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978-1-108-47635-5 — A Comprehensive Introduction to Sub-Riemannian Geometry

Andrei Agrachev , Davide Barilari , Ugo Boscain

Table of Contents

[More Information](#)

Contents

	<i>Preface</i>	<i>page xvii</i>
Introduction		1
1 Geometry of Surfaces in \mathbb{R}^3		11
1.1 Geodesics and Optimality		11
1.1.1 Existence and Minimizing Properties of Geodesics		16
1.1.2 Absolutely Continuous Curves		18
1.2 Parallel Transport		19
1.2.1 Parallel Transport and the Levi-Civita Connection		20
1.3 Gauss–Bonnet Theorems		23
1.3.1 Gauss–Bonnet Theorem: Local Version		23
1.3.2 Gauss–Bonnet Theorem: Global Version		27
1.3.3 Consequences of the Gauss–Bonnet Theorems		31
1.3.4 The Gauss Map		33
1.4 Surfaces in \mathbb{R}^3 with the Minkowski Inner Product		36
1.5 Model Spaces of Constant Curvature		40
1.5.1 Zero Curvature: The Euclidean Plane		40
1.5.2 Positive Curvature: The Sphere		41
1.5.3 Negative Curvature: The Hyperbolic Plane		43
1.6 Bibliographical Note		44
2 Vector Fields		45
2.1 Differential Equations on Smooth Manifolds		45
2.1.1 Tangent Vectors and Vector Fields		45
2.1.2 Flow of a Vector Field		47
2.1.3 Vector Fields as Operators on Functions		48
2.1.4 Nonautonomous Vector Fields		49
2.2 Differential of a Smooth Map		51
2.3 Lie Brackets		53

Cambridge University Press

978-1-108-47635-5 — A Comprehensive Introduction to Sub-Riemannian Geometry

Andrei Agrachev , Davide Barilari , Ugo Boscain

Table of Contents

[More Information](#)

2.4	Frobenius' Theorem	57
2.4.1	An Application of Frobenius' Theorem	59
2.5	Cotangent Space	60
2.6	Vector Bundles	62
2.7	Submersions and Level Sets of Smooth Maps	64
2.8	Bibliographical Note	66
3	Sub-Riemannian Structures	67
3.1	Basic Definitions	67
3.1.1	The Minimal Control and the Length of an Admissible Curve	71
3.1.2	Equivalence of Sub-Riemannian Structures	74
3.1.3	Examples	76
3.1.4	Every Sub-Riemannian Structure is Equivalent to a Free Sub-Riemannian Structure	77
3.2	Sub-Riemannian Distance and Rashevskii–Chow Theorem	80
3.2.1	Proof of the Rashevskii–Chow Theorem	81
3.2.2	Non-Bracket-Generating Structures	86
3.3	Existence of Length-Minimizers	87
3.3.1	On the Completeness of the Sub-Riemannian Distance	89
3.3.2	Lipschitz Curves with respect to d vs. Admissible Curves	91
3.3.3	Lipschitz Equivalence of Sub-Riemannian Distances	93
3.3.4	Continuity of d with respect to the Sub-Riemannian Structure	94
3.4	Pontryagin Extremals	97
3.4.1	The Energy Functional	99
3.4.2	Proof of Theorem 3.59	100
3.5	Appendix: Measurability of the Minimal Control	104
3.5.1	A Measurability Lemma	104
3.5.2	Proof of Lemma 3.12	106
3.6	Appendix: Lipschitz vs. Absolutely Continuous Admissible Curves	106
3.7	Bibliographical Note	108
4	Pontryagin Extremals: Characterization and Local Minimality	109
4.1	Geometric Characterization of Pontryagin Extremals	109
4.1.1	Lifting a Vector Field from M to T^*M	110
4.1.2	The Poisson Bracket	111
4.1.3	Hamiltonian Vector Fields	114

