

ASYMPTOTIC MEAN VALUE PROPERTIES FOR FRACTIONAL ANISOTROPIC OPERATORS

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ABSTRACT. We obtain an asymptotic representation formula for harmonic functions with respect to a linear anisotropic nonlocal operator. Furthermore we get a Bourgain-Brezis-Mironescu type limit formula for a related class of anisotropic nonlocal norms.

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1. INTRODUCTION

This paper presents an asymptotic mean value property for harmonic functions for a class of anisotropic nonlocal operators. To introduce the argument, we notice that as known from elementary PDEs facts, a C^2 function $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is harmonic in Ω (i.e. it holds that $\Delta u = 0$ in Ω) if and only if it satisfies the mean value property, that is

$$u(x) = \int_{B_r(x)} u(y) dy, \quad \text{whenever } B_r(x) \subset \Omega.$$

As a matter of fact, this condition can be relaxed to a pointwise formulation by saying that $u \in C^2(\Omega)$ satisfies $\Delta u(x) = 0$ at a point $x \in \Omega$ if and only if

$$(1.1) \quad u(x) = \int_{B_r(x)} u(y) dy + o(r^2), \quad \text{as } r \rightarrow 0.$$

This asymptotic formula holds true also in the viscosity sense for any continuous function. A similar property can be proved for quasi-linear elliptic operators such as the p -Laplace operator $-\Delta_p u$ in the asymptotic form, as the radius r of the ball vanishes. More precisely, Manfredi, Parviainen and Rossi proved [18] that, if $p \in (1, \infty]$, a continuous function $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is p -harmonic in Ω in viscosity sense if and only if

$$(1.2) \quad \varphi(x) \geq (\leq) \frac{p-2}{2p+2n} \left(\max_{B_r(x)} \varphi + \min_{B_r(x)} \varphi \right) + \frac{2+n}{p+n} \int_{B_r(x)} \varphi(y) dy + o(r^2),$$

for any $\varphi \in C^2$ such that $u - \varphi$ has a strict minimum (strict maximum for \leq) at $x \in \overline{\Omega}$ at the zero level. Notice that formula (1.2) reduces to (1.1) for $p = 2$. Formula (1.2) holds in the classical sense for smooth functions, at those points $x \in \overline{\Omega}$ such that $\nabla u(x) \neq 0$. On the other hand, the

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case $p = \infty$ offers a counterexample for the validity of (1.2) in the classical sense, since the function $|x|^{4/3} - |y|^{4/3}$ is ∞ -harmonic in \mathbb{R}^2 in the viscosity sense but (1.2) fails to hold pointwisely. If $p \in (1, \infty)$ and $n = 2$ Arroyo and LLorente [3] (see also [16]) proved that the characterization holds in the classical sense. The limit case $p = 1$ was finally investigated in [14].

Since the local (linear and nonlinear) case is well understood, it is natural to wonder about the validity of some kind of asymptotic mean value property in the *nonlocal* case. As a first approach, we want to investigate this type of property for a nonlocal, linear, anisotropic operator, defined as

$$(1.3) \quad \mathcal{L}u(x) = \int_0^\infty d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{\delta(u, x, \rho\omega)}{\rho^{1+2s}},$$

where

$$\delta(u, x, y) := 2u(x) - u(x - y) - u(x + y).$$

Here, a is a non-negative measure on \mathbb{S}^{n-1} , finite i.e.

$$\int_{\mathbb{S}^{n-1}} da \leq \Lambda$$

for some real number $\Lambda > 0$. We refer to this type of measure as *spectral measure*, as it is common in the literature. We notice that when the measure a is absolutely continuous with respect to the Lebesgue measure, i.e. when

$$da(\omega) = a(\omega) d\mathcal{H}^{n-1}(\omega)$$

for a suitable, non-negative function $a \in L^1(\mathbb{S}^{n-1})$, the operator can be represented (using polar coordinates) as

$$(1.4) \quad \mathcal{L}u(x) = \int_{\mathbb{R}^n} \delta(u, x, y) a\left(\frac{y}{|y|}\right) \frac{dy}{|y|^{n+2s}}.$$

Moreover, if $a \equiv 1$ then the formula gets more familiar, as we obtain the well-known fractional Laplace operator (see, e.g. [9, 11, 22] and other references therein).

We remark also that the operator \mathcal{L} is pointwise defined in an open set $\Omega \subset \mathbb{R}^n$ when, for instance, $u \in C^{2s+\varepsilon}(\Omega) \cap L^\infty(\mathbb{R}^n)$. (Here, $C^{2s+\varepsilon}(\Omega)$ denotes $C^{0,2s+\varepsilon}(\Omega)$ or $C^{1,2s+\varepsilon-1}(\Omega)$ for a small $\varepsilon > 0$, when $2s + \varepsilon < 1$, respectively when $2s + \varepsilon \geq 1$.)

As a matter of fact, the operator (1.3) has been widely studied in the literature, being \mathcal{L} the generator of any stable, symmetric Levy process. In particular, regularity issues for harmonic functions of \mathcal{L} have been studied in papers like [4–6, 13, 23] (check also other numerous references therein). There, some additional condition are required to the measure, in order to assure regularity. A typical assumption when \mathcal{L} is of the form (1.4) is

$$0 < c \leq a(y) \leq C \quad \text{in } \mathbb{S}^{n-1},$$

or less restrictively

$$a(y) \geq c > 0 \quad \text{in a subset of positive measure } \Sigma \subset \mathbb{S}^{n-1}.$$

Furthermore, for instance in [4] the measure needs not to be absolutely continuous with respect to the Lebesgue measure. In [20, 21], the optimal regularity is proved for general operators of the form (1.3), with the “ellipticity” assumption

$$(1.5) \quad \inf_{\bar{\omega} \in \mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} |\omega \cdot \bar{\omega}|^{2s} da(\omega) \geq \lambda > 0$$

for some real number λ . We note that the assumptions (1.5) are satisfied by any stable operator with the spectral measure which is n -dimensional (that is, when the measure is not supported on any proper hyperplane of \mathbb{R}^n). We will discuss some details related to this ellipticity requirement

in Section 4.

