On the asymptotic behaviour of nonlocal perimeters

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Abstract

We study a class of integral functionals known as nonlocal perimeters, which, intuitively, express a weighted interaction between a set and its complement. The weight is provided by a positive kernel K, which might be singular.

In the first part of the paper, we show that these functionals are indeed perimeters in an generalised sense and we establish existence of minimisers for the corresponding Plateau's problem; also, when K is radial and strictly decreasing, we prove that halfspaces are minimisers if we prescribe "flat" boundary conditions.

A Γ-convergence result is discussed in the second part of the work. We study the limiting behaviour of the nonlocal perimeters associated with certain rescalings of a given kernel that has faster-than-L^1 decay at infinity and we show that the Γ-limit is the classical perimeter, up to a multiplicative constant that we compute explicitly.

Contents

1 Introduction

In a qualitative way, we might think of the perimeter of a set in the Euclidean space $\mathbb{R}^d$ as a measure of the locus that circumscribes the set itself. This intuition is captured by the analytical theory of finite perimeters sets in use nowadays, which is grounded on Caccioppoli's seminal works and on the ideas developed by De Giorgi in the 1950's. In a sloppy manner, we may summarise as follows the gist of this theory: identify a set with its characteristic function, consider the distributional gradient of the latter and define its total variation as perimeter of the given set. A fundamental result by De Giorgi and Federer shows that if we retain this definition the perimeter coincides with the $(d-1)$-dimensional Hausdorff measure of a certain subset of the topological boundary, so that consistency with the naïve idea is guaranteed. Besides, the class of sets such that their perimeter is finite has good compactness properties, thus it is possible to tackle various problems that are formulated in geometric terms via the direct method of calculus of variations; among all the possible examples of this that could be listed, we cite only Plateau's problem, because we shall deal with it later on (see Theorem 2.10).

Going beyond this by now well-established theory, recently several authors have grown interested in some set functionals that are globally referred to as *nonlocal perimeters*: for instance, a prominent case is offered by fractional perimeters that were introduced by Caffarelli, Roquejoffre and Savin in [4] and that were later extended and largely investigated (see for instance [2, 9, 5, 12, 15]). The study of nonlocal perimeters is motivated by both theory and application, as described in the brief account given by Cinti, Serra and Valdinoci in [9]. Moreover, although the definition of these functionals might seem distant from De Giorgi's one (confront (1.2) and (1.5)), nonlocal perimeters resemble the classical one from various perspectives. Actually, one can prove that they are indeed perimeters in the sense proposed in [8], where Chambolle, Morini and Ponsiglione collect some properties that a set functional should have in order to deserve such label; up to minor changes, the axiomatic definition they propose is this one:

1.1 - Definition. *Let $\mathscr{M}$ be the collection of all Lebesgue measurable sets in $\mathbb{R}^d$ and let $\Omega \in \mathscr{M}$ be a fixed set with strictly positive Lebesgue measure. Choose arbitrarily $E, F \in \mathscr{M}$. A functional $p_\Omega \colon \mathscr{M} \to [0,+\infty]$ is a* perimeter in Ω *if*

(i) $p_\Omega(\emptyset) = 0$;

(ii) $p_\Omega(E) = p_\Omega(F)$ *whenever* $|(E \triangle F) \cap \Omega| = 0$;

(iii) *it is invariant under translations, that is* $p_{\Omega+h}(E+h) = p_\Omega(E)$ *for any* $h \in \mathbb{R}^d$;

(iv) *it is finite on any set that is the closure of an open set with compact C^2 boundary;*

(v) *it is lower semicontinuous w.r.t.* $L^1_{\mathrm{loc}}(\mathbb{R}^d)$*-convergence;*

(vi) *it is submodular, that is*

$$p_\Omega(E \cap F) + p_\Omega(E \cup F) \le p_\Omega(E) + p_\Omega(F). \tag{1.1}$$

The authors also provide examples of functionals that fit in this framework; naturally, De Giorgi's perimeter is one of them, but we also find the functional (2.7), which stands as an instance of the analysis we carry out here. Indeed, this paper is devoted to the study of functionals of the form

$$\begin{aligned}\mathrm{Per}_K(E,\Omega) \coloneqq & \int_{E\cap\Omega}\int_{E^c\cap\Omega} K(y-x)\mathrm{d}y\mathrm{d}x \\ & + \int_{E\cap\Omega}\int_{E^c\cap\Omega^c} K(y-x)\mathrm{d}y\mathrm{d}x + \int_{E\cap\Omega^c}\int_{E^c\cap\Omega} K(y-x)\mathrm{d}y\mathrm{d}x,\end{aligned} \tag{1.2}$$

where E and Ω are Lebesgue measurable sets in $\mathbb{R}^d$ and $K\colon \mathbb{R}^d \to [0,+\infty]$ is a Lebesgue measurable function on which we prescribe suitable conditions.

In Section 2, as a preliminary step, we consider a general interaction functional between two sets:

$$L_K(E,F) = \int_F\int_E K(y-x)\mathrm{d}y\mathrm{d}x$$

and we describe some of its basic properties. We show that, when one of the sets of the couple has finite classical perimeter the interaction is bounded by the BV norm of that set. This, combined with the expression of $\mathrm{Per}_K(\,\cdot\,,\Omega)$ in terms of suitable couplings L_K, comes in handy to prove that the functional in (1.2) is a perimeter according to Definition 1.1.

Once we know that 1.2 defines a perimeter, in Subsection 2.2 we provide an existence result for nonlocal minimal surfaces, i.e. sets that minimise $\mathrm{Per}_K(\,\cdot\,,\Omega)$ among all the sets that coincide with a given one outside Ω. The proof of this takes into account the extension of the perimeter functional to measurable functions that range in $[0,1]$ and it exploits the convexity of this extension; in turn, convexity relies on the submodularity of $\mathrm{Per}(\,\cdot\,,\Omega)$ and on the validity of a generalised Coarea Formula. Moreover, when the perimeter is built from a radial, strictly decreasing kernel we are able to show that minimisers for Plateau's problem with "flat" boundary conditions are halfspaces.

Section 3 is devoted to a Γ-convergence argument. We let Ω be an open bounded set with Lipschitz boundary and we focus on the family of perimeter functionals $J_\varepsilon(\,\cdot\,,\Omega)$ induced by mass preserving rescalings of a fixed kernel K, that is

$$K_\varepsilon(h) \coloneqq \frac{1}{\varepsilon^d} K\left(\frac{h}{\varepsilon}\right). \tag{1.3}$$

We are interested in the limiting behaviour of the ratios $\frac{1}{\varepsilon}J_\varepsilon(\,\cdot\,,\Omega)$; precisely, our intent is showing that they Γ-converge w.r.t. the $L^1_{\mathrm{loc}}(\mathbb{R}^d)$-metric to the classical perimeter in Ω, up to the multiplicative constant

$$c_K \coloneqq \frac{1}{2}\int_{\mathbb{R}^d} K(h)\,|h_d|\,\mathrm{d}h \tag{1.4}$$

(h_d is the last component of the vector h). Notice that the scaling factor $\frac{1}{\varepsilon}$ is necessary to rule out trivial conclusions: indeed, we have $\lim_{\varepsilon\to 0} J_\varepsilon(E,\Omega) = 0$. For the sake of completeness, we recall the notion of Γ-convergence:

1.2 - Definition. *Let (X,d) be a metric space. The family $f_\varepsilon\colon X \to [-\infty,+\infty]$ Γ-converges w.r.t. the metric d to the function $f_0\colon X \to [-\infty,+\infty]$ as $\varepsilon \to 0$ if*

(i) *for any $x_0 \in X$ and for any $\{x_\varepsilon\} \subset X$ such that $x_\varepsilon \to x_0$ it holds*

$$f_0(x_0) \le \liminf_{\varepsilon\to 0} f_\varepsilon(x_\varepsilon)$$

(ii) *for any $x_0 \in X$ there exists $\{x_\varepsilon\} \subset X$ such that $x_\varepsilon \to x_0$ and*

$$\limsup_{\varepsilon\to 0} f_\varepsilon(x_\varepsilon) \le f_0(x_0).$$

The proofs of the inferior and of the superior limit inequality have very different natures: in the first case (Subsection 3.3), we use a compactness criterion to reduce the desired inequality to a density estimate, while in the second (Subsection 3.2) we give a pointwise convergence result and then we conclude by a density lemma. In spite of this diversity, there is a key point which is shared by the two arguments, namely the possibility of controlling the rescaled interactions

$$\frac{1}{\varepsilon}\int_F\int_E K_\varepsilon(y-x)\mathrm{d}y\mathrm{d}x$$

in the limit $\varepsilon \to 0$: when E and F do not overlap, Proposition 3.2 shows that asymptotically these functionals either vanish or they are uniformly bounded, depending on the mutual position of E and F.

Before setting off the analysis, we fix the notation we adopt throughout the paper and we premise some reminders about the theory of finite perimeter sets. All the study is carried out in the vector space $\mathbb{R}^d, d \geq 1$, endowed with the Euclidean inner product $\cdot$ and the Euclidean norm $|\cdot|$. We shall often consider a reference set $\Omega \subset \mathbb{R}^d$, assuming that it is open, connected and bounded. When $\lambda > 0$, $h \in \mathbb{R}^d$ and $E \subset \mathbb{R}^d$ we write $\lambda E + h$ to denote the set obtained from E firstly by the dilation of factor λ and then by the translation by h. For any set $E \subset \mathbb{R}^d$, E^{c} is the complement of E in $\mathbb{R}^d$ and χ_E denotes its characteristic function, while $|E|$ stands as its d-dimensional Lebesgue measure. We use the symbols $\mathscr{L}^d$ and $\mathscr{H}^{d-1}$ to denote respectively the d-dimensional Lebesgue and the $(d-1)$-dimensional Hausdorff measure. $\mathscr{M}$ is the collection of all Lebesgue measurable sets in $\mathbb{R}^d$. We shall systematically identify sets with their characteristic functions; in particular, by saying that a sequence $\{E_n\}$ converges in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ to E we mean that for any compact set C, the measure of the intersection $(E_n \triangle E) \cap C$ tends to 0 as n diverges. There are two sets that play a distinguished role in what follows: the halfspace

$$H \coloneqq \left\{x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d : x_d < 0\right\}$$

and the open unit cube

$$U \coloneqq \left(-\frac{1}{2}, \frac{1}{2}\right)^d.$$

If u is a function, we use the symbols ∇u and $\mathrm{D}u$ to denote respectively the classical and the distributional gradient of u; in particular, $\mathrm{D}u$ is a $\mathbb{R}^d$-valued measure. If $\Omega \subset \mathbb{R}^d$ is open, we say that u is a *function of bounded variation* when it belongs to $L^1(\Omega)$ and the total variation of the distributional gradient $|\mathrm{D}u|$ is finite on Ω:

$$\mathrm{BV}(\Omega) \coloneqq \left\{u \in L^1(\Omega) : |\mathrm{D}u|(\Omega) \text{ is finite}\right\}.$$

This space can be characterised in terms of the L^1-norm of difference quotients:

1.3 - PROPOSITION. *Let $\Omega \subset \mathbb{R}^d$ be an open subset. Then, $u\colon \Omega \to \mathbb{R}$ is a function of bounded variation in Ω if and only if there exists a constant $c \geq 0$ such that for any Ω' compactly contained in Ω and for any $h \in \mathbb{R}^d$ with $|h| < \operatorname{dist}(\Omega', \Omega^{\mathrm{c}})$ it holds*

$$\|\tau_h u - u\|_{L^1(\Omega')} \leq c\,|h|,$$

where $\tau_h u(x) \coloneqq u(x+h)$. In particular, it is possible to choose $c = |\mathrm{D}u|(\Omega)$.

