\partial B_i^\pm) = 0 \qquad \text{for every } i \in \mathbb{N}. \tag{4.10}$$

Step 2: choice of $N^\pm$ *and of* n_0. We choose $N^\pm$ satisfying

$$\mu^\pm\Big(\bigcup_{i=1}^{N^\pm} B_i^\pm\Big) > 1 - \bar{\varepsilon}.$$

For every $i \in \mathbb{N}$ we define
$$C_i^\pm := B_i^\pm \setminus \left(\cup_{k=1}^{i-1} B_k^\pm \right).$$

Without loss of generality, the coverings can be assumed not redundant, that is
$$\mu^-(C_i^-) > 0, \qquad \mu^+(C_i^+) > 0 \qquad \text{for every } i \in \mathbb{N}. \tag{4.11}$$

We can fix n_0 large enough so that the following properties hold:
$$\mu_n^\pm(C_i^\pm) \le 2\mu^\pm(C_i^\pm) \qquad \forall n \ge n_0 \text{ and } i \in \{1, ..., N^\pm\}, \tag{4.12}$$

$$\mu_n^\pm\left(\left(\bigcup_{i=1}^{N^\pm} B_i^\pm \right)^c \right) \le 2\bar{\varepsilon}, \tag{4.13}$$

$$\pi_n(C_i^- \times C_j^+) - \pi_m(C_i^- \times C_j^+) \le \frac{\bar{\varepsilon}}{N^- N^+} \qquad \forall m, n \ge n_0, \, (i,j) \in \{1, ..., N^-\} \times \{1, ..., N^+\}. \tag{4.14}$$

Indeed, (4.14) follows from assumptions (4.1) and (4.10). Moreover, since (4.1) implies that $\mu_n^\pm$ weakly converges to $\mu^\pm$, then (4.12) follows from (4.11) and (4.13) follows from (4.8).

Step 3: decomposition of the traffic plans according to initial and final points of each curve.
For every $n \in \mathbb{N}$ we define $\mathbf{P}_n^{i,j}$ to be the part of traffic plan $\mathbf{P}_n$ made by curves that start in C_i^- and end in C_j^+, namely
$$\mathbf{P}_n^{i,j} := \mathbf{P}_n \llcorner \{\gamma : \gamma(0) \in C_i^-, \, \gamma(\infty) \in C_j^+\},$$

observing that the coupling induced by $\mathbf{P}_n^{i,j}$ is
$$(e_0, e_\infty)_\# \mathbf{P}_n^{i,j} = \pi_n \llcorner (C_i^- \times C_j^+).$$

For any $i \in \{1, ..., N^-\}$ and $j \in \{1, ..., N^+\}$, we consider
$$\mu_{n,\pm}^{i,j} := \mu_{\mathbf{P}_n^{i,j}}^\pm,$$

$$\alpha_{m,n}^{i,j} := \min \left\{ 1, \frac{\mathbf{P}_m^{i,j}(\text{Lip}_1)}{\mathbf{P}_n^{i,j}(\text{Lip}_1)} \right\}, \qquad \alpha_{n,m}^{i,j} := \min \left\{ 1, \frac{\mathbf{P}_n^{i,j}(\text{Lip}_1)}{\mathbf{P}_m^{i,j}(\text{Lip}_1)} \right\}$$

(with the agreement that the fractions are infinite if the denominator is 0). Let us consider rescalings of $\mathbf{P}_n^{i,j}$ and $\mathbf{P}_m^{i,j}$, namely
$$\tilde{\mathbf{P}}_n^{i,j} := \alpha_{m,n}^{i,j} \mathbf{P}_n^{i,j}, \qquad \tilde{\mathbf{P}}_m^{i,j} := \alpha_{n,m}^{i,j} \mathbf{P}_m^{i,j}$$

and we consider their marginals
$$\tilde{\mu}_n^{\pm,i,j} := \mu_{\tilde{\mathbf{P}}_n^{i,j}}^\pm \quad \text{and} \quad \tilde{\mu}_m^{\pm,i,j} := \mu_{\tilde{\mathbf{P}}_m^{i,j}}^\pm.$$

With this definition, for every $(i,j) \in \{1, ..., N^-\} \times \{1, ..., N^+\}$ we have $\tilde{\mathbf{P}}_n^{i,j} \le \mathbf{P}_n^{i,j}$, $\tilde{\mathbf{P}}_m^{i,j} \le \mathbf{P}_m^{i,j}$,
$$\tilde{\mathbf{P}}_n^{i,j}(\text{Lip}_1) = \tilde{\mathbf{P}}_m^{i,j}(\text{Lip}_1), \qquad \tilde{\mu}_n^{-,i,j}(\mathbb{R}^d) = \tilde{\mu}_m^{-,i,j}(\mathbb{R}^d) = \tilde{\mu}_n^{+,i,j}(\mathbb{R}^d) = \tilde{\mu}_m^{+,i,j}(\mathbb{R}^d).$$

Moreover, it holds that $\tilde{\mathbf{P}}_n \le \mathbf{P}_n$, defining
$$\tilde{\mathbf{P}}_n := \sum_{i=1}^{N^-} \sum_{j=1}^{N^+} \tilde{\mathbf{P}}_n^{i,j}.$$

Next we introduce the residuum, namely $\mathbf{P}_n - \tilde{\mathbf{P}}_n$, and we split it into two pieces as follows
$$\mathbf{P}_n = \tilde{\mathbf{P}}_n + \mathbf{P}_n^{res} = \tilde{\mathbf{P}}_n + \mathbf{P}_n^{1,res} + \mathbf{P}_n^{2,res},$$

where
$$\mathbf{P}_n^{1,res} = \sum_{i=1}^{N^-} \sum_{j=1}^{N^+} (1 - \alpha_{m,n}^{i,j}) \mathbf{P}_n^{i,j}, \qquad \mathbf{P}_n^{2,res} = \mathbf{P}_n - \sum_{i=1}^{N^-} \sum_{j=1}^{N^+} \mathbf{P}_n^{i,j}.$$

We show that $\mathbf{P}_n^{res}$ has small mass, namely

$$\mathbf{P}_n^{res}(\mathrm{Lip}_1) \leq 5\bar{\varepsilon}. \tag{4.15}$$

Indeed, since $\mathbf{P}_n^{i,j}(\mathrm{Lip}_1) = \mathbf{P}_n(\{\gamma : \gamma(0) \in C_i^-, e_\infty(\gamma) \in C_j^+\}) = \pi_n(C_i^- \times C_j^+)$ and similarly for $\mathbf{P}_m^{i,j}(\mathrm{Lip}_1)$, and by (4.14), we have

$$
\begin{aligned}
\mathbf{P}_n^{1,res}(\mathrm{Lip}_1) &= \sum_{i=1}^{N^-}\sum_{j=1}^{N^+}(1 - \alpha_{m,n}^{i,j})\mathbf{P}_n^{i,j}(\mathrm{Lip}_1) \leq \sum_{i=1}^{N^-}\sum_{j=1}^{N^+}\max\left\{0, \mathbf{P}_n^{i,j}(\mathrm{Lip}_1) - \mathbf{P}_m^{i,j}(\mathrm{Lip}_1)\right\} \\
&\leq \sum_{i=1}^{N^-}\sum_{j=1}^{N^+}|\pi_n(C_i^- \times C_j^+) - \pi_m(C_i^- \times C_j^+)| \leq \sum_{i=1}^{N^-}\sum_{j=1}^{N^+}\frac{\bar{\varepsilon}}{N^-N^+} = \bar{\varepsilon}.
\end{aligned} \tag{4.16}
$$

As regards $\mathbf{P}_n^{2,res}$, it is supported on curves which either begin outside any C_i^-, $i = 1, ..., N^-$, or end outside any C_j^+, $j = 1, ..., N^+$, and therefore has small mass thanks to (4.13)

$$
\begin{aligned}
\mathbf{P}_n^{2,res}(\mathrm{Lip}_1) &= 1 - \mathbf{P}_n\Big(\{\gamma : \gamma(0) \in \bigcup_{i=1}^{N^-}C_i^-, e_\infty(\gamma) \in \bigcup_{j=1}^{N^+}C_j^+\}\Big) \\
&\leq \mathbf{P}_n\Big(\Big\{\gamma : \gamma(0) \notin \bigcup_{i=1}^{N^-}C_i^-\Big\}\Big) + \mathbf{P}_n\Big(\Big\{\gamma : e_\infty(\gamma) \notin \bigcup_{j=1}^{N^+}C_j^+\Big\}\Big) \\
&\leq \mu_n^-\Big(\Big(\bigcup_{i=1}^{N^-}C_i^-\Big)^c\Big) + \mu_n^+\Big(\Big(\bigcup_{j=1}^{N^+}C_j^+\Big)^c\Big) \leq 4\bar{\varepsilon}.
\end{aligned} \tag{4.17}
$$