In this paper, for (1.3) and (1.5), we will adopt a potential theory approach, by using a “mean kernel”, and provide a necessary and sufficient condition for a function to be harmonic for \mathcal{L} , in the viscosity sense. To be more precise, we denote for some $r > 0$

$$\mathcal{M}_r^s u(x) := c(n, s, a) r^{2s} \int_r^\infty d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{u(x + \rho\omega) + u(x - \rho\omega)}{(\rho^2 - r^2)^s \rho},$$

with

$$c(n, s, a) := \frac{\sin \pi s}{\pi} \left(\int_{\mathbb{S}^{n-1}} da \right)^{-1}.$$

The following is the main result of the paper.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in L^\infty(\mathbb{R}^n)$. The asymptotic expansion*

$$(1.6) \quad u(x) = \mathcal{M}_r^s u(x) + \mathcal{O}(r^2), \quad \text{as } r \rightarrow 0$$

holds for all $x \in \Omega$ in the viscosity sense if and only if

$$\mathcal{L}u(x) = 0$$

in the viscosity sense.

We point out here the paper [12], where the author studies a general type of nonlocal operators defined by means of mean value kernels. Furthermore, the readers can check [1, 8, 15] for details on the mean kernel and the mean value property for the fractional Laplacian.

As further results, we provide some asymptotics for $s \nearrow 1$ of the operator \mathcal{L} and of the mean value \mathcal{M}_r^s . We also prove a Bourgain-Brezis-Mironescu type of formula, for a nonlocal norm related to the operator \mathcal{L} . In fact, as $s \nearrow 1$, we obtain the “integer”, local counterpart of the objects under study.

This paper is organized as follows: in the next section we introduce some notations and some preliminary results. Section 3 deals with the viscosity setting and with the proof of Theorem 1.1. In Section 4 we make some remarks about the weak setting. In the last Section 5 we study the asymptotic behavior as $s \nearrow 1$ of our fractional operators and prove a Bourgain-Brezis-Mironescu type of formula.

2. PRELIMINARY RESULTS AND NOTATIONS

Notations

We use the following notations throughout this paper.

- For some $r > 0$ and any $x \in \mathbb{R}^n$

$$B_r(x) := \{y \in \mathbb{R}^n \mid |x - y| < r\}, \quad B_r := B_r(0). \\ \mathbb{S}^{n-1} = \partial B_1.$$

- For any $x > 0$, the Gamma function is (see [2], Chapter 6):

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

- For any $x, y > 0$, the Beta function is (see [2], Chapter 6):

$$\beta(x, y) = \int_0^\infty \frac{t^{x-1}}{(t+1)^{x+y}} dt.$$

We remark as a first thing the following integral identity.

Lemma 2.1. *For any $r > 0$*

$$\frac{2 \sin \pi s}{\pi} r^{2s} \int_r^\infty \frac{d\rho}{(\rho^2 - r^2)^s \rho} = 1.$$

Proof. Changing coordinates, we get that

$$r^{2s} \int_r^\infty \frac{d\rho}{(\rho^2 - r^2)^s \rho} = \frac{1}{2} \int_0^\infty \frac{dt}{t^s(t+1)} = \frac{\beta(1-s, s)}{2} = \frac{\pi}{2 \sin(\pi s)},$$

where we have used formulas 6.2.2 and 6.1.17 in [2]. \square

We obtain the asymptotic mean value property for smooth functions, as follows.

Theorem 2.2. *Let $u \in C^2(\Omega) \cap L^\infty(\mathbb{R}^n)$. Then*

$$u(x) = \mathcal{M}_r^s u(x) + c(n, s, a) r^{2s} \mathcal{L}u(x) + \mathcal{O}(r^2)$$

as $r \rightarrow 0$.

Proof. We fix $x \in \Omega$ and $\delta > 0$ such that $B_{2\delta}(x) \subset \Omega$. For any $0 < r < \delta$, by Lemma 2.1 we have that

$$\begin{aligned} u(x) - \mathcal{M}_r^s u(x) &= c(n, s, a) r^{2s} \int_r^\infty d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{2u(x)}{(\rho^2 - r^2)^s \rho} - \mathcal{M}_r^s u(x) \\ &= c(n, s, a) r^{2s} \int_r^\infty d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{\delta(u, x, \rho\omega)}{(\rho^2 - r^2)^s \rho} \\ &= c(n, s, a) \int_1^\infty d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{\delta(u, x, r\rho\omega)}{(\rho^2 - 1)^s \rho}. \end{aligned}$$

Notice that $1 < \delta/r$, so we write

$$(2.1) \quad \begin{aligned} \frac{u(x) - \mathcal{M}_r^s u(x)}{c(n, s, a)} &= \int_{\frac{\delta}{r}}^\infty d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{\delta(u, x, r\rho\omega)}{(\rho^2 - 1)^s \rho} + \int_1^{\frac{\delta}{r}} d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{\delta(u, x, r\rho\omega)}{(\rho^2 - 1)^s \rho} \\ &=: \mathcal{I}_1 + \mathcal{I}_2. \end{aligned}$$

We have that

$$\mathcal{I}_1 = \int_{\frac{\delta}{r}}^\infty d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{\delta(u, x, r\rho\omega)}{\rho^{1+2s}} \frac{1}{\left(1 - \frac{1}{\rho^2}\right)^s}.$$

Denote by

$$t := \frac{1}{\rho} \in \left(0, \frac{r}{\delta}\right)$$

and for r small enough with a Taylor expansion we have that

$$(1 - t^2)^{-s} = 1 + st^2 + \mathfrak{o}(t^2).$$

So

$$\begin{aligned} \mathcal{I}_1 &= \int_{\frac{\delta}{r}}^\infty d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{\delta(u, x, r\rho\omega)}{\rho^{1+2s}} + s \int_{\frac{\delta}{r}}^\infty d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{\delta(u, x, r\rho\omega)}{\rho^{3+2s}} \\ &\quad + \int_{\frac{\delta}{r}}^\infty d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \delta(u, x, r\rho\omega) \mathfrak{o}(\rho^{-3-2s}). \end{aligned}$$

