New characterizations of Ricci curvature on RCD metric measure spaces

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Abstract

We prove that on a large family of metric measure spaces, if the 'carré du champ' Γ satisfies a L^p-gradient estimate of heat flows for some $p > 2$, then the L^1-gradient estimate holds. This result extends Savaré's theorem on metric measure space in [15], and also provides a new proof to von Renesse-Sturm's theorem on smooth metric measure space in [14]. As a consequence, we also show that Gigli's Ricci tensor ([10]) could characterize the Ricci curvature of RCD space in a local way.

The argument is based on iteration use of non-smooth Bakry-Émery's theory, which is a new method to study metric measure spaces which lack of local regularity.

Keywords: Bakry-Émery theory, gradient estimate, heat flow, metric measure space, Ricci curvature

Contents

1 Introduction

For any smooth Riemannian manifold M and any $K \in \mathbb{R}$, it is proved by von Renesse and Sturm in [14] that the following properties are equivalent

1) $\mathrm{Ricci}_M \geq K$

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2) there exists $p \in (1, \infty)$ such that for all $f \in C_c^\infty(M)$, all $x \in M$ and $t \geq 0$

$$|\mathrm{DH}_t f|^p(x) \leq e^{-pKt} \mathrm{H}_t |\mathrm{D}f|^p(x). \tag{1.1}$$

3) for all $f \in C_c^\infty(M)$, all $x \in M$ and $t \geq 0$

$$|\mathrm{DH}_t f|(x) \leq e^{-Kt} \mathrm{H}_t |\mathrm{D}f|(x), \tag{1.2}$$

where $\mathrm{H}_t f$ is the solution to the heat equation with initial datum f.

In non-smooth setting, the notion of synthetic Ricci curvature bounds, or non-smooth curvature-dimension conditions, were proposed by Lott-Sturm-Villani (see [13] and [16]) using optimal transport theory. Later on, by assuming the infinitesimally Hilbertianity (i.e. the Sobolev space $W^{1,2}$ is a Hilbert space), RCD condition (or $\mathrm{RCD}(K, \infty)$ condtion if we want to specify the curvature) which is a refinement of curvature-dimension condition, was proposed by Ambrosio-Gigli-Savaré (see [4] and [1]). It is known that $\mathrm{RCD}(K, \infty)$ spaces are generalizations of Riemannian manifolds with lower Ricci curvature bound and their limit spaces, as well as Alexandrov spaces with lower curvature bound.

Then we would like to know the relationship between Lott-Sturm-Villani's synthetic Ricci bound and Bakry-Émery's gradient estimate in the non-smooth setting. Let $(X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}(K, \infty)$ space, it is proved (in [4]) that

$$|\mathrm{DH}_t f|^2 \leq e^{-2Kt} \mathrm{H}_t |\mathrm{D}f|^2, \quad \mathfrak{m} - \text{a.e.} \tag{1.3}$$

for any $f \in W^{1,2}$ and $t > 0$, where $\mathrm{H}_t f$ is the heat flow from f and $|\mathrm{D}f|$ is the minimal weak upper gradient (or weak gradient for simplicity) of f. In particular, by Hölder inequality we know

$$|\mathrm{DH}_t f|^p \leq e^{-pKt} \mathrm{H}_t |\mathrm{D}f|^p, \quad \mathfrak{m} - \text{a.e.} \tag{1.4}$$

for any $p \geq 2$. Furthermore, it is proved in [15] that the inequality (1.3) can be improved as:

$$|\mathrm{DH}_t f| \leq e^{-Kt} \mathrm{H}_t |\mathrm{D}f|, \quad \mathfrak{m} - \text{a.e.}. \tag{1.5}$$

In other words, inequality (1.4) holds for any $p \in [1, \infty]$.

Conversely, it is shown in [5] that the inequality (1.3) is sufficient to characterize $\mathrm{RCD}(K, \infty)$ condition in the following way. Let $(X, \mathrm{d}, \mathfrak{m})$ be an infinitesimally Hilbertian space, we have a well-defined Dirichlet energy:

$$E(f) := \int |\mathrm{D}f|^2 \, \mathrm{d}\mathfrak{m}$$

for any $f \in W^{1,2}(X, \mathrm{d}, \mathfrak{m})$. We denote the L^2-gradient flow of $E(\cdot)$ starting from f by $(\mathrm{H}_t f)_t$. Assume further that the space $(X, \mathrm{d}, \mathfrak{m})$ has Sobolev-to-Lipschitz property, i.e. for any function $f \in W^{1,2}$ such that $|\mathrm{D}f| \in L^\infty$, we can find a Lipschitz continuous function $\bar{f}$ such that $f = \bar{f}$ $\mathfrak{m}$-a.e. and $\mathrm{Lip}(\bar{f}) = \operatorname{ess\,sup} |\mathrm{D}f|$. If

$$|\mathrm{DH}_t f|^2 \leq e^{-2Kt} \mathrm{H}_t |\mathrm{D}f|^2, \quad \mathfrak{m} - \text{a.e.} \tag{1.6}$$

for any $f \in W^{1,2}$, and $t > 0$. Then $(X, \mathrm{d}, \mathfrak{m})$ is a $\mathrm{RCD}(K, \infty)$ space.

The main goal of this paper is to prove that inequality (1.3) could also characterize the curvature-dimension condition of metric measure spaces. Equivalently, we prove a non-smooth version of 2) $\Rightarrow$ 3) in von Renesse-Sturm's theorem, thus we complete the circle 1) $\Leftrightarrow$ 2) $\Leftrightarrow$ 3) in non-smooth setting.

Now, we introduce our main result in this paper. Under Assumption 3.5 (i.e. the existence of a dense subspace $\mathcal{A}$ of TestF such that $\Gamma(f) \in \mathbb{M}_\infty$ for any $f \in \mathcal{A}$) we prove:

Theorem 1.1 (Theorem 3.6, Improved Bakry-Émery theory). *Let $M := (X, \mathrm{d}, \mathfrak{m})$ be a metric measure space fulfills Assumption 3.5. If for any $f \in W^{1,2}(X) \cap \mathrm{Lip}(X) \cap L^\infty(X)$ we have the gradient estimate*

$$|\mathrm{DH}_t f|^p \le e^{-pKt}\mathrm{H}_t|\mathrm{D}f|^p, \quad \mathfrak{m} - a.e. \tag{1.7}$$

for some $p \in (1, \infty)$. Then (1.7) holds for $p = 1$. In particular, M is a $\mathrm{RCD}(K, \infty)$ space.

Since we do not have second order differentiation formula for relative entropy along Wasserstein geodesics, or Taylor's expansion in non-smooth setting. We can not simply use the argument in smooth metric measure space (see e.g. the proofs in [14]). The argument we adopt here is the so-called 'self-improved' method in Bakry-Émery's Γ-calculus, which was used in [15] to deal with the non-smooth problems. We remark that we not only use 'self-improved' technique, but an improved iteration use of 'self-improved' argument. We believe that this method also has potential application in the future.