Step 4: initial connection. Let us apply the Corollary 3.8 to the coupling

$$\pi^{I,i} = \sum_{j=1}^{N^+}\tilde{\mu}_n^{-,i,j} \times \tilde{\mu}_m^{-,i,j}.$$

We find a traffic plan $\mathbf{P}^{I,i} \in \mathbf{TP}(\pi^{I,i})$ with energy controlled by

$$\mathcal{E}^\alpha(\mathbf{P}^{I,i}) \leq c(d,\alpha)\pi^{I,i}(\mathbb{R}^d \times \mathbb{R}^d)^\alpha r_i^- \leq c(d,\alpha)\Big(\sum_{j=1}^{N^+}\tilde{\mu}_n^{-,i,j}(\mathbb{R}^d)\Big)^\alpha r_i^-.$$

Finally, recalling that $\sum_{j=1}^{N^+}\tilde{\mu}_n^{-,i,j}(\mathbb{R}^d) \leq \sum_{j=1}^{N^+}\mu_n^{-,i,j}(C_i^-) \leq \mu_n^-(C_i^-)$ and (4.12), we obtain

$$\mathcal{E}^\alpha(\mathbf{P}^{I,i}) \leq c(d,\alpha)\mu_n^-(C_i^-)^\alpha r_i^- \leq 2c(d,\alpha)\mu^-(C_i^-)^\alpha r_i^-.$$

By means of Lemma 3.2, we can write $\mathbf{P}^{I,i}$ as the following sum

$$\mathbf{P}^{I,i} := \sum_{j=1}^{N^+}\mathbf{P}^{I,i,j} \qquad \text{where } \mathbf{P}^{I,i,j} \in \mathbf{TP}(\tilde{\mu}_n^{-,i,j} \times \tilde{\mu}_m^{-,i,j}).$$

Setting

$$\mathbf{P}^I := \sum_{i=1}^{N^-}\mathbf{P}^{I,i}$$

we obtain that its energy is small thanks to (4.9)

$$\mathcal{E}^\alpha(\mathbf{P}^I) \leq \sum_{i=1}^{N^-}\mathcal{E}^\alpha(\mathbf{P}^{I,i}) \leq 2c(d,\alpha)\sum_{i=1}^{N^-}\mu^-(C_i^-)^\alpha r_i^- \leq 2c(d,\alpha)\bar{\varepsilon}^\alpha. \tag{4.18}$$

Step 5: final connection: some useful traffic plans. Let us finally consider $\mathbf{P}^{II,j} \in$ $\mathbf{TP}\big(\sum_{i=1}^{N^-} \tilde{\mu}_m^{+,i,j}, |\sum_{i=1}^{N^-} \tilde{\mu}_m^{+,i,j}|\delta_{x_j}\big)$ (where we recall that $|\mu|$ denotes the total mass of any non-negative measure μ) with

$$\mathcal{E}^\alpha(\mathbf{P}^{II,j}) \leq c(d,\alpha)\Big(\sum_{i=1}^{N^-} \tilde{\mu}_m^{+,i,j}(\mathbb{R}^d)\Big)^\alpha r_j^+ \leq c(d,\alpha)\mu_m^+(C_j^+)^\alpha r_j^+ \leq 2c(d,\alpha)\mu^+(C_j^+)^\alpha r_j^+.$$

Setting

$$\mathbf{P}^{II} := \sum_{j=1}^{N^+} \mathbf{P}^{II,j}$$

we obtain that its energy is small thanks to (4.9)

$$\mathcal{E}^\alpha(\mathbf{P}^{II}) \leq \sum_{j=1}^{N^+} \mathcal{E}^\alpha(\mathbf{P}^{II,j}) \leq 2c(d,\alpha)\sum_{j=1}^{N^+} \mu^+(C_j^+)^\alpha r_j^+ \leq 2c(d,\alpha)\bar{\varepsilon}^\alpha. \tag{4.19}$$

By Lemma 3.2, we can write

$$\mathbf{P}^{II,j} := \sum_{i=1}^{N^-} \mathbf{P}^{II,i,j} \qquad \text{where } \mathbf{P}^{II,i,j} \in \mathbf{TP}\big(\tilde{\mu}_m^{+,i,j} \times (|\tilde{\mu}_m^{+,i,j}|\delta_{x_j})\big)$$

In a completely analogous way, we define

$$\mathbf{P}^{III} = \sum_{i=1}^{N^-}\sum_{j=1}^{N^+} \mathbf{P}^{III,i,j} \qquad \text{where } \mathbf{P}^{III,i,j} \in \mathbf{TP}\big((|\tilde{\mu}_n^{+,i,j}|\delta_{x_j}) \times \tilde{\mu}_n^{+,i,j}\big)$$

with small energy

$$\mathcal{E}^\alpha(\mathbf{P}^{III}) \leq 2c(d,\alpha)\bar{\varepsilon}^\alpha. \tag{4.20}$$

Step 6: Concatenation with correct coupling for fixed i,j. For any $i \in \{1,...,N^-\}$ and $j \in \{1,...,N^+\}$, we apply Lemma 3.10 with $\mathbf{P}_0 = \tilde{\mathbf{P}}_n^{i,j}$, $\mathbf{P}_1$ a concatenation between $\mathbf{P}^{I,i,j}$ and $\tilde{\mathbf{P}}_m^{i,j}$, $\mathbf{P}_2 = \mathbf{P}^{II,i,j}$, $\mathbf{P}_3 = \mathbf{P}^{III,i,j}$ to find a concatenation $\mathbf{P}^{IV,i,j}$ between $\mathbf{P}^{II,i,j}$ and $\mathbf{P}^{III,i,j}$ and a further concatenation $\mathbf{P}^{V,i,j}$ between $\mathbf{P}_1$ and $\mathbf{P}^{IV,i,j}$ with the same coupling as $\tilde{\mathbf{P}}_n^{i,j}$

$$(e_0, e_\infty)_\# \mathbf{P}^{V,i,j} = (e_0, e_\infty)_\# \tilde{\mathbf{P}}_n^{i,j}. \tag{4.21}$$

We next set

$$\mathbf{P}^{IV} := \sum_{i=1}^{N^-}\sum_{j=1}^{N^+} \mathbf{P}^{IV,i,j}, \qquad \mathbf{P}^V := \sum_{i=1}^{N^-}\sum_{j=1}^{N^+} \mathbf{P}^{V,i,j}$$

and we notice that by Lemma 3.6(5) $\mathbf{P}^{IV}$ is a concatenation between $\mathbf{P}^{II}$ and $\mathbf{P}^{III}$. Hence, by Lemma 3.6(3) together with the smallness of the energies of $\mathbf{P}^{II}$ and $\mathbf{P}^{III}$ we have

$$\mathcal{E}^\alpha(\mathbf{P}^{IV}) \leq \mathcal{E}^\alpha(\mathbf{P}^{II}) + \mathcal{E}^\alpha(\mathbf{P}^{III}) \leq 4c(d,\alpha)\bar{\varepsilon}^\alpha. \tag{4.22}$$

Thanks to the computation of the marginals of $\mathbf{P}^{V,i,j}$ in (4.21), we deduce that

$$(e_0, e_\infty)_\# \mathbf{P}^V = (e_0, e_\infty)_\# \tilde{\mathbf{P}}_n \tag{4.23}$$

Step 7: residuals. Let $\mathbf{P}^{I,res}$ be an optimal traffic plan with coupling $(e_0)_\# \mathbf{P}_n^{res} \times (e_0)_\# \mathbf{P}_m^{res}$. By Corollary 3.8 and (4.15) we have the energy bound

$$\mathcal{E}^\alpha(\mathbf{P}^{I,res}) \leq c(d,\alpha,\text{diam}(X))\mathbf{P}_n^{res}(\text{Lip}_1)^\alpha \leq 5c(d,\alpha,\text{diam}(X))\bar{\varepsilon}^\alpha. \tag{4.24}$$