Notice that

$$\begin{aligned} \left| \int_{\frac{\delta}{r}}^{\infty} d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{\delta(u, x, r\rho\omega)}{\rho^{3+2s}} \right| &\leq 4\|u\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{S}^{n-1}} da \int_{\frac{\delta}{r}}^{\infty} \frac{d\rho}{\rho^{3+2s}} \\ &\leq 2\Lambda\|u\|_{L^\infty(\mathbb{R}^n)} \frac{r^{2+2s}\delta^{-2-2s}}{1-s}. \end{aligned}$$

It follows that

$$\mathcal{I}_1 = \int_{\frac{\delta}{r}}^{\infty} d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{\delta(u, x, r\rho\omega)}{\rho^{1+2s}} + \mathcal{O}(r^{2+2s}).$$

On the other hand we write

$$\begin{aligned} \mathcal{I}_2 &= \int_1^{\frac{\delta}{r}} d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{\delta(u, x, r\rho\omega)}{\rho^{1+2s}} \\ &\quad + \int_1^{\frac{\delta}{r}} d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{\delta(u, x, r\rho\omega)}{\rho^{1+2s}} \left(\frac{\rho^{2s}}{(\rho^2 - 1)^s} - 1 \right). \end{aligned}$$

Then putting together \mathcal{I}_1 and \mathcal{I}_2 into (2.1) we have that

$$\begin{aligned} \frac{u(x) - \mathcal{M}_r^s u(x)}{c(n, s, a)} &= \int_1^{\infty} d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{\delta(u, x, r\rho\omega)}{\rho^{1+2s}} \\ &\quad + \int_1^{\frac{\delta}{r}} d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{\delta(u, x, r\rho\omega)}{\rho^{1+2s}} \left(\frac{\rho^{2s}}{(\rho^2 - 1)^s} - 1 \right) + \mathcal{O}(r^{2+2s}) \\ (2.2) \quad &= \int_0^{\infty} d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{\delta(u, x, r\rho\omega)}{\rho^{1+2s}} \\ &\quad + \int_1^{\frac{\delta}{r}} d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{\delta(u, x, r\rho\omega)}{\rho^{1+2s}} \left(\frac{\rho^{2s}}{(\rho^2 - 1)^s} - 1 \right) \\ &\quad - \int_0^1 d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{\delta(u, x, r\rho\omega)}{\rho^{1+2s}} + \mathcal{O}(r^{2+2s}) \\ &=: r^{2s} \mathcal{L}u(x) + \mathcal{J} + \mathcal{O}(r^{2+2s}). \end{aligned}$$

Recalling that $u \in C^2(\Omega)$, both for $\rho \in (1, \delta/r)$ and for $\rho \in (0, 1)$ we have that

$$|\delta(u, x, r\rho\omega)| \leq r^2 \rho^2 \|u\|_{C^2(B_\delta(x))}.$$

We thus obtain

$$\begin{aligned} |\mathcal{J}| &\leq \|u\|_{C^2(\Omega)} r^2 \int_{\mathbb{S}^{n-1}} da(\omega) \left(\int_1^{\frac{\delta}{r}} d\rho \rho^{1-2s} \left(\frac{\rho^{2s}}{(\rho^2 - 1)^s} - 1 \right) + \int_0^1 d\rho \rho^{1-2s} \right) \\ &= C \frac{r^2 (1 + \mathcal{O}(r^{2s}))}{2(1-s)}. \end{aligned}$$

The last line follows since

$$\int \rho^{1-2s} \left(\frac{\rho^{2s}}{(\rho^2 - 1)^s} - 1 \right) d\rho = -\frac{\rho^{2-2s} - (\rho^2 - 1)^{1-s}}{2-2s}.$$

Hence in (2.2) we finally get that

$$u(x) - \mathcal{M}_r^s u(x) = c(n, s, a) r^{2s} \mathcal{L}u(x) + \mathcal{O}(r^2).$$

This concludes the proof of the theorem. \square

3. VISCOSITY SETTING

We begin by giving the definitions for the viscosity setting. First of all (as in [10]), we define the notion of viscosity solution.

Definition 3.1. A function $u \in L^\infty(\mathbb{R}^n)$, lower (upper) semi-continuous in $\overline{\Omega}$ is a viscosity supersolution (subsolution) to

$$\mathcal{L}u = 0, \quad \text{and we write } \mathcal{L}u \leq (\geq) 0$$

if for every $x \in \Omega$, any neighborhood $U = U(x) \subset \Omega$ and any $\varphi \in C^2(\overline{U})$ such that

$$\begin{aligned} \varphi(x) &= u(x) \\ \varphi(y) &< (>) u(y), \quad \text{for any } y \in U \setminus \{x\}, \end{aligned}$$

if we let

$$(3.1) \quad v = \begin{cases} \varphi, & \text{in } U \\ u, & \text{in } \mathbb{R}^n \setminus U \end{cases}$$

then

$$\mathcal{L}v(x) \geq (\leq) 0.$$

A viscosity solution of $\mathcal{L}u = 0$ is a (continuous) function that is both a subsolution and a supersolution.

Definition 3.2. A function $u \in L^\infty(\mathbb{R}^n)$, lower (upper) semi-continuous in $\overline{\Omega}$, verifies for any $x \in \Omega$

$$u(x) = \mathcal{M}_r^s u(x) + \mathcal{O}(r^2)$$

as $r \rightarrow 0$, in a viscosity sense, if for any neighborhood $U = U(x) \subset \Omega$ and any $\varphi \in C^2(\overline{U})$ such that

$$\begin{aligned} \varphi(x) &= u(x) \\ \varphi(y) &< (>) u(y), \quad \text{for any } y \in U \setminus \{x\}, \end{aligned}$$

if we let

$$v = \begin{cases} \varphi, & \text{in } U \\ u, & \text{in } \mathbb{R}^n \setminus U \end{cases}$$

then

$$(3.2) \quad v(x) \geq (\leq) \mathcal{M}_r^s v(x) + \mathcal{O}(r^2).$$

We can now prove the main theorem of this paper.