When $E \subset \mathbb{R}^d$ is a measurable set such that χ_E is a function of bounded variation in a certain reference set Ω, we say that E has *finite perimeter* in Ω or that it is a *Caccioppoli set* in Ω and we put

$$\mathrm{Per}(E, \Omega) \coloneqq |\mathrm{D}\chi_E|(\Omega). \tag{1.5}$$

To recall the key result by De Giorgi and Federer, let us consider for any $x \in \operatorname{supp} |D\chi_E|$ the Radon-Nikodym derivative

$$\hat{n}(x) \coloneqq \frac{\mathrm{d}D\chi_E}{\mathrm{d}\,|D\chi_E|}(x) = \lim_{r \to 0^+} \frac{D\chi_E(B(x,r))}{|D\chi_E|\,(B(x,r))};$$

here $B(x,r)$ is the open ball of centre x and radius $r > 0$. We call the set

$$\partial^* E \coloneqq \left\{x \in \mathbb{R}^d : \hat{n}(x) \text{ exists and has norm } 1\right\}$$

the *reduced boundary* of E and when $x \in \partial^* E$ we also say that $\hat{n}(x)$ is the measure theoretic *inner normal* to E in x. Now, the cited theorem states that if E is a measurable set, then $\partial^* E$ is $(d-1)$-rectifiable and $D\chi_E = \hat{n}\chi_{\partial^* E}\mathscr{H}^{d-1}$ so that

$$\operatorname{Per}(E,\Omega) = \mathscr{H}^{d-1}(\partial^* E \cap \Omega); \tag{1.6}$$

for this reason we shall call $\mathscr{H}^{d-1} \llcorner \partial^* E$ the *perimeter measure of* E. In addition, for any $x \in \partial^* E$ there exists $R_x \in SO(d)$ such that

$$\frac{E - x}{r} \to R_x H \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^d) \text{ as } r \to 0^+. \tag{1.7}$$

We shall invoke these properties when proving the inferior limit inequality of the Γ-convergence theorem.

For further details about the theory of functions of bounded variations and finite perimeter sets we refer to the monographs by Ambrosio, Fusco and Pallara [3] and by Maggi [13].

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2 An overview of nonlocal perimeters

2.1 The nonlocal perimeter associated with an integral kernel

Let $K\colon \mathbb{R}^d \to [0,+\infty]$ be a measurable function and E, F be sets in $\mathscr{M}$; we define the *nonlocal K-interaction* between E and F as

$$L_K(E,F) \coloneqq \int_F \int_E K(y-x)\mathrm{d}y\mathrm{d}x.$$

We can view this functional as the quantity of energy that is stored in the couple of sets because of the interaction expressed by the kernel K.

Notice that by Tonelli's Theorem

$$L_K(E,F) = L_K(F,E) = \int_{E\times F} K(y-x)\mathrm{d}y\mathrm{d}x,$$

thus it is not restrictive to assume K to be even:

$$K(h) = K(-h) \quad \text{for any } h \in \mathbb{R}^d. \tag{2.1}$$

The following facts can be derived in a straightforward manner:

2.1 - LEMMA. *Let $K\colon \mathbb{R}^d \to [0, +\infty]$ be a measurable function such that (2.1) is satisfied. Then,*

(i) *L_K ranges in $[0, +\infty]$, and it vanishes if one its arguments has zero Lebesgue measure;*

(ii) *for any $E_1, E_2, F \in \mathscr{M}$, we have*

$$L_K(E_1, F) = L_K(E_2, F) \quad \textit{if } |E_1 \triangle E_2| = 0 \quad \textit{and}$$
$$L_K(E_1 \cup E_2, F) = L_K(E_1, F) + L_K(E_2, F) \quad \textit{if } |E_1 \cap E_2| = 0;$$

(iii) *for any $\lambda > 0$ and $h \in \mathbb{R}^d$,*

$$L_K(\lambda E + h, F) = \lambda^{2d} \int_E \int_{\frac{1}{\lambda}(F-h)} K(\lambda(y - x))\mathrm{d}y\mathrm{d}x;$$

in particular, L_K is left unchanged if both arguments are translated by the same vector;

(iv) *the following equality holds:*

$$L_K(E, F) = \int_{\mathbb{R}^d} K(h) \, |E \cap (F - h)| \, \mathrm{d}h. \tag{2.2}$$

One may ask when the interaction L_K is finite; clearly, the answer heavily depends on the summability assumptions on K. For instance, let us admit provisionally that K is $L^1(\mathbb{R}^d)$; then, from (2.2) we see that $L_K(E, F)$ is finite as soon as one of either E or F has finite Lebesgue measure and it holds

$$L_K(E, F) \leq \|K\|_{L^1(\mathbb{R}^d)} \min\{|E|, |F|\}. \tag{2.3}$$

Further, assume that the support of K is contained in a ball of radius r: we get

$$L_K(E, F) = \int_{\{x \in E : \mathrm{dist}(x,F) < r\}} \int_{\{y \in F : \mathrm{dist}(y,E) < r\}} K(y - x)\mathrm{d}y\mathrm{d}x,$$

which shows that, for each set in the couple, the points that play a major role are the ones that lie near to the other set; similarly, in the general case, we expect that points that are separated by a large distance have smaller influence on the total interaction L_K. We shall come back to this point later on, when we consider the behaviour of functionals induced by mass-preserving rescalings of K.

From now on, we assume that

$$\begin{gathered} K\colon \mathbb{R}^d \to [0, +\infty) \text{ is a measurable even function such that} \\ h \mapsto K(h) \min\{1, |h|\} \text{ belongs to } L^1(\mathbb{R}^d). \end{gathered} \tag{2.4}$$

If (2.4) is fulfilled, we are in position to prove that the nonlocal interaction between a couple of sets is finite provided we have some information on the mutual positions. Indeed, if two sets overlap on a region of full Lebesgue measure, we cannot expect L_K to be finite because K might not be summable around the origin.

2.2 - PROPOSITION. *Let E and F be sets with strictly positive Lebesgue measure and let us assume that E has finite perimeter in $\mathbb{R}^d$ and that $|\, E \cap F \,| = 0$. If (2.4) holds, then*

$$L_K(E,F) \leq c(E) \int_{\mathbb{R}^d} K(h) \min\{1, |h|\} \, \mathrm{d}h, \tag{2.5}$$

where $c(E) := \max\left\{|E|, \frac{\mathrm{Per}(E)}{2}\right\}$.

Proof. Up to Lebesgue negligible sets, $F \subset E^{\mathrm{c}}$; therefore,

$$\begin{aligned} L_K(E,F) \leq & L_K(E,E^{\mathrm{c}}) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(h)\,|\chi_E(x+h) - \chi_E(x)| \,\mathrm{d}x\mathrm{d}h \\ = & \frac{1}{2}\int_{\{|h|<1\}} K(h) \int_{\mathbb{R}^d} |\chi_E(x+h) - \chi_E(x)| \,\mathrm{d}x\mathrm{d}h \\ & + \frac{1}{2}\int_{\{|h|\geq 1\}} K(h) \int_{\mathbb{R}^d} |\chi_E(x+h) - \chi_E(x)| \,\mathrm{d}x\mathrm{d}h; \end{aligned}$$

we estimate the last integral by the triangle inequality, while the assumption that χ_E is a function of bounded variation on $\mathbb{R}^d$ provides the upper bound

$$\int_{\mathbb{R}^d} |\chi_E(x+h) - \chi_E(x)| \,\mathrm{d}x \leq \mathrm{Per}(E)\,|h|$$

(recall Proposition 1.3) and hence, on the whole, we get

$$L_K(E,F) \leq \frac{1}{2}\,\mathrm{Per}(E) \int_{\{|h|<1\}} K(h)\,|h| \,\mathrm{d}h + |E| \int_{\{|h|\geq 1\}} K(h)\mathrm{d}h.$$

□

Now, we use the functional L_K to recall the definition of nonlocal perimeter. We firstly fix a reference set $\Omega \in \mathscr{M}$ and, to avoid trivialities, hereafter we always assume that it has strictly positive measure. Let us define the *nonlocal perimeter* of a set $E \in \mathscr{M}$ in Ω:

$$\begin{aligned} \mathrm{Per}_K(E,\Omega) := & L_K(E\cap\Omega, E^{\mathrm{c}}\cap\Omega) \\ & + L_K(E\cap\Omega, E^{\mathrm{c}}\cap\Omega^{\mathrm{c}}) + L_K(E\cap\Omega^{\mathrm{c}}, E^{\mathrm{c}}\cap\Omega); \end{aligned} \tag{2.6}$$

as a particular case, we set

$$\mathrm{Per}_K(E) := \mathrm{Per}_K(E,\mathbb{R}^d) = L_K(E,E^{\mathrm{c}}) \tag{2.7}$$

and we observe that $\mathrm{Per}_K(E,\Omega) = L_K(E,E^{\mathrm{c}}) = \mathrm{Per}_K(E)$ whenever $|\, E\cap\Omega \,| = 0$. These positions rely on the intuitive notion of perimeter that we discussed in the introductory section: we attempt to identify the locus that divides a set E from its complement and we do this by considering suitable K-couplings between E and E^{c}. On one hand, this is evident from Definition (2.7), on the other this is true for Definition (2.6) as well, the only difference being the omission of the interactions that arise inside Ω^{c}. More precisely, one can understand the nonlocal perimeter of a set E in Ω as being made of two contributions: the former is expressed by the summand $L_K(E\cap\Omega, E^{\mathrm{c}}\cap\Omega)$ and it encodes the energy that is located in Ω, while the latter is provided by $L_K(E\cap\Omega, E^{\mathrm{c}}\cap\Omega^{\mathrm{c}}) + L_K(E\cap\Omega^{\mathrm{c}}, E^{\mathrm{c}}\cap\Omega)$ and it captures the energy that "flows" through the portions of the boundaries that E and Ω share. When treating the Γ-convergence of the perimeter, we shall see that these different natures give birth to distinct asymptotics.

We gather here some examples of kernels that fulfil the assumptions in (2.4)

2.3 - Examples. Of course, perimeters associated to L^1 kernels fit into our theory. Outside this class, a relevant example is given by fractional kernels ([4, 12]), that is

$$K(h) = \frac{a(h)}{|h|^{d+s}},$$

where $s \in (0,1)$ and $a\colon \mathbb{R}^d \to \mathbb{R}$ is a measurable even function such that $0 < m \le a(h) \le M$ for any $h \in \mathbb{R}^d$ for some positive m and M. A third case is represented by the kernels we shall deal with most of the times in the sequel, namely the functions $K\colon \mathbb{R}^d \to [0,+\infty)$ such that the map $h \mapsto K(h)\,|h|$ is L^1; observe that this summability assumption allows for a fractional-type behaviour near the origin, but it also implies faster-than-L^1 decay at infinity.

By now, the literature concerning nonlocal-perimeter-like functionals is expanding. For instance, the mentioned class fractional perimeters, i.e.

$$\begin{aligned}\mathrm{Per}_s(E,\Omega) \coloneqq & \int_{E\cap\Omega}\int_{E^{\mathsf{c}}\cap\Omega} \frac{\mathrm{d}y\mathrm{d}x}{|y-x|^{d+s}} \\ & + \int_{E\cap\Omega}\int_{E^{\mathsf{c}}\cap\Omega^{\mathsf{c}}} \frac{\mathrm{d}y\mathrm{d}x}{|y-x|^{d+s}} + \int_{E\cap\Omega^{\mathsf{c}}}\int_{E^{\mathsf{c}}\cap\Omega} \frac{\mathrm{d}y\mathrm{d}x}{|y-x|^{d+s}}\end{aligned}$$

has been extensively studied; here, we wish to mention just [4], where existence and regularity of solutions to Plateau's problem are dealt with, and the papers [5] by Caffarelli and Valdinoci and [2] by Ambrosio, De Philippis and Martinazzi, where the limiting behaviour as $s \to 1^-$ of $\mathrm{Per}_s(\,\cdot\,,\Omega)$ and of the related minimal surfaces are discussed. The analysis for general kernels K has been carried out in several directions as well and, as a short selection of known results, we cite the flatness properties for minimal surfaces in [9], the existence of isoperimetric profiles established by Cesaroni and Novaga in [6] and the study of nonlocal curvatures by Mazón, Rossi and Toledo in [14].