Let $\mathbf{P}^{II,res}$ and $\mathbf{P}^{III,res}$ be optimal traffic plans with coupling $(e_\infty)_\# \mathbf{P}_m^{res} \times \delta_0$ and $\delta_0 \times (e_\infty)_\# \mathbf{P}_n^{res}$ respectively. By Corollary 3.8 we have the energy bound

$$\mathcal{E}^\alpha(\mathbf{P}^{II,res}) + \mathcal{E}^\alpha(\mathbf{P}^{III,res}) \leq 2c(d,\alpha,\text{diam}(X))\mathbf{P}_n^{res}(\text{Lip}_1)^\alpha \leq 10c(d,\alpha,\text{diam}(X))\bar{\varepsilon}^\alpha. \tag{4.25}$$

We apply Lemma 3.10 with $\mathbf{P}_0 = \mathbf{P}_n^{res}$, $\mathbf{P}_1$ a concatenation between $\mathbf{P}^{I,res}$ and $\mathbf{P}_m^{res}$, $\mathbf{P}_2 = \mathbf{P}^{II,res}$, $\mathbf{P}_3 = \mathbf{P}^{III,res}$ to find a concatenation $\mathbf{P}^{IV,res}$ between $\mathbf{P}^{II,res}$ and $\mathbf{P}^{III,res}$ and a further concatenation $\mathbf{P}^{V,res}$ between $\mathbf{P}_1$ and $\mathbf{P}^{IV,res}$ with the same coupling as $\mathbf{P}_n^{res}$

$$(e_0, e_\infty)_\# \mathbf{P}^{V,res} = (e_0, e_\infty)_\# \mathbf{P}_n^{res}. \tag{4.26}$$

Step 8: Conclusion. We set

$$\mathbf{P}_{n,m}^1 := \mathbf{P}^{I,res} + \mathbf{P}^I, \qquad \mathbf{P}_{n,m}^2 := \mathbf{P}^{IV,res} + \mathbf{P}^{IV}, \qquad \tilde{\mathbf{P}}_{n,m} := \mathbf{P}^{V,res} + \mathbf{P}^V$$

and we claim that they satisfy the properties in the statement of the theorem.

Indeed, the energy of $\mathbf{P}_{n,m}^1$ is estimated using (4.24) and (4.18) by

$$\mathcal{E}^\alpha(\mathbf{P}_{n,m}^1) \leq \mathcal{E}^\alpha(\mathbf{P}^{I,res}) + \mathcal{E}^\alpha(\mathbf{P}^I) \leq 7c(d, \alpha, \operatorname{diam}(X))\bar{\varepsilon}^\alpha \leq \varepsilon,$$

and the energy of $\mathbf{P}_{n,m}^2$ is estimated using (4.22) and (4.25) by

$$\mathcal{E}^\alpha(\mathbf{P}_{n,m}^2) \leq \mathcal{E}^\alpha(\mathbf{P}^{IV,res}) + \mathcal{E}^\alpha(\mathbf{P}^{IV}) \leq 14c(d, \alpha, \operatorname{diam}(X))\bar{\varepsilon}^\alpha \leq \varepsilon$$

where the last inequality in both lines follows by choosing $\bar{\varepsilon}$ suitably small in terms of ε.

By definition, $\mathbf{P}^V$ is a concatenation of $\mathbf{P}^I$ with a concatenation between $\tilde{\mathbf{P}}_m$ and $\mathbf{P}^{IV}$. Hence, adding term by term and employing Lemma 3.6(5), $\tilde{\mathbf{P}}_{n,m}$ is a concatenation of $\mathbf{P}_{n,m}^1 = \mathbf{P}^{I,res} + \mathbf{P}^I$ with a concatenation between $\mathbf{P}_m = \mathbf{P}_m^{res} + \tilde{\mathbf{P}}_m$ and $\mathbf{P}_{n,m}^2 = \mathbf{P}^{IV,res} + \mathbf{P}^{IV}$. This implies in particular that $\mathbf{P}_{n,m}^1$ has the same right-marginal as the initial marginal of $\mathbf{P}_m$, that is $\mathbf{P}_{n,m}^1 \in \mathbf{TP}(\mu_n^-, \mu_m^-)$. Analogously, $\mathbf{P}_{n,m}^2 \in \mathbf{TP}(\mu_m^+, \mu_n^+)$.

Finally, the coupling identified by $\tilde{\mathbf{P}}_{n,m}$ is the same as the coupling of $\mathbf{P}_n$: indeed, we have already computed in (4.26) and (4.23) the couplings of $\mathbf{P}^{V,res}$ and $\mathbf{P}^V$. So we get

$$(e_0, e_\infty)_\# \tilde{\mathbf{P}}_{n,m} = (e_0, e_\infty)_\# \mathbf{P}^{V,res} + (e_0, e_\infty)_\# \mathbf{P}^V = (e_0, e_\infty)_\# \mathbf{P}_n^{res} + (e_0, e_\infty)_\# \tilde{\mathbf{P}}_n = (e_0, e_\infty)_\# \mathbf{P}_n.$$

$\square$

5. Stability in the exponent

In the work [BF08], the authors study stability for the irrigation problem with respect to varying exponent α. With the same proof, we obtain the following results for the mailing problem, which are analogous respectively to [BF08, Theorem 6.5] and [BF08, Corollary 6.3]:

5.1. Theorem. *Let $\alpha_n \in [1 - \frac{1}{d}, 1]$ be a sequence converging to α. If the traffic plans $\mathbf{P}_n \in \mathbf{TP}(\pi)$ are optimal for $\mathcal{E}^{\alpha_n}$ and $\mathbf{P}_n \overset{*}{\rightharpoonup} \mathbf{P}$, then $\mathbf{P}$ is optimal for $\mathcal{E}^\alpha$.*

5.2. Theorem. *Let $\alpha_n \in [0, 1]$ be a decreasing sequence converging to α. If the traffic plans $\mathbf{P}_n \in \mathbf{TP}(\pi)$ are optimal for $\mathcal{E}^{\alpha_n}$ and $\mathbf{P}_n \overset{*}{\rightharpoonup} \mathbf{P}$, then $\mathbf{P}$ is optimal for $\mathcal{E}^\alpha$.*

In order to prove them, we can use [BF08, Lemma 6.1] as it is. We need instead a counterpart of [BF08, Proposition 6.2] for the mailing problem as stated below:

5.3. Proposition. *Let $\pi \in \mathscr{P}(X \times X)$. The function $[0, 1] \ni \alpha \mapsto \mathcal{E}^\alpha(\pi) \in [0, +\infty]$ is non-increasing, right continuous and left lower semicontinuous.*

This can be proved as [BF08, Proposition 6.2], replacing in the proof of the latter the definition of $f(\alpha)$ with $f(\alpha) := \mathcal{E}^\alpha(\pi)$. These two results, combined, allow to prove Theorem 5.2 as in the proof of [BF08, Corollary 6.3]. In order to get Theorem 5.1, we need to replace [BF08, Lemma 6.4] with the following statement, which holds true by [BCM09, Proposition 9.1]:

5.4. Lemma. *Let $\pi := \sum_{i=1}^N \sum_{j=1}^M a_{ij} \delta_{x_i} \otimes \delta_{y_j}$. Then $\alpha \mapsto \mathcal{E}^\alpha(\pi)$ is continuous on $[0, 1]$.*

We can consequently prove Theorem 5.1 with the same proof of [BF08, Theorem 6.5].

6. FINITENESS OF THE NUMBER OF CONNECTED COMPONENTS

We recall the definition of connected components of a traffic plan. This requires some preliminary notation. Given $t_1 \leq t_2 \in [0, \infty)$ we define the map $res_{t_1, t_2} : \mathrm{Lip}_1 \to \mathrm{Lip}_1$ as

$$[res_{t_1, t_2}(\gamma)](t) := \begin{cases} \gamma(t_1) & \text{if } t \leq t_1, \\ \gamma(t) & \text{if } t_1 < t < t_2, \\ \gamma(t_2) & \text{if } t \geq t_2. \end{cases}$$

We say that a curve $\gamma \in \mathrm{Lip}_1$ is *simple* if there exists no triple of times $t_1 < t_2 < t_3$ such that $\gamma(t_1) = \gamma(t_3) \neq \gamma(t_2)$. For every $x, y \in X$,

(1) we consider the set $A_{x,y} \subset \mathrm{Lip}_1$

$$A_{x,y} := \{\gamma \in \mathrm{Lip}_1 : \gamma \text{ is simple and } \exists t_1, t_2 \in \mathbb{R}^+ : t_1 \leq t_2, \ \gamma(t_1) = x, \ \gamma(t_2) = y\};$$

(2) we denote t_p the first time for which $\gamma(t_p) = p$ and we observe that res_{t_x, t_y} is well defined on $A_{x,y}$; moreover, fixing an $x_0 \in X$, we can extend res_{t_x, t_y} to be the constant curve in x_0;

(3) we define the traffic plan $\mathbf{P}_{x,y} := (res_{t_x, t_y})_\# (\mathbf{P} \llcorner A_{x,y})$;

(4) we denote

$$\Gamma_{x,y} := \{p : |p|_{\mathbf{P}_{x,y}} > 0\} \cup \{p : |p|_{\mathbf{P}_{y,x}} > 0\}.$$

We say that a traffic plan $\mathbf{P}$ is *loop free* if $\mathbf{P}$ is supported on simple curves. Optimal traffic plans for the mailing problem are loop free (see Proposition 4.9 [BCM09]), and we recall that they enjoy the single path property (see [BCM09, Proposition 7.4]). Hence if $\mathbf{P} \in \mathbf{OTP}(\pi_{\mathbf{P}})$, then for every x, y such that $\Gamma_{x,y} \neq \emptyset$, we have that $\mathbf{P}_{x,y}$ is supported on a single, simple curve γ (up to reparametrizations of the curve) and $\mathbf{P}_{y,x}$ is supported on the same curve (parametrized backwards in time). In particular $\Gamma_{x,y}$ is the image of such curve γ.

