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NEW CHARACTERIZATIONS OF RICCI CURVATURE ON RCD METRIC MEASURE SPACES

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Abstract. We prove that on a large family of metric measure spaces, if the L^p-gradient estimate for heat flows holds for some $p > 2$, then the L^1-gradient estimate also holds. This result extends Savaré's result on metric measure spaces, and provides a new proof to von Renesse-Sturm theorem on smooth metric measure spaces. As a consequence, we propose a new analysis object based on Gigli's measure-valued Ricci tensor, to characterize the Ricci curvature of RCD space in a local way. In the proof we adopt an iteration technique based on non-smooth Bakry-Émery theory, which is a new method to study the curvature dimension condition of metric measure spaces.

1. **Introduction.** For any smooth Riemannian manifold M and any $K \in \mathbb{R}$, it is proved by von Renesse and Sturm in [14] that the following properties are equivalent

1) $\mathrm{Ricci}_M \geq K$,
2) there exists $p \in (1, \infty)$ such that for all $f \in C_c^\infty(M)$, all $x \in M$ and $t \geq 0$

$$|\mathrm{DH}_t f|^p(x) \leq e^{-pKt}\mathrm{H}_t|\mathrm{D}f|^p(x), \tag{1}$$

3) for all $f \in C_c^\infty(M)$, all $x \in M$ and $t \geq 0$

$$|\mathrm{DH}_t f|(x) \leq e^{-Kt}\mathrm{H}_t|\mathrm{D}f|(x), \tag{2}$$

where $\mathrm{H}_t f$ is the solution of the heat equation with initial datum f.

In non-smooth setting, the notions of synthetic Ricci curvature bounds, or non-smooth curvature-dimension conditions, were proposed by Lott-Villani and Sturm (see [13] and [16]) using optimal transport theory. Later on, by assuming the infinitesimally Hilbertianity (i.e. the Sobolev space $W^{1,2}$ is a Hilbert space), RCD condition (or $\mathrm{RCD}(K,\infty)$ condtion to emphasize the curvature) which is a refinement of Lott-Sturm-Villani's curvature-dimension condition, was proposed by Ambrosio-Gigli-Savaré (see [4] and [1]). It is known that $\mathrm{RCD}(K,\infty)$ spaces are generalizations of Riemannian manifolds with lower Ricci curvature bound and their limit spaces, as well as Alexandrov spaces with lower curvature bound.

Is is known that Lott-Sturm-Villani's synthetic Ricci bound and 2-gradient estimate (for heat flows) are equivalent in non-smooth setting. Let $(X, \mathrm{d}, \mathfrak{m})$ be a

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$\mathrm{RCD}(K,\infty)$ space, it is proved in [4] that

$$|\mathrm{DH}_t f|^2 \le e^{-2Kt}\mathrm{H}_t|\mathrm{D}f|^2,\ \mathfrak{m}-\text{a.e.} \tag{3}$$

for any $f \in W^{1,2}$ and $t > 0$, where $\mathrm{H}_t f$ is the heat flow from f and $|\mathrm{D}f|$ is the minimal weak upper gradient (or weak gradient for simplicity) of f. In particular, by Hölder inequality we know

$$|\mathrm{DH}_t f|^p \le e^{-pKt}\mathrm{H}_t|\mathrm{D}f|^p,\ \mathfrak{m}-\text{a.e.} \tag{4}$$

for any $p \ge 2$. Furthermore, it is proved in [15] that inequality (3) can be improved as

$$|\mathrm{DH}_t f| \le e^{-Kt}\mathrm{H}_t|\mathrm{D}f|,\ \mathfrak{m}-\text{a.e.}. \tag{5}$$

In conclusion, inequality (4) holds for any $p \in [1,\infty]$.

Conversely, it is shown in [5] that a space satisfying inequality (3) is $\mathrm{RCD}(K,\infty)$. Let $(X,\mathrm{d},\mathfrak{m})$ be an infinitesimally Hilbertian space, we have a well-defined Dirichlet energy:

$$E(f) := \frac{1}{2}\int |\mathrm{D}f|^2\,\mathrm{d}\mathfrak{m}$$

for any $f \in W^{1,2}(X,\mathrm{d},\mathfrak{m})$. We denote the L^2-gradient flow of $E(\cdot)$ starting from f by $(\mathrm{H}_t f)_t$. Assume further that the space $(X,\mathrm{d},\mathfrak{m})$ has Sobolev-to-Lipschitz property: for any function $f \in W^{1,2}$ with$|\mathrm{D}f| \in L^\infty$, there exists a Lipschitz continuous function $\bar{f}$ such that $f = \bar{f}$ $\mathfrak{m}$-a.e. and $\mathrm{Lip}(\bar{f}) = \operatorname{ess\,sup}|\mathrm{D}f|$. If

$$|\mathrm{DH}_t f|^2 \le e^{-2Kt}\mathrm{H}_t|\mathrm{D}f|^2,\ \mathfrak{m}-\text{a.e.} \tag{6}$$

for any $f \in W^{1,2}$ and $t > 0$, then $(X,\mathrm{d},\mathfrak{m})$ is $\mathrm{RCD}(K,\infty)$.

The main goal of this paper is to prove that for any $p > 2$, p-gradient estimate (4) also characterizes the curvature-dimension condition. We prove a non-smooth version of 2) $\Rightarrow$ 3) in von Renesse-Sturm's result, thus we complete the circle 1) $\Leftrightarrow$ 2) $\Leftrightarrow$ 3) in non-smooth setting.

Now, we introduce our main result in this paper. When $p = 2$, it is proved in [15] that there exists a space of test functions $\mathrm{TestF}(X,\mathrm{d},\mathfrak{m})$ which is a dense subspace of $W^{1,2}(X)$ defined as

$$\mathrm{TestF}(X,\mathrm{d},\mathfrak{m}) := \Big\{f \in \mathrm{D}(\mathbf{\Delta}) \cap L^\infty : |\mathrm{D}f| \in L^\infty \text{ and } \Delta f \in W^{1,2} \cap L^\infty(X,\mathfrak{m})\Big\},$$

such that $\mathbf{\Delta}|\mathrm{D}f|^2$ is a well-defined measure (see Definition 3.1) for any $f \in \mathrm{TestF}$. So it is reasonable to the following assumption (**Assumption** 2, see a similar assumption in [17]): there exists a dense subspace $\mathcal{A}$ in TestF with respect to the graph norm

$$f \mapsto \Big[\|(-\Delta)^{\frac{3}{2}}f\|_{L^2}^2 + \|f\|_{W^{1,2}}^2\Big]^{\frac{1}{2}} = \Big[E(\Delta f) + \|f\|_{W^{1,2}}^2\Big]^{\frac{1}{2}}$$

such that $|\mathrm{D}f|^2 \in \mathbb{M}_\infty$ for any $f \in \mathcal{A}$. We remark that we do not need to assume the density of $\mathcal{A}$ in $W^{1,2}$.