Cambridge University Press

978-1-108-47635-5 — A Comprehensive Introduction to Sub-Riemannian Geometry

Andrei Agrachev , Davide Barilari , Ugo Boscain

Table of Contents

[More Information](#)*Contents*

ix

4.2	The Symplectic Structure	116
4.2.1	Symplectic Form vs. Poisson Bracket	117
4.3	Characterization of Normal and Abnormal Pontryagin Extremals	119
4.3.1	Normal Extremals	120
4.3.2	Abnormal Extremals	124
4.3.3	Codimension-1 and Contact Distributions	126
4.4	Examples	127
4.4.1	2D Riemannian Geometry	128
4.4.2	Isoperimetric Problem	130
4.4.3	Heisenberg Group	134
4.5	Lie Derivative	136
4.6	Symplectic Manifolds	138
4.7	Local Minimality of Normal Extremal Trajectories	140
4.7.1	The Poincaré–Cartan 1-Form	140
4.7.2	Normal Pontryagin Extremal Trajectories are Geodesics	142
4.8	Bibliographical Note	148
5	First Integrals and Integrable Systems	149
5.1	Reduction of Hamiltonian Systems with Symmetries	149
5.1.1	An Example of Symplectic Reduction: the Space of Affine Lines in \mathbb{R}^n	152
5.2	Riemannian Geodesic Flow on Hypersurfaces	153
5.2.1	Geodesics on Hypersurfaces	153
5.2.2	Riemannian Geodesic Flow and Symplectic Reduction	154
5.3	Sub-Riemannian Structures with Symmetries	157
5.4	Completely Integrable Systems	159
5.5	Arnold–Liouville Theorem	163
5.6	Geodesic Flows on Quadrics	166
5.7	Bibliographical Note	170
6	Chronological Calculus	171
6.1	Motivation	171
6.2	Duality	172
6.2.1	On the Notation	174
6.3	Topology on the Set of Smooth Functions	174
6.3.1	Family of Functionals and Operators	175
6.4	Operator ODEs and Volterra Expansion	176
6.4.1	Volterra Expansion	177
6.4.2	Adjoint Representation	180
6.5	Variation Formulas	182

Cambridge University Press

978-1-108-47635-5 — A Comprehensive Introduction to Sub-Riemannian Geometry

Andrei Agrachev , Davide Barilari , Ugo Boscain

Table of Contents

[More Information](#)

x

Contents

6.6	Appendix: Estimates and Volterra Expansion	183
6.7	Appendix: Remainder Term of the Volterra Expansion	187
6.8	Bibliographical Note	190
7	Lie Groups and Left-Invariant Sub-Riemannian Structures	191
7.1	Subgroups of $\text{Diff}(M)$ Generated by a Finite-Dimensional Lie Algebra of Vector Fields	191
7.1.1	A Finite-Dimensional Approximation	193
7.1.2	Passage to Infinite Dimension	196
7.1.3	Proof of Proposition 7.2	197
7.2	Lie Groups and Lie Algebras	198
7.2.1	Lie Groups as Groups of Diffeomorphisms	200
7.2.2	Matrix Lie Groups and Matrix Notation	202
7.2.3	Bi-Invariant Pseudo-Metrics and Haar Measures	205
7.2.4	The Levi–Malcev Decomposition	207
7.3	Trivialization of TG and T^*G	208
7.4	Left-Invariant Sub-Riemannian Structures	209
7.5	Example: Carnot Groups of Step 2	210
7.5.1	Normal Pontryagin Extremals for Carnot Groups of Step 2	213
7.6	Left-Invariant Hamiltonian Systems on Lie Groups	216
7.6.1	Vertical Coordinates in TG and T^*G	216
7.6.2	Left-Invariant Hamiltonians	218
7.7	Normal Extremals for Left-Invariant Sub-Riemannian Structures	221
7.7.1	Explicit Expression for Normal Pontryagin Extremals in the $\mathbf{d} \oplus \mathbf{s}$ Case	221
7.7.2	Example: The $\mathbf{d} \oplus \mathbf{s}$ Problem on $SO(3)$	223
7.7.3	Further Comments on the $\mathbf{d} \oplus \mathbf{s}$ Problem: $SO(3)$ and $SO_+(2, 1)$	225
7.7.4	Explicit Expression for Normal Pontryagin Extremals in the $\mathbf{k} \oplus \mathbf{z}$ Case	228
7.8	Rolling Spheres	232
7.8.1	Rolling with Spinning	232
7.8.2	Rolling without Spinning	235
7.8.3	Euler’s “Curvae Elasticae”	240
7.8.4	Rolling Spheres: Further Comments	243
7.9	Bibliographical Note	244
8	Endpoint Map and Exponential Map	246
8.1	The Endpoint Map	247
8.1.1	Regularity of the Endpoint Map: Proof of Proposition 8.5	248