For a traffic plan $\mathbf{P}$ we define $\Gamma_{\mathbf{P}} := \{x : |x|_{\mathbf{P}} > 0\}$. We say that a (finite or countable) family of traffic plans $\{\mathbf{P}_n\}_{n \in \mathbb{N}}$ is *disjoint* if the measures $\mathscr{H}^1 \llcorner \Gamma_{\mathbf{P}_n}$ are mutually singular.

6.1. Definition (Connected components). *Let $\mathbf{P}$ be an optimal traffic plan for the mailing problem, i.e. $\mathbf{P} \in \mathbf{OTP}(\pi_{\mathbf{P}})$. Given an open set $U \subset \mathbb{R}^d$, two points $x, y \in U$ are said to be connected in U if there is a chain $x_1 = x, ..., x_l = y$ such that $\emptyset \neq \Gamma_{x_i, x_{i+1}} \subset U$. One can see that connectedness is an equivalence relation on $\Gamma_{\mathbf{P}} \cap U$.*

We say that $F \subset \Gamma_{\mathbf{P}} \cap U$ is a connected component of $\mathbf{P}$ in U if F is an equivalence class with respect to such equivalence relation.

The present section is devoted to show the finiteness of the number of connected components of an optimal traffic plan, supported on curves whose length is uniformly bounded from below.

6.2. Theorem. *Let $\alpha > 1 - \frac{1}{d}$ and $\mathbf{P}$ be an optimal traffic plan for the mailing problem, i.e. $\mathbf{P} \in \mathbf{OTP}(\pi_{\mathbf{P}})$ and assume that $\mathcal{E}^\alpha(\mathbf{P}) < \infty$ and there exists $C > 0$ such that*

$$\int_0^\infty |\dot\gamma| \, dt \geq C \quad \text{for } \mathbf{P}\text{-a.e. } \gamma. \tag{6.1}$$

Then

(1) *the traffic plan $\mathbf{P}$ has a finite number of connected components in $\mathbb{R}^d$;*

(2) *for every $x \in \mathbb{R}^d \setminus (\mathrm{supp}(\mu_{\mathbf{P}}^-) \cup \mathrm{supp}(\mu_{\mathbf{P}}^+))$, $\mathbf{P}$ has a finite number of connected components in $\mathbb{R}^d \setminus \{x\}$.*

We observe that conclusion (2) in Theorem 6.2 is stronger than conclusion (1), since a connected component of $\mathbf{P}$ in $\mathbb{R}^d \setminus \{x\}$ is a subset of a connected component of $\mathbf{P}$ in $\mathbb{R}^d$. The proof of (2) is by contradiction. Firstly we will associate a canonical traffic plan to every connected component in $\mathbb{R}^d \setminus \{x\}$ of an optimal traffic plan (Lemma 6.6) and we will observe

that those traffic plans are disjoint and satisfy (6.1). Moreover we will show that the traffic plan associated to every connected component is optimal as well as the sum of two of them (Lemma 6.9). Lastly we will show, using Theorem 1.1, that for every infinite sequence of disjoint, optimal traffic plans satisfying (6.1), there are two elements of the sequence whose sum is not optimal (Lemma 6.10).

6.3. Remark. The hypothesis (6.1) is clearly implied by the stronger assumption, recurrent in the literature, that

$$d(\mathrm{supp}(\mu_{\mathbf{P}}^-), \mathrm{supp}(\mu_{\mathbf{P}}^+)) > 0. \tag{6.2}$$

A similar situation to the one presented in Theorem 6.2 happens for the irrigation problem (when the coupling between the initial and final measure is not assigned; see [BCM09, Lemma 8.10 and 8.11]) and the general strategies of proof are related in the two cases. On the other hand, in the irrigation problem one can push this result further and show that the traffic plan has the finite tree structure far from the support of the initial and target measures. For the mailing problem, this result is not true, indeed one could build examples where the optimal traffic plan contains cycles (far from the support of the initial and target measures) even when the marginals are given by a finite number of Dirac deltas; this introduces some additional difficulties to the arguments performed below.

In this section we will use the following

6.4. Lemma. *For every $\gamma \in \mathrm{Lip}_1$ let M_γ be a subset of $Im(\gamma)$ (chosen in a measurable fashion). Let M be a 1-rectifiable set and $\theta \in L^1(\mathscr{H}^1 \llcorner M)$ and consider the measure $\nu := \theta \mathscr{H}^1 \llcorner M$. Assume that ν can be written in the form $\nu = \int \nu_\gamma d\mathbf{P}(\gamma)$, where $\nu_\gamma := \mathscr{H}^1 \llcorner M_\gamma$ and $\mathbf{P} \in \mathscr{M}_+(\mathrm{Lip}_1)$. Then for ν-a.e. x it holds*

$$\theta(x) = \mathbf{P}(\{\gamma : x \in M_\gamma\}).$$

Proof. We can easily compute for every $E \subset M$

$$\int_E \theta d\mathscr{H}^1 = \nu(E) = \int_{\mathrm{Lip}_1} \nu_\gamma(E) d\mathbf{P}(\gamma) = \int_{\mathrm{Lip}_1} \int_{M_\gamma \cap E} d\mathscr{H}^1 d\mathbf{P}(\gamma)$$

$$= \int_E \int_{\{\gamma : x \in M_\gamma\}} d\mathbf{P}(\gamma) d\mathscr{H}^1(x) = \int_E \mathbf{P}(\{\gamma : x \in M_\gamma\}) d\mathscr{H}^1(x). \qquad \square$$

With the following two lemmata we associate a canonical traffic plan to every connected component of a given optimal traffic plan. For a traffic plan $\mathbf{P}$ and a connected component F of $\mathbf{P}$, we denote

$$A_F := \{\gamma \in \mathrm{Lip}_1 : \mathscr{H}^1(\mathrm{Im}(\gamma) \cap F) > 0\}. \tag{6.3}$$

6.5. Lemma. *Let $\mathbf{P}$ be an optimal traffic plan for the mailing problem, with finite energy, satisfying (6.1) and let F be a connected component of $\mathbf{P}$ in $\mathbb{R}^d$. Then*

   (i) $|y|_{\mathbf{P}} = |y|_{\mathbf{P} \llcorner A_F}$ *for every $y \in F$;*

   (ii) $|y|_{\mathbf{P} \llcorner A_F} = 0$ *for every $y \in \mathbb{R}^d \setminus F$.*

We call $\mathbf{P} \llcorner A_F$ the traffic plan associated to F.

Proof. The inequality $|y|_{\mathbf{P}} \geq |y|_{\mathbf{P} \llcorner A_F}$ in (i) follows from the fact that $\mathbf{P} \llcorner A_F \leq \mathbf{P}$. To prove the opposite inequality, assume by contradiction that there exists $y \in F$ such that $|y|_{\mathbf{P}} > |y|_{\mathbf{P} \llcorner A_F}$. Then, denoting

$$B_y := \{\gamma \in \mathrm{Lip}_1 \setminus A_F : y \in \mathrm{Im}(\gamma)\},$$

we have $\mathbf{P}(B_y) > 0$. By (6.1), the rectifiable measure μ defined for every Borel set E by

$$\mu(E) := \int_{B_y} \left(\int_{\mathrm{Im}(\gamma) \setminus F} \mathbf{1}_E(z) d\mathscr{H}^1(z) \right) d\mathbf{P}(\gamma)$$

is non-trivial. Observe also that μ is supported on $\mathbb{R}^d \setminus F$. Let $z \in \mathbb{R}^d \setminus F$ be a point of positive density for μ with respect to $\mathcal{H}^1$. By Lemma 6.4, we deduce that $\mathbf{P}_{y,z} + \mathbf{P}_{z,y} \neq 0$, hence, by the single path property $\Gamma_{y,z} \neq \emptyset$, hence $z \in F$, which is a contradiction.