Theorem 1.1 (Theorem 3.5, Improved Bakry-Émery theory)**.** *Let $M := (X,\mathrm{d},\mathfrak{m})$ be a metric measure space such that there exists an algebra $\mathcal{A}$ as described above. If for any $f \in W^{1,2}(X) \cap \mathrm{Lip}(X) \cap L^\infty(X)$ we have the gradient estimate*

$$|\mathrm{DH}_t f|^p \le e^{-pKt}\mathrm{H}_t|\mathrm{D}f|^p,\ \mathfrak{m}-a.e. \tag{7}$$

for some $p \in (1,\infty)$. Then (7) holds for $p = 1$. In particular, M is a $\mathrm{RCD}(K,\infty)$ space.

Since we do not have second order differentiation formula for relative entropy along Wasserstein geodesics, or Taylor's expansion in non-smooth setting, we can not simply use the argument in smooth metric measure space (see the proofs in [14]). The argument we adopt here is the so-called 'self-improvement' method in Bakry-Émery's Γ-calculus, which was used in [15] to deal with the non-smooth problems. We remark that we not only use 'self-improvement' technique, but an improved iteration method based on this technique. We believe that this method also has potential application in the future.

It can be seen that Assumption 2 is satisfied in the following cases, where we can apply our main result.

Example 1. Smooth metric measure space: obviously, $C_c^\infty(M)$, the space of s-mooth functions with compact support is a good algebra in Assumption 2. Hence we obtain a new quick proof to von Renesse-Sturm's theorem, without using Taylor's expansion method.

Example 2. $\mathrm{RCD}(K,\infty)$ metric measure space: it is proved in Lemma 3.2 [15] that $|\mathrm{D}f|^2 \in \mathbb{M}_\infty$ for any $f \in \mathrm{TestF}$. By Theorem 1.1 we obtain the following proposition which deals with the optimal comstant K in the curvature-dimension condition. It is also a complement to Savaré's result in [15].

Proposition 1 (Self-improvement of gradient estimate)**.** *Let $(X,\mathrm{d},\mathfrak{m})$ be a $\mathrm{RCD}(K,\infty)$ metric measure space. If for any $f \in W^{1,2}\cap \mathrm{Lip}(X)\cap L^\infty(X)$ we have the gradient estimate*

$$|\mathrm{D}\mathrm{H}_t f|^p \leq e^{-pK't}\mathrm{H}_t|\mathrm{D}f|^p,\ \mathfrak{m}-a.e. \tag{8}$$

for some $p\in[1,\infty)$ and $K'>K$. Then $(X,\mathrm{d},\mathfrak{m})$ is a $\mathrm{RCD}(K',\infty)$ space. In particular, we know

$$|\mathrm{D}\mathrm{H}_t f| \leq e^{-K't}\mathrm{H}_t|\mathrm{D}f|,\ \mathfrak{m}-a.e.. \tag{9}$$

In [10], Gigli defines measure valued Ricci tensor on RCD metric measure space (see also [12]) as

$$\mathbf{Ricci}(\nabla f,\nabla f) := \mathbf{\Gamma}_2(f) - |\mathrm{Hess}[f]|^2_{\mathrm{HS}}\,\mathfrak{m}$$

where $\mathbf{\Gamma}_2(f) := \frac{1}{2}\mathbf{\Delta}|\mathrm{D}f|^2 - \langle\nabla f,\nabla\Delta f\rangle\,\mathfrak{m}$ and $|\mathrm{Hess}[f]|_{\mathrm{HS}}$ is the Hilbert-Schmidt norm of the Hessian $\mathrm{Hess}[f]$ as a module (see [10] for details). He shows that $\mathbf{Ricci}(\nabla f,\nabla f)\geq K|\mathrm{D}f|^2\,\mathfrak{m}$ if and only if the space is $\mathrm{RCD}(K,\infty)$. However, we do not know if $\mathbf{Ricci}$ has locality in the sense that $\mathbf{Ricci}(\nabla f,\nabla f)|_{\{|\mathrm{D}f|=0\}}=0$.

From the proof of Theorem 1.1 we have the following new characterization of curvature bound which extends Gigli's result:

Proposition 2 (Proposition 3)**.** *Let $(X,\mathrm{d},\mathfrak{m})$ be a RCD space. For any f such that $\mathbf{Ricci}(\nabla f,\nabla f)$ is well-defined, we denote the Lebesgue decomposition of $\mathbf{Ricci}(\nabla f, \nabla f)$ with respect to $\mathfrak{m}$ by*

$$\mathbf{Ricci}(\nabla f,\nabla f) = \mathrm{Ricci}_{ac}(\nabla f,\nabla f)\,\mathfrak{m} + \mathbf{Ricci}_{sing}(\nabla f,\nabla f).$$

Then the following characterizations are equivalent.

1) $(X,\mathrm{d},\mathfrak{m})$ *is* $\mathrm{RCD}(K,\infty)$,
2) *for any test function $f\in\mathrm{TestF}$ we have $\mathbf{Ricci}(\nabla f,\nabla f)\geq K|\mathrm{D}f|^2\,\mathfrak{m}$ in the sense that*

$$\mathrm{Ricci}_{ac}(\nabla f,\nabla f)\geq K|\mathrm{D}f|^2\ \mathfrak{m}-a.e.$$

and $\mathbf{Ricci}_{sing}(\nabla f,\nabla f)\geq 0$, *3) for any test function* $f \in \mathrm{TestF}$ *we have*

$$|\mathrm{D}f|^2 \mathrm{Ricci}_{ac}(\nabla f, \nabla f) \geq K|\mathrm{D}f|^4 \ \mathfrak{m} - a.e.$$

and $\mathbf{Ricci}_{sing}(\nabla f, \nabla f) \geq 0$.

We remark that this naive extension is non-trivial, because 2) is not a direct consequence of 3) due to lack of the locality of $\mathbf{Ricci}(\cdot,\cdot)$. From this proposition, we know that $\overline{\mathrm{Ricci}}(\nabla f, \nabla f) := |\mathrm{D}f|^2 \mathrm{Ricci}_{ac}(\nabla f, \nabla f)\, \mathfrak{m}$ characterizes the Ricci curvature of $(X, \mathrm{d}, \mathfrak{m})$ and $\overline{\mathrm{Ricci}}$ has locality in the sense that

$$\overline{\mathrm{Ricci}}(\nabla f, \nabla f)|_{\{|\mathrm{D}f|=0\}} = 0.$$

2. **Preliminaries.** First of all, we summarize the basic hypothesis on the metric measure space $(X, \mathrm{d}, \mathfrak{m})$ below, the notions and concepts in this assumption will be explained later.

Assumption 1. *We assume that:*

(1) (X, d) *is a complete, separable geodesic space,*
(2) $\mathrm{supp}\, \mathfrak{m} = X$, $\mathfrak{m}(B_r(x)) < c_1 \exp(c_2 r^2)$ *for every* $r > 0$,
(3) $W^{1,2}(X)$ *is a Hilbert space,*
(4) $(X, \mathrm{d}, \mathfrak{m})$ *has Sobolev-to-Lipschitz property,*
(5) there exits a unique heat kernel $p_t(x, y)$.

