

Second order differentiation formula on $\text{RCD}(K, N)$ spaces

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Abstract

We prove the second order differentiation formula along geodesics in finite-dimensional $\text{RCD}(K, N)$ spaces. Our approach strongly relies on the approximation of W_2 -geodesics by entropic interpolations and, in order to implement this approximation procedure, on the proof of new (even in the smooth setting) estimates for such interpolations.

Contents

| | | |
|----------|--|----------|
| 1 | Main result and comments | 1 |
| 2 | Strategy of the proof | 4 |
| 2.1 | The need of an approximation procedure | 4 |
| 2.2 | Entropic interpolation: definition | 5 |
| 2.3 | Entropic interpolations: uniform control and convergence | 7 |

1 Main result and comments

This work is about the development of calculus tools in the setting of $\text{RCD}(K, N)$ spaces $(X, \mathbf{d}, \mathbf{m})$ with $K \in \mathbb{R}$ and $N \in [1, \infty)$ (see [2] for the original definition with $N = \infty$ and [9] for the case $N < \infty$). The proofs of the announced results are contained in [12] and, up to technical difficulties, they rely on [11], where the same results are obtained for compact $\text{RCD}(K, N)$ spaces.

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Recall that an optimal geodesic test plan π on X is a probability measure on $C([0, 1], X)$ such that $(e_t)_*\pi \leq C\mathbf{m}$ for every $t \in [0, 1]$ and some $C > 0$ and satisfying

$$\iint_0^1 |\dot{\gamma}_t|^2 dt d\pi(\gamma) = W_2^2((e_0)_*\pi, (e_1)_*\pi).$$

Here $e_t : C([0, 1], X) \rightarrow X$ is the evaluation map sending γ to γ_t . Any such π is concentrated on constant speed geodesics and for any couple of measures $\mu_0, \mu_1 \in \mathcal{P}(X)$ with bounded densities and supports, there is a unique optimal geodesic test plan such that $(e_0)_*\pi = \mu_0, (e_1)_*\pi = \mu_1$.

From the point of view of calculus on metric measure spaces as developed in [1], the relation between optimal geodesic test plans and standard geodesics is in some sense the same that there is between Sobolev functions and Lipschitz ones. An example of this phenomenon is the following result (a minor variant of a statement in [7]), which says that we can safely take one derivative of a $W^{1,2}(X)$ function along an optimal geodesic test plan:

Theorem 1. *Let (X, d, \mathbf{m}) be a $\text{RCD}(K, \infty)$ space, π an optimal geodesic test plan with bounded support (equivalently: such that $\{\gamma_t : t \in [0, 1], \gamma \in \text{supp}(\pi)\} \subset X$ is bounded) and $h \in W^{1,2}(X)$.*

Then the map $[0, 1] \ni t \mapsto h \circ e_t \in L^2(\pi)$ is in $C^1([0, 1], L^2(\pi))$ and we have

$$\frac{d}{dt}(h \circ e_t) = \langle \nabla h, \nabla \phi_t \rangle \circ e_t,$$

for every $t \in [0, 1]$, where ϕ_t is any function such that for some $s \neq t, s \in [0, 1]$, the function $-(s - t)\phi_t$ is a Kantorovich potential from $(e_t)_\pi$ to $(e_s)_*\pi$.*

Our main result here is the extension of the above to second order derivatives. Recalling that the second order Sobolev space $H^{2,2}(X)$ and the corresponding Hessian are defined in [8], we have:

Theorem 2. *Let (X, d, \mathbf{m}) be a $\text{RCD}(K, N)$ space, $N < \infty$, π an optimal geodesic test plan with bounded support and $h \in H^{2,2}(X)$.*

Then the map $[0, 1] \ni t \mapsto h \circ e_t \in L^2(\pi)$ is in $C^2([0, 1], L^2(\pi))$ and we have

$$\frac{d^2}{dt^2}(h \circ e_t) = \text{Hess}(h)(\nabla \phi_t, \nabla \phi_t) \circ e_t, \quad (1)$$

for every $t \in [0, 1]$, where ϕ_t is as in Theorem 1.