The proof of (ii) is similar. Assume by contradiction that there exists $y \in \mathbb{R}^d \setminus F$ such that $|y|_{\mathbf{P} \llcorner A_F} \neq 0$. This means that, denoting

$$A_y := \{\gamma \in A_F : y \in \text{Im}(\gamma)\},$$

it holds $\mathbf{P}(A_y) > 0$. Again, we deduce that the rectifiable measure ν defined for every Borel set E by

$$\nu(E) := \int_{A_y} \left(\int_{\text{Im}(\gamma) \cap F} \mathbf{1}_E(z) d\mathcal{H}^1(z) \right) d\mathbf{P}(\gamma)$$

is non-trivial. Moreover ν is supported on F. Let $z \in F$ be a point of positive density for ν with respect to $\mathcal{H}^1$. As above, we deduce by Lemma 6.4 that $\Gamma_{y,z} \neq \emptyset$, hence $y \in F$, which is a contradiction. $\square$

Now consider an optimal traffic plan $\mathbf{P}$ and let us fix a point

$$x \in \Gamma_{\mathbf{P}} \setminus (\text{supp}(\mu_{\mathbf{P}}^-) \cup \text{supp}(\mu_{\mathbf{P}}^+)).$$

Let F be a connected component of $\mathbf{P}$ in $\mathbb{R}^d \setminus \{x\}$. We define the set

$$L_x := \{\gamma \in \text{Lip}_1 : x \in \text{Im}(\gamma)\}.$$

For $\mathbf{P}$-a.e. curve $\gamma \in L_x$ it holds $0 < t_x < T(\gamma)$. We define a new traffic plan $\mathbf{P}_x$ obtained "splitting in two" each curve in L_x, namely

$$\mathbf{P}_x := \mathbf{P} \llcorner (\text{Lip}_1 \setminus L_x) + (res_{0,t_x(\cdot)})_\sharp (\mathbf{P} \llcorner L_x) + (res_{t_x(\cdot),T(\cdot)})_\sharp (\mathbf{P} \llcorner L_x). \tag{6.4}$$

Observe that

$$|\mathbf{P}_x| = |\mathbf{P}| + \mathbf{P}(L_x). \tag{6.5}$$

6.6. Lemma. *Let $\mathbf{P}$ be an optimal traffic plan for the mailing problem, with finite energy, satisfying* (6.1) *and let us fix a point*

$$x \in \Gamma_{\mathbf{P}} \setminus (\text{supp}(\mu_{\mathbf{P}}^-) \cup \text{supp}(\mu_{\mathbf{P}}^+)). \tag{6.6}$$

Let F be a connected component of $\mathbf{P}$ in $\mathbb{R}^d \setminus \{x\}$ and let A_F be as in (6.3) *and $\mathbf{P}_x$ be as in* (6.4). *Then*

(i) $|y|_{\mathbf{P}} = |y|_{\mathbf{P}_x \llcorner A_F}$ *for every $y \in F$;*
(ii) $|y|_{\mathbf{P}_x \llcorner A_F} = 0$ *for every $y \in \mathbb{R}^d \setminus F$ with $y \neq x$.*

We call $\mathbf{P}_x \llcorner A_F$ the traffic plan associated to F.

Proof. Observe that by the loop free property of optimal traffic plans, for $\mathbf{P}$-a.e. curve $\gamma \in L_x$ and for every $y \in \text{Im}(\gamma) \setminus \{x\}$, exactly one curve among $res_{0,t_x(\gamma)}(\gamma)$ and $res_{t_x(\gamma),T(\gamma)}(\gamma)$ passes through y. This implies, together with (6.4), that for every $y \neq x$ it holds $|y|_{\mathbf{P}} = |y|_{\mathbf{P}_x}$ and in particular $|y|_{\mathbf{P}} \geq |y|_{\mathbf{P}_x \llcorner A_F}$. To complete the proof of (i), we proceed as in Lemma 6.5. Assume by contradiction that there exists $y \in F$ such that $|y|_{\mathbf{P}} > |y|_{\mathbf{P}_x \llcorner A_F}$. Then, denoting

$$B_y := \{\gamma \in \text{Lip}_1 \setminus A_F : y \in \text{Im}(\gamma)\},$$

we have $\mathbf{P}_x(B_y) > 0$. By (6.1) and (6.6), $\mathbf{P}_x$ is also supported on curves of positive length, hence the rectifiable measure μ defined for every Borel set E by

$$\mu(E) := \int_{B_y} \left(\int_{\text{Im}(\gamma) \setminus F} \mathbf{1}_E(z) d\mathcal{H}^1(z) \right) d\mathbf{P}_x(\gamma)$$

is non-trivial. Again μ is supported on $\mathbb{R}^d \setminus F$. Let $z \in \mathbb{R}^d \setminus F, z \neq x$ be a point of positive density for μ with respect to $\mathscr{H}^1$. By Lemma 6.4, we deduce that $(\mathbf{P}_x)_{y,z} + (\mathbf{P}_x)_{z,y} \neq 0$. Since by the definition of $\mathbf{P}_x$ we have

$$\mathbf{P}_x(\gamma \in \mathrm{Lip}_1 : x, y, z \in \mathrm{Im}(\gamma) \text{ and } t_y(\gamma) < t_x(\gamma) < t_z(\gamma) \text{ or } t_z(\gamma) < t_x(\gamma) < t_y(\gamma)) = 0,$$

then, denoting $(\Gamma_x)_{y,z}$ the set $\Gamma_{y,z}$ relative to $\mathbf{P}_x$, the fact that $(\mathbf{P}_x)_{y,z} + (\mathbf{P}_x)_{z,y} \neq 0$ implies that $(\Gamma_x)_{y,z}$ is non-empty and it is contained in $\mathbb{R}^d \setminus \{x\}$. Therefore z is connected to y in $\mathbb{R}^d \setminus \{x\}$, which implies the contradiction $z \in F$.

To prove (ii), assume by contradiction that there exists $y \in \mathbb{R}^d \setminus F$, with $y \neq x$, such that $|y|_{\mathbf{P}_x \llcorner A_F} \neq 0$. This means that, denoting

$$A_y := \{\gamma \in A_F : y \in \mathrm{Im}(\gamma)\},$$

it holds $\mathbf{P}_x(A_y) > 0$. Again, we deduce that the rectifiable measure ν defined for every Borel set E by

$$\nu(E) := \int_{A_y} \left(\int_{\mathrm{Im}(\gamma) \cap F} \mathbf{1}_E(z) d\mathscr{H}^1(z) \right) d\mathbf{P}_x(\gamma)$$

is non-trivial. Moreover ν is supported on F. Let $z \in F$ be a point of positive density for ν with respect to $\mathscr{H}^1$. As above, we deduce by Lemma 6.4 that $(\Gamma_x)_{y,z} \neq \emptyset$ and it is contained in $\mathbb{R}^d \setminus \{x\}$, hence $y \in F$, which is a contradiction. $\qquad\square$

6.7. Remark. Given $\mathbf{P}$ as in Lemma 6.5 and given F_1, F_2 distinct connected components of $\mathbf{P}$ in $\mathbb{R}^d$, it holds $\mathbf{P}(A_{F_1} \cap A_{F_2}) = 0$. This follows from the fact that, since $\mathbf{P}$ satisfies (6.1), then, by Remark 2.9, we have $\mathbb{M}^\alpha(\mathbf{P} \llcorner E) = \mathcal{E}^\alpha(\mathbf{P} \llcorner E) > 0$ for every E such that $\mathbf{P}(E) > 0$. Hence, assuming by contradiction that $\mathbf{P}(A_{F_1} \cap A_{F_2}) > 0$, Lemma 6.5 implies $F_1 \cap F_2 \neq \emptyset$. Similarly, if $\mathbf{P}$ and x are as in Lemma 6.6, F_1, F_2 are distinct connected components of $\mathbf{P}$ in $\mathbb{R}^d \setminus \{x\}$, it holds $\mathbf{P}_x(A_{F_1} \cap A_{F_2}) = 0$. In this case the fact that $\mathbb{M}^\alpha(\mathbf{P}_x \llcorner E) > 0$ whenever $\mathbf{P}(E) > 0$ follows from (6.1) and (6.6). As a consequence, both in the setting of Lemma 6.5 and of Lemma 6.6, there are at most countably many connected components and the associated traffic plans are disjoint. Moreover we remark that, since $\mathbf{P}$ is optimal, then it satisfies the single path and loop free properties and consequently the traffic plans associated to the connected components of $\mathbf{P}$ with respect to $\mathbb{R}^d$ or $\mathbb{R}^d \setminus \{x\}$ satisfy the single path and loop free properties as well.