Contents

xi

8.2	Lagrange Multiplier Rule	251
8.3	Pontryagin Extremals via Lagrange Multipliers	251
8.4	Critical Points and Second-Order Conditions	253
8.4.1	The Manifold of Lagrange Multipliers	256
8.5	Sub-Riemannian Case	261
8.6	Exponential Map and Gauss' lemma	266
8.7	Conjugate Points	270
8.8	Minimizing Properties of Extremal Trajectories	274
8.8.1	Local Length-Minimality in the $W^{1,2}$ Topology. Proof of Theorem 8.52	275
8.8.2	Local Length-Minimality in the C^0 Topology	278
8.9	Compactness of Length-Minimizers	283
8.10	Cut Locus and Global Length-Minimizers	285
8.11	An Example: First Conjugate Locus on a Perturbed Sphere	290
8.12	Bibliographical Note	293
9	2D Almost-Riemannian Structures	295
9.1	Basic Definitions and Properties	295
9.1.1	How Large is the Singular Set?	301
9.1.2	Genuinely 2D Almost-Riemannian Structures Always Have Infinite Area	303
9.1.3	Pontryagin Extremals	304
9.2	The Grushin Plane	306
9.2.1	Geodesics on the Grushin Plane	307
9.3	Riemannian, Grushin and Martinet Points	310
9.3.1	Normal Forms	313
9.4	Generic 2D Almost-Riemannian Structures	315
9.4.1	Proof of the Genericity Result	316
9.5	A Gauss–Bonnet Theorem	318
9.5.1	Integration of the Curvature	318
9.5.2	The Euler Number	319
9.5.3	Gauss–Bonnet Theorem	320
9.5.4	Every Compact Orientable 2D Manifold can be Endowed with a Free Almost-Riemannian Structure with only Riemannian and Grushin Points	328
9.6	Bibliographical Note	329
10	Nonholonomic Tangent Space	331
10.1	Flag of the Distribution and Carnot Groups	331
10.2	Jet Spaces	333
10.2.1	Jets of Curves	333
10.2.2	Jets of Vector Fields	336
10.3	Admissible Variations and Nonholonomic Tangent Space	338

Cambridge University Press

978-1-108-47635-5 — A Comprehensive Introduction to Sub-Riemannian Geometry

Andrei Agrachev , Davide Barilari , Ugo Boscain

Table of Contents

[More Information](#)

10.3.1	Admissible Variations	338
10.3.2	Nonholonomic Tangent Space	340
10.4	Nonholonomic Tangent Space and Privileged Coordinates	343
10.4.1	Privileged Coordinates	343
10.4.2	Description of the Nonholonomic Tangent Space in Privileged Coordinates	346
10.4.3	Existence of Privileged Coordinates: Proof of Theorem 10.32	354
10.4.4	Nonholonomic Tangent Spaces in Low Dimension	358
10.5	Metric Meaning	361
10.5.1	Convergence of the Sub-Riemannian Distance and the Ball–Box Theorem	362
10.6	Algebraic Meaning	367
10.6.1	Nonholonomic Tangent Space: The Equiregular Case	369
10.7	Carnot Groups: Normal Forms in Low Dimension	371
10.8	Bibliographical Note	375
11	Regularity of the Sub-Riemannian Distance	376
11.1	Regularity of the Sub-Riemannian Squared Distance	376
11.2	Locally Lipschitz Functions and Maps	385
11.2.1	Locally Lipschitz Map and Lipschitz Submanifolds	390
11.2.2	A Non-Smooth Version of the Sard Lemma	393
11.3	Regularity of Sub-Riemannian Spheres	396
11.4	Geodesic Completeness and the Hopf–Rinow Theorem	399
11.5	Bibliographical Note	400
12	Abnormal Extremals and Second Variation	402
12.1	Second Variation	402
12.2	Abnormal Extremals and Regularity of the Distance	404
12.3	Goh and Generalized Legendre Conditions	410
12.3.1	Proof of Goh Condition – Part (i) of Theorem 12.12	413
12.3.2	Proof of the Generalized Legendre Condition – Part (ii) of Theorem 12.12	420
12.3.3	More on the Goh and Generalized Legendre Conditions	422
12.4	Rank-2 Distributions and Nice Abnormal Extremals	424
12.5	Minimality of Nice Abnormal Extremals in Rank-2 Structures	427
12.5.1	Proof of Theorem 12.33	429
12.6	Conjugate Points along Abnormals	436
12.6.1	Abnormals in Dimension 3	439
12.6.2	Higher Dimensions	444