Notice that by Theorem 1 we have that such result is really a statement about the C^1 regularity of $t \mapsto \langle \nabla h, \nabla \phi_t \rangle \circ e_t$.

Let us collect a couple of equivalent formulations of Theorem 2. For the first recall that the space of Sobolev vector fields $H_C^{1,2}(TX)$ as well as the covariant derivative have been defined in [8]. Then we have:

Theorem 3. *Let (X, d, m) be a $\text{RCD}(K, N)$ space, $N < \infty$, π an optimal geodesic test plan with bounded support and $X \in H_C^{1,2}(X)$.*

Then the map $[0, 1] \ni t \mapsto \langle X, \nabla \phi_t \rangle \circ e_t \in L^2(\pi)$ is in $C^1([0, 1], L^2(\pi))$ and we have

$$\frac{d}{dt} (\langle X, \nabla \phi_t \rangle \circ e_t) = \nabla X(\nabla \phi_t, \nabla \phi_t) \circ e_t, \quad (2)$$

for every $t \in [0, 1]$, where ϕ_t is as in Theorem 1.

From the identity $\nabla(\nabla h) = \text{Hess}(h)$ (assuming to identify tangent and cotangent vector fields) we see that Theorem 3 implies Theorem 2. For the converse implication notice that Theorem 2 and the Leibniz rule easily provide the correct formula for the derivative of $t \mapsto \langle X, \nabla \phi_t \rangle \circ e_t$ for $X = \sum_i \tilde{h}_i \nabla h_i$, with $(\tilde{h}_i) \subset L^\infty \cap W^{1,2}(X)$ and $(h_i) \subset H^{2,2}(X)$, then conclude by the closure of the covariant derivative.

Another equivalent formulation of Theorem 2, which is the one we shall actually prove, is:

Theorem 4. *Let (X, d, m) be a $\text{RCD}(K, N)$ space, $N < \infty$, $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ be such that $\mu_0, \mu_1 \leq C m$ for some $C > 0$, with compact supports and let (μ_t) be the unique W_2 -geodesic connecting μ_0 to μ_1 . Also, let $h \in H^{2,2}(X)$.*

Then the map

$$[0, 1] \ni t \mapsto \int h d\mu_t \in \mathbb{R}$$

belongs to $C^2([0, 1])$ and it holds

$$\frac{d^2}{dt^2} \int h d\mu_t = \int \text{Hess}(h)(\nabla \phi_t, \nabla \phi_t) d\mu_t, \quad (3)$$

for every $t \in [0, 1]$, where ϕ_t is any function such that for some $s \neq t$, $s \in [0, 1]$, the function $-(s-t)\phi_t$ is a Kantorovich potential from μ_t to μ_s .

Since for any W_2 -geodesic as in the statement there is a (unique) optimal geodesic test plan π such that $\mu_t = (e_t)_* \pi$ for any t , we see that Theorem 4 follows from Theorem 2 by integration w.r.t. π . For the converse implication one notices that for any optimal geodesic test plan π with bounded support and $\Gamma \subset C([0, 1], X)$ Borel with $\pi(\Gamma) > 0$, the curve $t \mapsto \pi(\Gamma)^{-1}(e_t)_*(\pi|_\Gamma)$ fulfils the assumptions of Theorem 4 with the same ϕ_t 's as in Theorem 2. The conclusion then follows by the arbitrariness of Γ observing that $L^2(\pi)$ -derivatives exist for every t if and only if the difference quotients converge in the weak $L^2(\pi)$ -topology for every t .

Let us comment about the assumptions in Theorems 2, 3, 4:

- The first order differentiation formula is valid on general $\text{RCD}(K, \infty)$ spaces, while for the second order one we need to assume finite dimensionality. This is due to the strategy of our proof, which among other things uses the Li-Yau inequality.

- There exist optimal geodesic test plans without bounded support (if $K = 0$ or the densities of the initial and final marginals decay sufficiently fast) but in this case the functions ϕ_t appearing in the statement(s) are not Lipschitz. As such it seems hard to have $\text{Hess}(h)(\nabla\phi_t, \nabla\phi_t) \circ e_t \in L^1(\pi)$ and thus we can not really hope for anything like (1), (2), (3) to hold: this explains the need of the assumption on bounded supports.