We close this Subsection by proving that the functional Per_K is a perimeter in the axiomatic sense introduced in [8]. Starting from the properties of L_K that are shown in Lemma 2.1, it is easy to check that statements (i), (ii) and (iii) in Definition 1.1 hold true; in addition, once one has observed that

$$\begin{aligned}\mathrm{Per}_K(E,\Omega) = & \frac{1}{2}\int_{\Omega}\int_{\Omega} K(y-x)\,|\chi_E(y)-\chi_E(x)|\,\mathrm{d}y\mathrm{d}x \\ & + \int_{\Omega}\int_{\Omega^{\mathsf{c}}} K(y-x)\,|\chi_E(y)-\chi_E(x)|\,\mathrm{d}y\mathrm{d}x,\end{aligned} \tag{2.8}$$

semicontinuity (v) follows by Fatou's Lemma. To prove submodularity (vi) it suffices to decompose the involved sets in a suitable manner: for instance, one can find

$$\begin{aligned}& L_K((E\cup F)\cap\Omega, E^{\mathsf{c}}\cap F^{\mathsf{c}}\cap\Omega) \\ & \qquad = L_K(E\cap\Omega, E^{\mathsf{c}}\cap\Omega) + L_K(F\cap\Omega, F^{\mathsf{c}}\cap\Omega) \\ & \qquad\quad - L_K(E\cap\Omega, E^{\mathsf{c}}\cap F\cap\Omega) - L_K(F\cap\Omega, E\cap F^{\mathsf{c}}\cap\Omega) \\ & \qquad\quad - L_K(E\cap F\cap\Omega, E^{\mathsf{c}}\cap F^{\mathsf{c}}\cap\Omega)\end{aligned}$$

and

$$\begin{aligned}& L_K(E\cap F\cap\Omega,\ (E^{\mathsf{c}}\cup F^{\mathsf{c}})\cap\Omega) \\ & \qquad = L_K(E\cap F\cap\Omega, E^{\mathsf{c}}\cap F^{\mathsf{c}}\cap\Omega) \\ & \qquad\quad + L_K(E\cap F\cap\Omega, E^{\mathsf{c}}\cap F\cap\Omega) + L_K(E\cap F\cap\Omega, E\cap F^{\mathsf{c}}\cap\Omega),\end{aligned}$$

so that on the whole one gets

$$\begin{aligned}
\mathrm{Per}_K(E,\Omega)+&\,\mathrm{Per}_K(F,\Omega)\\
&= \mathrm{Per}_K(E\cap F,\Omega)+\mathrm{Per}_K(E\cup F,\Omega)\\
&\quad + 2L_K(E\cap F^{\mathrm{c}}\cap\Omega, E^{\mathrm{c}}\cap F\cap\Omega) + 2L_K(E\cap F^{\mathrm{c}}\cap\Omega, E^{\mathrm{c}}\cap F\cap\Omega^{\mathrm{c}})\\
&\quad + 2L_K(E\cap F^{\mathrm{c}}\cap\Omega^{\mathrm{c}}, E^{\mathrm{c}}\cap F\cap\Omega).
\end{aligned}$$

Eventually, we are left to show that also (iv) is satisfied.

2.4 - PROPOSITION. *Let us assume that* (2.4) *holds and suppose that* Ω *is an open set with finite Lebesgue measure. Then, if* E *is a Caccioppoli set in* $\mathbb{R}^d$*,*

$$\mathrm{Per}_K(E,\Omega) \le c(E,\Omega)\int_{\mathbb{R}^d} K(h)\min\{1,|h|\}\,\mathrm{d}h,$$

where $c(E) := \max\left\{\frac{\mathrm{Per}(E)}{2}, |\Omega|\right\}$*. In particular,*$E$ *has finite nonlocal* K*-perimeter in* Ω *as well and* $\mathrm{Per}_K(\,\cdot\,,\Omega)$ *is a perimeter in the sense of Definition 1.1.*

Proof. The conclusion can be obtained imitating the proof of Proposition 2.2; see also Proposition 2.5. ☐

2.2 Extension to functions and nonlocal minimal surfaces

Of course one is led to consider the perimeter as a geometric property attached to a set; nevertheless, we know that the classic notion by De Giorgi can be casted in the framework of functions of bounded variation. Here, we present a construction of the same flavour, whose aim is extending the functional Per_K to functions. This can be achieved in a natural way: grounding on identity (2.8), we are induced to set for any measurable $u\colon \mathbb{R}^d\to\mathbb{R}$

$$\begin{aligned}
J^1_K(u,\Omega) &:= \int_\Omega\int_\Omega K(y-x)\,|u(y)-u(x)|\,\mathrm{d}x\mathrm{d}y,\\
J^2_K(u,\Omega) &:= \int_\Omega\int_{\Omega^{\mathrm{c}}} K(y-x)\,|u(y)-u(x)|\,\mathrm{d}x\mathrm{d}y \quad\text{and}\\
J_K(u,\Omega) &:= \frac{1}{2}J^1_K(u,\Omega)+J^2_K(u,\Omega).
\end{aligned}\tag{2.9}$$

We shall refer to $J_K(\,\cdot\,,\Omega)$ as *nonlocal* K*-energy functional* and it can be easily seen that it is lower semicontinuous w.r.t. $L^1_{\mathrm{loc}}(\mathbb{R}^d)$-converge. By a small abuse of notation, we shall write $J^i_K(E,\Omega)$ for $i=1,2$ and $J_K(E,\Omega)$ when the functionals are evaluated on the characteristic function of E, so that

$$\begin{aligned}
\frac{1}{2}J^1_K(E,\Omega) &= L_K(E\cap\Omega, E^{\mathrm{c}}\cap\Omega)\\
J^2_K(E,\Omega) &= L_K(E\cap\Omega, E^{\mathrm{c}}\cap\Omega^{\mathrm{c}})+L_K(E^{\mathrm{c}}\cap\Omega, E\cap\Omega^{\mathrm{c}}) \quad\text{and}\\
\mathrm{Per}_K(E,\Omega) = J_K(E,\Omega) &= \frac{1}{2}J^1_K(E,\Omega)+J^2_K(E,\Omega).
\end{aligned}$$

In view of these equalities, we shall informally say that the functional $J^1_K(\,\cdot\,,\Omega)$ is the local contribution to the perimeter, while $J^2_K(\,\cdot\,,\Omega)$ is the nonlocal one.

In the previous subsection, we gave some heuristic justification to the definition of nonlocal perimeter and then we also proved that this object owns certain "reasonable" properties; amongst them, there is the finiteness of the K-perimeter for regular sets. Actually, if we suppose that

C1 Ω is an open, connected and bounded subset of $\mathbb{R}^d$ with Lipschitz boundary and that

C2 $K\colon \mathbb{R}^d \to [0,+\infty)$ is a measurable even function such that the quantity

$$c'_K := \int_{\mathbb{R}^d} K(h)\,|h|\,\mathrm{d}h \quad \text{is finite,} \tag{2.10}$$

then not only the theory we have developed so far applies, but we can also prove a broader result involving functions of bounded variation that yields a conclusion which is similar in spirit to the one of Proposition 2.4.

2.5 - Proposition. *Let us assume that conditions **C1** and **C2** are fulfilled.*

(i) *If Ω is convex and $u \in \mathrm{BV}(\Omega)$, then*

$$J^1_K(u,\Omega) \le c'_K \,|\mathrm{D}u|\,(\Omega). \tag{2.11}$$

(ii) *If $u \in C^1(\mathbb{R}^d) \cap \mathrm{BV}(\mathbb{R}^d)$, then*

$$J_K(u,\Omega) \le c'_K \int_{\mathbb{R}^d} |\nabla u|\,. \tag{2.12}$$

(iii) *If $u \in \mathrm{BV}(\mathbb{R}^d)$,* (2.12) *holds as well, on condition that one replaces the integral on the right-hand side with $|\mathrm{D}u|\,(\mathbb{R}^d)$.*

Proof. Let us firstly assume that Ω is a convex open subset in $\mathbb{R}^d$ and that $u \in \mathrm{BV}(\Omega)$. By the change of variables $h = y - x$ we find

$$J^1_K(u,\Omega) = \int_{\mathbb{R}^d} K(h) \int_{\{x\in\Omega : x+h\in\Omega\}} |u(x+h) - u(x)|\,\mathrm{d}x\mathrm{d}h$$

and, subsequently, by the characterisation of BV functions recalled in Proposition 1.3,

$$J^1_K(u,\Omega) \le \left(\int_{\mathbb{R}^d} K(h)\,|h|\,\mathrm{d}h \right) |\mathrm{D}u|\,(\Omega),$$

that is (2.11).

Next, suppose that $u \in C^1(\mathbb{R}^d) \cap \mathrm{BV}(\mathbb{R}^d)$; similarly to the previous lines, we infer

$$J_K(u,\Omega) \le \int_{\mathbb{R}^d} K(h) \int_{\Omega} |u(x+h) - u(x)|\,\mathrm{d}x\mathrm{d}h,$$

but under the current hypotheses we can no longer localise the points of the segment from x to $x+h$ and thus we integrate over the whole space:

$$J_K(u,\Omega) \le \int_{\mathbb{R}^d} K(h)\,|h|\,\mathrm{d}h \int_{\mathbb{R}^d} |\nabla u(\xi)|\,\mathrm{d}\xi.$$

Finally, recall that if $u \in \mathrm{BV}(\mathbb{R}^d)$, then there exist a sequence $\{u_n\} \subset C^\infty(\mathbb{R}^d) \cap \mathrm{BV}(\mathbb{R}^d)$ that converges to u in $L^1(\mathbb{R}^d)$ and that satisfies

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |\nabla u_n| = |\mathrm{D}u|\,(\mathbb{R}^d).$$

Also, thanks to $L^1_{\mathrm{loc}}(\mathbb{R}^d)$-lower semicontinuity, we deduce the last statement from the second by an approximation argument. □

The functional J_K and the K-perimeter are further linked by a coarea-type result (see [9] and also [2, 6, 14] for analogous statements).

2.6 - PROPOSITION (Coarea formula). *If $K\colon \mathbb{R}^d \to [0, +\infty)$ is measurable, then for any measurable function $u\colon \mathbb{R}^d \to [0, 1]$*

$$J_K^1(u, \Omega) = \int_0^1 J_K^1(\{u > t\}, \Omega)\mathrm{d}t \quad \textit{and} \quad J_K^2(u, \Omega) = \int_0^1 J_K^2(\{u > t\}, \Omega)\mathrm{d}t$$

and hence

$$J_K(u, \Omega) = \int_0^1 \mathrm{Per}_K(\{u > t\}, \Omega)\mathrm{d}t$$

Proof. Given $x, y \in \Omega$, let us suppose without loss of generality that $u(x) \le u(y)$; we consider the function $[0, 1] \ni t \mapsto \chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y)$ and we notice that it is different from 0 exactly when $t \in [u(x), u(y)]$. Consequently,

$$|u(x) - u(y)| = \int_0^1 \left|\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y)\right| \mathrm{d}t$$

and, by Tonelli's Theorem,

$$\begin{aligned} J_K^1(u, \Omega) &:= \int_\Omega \int_\Omega K(x - y)\, |u(x) - u(y)|\, \mathrm{d}x\mathrm{d}y \\ &= \int_0^1 \int_\Omega \int_\Omega K(x - y) \left|\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y)\right| \mathrm{d}x\mathrm{d}y\mathrm{d}t \\ &= \int_0^1 J_K^1(\{u > t\}, \Omega)\mathrm{d}t \end{aligned}$$

In a similar way, one also proves that the equality concerning $J_K^2(u, \Omega)$ holds. □

The validity of coarea formula is crucial for variational purposes. Indeed, it allows to invoke two abstract results proved in [7] by Chambolle, Giacomini and Lussardi:

2.7 - THEOREM. *If $J\colon L^1(\Omega) \to [0, +\infty]$ is a proper lower semicontinuous functional such that*

$$J(u) = \int_{-\infty}^{+\infty} J(\chi_{\{u>t\}})\mathrm{d}t \tag{2.13}$$

and that

$$J(\chi_{E\cap F}) + J(\chi_{E\cup F}) \le J(\chi_E) + J(\chi_F)$$

for any couple of measurable sets in Ω, then J is convex.