The Sobolev space $W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ is defined as in [2]. We say that $f \in L^2(X, \mathfrak{m})$ is a Sobolev function in $W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ if there exists a sequence of Lipschitz functions $(f_n) \subset L^2$, such that $f_n \to f$ and $\mathrm{lip}(f_n) \to G$ in L^2 for some $G \in L^2(X, \mathfrak{m})$, where $\mathrm{lip}(f_n)$ is the local Lipschitz constant of f_n. It is known that there exists a minimal function G in $\mathfrak{m}$-a.e. sense. We call the minimal G the minimal weak upper gradient (or weak gradient for simplicity) of the function f, and denote it by $|\mathrm{D}f|$. It is known that the locality holds for $|\mathrm{D}f|$, i.e. $|\mathrm{D}f| = |\mathrm{D}g|$ a.e. on the set $\{f = g\}$. Furthermore, we have the lower semi-continuity: if $\{f_n\}_n \subset W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ is a sequence converging to some f in $\mathfrak{m}$-a.e. sense and $(|\mathrm{D}f_n|)_n$ is bounded in $L^2(X, \mathfrak{m})$, then $f \in W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ and

$$\||\mathrm{D}f|\|_{L^2} \leq \varliminf_{n \to \infty} \||\mathrm{D}f_n|\|_{L^2}.$$

We equip $W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ with the norm

$$\|f\|^2_{W^{1,2}(X,\mathrm{d},\mathfrak{m})} := \|f\|^2_{L^2(X,\mathfrak{m})} + \||\mathrm{D}f|\|^2_{L^2(X,\mathfrak{m})}.$$

We say that $(X, \mathrm{d}, \mathfrak{m})$ is an infinitesimally Hilbertian space if $W^{1,2}$ is a Hilbert space (see [4], [11] for more discussions).

On an infinitesimally Hilbertian space, we have a 'carré du champ' operator $\Gamma(\cdot,\cdot) : [W^{1,2}(X, \mathrm{d}, \mathfrak{m})]^2 \mapsto L^1(X, \mathrm{d}, \mathfrak{m})$ defined by

$$\Gamma(f, g) := \frac{1}{4}\Big(|\mathrm{D}(f + g)|^2 - |\mathrm{D}(f - g)|^2\Big).$$

It can be seen that $\Gamma(\cdot,\cdot)$ is symmetric, bilinear and continuous. We denote $\Gamma(f, f)$ by $\Gamma(f)$. We have the following chain rule and Leibniz rule (Lemma 4.7 and Proposition 4.17 in [1], see also Corollary 7.1.2 in [8])

$$\Gamma(\Phi(f), g) = \Phi'(f)\Gamma(f, g) \text{ for every } f, g \in W^{1,2},\ \Phi \in \mathrm{Lip}\,(\mathbb{R}), \Phi(0) = 0$$

and

$$\Gamma(fg, h) = f\Gamma(g, h) + g\Gamma(f, h) \text{ for every } f, g, h \in W^{1,2} \cap L^\infty.$$

We say that a metric measure space $M = (X, \mathrm{d}, \mathfrak{m})$ has Sobolev-to-Lipschitz property if: for any function $f \in W^{1,2}$ with $|\mathrm{D}f| \in L^\infty$, we can find a Lipschitz continuous function $\bar{f}$ such that $f = \bar{f}$ $\mathfrak{m}$-a.e. and $\mathrm{Lip}(\bar{f}) = \operatorname{ess\,sup} |\mathrm{D}f|$.

We define the Dirichlet (energy) form $E : L^2 \mapsto [0, \infty]$ by

$$E(f) := \frac{1}{2} \int \Gamma(f) \,\mathrm{d}\mathfrak{m}.$$

It is proved (see [2, 3]) that Lipschitz functions are dense in energy: for any $f \in W^{1,2}$ there is a sequence of Lipschitz functions $(f_n)_n \subset L^2(X, \mathfrak{m})$ such that $f_n \to f$ and $\mathrm{lip}(f_n) \to |\mathrm{D}f|$ in L^2. Moreover, if $W^{1,2}$ is Hilbert we know Lipschitz functions are dense (strongly) in $W^{1,2}$.

It can be proved that E is a strongly local, symmetric, quasi-regular Dirichlet form (see [5, 2, 4]). The Markov semigroup $(\mathrm{H}_t)_{t\geq 0}$ generated by E is called the heat flow. There exists heat kernel which is a family of functions $p_t(x, y) : X \times X \times \mathbb{R} \mapsto \mathbb{R}$ such that $p_t(x, y)\,\mathrm{d}\mathfrak{m}(y)$ is a probability measure for any $x \in X, t \in \mathbb{R}$, and $\mathrm{H}_t f(x) = \int f(y)p_t(x, y)\,\mathrm{d}\mathfrak{m}(y)$ for any $f \in L^2(X, \mathfrak{m})$.

For any $f \in L^2(X, \mathfrak{m})$ we know that $(0, \infty) \ni t \mapsto \mathrm{H}_t f \in L^2 \cap D(\Delta)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{H}_t f = \Delta \mathrm{H}_t f \;\; \forall t \in (0, \infty),$$

and

$$\lim_{t\to 0} \mathrm{H}_t f = f \text{ in } L^2.$$

Here the Laplacian is defined in the following way (see [11] for alternative definitions):

Definition 2.1 (Measure valued Laplacian, [11, 10, 15])**.** The domain of the Laplacian $\mathrm{D}(\mathbf{\Delta}) \subset W^{1,2}$ consists of $f \in W^{1,2}$ such that there is a measure $\mu \in \mathrm{Meas}(M)$ satisfying

$$\int \varphi \,\mu = -\int \Gamma(\varphi, f)\,\mathfrak{m}, \forall \varphi : M \mapsto \mathbb{R}, \text{ Lipschitz with bounded support.}$$

In this case the measure μ is unique and we denote it by $\mathbf{\Delta} f$. If $\mathbf{\Delta} f \ll m$, we denote its density with respect to $\mathfrak{m}$ by Δf.

We define $\mathrm{TestF}(X, \mathrm{d}, \mathfrak{m}) \subset W^{1,2}(X, \mathrm{d}, \mathfrak{m})$, the space of test functions as

$$\mathrm{TestF}(X, \mathrm{d}, \mathfrak{m}) := \Big\{ f \in \mathrm{D}(\mathbf{\Delta}) \cap L^\infty : |\mathrm{D}f| \in L^\infty \text{ and } \Delta f \in W^{1,2} \cap L^\infty(X, \mathfrak{m}) \Big\}.$$

It is known from [15] and [4] that $\mathrm{TestF}(M)$ is an algebra and it is dense in $W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ when $(X, \mathrm{d}, \mathfrak{m})$ is a RCD metric measure space. We will see in Lemma 3.4 that TestF is dense in $W^{1,2}$ when L^p-gradient estimate for heat flow holds for some $p > 2$.