It can be seen that the Assumption 3.5 hold in the following cases, where our main result apply.

Example 1. Smooth metric measure space: obviously, $C_c^\infty(M)$, the space of smooth functions with bounded support is a good algebra in Assumption 3.5. Hence we obtain a new quick proof of von Renesse-Sturm's theorem, without using Taylor's expansion method.

Example 2. $\mathrm{RCD}(K, \infty)$ metric measure space: it is proved in Lemma 3.2, [15] that $f \in \mathbb{M}_\infty$ for any $f \in$ TestF. By Theorem 1.1 we obtain the following proposition which extends Savaré's result in [15].

Proposition 1.2 (Self-improvement of gradient estimate). *Let $(X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}(K, \infty)$ metric measure space. If for any $f \in W^{1,2} \cap \mathrm{Lip}(X) \cap L^\infty(X)$ we have the gradient estimate*

$$|\mathrm{DH}_t f|^p \le e^{-pK't}\mathrm{H}_t|\mathrm{D}f|^p, \quad \mathfrak{m} - a.e. \tag{1.8}$$

for some $p \in [1, \infty)$ and $K' > K$. Then $(X, \mathrm{d}, \mathfrak{m})$ is a $\mathrm{RCD}(K', \infty)$ space. In particular, we know

$$|\mathrm{DH}_t f| \le e^{-K't}\mathrm{H}_t|\mathrm{D}f|, \quad \mathfrak{m} - a.e.. \tag{1.9}$$

In [10], Gigli defines measure valued Ricci tensor on RCD metric measure space (see also [12]) as

$$\mathbf{Ricci}(\nabla f, \nabla f) := \mathbf{\Gamma}_2(f) - |\mathrm{H}_f|^2_{\mathrm{HS}}\,\mathfrak{m}$$

where $\mathbf{\Gamma}_2(f) := \frac{1}{2}\mathbf{\Delta}|\mathrm{D}f|^2 - \langle \nabla f, \nabla \Delta f\rangle\,\mathfrak{m}$. He shows that $\mathbf{Ricci}(\nabla f, \nabla f) \geq K|\mathrm{D}f|^2\,\mathfrak{m}$ if and only if the space is $\mathrm{RCD}(K, \infty)$. However, we do not know if **Ricci** has locality in the sense that $\mathbf{Ricci}(\nabla f, \nabla f)|_{\{|\mathrm{D}f|=0\}} = 0$.

From the proof of Theorem 1.1 we have the following new characterization of curvature bound which extends Gigli's result:

Proposition 1.3 (Proposition 3.7)**.** *Let* $(X, \mathrm{d}, \mathfrak{m})$ *be a* RCD *space. Then the following characterizations are equivalent.*

1) $(X, \mathrm{d}, \mathfrak{m})$ *is* $\mathrm{RCD}(K, \infty)$*,*

2) $\mathbf{Ricci}(\nabla f, \nabla f) \geq K|\mathrm{D}f|^2\,\mathfrak{m}$ *for any test function* f*,*

3) $|\mathrm{D}f|^2\mathbf{Ricci}(\nabla f, \nabla f) \geq K|\mathrm{D}f|^4\,\mathfrak{m}$ *for any test function* f*.*

We remark that this naive extension is non-trivial, because 2) is not a direct consequence of 3) due to lack of the locality of **Ricci**.

2 Preliminaries

The basic object we will study in this article is metric measure space $(X, \mathrm{d}, \mathfrak{m})$. First of all, we need the following basic assumptions on $(X, \mathrm{d}, \mathfrak{m})$, the notions and concepts in this assumption will be explained later.

Assumption 2.1. We assume that:

(1) (X, d) is a complete, separable geodesic space,

(2) $\mathrm{supp}\,\mathfrak{m} = X, \quad \mathfrak{m}(B_r(x)) < c_1 \exp(c_2 r^2) \quad \text{for every} \quad r > 0,$

(3) $W^{1,2}(X)$ is a Hilbert space,

(4) $(X, \mathrm{d}, \mathfrak{m})$ has Sobolev-to-Lipschitz property,

(5) existence of the heat kernel $p_t(x, y)$.

Remark 2.2. It is known that both smooth metric measure spaces and RCD spaces satisfy the properties above.

Remark 2.3. We can also use the language of Dirichlet form to study our problems, by assuming that the Dirichlet form is a so-called 'Riemannian energy measure space'. It is known from [5] that this way is compatible with the current approach using Sobolev space on metric measure space.

The Sobolev space $W^{1,2}(X,\mathrm{d},\mathfrak{m})$ is defined as in [2]. We say that $f \in L^2(X,\mathfrak{m})$ is a Sobolev function in $W^{1,2}(M)$ if there exists a sequence of Lipschitz functions functions $(f_n) \subset L^2$, such that $f_n \to f$ and $\mathrm{lip}(f_n) \to G$ in L^2 for some $G \in L^2(X,\mathfrak{m})$, where $\mathrm{lip}(f_n)$ is the local Lipschitz constant of f_n. It is known that there exists a minimal function G in $\mathfrak{m}$-a.e. sense. We call the minimal G the minimal weak upper gradient (or weak gradient for simplicity) of the function f, and denote it by $|\mathrm{D}f|$. It is known that the locality holds for $|\mathrm{D}f|$, i.e. $|\mathrm{D}f| = |\mathrm{D}g|$ a.e. on the set $\{f = g\}$. Furthermore, we have the lower semi-continuity: if $\{f_n\}_n \subset W^{1,2}(X,\mathrm{d},\mathfrak{m})$ is a sequence converging to some f in $\mathfrak{m}$-a.e. sense and such that $(|\mathrm{D}f_n|)_n$ is bounded in $L^2(X,\mathfrak{m})$, then $f \in W^{1,2}(X,\mathrm{d},\mathfrak{m})$ and

$$\||\mathrm{D}f|\|_{L^2} \le \varliminf_{n\to\infty} \||\mathrm{D}f_n|\|_{L^2}.$$

We equip $W^{1,2}(X,\mathrm{d},\mathfrak{m})$ with the norm

$$\|f\|^2_{W^{1,2}(X,\mathrm{d},\mathfrak{m})} := \|f\|^2_{L^2(X,\mathfrak{m})} + \||\mathrm{D}f|\|^2_{L^2(X,\mathfrak{m})}.$$

We say that $(X,\mathrm{d},\mathfrak{m})$ is an infinitesimally Hilbertian space if $W^{1,2}$ is a Hilbert space (see [4], [11] for more discussions).