Proof of Theorem 1.1. Let $x \in \Omega$ and any $R > 0$ be such that $\overline{B_R(x)} \subset \Omega$. Let $\varphi \in C^2(\overline{B_R(x)})$ be such that

$$\begin{aligned} \varphi(x) &= u(x) \\ \varphi(y) &< u(y), \quad \text{for any } y \in B_R(x) \setminus \{x\}. \end{aligned}$$

We let v be defined as in (3.1), hence $v \in C^2(B_R(x)) \cap L^\infty(\mathbb{R}^n)$. By Theorem 2.2 we have that

$$(3.3) \quad v(x) = \mathcal{M}_r^s v(x) + c(n, s, a)r^{2s}\mathcal{L}v(x) + \mathcal{O}(r^2).$$

We prove at first that if u satisfies (1.6) in the viscosity sense given by Definition 3.2 then u is a supersolution of $\mathcal{L}u(x) = 0$ in the viscosity sense. Since

$$v(x) \geq \mathcal{M}_r^s v(x) + \mathcal{O}(r^2)$$

dividing by r^{2s} in (3.3) and sending $r \rightarrow 0$, it follows that

$$\mathcal{L}v(x) \geq 0.$$

At the same manner, one proves that u is a subsolution of $\mathcal{L}u(x) = 0$ in the viscosity sense.

In order to prove the other implication, if u is a supersolution, one has from (3.3) that

$$\limsup_{r \rightarrow 0} \frac{v(x) - \mathcal{M}_r^s v(x)}{r^{2s}} \geq 0,$$

hence (3.2). In the same way, one gets the conclusion when u is a subsolution. \square

4. SOME REMARKS ABOUT THE WEAK SETTING

In this section we consider the weak setting, and in particular provide the condition for a weak solution to be a pointwise, and a viscosity solution. In this sense, Theorem 1.1 applies also to weak solutions.

Let us now define some new spaces by setting

$$[u]_{H_a^1(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} dx \int_{\mathbb{S}^{n-1}} da(\omega) (\nabla u(x) \cdot \omega)^2 \right)^{\frac{1}{2}}, \quad \|u\|_{H_a^1(\mathbb{R}^n)} := [u]_{H_a^1(\mathbb{R}^n)} + \|u\|_{L^2(\mathbb{R}^n)},$$

$$H_a^1(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n) \mid [u]_{H_a^1(\mathbb{R}^n)} < +\infty\}, \quad H_{a,0}^1(\mathbb{R}^n) := \overline{C_c^\infty(\mathbb{R}^n)}^{\|\cdot\|_{H_a^1(\mathbb{R}^n)}},$$

and

$$[u]_{H_a^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} dx \int_{\mathbb{R}} d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{(u(x) - u(x + \rho\omega))^2}{|\rho|^{1+2s}} \right)^{\frac{1}{2}}.$$

Taking into account that

$$\|v\| := \left(\int_{\mathbb{S}^{n-1}} (v \cdot \omega)^2 da(\omega) \right)^{\frac{1}{2}}, \quad v \in \mathbb{R}^n,$$

is a norm in \mathbb{R}^n , it is readily seen that $H_a^1(\mathbb{R}^n)$ is a Banach space.

We take the operator \mathcal{L} that satisfies the ellipticity assumption (1.5) and we consider weak solutions of the equation

$$\mathcal{L}u = 0 \quad \text{in } \Omega.$$

In particular, we consider $u \in L^\infty(\mathbb{R}^n)$ of finite energy, i.e. such that

$$(4.1) \quad [u]_{H_a^s(\mathbb{R}^n)} < \infty$$

that satisfies

$$\int_{\mathbb{R}^n} dx \int_{\mathbb{R}} d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{(u(x) - u(x + \rho\omega))(\varphi(x) - \varphi(x + \rho\omega))}{|\rho|^{1+2s}} = 0$$

for any $\varphi \in C_c^\infty(\Omega)$.

We notice that, from [20, Theorem 1.1, b)] we have that if $u \in L^\infty(\mathbb{R}^n) \cap C^\alpha(\mathbb{R}^n)$ for some $\alpha > 0$, we get that $u \in C^{\alpha+2s}(B_r(p))$ for all $r > 0$ and $p \in \Omega$ for which $B_{2r}(p) \subset \Omega$. As remarked in [20], one cannot remove the hypothesis that $u \in C^\alpha(\mathbb{R}^n)$, in order to obtain the $C^{2s+\alpha}$ regularity of u . So, in this way, a weak solution of $\mathcal{L}u = 0$ in Ω is a pointwise solution and a viscosity solution in say, Ω' , taking

$$\Omega' := \bigcup_{\substack{p \in \Omega, r > 0 \\ B_{2r}(p) \subset \Omega}} B_r(p).$$

Notice that when Ω has C^2 boundary, Ω' can be taken as any compactly contained subset of Ω . We have the next result.

Theorem 4.1. *Let $\alpha > 0$ be such that $\alpha + 2s$ is not an integer. Let $u \in L^\infty(\mathbb{R}^n) \cap C^\alpha(\mathbb{R}^n)$ be of finite energy, and be a weak solution of $\mathcal{L}u = 0$ in Ω . Then $u \in C^{2\alpha+s}(\Omega')$ is a pointwise and a viscosity solution of $\mathcal{L}u = 0$ in Ω' .*

Proof. We recall that, as just noticed, $u \in C^{2s+\alpha}(\overline{\Omega}')$. Hence $\mathcal{L}u(x)$ is well defined for any $x \in \Omega'$. We take any $\varphi \in C_c^\infty(\Omega')$ and since u is a weak solution, we have that