Lemma 2.2 (Lemma 3.2, [15])**.** *Let $M = (X, \mathrm{d}, \mathfrak{m})$ be a metric measure space satisfying Assumptions 1. Assume that the algebra generated by $\{f_1, ..., f_n\} \subset \mathrm{TestF}(M)$ is included in $\mathrm{TestF}(M)$. Let $\Phi \in C^\infty(\mathbb{R}^n)$ with $\Phi(0) = 0$. Put $\mathbf{f} = (f_1, ..., f_n)$, then $\Phi(\mathbf{f}) \in \mathrm{TestF}(M)$.*

Let $f \in \mathrm{TestF}(M)$. We define the Hessian $\mathrm{Hess}[f](\cdot, \cdot) : \{\mathrm{TestF}(M)\}^2 \mapsto L^0(M)$ by

$$2\mathrm{Hess}[f](g, h) = \Gamma(g, \Gamma(f, h)) + \Gamma(h, \Gamma(f, g)) - \Gamma(f, \Gamma(g, h)).$$

We have the following lemma.

Lemma 2.3 (Chain rules, [7], [15])**.** *Let* $f_1, ..., f_n \in \mathrm{TestF}(M)$ *and* $\Phi \in C^\infty(\mathbb{R}^n)$ *with* $\Phi(0) = 0$*. Assume that the algebra generated by* $\{f_1, ..., f_n\} \subset \mathrm{TestF}(M)$ *is included in* $\mathrm{TestF}(M)$*. Put* $\mathbf{f} = (f_1, ..., f_n)$*, then*

$$|\mathrm{D}\Phi(\mathbf{f})|^2\,\mathfrak{m} = \sum_{i,j=1}^{n} \Phi_i \Phi_j(\mathbf{f}) \Gamma(f_i, f_j)\,\mathfrak{m},$$

and

$$\mathbf{\Delta}\Phi(\mathbf{f}) = \sum_{i=1}^{n} \Phi_i(\mathbf{f}) \mathbf{\Delta} f_i + \sum_{i,j=1}^{n} \Phi_{ij}(\mathbf{f}) \Gamma(f_i, f_j)\,\mathfrak{m}.$$

The last lemma will be used in the proof of Theorem 3.5.

Lemma 2.4 (Lemma 3.3.6, [10])**.** *Let* $\mu_i = \rho_i\,\mathfrak{m} + \mu_i^s$ *be measures with* $\mu_i^s \perp \mathfrak{m}$*,* $i = 1, 2, 3$*. We assume that*

$$\lambda^2 \mu_1 + 2\lambda\mu_2 + \mu_3 \geq 0, \ \forall \lambda \in \mathbb{R}.$$

Then we have

$$\mu_1^s \geq 0, \ \mu_3^s \geq 0$$

and

$$|\rho_2|^2 \leq \rho_1 \rho_3, \ \mathfrak{m} - a.e..$$

3. **Main results.** Firstly, we discuss more about the measure-valued Laplacian. Since E is quasi-regular, we know (see Remark 1.3.9 (ii), [9]) that every function $f \in W^{1,2}$ has an quasi-continuous representative $\overline{f}$. And $\overline{f}$ is unique up to quasi-everywhere equality, i.e. if $\tilde{f}$ is another quasi-continuous representative, then $\tilde{f} = \overline{f}$ holds in a complement of an E-polar set. For more details, see Definition 2.1 in [15] and the references therein.

Definition 3.1. We define $\mathbb{M}_\infty$ the space of $f \in \mathrm{D}(\mathbf{\Delta}) \cap L^\infty$ such that there exists a measure decomposition $\mathbf{\Delta} f = \mu_+ - \mu_-$ with $\mu_\pm$ in the positive cone in $(W^{1,2})'$, such that:

$$\int \overline{\varphi}\,\mathrm{d}(\mathbf{\Delta} f) = -\int \Gamma(\varphi, f)\,\mathrm{d}\mathfrak{m}$$

for any $\varphi \in W^{1,2}$ and the quasi-continuous representative $\overline{\varphi} \in L^1(X, \mathbf{\Delta} f)$.

In particular, every E-polar set is $(\mathbf{\Delta} f)$-negligible and the measure $\overline{\varphi}\mathbf{\Delta} f$ is well-defined.

In the next lemma we study the measure $\mathbf{\Delta}\Gamma(f)^{\frac{p}{2}}$. Since $\Gamma(f)$ is not necessarily continuous, and $\Phi(x) = x^{\frac{p}{2}}$ is not $C^2(\mathbb{R})$, we can not use Lemma 2.3 directly.

Lemma 3.2. *Let* $(X, \mathrm{d}, \mathfrak{m})$ *be a metric measure space satisfying Assumptions 1. Let* $f \in \mathrm{TestF}$ *such that* $\Gamma(f), \Gamma(f)^{\frac{p}{2}} \in \mathbb{M}_\infty$*,* $p > 2$*. Then*

$$\frac{1}{p}\mathbf{\Delta}\Gamma(f)^{\frac{p}{2}} - \Gamma(f)^{\frac{p}{2}-1}\Gamma(\Delta f, f)\mathrm{d}\mathfrak{m} \geq K\Gamma(f)^{\frac{p}{2}}\mathrm{d}\mathfrak{m} \tag{10}$$

if and only if

$$\frac{1}{2}\Gamma(f)\mathbf{\Delta}_{ac}\Gamma(f) + \frac{1}{2}(\frac{p}{2} - 1)\Gamma(\Gamma(f))\mathrm{d}\mathfrak{m} \geq \Big(\Gamma(f)\Gamma(\Delta f, f) + K\Gamma(f)^2\Big)\mathrm{d}\mathfrak{m} \tag{11}$$

and $\overline{\Gamma(f)}\mathbf{\Delta}_{sing}\Gamma(f) \geq 0$ *as measures, where* $\mathbf{\Delta}_{ac}\Gamma(f)$ *is the absolutely continuous part in the measure decomposition* $\mathbf{\Delta}\Gamma(f) = \mathbf{\Delta}_{ac}\Gamma(f) + \mathbf{\Delta}_{sing}\Gamma(f)$ *with respect to* $\mathfrak{m}$*, and* $\overline{\Gamma(f)}$ *is the quasi-continuous representation of* $\Gamma(f)$*.*

Proof. Since $p>2$, it can be seen that (11) is equivalent to

$$\begin{aligned}&\frac{1}{2}\Gamma(f)^{\frac{p}{2}-1}\mathbf{\Delta}_{ac}\Gamma(f)+\frac{1}{2}(\frac{p}{2}-1)\Gamma(f)^{\frac{p}{2}-2}\Gamma(\Gamma(f))\mathrm{d}\mathfrak{m}\\ \geq&\Big(\Gamma(f)^{\frac{p}{2}-1}\Gamma(\Delta f,f)+K\Gamma(f)^{\frac{p}{2}}\Big)\mathrm{d}\mathfrak{m}.\end{aligned}\tag{12}$$

Assume that we have the decomposition of the measure $\frac{1}{p}\mathbf{\Delta}\Gamma(f)^{\frac{p}{2}}$ with respect to $\mathfrak{m}$: $\frac{1}{p}\mathbf{\Delta}\Gamma(f)^{\frac{p}{2}}=\frac{1}{p}\mathbf{\Delta}_{sing}\Gamma(f)^{\frac{p}{2}}+\frac{1}{p}\mathbf{\Delta}_{ac}\Gamma(f)^{\frac{p}{2}}$. From (10) we know the singular part $\frac{1}{p}\mathbf{\Delta}_{sing}\Gamma(f)^{\frac{p}{2}}$ of the measure $\frac{1}{p}\mathbf{\Delta}\Gamma(f)^{\frac{p}{2}}$ is non-negative.