Having at disposal the second order differentiation formula is interesting not only at the theoretical level, but also for applications to the study of the geometry of RCD spaces. For instance, the proofs of both the splitting theorem [7] and of the ‘volume cone implies metric cone’ [5] in this setting can be greatly simplified by using such formula (in this direction, see [14] for comments about the splitting). Also, one aspect of the theory of RCD spaces which is not yet clear is whether they have constant dimension: for Ricci-limit spaces this is known to be true by a result of Colding-Naber [4] which uses second order derivatives along geodesics in a crucial way. Thus our result is necessary to replicate Colding-Naber argument in the non-smooth setting (but not sufficient: they also use a calculus with Jacobi fields which as of today does not have a non-smooth counterpart).

2 Strategy of the proof

2.1 The need of an approximation procedure

Let us recall that a second order differentiation formula, valid for sufficiently regular curves, has been proved in [8]:

Theorem 5. *Let (μ_t) be a W_2 -absolutely continuous curve solving the continuity equation*

$$\frac{d}{dt}\mu_t + \text{div}(X_t\mu_t) = 0, \quad (4)$$

for some vector fields $(X_t) \subset L^2(TX)$ in the following sense: for every $f \in W^{1,2}(X)$ the map $t \mapsto \int f d\mu_t$ is absolutely continuous and it holds

$$\frac{d}{dt} \int f d\mu_t = \int \langle \nabla f, X_t \rangle d\mu_t.$$

Assume that

- (i) $t \mapsto X_t \in L^2(TX)$ is absolutely continuous,
- (ii) $\sup_t \{\|X_t\|_{L^2} + \|X_t\|_{L^\infty} + \|\nabla X_t\|_{L^2}\} < +\infty$.

Then for $f \in H^{2,2}(X)$ the map $t \mapsto \int f d\mu_t$ is $C^{1,1}$ and the formula

$$\frac{d^2}{dt^2} \int f d\mu_t = \int \text{Hess}(f)(X_t, X_t) + \langle \nabla f, \frac{d}{dt}X_t + \nabla_{X_t}X_t \rangle d\mu_t \quad (5)$$

holds for a.e. $t \in [0, 1]$.

If the vector fields X_t are of gradient type, so that $X_t = \nabla\phi_t$ for every t and the ‘acceleration’ a_t is defined as

$$\frac{d}{dt}\phi_t + \frac{|\nabla\phi_t|^2}{2} =: a_t$$

then (5) reads as

$$\frac{d^2}{dt^2} \int f d\mu_t = \int \text{Hess}(f)(\nabla\phi_t, \nabla\phi_t) d\mu_t + \int \langle \nabla f, \nabla a_t \rangle d\mu_t. \quad (6)$$

In the case of geodesics it is well-known that (4) holds exactly with $X_t = -\nabla\varphi_t$ for appropriate choices of Kantorovich potentials φ_t (see also [10] in this direction) and moreover the functions φ_t solve (in a sense which we will not make precise here) the Hamilton-Jacobi equation

$$\frac{d}{dt}\varphi_t = \frac{|\nabla\varphi_t|^2}{2}, \quad (7)$$

thus in this case the acceleration a_t is identically 0. Hence if the vector fields $(-\nabla\varphi_t)$ satisfied the regularity requirements (i), (ii) in the last theorem, we would easily be able to establish Theorem 2. However in general this is not the case; informally speaking this has to do with the fact that for solutions of the Hamilton-Jacobi equations we do not have sufficiently strong second order estimates.

In order to establish Theorem 2 it is therefore natural to look for suitable ‘smooth’ approximations of geodesics for which we can apply Theorem 5 above and then pass to the limit in formula (5). Given that the source of non-smoothness is in the Hamilton-Jacobi equation it is natural to think at viscous approximation as smoothing procedure: all in all viscous limit is ‘the’ way of approximating the ‘correct’ solution of Hamilton-Jacobi and the Laplacian is well behaved under lower Ricci curvature bounds. However, this does not really work: shortly said, the problem is that not every solution of Hamilton-Jacobi is linked to W_2 -geodesics, but only those for which shocks do not occur in the time interval $[0, 1]$. Since the conclusion of Theorem 2 can only hold along geodesics, we see that we cannot simply use viscous approximation and PDE estimates to conclude (one should incorporate in the estimates the fact that the starting function is c -concave, but this seems hard to do).