On an infinitesimally Hilbertian space, we have a natural 'carré du champ' operator $\Gamma(\cdot,\cdot) : [W^{1,2}(X,\mathrm{d},\mathfrak{m})]^2 \mapsto L^1(X,\mathrm{d},\mathfrak{m})$ defined by

$$\Gamma(f,g) := \frac{1}{4}\Big(|\mathrm{D}(f+g)|^2 - |\mathrm{D}(f-g)|^2\Big).$$

It can be seen that $\Gamma(\cdot,\cdot)$ is symmetric, bilinear and continuous. We denote $\Gamma(f,f)$ by $\Gamma(f)$. We have the following chain rule and Leibniz rule (Lemma 4.7 and Proposition 4.17 in [1], see also Corollary 7.1.2 in [8])

$$\Gamma(\Phi(f),g) = \Phi'(f)\Gamma(f,g) \ \text{ for every } f,g \in W^{1,2}, \quad \Phi \in \mathrm{Lip}\,(\mathbb{R}), \Phi(0) = 0$$

and

$$\Gamma(fg,h) = f\Gamma(g,h) + g\Gamma(f,h) \ \text{ for every } f,g,h \in W^{1,2} \cap L^\infty.$$

We say that a metric measure space $M = (X,\mathrm{d},\mathfrak{m})$ has Sobolev-to-Lipschitz property if: for any function $f \in W^{1,2}$ such that $|\mathrm{D}f| \in L^\infty$, we can find a Lipschitz continuous function $\bar{f}$ such that $f = \bar{f}$ $\mathfrak{m}$-a.e. and $\mathrm{Lip}(\bar{f}) = \operatorname{ess\,sup} |\mathrm{D}f|$. In particular, by applying this property to the functions $\{\mathrm{d}(z,\cdot) : z \in X\}$, we know the distance d is induced by Γ, i.e.

$$\mathrm{d}(x,y) = \sup\big\{f(x) - f(y) : f \in W^{1,2} \cap C_b(X), |\mathrm{D}f| \le 1, \ \ \mathfrak{m} \text{ a.e.}\big\}.$$

Then we define the Dirichlet (energy) form $E : L^2 \mapsto [0,\infty]$ by

$$E(f) := \int \Gamma(f)\,\mathrm{d}\mathfrak{m}.$$

It is proved (see [2,3]) Lipschitz functions are dense in energy in the sense that: for any $f \in W^{1,2}$ there is a sequence of Lipschitz functions $(f_n)_n \subset L^2(X,\mathfrak{m})$ such that

$f_n \to f$ and $\mathrm{lip}(f_n) \to |\mathrm{D}f|$ in L^2. Moreover, if $W^{1,2}$ is Hilbert we know Lipschitz functions are dense (strongly) in $W^{1,2}$.

It can be proved that E is a strongly local, symmetric, quasi-regular Dirichlet form (see [2,4,5]). The Markov semigroup $(\mathrm{H}_t)_{t\geq 0}$ generated by E is called the heat flow. There exists heat kernel which is a family of functions $p_t(x,y) : X \times X \times \mathbb{R} \mapsto \mathbb{R}$ such that $p_t(x,y)\,\mathrm{d}\mathfrak{m}(y)$ is a probability measure for any $x \in X, t \in \mathbb{R}$, and $\mathrm{H}_t f(x) = \int f(y) p_t(x,y)\,\mathrm{d}\mathfrak{m}(y)$ for any $f \in L^2(X,\mathfrak{m})$.

For any $f \in L^2(X,\mathfrak{m})$ we know $(0,\infty) \ni t \mapsto \mathrm{H}_t f \in L^2 \cap D(\Delta)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}_t f = \Delta \mathrm{H}_t f \quad \forall t \in (0,\infty),$$

and

$$\lim_{t\to 0} \mathrm{H}_t f = f \quad \text{in } L^2.$$

Here the Lapalcian is defined in the following way (see [11] for the compatibility of different definitions of Laplacian):

Definition 2.4 (Measure valued Laplacian, [10,11,15])**.** The space $\mathrm{D}(\boldsymbol{\Delta}) \subset W^{1,2}$ is the space of $f \in W^{1,2}$ such that there is a measure $\mu \in \mathrm{Meas}(M)$ satisfying

$$\int \varphi\,\mu = -\int \Gamma(\varphi, f)\,\mathfrak{m}, \forall \varphi : M \mapsto \mathbb{R}, \quad \text{Lipschitz with bounded support.}$$

In this case the measure μ is unique and we denote it by $\boldsymbol{\Delta} f$. If $\boldsymbol{\Delta} f \ll m$, we denote its density with respect to $\mathfrak{m}$ by Δf.

We define $\mathrm{TestF}(X,\mathrm{d},\mathfrak{m}) \subset W^{1,2}(X,\mathrm{d},\mathfrak{m})$, the space of test functions as

$$\mathrm{TestF}(X,\mathrm{d},\mathfrak{m}) := \Big\{ f \in \mathrm{D}(\boldsymbol{\Delta}) \cap L^\infty : |\mathrm{D}f| \in L^\infty \quad \text{and} \quad \Delta f \in W^{1,2} \cap L^\infty(X,\mathfrak{m}) \Big\}.$$

It is known from [15] and [4] that $\mathrm{TestF}(M)$ is an algebra and it is dense in $W^{1,2}(X,\mathrm{d},\mathfrak{m})$ when $(X,\mathrm{d},\mathfrak{m})$ is a $\mathrm{RCD}(K,\infty)$ metric measure space. We will see in Lemma 3.4 that TestF is dense in $W^{1,2}$ when L^p-gradient estimate holds.

Lemma 2.5 (Lemma 3.2, [15])**.** *Let $M = (X,\mathrm{d},\mathfrak{m})$ be a metric measure space satisfying Assumptions 2.1. Assume that the algebra generated by $\{f_1, ..., f_n\} \subset$ $\mathrm{TestF}(M)$ is included in* $\mathrm{TestF}(M)$. *Let $\Phi \in C^\infty(\mathbb{R}^n)$ be with $\Phi(0) = 0$. Put $\mathbf{f} = (f_1, ..., f_n)$, then $\Phi(\mathbf{f}) \in \mathrm{TestF}(M)$.*

Let $f \in \mathrm{TestF}(M)$. We define the Hessian $\mathrm{H}_f(\cdot,\cdot) : \{\mathrm{TestF}(M)\}^2 \mapsto L^0(M)$ by

$$2\mathrm{Hess}[f](g,h) = \Gamma(g, \Gamma(f,h)) + \Gamma(h, \Gamma(f,g)) - \Gamma(f, \Gamma(g,h)).$$

Then we have the following lemma.