From hypothesis we know $\Gamma(f),\Gamma(f)^{\frac{p}{2}}\in D(\mathbf{\Delta})$, by chain rule we know

$$\int\varphi\,\mathrm{d}\mathbf{\Delta}\Gamma(f)^{\frac{p}{2}}=-\int\Gamma(\varphi,\Gamma(f)^{\frac{p}{2}})\,\mathrm{d}\mathfrak{m}=-\int\frac{p}{2}\Gamma(f)^{\frac{p}{2}-1}\Gamma(\varphi,\Gamma(f))\,\mathrm{d}\mathfrak{m}\tag{13}$$

for any Lipschitz function φ with bounded support.

Denote by $\overline{\Gamma(f)}$ the quasi-continuous representation of $\Gamma(f)$. From Leibniz rule and chain rule we know $\varphi(\Gamma(f)+\epsilon)^{\frac{p}{2}-1}\in W^{1,2}$, for any $\epsilon>0$. According to Definition 3.1 we have

$$\begin{aligned}-\int\varphi(\overline{\Gamma(f)}+\epsilon)^{\frac{p}{2}-1}\,\mathrm{d}\mathbf{\Delta}\Gamma(f)&=\int\Gamma(\varphi(\Gamma(f)+\epsilon)^{\frac{p}{2}-1},\Gamma(f))\,\mathrm{d}\mathfrak{m}\\ =\int\varphi(\frac{p}{2}-1)(\Gamma(f)+\epsilon)^{\frac{p}{2}-2}\Gamma(\Gamma(f))\,\mathrm{d}\mathfrak{m}&+\int(\Gamma(f)+\epsilon)^{\frac{p}{2}-1}\Gamma(\varphi,\Gamma(f))\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

Letting $\epsilon\to0$, by monotone convergence theorem we obtain

$$-\int\varphi\overline{\Gamma(f)}^{\frac{p}{2}-1}\,\mathrm{d}\mathbf{\Delta}\Gamma(f)=\int\Big[\varphi(\frac{p}{2}-1)\Gamma(f)^{\frac{p}{2}-2}\Gamma(\Gamma(f))+\Gamma(f)^{\frac{p}{2}-1}\Gamma(\varphi,\Gamma(f))\Big]\,\mathrm{d}\mathfrak{m}.\tag{14}$$

Combining (13) and (14) we have

$$\frac{1}{p}\mathbf{\Delta}\Gamma(f)^{\frac{p}{2}}=\frac{1}{2}\overline{\Gamma(f)}^{\frac{p}{2}-1}\mathbf{\Delta}\Gamma(f)+\frac{1}{2}(\frac{p}{2}-1)\Gamma(f)^{\frac{p}{2}-2}\Gamma(\Gamma(f))\mathrm{d}\mathfrak{m}\tag{15}$$

as measures. Therefore, we know

$$\begin{aligned}\frac{1}{p}\mathbf{\Delta}_{ac}\Gamma(f)^{\frac{p}{2}}&=\frac{1}{2}\overline{\Gamma(f)}^{\frac{p}{2}-1}\mathbf{\Delta}_{ac}\Gamma(f)+\frac{1}{2}(\frac{p}{2}-1)\Gamma(f)^{\frac{p}{2}-2}\Gamma(\Gamma(f))\mathrm{d}\mathfrak{m}\\ &=\frac{1}{2}\Gamma(f)^{\frac{p}{2}-1}\mathbf{\Delta}_{ac}\Gamma(f)+\frac{1}{2}(\frac{p}{2}-1)\Gamma(f)^{\frac{p}{2}-2}\Gamma(\Gamma(f))\mathrm{d}\mathfrak{m}\end{aligned}$$

and

$$\frac{1}{p}\mathbf{\Delta}_{sing}\Gamma(f)^{\frac{p}{2}}=\frac{1}{2}\overline{\Gamma(f)}^{\frac{p}{2}-1}\mathbf{\Delta}_{sing}\Gamma(f).$$

In conclusion, we obtain

$$\begin{aligned}&\frac{1}{p}\mathbf{\Delta}\Gamma(f)^{\frac{p}{2}}\\ =&\frac{1}{2}\Gamma(f)^{\frac{p}{2}-1}\mathbf{\Delta}_{ac}\Gamma(f)+\frac{1}{2}(\frac{p}{2}-1)\Gamma(f)^{\frac{p}{2}-2}\Gamma(\Gamma(f))\mathrm{d}\mathfrak{m}+\frac{1}{2}\overline{\Gamma(f)}^{\frac{p}{2}-1}\mathbf{\Delta}_{sing}\Gamma(f).\end{aligned}$$

Hence (10) is equivalent to (12), we prove the lemma. □

The following lemma will be used in the proof of Theorem 3.5.

Lemma 3.3. *Let $P(r) : [0,\infty) \mapsto [-\frac{1}{4},\infty)$ be a function defined as*

$$P(r) = r - \frac{1}{4(r+1)},$$

and $a_0 \geq 0$ be an arbitrary initial datum, we define $(a_n)_{n\in\mathbb{N}}$ recursively by the formula

$$a_{n+1} = P(a_n).$$

Then there exists an integer N_0 such that $0 \leq a_{N_0} < 1$ and $-\frac{1}{4} \leq a_{N_0+1} < 0$.

Conversely, for any $a \in [0,1)$ and $b > a$, there exists a sequence $a_0, ..., a_{N_0}$ defined by the recursive function P such that $a_0 > b$ and $a_{N_0} = a$.

Proof. It can be seen that $a_{n+1} < a_n$. If $a_0 \geq 0$, by monotonicity we know $a_n - a_{n+1} \in [\frac{1}{4(a_0+1)}, \frac{1}{4}]$ for any $n \in \mathbb{N}$. So there exists a unique N_0 such that $0 \leq a_{N_0} < 1$ and $-\frac{1}{4} \leq a_{N_0+1} < 0$. Conversely, since $P(r)$ is strictly monotone on $[0,\infty)$, we know $P^{-1}(r) : [-\frac{1}{4},\infty) \mapsto [0,\infty)$ is well defined. And $(P^{-1})^{(n+1)}(a) - (P^{-1})^{(n)}(a) \in [\frac{1}{4((P^{-1})^{(n+1)}(a)+1)}, \frac{1}{4}]$ for any $n \in \mathbb{N}$. Thus there exists $N \in \mathbb{N}$ such that $(P^{-1})^{(N_0)}(a) \geq b$. Finally, we can pick $a_0 = (P^{-1})^{(N_0)}(a)$, so that $a_{N_0} = (P)^{(N_0)}(a_0) = a$ fulfils our request. □

As we mentioned in the Introduction, the space of test functions is dense in $W^{1,2}(X,\mathrm{d},\mathfrak{m})$ when L^p-gradient estimate for heat flow holds.