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} dx \int_{\mathbb{R}} d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{(u(x) - u(x + \rho\omega))(\varphi(x) - \varphi(x + \rho\omega))}{|\rho|^{1+2s}} \\ &= \int_{\mathbb{R}^n} dx \int_{\mathbb{R}} d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{(2u(x) - u(x + \rho\omega) - u(x - \rho\omega))\varphi(x)}{|\rho|^{1+2s}} \end{aligned}$$

with a change of variables, given the smoothness of u . Thus pointwise in Ω'

$$0 = \int_{\mathbb{R}} d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{2u(x) - u(x + \rho\omega) - u(x - \rho\omega)}{|\rho|^{1+2s}} = 2\mathcal{L}u(x).$$

Furthermore, consider v that touches u at $x_0 \in \Omega'$ from below (as defined in (3.1)) i.e. $u(x_0) = v(x_0)$, with $v - u \leq 0$ in \mathbb{R}^n . Then

$$\mathcal{L}v(x_0) = \int_0^\infty d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{2v(x_0) - v(x_0 + \rho\omega) - v(x_0 - \rho\omega)}{|\rho|^{1+2s}} \geq \mathcal{L}u(x_0) = 0.$$

This proves that u is a supersolution (in the same way, one can prove that u is a subsolution), and concludes the proof of the Theorem. \square

5. ASYMPTOTICS AS $s \nearrow 1$

In this section we provide some asymptotic properties on the operator \mathcal{L} , the mean value defined by \mathcal{M}_r^s and the semi-norm in (4.1). We study their limit behavior as s approaches the upper value 1. Indeed, re-normalizing (multiplying by $(1-s)$) and sending $s \nearrow 1$, we obtain the local counterpart of the operators under study. It is interesting in our opinion, from this point of view, to understand what is the influence of the non symmetric measure da in the limit, with respect to having the $d\mathcal{H}^{n-1}$ measure on the hypersphere.

We begin by showing the following.

Proposition 5.1. *Let $u \in C^2(\Omega) \cap L^\infty(\mathbb{R}^n)$. Then for all $x \in \Omega$*

$$\lim_{s \nearrow 1} (1-s)\mathcal{L}u(x) = -\frac{1}{2} \sum_{i,j=1}^n \left(\int_{\mathbb{S}^{n-1}} da(\omega) \omega_i \omega_j \right) \partial_{ij}^2 u(x).$$

Proof. We fix $x \in \Omega$. By a Taylor expansion, we have that for every $\rho > 0$ (such that $B_\rho(x) \subset \Omega$) and every $\omega \in \mathbb{S}^{n-1}$, there exists $\underline{h} := \underline{h}(\rho, \omega), \bar{h} := \bar{h}(\rho, \omega) \in [0, \rho]$ such that

$$\delta(u, x, \rho\omega) = -\frac{\rho^2}{2} \langle D^2 u(x + \bar{h}\omega) \omega, \omega \rangle - \frac{\rho^2}{2} \langle D^2 u(x - \underline{h}\omega) \omega, \omega \rangle.$$

Since $u \in C^2(\Omega)$, we have that for any $\varepsilon > 0$ there exists $r := r(\varepsilon) > 0$ such that

$$(5.1) \quad \left| \langle (D^2 u(x + h\omega) - D^2 u(x)) \omega, \omega \rangle \right| \leq \left| D^2 u(x + h\omega) - D^2 u(x) \right| |\omega|^2 < \varepsilon, \\ \text{whenever } |h| = |h\omega| \leq \rho < r.$$

Fixing an arbitrary ε and taking the corresponding $r := r(\varepsilon)$, we write

$$\begin{aligned}
\mathcal{L}u(x) &= \int_0^r d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{\delta(u, x\rho\omega)}{\rho^{1+2s}} + \int_r^\infty d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{\delta(u, x\rho\omega)}{\rho^{1+2s}} \\
&= -\frac{1}{2} \int_0^r d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \rho^{1-2s} \langle D^2u(x + \bar{h}\omega)\omega, \omega \rangle \\
&\quad - \frac{1}{2} \int_0^r d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \rho^{1-2s} \langle D^2u(x - \underline{h}\omega)\omega, \omega \rangle \\
&\quad + \int_r^\infty d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{\delta(u, x\rho\omega)}{\rho^{1+2s}} \\
&= : \left(-\frac{1}{2}\right) (I_{r,s}^1 + I_{r,s}^2) + J_{r,s}.
\end{aligned}$$

Now notice that

$$\begin{aligned}
I_{r,s}^1 &= \int_0^r d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \langle (D^2u(x + \bar{h}\omega) - D^2u(x))\omega, \omega \rangle \rho^{1-2s} \\
&\quad + \int_0^r d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \langle D^2u(x)\omega, \omega \rangle \rho^{1-2s}.
\end{aligned}$$

By using (5.1) we notice that

$$\begin{aligned}
\left| \int_0^r d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \langle (D^2u(x + \bar{h}\omega) - D^2u(x))\omega, \omega \rangle \rho^{1-2s} \right| &\leq \int_0^r d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \varepsilon \rho^{1-2s} \\
&\leq \varepsilon \Lambda \frac{r^{2-2s}}{2(1-s)}.
\end{aligned}$$

On the other hand, we get that

$$\begin{aligned}
\int_0^r d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \langle D^2u(x)\omega, \omega \rangle \rho^{1-2s} &= \sum_{i,j=1}^n \partial_{ij}^2 u(x) \int_0^r d\rho \rho^{1-2s} \int_{\mathbb{S}^{n-1}} da(\omega) \omega_i \omega_j \\
(5.2) \qquad \qquad \qquad &= \frac{r^{2-2s}}{2(1-s)} \sum_{i,j=1}^n \partial_{ij}^2 u(x) \int_{\mathbb{S}^{n-1}} da(\omega) \omega_i \omega_j \\
&= \frac{r^{2-2s}}{2(1-s)} \sum_{i,j=1}^n m_{ij} \partial_{ij}^2 u(x),
\end{aligned}$$

using the notation

$$m_{ij} = \int_{\mathbb{S}^{n-1}} \omega_i \omega_j da(\omega).$$

Multiplying by $(1-s)$, letting $s \nearrow 1$ we get that

$$\lim_{s \nearrow 1} (1-s) I_{r,s}^1 = \frac{1}{2} \sum_{i,j=1}^n m_{ij} \partial_{ij}^2 u(x) + \mathcal{O}(\varepsilon).$$