Lemma 2.6 (Chain rules, [7], [15])**.** *Let $f_1, ..., f_n \in \mathrm{TestF}(M)$ and $\Phi \in C^\infty(\mathbb{R}^n)$ be with $\Phi(0) = 0$. Assume that the algebra generated by $\{f_1, ..., f_n\} \subset \mathrm{TestF}(M)$ is included in $\mathrm{TestF}(M)$. Put $\mathbf{f} = (f_1, ..., f_n)$, then*

$$|\mathrm{D}\Phi(\mathbf{f})|^2\,\mathfrak{m} = \sum_{i,j=1}^n \Phi_i \Phi_j(\mathbf{f}) \Gamma(f_i, f_j)\,\mathfrak{m},$$

and

$$\mathbf{\Delta}\Phi(\mathbf{f}) = \sum_{i=1}^{n} \Phi_i(\mathbf{f})\mathbf{\Delta} f_i + \sum_{i,j=1}^{n} \Phi_{ij}(\mathbf{f})\Gamma(f_i, f_j)\,\mathfrak{m}.$$

The last lemma will be used in the proof of Theorem 3.6.

Lemma 2.7 (Lemma 3.3.6, [10]). *Let $\mu_i = \rho_i\,\mathfrak{m} + \mu_i^s, i = 1,2,3$ be measures with $\mu_i^s \perp \mathfrak{m}$. We assume that*

$$\lambda^2\mu_1 + 2\lambda\mu_2 + \mu_3 \geq 0, \qquad \forall \lambda \in \mathbb{R}.$$

Then we have

$$\mu_1^s \geq 0, \quad \mu_3^s \geq 0$$

and

$$|\rho_2|^2 \leq \rho_1\rho_3, \quad \mathfrak{m} - a.e..$$

3 Main Results

Firstly, we will discuss more about the measure valued Laplacian. Since E is quasi-regular, we know (see Remark 1.3.9(ii), [9]) that every function $f \in W^{1,2}$ has an quasi-continuous representative $\overline{f}$. And $\overline{f}$ is unique up to quasi-everywhere equality, i.e. if $\overline{f}'$ is another quasi-continuous representative, then $\overline{f}' = \overline{f}$ holds in a complement of an E-polar set. For more details, see Definition 2.1 in [15] and the references therein.

Definition 3.1. We define $\mathbb{M}_\infty$ the space of $f \in D(\mathbf{\Delta}) \cap L^\infty$ such that that there exists a measure decomposition $\mathbf{\Delta} f = \mu_+ - \mu_-$ with $\mu_\pm \in (W_+^{1,2})'$ where $W_+^{1,2} := \{\varphi \in W^{1,2} : \varphi \geq 0, \mathfrak{m} - \text{a.e.}\}$, such that: every E-polar set is $(\mathbf{\Delta} f)$-negligible and

$$\int \overline{\varphi}\,\mathrm{d}(\mathbf{\Delta} f) = -\int \Gamma(\varphi, f)\,\mathrm{d}\mathfrak{m}$$

for any $\varphi \in W^{1,2}$, the quasi-continuous representative $\overline{\varphi} \in L^1(X, \mathbf{\Delta} f)$.

In this case, the measure $\overline{\varphi}\mathbf{\Delta} f$ is well-defined.

In the next lemma we study the measure $\mathbf{\Delta}\Gamma(f)^{\frac{p}{2}}$. Since $\Gamma(f)$ is not necessarily continuous, and $\Phi(x) = x^{\frac{p}{2}}$ is not $C^2(\mathbb{R})$, we can not use Lemma 2.6 directly.

Lemma 3.2. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space satisfying assumptions 2.1. Let $f \in \mathrm{TestF}$ such that $\Gamma(f), \Gamma(f)^{\frac{p}{2}} \in \mathbb{M}_\infty$, $p > 2$. Then*

$$\frac{1}{p}\mathbf{\Delta}\Gamma(f)^{\frac{p}{2}} - \Gamma(f)^{\frac{p}{2}-1}\Gamma(\Delta f, f)\mathrm{d}\mathfrak{m} \geq K\Gamma(f)^{\frac{p}{2}}\mathrm{d}\mathfrak{m} \tag{3.1}$$

if and only if

$$\frac{1}{2}\Gamma(f)\mathbf{\Delta}_{ac}\Gamma(f) + \frac{1}{2}(\frac{p}{2} - 1)\Gamma(\Gamma(f))\mathrm{d}\mathfrak{m} \geq \Big(\Gamma(f)\Gamma(\Delta f, f) + K\Gamma(f)^2\Big)\mathrm{d}\mathfrak{m} \tag{3.2}$$

and $\overline{\Gamma(f)}\mathbf{\Delta}_{sing}\Gamma(f) \geq 0$ as measures, where $\mathbf{\Delta}_{ac}\Gamma(f)$ is the absolutely continuous part in the measure decomposition $\mathbf{\Delta}\Gamma(f) = \mathbf{\Delta}_{ac}\Gamma(f) + \mathbf{\Delta}_{sing}\Gamma(f)$ with respect to $\mathfrak{m}$, and $\overline{\Gamma(f)}$ is the quasi-continuous representation of $\Gamma(f)$.

Proof. Since $p > 2$, it can be seen that (3.2) is equivalent to

$$\frac{1}{2}\Gamma(f)^{\frac{p}{2}-1}\mathbf{\Delta}_{ac}\Gamma(f)+\frac{1}{2}(\frac{p}{2}-1)\Gamma(f)^{\frac{p}{2}-2}\Gamma(\Gamma(f))\mathrm{d}\mathfrak{m} \geq \Big(\Gamma(f)^{\frac{p}{2}-1}\Gamma(\Delta f, f)+K\Gamma(f)^{\frac{p}{2}}\Big)\mathrm{d}\mathfrak{m}. \tag{3.3}$$

Assume that we have the decomposition of the measure $\frac{1}{p}\mathbf{\Delta}\Gamma(f)^{\frac{p}{2}}$ with respect to $\mathfrak{m}$: $\frac{1}{p}\mathbf{\Delta}\Gamma(f)^{\frac{p}{2}} = \frac{1}{p}\mathbf{\Delta}_{sing}\Gamma(f)^{\frac{p}{2}} + \frac{1}{p}\mathbf{\Delta}_{ac}\Gamma(f)^{\frac{p}{2}}$. From (3.1) we know the singular part $\frac{1}{p}\mathbf{\Delta}_{sing}\Gamma(f)^{\frac{p}{2}}$ of the measure $\frac{1}{p}\mathbf{\Delta}\Gamma(f)^{\frac{p}{2}}$ is non-negative.

From hypothesis we know $\Gamma(f), \Gamma(f)^{\frac{p}{2}} \in D(\mathbf{\Delta})$, by chain rule we know

$$\int \varphi \, \mathrm{d}\mathbf{\Delta}\Gamma(f)^{\frac{p}{2}} = -\int \Gamma(\varphi, \Gamma(f)^{\frac{p}{2}}) \, \mathrm{d}\mathfrak{m} = -\int \frac{p}{2}\Gamma(f)^{\frac{p}{2}-1}\Gamma(\varphi, \Gamma(f)) \, \mathrm{d}\mathfrak{m} \tag{3.4}$$

for any Lipschitz function φ with bounded support.