We shall instead use entropic interpolation, which we now introduce.

2.2 Entropic interpolation: definition

Fix two probability measures $\mu_0 = \rho_0 \mathbf{m}$, $\mu_1 = \rho_1 \mathbf{m}$ on X . The Schrödinger functional equations are

$$\rho_0 = f \mathbf{h}_1 g \qquad \rho_1 = g \mathbf{h}_1 f, \quad (8)$$

the unknown being the Borel functions $f, g : X \rightarrow [0, \infty)$, where $h_t f$ is the heat flow starting at f evaluated at time t . It turns out that in great generality these equations admit a solution which is unique up to the trivial transformation $(f, g) \mapsto (cf, g/c)$ for some constant $c > 0$. Such solution can be found in the following way: let R be the measure on X^2 whose density w.r.t. $\mathbf{m} \otimes \mathbf{m}$ is given by the heat kernel $r_t(x, y)$ at time $t = 1$ and minimize the Boltzmann-Shannon entropy $H(\gamma | R)$ among all transport plans γ from μ_0 to μ_1 . The Euler equation for the minimizer forces it to be of the form $f \otimes g R$ for some Borel functions $f, g : X \rightarrow [0, \infty)$, where $f \otimes g(x, y) := f(x)g(y)$. Then the fact that $f \otimes g R$ is a transport plan from μ_0 to μ_1 is equivalent to (f, g) solving (8).

Once we have found the solution of (8) we can use it in conjunction with the heat flow to interpolate from ρ_0 to ρ_1 by defining

$$\rho_t := h_t f h_{1-t} g.$$

This is called **entropic interpolation**. Now we slow down the heat flow: fix $\varepsilon > 0$ and by mimicking the above find $f^\varepsilon, g^\varepsilon$ such that

$$\rho_0 = f^\varepsilon h_{\varepsilon/2} g^\varepsilon \quad \rho_1 = g^\varepsilon h_{\varepsilon/2} f^\varepsilon, \quad (9)$$

(the factor $1/2$ plays no special role, but is convenient in computations). Then define

$$\rho_t^\varepsilon := h_{t\varepsilon/2} f^\varepsilon h_{(1-t)\varepsilon/2} g^\varepsilon.$$

The remarkable and non-trivial fact here is that as $\varepsilon \downarrow 0$ the curves of measures $(\rho_t^\varepsilon \mathbf{m})$ converge to the W_2 -geodesic from μ_0 to μ_1 . In order to state our results, it is convenient to introduce the (interpolated) **Schrödinger potentials** $\varphi_t^\varepsilon, \psi_t^\varepsilon$ as

$$\varphi_t^\varepsilon := \varepsilon \log h_{t\varepsilon/2} f^\varepsilon \quad \psi_t^\varepsilon := \varepsilon \log h_{(1-t)\varepsilon/2} g^\varepsilon.$$

In the limit $\varepsilon \downarrow 0$ these will converge to forward and backward Kantorovich potentials along the limit geodesic (μ_t) (see below). In this direction, it is worth to notice that while for $\varepsilon > 0$ there is a tight link between potentials and densities, as we trivially have

$$\varphi_t^\varepsilon + \psi_t^\varepsilon = \varepsilon \log \rho_t^\varepsilon,$$

in the limit this becomes the well known (weaker) relation that is in place between forward/backward Kantorovich potentials and measures (μ_t) :

$$\begin{aligned} \varphi_t + \psi_t &= 0 && \text{on } \text{supp}(\mu_t), \\ \varphi_t + \psi_t &\leq 0 && \text{on } X, \end{aligned}$$

see e.g. Remark 7.37 in [15] (paying attention to the different sign convention). By direct computation one can verify that $(\varphi_t^\varepsilon), (\psi_t^\varepsilon)$ solve the Hamilton-Jacobi-Bellman equations