2.8 - THEOREM. *Let $\{J_n\}_{n\in\mathbb{N}}$ be a sequence of convex functionals such that* (2.13) *holds and let us suppose that there exists a functional $\tilde{J}$ defined on measurable sets of Ω such that the sequence obtained by restriction of the functionals J_n to measurable sets Γ-converges to $\tilde{J}$ w.r.t. the L^1-convergence. Then, the sequence $\{J_n\}$ Γ-converges to J w.r.t. the same norm if we put*

$$J(u) = \int_{-\infty}^{+\infty} \tilde{J}(\chi_{\{u>t\}})\mathrm{d}t.$$

The latter of the two theorems above is relevant for the discussion contained in Section 3 concerning the limiting properties of nonlocal perimeters. For the moment being, we take advantage of the former and we infer

2.9 - COROLLARY. *If $K\colon \mathbb{R}^d \to [0,+\infty)$ is measurable, the functional $J_K(\,\cdot\,,\Omega)$ is convex on $L^1(\mathbb{R}^d;[0,1])$.*

At this stage, we are in position to solve a Plateau-type problem for nonlocal perimeters through the direct method of calculus of variations. Notice that strong convergence of minimising sequences in L^1 is not guaranteed in principle, because a uniform bound on the nonlocal perimeter is very weak information; for example, if the kernel K is L^1 and Ω is bounded, then any measurable E satisfies $\mathrm{Per}_K(E,\Omega) \leq 3\,\|K\|_{L^1(\mathbb{R}^d)}\,|\Omega|$. We circumvent this obstacle by making use of convexity, which permits to draw the conclusion from weak compactness only.

2.10 - THEOREM (Existence of solutions to Plateau's problem). *Let $K\colon \mathbb{R}^d \to [0,+\infty)$ be measurable and let $\Omega \subset \mathbb{R}^d$ be open and bounded. Suppose that $E_0 \in \mathscr{M}$ has finite K-perimeter in Ω and define*

$$\mathscr{F} \coloneqq \{F \in \mathscr{M} : \mathrm{Per}_K(F,\Omega) < +\infty \text{ and } F\cap\Omega^{\mathrm{c}} = E_0 \cap \Omega^{\mathrm{c}}\}.$$

Then, there exists $E \in \mathscr{F}$ such that

$$\mathrm{Per}_K(E,\Omega) \leq \mathrm{Per}_K(F,\Omega) \quad \text{for any } F \in \mathscr{F}.$$

Also, any minimiser satisfies

$$L_K(E,F) \leq L_K(E^{\mathrm{c}}\cap F^{\mathrm{c}},F) \quad \text{whenever } F \subset E^{\mathrm{c}}\cap\Omega \quad \text{and} \tag{2.14}$$
$$L_K(E^{\mathrm{c}},F) \leq L_K(E\cap F^{\mathrm{c}},F) \quad \text{whenever } F \subset E\cap\Omega \tag{2.15}$$

Proof. Let us consider a minimising sequence $\{u_n\}$ for the more general minimisation problem

$$\inf\left\{J_K(v,\Omega) : v\colon \mathbb{R}^d \to [0,1], v \text{ measurable}, J_K(v,\Omega) < +\infty \text{ and } v|_{\Omega^{\mathrm{c}}} = u_0\right\},$$

where $u_0 \coloneqq \chi_{E_0\cap\Omega^{\mathrm{c}}}$; notice that the set of competitors is non-empty, because it contains at least χ_{E_0}. We also observe that, for any choice of $p \in (1,+\infty)$, $\{u_n\}$ is bounded in $L^p(\Omega;[0,1])$ and therefore there exists $u \in L^p(\Omega;[0,1])$ such that $u_n|_\Omega$ weakly converges to it, up to subsequences. We extend u outside Ω setting $u|_{\Omega^{\mathrm{c}}} = u_0$ and with this choice we get

$$\lim_{n\to\infty} J_K(u_n,\Omega) \geq J_K(u,\Omega).$$

Indeed, J_K is convex and lower semicontinuous w.r.t. strong convergence in $L^1(\mathbb{R}^d;[0,1])$ and hence it is also weakly lower semicontinuous in $L^p(\Omega;[0,1])$ for any $p \in [1,+\infty)$, which implies immediately $\liminf_{n\to\infty} J^1_K(u_n,\Omega) \geq J^1_K(u,\Omega)$; the analogous inequality for the nonlocal term follows as well noticing that $u_n = u = u_0$ in Ω^{c}. Hence, u is a minimiser for $J_K(\,\cdot\,,\Omega)$.

At this stage, the statement concerning existence is proved once we show that from any function that minimises $J_K(\,\cdot\,,\Omega)$ one can recover a set E that minimises $\mathrm{Per}_K(\,\cdot\,,\Omega)$. To this purpose, we apply the Coarea formula: given that

$$J_K(u,\Omega) = \int_0^1 \mathrm{Per}_K(\{u>t\},\Omega)\mathrm{d}t,$$

for some $t^* \in (0,1)$ it must hold $J_K(u,\Omega) \geq \mathrm{Per}_K(\{u > t^*\},\Omega)$; then, just set $E = \chi_{\{u>t^*\}}$.

Eventually, we prove inequalities (2.14) and (2.15). Suppose that E minimises the perimeter and that $F \subset E^c \cap \Omega$; then, the inequality $\mathrm{Per}_K(E,\Omega) \leq \mathrm{Per}_K(E\cup F,\Omega)$ holds and we rewrite it as

$$L_K(E\cap\Omega, E^c)+L_K(E\cap\Omega^c, E^c\cap\Omega) \leq L_K((E\cup F)\cap\Omega, E^c\cap F^c)+L_K(E\cap\Omega^c, E^c\cap F^c\cap\Omega).$$

We decompose the first term in the left-hand side and we confront the second summands on each side, getting

$$L_K(E\cap\Omega, F) + L_K(E\cap\Omega, E^c\cap F^c) + L_K(E\cap\Omega^c, F) \leq L_K((E\cup F)\cap\Omega, E^c\cap F^c)$$

and therefore we find

$$L_K(E\cap\Omega, F) + L_K(E\cap\Omega^c, F) \leq L_K(F, E^c\cap F^c),$$

which is (2.14). The other inequality can be proved similarly starting from $\mathrm{Per}_K(E,\Omega) \leq \mathrm{Per}_K(E\cap F^c,\Omega)$. ☐

We borrowed the proof of optimality conditions from [2, 4], where analogous results are stated for fractional perimeters; notice that to the validity of (2.14) and (2.15) no restriction on K is needed. On the contrary, to deduce some extra information on minimisers, still following the same papers,we shall require that

C3 $\bar{K}\colon [0,+\infty) \to [0,+\infty)$ is a measurable function and for any $h \in \mathbb{R}^d$, $K(h) := \bar{K}(r)$ if $|h| = r$;

C4 $\bar{K}$ is strictly decreasing.

When K satisfies condition **C3**, the coupling L_K is left unchanged by isometries:

$$L_K(R(E),R(F)) = \int_E\int_F K(R(y-x))\mathrm{d}y\mathrm{d}x = L_K(E,F) \quad \text{for any isometry } R \tag{2.16}$$

2.11 - Proposition (Flatness of minimisers). *Let us assume that **C3** and **C4** hold and let $E \in \mathscr{M}$.*

(i) If (2.14) holds for E with $\Omega = U$ and $H\cap U^c \subset E$, then $H \subset E$ up to a set of measure zero.

(ii) If (2.15) holds for E with $\Omega = U$ and $E\cap U^c \subset H$, then $E \subset H$ up to a set of measure zero.

Also, the same statements hold true replacing H by H^c and if E is a minimiser for the problem $\inf\{\mathrm{Per}_K(F,U) : F\cap U^c = H\cap U^c\}$, then, $|E\triangle H| = 0$.

Proof. Let us provisionally assume that (i) holds both for H and H^c; then (ii) follows. Indeed, if E fulfils (2.15), then E^c satisfies (2.14) and hence, by applying (i) with H^c, we get $|H^c\cap U\cap E| = 0$, as desired.

Consequently, if E is a solution to Plateau's problem

$$\inf\{\mathrm{Per}_K(F,U) : F\cap U^c = H\cap U^c\},$$

by Theorem 2.10, (2.14) and (2.15) hold and thanks to the constraint $E\cap U^c = H\cap U^c$ we can invoke both (i) and (ii), thus concluding $|E\triangle H| = 0$.

Finally, we turn to the proof of (i). The idea is to apply (2.14) with a suitable competitor. Since we suppose $H \cap U^{\mathrm{c}} \subset E$, $F^- := H \cap E^{\mathrm{c}}$ is contained in Q. Let us put $F^+ := R(F^-) \cap E^{\mathrm{c}}$ and $F := F^- \cup F^+$, where $R(x_1, \dots, x_{d-1}, x_d) = (x_1, \dots, x_{d-1}, -x_d)$. Notice that $F \subset E^{\mathrm{c}} \cap U$ and hence, taking advantage of (2.16), we have

$$L_K(E, F) \leq L_K(E^{\mathrm{c}} \cap F^{\mathrm{c}}, F) = L_K(G, R(F)),$$

where $G := R(E^{\mathrm{c}} \cap F^{\mathrm{c}})$. F can be decomposed as the disjoint union of $F' := F^- \setminus R(F^+)$ and $F'' := F^+ \cup R(F^+)$, so that the inequality above becomes

$$\begin{aligned} L_K(E, F) &\leq L_K(G, R(F'))) + L_K(G, F'') \\ &= L_K(G, R(F')) - L_K(G, F') + L_K(G, F), \end{aligned}$$

that is

$$L_K(E, F) - L_K(G, F) \leq L_K(G, R(F')) - L_K(G, F').$$

We observe that the left-hand side can be rewritten as $L_K(E \cap G^{\mathrm{c}}, F)$, yielding

$$0 \leq L_K(E \cap G^{\mathrm{c}}, F) \leq L_K(G, R(F')) - L_K(G, F').$$

Nevertheless, if F' is not negligible, the last quantity is always strictly negative because, for any $x \in G$, $|y - x| < |R(y) - x|$ if $y \in F' \cap \{y : y_d \neq 0\}$ and K is a radially strictly decreasing function; it follows that $|F'| = 0$ and either $|E \cap G^{\mathrm{c}}| = 0$ or $|F| = 0$. The latter of these conditions immediately implies the conclusion since $H \cap E^{\mathrm{c}} = F^- \subset F$.

Let us assume instead that $|E \cap G^{\mathrm{c}}| = 0$. We repeat the argument that we have just outlined above to a perturbation of E; namely, for any $\varepsilon > 0$, we set $E_\varepsilon := E + (0, 0, \dots, \varepsilon)$ and we observe that E_ε satisfies (2.14) with $\Omega = Q_\varepsilon := Q + (0, 0, \dots, \varepsilon)$ and thus also with $\Omega = \tilde{Q}_\varepsilon := Q_\varepsilon \cap R(Q_\varepsilon)$. We next define $F_\varepsilon^-, F_\varepsilon^+, F_\varepsilon, F_\varepsilon'$ and F_ε'' in complete analogy with the sets F^-, F^+, F, F' and F'' introduced in the previous lines and we infer that $|F_\varepsilon'| = 0$ and either $|E_\varepsilon \cap G_\varepsilon^{\mathrm{c}}| = 0$ or $|F_\varepsilon| = 0$. The point is that now it holds $|E_\varepsilon \cap G_\varepsilon^{\mathrm{c}}| = \infty$, thus F_ε in necessarily negligible and $|H \cap E_\varepsilon| = 0$; finally, let ε tend to 0.

Similarly to the lines above, one can prove that the conclusions are not compromised if H is replaced by H^{c} and in this way the proof is concluded. $\square$

We shall exploit the flatness result for minimisers to prove a useful characterisation of the constant c_K appearing in Theorem 3.1, similarly to what is done in [2].