In the same way, one gets the same limit for $I_{r,s}^2$. Notice also that for s close to 1 (hence when for instance $s > 1/2$)

$$|J_{r,s}| \leq \frac{2r^{-2s} \|u\|_{L^\infty(\mathbb{R}^n)} \Lambda}{s}.$$

Thus we obtain

$$\lim_{s \nearrow 1} (1-s) J_{r,s} = 0.$$

Using the arbitrariness of ε , it follows that

$$\lim_{s \nearrow 1} (1-s)\mathcal{L}u(x) = -\frac{1}{2} \sum_{i,j=1}^n m_{ij} \partial_{ij}^2 u(x)$$

hence the conclusion. \square

Remark 5.2. Notice that the matrix associated to the local operator, given by the constant coefficients $\int_{\mathbb{S}^{n-1}} da(\omega) \omega_i \omega_j$, is symmetric. We observe then that the local operator that we have obtained is the classical Laplacian, up to a change of coordinates, provided that the matrix is also positive definite.

Furthermore, we have the next result.

Proposition 5.3. *Let $u \in C^1(\Omega) \cap L^\infty(\mathbb{R}^n)$. For all $x \in \Omega$ and any $r > 0$ with $B_{2r}(x) \subset \Omega$*

$$\lim_{s \nearrow 1} \mathcal{M}_r^s u(x) = \frac{1}{2} \left(\int_{\mathbb{S}^{n-1}} da \right)^{-1} \int_{\mathbb{S}^{n-1}} da(\omega) (u(x-r\omega) + u(x+r\omega)).$$

Proof. We fix $\varepsilon \in (0, 1)$, which we will take arbitrarily small in the sequel. We have that

$$\begin{aligned} \frac{\mathcal{M}_r^s u(x)}{c(n, s, a)} &= r^{2s} \int_r^\infty d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{u(x+\rho\omega) + u(x-\rho\omega)}{(\rho^2 - r^2)^s \rho} \\ &= \int_1^\infty d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{u(x+r\rho\omega) + u(x-r\rho\omega)}{(\rho^2 - 1)^s \rho} \\ (5.3) \quad &= \int_{1+\varepsilon}^\infty d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{u(x+r\rho\omega) + u(x-r\rho\omega)}{(\rho^2 - 1)^s \rho} \\ &\quad + \int_1^{1+\varepsilon} d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{u(x+r\rho\omega) + u(x-r\rho\omega)}{(\rho^2 - 1)^s \rho} \\ &=: \mathcal{I}_1 + \mathcal{I}_2. \end{aligned}$$

Now

$$\begin{aligned} |\mathcal{I}_1| &\leq \int_{1+\varepsilon}^2 d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{|u(x+r\rho\omega) + u(x-r\rho\omega)|}{(\rho^2 - 1)^s \rho} \\ &\quad + \int_2^\infty d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{|u(x+r\rho\omega) + u(x-r\rho\omega)|}{(\rho^2 - 1)^s \rho} \\ &\leq 2\|u\|_{C(B_{2r}(x))} \int_{\mathbb{S}^{n-1}} da \int_{1+\varepsilon}^2 \frac{d\rho}{(\rho-1)^s (\rho+1)^s \rho} \\ &\quad + 2\|u\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{S}^{n-1}} da \int_2^\infty \frac{d\rho}{(\rho^2 - 1)^s \rho} \\ &\leq \frac{2\Lambda\|u\|_{C(B_{2r}(x))}}{(1+\varepsilon)(2+\varepsilon)^s} \int_{1+\varepsilon}^2 \frac{d\rho}{(\rho-1)^s} + 4\Lambda\|u\|_{L^\infty(\mathbb{R}^n)} \int_2^\infty \frac{d\rho}{\rho^{1+2s}} \\ &\leq \frac{2\Lambda\|u\|_{C(B_{2r}(x))}(1-\varepsilon^{1-s})}{1-s} + \frac{4\Lambda\|u\|_{L^\infty(\mathbb{R}^n)}}{s}. \end{aligned}$$

Notice that since

$$\lim_{s \nearrow 1} \frac{c(n, s, a)}{1-s} = \left(\int_{\mathbb{S}^{n-1}} da \right)^{-1}$$

we obtain

$$\lim_{s \rightarrow 1} c(n, s, a) \mathcal{I}_1 = 0.$$

On the other hand, integrating by parts, we get that

$$\int_1^{1+\varepsilon} d\rho \frac{u(x-r\rho\omega)}{(\rho^2-1)^s\rho} = \frac{\varepsilon^{1-s}u(x-r(1+\varepsilon)\omega)}{(1-s)(\varepsilon+2)^s(1+\varepsilon)} - \frac{1}{1-s} \int_1^{1+\varepsilon} d\rho(\rho-1)^{1-s} \frac{d}{d\rho} \frac{u(x-r\rho\omega)}{(\rho+1)^s\rho}.$$

We have that

$$\left| \frac{d}{d\rho} \frac{u(x-r\rho\omega)}{(\rho+1)^s\rho} \right| \leq c(r)\|u\|_{C^1(B_{2r}(x))},$$

hence

$$\left| \int_1^{1+\varepsilon} d\rho \frac{u(x-r\rho\omega)}{(\rho^2-1)^s\rho} - \frac{\varepsilon^{1-s}u(x-r(1+\varepsilon)\omega)}{(1-s)(\varepsilon+2)^s(1+\varepsilon)} \right| \leq \frac{\varepsilon^{2-s}}{1-s} c(r)\|u\|_{C^1(B_{2r}(x))}.$$

In the same way, we get that

$$\left| \int_1^{1+\varepsilon} d\rho \frac{u(x+r\rho\omega)}{(\rho^2-1)^s\rho} - \frac{\varepsilon^{1-s}u(x+r(1+\varepsilon)\omega)}{(1-s)(\varepsilon+2)^s(1+\varepsilon)} \right| \leq \frac{\varepsilon^{2-s}}{1-s} c(r)\|u\|_{C^1(B_{2r}(x))}.$$