Lemma 3.4 (Density of test functions in $W^{1,2}(X,\mathrm{d},\mathfrak{m})$, Remark 2.5 [5])**.** *Let $(X,\mathrm{d},\mathfrak{m})$ be a metric measure space satisfying Assumption 1. Assume that for any $f \in W^{1,2} \cap \mathrm{Lip} \cap L^\infty(X,\mathrm{d},\mathfrak{m})$ we have the L^p-gradient estimate*

$$|\mathrm{DH}_t f|^p \leq e^{-pKt}\mathrm{H}_t|\mathrm{D}f|^p \ \ \mathfrak{m} - a.e. \tag{16}$$

for some $p \in [1,\infty)$. Then the space of test functions $\mathrm{TestF}(X,\mathrm{d},\mathfrak{m})$ is dense in $W^{1,2}$.

Proof. As we discussed in the preliminary section, the space

$$\mathbb{V}^1 := \Big\{\varphi \in W^{1,2} : \Gamma(\varphi) \in L^\infty(X,\mathfrak{m})\Big\}$$

is dense in $W^{1,2}$. We also know that the

$$\mathbb{V}^1_\infty := \Big\{\varphi \in W^{1,2} \cap L^\infty : \Gamma(\varphi) \in L^\infty(X,\mathfrak{m})\Big\}$$

in dense in L^2, and $\mathbb{V}^1_\infty$ is invariant under the action $(\mathrm{H}_t)_t$ by (16) and Sobolev-to-Lipschitz property. Hence by an approximation argument (see e.g. Lemma 4.9 in [4]), we know $\mathbb{V}^1_\infty$ is dense in $W^{1,2}$. Similarly, by a semigroup mollification (see e.g. page 351, [5]) we can prove that

$$\mathbb{V}^2_\infty := \Big\{\varphi \in \mathbb{V}^1_\infty : \Delta\varphi \in W^{1,2} \cap L^\infty(X,\mathfrak{m})\Big\}$$

is dense in $W^{1,2}$. □

We now introduce the following technical assumption, which is important in our proof. It can be proved that Riemannian manifolds and $\mathrm{RCD}(K,\infty)$ spaces satisfy this assumption.

Assumption 2 (Existence of good algebra)**.** *We assume the existence of a dense subspace* $\mathcal{A}$ *in* $\mathrm{TestF}(X,\mathrm{d},\mathfrak{m})$ *with respect to the graph norm*

$$f \mapsto \left[\|(-\Delta)^{\frac{3}{2}} f\|_{L^2}^2 + \|f\|_{W^{1,2}}^2\right]^{\frac{1}{2}} = \left[\|\Gamma(\Delta f)\|_{L^2}^2 + \|f\|_{W^{1,2}}^2\right]^{\frac{1}{2}}$$

such that $\Gamma(f) \in \mathbb{M}_\infty$ *for any* $f \in \mathcal{A}$.

It can be seen that $\mathcal{A}$ is an algebra (i.e. $\mathcal{A}$ is closed w.r.t. pointwise multiplication), if it is non-trivial. In particular, by Lemma 3.4 we know that $\mathcal{A}$ is dense in $W^{1,2}$ if L^p gradient estimate holds.

Theorem 3.5 (Improved Bakry-Émery theory)**.** *Let* $(X,\mathrm{d},\mathfrak{m})$ *be a metric measure space satisfying Assumption 1 and Assumption 2. If for any* $f \in W^{1,2} \cap \mathrm{Lip} \cap L^\infty(X,\mathrm{d},\mathfrak{m})$ *we have the gradient estimate*

$$|\mathrm{DH}_t f|^p \leq e^{-pKt}\mathrm{H}_t |\mathrm{D}f|^p,\ \ \mathfrak{m}-a.e. \tag{17}$$

for some $p \in [1,\infty)$*. Then* $(X,\mathrm{d},\mathfrak{m})$ *is a* $\mathrm{RCD}(K,\infty)$ *space.*

Proof. If $p \leq 2$, by the result in [5] we know $(X,\mathrm{d},\mathfrak{m})$ is a $\mathrm{RCD}(K,\infty)$. Now we assume $p > 2$.

Part 1. Firstly, we prove

$$\Gamma(f)\mathbf{\Delta}_{ac}\Gamma(f) + \epsilon\Gamma(\Gamma(f)) \geq \Gamma(f)\Gamma(\Delta f, f) + K\Gamma(f)^2, \tag{18}$$

and $\overline{\Gamma(f)}\mathbf{\Delta}_{sing}\Gamma(f) \geq 0$, for any $f \in \mathcal{A}$ and $\epsilon > 0$.

For any $f \in \mathcal{A}, \varphi \in \mathrm{TestF}(X,\mathrm{d},\mathfrak{m}), \varphi \geq 0$ and $t > 0$, we define $F : [0,t] \mapsto \mathbb{R}$ by

$$F(s) = \int e^{-pKs}\mathrm{H}_s\varphi\Gamma(\mathrm{H}_{t-s}f)^{\frac{p}{2}}.$$

It can be seen that F is a C^1 function (see Lemma 2.1, [5]). From (17) we know $F(s) \leq F(t)$ holds for any $s \in [0,t]$. Hence $F'(s)_{|s=t} \geq 0$, and so

$$\begin{aligned}&\int e^{-pKs}\Delta\mathrm{H}_s\varphi\Gamma(\mathrm{H}_{t-s}f)^{\frac{p}{2}}{}_{|s=t} - p\int e^{-pKs}\mathrm{H}_s\varphi\Gamma(\mathrm{H}_{t-s}f)^{\frac{p}{2}-1}\Gamma(\Delta\mathrm{H}_{t-s}f,\mathrm{H}_{t-s}f)_{|s=t}\\ \geq\ & pK\int e^{-pKs}\mathrm{H}_s\varphi\Gamma(\mathrm{H}_{t-s}f)^{\frac{p}{2}}{}_{|s=t}.\end{aligned}$$

Letting $t \to 0$ we obtain

$$\int \Delta\varphi\Gamma(f)^{\frac{p}{2}} - p\int \varphi\Gamma(f)^{\frac{p}{2}-1}\Gamma(\Delta f, f) \geq pK\int \varphi\Gamma(f)^{\frac{p}{2}}.$$

In particular, from Lemma 2.6 and Lemma 3.2 in [15] we know $\Gamma(f)^{\frac{p}{2}} \in \mathrm{D}(\mathbf{\Delta})$ and

$$\frac{1}{p}\mathbf{\Delta}\Gamma(f)^{\frac{p}{2}} - \Gamma(f)^{\frac{p}{2}-1}\Gamma(\Delta f, f)\mathrm{d}\mathfrak{m} \geq K\Gamma(f)^{\frac{p}{2}}\mathrm{d}\mathfrak{m}. \tag{19}$$

By Lemma 3.2, we get that

$$\frac{1}{2}\Gamma(f)\Delta_{ac}\Gamma(f) + (\frac{p}{4} - \frac{1}{2})\Gamma(\Gamma(f)) \geq \Gamma(f)\Gamma(\Delta f, f) + K\Gamma(f)^2 \tag{20}$$

holds $\mathfrak{m}$-a.e., and $\overline{\Gamma(f)}\mathbf{\Delta}_{sing}\Gamma(f) \geq 0$.