6.8. Lemma. *Let $\mathbf{P}$ be an optimal traffic plan for the mailing problem, with finite energy, satisfying (6.1). Let $\{F_n\}_n$ be the (finite or countable) family of connected components of $\mathbf{P}$ in $\mathbb{R}^d$. Then the traffic plans $\mathbf{P}_n := \mathbf{P} \llcorner A_{F_n}$ associated to $\{F_n\}_n$ are optimal. Moreover for every couple $(n, m) \in \mathbb{N}^2$, the traffic plan $\mathbf{P}_n + \mathbf{P}_m$ is optimal.*

Proof. Up to reordering, we may assume by contradiction that $\mathbf{P}_1$ is not optimal for the pairing $\pi_{\mathbf{P} \llcorner A_{F_1}}$. Let $\mathbf{P}_1' \in \mathbf{OTP}(\pi_{\mathbf{P} \llcorner A_{F_1}})$ be a better competitor, i.e. by (2.7) $\mathbb{M}^\alpha(\mathbf{P}_1') < \mathbb{M}^\alpha(\mathbf{P}_1)$. Denoting $\mathbf{P}' := \mathbf{P}_1' + \sum_{n>1} \mathbf{P}_n$, it holds $\pi_{\mathbf{P}'} = \pi_{\mathbf{P}}$. Since $\mathbf{P} = \mathbf{P}_1 + \sum_{n>1} \mathbf{P}_n$ and since the traffic plans $\{\mathbf{P}_n\}_n$ are mutually disjoint by Remark 6.7, we can estimate

$$\mathbb{M}^\alpha(\mathbf{P}) = \mathbb{M}^\alpha(\mathbf{P}_1) + \mathbb{M}^\alpha(\sum_{n>1} \mathbf{P}_n) > \mathbb{M}^\alpha(\mathbf{P}_1') + \mathbb{M}^\alpha(\sum_{n>1} \mathbf{P}_n) \geq \mathbb{M}^\alpha(\mathbf{P}_1' + \sum_{n>1} \mathbf{P}_n) = \mathbb{M}^\alpha(\mathbf{P}').$$

Thanks to (2.7), the previous inequality contradicts the optimality of $\mathbf{P}$. The proof of the optimality of $\mathbf{P}_n + \mathbf{P}_m$ is analogous. $\qquad\square$

6.9. Lemma. *Let $\mathbf{P}$ be an optimal traffic plan for the mailing problem, with finite energy, satisfying (6.1) and let x be as in (6.6). Let $\{F_n\}_n$ be the (finite or countable) family of connected components of $\mathbf{P}$ in $\mathbb{R}^d \setminus \{x\}$. Then the traffic plans $\mathbf{P}_n := \mathbf{P}_x \llcorner A_{F_n}$ associated to $\{F_n\}_n$ are optimal. Moreover for every couple $(n, m) \in \mathbb{N}^2$, the traffic plan $\mathbf{P}_n + \mathbf{P}_m$ is optimal.*

Proof. The first part of the proof is entirely analogous to the proof of Lemma 6.8. Up to reordering, we may assume by contradiction that $\mathbf{P}_1$ is not optimal for the pairing $\pi_{\mathbf{P}_x \llcorner A_{F_1}}$. Let $\mathbf{P}_1'$ be a better competitor, i.e. by (2.7) $\mathbb{M}^\alpha(\mathbf{P}_1') < \mathbb{M}^\alpha(\mathbf{P}_1)$. Denoting $\mathbf{P}_x' := \mathbf{P}_1' + \sum_{n>1} \mathbf{P}_n$, it holds $\pi_{\mathbf{P}_x'} = \pi_{\mathbf{P}_x}$. Since $\mathbf{P}_x = \mathbf{P}_1 + \sum_{n>1} \mathbf{P}_n$ and since the traffic plans $\{\mathbf{P}_n\}_n$ are mutually disjoint by Remark 6.7, we can estimate

$$\mathbb{M}^\alpha(\mathbf{P}_x) = \mathbb{M}^\alpha(\mathbf{P}_1) + \mathbb{M}^\alpha(\sum_{n>1} \mathbf{P}_n) > \mathbb{M}^\alpha(\mathbf{P}_1') + \mathbb{M}^\alpha(\sum_{n>1} \mathbf{P}_n) \geq \mathbb{M}^\alpha(\mathbf{P}_1' + \sum_{n>1} \mathbf{P}_n) = \mathbb{M}^\alpha(\mathbf{P}_x'),$$

which would imply that $\mathbf{P}_x$ is not optimal for the coupling $\pi_{\mathbf{P}_x}$. Now we have to prove that this is a contradiction to the optimality of $\mathbf{P}$. Lemma 6.6 implies that

$$\mathbb{M}^\alpha(\mathbf{P}) = \mathbb{M}^\alpha(\sum_{n\in\mathbb{N}} \mathbf{P} \llcorner A_{F_n}) = \mathbb{M}^\alpha(\sum_{n\in\mathbb{N}} \mathbf{P}_x \llcorner A_{F_n}) = \mathbb{M}^\alpha(\mathbf{P}_x).$$

Hence, it is sufficient to show that we can construct a traffic plan $\mathbf{P}' \in \mathbf{TP}(\pi_\mathbf{P})$ such that $\mathbb{M}^\alpha(\mathbf{P}') \leq \mathbb{M}^\alpha(\mathbf{P}_x')$. Roughly speaking, $\mathbf{P}'$ will be obtained as follows: all the curves of $\mathbf{P}_x'$ with final point in x will be sewed with the curves of $\mathbf{P}_x'$ with initial point in x, while the remaining curves are left unchanged. Since $\pi_{\mathbf{P}_x'} = \pi_{\mathbf{P}_x}$, by (6.4) we have that

$$\pi_{\mathbf{P}_x'} = \pi_{\mathbf{P}_x} = \pi_{\mathbf{P} \llcorner (\mathrm{Lip}_1 \setminus L_x)} + \left(\mu_{\mathbf{P} \llcorner L_x}^- \times \mathbf{P}(L_x)\delta_x \right) + \left(\mathbf{P}(L_x)\delta_x \times \mu_{\mathbf{P} \llcorner L_x}^+ \right).$$

We disintegrate $\mathbf{P}_x'$ with respect to $\pi_{\mathbf{P}_x'}$:

$$\mathbf{P}_x' = \pi_{\mathbf{P}_x'}(y,z) \otimes \mathbf{P}_x'^{y,z},$$

so that we can define the three plans

$$\mathbf{P}_x'^s(A) := \int_{X \times X} \mathbf{P}_x'^{y,z}(A) d\left(\mu_{\mathbf{P} \llcorner L_x}^- \times \mathbf{P}(L_x)\delta_x \right)(y,z), \quad \forall A \subset \mathrm{Lip}_1 \text{ Borel},$$

$$\mathbf{P}_x'^f(A) := \int_{X \times X} \mathbf{P}_x'^{y,z} d\left(\mathbf{P}(L_x)\delta_x \times \mu_{\mathbf{P} \llcorner L_x}^+ \right)(y,z), \quad \forall A \subset \mathrm{Lip}_1 \text{ Borel},$$

$$\mathbf{P}_x'^r(A) := \int_{X \times X} \mathbf{P}_x'^{y,z} d\pi_{\mathbf{P} \llcorner (\mathrm{Lip}_1 \setminus L_x)}(y,z), \quad \forall A \subset \mathrm{Lip}_1 \text{ Borel}.$$