3 Γ-convergence of nonlocal perimeters

In this section we turn to a Γ-convergence result of mass preserving rescalings of the K-perimeter. Hereafter we assume that **C1** and **C3** hold. Let us suppose in addition that

C2' the quantity

$$\int_0^{+\infty} \bar{K}(r) r^d \mathrm{d}r \quad \text{is finite.}$$

The combination of **C2'** and **C3** guarantees that **C2** holds as well: indeed, the implication is trivial when $d = 1$, while when $d \geq 2$

$$c_K' = \int_{\mathbb{R}^d} K(h)\, |h| \, \mathrm{d}h = \int_0^{+\infty} \int_{\partial B(0,r)} \bar{K}(r) r \mathrm{d}\mathscr{H}^{d-1}(z) \mathrm{d}r = d\omega_d \int_0^{+\infty} \bar{K}(r) r^d \mathrm{d}r.$$

Besides, thanks to radial symmetry, if $d \geq 2$, we have the following chain of equalities:

$$\begin{aligned}\int_{\mathbb{R}^d} K(h)\,|h_d|\,\mathrm{d}h &= \int_0^{+\infty}\int_{\partial B(0,r)} \bar{K}(r)\,|e_d \cdot z|\,\mathrm{d}\mathscr{H}^{d-1}(z)\mathrm{d}r \\ &= \int_{\partial B(0,1)} |e_d \cdot z|\,\mathrm{d}\mathscr{H}^{d-1}(z)\int_0^{+\infty}\bar{K}(r)r^d\mathrm{d}r \\ &= \frac{\int_{\partial B(0,1)} |e_d \cdot z|\,\mathrm{d}\mathscr{H}^{d-1}(z)}{d\omega_d}\int_{\mathbb{R}^d} K(h)\,|h|\,\mathrm{d}h;\end{aligned}$$

thus, recalling (1.4), we have

$$c_K = \frac{\alpha_{1,d}}{2}c'_K, \quad \text{with } \alpha_{1,d} \coloneqq \frac{\int_{\partial B(0,1)} |e_d \cdot z|\,\mathrm{d}\mathscr{H}^{d-1}(z)}{d\omega_d}. \tag{3.1}$$

Summing up, if we assume the validity of **C1**, **C2'** and **C3**, then the theory of Section 2 applies, the only exception being Proposition 2.11, which also requires **C4**.

In view of the forthcoming analysis, it is convenient to fix some further notation. For $\varepsilon > 0$ and $h \in \mathbb{R}^d$ recall position (1.3) and for $E, F \in \mathscr{M}$ let us define the functionals

$$\begin{gathered}L_\varepsilon(E,F) \coloneqq L_{K_\varepsilon}(E,F), \\ J^1_\varepsilon(E,\Omega) \coloneqq J^1_{K_\varepsilon}(E,\Omega), \quad J^2_\varepsilon(E,\Omega) \coloneqq J^2_{K_\varepsilon}(E,\Omega) \quad \text{and} \\ J_\varepsilon(E,\Omega) \coloneqq \frac{1}{2}J^1_\varepsilon(E,\Omega) + J_\varepsilon(E,\Omega).\end{gathered}$$

Our main goal is proving the following result:

3.1 - Theorem. *Let us suppose that **C1**, **C2'** and **C3** are fulfilled and let $E \in \mathscr{M}$; then,*

(i) there exist a family $\{E_\varepsilon\}_{\varepsilon>0}$ that converges to E in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ with the property that

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}J_\varepsilon(E_\varepsilon,\Omega) \leq c_K \operatorname{Per}(E,\Omega);$$

*(ii) if **C4** holds too, for any family $\{E_\varepsilon\}_{\varepsilon>0}$ that converges to E in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$,*

$$c_K \operatorname{Per}(E,\Omega) \leq \liminf_{\varepsilon\to 0}\frac{1}{\varepsilon}J^1_\varepsilon(E_\varepsilon,\Omega).$$

The functionals $J^2_\varepsilon(\,\cdot\,,\Omega)$ are positive and thus, evidently, the Theorem above implies the Γ-converge of the ratios $\frac{1}{\varepsilon}J_\varepsilon(\,\cdot\,,\Omega)$ to $c_K \operatorname{Per}(\,\cdot\,,\Omega)$ w.r.t. the $L^1_{\mathrm{loc}}(\mathbb{R}^d)$-distance. The two contributions J^1_ε and J^2_ε that compound the rescaled perimeter functional J_ε play different roles: qualitatively, when ε is small, the former is concentrated near the portions of the boundary of E inside Ω, the latter instead gathers around the portions that are close to the boundary of Ω; this is made precise by Proposition3.4, which shows that the pointwise limit and the Γ-limit do not agree in general.

The analogous of Theorem 3.1 for the case of fractional perimeters was established by Ambrosio, De Philippis and Martinazzi in [2]; notice that, however, the scaling used in that work is different, even if we can still adopt similar techniques. In particular, following [2] and the work [1] by Alberti and Bellettini concerned with anisotropic phase transitions, we prove the lower limit inequality via the strategy introduced by Fonseca and Müller in [11], which amounts to turn the proof of (ii) into an inequality of Radon-Nikodym derivatives.

On the other hand, proofs of upper limit inequalities are generally achieved through density arguments. Here, we avoid this by invoking an approximation result of the total variation due to Dávila [10], as it is also done by Mazón, Rossi and Toledo in [14].

Combining Theorems 2.8 and 3.1 we obtain a second Γ-convergence result:

3.1 - COROLLARY. *Let us assume that **C1, C2', C3** and **C4** hold. If for any measurable* $u\colon \mathbb{R}^d \to [0,1]$ *we define the functionals*

$$\frac{1}{\varepsilon} J_\varepsilon(u,\Omega) \coloneqq \frac{1}{\varepsilon} J_{K_\varepsilon}(u,\Omega) \quad \text{and} \quad J_0(u,\Omega) \coloneqq c_K \left|Du\right|(\Omega)$$

then, as ε *approaches* 0*, the family* $\left\{\frac{1}{\varepsilon} J_\varepsilon(\,\cdot\,,\Omega)\right\}$ Γ*-converges to* $J_0(\,\cdot\,,\Omega)$ *w.r.t. the* $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ *distance.*

3.1 Rescaled nonlocal interactions and compactness

To deal with the proof of Theorem 3.1, we need some preliminary tools. One of them is a compactness result which appears rather natural in a Γ-convergence framework; a second one is in fact more related to the peculiarities of our problem and we discuss it in the lines that follow.

We point out that the functional $J^1_K(\,\cdot\,,\Omega)$ is not additive on disjoint subsets w.r.t. its second argument and this missing property accounts exactly for nonlocality. Indeed, if F is any measurable set and we split the domain Ω in the disjoint regions $\Omega \cap F$ and $\Omega \cap F^{\mathsf{c}}$, for any measurable $u\colon \Omega \to \mathbb{R}$, we get

$$\begin{aligned} J^1_K(u,\Omega) =& J^1_K(u,\Omega\cap F) + J^1_K(u,\Omega\cap F^{\mathsf{c}}) \\ &+ 2\int_{\Omega\cap F}\int_{\Omega\cap F^{\mathsf{c}}} K(y-x)\left|u(y)-u(x)\right| \mathrm{d}x\mathrm{d}y. \end{aligned} \tag{3.2}$$

The formula above shows that the energy that is stored in two disjoint sets is smaller than the energy of their union and that the difference is precisely given by the mutual interaction, which, following the terminology suggested in [1], we shall call *locality defect*.

When one considers characteristic functions only, it can be easily seen that the locality defect is the sum of certain nonlocal couplings; hence, we are induced to analyse the limiting behaviour of nonlocal rescaled interactions. Intuitively, since the kernel K decays fast at infinity and the operation of rescaling and letting the scaling parameter tend to 0 amounts to "concentrate" the mass close to the origin, we expect some control of the limit in terms of the portion of boundary shared by the two interacting sets. The next statement puts this heuristic picture in precise terms:

3.2 - PROPOSITION (Asymptotic behaviour of nonlocal interactions). *Let us consider* $E, F \in \mathscr{M}$.

(i) *If there exists a Caccioppoli set* E' *in* $\mathbb{R}^d$ *such that* $E \subset E'$ *and* $F \subset (E')^{\mathsf{c}}$*, then*

$$\limsup_{\varepsilon\to 0} \frac{1}{\varepsilon} L_\varepsilon(E,F) \leq \frac{c'_K}{2}\operatorname{Per}(E').$$

(ii) *If* $\delta \coloneqq \operatorname{dist}(E,F) > 0$*, then*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} L_\varepsilon(E,F) = 0.$$

Proof. To prove the first estimate, we bound the interaction between E and F by means of the interaction between E' and its complement, that is, the nonlocal K_ε-perimeter of E':

$$\begin{aligned}\frac{1}{\varepsilon}L_\varepsilon(E,F) &\le \frac{1}{\varepsilon}\int_{E'}\int_{(E')^c} K_\varepsilon(y-x)\mathrm{d}y\mathrm{d}x \\ &= \frac{1}{2\varepsilon}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(h)\,|\chi_{E'}(x+\varepsilon h)-\chi_{E'}(x)|\,\mathrm{d}h\mathrm{d}x \\ &\le \frac{1}{2}\int_{\mathbb{R}^d} K(h)\,|h|\,\mathrm{d}h\,\mathrm{Per}(E'),\end{aligned}$$

where the last inequality is obtained by Proposition 1.3.

Now, let us suppose that $\delta \coloneqq \mathrm{dist}(E,F) > 0$. Then

$$\begin{aligned}\frac{1}{\varepsilon}L_\varepsilon(E,F) &\le \frac{1}{\varepsilon\delta}\int_E\int_F K_\varepsilon(y-x)\,|y-x|\,\mathrm{d}y\mathrm{d}x \\ &= \frac{1}{\delta}\int_E\int_{\mathbb{R}^d} K(h)\,|h|\,\chi_F(x+\varepsilon h)\mathrm{d}h\mathrm{d}x\end{aligned}$$

and we draw the conclusion applying Lebesgue's dominated convergence Theorem. □

The other tool we mentioned is a compactness criterion. Before stating it, we premise a Lemma, whose proof consists of direct computations:

3.3 - Lemma. *Let $G \in L^1(\mathbb{R}^d)$ be a positive function. Then, for any $u \in L^\infty(\mathbb{R}^d)$ it holds*

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} (G*G)(h)\,|u(x+h)-u(x)|\,\mathrm{d}h\mathrm{d}x \le 2\,\|G\|_{L^1(\mathbb{R}^d)}\,J_G(u,\mathbb{R}^d).$$

In particular, when u is the characteristic function of a measurable set E,

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} (G*G)(h)\,|\chi_E(x+h)-\chi_E(x)|\,\mathrm{d}h\mathrm{d}x \le 4\,\|G\|_{L^1(\mathbb{R}^d)}\,\mathrm{Per}_G(E). \tag{3.3}$$

3.2 - Theorem (Compactness criterion). *For any $n \in \mathbb{N}$, let us consider $\varepsilon_n > 0$ and a measurable $E_n \subset \Omega$. If $\varepsilon_n \to 0$ and*

$$\frac{1}{\varepsilon_n}J^1_{\varepsilon_n}(E_n,\Omega) \quad \textit{is uniformly bounded,}$$

there exist a subsequence $\{E_{n_k}\}$ and a set E with finite perimeter in Ω such that $\{E_{n_k}\}$ converges to E in $L^1(\Omega)$.

Proof. To avoid inconvenient notation, in what follows we omit the index n and we write E_ε in place of E_n.

The idea is to build a second sequence $\{v_\varepsilon\}$ that is asymptotically equivalent to $\{E_\varepsilon\}$ in $L^1(\mathbb{R}^d)$, i.e. $\|v_\varepsilon - \chi_{E_\varepsilon}\|_{L^1(\mathbb{R}^d)} = O(\varepsilon)$, but that in addition has better compactness properties. To this purpose, we consider a positive function $\varphi \in C^\infty_c(\mathbb{R}^d)$ and we set $v_\varepsilon \coloneqq \varphi_\varepsilon * \chi_{E_\varepsilon}$, where

$$\varphi_\varepsilon(x) \coloneqq \frac{1}{\left(\int_{\mathbb{R}^d}\varphi(h)\mathrm{d}h\right)\varepsilon^d}\varphi\left(\frac{x}{\varepsilon}\right).$$

Notice that any v_ε is supported in some ball B containing Ω. Easy computations show that

$$\int_{\mathbb{R}^d} |v_\varepsilon(x)-\chi_{E_\varepsilon}(x)|\,\mathrm{d}x \le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\varphi_\varepsilon(h)|\,|\chi_{E_\varepsilon}(x+h)-\chi_{E_\varepsilon}(x)|\,\mathrm{d}h\mathrm{d}x \tag{3.4}$$

and

$$\begin{aligned}\int_B |\nabla v_\varepsilon(x)|\,\mathrm{d}x &= \int_{\mathbb{R}^d} |\nabla v_\varepsilon(x)|\,\mathrm{d}x \\ &\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\nabla\varphi_\varepsilon(h)|\,|\chi_{E_\varepsilon}(x+h)-\chi_{E_\varepsilon}(x)|\,\mathrm{d}h\mathrm{d}x\end{aligned} \tag{3.5}$$

(to get the last bound we took advantage of the equality $\int_{\mathbb{R}^d}|\nabla\varphi_\varepsilon| = 0$); we claim that it is possible to choose φ in such a way that (3.4) yields asymptotic equivalence of the two sequences and that (3.5) provides a uniform bound on the BV-norm of $\{v_\varepsilon\}$. If our claim is true, on one hand, up to extraction of subsequences, v_ε converges to some $v \in \mathrm{BV}(B)$ in $L^1(B)$; on the other, this v must be the characteristic function of some $E \subset \Omega$, because it is a $L^1(\mathbb{R}^d)$ cluster point of χ_{E_ε}. This concludes the proof.