$$\frac{d}{dt} \varphi_t^\varepsilon = \frac{1}{2} |\nabla \varphi_t^\varepsilon|^2 + \frac{\varepsilon}{2} \Delta \varphi_t^\varepsilon \quad - \frac{d}{dt} \psi_t^\varepsilon = \frac{1}{2} |\nabla \psi_t^\varepsilon|^2 + \frac{\varepsilon}{2} \Delta \psi_t^\varepsilon, \quad (10)$$

thus introducing the functions

$$\vartheta_t^\varepsilon := \frac{\psi_t^\varepsilon - \varphi_t^\varepsilon}{2}$$

it is not hard to check that it holds

$$\frac{d}{dt}\rho_t^\varepsilon + \operatorname{div}(\nabla\vartheta_t^\varepsilon\rho_t^\varepsilon) = 0 \quad (11)$$

and

$$\frac{d}{dt}\vartheta_t^\varepsilon + \frac{|\nabla\vartheta_t^\varepsilon|^2}{2} = a_t^\varepsilon, \quad \text{where} \quad a_t^\varepsilon := -\frac{\varepsilon^2}{8}\left(2\Delta\log\rho_t^\varepsilon + |\nabla\log\rho_t^\varepsilon|^2\right).$$

2.3 Entropic interpolations: uniform control and convergence

With this said, our main results about entropic interpolations can be summarized as follows. Under the assumptions that the metric measure space (X, d, m) is $\operatorname{RCD}(K, N)$, $N < \infty$, and that ρ_0, ρ_1 belong to $L^\infty(X)$ with bounded supports it holds:

- Zeroth order

- *bound* For some $C > 0$ we have $\rho_t^\varepsilon \leq C$ for every $\varepsilon \in (0, 1)$ and $t \in [0, 1]$.
- *convergence* The curves $(\rho_t^\varepsilon m)$ W_2 -uniformly converge to the unique W_2 -geodesic (μ_t) from μ_0 to μ_1 and setting $\rho_t := \frac{d\mu_t}{dm}$ it holds $\rho_t^\varepsilon \xrightarrow{*} \rho_t$ in $L^\infty(X)$ for all $t \in [0, 1]$.

- First order

- *bound* For any $t \in (0, 1]$ the functions $\{\varphi_t^\varepsilon\}_{\varepsilon \in (0, 1)}$ are locally equi-Lipschitz. Similarly for the ψ 's.
- *convergence* For every sequence $\varepsilon_n \downarrow 0$ there is a subsequence - not relabeled - such that for any $t \in (0, 1]$ the functions φ_t^ε converge both locally uniformly and in $W_{loc}^{1,2}(X)$ to a function φ_t such that $-t\varphi_t$ is a Kantorovich potential from μ_t to μ_0 . Similarly for the ψ 's.

- Second order For every $\delta \in (0, 1/2)$ we have

- *bound*

$$\begin{aligned} \sup_{\varepsilon \in (0, 1)} \iint_{\delta}^{1-\delta} (|\operatorname{Hess}(\vartheta_t^\varepsilon)|_{\operatorname{HS}}^2 + \varepsilon^2 |\operatorname{Hess}(\log\rho_t^\varepsilon)|_{\operatorname{HS}}^2) \rho_t^\varepsilon dt dm < \infty, \\ \sup_{\varepsilon \in (0, 1)} \iint_{\delta}^{1-\delta} (|\Delta\vartheta_t^\varepsilon|^2 + \varepsilon^2 |\Delta\log\rho_t^\varepsilon|^2) \rho_t^\varepsilon dt dm < \infty. \end{aligned} \quad (12)$$

Notice that since in general the Laplacian is not the trace of the Hessian, there is no direct link between these two bounds.

– *convergence* For every function $h \in W^{1,2}(X)$ with $\Delta h \in L^\infty(X)$ it holds

$$\lim_{\varepsilon \downarrow 0} \iint_{\delta}^{\varepsilon^{1-\delta}} \langle \nabla h, \nabla a_t^\varepsilon \rangle \rho_t^\varepsilon dt d\mathbf{m} = 0. \quad (13)$$

With the exception of the convergence $\rho_t^\varepsilon \mathbf{m} \rightarrow \mu_t$, all these results are new even on compact smooth manifolds (in fact, even on \mathbb{R}^d).