Let $\overline{\Gamma(f)}$ be the quasi-continuous representation of $\Gamma(f)$, by Leibniz rule and chain rule we know $\varphi(\Gamma(f)+\epsilon)^{\frac{p}{2}-1} \in W^{1,2}$, for any $\epsilon > 0$. According to Definition 3.1 we have

$$\begin{aligned} -\int \varphi(\overline{\Gamma(f)}+\epsilon)^{\frac{p}{2}-1} \, \mathrm{d}\mathbf{\Delta}\Gamma(f) &= \int \Gamma(\varphi(\Gamma(f)+\epsilon)^{\frac{p}{2}-1}, \Gamma(f)) \, \mathrm{d}\mathfrak{m} \\ = \int \varphi(\frac{p}{2}-1)(\Gamma(f)+\epsilon)^{\frac{p}{2}-2}\Gamma(\Gamma(f)) \, \mathrm{d}\mathfrak{m} &+ \int (\Gamma(f)+\epsilon)^{\frac{p}{2}-1}\Gamma(\varphi, \Gamma(f)) \, \mathrm{d}\mathfrak{m}. \end{aligned}$$

Letting $\epsilon \to 0$, by monotone convergence theorem we obtain

$$-\int \varphi\overline{\Gamma(f)}^{\frac{p}{2}-1} \, \mathrm{d}\mathbf{\Delta}\Gamma(f) = \int \Big[\varphi(\frac{p}{2}-1)\Gamma(f)^{\frac{p}{2}-2}\Gamma(\Gamma(f)) + \Gamma(f)^{\frac{p}{2}-1}\Gamma(\varphi, \Gamma(f))\Big] \, \mathrm{d}\mathfrak{m}. \tag{3.5}$$

Combining (3.4) and (3.5) we have

$$\frac{1}{p}\mathbf{\Delta}\Gamma(f)^{\frac{p}{2}} = \frac{1}{2}\overline{\Gamma(f)}^{\frac{p}{2}-1}\mathbf{\Delta}\Gamma(f) + \frac{1}{2}(\frac{p}{2}-1)\Gamma(f)^{\frac{p}{2}-2}\Gamma(\Gamma(f))\mathrm{d}\mathfrak{m} \tag{3.6}$$

as measures. Therefore, we know

$$\begin{aligned} \frac{1}{p}\mathbf{\Delta}_{ac}\Gamma(f)^{\frac{p}{2}} &= \frac{1}{2}\overline{\Gamma(f)}^{\frac{p}{2}-1}\mathbf{\Delta}_{ac}\Gamma(f) + \frac{1}{2}(\frac{p}{2}-1)\Gamma(f)^{\frac{p}{2}-2}\Gamma(\Gamma(f))\mathrm{d}\mathfrak{m} \\ &= \frac{1}{2}\Gamma(f)^{\frac{p}{2}-1}\mathbf{\Delta}_{ac}\Gamma(f) + \frac{1}{2}(\frac{p}{2}-1)\Gamma(f)^{\frac{p}{2}-2}\Gamma(\Gamma(f))\mathrm{d}\mathfrak{m} \end{aligned}$$

and

$$\frac{1}{p}\mathbf{\Delta}_{sing}\Gamma(f)^{\frac{p}{2}} = \frac{1}{2}\overline{\Gamma(f)}^{\frac{p}{2}-1}\mathbf{\Delta}_{sing}\Gamma(f)$$

In conclusion, we obtain

$$\frac{1}{p}\mathbf{\Delta}\Gamma(f)^{\frac{p}{2}} = \frac{1}{2}\Gamma(f)^{\frac{p}{2}-1}\mathbf{\Delta}_{ac}\Gamma(f)+\frac{1}{2}(\frac{p}{2}-1)\Gamma(f)^{\frac{p}{2}-2}\Gamma(\Gamma(f))\mathrm{d}\mathfrak{m}+\frac{1}{2}\overline{\Gamma(f)}^{\frac{p}{2}-1}\mathbf{\Delta}_{sing}\Gamma(f).$$

Hence (3.1) is equivalent to (3.3), we prove the lemma. $\square$

The following lemma will be used in the proof in Theorem 3.6.

Lemma 3.3. *Let $P(r) : [0,\infty) \mapsto [-\frac{1}{4},\infty)$ be a function defined as*

$$P(r) = r - \frac{1}{4(r+1)}.$$

Let $a_0 \geq 0$ be an arbitrary initial datum, we define a_n recursively by the formula

$$a_{n+1} = P(a_n).$$

Then there exists a integer N_0 such that $0 \leq a_{N_0} < 1$ and $-\frac{1}{4} \leq a_{N_0+1} < 0$.

Conversely, for any $a \in [0,1)$ and $b > a$, there exists a sequence $a_0, ..., a_{N_0}$ defined by the recursive function P such that $a_0 > b$ and $a_{N_0} = a$.

Proof. It can be seen that $a_{n+1} < a_n$. If $a_0 \geq 0$, by monotonicity we know $a_n - a_{n+1} \in [\frac{1}{4(a_0+1)}, \frac{1}{4}]$ for any $n \in \mathbb{N}$. So there must exists a unique N_0 such that $0 \leq a_{N_0} < 1$ and $-\frac{1}{4} \leq a_{N_0+1} < 0$. Conversely, since $P(r)$ is strictly monotone on $[0,\infty)$, we know $P^{-1}(r) : [-\frac{1}{4},\infty) \mapsto [0,\infty)$ is well defined. And $(P^{-1})^{(n+1)}(a) - (P^{-1})^{(n)}(a) \in [\frac{1}{4((P^{-1})^{(n+1)}(a)+1)}, \frac{1}{4}]$ for any $n \in \mathbb{N}$. Thus there exists $N_\in \mathbb{N}$ such that $(P^{-1})^{(N_0)}(a) \geq b$. Therefore, we can pick $a_0 = (P^{-1})^{(N_0)}(a)$, and $a_{N_0} = (P)^{(N_0)}(a_0) = a$ fulfills our request. $\square$

As we mentioned in the Introduction, the space of test functions is dense in $W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ when L^p-gradient estimate of heat flow holds.