Now let us show that the claim holds. Define the truncation operator

$$T_1(s) := \begin{cases} s & \text{if } |s| \le 1 \\ 1 & \text{otherwise}\end{cases}$$

and the truncated kernel $G := T_1 \circ K \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. We observe that the convolution $G * G$ is positive and continuous and, therefore, we can build a positive $\varphi \in C_c^\infty(\mathbb{R}^d) \setminus \{0\}$ such that

$$\varphi \le G * G \quad \text{and} \quad |\nabla\varphi| \le G * G. \tag{3.6}$$

Let us set

$$G_\varepsilon(h) := \frac{1}{\varepsilon^d} G\left(\frac{h}{\varepsilon}\right).$$

With this choice, from (3.4) and (3.5) we obtain

$$\int_{\mathbb{R}^d} |v_\varepsilon(x) - \chi_{E_\varepsilon}(x)|\,\mathrm{d}x \le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |G_\varepsilon * G_\varepsilon(h)|\,|\chi_{E_\varepsilon}(x+h)-\chi_{E_\varepsilon}(x)|\,\mathrm{d}h\mathrm{d}x \tag{3.7}$$

and

$$\int_{\mathbb{R}^d} |\nabla v_\varepsilon(x)|\,\mathrm{d}x \le \frac{1}{\varepsilon}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |G_\varepsilon * G_\varepsilon(h)|\,|\chi_{E_\varepsilon}(x+h)-\chi_{E_\varepsilon}(x)|\,\mathrm{d}h\mathrm{d}x.$$

Both the right-hand sides of these inequalities can be bounded above by Lemma 3.3; we detail the estimates for (3.7) only, the others being identical. Thanks to (3.3), we have

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} &|G_\varepsilon * G_\varepsilon(h)|\,|\chi_{E_\varepsilon}(x+h)-\chi_{E_\varepsilon}(x)|\,\mathrm{d}h\mathrm{d}x \\ &\le 4\,\|G\|_{L^1(\mathbb{R}^d)}\,\mathrm{Per}_{G_\varepsilon}(E_\varepsilon) \\ &\le 4\,\|G\|_{L^1(\mathbb{R}^d)}\,\mathrm{Per}_{K_\varepsilon}(E_\varepsilon) \\ &= 4\,\|G\|_{L^1(\mathbb{R}^d)}\left(\frac{1}{2}J^1_\varepsilon(E_\varepsilon,\Omega) + J^2_\varepsilon(E_\varepsilon,\Omega)\right) \\ &= 4\,\|G\|_{L^1(\mathbb{R}^d)}\left(\frac{1}{2}J^1_\varepsilon(E_\varepsilon,\Omega) + L_\varepsilon(E_\varepsilon,E_\varepsilon^{\mathsf{c}}\cap\Omega^{\mathsf{c}})\right)\end{aligned}$$

and hence, in view of the current hypotheses and of Proposition 3.2, we deduce

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |G_\varepsilon * G_\varepsilon(h)|\,|\chi_{E_\varepsilon}(x+h)-\chi_{E_\varepsilon}(x)|\,\mathrm{d}h\mathrm{d}x = O(\varepsilon),$$

as desired. □

3.2 Asymptotic behaviour on finite perimeter sets: the upper limit inequality

It is possible to describe the asymptotic behaviour of the functional $J_\varepsilon(\,\cdot\,,\Omega)$ when it is evaluated on finite perimeter sets. This also provides an insight about the upper limit inequality that is to be discussed later on in this subsection.

We extend a result by Mazón, Rossi and Toledo contained in [14]: differently from that work, here we are able to cope also with unbounded domains.

3.4 - PROPOSITION. *Let $\tilde\Omega$ be an open subset of $\mathbb{R}^d$ with Lipschitz boundary, not necessarily bounded, or the whole space $\mathbb{R}^d$ and let conditions **C2'** and **C3** be satisfied. Then, if E is a finite perimeter set in $\tilde\Omega$ such that $E\cap\tilde\Omega$ is bounded, it holds*

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}J^1_\varepsilon(E,\tilde\Omega) = c_K\,\mathrm{Per}(E,\tilde\Omega) \tag{3.8}$$

and if E is also a finite perimeter set in $\mathbb{R}^d$ we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}J^2_\varepsilon(E,\tilde\Omega) = c_K\mathscr{H}^{d-1}(\partial^* E\cap\partial\tilde\Omega). \tag{3.9}$$

The proof of the analogous result proposed in [14] only relies on the approximation of the total variation of the gradient of a function by means of weighted integrals of the difference quotient. Precisely, for $\varepsilon > 0$, consider a collection of positive functions $\bar\rho_\varepsilon\colon [0,+\infty)\to[0,+\infty)$ such that

$$\lim_{\varepsilon\to 0}\int_\delta^{+\infty}\bar\rho_\varepsilon(r)r^{d-1}\mathrm{d}r = 0 \quad \text{for all } \delta>0;$$

also, define $\rho_\varepsilon(h) := \bar\rho_\varepsilon(r)$ whenever $h\in\mathbb{R}^d$ and $|h| = r$ and assume that

$$\int_{\mathbb{R}^d}\rho_\varepsilon(h)\mathrm{d}h = 1.$$

In [10], Dávila proved the following:

3.3 - THEOREM. *Let Ω and $\{\rho_\varepsilon\}_{\varepsilon>0}$ be as above. Then, for any $u\in\mathrm{BV}(\Omega)$*

$$\lim_{\varepsilon\to 0}\int_\Omega\int_\Omega\rho_\varepsilon(y-x)\frac{|u(y)-u(x)|}{|y-x|}\mathrm{d}y\mathrm{d}x = \alpha_{1,d}\,|\,\mathrm{D}u\,|\,(\Omega). \tag{3.10}$$

with $\alpha_{1,d}$ as in (3.1).

Proof of Proposition 3.4. Since $E\cap\tilde\Omega$ is bounded, there exists an open ball B such that the closure of $E\cap\tilde\Omega$ is contained in B and in particular $\mathrm{dist}(E\cap\tilde\Omega, B^\mathrm{c})>0$. By (3.2),

$$\frac{1}{\varepsilon}J^1_\varepsilon(E,\tilde\Omega) = \frac{1}{\varepsilon}J^1_\varepsilon(E,\tilde\Omega\cap B) + \frac{1}{\varepsilon}L_\varepsilon(E\cap\tilde\Omega, E^\mathrm{c}\cap\tilde\Omega\cap B^\mathrm{c});$$

and the second summand in the right-hand side is negligible when $\varepsilon\to 0$ thanks to Proposition 3.2.

The previous reasoning shows that, as far as J^1_ε is concerned, we can always suppose that $\tilde\Omega$ is an open and bounded subset with Lipschitz boundary, so that we may invoke Theorem 3.3: if we set

$$\rho_\varepsilon(h) = \frac{\alpha_{1,d}}{2c_K}K_\varepsilon(h)\left|\frac{h}{\varepsilon}\right|$$

the conclusion of that result reads

$$\lim_{\varepsilon\to 0}\frac{\alpha_{1,d}}{2c_K\varepsilon}J^1_\varepsilon(E,\tilde\Omega)=\alpha_{1,d}\operatorname{Per}(E,\tilde\Omega),$$

that is (3.8).

To show that (3.9) holds, we consider the nonlocal interaction associated with K_ε between E and E^c and we decompose it according to the partition $\{\tilde\Omega,\tilde\Omega^\mathsf{c}\}$:

$$\int_E\int_{E^\mathsf{c}}K_\varepsilon(y-x)\mathrm{d}x\mathrm{d}y=J^1_\varepsilon(E,\tilde\Omega)+J^1_\varepsilon(E,\tilde\Omega^\mathsf{c})+2J^2_\varepsilon(E,\tilde\Omega)$$

since the topological boundary of $\tilde\Omega$ is $\mathscr{L}^d$-negligible, we are allowed to apply (3.8) to each summand: we obtain

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}J^2_\varepsilon(E,\tilde\Omega)=c_K\operatorname{Per}(E)-c_K\operatorname{Per}(E,\tilde\Omega)-c_K\operatorname{Per}(E,\tilde\Omega^\mathsf{c})$$

and the conclusion follows. □

3.5 - REMARK. *As an immediate corollary of the last result, we get the characterisation*

$$c_K=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}J^1_\varepsilon(H,U). \tag{3.11}$$

Actually, we shall need further equivalent descriptions of c_K, see Lemma 3.8.

Now, we turn to the upper limit inequality. We need to show that whenever E is a measurable set there exists a recovery family $\{E_\varepsilon\}$, i.e. a family that converges to E in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ such that

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}J_\varepsilon(E_\varepsilon,\Omega)\le c_K\operatorname{Per}(E,\Omega).$$

First of all, we can assume that E has finite perimeter in Ω, otherwise any family that converges to E is a recovery one. Secondly, let us assume that E is a Caccioppoli set in the whole space $\mathbb{R}^d$; if we retain a *transversality condition* for E and Ω, that is $\mathscr{H}^{d-1}(\partial^*E\cap\partial\Omega)=0$, then we can invoke Proposition 3.4 to deduce that the choice $E_\varepsilon=E$ for all $\varepsilon>0$ defines a recovery family. Hence, the proof of statement (i) in Theorem 3.1 is concluded if we show that the class of finite perimeter sets in $\mathbb{R}^d$ that are transversal to Ω is dense in energy. This is the content of the next Lemma:

3.6 - LEMMA. *Let E be a finite perimeter set in Ω. Then, there exists a family $\{E_\varepsilon\}_{\varepsilon>0}$ of sets with smooth boundaries such that $\mathscr{H}^{d-1}(\partial E_\varepsilon\cap\partial\Omega)=0$ and that $E_\varepsilon\to E$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $\operatorname{Per}(E_\varepsilon,\Omega)\to\operatorname{Per}(E,\Omega)$.*

Proof. By standard approximation results for finite perimeter sets, there exists a family $\{F_\varepsilon\}_{\varepsilon>0}$ of open sets with smooth boundaries such that F_ε converges to E in $L^1(\mathbb{R}^d)$ and that $\operatorname{Per}(F_\varepsilon,\Omega)$ converges to $\operatorname{Per}(E,\Omega)$. Also, notice that in the family

$$F_{\varepsilon,t}\coloneqq\{x:\operatorname{dist}(x,F_\varepsilon)-\operatorname{dist}(x,F^\mathsf{c}_\varepsilon)\le t\}$$

there must be some $E_\varepsilon\coloneqq F_{\varepsilon,t^*}$ which is smooth, transversal to Ω and close to F_ε in L^1 and in perimeter. □

3.3 Density estimates: the lower limit inequality

Let us focus on the proof of statement (ii) in Theorem 3.1. Given $E \in \mathscr{M}$ and any family $\{E_\varepsilon\}_{\varepsilon>0}$ that converges to E in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$, we want to show that

$$c_K \operatorname{Per}(E,\Omega) \leq \liminf_{\varepsilon\to 0} \frac{1}{2\varepsilon} J^1_\varepsilon(E_\varepsilon,\Omega). \tag{3.12}$$

Observe that we can assume that the right-hand side is finite, otherwise the inequality holds trivially, and that the lower limit is a limit. Therefore, the ratios $\frac{1}{\varepsilon}J^1_\varepsilon(E_\varepsilon,\Omega)$ are bounded and, in view of Theorem 3.2, E is a Caccioppoli set in Ω.