From now on, all the inequalities are considered in $\mathfrak{m}$-a.e. sense. We denote $\frac{1}{2}\Delta_{ac}\Gamma(f) - \Gamma(\Delta f, f)$ by $\Gamma_2(f)$, and $\frac{1}{2}\Delta_{ac}\Gamma(f) - \Gamma(\Delta f, f) - K\Gamma(f)$ by $\Gamma_{2,K}(f)$, then (20) becomes

$$\Gamma_{2,K}(f)\Gamma(f) + (\frac{p}{4} - \frac{1}{2})\Gamma(\Gamma(f)) \geq 0.$$

For any real number $r \geq 0$, we say that the property $B(r)$ holds if

$$\Gamma_{2,K,r}(f) := \Gamma_{2,K}(f)\Gamma(f) + r\Gamma(\Gamma(f)) \geq 0$$

for any $f \in \mathrm{TestF}$. For example, (20) means $B(\frac{p}{4} - \frac{1}{2})$.

Now we define

$$P(r) = r - \frac{1}{4(r+1)}.$$

Then we will prove that $B(r)$ implies $B(P(r))$. We choose the smooth function $\Phi : \mathbb{R}^3 \mapsto \mathbb{R}$ defined by

$$\Phi(\mathbf{f}) := \lambda f_1 + (f_2 - a)(f_3 - b) - ab, \ a, b, \lambda \in \mathbb{R}.$$

Then we know

$$\begin{aligned} \Phi_{23}(\mathbf{f}) &= \Phi_{32} = a, \ \Phi_{ij}(\mathbf{f}) = 0, \ \text{if } (i,j) \notin \{(2,3),(3,2)\} \\ \Phi_1(\mathbf{f}) &= \lambda, \ \Phi_2(\mathbf{f}) = f_3 - b, \ \Phi_3(\mathbf{f}) = f_2 - a. \end{aligned}$$

If $\mathbf{f} := (f, g, h) \in \mathcal{A}^3$, we know $\Phi(\mathbf{f}) \in \mathcal{A}$ by Lemma 2.2. Hence we know

$$\Gamma_{2,K}(\Phi(\mathbf{f}))\Gamma(\Phi(\mathbf{f})) + r\Gamma(\Gamma(\Phi(\mathbf{f}))) \geq 0. \tag{21}$$

By direct computation using Lemma 2.3 (see also Theorem 3.4, [15]), we have

$$\begin{aligned} \Gamma(\Phi(\mathbf{f})) &= g^{ij}\Phi_i\Phi_j(\mathbf{f}) \\ &= \lambda^2\Gamma(f) + (g-a)A_1 + (h-b)B_1 \end{aligned}$$

where $g^{ij} = \Gamma(f_i, f_j)$, A_1, A_2 are some additional terms.

Similarly, we have

$$\begin{aligned} & \Gamma(\Gamma(\Phi(\mathbf{f}))) \\ = \ & \Gamma(g^{ij}\Phi_i\Phi_j(\mathbf{f})) \\ = \ & (g^{ij})^2\Gamma(\Phi_i\Phi_j) + (\Phi_i\Phi_j)^2\Gamma(g^{ij}) + 2g^{ij}\Phi_i\Phi_j\Gamma(g^{ij}, \Phi_i\Phi_j) \\ = \ & (g^{ij})^2\Big[\Phi_i^2\Gamma(\Phi_j) + \Phi_j^2\Gamma(\Phi_i) + 2\Phi_i\Phi_j\Gamma(\Phi_i, \Phi_j)\Big] \\ & +(\Phi_i\Phi_j)^2\Gamma(g^{ij}) + 2g^{ij}\Phi_i\Phi_j\Gamma(g^{ij}, \Phi_i\Phi_j) \\ = \ & 2(g^{12})^2\lambda^2\Gamma(h) + 2(g^{13})^2\lambda^2\Gamma(g) + \lambda^4\Gamma(g^{11}) + (g-a)A_2 + (h-b)B_2 \\ = \ & 2\Gamma(f,g)^2\lambda^2\Gamma(h) + 2\Gamma(f,h)^2\lambda^2\Gamma(g) + \lambda^4\Gamma(\Gamma(f)) + (g-a)A_2 + (h-b)B_2. \end{aligned}$$

We also know (see Theorem 3.4, [15] or Lemma 3.3.7, [10]) that

$$\begin{aligned} \Gamma_2(\mathbf{f}) - K\Gamma(\Phi(\mathbf{f})) &= \lambda^2\Gamma_2(f) + 4\lambda\mathrm{Hess}[f](g,h) + 2\Big(\Gamma(g)\Gamma(h) + \Gamma(g,h)^2\Big) \\ &+ \ (g-a)A_3 + (h-b)B_3 - K\lambda^2\Gamma(f). \end{aligned}$$

Combining the computations above, (21) becomes an inequality with parameters a, b, λ. By locality of weak gradients and density of simple functions, we can replace b by h and replace a by g (similar arguments are used in Theorem 3.4 [15] and Lemma 3.3.7 [10]). Then we obtain the following inequality from (21)

$$\begin{aligned} & \lambda^2\Gamma(f)\Big[\lambda^2\Gamma_2(f) + 4\lambda\mathrm{Hess}[f](g,h) + 2\Big(\Gamma(g)\Gamma(h) + \Gamma(g,h)^2\Big) - K\lambda^2\Gamma(f)\Big] \\ + \ & r\left[2\Gamma(f,g)^2\lambda^2\Gamma(h) + 2\Gamma(f,h)^2\lambda^2\Gamma(g) + \lambda^4\Gamma(\Gamma(f))\right] \\ \geq \ & 0. \end{aligned}$$

Since $r \geq 0$ and

$$\Gamma(g)\Gamma(h) \geq \Gamma(g,h)^2,$$

we know

$$\begin{aligned} & \Gamma(f)\left[\lambda^2\Gamma_2(f)+4\lambda\mathrm{Hess}[f](g,h)+4\big(\Gamma(g)\Gamma(h)\big)-K\lambda^2\Gamma(f)\right] \\ + \quad & r\left[4\Gamma(f)\Gamma(g)\Gamma(h)+\lambda^2\Gamma(\Gamma(f))\right] \\ \geq \quad & 0. \end{aligned}$$

Then we have

$$(\Gamma_2(f)\Gamma(f)+r\Gamma(\Gamma(f))-K\Gamma(f)^2)\lambda^2+4\lambda\Gamma(f)\mathrm{Hess}[f](g,h)+4(r+1)\Gamma(f)\Gamma(g)\Gamma(h)\geq 0.$$

Applying Lemma 2.4 we obtain

$$(1+r)\Gamma_{2,K,r}\Gamma(f)\Gamma(g)\Gamma(h)\geq \Gamma(f)^2\mathrm{Hess}[f](g,h).$$

Since $B(r)$ means $\Gamma_{2,K,r}\geq 0$, this inequality is equivalent to

$$(1+r)\Gamma_{2,K,r}(f)\Gamma(g)\Gamma(h)\geq \Gamma(f)\mathrm{Hess}[f](g,h). \tag{22}$$