Lemma 3.4 (Density of test functions in $W^{1,2}(X, \mathrm{d}, \mathfrak{m})$, see Remark 2.5, [5])**.** *Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space satisfying Assumption 2.1. Assume that for any $f \in W^{1,2} \cap \mathrm{Lip} \cap L^\infty(X, \mathrm{d}, \mathfrak{m})$ we have the L^p-gradient estimate*

$$|\mathrm{D}\mathrm{H}_t f|^p \leq e^{-pKt} \mathrm{H}_t |\mathrm{D}f|^p, \quad a.e. \tag{3.7}$$

for some $p \in [1,\infty)$. Then the space of test functions $\mathrm{TestF}(X, \mathrm{d}, \mathfrak{m})$ *is dense in* $W^{1,2}$*.*

Proof. As we discussed in the preliminary section, the space

$$\mathbb{V}^1 := \Big\{\varphi \in W^{1,2} : \Gamma(\varphi) \in L^\infty(X, \mathfrak{m})\Big\}$$

is dense in $W^{1,2}$. We also know that the

$$\mathbb{V}^1_\infty := \Big\{\varphi \in W^{1,2} \cap L^\infty : \Gamma(\varphi) \in L^\infty(X, \mathfrak{m})\Big\}$$

in dense in L^2, and $\mathbb{V}^1_\infty$ is invariant under the action $(\mathrm{H}_t)_t$ by (3.7) and Sobolev-to-Lipschitz property. Hence by an approximation argument (see e.g. Lemma 4.9 in [4]), we know $\mathbb{V}^1_\infty$ is dense in $W^{1,2}$. Similarly, by a semigroup mollification (see e.g. page 351, [5]) we can prove that

$$\mathbb{V}^2_\infty := \Big\{\varphi \in \mathbb{V}^1_\infty : \Delta\varphi \in W^{1,2} \cap L^\infty(X, \mathfrak{m})\Big\}$$

is dense in $W^{1,2}$. $\square$

We now introduce the following assumption, which is basic and necessary in Bakry-Émery theory.

Assumption 3.5 (Existence of good algebra). We assume the existence of a dense subspace $\mathcal{A}$ in $\mathrm{TestF}(X, \mathrm{d}, \mathfrak{m})$ w.r.t $W^{1,2}$ norm, such that $\Gamma(f) \in \mathbb{M}_\infty$ for any $f \in \mathcal{A}$.

It can be seen that $\mathcal{A}$ is an algebra (i.e. $\mathcal{A}$ is closed w.r.t. pointwise multiplication), if it exists. In particular, by Lemma 3.4 we know $\mathcal{A}$ is dense in $W^{1,2}$ if L^p gradient estimate holds.

Theorem 3.6 (Improved Bakry-Émery theory)**.** *Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space satisfying Assumption 2.1. Assume also the existence of an algebra $\mathcal{A}$ in Assumption 3.5. If for any $f \in W^{1,2} \cap \mathrm{Lip} \cap L^\infty(X, \mathrm{d}, \mathfrak{m})$ we have the gradient estimate*

$$|\mathrm{DH}_t f|^p \leq e^{-pKt} \mathrm{H}_t |\mathrm{D}f|^p, \quad \mathfrak{m} - a.e. \tag{3.8}$$

for some $p \in [1, \infty)$. Then $(X, \mathrm{d}, \mathfrak{m})$ is a $\mathrm{RCD}(K, \infty)$ space.

Proof. If $p \leq 2$, by the result of [5] we know $(X, \mathrm{d}, \mathfrak{m})$ is a $\mathrm{RCD}(K, \infty)$. So we assume $p > 2$.

Part 1. Firstly, we prove

$$\Gamma(f) \mathbf{\Delta}_{ac} \Gamma(f) + \epsilon \Gamma(\Gamma(f)) \geq \Gamma(f) \Gamma(\Delta f, f) + K \Gamma(f)^2, \tag{3.9}$$

and $\overline{\Gamma(f)} \mathbf{\Delta}_{sing} \Gamma(f) \geq 0$, for any $f \in \mathcal{A}$ and $\epsilon > 0$.

For any $\varphi \in \mathrm{TestF}(X, \mathrm{d}, \mathfrak{m}), \varphi \geq 0$ and $t > 0$, we define $F : [0, t] \mapsto \mathbb{R}$ by

$$F(s) = \int e^{-pKs} \mathrm{H}_s \varphi \Gamma(\mathrm{H}_{t-s} f)^{\frac{p}{2}}.$$

It can be seen that F is a C^1 function (see Lemma 2.1, [5]). From (3.8) we know $F(s) \leq F(t)$ holds for any $s \in [0, t]$. Hence $F'(s)_{|s=t} \geq 0$, which is to say

$$\begin{aligned} &\int e^{-pKs} \Delta \mathrm{H}_s \varphi \Gamma(\mathrm{H}_{t-s} f)^{\frac{p}{2}} - p \int e^{-pKs} \mathrm{H}_s \varphi \Gamma(\mathrm{H}_{t-s} f)^{\frac{p}{2}-1} \Gamma(\Delta \mathrm{H}_{t-s} f, \mathrm{H}_{t-s} f) \\ \geq \;& pK \int e^{-pKs} \mathrm{H}_s \varphi \Gamma(\mathrm{H}_{t-s} f)^{\frac{p}{2}} \end{aligned}$$

when $s = t$. Letting $t \to 0$ we obtain

$$\int \Delta \varphi \Gamma(f)^{\frac{p}{2}} - p \int \varphi \Gamma(f)^{\frac{p}{2}-1} \Gamma(\Delta f, f) \geq pK \int \varphi \Gamma(f)^{\frac{p}{2}}.$$

In particular, from Lemma 2.6 and Lemma 3.2 in [15] we know $\Gamma(f)^{\frac{p}{2}} \in D(\mathbf{\Delta})$ and

$$\frac{1}{p} \mathbf{\Delta} \Gamma(f)^{\frac{p}{2}} - \Gamma(f)^{\frac{p}{2}-1} \Gamma(\Delta f, f) \mathrm{d}\mathfrak{m} \geq K \Gamma(f)^{\frac{p}{2}} \mathrm{d}\mathfrak{m}. \tag{3.10}$$

By Lemma 3.2, we have the inequality

$$\frac{1}{2} \Gamma(f) \Delta_{ac} \Gamma(f) + (\frac{p}{4} - \frac{1}{2}) \Gamma(\Gamma(f)) \geq \Gamma(f) \Gamma(\Delta f, f) + K \Gamma(f)^2 \tag{3.11}$$

holds $\mathfrak{m}$-a.e., and $\overline{\Gamma(f)}\mathbf{\Delta}_{sing}\Gamma(f) \geq 0$.

From now, all the inequalities are considered in $\mathfrak{m}$-a.e. sense. We denote $\frac{1}{2}\Delta_{ac}\Gamma(f) - \Gamma(\Delta f, f)$ by $\Gamma_2(f)$, and $\frac{1}{2}\Delta_{ac}\Gamma(f) - \Gamma(\Delta f, f) - K\Gamma(f)$ by $\Gamma_{2,K}(f)$, then (3.11) becomes

$$\Gamma_{2,K}(f)\Gamma(f) + (\frac{p}{4} - \frac{1}{2})\Gamma(\Gamma(f)) \geq 0.$$

For any real number $r \geq 0$, we say that the property $B(r)$ holds if

$$\Gamma_{2,K,r}(f) := \Gamma_{2,K}(f)\Gamma(f) + r\Gamma(\Gamma(f)) \geq 0$$

for any $f \in \text{TestF}$. For example, (3.11) means $B(\frac{p}{4} - \frac{1}{2})$.