The first step of the approach *à la* Fonseca-Müller amounts to reducing the proof of (3.12) to the validity of a suitable density estimate. To this aim, we introduce a family of positive measures such that the total variation on Ω of each of them is equal to $\frac{1}{2\varepsilon}J^1_\varepsilon(E_\varepsilon,\Omega)$. Namely, when $x \in \Omega$, let us set

$$f_\varepsilon(x) \coloneqq \begin{cases} \dfrac{1}{2\varepsilon}\displaystyle\int_{E^{\mathrm{c}}_\varepsilon\cap\Omega} K_\varepsilon(y-x)\mathrm{d}y & \text{if } x \in E_\varepsilon \\[2ex] \dfrac{1}{2\varepsilon}\displaystyle\int_{E_\varepsilon\cap\Omega} K_\varepsilon(y-x)\mathrm{d}y & \text{if } x \in E^{\mathrm{c}}_\varepsilon \end{cases}$$

and $\nu_\varepsilon \coloneqq f_\varepsilon \mathscr{L}^d \llcorner \Omega$. In this way,

$$\|\nu_\varepsilon\| \coloneqq |\nu_\varepsilon|(\Omega) = \frac{1}{2\varepsilon}J^1_\varepsilon(E_\varepsilon,\Omega);$$

since the right-hand side in the latter equality is uniformly bounded w.r.t. ε, we deduce that there exists a finite positive measure ν on Ω such that $\nu_\varepsilon \stackrel{*}{\rightharpoonup} \nu$ as $\varepsilon \to 0$ and hence

$$\liminf_{\varepsilon\to 0}\|\nu_\varepsilon\| \geq \|\nu\|.$$

Because of the inequality above, the conclusion (3.12) follows if we prove that

$$\|\nu\| \geq c_K \operatorname{Per}(E,\Omega); \tag{3.13}$$

recalling (1.6), if we denote by μ the perimeter measure of E, we see that (3.13) in turn is implied by

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) \geq c_K \quad \text{for } \mu\text{-a.e. } x \in \Omega, \tag{3.14}$$

where the left-hand side is the Radon-Nikodym derivative of ν w.r.t. μ. Summing up, the proof is concluded if we show that (3.14) holds. This can be done by recovering at first a "natural" bound for the derivative (Lemma 3.7) and then by proving that this bound is indeed the desired one (Lemmma 3.8).

3.7 - Lemma. *Keeping the assumptions and the notation above, it holds*

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) \geq b_K \quad \textit{for every } x \in \partial^* E \cap \Omega.$$

where

$$b_K = \inf\left\{\liminf_{\varepsilon\to 0}\frac{1}{2\varepsilon}J^1_\varepsilon(E_\varepsilon,U) : E_\varepsilon \to H \textit{ in } L^1(U)\right\}. \tag{3.15}$$

Proof. Let us fix $x \in \partial^* E$. By (1.6), we have

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) = \lim_{r\to 0} \frac{\nu(Q(x,r))}{r^{d-1}},$$

where $Q(x,r) \coloneqq x + rR_x U$ and R_x is chosen as to satisfy (1.7). Also, since the sequence ν_ε weakly-$*$ converges to ν, we have that $\nu(Q(x,r)) = \lim_{\varepsilon\to 0} \nu_\varepsilon(Q(x,r))$ for all $r > 0$ except at most a countable set Z and hence

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) = \lim_{r\to 0, r\notin Z} \left[\lim_{\varepsilon\to 0} \frac{\nu_\varepsilon(Q(x,r))}{r^{d-1}}\right].$$

Via a diagonal process, it is possible to choose two sequences $\{\varepsilon_n\}$ and $\{r_n\}$ such that

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} \frac{\varepsilon_n}{r_n} = 0$$

and that

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) = \lim_{n\to\infty} \frac{\nu_{\varepsilon_n}(Q(x,r_n))}{r_n^{d-1}},$$

or, explicitly,

$$\begin{aligned}\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) = \lim_{n\to\infty} \frac{1}{2\varepsilon_n r_n^{d-1}} \Bigg[&\int_{E_{\varepsilon_n}\cap Q(x,r_n)\cap\Omega} \int_{E^c_{\varepsilon_n}\cap\Omega} K_{\varepsilon_n}(y-x)\mathrm{d}y\mathrm{d}x \\ &+ \int_{E^c_{\varepsilon_n}\cap Q(x,r_n)\cap\Omega} \int_{E_{\varepsilon_n}\cap\Omega} K_{\varepsilon_n}(y-x)\mathrm{d}y.\mathrm{d}x\Bigg]\end{aligned}$$

From this equality we infer the lower bound

$$\begin{aligned}\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) &\geq \limsup_{n\to\infty} \frac{1}{2\varepsilon_n r_n^{d-1}} J^1_{\varepsilon_n}(E_{\varepsilon_n}, Q(x,r_n)\cap\Omega) \\ &= \limsup_{n\to\infty} \frac{1}{2\varepsilon_n r_n^{d-1}} J^1_{\varepsilon_n}(E_{\varepsilon_n}, Q(x,r_n))\end{aligned}$$

(when r_n is small enough, $Q(x,r_n) \subset \Omega$); moreover, by means of a change of variables and (2.16), we find

$$J^1_{\varepsilon_n}(E_{\varepsilon_n}, Q(x,r_n)) = r_n^d J^1_{\frac{\varepsilon_n}{r_n}}\left(R_x^{-1}\left(\frac{E_{\varepsilon_n} - x}{r_n}\right), U\right)$$

and this, plugged in the last inequality, yields

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu}(x) \geq \limsup_{n\to\infty} \frac{r_n}{2\varepsilon_n} J^1_{\frac{\varepsilon_n}{r_n}}\left(R_x^{-1}\left(\frac{E_{\varepsilon_n} - x}{r_n}\right), U\right).$$

Now, thanks to our choice of R_x we have that

$$R_x^{-1}\left(\frac{E_{\varepsilon_n} - x}{r_n}\right) \to H \quad \text{in } L^1(U) \text{ as } n\to\infty,$$

and by definition of b_K we conclude. $\square$

To accomplish the proof of the inferior limit inequality in Theorem 3.1 we have to show that $c_K = b_K$. Imitating the approach of [2], we do this by introducing a third constant b'_K:

3.8 - LEMMA. *For any $\delta > 0$, set $U^\delta := \{x \in U, \operatorname{dist}(x, U^{\mathsf{c}}) \leq \delta\}$ and*

$$b'_K := \inf\left\{\liminf_{\varepsilon\to 0} \frac{1}{2\varepsilon} J^1_\varepsilon\left(E_\varepsilon, U\right) : E_\varepsilon \to H \text{ in } L^1(U) \text{ and } E_\varepsilon \cap U^\delta = H \cap U^\delta\right\}.$$

Under the previous assumptions, $c_K = b'_K = b_K$.

Because of (3.11), one clearly has $c_K \geq b'_K \geq b_K$. The conclusion of Lemma 3.8 follows if we prove that the reverse inequalities hold as well; this can be achieved invoking Proposition 2.11 and the next result, which extends a similar one proved in [2] for s-perimeters:

3.9 - PROPOSITION (Gluing). *Let us consider $E_1, E_2 \in \mathscr{M}$ and $\delta_1, \delta_2 \in \mathbb{R}$ such that $\delta_1 > \delta_2 > 0$. For $\delta > 0$, we set*

$$\Omega^\delta := \{x \in \Omega : \operatorname{dist}(x, \Omega^{\mathsf{c}}) \leq \delta\}.$$

If $J^1_K(E_i, \Omega)$ is finite for both $i = 1, 2$, then there exists $F \in \mathscr{M}$ such that

- *(i)* $F \cap (\Omega \setminus \Omega^{\delta_1}) = E_1 \cap (\Omega \setminus \Omega^{\delta_1})$ *and* $F \cap \Omega^{\delta_2} = E_2 \cap \Omega^{\delta_2}$;
- *(ii)* $|(E_1 \triangle F) \cap \Omega| \leq |(E_1 \triangle E_2) \cap \Omega|$;
- *(iii)* *for all $\eta > 0$:*

$$\begin{aligned} J^1_\varepsilon(F, \Omega) \leq & J^1_\varepsilon(E_1, \Omega) + J^1_\varepsilon(E_2, \Omega^{\delta_1+\eta}) + \frac{2\varepsilon c'_K}{\delta_1 - \delta_2} \,|\, (E_1 \triangle E_2) \cap \Omega \,| \\ & + \frac{2\varepsilon}{\eta} \int_{\Omega\setminus\Omega^{\delta_1+\eta}} \int_{\mathbb{R}^d} K(h) \,|\, h \,|\, \chi_{\Omega^{\delta_1}}(x + \varepsilon h) \mathrm{d}h \mathrm{d}x. \end{aligned} \tag{3.16}$$

Proof. Suppose that for some function $w \colon \Omega \to [0, 1]$ it holds

$$\begin{aligned} J^1_\varepsilon(w, \Omega) \leq & J^1_\varepsilon(E_1, \Omega) + J^1_\varepsilon(E_2, \Omega^{\delta_1+\eta}) + \frac{2\varepsilon c'_K}{\delta_1 - \delta_2} \,|\, (E_1 \triangle E_2) \cap \Omega \,| \\ & + \frac{2\varepsilon}{\eta} \int_{\Omega\setminus\Omega^{\delta_1+\eta}} \int_{\mathbb{R}^d} K(h) \,|\, h \,|\, \chi_{\Omega^{\delta_1}}(x + \varepsilon h) \mathrm{d}h \mathrm{d}x; \end{aligned} \tag{3.17}$$

then, thanks to Coarea formula there exists $t^* \in (0, 1)$ such that (3.16) holds for the superlevel $F := \{w > t^*\}$. Let us exhibit a function w that fulfils (3.17).

Loosely speaking, we choose w to be a convex combination of the data χ_{E_1} and χ_{E_2}. Precisely, for any $u, v \colon \Omega \to [0, 1]$ such that $J^1_K(u, \Omega)$ and $J^1_K(v, \Omega)$ are finite, let us set $w := \varphi u + (1 - \varphi) v$, where $\varphi \in C^\infty_c(\mathbb{R}^d)$ satisfies

$$0 \leq \varphi \leq 1 \text{ in } \Omega, \quad \varphi = 0 \text{ in } \Omega^{\delta_2}, \quad \varphi = 1 \text{ in } \Omega \setminus \Omega^{\delta_1} \quad \text{and} \quad |\nabla \varphi| \leq \frac{2}{\delta_1 - \delta_2}.$$

We explicit the integrand appearing in $J^1_K(w, \Omega)$ and for $x, y \in \Omega$ we get the bounds

$$\begin{aligned} |\, w(y) - w(x) \,| \leq & \varphi(y) \,|\, u(y) - u(x) \,| + (1 - \varphi(y)) \,|\, v(y) - v(x) \,| \\ & + |\, \varphi(y) - \varphi(x) \,|\, |\, v(x) - u(x) \,| \\ \leq & \chi_{\{\varphi\neq 0\}}(y) \,|\, u(y) - u(x) \,| + \chi_{\{\varphi\neq 1\}}(y) \,|\, v(y) - v(x) \,| \\ & + |\, \varphi(y) - \varphi(x) \,|\, |\, v(x) - u(x) \,|. \end{aligned}$$

By our choice of φ, $\{\varphi \neq 0\} \subset \Omega \setminus \Omega^{\delta_2}$ and $\{\varphi \neq 1\} \subset \Omega^{\delta_1}$, therefore

$$\begin{aligned} J^1_\varepsilon(w,\Omega) \leq & \int_\Omega \int_{\Omega\setminus\Omega^{\delta_2}} K_\varepsilon(y-x)\,|\,u(y)-u(x)\,|\,\mathrm{d}y\mathrm{d}x \\ & + \int_\Omega \int_{\Omega^{\delta_1}} K_\varepsilon(y-x)\,|\,v(y)-v(x)\,|\,\mathrm{d}y\mathrm{d}x \\ & + \int_\Omega \int_\Omega K_\varepsilon(y-x)\,|\,\varphi(y)-\varphi(x)\,|\,|\,v(x)-u(x)\,|\,\mathrm{d}y\mathrm{d}x \end{aligned}$$