Cambridge University Press

978-1-108-47635-5 — A Comprehensive Introduction to Sub-Riemannian Geometry

Andrei Agrachev , Davide Barilari , Ugo Boscain

Table of Contents

[More Information](#)*Contents*

xiii

12.7 Equivalence of Local Minimality with Respect to the $W^{1,2}$ and C^0 Topologies	445
12.8 Non-Minimality of Corners	449
12.9 Bibliographical Note	454
13 Some Model Spaces	456
13.1 Carnot Groups of Step 2	457
13.2 Multidimensional Heisenberg Groups	460
13.2.1 Pontryagin Extremals in the Contact Case	461
13.2.2 Optimal Synthesis	463
13.3 Free Carnot Groups of Step 2	466
13.3.1 Intersection of the Cut Locus with the Vertical Subspace	470
13.3.2 The Cut Locus for the Free Step-2 Carnot Group of Rank 3	471
13.4 An Extended Hadamard Technique to Compute the Cut Locus	472
13.5 The Grushin Structure	478
13.5.1 Optimal Synthesis Starting from a Riemannian Point	479
13.5.2 Optimal Synthesis Starting from a Singular Point	482
13.6 The Standard Sub-Riemannian Structure on $SU(2)$	486
13.7 Optimal Synthesis on the Groups $SO(3)$ and $SO_+(2, 1)$	490
13.8 Synthesis for the Group of Euclidean Transformations of the Plane $SE(2)$	494
13.8.1 Mechanical Interpretation	495
13.8.2 Geodesics	496
13.9 The Martinet Flat Sub-Riemannian Structure	502
13.9.1 Abnormal Extremals	503
13.9.2 Normal Extremals	504
13.10 Bibliographical Note	509
14 Curves in the Lagrange Grassmannian	513
14.1 The Geometry of the Lagrange Grassmannian	513
14.1.1 The Lagrange Grassmannian	517
14.2 Regular Curves in the Lagrange Grassmannian	519
14.3 Curvature of a Regular Curve	523
14.4 Reduction of Non-Regular Curves in Lagrange Grassmannian	527
14.5 Ample Curves	529
14.6 From Ample to Regular	530
14.7 Conjugate Points in $L(\Sigma)$	536
14.8 Comparison Theorems for Regular Curves	538
14.9 Bibliographical Note	540

Cambridge University Press

978-1-108-47635-5 — A Comprehensive Introduction to Sub-Riemannian Geometry

Andrei Agrachev , Davide Barilari , Ugo Boscain

Table of Contents

[More Information](#)

xiv

Contents

15 Jacobi Curves	542
15.1 From Jacobi Fields to Jacobi Curves	542
15.1.1 Jacobi Curves	543
15.2 Conjugate Points and Optimality	545
15.3 Reduction of the Jacobi Curves by Homogeneity	547
15.4 Bibliographical Note	550
16 Riemannian Curvature	551
16.1 Ehresmann Connection	551
16.1.1 Curvature of an Ehresmann Connection	552
16.1.2 Linear Ehresmann Connections	554
16.1.3 Covariant Derivative and Torsion for Linear Connections	555
16.2 Riemannian Connection	557
16.3 Relation to Hamiltonian Curvature	563
16.4 Comparison Theorems for Conjugate Points	565
16.5 Locally Flat Spaces	567
16.6 Curvature of 2D Riemannian Manifolds	568
16.7 Bibliographical Note	570
17 Curvature in 3D Contact Sub-Riemannian Geometry	571
17.1 A Worked-Out Example: The 2D Riemannian Case	571
17.2 3D Contact Sub-Riemannian Manifolds	576
17.3 Canonical Frames	579
17.4 Curvature of a 3D Contact Structure	582
17.4.1 Geometric Interpretation	588
17.5 Local Classification of 3D Left-Invariant Structures	589
17.5.1 A Description of the Classification	591
17.5.2 A Sub-Riemannian Isometry Between Non-Isomorphic Lie groups	594
17.5.3 Canonical Frames and Classification. Proof of Theorem 17.29	596
17.5.4 An Explicit Isometry. Proof of Theorem 17.32	599
17.6 Appendix: Remarks on Curvature Coefficients	604
17.7 Bibliographical Note	605
18 Integrability of the Sub-Riemannian Geodesic Flow on 3D Lie Groups	607
18.1 Poisson Manifolds and Symplectic Leaves	607
18.1.1 Poisson Manifolds	607
18.1.2 The Poisson Bi-Vector	608
18.1.3 Symplectic Manifolds	609