Now we define

$$P(r) = r - \frac{1}{4(r+1)}.$$

Then we will prove that $B(r)$ implies $B(P(r))$. We choose the smooth function $\Phi : \mathbb{R}^3 \mapsto \mathbb{R}$ defined by

$$\Phi(\mathbf{f}) := \lambda f_1 + (f_2 - a)(f_3 - b) - ab, \quad a, b, \lambda \in \mathbb{R}.$$

Then we know

$$\begin{aligned}
\Phi_{23}(\mathbf{f}) &= \Phi_{32} = a, \qquad\qquad \Phi_{ij}(\mathbf{f}) = 0, \quad \text{i}f\ (i,j) \notin \{(2,3),(3,2)\} \\
\Phi_1(\mathbf{f}) &= \lambda, \qquad \Phi_2(\mathbf{f}) = f_3 - b, \qquad \Phi_3(\mathbf{f}) = f_2 - a.
\end{aligned}$$

If $\mathbf{f} := (f, g, h) \in \mathcal{A}^3$, we know $\Phi(\mathbf{f}) \in \mathcal{A}$ by Lemma 2.5. Hence we know

$$\Gamma_{2,K}(\Phi(\mathbf{f}))\Gamma(\Phi(\mathbf{f})) + r\Gamma(\Gamma(\Phi(\mathbf{f}))) \geq 0. \tag{3.12}$$

By direct computation using Lemma 2.6 (see also Theorem 3.4, [15]), we have

$$\begin{aligned}
\Gamma(\Phi(\mathbf{f})) &= g^{ij}\Phi_i\Phi_j(\mathbf{f}) \\
&= \lambda^2\Gamma(f) + (g-a)A_1 + (h-b)B_1
\end{aligned}$$

where $g^{ij} = \Gamma(f_i, f_j)$, A_1, A_2 are some additional terms.

Similarly, we have

$$\begin{aligned}
\Gamma(\Gamma(\Phi(\mathbf{f}))) &= \Gamma(g^{ij}\Phi_i\Phi_j(\mathbf{f})) \\
&= (g^{ij})^2\Gamma(\Phi_i\Phi_j) + (\Phi_i\Phi_j)^2\Gamma(g^{ij}) + 2g^{ij}\Phi_i\Phi_j\Gamma(g^{ij}, \Phi_i\Phi_j) \\
&= (g^{ij})^2\Big[\Phi_i^2\Gamma(\Phi_j) + \Phi_j^2\Gamma(\Phi_i) + 2\Phi_i\Phi_j\Gamma(\Phi_i, \Phi_j)\Big] \\
&+ (\Phi_i\Phi_j)^2\Gamma(g^{ij}) + 2g^{ij}\Phi_i\Phi_j\Gamma(g^{ij}, \Phi_i\Phi_j) \\
&= 2(g^{12})^2\lambda^2\Gamma(h) + 2(g^{13})^2\lambda^2\Gamma(g) + \lambda^4\Gamma(g^{11}) + (g-a)A_2 + (h-b)B_2 \\
&= 2\Gamma(f,g)^2\lambda^2\Gamma(h) + 2\Gamma(f,h)^2\lambda^2\Gamma(g) + \lambda^4\Gamma(\Gamma(f)) + (g-a)A_2 + (h-b)B_2.
\end{aligned}$$

We also know (see Theorem 3.4, [15] or Lemma 3.3.7, [10]) that

$$\begin{aligned}
\Gamma_2(\mathbf{f}) - K\Gamma(\Phi(\mathbf{f})) &= \lambda^2\Gamma_2(f) + 4\lambda\text{Hess}[f](g,h) + 2\Big(\Gamma(g)\Gamma(h) + \Gamma(g,h)^2\Big) \\
&+ (g-a)A_3 + (h-b)B_3 - K\lambda^2\Gamma(f).
\end{aligned}$$

Combining the computations above, (3.12) becomes an inequality with parameters a, b, λ. Then by locality of weak gradients and density of simple functions, we can replace b by h and replace a by g (similar arguments are used in Theorem 3.4, [15] and Lemma 3.3.7, [10]). Then we obtain the following inequality from (3.12):

$$\begin{aligned} & \lambda^2\Gamma(f)\left[\lambda^2\Gamma_2(f)+4\lambda\mathrm{Hess}[f](g,h)+2\Big(\Gamma(g)\Gamma(h)+\Gamma(g,h)^2\Big)-K\lambda^2\Gamma(f)\right] \\ + & \; r\left[2\Gamma(f,g)^2\lambda^2\Gamma(h)+2\Gamma(f,h)^2\lambda^2\Gamma(g)+\lambda^4\Gamma(\Gamma(f))\right] \\ \geq & \; 0. \end{aligned}$$

Since $r \geq 0$ and

$$\Gamma(g)\Gamma(h) \geq \Gamma(g,h)^2,$$

we have

$$\begin{aligned} & \Gamma(f)\left[\lambda^2\Gamma_2(f)+4\lambda\mathrm{Hess}[f](g,h)+4\big(\Gamma(g)\Gamma(h)\big)-K\lambda^2\Gamma(f)\right] \\ + & \; r\left[4\Gamma(f)\Gamma(g)\Gamma(h)+\lambda^2\Gamma(\Gamma(f))\right] \\ \geq & \; 0. \end{aligned}$$

Then we have

$$(\Gamma_2(f)\Gamma(f)+r\Gamma(\Gamma(f))-K\Gamma(f)^2)\lambda^2+4\lambda\Gamma(f)\mathrm{Hess}[f](g,h)+4(r+1)\Gamma(f)\Gamma(g)\Gamma(h) \geq 0.$$

Applying Lemma 2.7 we obtain

$$(1+r)\Gamma_{2,K,r}\Gamma(f)\Gamma(g)\Gamma(h) \geq \Gamma(f)^2\mathrm{Hess}[f](g,h).$$

Since $B(r)$ means $\Gamma_{2,K,r} \geq 0$, this inequality is equivalent to

$$(1+r)\Gamma_{2,K,r}(f)\Gamma(g)\Gamma(h) \geq \Gamma(f)\mathrm{Hess}[f](g,h). \tag{3.13}$$

From the definition of $\mathrm{Hess}[\cdot]$, we know

$$\mathrm{Hess}[f](g,h)+\mathrm{Hess}[g](f,h) = \Gamma(\Gamma(f,g),h).$$

Combining with inequality (3.13) we have

$$\begin{aligned} \sqrt{\frac{1}{1+r}}\Gamma(\Gamma(f,g),h)\sqrt{\Gamma(f)} & \leq \sqrt{\Gamma_{2,K,r}(f)\Gamma(g)\Gamma(h)}+\sqrt{\Gamma_{2,K,r}(g)\Gamma(f)\Gamma(h)} \\ & = \Big(\sqrt{\Gamma_{2,K,r}(f)\Gamma(g)}+\sqrt{\Gamma_{2,K,r}(g)\Gamma(f)}\Big)\sqrt{\Gamma(h)}. \end{aligned}$$