Obviously

$$\mathbf{P}_x' := \mathbf{P}_x'^r + \mathbf{P}_x'^s + \mathbf{P}_x'^f.$$

By Lemma 3.7, there exists a concatenation $\tilde{\mathbf{P}}' \in \mathbf{TP}(\pi_{\mathbf{P} \llcorner L_x})$ between $\mathbf{P}_x'^s$ and $\mathbf{P}_x'^f$. Defining $\mathbf{P}' := \mathbf{P}_x'^r + \tilde{\mathbf{P}}'$, we can easily observe that

$$(e_0, e_\infty)_\sharp \mathbf{P}' = (e_0, e_\infty)_\sharp \mathbf{P}_x'^r + (e_0, e_\infty)_\sharp \tilde{\mathbf{P}}' = \pi_{\mathbf{P} \llcorner (\mathrm{Lip}_1 \setminus L_x)} + \pi_{\mathbf{P} \llcorner L_x} = (e_0, e_\infty)_\sharp \mathbf{P},$$

so that $\mathbf{P}' \in \mathbf{TP}(\pi_\mathbf{P})$. Moreover we can estimate

$$\mathbb{M}^\alpha(\mathbf{P}') = \mathbb{M}^\alpha(\mathbf{P}_x'^r + \tilde{\mathbf{P}}') = \int (|z|_{\mathbf{P}_x'^r} + |z|_{\tilde{\mathbf{P}}'})^\alpha d\mathscr{H}^1(z) \overset{(3.4)}{\leq} \int (|z|_{\mathbf{P}_x'^r} + |z|_{\mathbf{P}_x'^s} + |z|_{\mathbf{P}_x'^f})^\alpha d\mathscr{H}^1(z)$$

$$= \mathbb{M}^\alpha(\mathbf{P}_x'^r + \mathbf{P}_x'^s + \mathbf{P}_x'^f) = \mathbb{M}^\alpha(\mathbf{P}_x') < \mathbb{M}^\alpha(\mathbf{P}_x) = \mathbb{M}^\alpha(\mathbf{P}).$$

$$(6.7)$$

This is a contradiction to the optimality of $\mathbf{P}$. The proof of the optimality of $\mathbf{P}_n + \mathbf{P}_m$ is analogous. Indeed, assuming the contrary, the very same proof above would imply that $\mathbf{P}_x$ is not optimal, which is a contradiction to the optimality of $\mathbf{P}$, as proved above. $\qquad\square$

6.10. Lemma. *Let $\alpha > 1 - \frac{1}{d}$ and $\{\mathbf{P}_n\}_{n=1}^\infty$ a sequence of disjoint traffic plans such that $\sum_{n\in\mathbb{N}} |\mathbf{P}_n| < +\infty$. If there exists $C > 0$ such that (6.1) is in force for every $n \in \mathbb{N}$, then there exists $n \in \mathbb{N}$ or a couple $(n,m) \in \mathbb{N}^2$ such that either $\mathbf{P}_n$ or $\mathbf{P}_n + \mathbf{P}_m$ is not optimal, i.e. either $\mathbf{P}_n \notin \mathbf{OTP}(\pi_{\mathbf{P}_n})$ or $\mathbf{P}_n + \mathbf{P}_m \notin \mathbf{OTP}(\pi_{\mathbf{P}_n + \mathbf{P}_m})$.*

Proof. Without loss of generality, we can assume that all traffic plans are supported on curves parametrized by arc-length. Indeed, assume we are able to prove Lemma 6.10 in case the elements of the sequence are supported on curves parametrized by arc-length. If $(\mathbf{P}_n)_{n=1}^\infty$ does not satisfy this assumption, since (6.1) is invariant under reparametrization, we can apply Lemma 6.10 to a sequence $(\tilde{\mathbf{P}}_n)_{n=1}^\infty$, obtained reparametrizing by arc-length the curves on which the elements of $(\mathbf{P}_n)_{n=1}^\infty$ are supported. Then there exists $n \in \mathbb{N}$ or a couple $(n,m) \in \mathbb{N}^2$ such that either $\tilde{\mathbf{P}}_n \notin \mathbf{OTP}(\pi_{\tilde{\mathbf{P}}_n})$ or $\tilde{\mathbf{P}}_n + \tilde{\mathbf{P}}_m \notin \mathbf{OTP}(\pi_{\tilde{\mathbf{P}}_n + \tilde{\mathbf{P}}_m})$. But, since

$$\mathcal{E}^\alpha(\tilde{\mathbf{P}}_n) = \mathcal{E}^\alpha(\mathbf{P}_n), \quad \mathcal{E}^\alpha(\tilde{\mathbf{P}}_n + \tilde{\mathbf{P}}_m) = \mathcal{E}^\alpha(\mathbf{P}_n + \mathbf{P}_m), \quad \pi_{\tilde{\mathbf{P}}_n} = \pi_{\mathbf{P}_n}, \quad \pi_{\tilde{\mathbf{P}}_n + \tilde{\mathbf{P}}_m} = \pi_{\mathbf{P}_n + \mathbf{P}_m},$$

then either $\mathbf{P}_n \notin \mathbf{OTP}(\pi_{\mathbf{P}_n})$ or $\mathbf{P}_n + \mathbf{P}_m \notin \mathbf{OTP}(\pi_{\mathbf{P}_n + \mathbf{P}_m})$.

Now we prove Lemma 6.10 in case the elements of $(\mathbf{P}_n)_{n=1}^\infty$ are supported on curves parametrized by arc-length. If there exists $n \in \mathbb{N}$ such that $\mathbf{P}_n$ is not optimal, the statement is true; we can consequently assume that all $\mathbf{P}_n$ are optimal. We first renormalize every plan $\mathbf{P}_n$, defining

$$\mathbf{P}'_n := \frac{\mathbf{P}_n}{|\mathbf{P}_n|}, \tag{6.8}$$

which is an optimal traffic plan of unit mass supported on curves parametrized by arc-length. Moreover, by Remark 3.9, there exists $C > 0$ such that $\mathbf{P}'_n \in \mathbf{TP}_C$ for every $n \in \mathbb{N}$. Up to extracting a subsequence, there exists $\mathbf{P}' \in \mathscr{P}(\mathrm{Lip}_1)$ such that

$$\mathbf{P}'_n \rightharpoonup \mathbf{P}'.$$

Taking the push-forward through the maps (e_0, e_∞), e_0 and e_∞, we deduce respectively that $\pi_{\mathbf{P}'_n} = \frac{\pi_{\mathbf{P}_n}}{|\mathbf{P}_n|} \rightharpoonup \pi_{\mathbf{P}'}$ and $\mu_{\mathbf{P}'_n}^\pm = \frac{\mu_{\mathbf{P}_n}^\pm}{|\mathbf{P}_n|} \rightharpoonup \mu_{\mathbf{P}'}^\pm$. We can apply Theorem 4.1 to deduce that, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n, m \geq n_0$, there exist two traffic plans $\mathbf{P}^1_{n,m} \in \mathbf{TP}(\mu_{\mathbf{P}'_n}^-, \mu_{\mathbf{P}'_m}^-)$, $\mathbf{P}^2_{n,m} \in \mathbf{TP}(\mu_{\mathbf{P}'_m}^+, \mu_{\mathbf{P}'_n}^+)$ such that

$$\mathcal{E}^\alpha(\mathbf{P}^1_{n,m}) \leq \varepsilon, \qquad \mathcal{E}^\alpha(\mathbf{P}^2_{n,m}) \leq \varepsilon \tag{6.9}$$

and there exists a traffic plan obtained as a concatenation between $\mathbf{P}'_m$ and $\mathbf{P}^2_{n,m}$, and a further concatenation $\mathbf{P}''$ between $\mathbf{P}^1_{n,m}$ and the latter concatenation such that $\mathbf{P}'' \in \mathbf{TP}(\pi_{\mathbf{P}'_n})$.

We claim that for n large enough, the following estimates hold

$$\mathcal{E}^\alpha(\mathbf{P}_n) \geq C|\mathbf{P}_n|^\alpha. \tag{6.10}$$

Indeed, since by assumption the traffic plans $\mathbf{P}_n$ are optimal, then $\mathbf{P}'_n \in \mathbf{OTP}(\pi_{\mathbf{P}'_n})$. Moreover, thanks to (6.1) for every $n \in \mathbb{N}$ we have that $\mathbf{P}_n$-a.e.γ (and hence $\mathbf{P}'_n$-a.e.γ) has length uniformly bounded from below by C. We compute

$$\mathcal{E}^\alpha(\mathbf{P}'_n) = \int_{\mathrm{Lip}_1} \int_{\mathbb{R}^+} |\gamma(t)|_{\mathbf{P}'_n}^{\alpha-1} |\dot{\gamma}(t)| dt \, d\mathbf{P}'_n(\gamma) \geq \int_{\mathrm{Lip}_1} T(\gamma) \, d\mathbf{P}'_n(\gamma) \geq C, \qquad \forall n \in \mathbb{N},$$

and renormalizing the traffic plans we deduce $\mathcal{E}^\alpha(\mathbf{P}_n) = \mathcal{E}^\alpha(\mathbf{P}'_n)|\mathbf{P}_n|^\alpha \geq C|\mathbf{P}_n|^\alpha$, namely (6.10). Moreover, we claim that for every $\varepsilon > 0$ (to be chosen later), for n and m large enough (depending on ε)

$$\mathcal{E}^\alpha(\mathbf{P}_n) \geq \frac{|\mathbf{P}_n|^\alpha}{|\mathbf{P}_m|^\alpha} \mathcal{E}^\alpha(\mathbf{P}_m) - \varepsilon|\mathbf{P}_n|^\alpha. \tag{6.11}$$