We treat each of the integrals appearing in the right-hand side separately. Of course one has

$$\int_\Omega \int_{\Omega\setminus\Omega^{\delta_2}} K_\varepsilon(y-x)\,|\,u(y)-u(x)\,|\,\mathrm{d}y\mathrm{d}x \leq J^1_\varepsilon(u,\Omega). \tag{3.18}$$

To estimate the second integral, we split the reference set Ω in the regions $\Omega^{\delta_1+\eta}$ and $\Omega \setminus \Omega^{\delta_1+\eta}$ and this yields

$$\begin{aligned} & \int_\Omega \int_{\Omega^{\delta_1}} K_\varepsilon(y-x)\,|\,v(y)-v(x)\,|\,\mathrm{d}y\mathrm{d}x = \\ & \qquad \int_{\Omega^{\delta_1+\eta}} \int_{\Omega^{\delta_1}} K_\varepsilon(y-x)\,|\,v(y)-v(x)\,|\,\mathrm{d}y\mathrm{d}x \\ & \qquad + \int_{\Omega\setminus\Omega^{\delta_1+\eta}} \int_{\Omega^{\delta_1}} K_\varepsilon(y-x)\,|\,v(y)-v(x)\,|\,\mathrm{d}y\mathrm{d}x; \end{aligned}$$

evidently

$$\int_{\Omega^{\delta_1+\eta}} \int_{\Omega^{\delta_1}} K_\varepsilon(y-x)\,|\,v(y)-v(x)\,|\,\mathrm{d}y\mathrm{d}x \leq J^1_\varepsilon(v,\Omega^{\delta_1+\eta})$$

and, further,

$$\begin{aligned} & \int_{\Omega\setminus\Omega^{\delta_1+\eta}} \int_{\Omega^{\delta_1}} K_\varepsilon(y-x)\,|\,v(y)-v(x)\,|\,\mathrm{d}y\mathrm{d}x \\ & \qquad \leq \frac{2\varepsilon}{\eta} \int_{\Omega\setminus\Omega^{\delta_1+\eta}} \int_{\Omega^{\delta_1}} K_\varepsilon(y-x)\frac{|\,y-x\,|}{\varepsilon}\mathrm{d}y\mathrm{d}x \\ & \qquad \leq \frac{2\varepsilon}{\eta} \int_{\Omega\setminus\Omega^{\delta_1+\eta}} \int_{\mathbb{R}^d} K(h)\,|\,h\,|\,\chi_{\Omega^{\delta_1}}(x+\varepsilon h)\mathrm{d}h\mathrm{d}x, \end{aligned}$$

so that, all in all,

$$\begin{aligned} & \int_\Omega \int_{\Omega^{\delta_1}} K_\varepsilon(y-x)\,|\,v(y)-v(x)\,|\,\mathrm{d}y\mathrm{d}x \\ & \qquad \leq J^1_\varepsilon(v,\Omega^{\delta_1+\eta}) + \frac{2\varepsilon}{\eta} \int_{\Omega\setminus\Omega^{\delta_1+\eta}} \int_{\mathbb{R}^d} K(h)\,|\,h\,|\,\chi_{\Omega^{\delta_1}}(x+\varepsilon h)\mathrm{d}h\mathrm{d}x. \end{aligned} \tag{3.19}$$

Lastly, we observe that

$$|\,\varphi(y)-\varphi(x)\,| \leq \frac{2}{\delta_1-\delta_2}\,|\,y-x\,|$$

and hence

$$\int_\Omega \int_\Omega K_\varepsilon(y-x)\,|\,\varphi(y)-\varphi(x)\,|\,|\,v(x)-u(x)\,|\,\mathrm{d}y\mathrm{d}x \leq \frac{2\varepsilon c'_K}{\delta_1-\delta_2} \int_\Omega |\,v(x)-u(x)\,|\,\mathrm{d}x \tag{3.20}$$

Combining (3.18), (3.19) and (3.20) we obtain

$$\begin{aligned}J^1_\varepsilon(w,\Omega) \le& J^1_\varepsilon(u,\Omega) + J^1_\varepsilon(v,\Omega^{\delta_1+\eta}) + \frac{2\varepsilon c'_K}{\delta_1-\delta_2}\,\|v-u\|_{L^1(\Omega)}\\ &+\frac{2\varepsilon}{\eta}\int_{\Omega\setminus\Omega^{\delta_1+\eta}}\int_{\mathbb{R}^d} K(h)\,|\,h\,|\,\chi_{\Omega^{\delta_1}}(x+\varepsilon h)\mathrm{d}h\mathrm{d}x;\end{aligned}$$

thus, if we pick $u=\chi_{E_1}$ and $v=\chi_{E_2}$ we have (3.17).

It remains to check that conditions (i) and (ii) hold true for the set F defined above. As for the former, it suffices to recall that φ is supported in $\Omega\setminus\Omega^{\delta_2}$ and that it is constantly 1 in $\Omega\setminus\Omega^{\delta_1}$. On the other hand, to prove the second condition we remark that $x\in E_1\cap F^{\mathrm{c}}$ and $x\in E_1^{\mathrm{c}}\cap F$ imply respectively the equalities $w(x)=\varphi(x)+(1-\varphi(x))\chi_{E_2}(x)\le t<1$ and $w(x)=(1-\varphi)\chi_{E_2}(x)>t>0$, which in turn entail $x\in E_1\cap E^{\mathrm{c}}$ and $x\in E_1^{\mathrm{c}}\cap E_2$. $\square$

Proof of Lemma 3.8. We firstly show that $c_K\le b'_K$. Choose arbitrarily $\delta>0$ and consider a family $\{E_\varepsilon\}$ such that $E_\varepsilon\to H$ in $L^1(U)$ and $E_\varepsilon\cap U^\delta=H\cap U^\delta$. We may extend each E_ε outside the unit cube putting $E_\varepsilon\cap U^{\mathrm{c}}=H\cap U^{\mathrm{c}}$. Then, we invoke the third statement of Corollary 2.11 to infer $J_\varepsilon(H,U)\le J_\varepsilon(E_\varepsilon,U)$, which gets

$$\frac{1}{2\varepsilon}J^1_\varepsilon(H,U)\le\frac{1}{2\varepsilon}J^1_\varepsilon(E_\varepsilon,U)+\frac{1}{\varepsilon}\left(J^2_\varepsilon(E_\varepsilon,U)-J^2_\varepsilon(H,U)\right).$$

The desired inequality follows if we show that the second summand in the right-hand side vanishes as ε tends to 0. To this aim, we exploit our information about $\{E_\varepsilon\}$, which provides the equalities

$$\begin{aligned}&J^2_\varepsilon(E_\varepsilon,U)-J^2_\varepsilon(H,U)\\ &\qquad= L_\varepsilon(E_\varepsilon\cap U,H^{\mathrm{c}}\cap U^{\mathrm{c}})+L_\varepsilon(H\cap U^{\mathrm{c}},E_\varepsilon^{\mathrm{c}}\cap U)\\ &\qquad\quad- L_\varepsilon(H\cap U,H^{\mathrm{c}}\cap U^{\mathrm{c}})-L_\varepsilon(H\cap U^{\mathrm{c}},H^{\mathrm{c}}\cap U)\\ &\qquad= L_\varepsilon(E_\varepsilon\cap(U\setminus U^\delta),H^{\mathrm{c}}\cap U^{\mathrm{c}})+L_\varepsilon(H\cap U^{\mathrm{c}},E_\varepsilon^{\mathrm{c}}\cap(U\setminus U^\delta))\\ &\qquad\quad- L_\varepsilon(H\cap(U\setminus U^\delta),H^{\mathrm{c}}\cap U^{\mathrm{c}})-L_\varepsilon(H\cap U^{\mathrm{c}},H^{\mathrm{c}}\cap(U\setminus U^\delta));\end{aligned}$$

and from this, in view of Proposition 3.2, the claim is proved.

Now, we show that $b'_K\le b_K$. We let $\{E_\varepsilon\}$ be such that $E_\varepsilon\to H$ in $L^1(U)$ as ε approaches 0 and, without loss of generality, that $J^1_\varepsilon(E_\varepsilon,U)$ is finite. For any ε, we apply Proposition 3.9 to E_ε and H and this yields a family $\{F_\varepsilon\}$ with the properties that it L^1-converges to H in U, that $F_\varepsilon\cap U^\delta=H\cap U^\delta$ and that for any $\eta>0$

$$\begin{aligned}\frac{1}{2\varepsilon}J^1_\varepsilon(F_\varepsilon,U)\le&\frac{1}{2\varepsilon}J^1_\varepsilon(E_\varepsilon,U)+\frac{1}{2\varepsilon}J^1_\varepsilon(H,U^{\delta+\eta})+\frac{2c'_K}{\eta}\,|\,(E_\varepsilon\triangle H)\cap U\,|\\ &+\frac{1}{\eta}\int_{U\setminus U^{\delta_1+\eta}}\int_{\mathbb{R}^d}K(h)\,|\,h\,|\,\chi_{U^{\delta_1}}(x+\varepsilon h)\mathrm{d}h\mathrm{d}x.\end{aligned}$$

We notice that in view of (3.8) it holds

$$\lim_{\varepsilon\to0}\frac{1}{2\varepsilon}J^1_\varepsilon(H,U^{\delta+\eta})=c_K\operatorname{Per}(H,U^{\delta+\eta})$$

and consequently, taking the limit as $\varepsilon\to0$, we get

$$\begin{aligned}b'_K\le&\liminf_{\varepsilon\to0}\frac{1}{2\varepsilon}J^1_\varepsilon(F_\varepsilon,U)\\ \le&\liminf_{\varepsilon\to0}\frac{1}{2\varepsilon}J^1_\varepsilon(E_\varepsilon,U)+c_K\operatorname{Per}(H,U^{\delta+\eta});\end{aligned}$$

next, we let η and δ vanish and, eventually, we take advantage of the arbitrariness of the family $\{E_\varepsilon\}$ to conclude. $\square$

3.10 - Remark (Outlook). *The Γ-convergence result of Theorem 3.1 might be extended to different classes of kernels. The first step would be dropping the assumption of strict monotonicity; this entails the use of a different strategy for the proof of the Γ-inferior limit inequality, but the same conclusions would hold. Similarly, we expect an analogous result without the radial symmetry hypothesis; nevertheless, if K is anisotropic, De Giorgi's perimeter has to be replaced with the functional*

$$\int_{\partial^* E} \sigma_K(\hat{n}(z)) \mathrm{d}\mathscr{H}^{d-1},$$

where the function $\sigma_K \colon \partial B(0,1) \to \mathbb{R}$ is a weight that depends on K and that corresponds to the constant c_K appearing in this paper, see the analysis carried out in [1] for "localised" functionals.

The cases when K changes sign or when it is substituted by a Radon measure μ are also possible subjects of further study, but the conclusions of Theorem 3.1 might be affected..

Finally, it would be interesting to obtain a Γ-convergence result for multi-phase systems, i.e. for functionals of the form

$$J_\varepsilon(E_1, \dots, E_N) \coloneqq \sum_{0 \le i < j \le N} a_{i,j} L_\varepsilon(E_i, E_j),$$

where $E_1, \dots, E_N$ are measurable sets such that $|\, E_i \,| > 0$ and $|\, E_i \cap E_i \,| = 0$ for any $i, j = 1, \dots, N$, $E_0 \coloneqq (\bigcup_{i=1}^N E_i)^{\mathrm{c}}$ and the coefficients $a_{i,j}$ are positive and they satisfy $a_{i,j} \le a_{i,k} + a_{k,j}$ for every $i, j, k = 0, \dots, N$. As a particular instance, if for any $i = 0, \dots, N$ there exists $a_i \ge 0$ such that $a_i = a_{i,j}$ for all $j \ne i$, we get

$$J_\varepsilon(E_1, \dots, E_N) = \frac{1}{2} \sum_{i=0}^N a_i J_\varepsilon(E_i, \mathbb{R}^d);$$

in this case, it easy to recover a Γ-convergence result from the theory we developed in this paper.

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