Contents

xv

18.1.4 Casimir Functions	609
18.1.5 Symplectic Leaves	611
18.2 Integrability of Hamiltonian Systems on Lie Groups	612
18.2.1 The Poisson Manifold \mathfrak{g}^*	612
18.2.2 The Casimir First Integral	614
18.2.3 First Integrals Associated with a Right-Invariant Vector Field	615
18.2.4 Complete Integrability on Lie Groups	615
18.3 Left-Invariant Hamiltonian Systems on 3D Lie Groups	616
18.3.1 Rank-2 Sub-Riemannian Structures on 3D Lie Groups	621
18.3.2 Classification of Symplectic Leaves on 3D Lie Groups	623
18.4 Bibliographical Note	632
19 Asymptotic Expansion of the 3D Contact Exponential Map	633
19.1 The Exponential Map	633
19.1.1 The Nilpotent Case	634
19.2 General Case: Second-Order Asymptotic Expansion	636
19.2.1 Proof of Proposition 19.2: Second-Order Asymptotics	637
19.3 General Case: Higher-Order Asymptotic Expansion	641
19.3.1 Proof of Theorem 19.6: Asymptotics of the Exponential Map	643
19.3.2 Asymptotics of the Conjugate Locus	648
19.3.3 Asymptotics of the Conjugate Length	650
19.3.4 Stability of the Conjugate Locus	651
19.4 Bibliographical Note	653
20 Volumes in Sub-Riemannian Geometry	654
20.1 Equiregular Sub-Riemannian Manifolds	654
20.2 The Popp Volume	655
20.3 A Formula for the Popp Volume in Terms of Adapted Frames	657
20.4 The Popp Volume and Smooth Isometries	661
20.5 Hausdorff Dimension and Hausdorff Volume	664
20.6 Hausdorff Volume on Sub-Riemannian Manifolds	664
20.6.1 Hausdorff Dimension	665
20.6.2 On the Metric Convergence	668
20.6.3 Induced Volumes and Estimates	669
20.7 Density of the Spherical Hausdorff Volume with respect to a Smooth Volume	671
20.8 Bibliographical Note	672

21 The Sub-Riemannian Heat Equation	674
21.1 The Heat Equation	674
21.1.1 The Heat Equation in the Riemannian Context	674
21.1.2 The Heat Equation in the Sub-Riemannian Context	678
21.1.3 The Hörmander Theorem and the Existence of the Heat Kernel	681
21.1.4 The Heat Equation in the Non-Bracket-Generating Case	683
21.2 The Heat Kernel on the Heisenberg Group	684
21.2.1 The Heisenberg Group as a Group of Matrices	684
21.2.2 The Heat Equation on the Heisenberg Group	686
21.2.3 Construction of the Gaveau–Hulanicki Fundamental Solution	687
21.2.4 Small-Time Asymptotics for the Gaveau–Hulanicki Fundamental Solution	695
21.3 Bibliographical Note	696
Appendix Geometry of Parametrized Curves in Lagrangian Grassmannians Igor Zelenko	698
A.1 Preliminaries	698
A.2 Algebraic Theory of Curves in Grassmannians and Flag Varieties	704
A.3 Application to the Differential Geometry of Monotonic Parametrized Curves in Lagrangian Grassmannians	715
<i>References</i>	725
<i>Index</i>	740