The zeroth and first order bounds are obtained via a combination of Hamilton's gradient estimate and Li-Yau's Laplacian estimate. Similar bounds can also be obtained for the viscous approximation.

The fact that the limit curve (μ_t) is the W_2 -geodesic and that the limit potentials are Kantorovich potentials are consequence of the fact that we can pass to the limit in the continuity equation (11) and that the limit potentials satisfy the Hamilton-Jacobi equation. Notice that these zeroth and first order convergences are sufficient to pass to the limit in the term with the Hessian in (6).

The crucial advantage of dealing with entropic interpolations (which has no counterpart in viscous approximation) is in the second order bounds and convergence results. The key ingredient that allows to obtain these is a formula due to Léonard [13], who realized that there is a connection between entropic interpolation and lower Ricci bounds; our contribution is the rigorous proof in the RCD framework of his formal computations:

Proposition 6. *For any $\varepsilon > 0$ the map $t \mapsto H(\mu_t^\varepsilon | \mathbf{m})$ belongs to $C([0, 1]) \cap C^2(0, 1)$ and for every $t \in (0, 1)$ it holds*

$$\frac{d}{dt} H(\mu_t^\varepsilon | \mathbf{m}) = \int \langle \nabla \rho_t^\varepsilon, \nabla \vartheta_t^\varepsilon \rangle d\mathbf{m} = \frac{1}{2\varepsilon} \int (|\nabla \psi_t^\varepsilon|^2 - |\nabla \varphi_t^\varepsilon|^2) \rho_t^\varepsilon d\mathbf{m}, \quad (14a)$$

$$\frac{d^2}{dt^2} H(\mu_t^\varepsilon | \mathbf{m}) = \int \rho_t^\varepsilon d(\mathbf{\Gamma}_2(\vartheta_t^\varepsilon) + \frac{\varepsilon^2}{4} \mathbf{\Gamma}_2(\log(\rho_t^\varepsilon))) = \frac{1}{2} \int \rho_t^\varepsilon d(\mathbf{\Gamma}_2(\varphi_t^\varepsilon) + \mathbf{\Gamma}_2(\psi_t^\varepsilon)). \quad (14b)$$

Let us see how to use (14b) in the simplified case $K = 0$ and $\mathbf{m}(X) = 1$ to obtain (12). Observe that if $h : [0, 1] \rightarrow \mathbb{R}^+$ is a convex function, then $-\frac{h(0)}{t} \leq h'(t) \leq \frac{h(1)}{1-t}$ for any $t \in (0, 1)$ and thus

$$\int_{\delta}^{1-\delta} h''(t) dt = h'(1-\delta) - h'(\delta) \leq \frac{h(1)}{1-\delta} + \frac{h(0)}{\delta}. \quad (15)$$

If $K = 0$ we have $\mathbf{\Gamma}_2 \geq 0$, so that (14b) tells in particular that $t \mapsto H(\mu_t^\varepsilon | \mathbf{m})$ is convex for any $\varepsilon > 0$, and if $\mathbf{m}(X) = 1$ such function is non-negative. Therefore (15) gives that for any $\delta \in (0, 1/2)$ it holds

$$\sup_{\varepsilon \in (0, 1)} \int_{\delta}^{1-\delta} \rho_t^\varepsilon d(\mathbf{\Gamma}_2(\vartheta_t^\varepsilon) + \frac{\varepsilon^2}{4} \mathbf{\Gamma}_2(\log(\rho_t^\varepsilon))) dt \leq \frac{H(\mu_1 | \mathbf{m})}{1-\delta} + \frac{H(\mu_0 | \mathbf{m})}{\delta} < \infty. \quad (16)$$

Recalling the Bochner inequalities ([6], [3],[8])

$$\Gamma_2(\eta) \geq |\text{Hess}(\eta)|_{\text{HS}}^2 \mathbf{m}, \quad \Gamma_2(\eta) \geq \frac{(\Delta\eta)^2}{N} \mathbf{m},$$

we see that (12) follows from (16). Then with some work (see [11] for the details) starting from (16) we can deduce (13) which in turn ensures that the term with the acceleration in (6) vanishes in the limit $\varepsilon \downarrow 0$, thus leading to our main result Theorem 4.

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