Recall that $2\mathrm{Hess}[f](g,h)=\Gamma(g,\Gamma(f,h))+\Gamma(h,\Gamma(f,g))-\Gamma(f,\Gamma(g,h))$, we know

$$\mathrm{Hess}[f](g,h)+\mathrm{Hess}[g](f,h)=\Gamma(\Gamma(f,g),h).$$

Combining with inequality (22) we have

$$\begin{aligned} \sqrt{\frac{1}{1+r}\Gamma(\Gamma(f,g),h)\sqrt{\Gamma(f)}} \quad & \leq \quad \sqrt{\Gamma_{2,K,r}(f)\Gamma(g)\Gamma(h)}+\sqrt{\Gamma_{2,K,r}(g)\Gamma(f)\Gamma(h)} \\ & = \quad \Big(\sqrt{\Gamma_{2,K,r}(f)\Gamma(g)}+\sqrt{\Gamma_{2,K,r}(g)\Gamma(f)}\Big)\sqrt{\Gamma(h)}. \end{aligned}$$

Then we fix $f,g\in\mathcal{A}$, and approximate any $h\in W^{1,2}\cap L^\infty$ with a sequence $(h_n)\subset\mathcal{A}$ converging to h strongly in $W^{1,2}$ such that

$$\Gamma(h_n)\to\Gamma(h),\ \Gamma(h_n,\Gamma(f,g))\to\Gamma(h,\Gamma(f,g))$$

pointwise and in $L^1(X,\mathfrak{m})$. Thus we can replace h by $\Gamma(f,g)$ in the last inequality and obtain

$$\sqrt{\frac{1}{1+r}}\sqrt{\Gamma(\Gamma(f,g))\Gamma(f)}=\Big(\sqrt{\Gamma_{2,K,r}(f)\Gamma(g)}+\sqrt{\Gamma_{2,K,r}(g)\Gamma(f)}\Big). \tag{23}$$

Let $g=f$ in (23), we obtain

$$\frac{1}{1+r}\Gamma(\Gamma(f))\Gamma(f)\leq 4\Gamma_{2,K,r}(f)\Gamma(f).$$

Therefore,

$$(\frac{1}{4}\frac{1}{1+r}-r)\Gamma(\Gamma(f))\Gamma(f)\leq\Gamma_{2,K}(f)\Gamma(f).$$

In other words, we have $B(P(r))$.

From Lemma 3.3 we know there exists $a_0\geq\frac{p}{4}-\frac{1}{2}$ and $N_0\in\mathbb{N}$ such that $a_{N_0}=\epsilon$, where $a_{n+1}=P(a_n)$, $n=0,...,N_0-1$. Then we know $B(a_0)$ from (20). From the result above, we can see that $B(a_{N_0})$ holds by induction. So we prove (18).

Part 2. From (18) and Lemma 3.2 we know

$$\frac{1}{p_n}\mathbf{\Delta}\Gamma(f)^{\frac{p_n}{2}}-\Gamma(f)^{\frac{p_n}{2}-1}\Gamma(\Delta f,f)\,\mathfrak{m}\geq K\Gamma(f)^{\frac{p_n}{2}}\,\mathfrak{m} \tag{24}$$

for any $p_n=2+\frac{1}{2^n}$, $n\in\mathbb{N}$.

Let $f\in\mathcal{A}$, $\varphi\in\mathrm{TestF}$ and $\varphi\geq 0$. From (24) we know

$$\int\frac{1}{p_n}\Delta\varphi\Gamma(f)^{\frac{p_n}{2}}\,\mathrm{d}\mathfrak{m}-\int\varphi\Gamma(f)^{\frac{p_n}{2}-1}\Gamma(\Delta f,f)\mathrm{d}\mathfrak{m}\geq K\int\varphi\Gamma(f)^{\frac{p_n}{2}}\,\mathrm{d}\mathfrak{m}.$$

Letting $n \to \infty$, by dominated convergence theorem and monotone convergence theorem we know

$$\frac{1}{2}\int \Delta\varphi\Gamma(f)\,\mathrm{d}\mathfrak{m} - \int \varphi\Gamma(\Delta f, f)\,\mathrm{d}\mathfrak{m} \geq K\int \varphi\Gamma(f)\,\mathrm{d}\mathfrak{m}. \tag{25}$$

Combining with the density of $\mathcal{A}$ in TestF, we know (25) holds for all $f \in \mathrm{TestF}$. Finally, by Theorem 4.17 [5] we know that $(X, \mathrm{d}, \mathfrak{m})$ is a $\mathrm{RCD}(K, \infty)$ space. □

As a corollary, we have the following proposition. We recall (see [10]) that the measure-valued Ricci tensor on RCD metric measure space is defined as

$$\mathbf{Ricci}(\nabla f, \nabla f) := \mathbf{\Gamma}_2(f) - |\mathrm{Hess}[f]|^2_{\mathrm{HS}}\,\mathfrak{m},$$

where $\mathbf{\Gamma}_2(f) := \frac{1}{2}\mathbf{\Delta}\Gamma(f) - \Gamma(f, \Delta f)\,\mathfrak{m}$ and $|\mathrm{Hess}[f]|_{\mathrm{HS}}$ is the minimal L^2 function G such that $|\sum_{i,j}\mathrm{Hess}[f](g_i, h_j)| \leq G\sqrt{\sum_{i,j}\Gamma^2(g_i, h_j)}$ for any $(g_i), (h_j) \subset \mathrm{TestF}$ (see [10] and [15] for details). It is proved that **Ricci** is well defined for any $f \in \mathrm{TestF}(X, \mathrm{d}, \mathfrak{m})$ when $(X, \mathrm{d}, \mathfrak{m})$ is RCD.

Proposition 3. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a* RCD *space. Then the following characterizations are equivalent.*

1) $(X, \mathrm{d}, \mathfrak{m})$ *is* $\mathrm{RCD}(K, \infty)$,
2) *for any test function* $f \in \mathrm{TestF}$ *we have* $\mathbf{Ricci}(\nabla f, \nabla f) \geq K|\mathrm{D}f|^2\,\mathfrak{m}$ *in the sense that*
$$\mathrm{Ricci}_{ac}(\nabla f, \nabla f) \geq K|\mathrm{D}f|^2\ \mathfrak{m} - a.e.$$
and $\mathbf{Ricci}_{sing}(\nabla f, \nabla f) \geq 0$.
3) *for any test function* $f \in \mathrm{TestF}$ *we have*
$$|\mathrm{D}f|^2\mathrm{Ricci}_{ac}(\nabla f, \nabla f) \geq K|\mathrm{D}f|^4\ \mathfrak{m} - a.e.$$
and $\mathbf{Ricci}_{sing}(\nabla f, \nabla f) \geq 0$.

Proof. 1) ⇒ 2) is Lemma 3.6.2 [10], 2) ⇒ 3) is trivial. So we just need to prove 3) ⇒ 1).

From 3) we know $\Gamma_{2,K,0}(f) \geq 0$, $\mathfrak{m}$-a.e. for any $f \in \mathrm{TestF}$. Therefore $\Gamma_{2,K,r}(f) \geq 0$ for any $r > 0$. Using the same argument as in the proof of Theorem 3.5, we know $(X, \mathrm{d}, \mathfrak{m})$ is $\mathrm{RCD}(K, \infty)$. □

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