Then we fix $f, g \in \mathcal{A}$, and approximate any $h \in W^{1,2} \cap L^\infty$ with a sequence $(h_n) \subset \mathcal{A}$ converging to h strongly in $W^{1,2}$ such that

$$\Gamma(h_n) \to \Gamma(h), \quad \Gamma(h_n, \Gamma(f,g)) \to \Gamma(h, \Gamma(f,g))$$

pointwise and in $L^1(X, \mathfrak{m})$. Thus we can replace h by $\Gamma(f,g)$ in the last inequality and obtain

$$\sqrt{\frac{1}{1+r}}\sqrt{\Gamma(\Gamma(f,g))\Gamma(f)} = \Big(\sqrt{\Gamma_{2,K,r}(f)\Gamma(g)}+\sqrt{\Gamma_{2,K,r}(g)\Gamma(f)}\Big). \tag{3.14}$$

Let $f = g$ in (3.14) we have

$$\frac{1}{1+r}\Gamma(\Gamma(f))\Gamma(f) \le 4\Gamma_{2,K,r}(f)\Gamma(f).$$

Therefore,

$$(\frac{1}{4}\frac{1}{1+r} - r)\Gamma(\Gamma(f))\Gamma(f) \le \Gamma_{2,K}(f)\Gamma(f).$$

In other words, we have $B(P(r))$.

From Lemma 3.3 we know there exists $a_0 \ge \frac{p}{4} - \frac{1}{2}$ and $N_0 \in \mathbb{N}$ such that $a_{N_0} = \epsilon$, where $a_{n+1} = P(a_n)$, $n = 0, ..., N_0 - 1$. Then we know $B(a_0)$ from (3.11). From the result above, we can see that $B(a_{N_0})$ holds by induction. So we prove (3.9).

Part 2. From (3.9) and Lemma 3.2 we know

$$\frac{1}{p_n}\mathbf{\Delta}\Gamma(f)^{\frac{p_n}{2}} - \Gamma(f)^{\frac{p_n}{2}-1}\Gamma(\Delta f, f)\mathrm{d}\mathfrak{m} \ge K\Gamma(f)^{\frac{p_n}{2}}\mathrm{d}\mathfrak{m}.$$

for any $p_n = 2 + \frac{1}{2^n}$, where $n \in \mathbb{N}$.

Let $f \in \mathcal{A}$, $\varphi \in \mathrm{TestF}$, and $t > 0$, we define $F : [0, t] \mapsto \mathbb{R}$ by

$$F(s) = \int e^{-p_n K s}\mathrm{H}_s\varphi\Gamma(\mathrm{H}_{t-s}f)^{\frac{p_n}{2}}.$$

We know:

$$\begin{aligned} F'(s) &= \int e^{-p_n K s}\Delta\mathrm{H}_s\varphi\Gamma(\mathrm{H}_{t-s}f)^{\frac{p_n}{2}} - p_n\int e^{-p_n K s}\mathrm{H}_s\varphi\Gamma(\mathrm{H}_{t-s}f)^{\frac{p_n}{2}-1}\Gamma(\Delta\mathrm{H}_{t-s}f, \mathrm{H}_{t-s}f) \\ &- p_n K\int e^{-p_n K s}\mathrm{H}_s\varphi\Gamma(\mathrm{H}_{t-s}f)^{\frac{p_n}{2}} \ge 0 \end{aligned}$$

for any $s \in [0, t]$. Hence $F(t) \ge F(0)$, i.e.

$$\int \varphi e^{-p_n K t}\mathrm{H}_t\Gamma(f)^{\frac{p_n}{2}} \ge \int \varphi\Gamma(\mathrm{H}_t f)^{\frac{p_n}{2}}.$$

Since φ is arbitrary, by Lemma 3.4 we know

$$\Gamma(\mathrm{H}_t f)^{1+\frac{1}{2^{n+1}}} \le e^{-(2+\frac{1}{2^n})Kt}\mathrm{H}_t\Gamma(f)^{1+\frac{1}{2^{n+1}}} \quad \mathfrak{m} - \text{a.}e..$$

Letting $n \to \infty$, by dominated convergence theorem we know

$$\Gamma(\mathrm{H}_t f) \le e^{-2Kt}\mathrm{H}_t\Gamma(f) \quad \mathfrak{m} - \text{a.}e. \tag{3.15}$$

for all $f \in \mathcal{A}$. At last, combining the density of $\mathcal{A}$ in TestF and Lemma 3.4, by lower semi-continuity (see preliminary section), we know (3.15) holds for all $f \in W^{1,2}$.

Then, by Theorem 4.17 in [5] we know $(X, \mathrm{d}, \mathfrak{m})$ is a $\mathrm{RCD}(K, \infty)$ space. □

As a corollary, we have the following proposition. We recall (see [10]) that the measure valued Ricci tensor on RCD metric measure space is defined as

$$\mathbf{Ricci}(\nabla f, \nabla f) := \mathbf{\Gamma}_2(f) - |\mathrm{Hess}[f]|_{\mathrm{HS}}^2\,\mathfrak{m}$$

where $\mathbf{\Gamma}_2(f) := \frac{1}{2}\mathbf{\Delta}|\mathrm{D}f|^2 - \Gamma(f, \Delta f)\,\mathfrak{m}$. It can be seen that **Ricci** is well defined for any $f \in \mathrm{TestF}(X, \mathrm{d}, \mathfrak{m})$ when $(X, \mathrm{d}, \mathfrak{m})$ is RCD.

Proposition 3.7. *Let* $(X, \mathrm{d}, \mathfrak{m})$ *be a* RCD *space. Then the following characterizations are equivalent.*

1) $(X, \mathrm{d}, \mathfrak{m})$ *is* $\mathrm{RCD}(K, \infty)$*,*

2) $\mathbf{Ricci}(\nabla f, \nabla f) \geq K\Gamma(f)\,\mathfrak{m}$ *for any test function* f*,*

3) $\Gamma(f)\mathbf{Ricci}_{ac}(\nabla f, \nabla f) \geq K\Gamma(f)^2\,\mathfrak{m}$ *and* $\mathbf{Ricci}_{sing}(\nabla f, \nabla f) \geq 0$ *for any test function* f*.*

Proof. 1) $\Rightarrow$ 2) is Lemma 3.6.2 in [10], 2) $\Rightarrow$ 3) is trivial. So we just need to prove 3) $\Rightarrow$ 1).

From 3) we know $\Gamma_{2,K,0}(f) \geq 0$, $\mathfrak{m}$-a.e. for any $f \in \mathrm{TestF}$. Therefore $\Gamma_{2,K,r}(f) \geq 0$ for any $r > 0$. Using the same argument as **Part 2.** of the proof of Theorem 3.6, we know $(X, \mathrm{d}, \mathfrak{m})$ is $\mathrm{RCD}(K, \infty)$. □

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