It follows that

$$\begin{aligned} & \left| \mathcal{I}_2 - \frac{\varepsilon^{1-s}}{(1-s)(\varepsilon+2)^s(1+\varepsilon)} \int_{\mathbb{S}^{n-1}} da(\omega) (u(x-r(1+\varepsilon)\omega) + u(x+r(1+\varepsilon)\omega)) \right| \\ & \leq \frac{\varepsilon^{2-s}}{1-s} c(r)\Lambda\|u\|_{C^1(B_{2r}(x))}. \end{aligned}$$

Multiplying by $c(n, s, a)$ and sending $s \nearrow 1$ we get that

$$\lim_{s \nearrow 1} c(n, s, a)\mathcal{I}_2 = \frac{(\int_{\mathbb{S}^{n-1}} da)^{-1}}{(\varepsilon+2)(\varepsilon+1)} \int_{\mathbb{S}^{n-1}} da(\omega) (u(x-r(1+\varepsilon)\omega) + u(x+r(1+\varepsilon)\omega)) + \mathcal{O}(\varepsilon).$$

For $\varepsilon \rightarrow 0$ we get that

$$\lim_{s \nearrow 1} c(n, s, a)\mathcal{I}_2 = \frac{1}{2} \left(\int_{\mathbb{S}^{n-1}} da \right)^{-1} \int_{\mathbb{S}^{n-1}} da(\omega) (u(x-r\omega) + u(x+r\omega)).$$

So putting together the limits involving $\mathcal{I}_1, \mathcal{I}_2$ into (5.3) we obtain the conclusion. \square

We use now the norms introduced at the beginning of Section 4. We have the next inequality.

Proposition 5.4. *Let $u \in H_a^1(\mathbb{R}^n)$. Then there exists $C > 0$ independent of $s \in (1/2, 1)$ with*

$$(1-s)[u]_{H_a^s(\mathbb{R}^n)}^2 \leq C\|u\|_{H_a^1(\mathbb{R}^n)}^2.$$

Proof. We have that

$$\int_{\mathbb{R}^n} dx \int_{\mathbb{S}^{n-1}} da(\omega) (u(x) - u(x+\rho\omega))^2 \leq \rho^2 [u]_{H_a^1(\mathbb{R}^n)}^2.$$

Indeed, for all $\rho \in \mathbb{R}$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} dx \int_{\mathbb{S}^{n-1}} da(\omega) (u(x) - u(x+\rho\omega))^2 & \leq \rho^2 \int_{\mathbb{R}^n} dx \int_{\mathbb{S}^{n-1}} da(\omega) \left(\int_0^1 \nabla u(x+t\rho\omega) \cdot \omega dt \right)^2 \\ & \leq \rho^2 \int_{\mathbb{R}^n} dx \int_{\mathbb{S}^{n-1}} da(\omega) \int_0^1 dt (\nabla u(x+t\rho\omega) \cdot \omega)^2 \\ & \leq \rho^2 \int_0^1 dt \int_{\mathbb{R}^n} dx \int_{\mathbb{S}^{n-1}} da(\omega) (\nabla u(x+t\rho\omega) \cdot \omega)^2 \\ & = \rho^2 [u]_{H_a^1(\mathbb{R}^n)}^2. \end{aligned}$$

Therefore

$$\begin{aligned}
(1-s)\|u\|_{H_a^s(\mathbb{R}^n)}^2 &\leq 2(1-s)[u]_{H_a^1(\mathbb{R}^n)}^2 \int_0^1 \rho^{1-2s} d\rho \\
&\quad + (1-s) \int_1^\infty d\rho \rho^{-1-2s} \int_{\mathbb{R}^n} dx \int_{\mathbb{S}^{n-1}} da(\omega) (u(x) - u(x + \rho\omega))^2 \\
&\quad + (1-s) \int_1^\infty d\rho \rho^{-1-2s} \int_{\mathbb{R}^n} dx \int_{\mathbb{S}^{n-1}} da(\omega) (u(x) - u(x - \rho\omega))^2 \\
&\leq C([u]_{H_a^1(\mathbb{R}^n)}^2 + \|u\|_{L^2(\mathbb{R}^n)}^2),
\end{aligned}$$

for some positive constant C . \square

In what follows, we prove a Bourgain-Brezis-Mironescu type property [7] for anisotropic norms. A different type of anisotropy in the formula was recently investigated in [19] and in [17].

Proposition 5.5 (BBM type formula). *Let $u \in H_{a,0}^1(\mathbb{R}^n)$. Then, we have the formula*

$$(5.4) \quad \lim_{s \rightarrow 1} (1-s)[u]_{H_a^s(\mathbb{R}^n)}^2 = [u]_{H_a^1(\mathbb{R}^n)}^2.$$

Proof. We prove (5.4) first for any $u \in C_c^1(\mathbb{R}^n)$. We write

$$\begin{aligned}
(5.5) \quad &\int_{\mathbb{R}^n} dx \int_{\mathbb{R}} d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{(u(x) - u(x + \rho\omega))^2}{\rho^{1+2s}} \\
&= \int_{\mathbb{R}^n} dx \int_0^\infty d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{(u(x) - u(x + \rho\omega))^2}{\rho^{1+2s}} \\
&\quad + \int_{\mathbb{R}^n} dx \int_0^\infty d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{(u(x) - u(x - \rho\omega))^2}{\rho^{1+2s}}.
\end{aligned}$$

Since $u \in C^1$ by the mean value theorem, there exist $\underline{h} := \underline{h}(\rho, \omega), \bar{h} := \bar{h}(\rho, \omega) \in [0, \rho]$ such that

$$\begin{aligned}
u(x + \rho\omega) - u(x) &= \rho\omega \cdot \nabla u(x + \bar{h}\omega) \quad \text{and} \\
u(x - \rho\omega) - u(x) &= -\rho\omega \cdot \nabla u(x + \underline{h}\omega).
\end{aligned}$$