Indeed, we observe that Theorem 1.1 implies that the limit $\mathbf{P}'$ of the sequence of optimal plans $\{\mathbf{P}'_n\}_{n=1}^\infty$ is optimal and that $\mathcal{E}^\alpha(\mathbf{P}') = \lim_{n\to\infty} \mathcal{E}^\alpha(\mathbf{P}'_n)$. In particular $\mathcal{E}^\alpha(\mathbf{P}'_n)$ is a Cauchy sequence, that is for every $\varepsilon > 0$ we have

$$\varepsilon > |\mathcal{E}^\alpha(\mathbf{P}'_n) - \mathcal{E}^\alpha(\mathbf{P}'_m)| = \left| \frac{\mathcal{E}^\alpha(\mathbf{P}_n)}{|\mathbf{P}_n|^\alpha} - \frac{\mathcal{E}^\alpha(\mathbf{P}_m)}{|\mathbf{P}_m|^\alpha} \right|, \tag{6.12}$$

for every m, n sufficiently large (the latter equality follows by the definition (6.8) of $\mathbf{P}'_n$). This gives in particular (6.11).

Now we prove that $\mathbf{P}_n + \mathbf{P}_m$ is not optimal, exhibiting as better competitor the following traffic plan

$$\mathbf{P}''|\mathbf{P}_n| + \mathbf{P}'_m|\mathbf{P}_m| \in \mathbf{TP}(\pi_{\mathbf{P}_n + \mathbf{P}_m}).$$

Indeed, if we denote with $\mathbf{P}^1_m \in \mathbf{OTP}(\mu^-_{\mathbf{P}_m}, \mu^-_{\mathbf{P}_m})$ and $\mathbf{P}^2_m \in \mathbf{OTP}(\mu^+_{\mathbf{P}_m}, \mu^+_{\mathbf{P}_m})$ the unique plans supported on constant curves, we have that

$$\mathcal{E}^\alpha(\mathbf{P}^1_m) = \mathcal{E}^\alpha(\mathbf{P}^2_m) = 0,$$

and by Lemma 3.6 (we think the traffic plan $\mathbf{P}''|\mathbf{P}_n| + \mathbf{P}'_m|\mathbf{P}_m|$ as a concatenation between $\mathbf{P}^1_{n,m}|\mathbf{P}_n| + \mathbf{P}^1_m$, $(|\mathbf{P}_n| + |\mathbf{P}_m|)\mathbf{P}'_m$ and $\mathbf{P}^2_{n,m}|\mathbf{P}_n| + \mathbf{P}^2_m$), we can estimate its energy as follows:

$$\mathcal{E}^\alpha(\mathbf{P}''|\mathbf{P}_n| + \mathbf{P}'_m|\mathbf{P}_m|) \le \mathcal{E}^\alpha(\mathbf{P}^1_{n,m}|\mathbf{P}_n|) + \mathcal{E}^\alpha(\mathbf{P}^2_{n,m}|\mathbf{P}_n|) + \mathcal{E}^\alpha((|\mathbf{P}_n| + |\mathbf{P}_m|)\mathbf{P}'_m)$$

$$\overset{(6.9)}{\le} 2\varepsilon|\mathbf{P}_n|^\alpha + \mathcal{E}^\alpha\left(\left(1 + \frac{|\mathbf{P}_n|}{|\mathbf{P}_m|}\right)\mathbf{P}_m\right)$$

$$= 2\varepsilon|\mathbf{P}_n|^\alpha + \mathcal{E}^\alpha(\mathbf{P}_m)\left(1 + \frac{|\mathbf{P}_n|}{|\mathbf{P}_m|}\right)^\alpha \tag{6.13}$$

$$\le 2\varepsilon|\mathbf{P}_n|^\alpha + \mathcal{E}^\alpha(\mathbf{P}_m)\left(1 + \alpha\frac{|\mathbf{P}_n|}{|\mathbf{P}_m|}\right).$$

On the other hand, since $\mathbf{P}_n$ and $\mathbf{P}_m$ are disjoint and optimal traffic plans, by Proposition 2.7 we compute

$$\mathcal{E}^\alpha(\mathbf{P}_n + \mathbf{P}_m) \ge \mathbb{M}^\alpha(\mathbf{P}_n + \mathbf{P}_m) = \mathbb{M}^\alpha(\mathbf{P}_n) + \mathbb{M}^\alpha(\mathbf{P}_m) = \mathcal{E}^\alpha(\mathbf{P}_n) + \mathcal{E}^\alpha(\mathbf{P}_m)$$

$$\overset{(6.11)}{\ge} \mathcal{E}^\alpha(\mathbf{P}_m) + \frac{|\mathbf{P}_n|^\alpha}{|\mathbf{P}_m|^\alpha}\mathcal{E}^\alpha(\mathbf{P}_m) - \varepsilon|\mathbf{P}_n|^\alpha. \tag{6.14}$$

Combining the previous inequalities (6.13) and (6.14) we have

$$\mathcal{E}^\alpha(\mathbf{P}''|\mathbf{P}_n| + \mathbf{P}'_m|\mathbf{P}_m|) - \mathcal{E}^\alpha(\mathbf{P}_n + \mathbf{P}_m) \le \left(\alpha\frac{|\mathbf{P}_n|}{|\mathbf{P}_m|} - \frac{|\mathbf{P}_n|^\alpha}{|\mathbf{P}_m|^\alpha}\right)\mathcal{E}^\alpha(\mathbf{P}_m) + 3\varepsilon|\mathbf{P}_n|^\alpha$$

$$\overset{(6.10)}{\le} \left(\alpha\frac{|\mathbf{P}_n|}{|\mathbf{P}_m|} - \frac{|\mathbf{P}_n|^\alpha}{|\mathbf{P}_m|^\alpha}\right)\mathcal{E}^\alpha(\mathbf{P}_m) + \frac{3}{C}\varepsilon\frac{|\mathbf{P}_n|^\alpha}{|\mathbf{P}_m|^\alpha}\mathcal{E}^\alpha(\mathbf{P}_m) \tag{6.15}$$

$$\le \left(\alpha\frac{|\mathbf{P}_n|}{|\mathbf{P}_m|} - \left(1 - \frac{3}{C}\varepsilon\right)\frac{|\mathbf{P}_n|^\alpha}{|\mathbf{P}_m|^\alpha}\right)\mathcal{E}^\alpha(\mathbf{P}_m) < 0,$$

where in the last inequality we chose ε small enough to get $1 - \frac{3}{C}\varepsilon > \alpha$ and n, m in order to guarantee that $\frac{|\mathbf{P}_n|}{|\mathbf{P}_m|} < 1$ and consequently $\frac{|\mathbf{P}_n|}{|\mathbf{P}_m|} < \frac{|\mathbf{P}_n|^\alpha}{|\mathbf{P}_m|^\alpha}$. We deduce from inequality (6.15) that $\mathbf{P}_n + \mathbf{P}_m$ is not optimal. $\square$

Proof of Theorem 6.2. As already observed after the statement of the theorem, it is sufficient to prove (2). Assume by contradiction that the sequence $\{\mathbf{P}_n\}_{n=1}^\infty$ of the traffic plans canonically associated to the connected components of $\mathbf{P}$ in $\mathbb{R}^d \setminus \{x\}$ by Lemma 6.6 is infinite. Since (6.1) is in force for the traffic plan $\mathbf{P}$ and x satisfies (6.6), then by Lemma 6.6, there exists a constant C such that (6.1) holds for every $\mathbf{P}_n$. Moreover, by Lemma 6.6 and (6.5), we have that

$$\sum_{n \in \mathbb{N}} |\mathbf{P}_n| = |\mathbf{P}_x| \le 2|\mathbf{P}|.$$

We are in the hypothesis to apply Lemma 6.10, hence there exists $n \in \mathbb{N}$ or a couple $(n, m) \in \mathbb{N}^2$ such that $\mathbf{P}_n$ or $\mathbf{P}_n + \mathbf{P}_m$ is not optimal. This is a contradiction to Lemma 6.9. $\square$

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