Furthermore, for any $\varepsilon > 0$ there exists $r := r(\varepsilon) > 0$ such that

$$(5.6) \quad |\nabla u(x + h\omega) - \nabla u(x)| < \varepsilon \quad \text{whenever } |h| = |h\omega| < \rho < r.$$

We fix $\varepsilon > 0$ (to be taken arbitrarily small in the sequel) and consider the correspondent $r := r(\varepsilon)$. We then write

$$\begin{aligned}
&\int_{\mathbb{R}^n} dx \int_0^\infty d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{(u(x) - u(x + \rho\omega))^2}{\rho^{1+2s}} \\
&= \int_{\mathbb{R}^n} dx \int_0^r d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \rho^{1-2s} (\nabla u(x + \bar{h}\omega) \cdot \omega)^2 \\
&\quad + \int_{\mathbb{R}^n} dx \int_r^\infty d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{(u(x) - u(x + \rho\omega))^2}{\rho^{1+2s}} \\
&=: I_{r,s}^1 + I_{r,s}^2.
\end{aligned}$$

Notice that

$$\begin{aligned} I_{r,s}^1 &= \int_{\mathbb{R}^n} dx \int_0^r d\rho \rho^{1-2s} \int_{\mathbb{S}^{n-1}} da(\omega) (\nabla u(x + \bar{h}\omega) \cdot \omega)^2 - (\nabla u(x) \cdot \omega)^2 \\ &\quad + \int_{\mathbb{R}^n} dx \int_0^r d\rho \rho^{1-2s} \int_{\mathbb{S}^{n-1}} da(\omega) (\nabla u(x) \cdot \omega)^2 \\ &= J_{r,s}^1 + J_{r,s}^2. \end{aligned}$$

From (5.6), we have that

$$\begin{aligned} &\left| (\nabla u(x + \bar{h}\omega) \cdot \omega)^2 - (\nabla u(x) \cdot \omega)^2 \right| \\ &\leq \left| (\nabla u(x + \bar{h}\omega) - \nabla u(x)) \cdot \omega \right| \left| (\nabla u(x + \bar{h}\omega) + \nabla u(x)) \cdot \omega \right| \leq 2\varepsilon \|u\|_{C^1(\mathbb{R}^n)}. \end{aligned}$$

Therefore, for some compact set $K \subset \mathbb{R}^n$ independent of ε , there holds

$$|J_{r,s}^1| \leq \varepsilon \Lambda \|u\|_{C^1(\mathbb{R}^n)} \frac{r^{2-2s}}{2(1-s)} |K|.$$

Also

$$J_{r,s}^2 = \frac{r^{2-2s}}{2(1-s)} \int_{\mathbb{R}^n} dx \int_{\mathbb{S}^{n-1}} da(\omega) (\nabla u(x) \cdot \omega)^2.$$

It follows that

$$\lim_{s \rightarrow 1} (1-s) I_{r,s}^1 = \frac{1}{2} \int_{\mathbb{R}^n} dx \int_{\mathbb{S}^{n-1}} da(\omega) (\nabla u(x) \cdot \omega)^2 + \mathcal{O}(\varepsilon).$$

Furthermore we get that

$$|I_{r,s}^2| \leq 2 \|u\|_{L^2(\mathbb{R}^n)}^2 \Lambda \int_r^\infty \rho^{-1-2s} d\rho = \frac{r^{2s} \|u\|_{L^2(\mathbb{R}^n)}^2 \Lambda}{s},$$

hence

$$\lim_{s \rightarrow 1} (1-s) I_{r,s}^2 = 0.$$

We finally get that

$$\lim_{s \rightarrow 1} (1-s) \int_{\mathbb{R}^n} dx \int_0^\infty d\rho \int_{\mathbb{S}^{n-1}} da(\omega) \frac{(u(x) - u(x + \rho\omega))^2}{\rho^{1+2s}} = \frac{1}{2} \int_{\mathbb{S}^{n-1}} da(\omega) (\nabla u(x) \cdot \omega)^2 + \mathcal{O}(\varepsilon).$$

We obtain the same limit for the second term in (5.5) and get (5.4) for $u \in C_c^1(\mathbb{R}^n)$ by sending $\varepsilon \rightarrow 0$. Let now $u \in H_{a,0}^1(\mathbb{R}^n)$. There exists $\{u_j\}_j \in C_c^\infty(\mathbb{R}^n)$ such that

$$\|u - u_j\|_{H_a^1(\mathbb{R}^n)} \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

Then, according to Proposition 5.4 we have that

$$(1-s) \left([u]_{H_a^s(\mathbb{R}^n)} - [u_j]_{H_a^s(\mathbb{R}^n)} \right)^2 \leq (1-s) [u - u_j]_{H_a^s(\mathbb{R}^n)}^2 \leq C \|u - u_j\|_{H_a^1(\mathbb{R}^n)}^2 \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

The conclusion (5.4) for $u \in H_{a,0}^1(\mathbb{R}^n)$ immediately follows. \square

Remark 5.6. If $da = d\mathcal{H}^{n-1}$, the left-hand side of the formula in Proposition 5.4 boils down to

$$\lim_{s \rightarrow 1} (1-s) \int_{\mathbb{R}^n} dx \int_0^\infty d\rho \int_{\mathbb{S}^{n-1}} d\mathcal{H}^{n-1}(\omega) \frac{(u(x) - u(x + \rho\omega))^2}{|\rho|^{1+2s}},$$

while the right-hand side to

$$\frac{1}{2} \int_{\mathbb{R}^n} dx \int_{\mathbb{S}^{n-1}} d\mathcal{H}^{n-1}(\omega) (\omega \cdot \nabla u(x))^2 = \frac{Q_{n,2}}{2} \int_{\Omega} |\nabla u(x)|^2 dx$$

where

$$Q_{n,2} = \int_{\mathbb{S}^{n-1}} |\sigma \cdot \omega|^2 d\mathcal{H}^{n-1}(\omega)$$

for some $\sigma \in \mathbb{S}^{n-1}$. This is consistent with the usual Brezis-Bourgain-Mironescu formula (see [7]).

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