Characterizations of monotonicity of vector fields on metric measure space

Bang-Xian Han *

November 8, 2017

Abstract

There are two main aims of this paper. The first aim is to characterize the convexity of functions on metric measure space, so that we could link the existence of some special K-convex functions to the particular metric structure of the space, which is a new approach to deal with some rigidity theorems such as "splitting theorem" and "volume cone implies metric cone theorem". The second aim is to study the convexity/monotonicity of non-smooth vector fields on metric measure space. We introduce the notion of K-monotonicity which is stable under measured Gromov-Hausdorff convergence, then characterize the K-monotone vector fields in several equivalent ways.

Keywords: continuity equation, convex function, metric measure space, metric rigidity, monotone vector field, optimal transport.

Contents

*University of Bonn, Institute for Applied Mathematics, han@iam.uni-bonn.de

1 Introduction

In the past twenty years, the displacement convexity of functionals on Wasserstein space $\mathcal{W}_2 = (\mathcal{P}_2(X), W_2)$, i.e. geodesically convex functionals on the space of probability measures equipped with L^2- transportation distance, have been deeply studied and applied in many fields such as differential equation, probability theory, differential and metric geometry (see e.g. Ambrosio-Gigli-Savaré's "green book" [4] and Villani's encyclopedia [45] for an overview of related theories).

One of the most interesting functionals is the Boltzmann entropy. On a Riemannian manifold $(M, \mathrm{d}, \mathfrak{m})$, the convexity of the Boltzmann entropy (or relative entropy) $\mathrm{Ent}_{\mathfrak{m}}(\cdot)$ defined by

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) := \begin{cases} \int \rho \ln \rho \,\mathrm{d}\mathfrak{m} & \text{if } \ \mu = \rho\,\mathfrak{m}, \\ \infty & \text{otherwise} \end{cases}$$

characterizes the lower Ricci curvature bound of M (see [46]). It is proven (by Erbar in [20]) that the gradient flow of $\mathrm{Ent}_{\mathfrak{m}}$ in Wasserstein space could be identified with the heat flow in the following sense: let $\mathcal{H}_t(f)$ be the solution to the heat equation with initial datum f, $\tilde{\mathcal{H}}_t(f\mathfrak{m})$ be the Wasserstein gradient flow of $\mathrm{Ent}_{\mathfrak{m}}$ from $f\mathfrak{m} \in \mathcal{P}_2(M)$. then $\mathcal{H}_t(f)\,\mathfrak{m} = \tilde{\mathcal{H}}_t(f\mathfrak{m})$.

Moreover, the following well-known characterizations are equivalent (see [46]):

1) The uniform lower Ricci curvature bound: $\mathrm{Ric}_M \geq K$ for some $K \in \mathbb{R}$.

2) $\mathrm{Ent}_{\mathfrak{m}}$ is K-convex in Wasserstein space.

3) The existence of EVI_K-gradient flow of $\mathrm{Ent}_{\mathfrak{m}}$ from any initial measure.

4) The exponential contraction of the heat flows in Wasserstein distance:

$$W_2\big(\mu_t^1, \mu_t^2\big) \leq e^{-Kt} W_2(\mu_0^1, \mu_0^2), \quad \forall t > 0$$

holds for any two heat flows $\mu_t^i := \mathcal{H}_t(f^i)\mathfrak{m}, i = 1, 2$.

5) The existence of the heat kernel, and exponential contraction of the heat kernels in Wasserstein distance :

$$W_2\big(\rho_t(x, \mathrm{d}z)\mathfrak{m}(z), \rho_t(y, \mathrm{d}z)\mathfrak{m}(z)\big) \leq e^{-Kt}\mathrm{d}(x, y)$$

for any $x, y \in X$ and $t > 0$, where $\rho_t(x, \mathrm{d}z)\mathfrak{m}(z) = \tilde{\mathcal{H}}_t(\delta_x)$.

6) The gradient estimate of heat flow:

$$|\mathrm{D}\mathcal{H}_t(f)|^2(x) \leq e^{-2Kt}\mathcal{H}_t(|\mathrm{D}f|^2)(x), \quad \mathfrak{m} - \text{a.e. } x \in X$$

for any $f \in W^{1,2}(M)$.

In the last few years, the notion of curvature-dimension condition of (non-smooth) metric measure space, was proposed by Lott-Sturm-Villani (see [37] and [42,43]) . They use the characterization 2) above as a definition of synthetic lower Ricci curvature bound. Later on, the curvature-dimension condition was refined by Ambrosio-Gigli-Savaré (see [6] and [24]), which we call Riemannian curvature-dimension condition or RCD condition for short. It is known that the class of $\mathrm{RCD}(k,\infty)$ spaces includes weighted Riemannian manifolds satisfying curvature-dimension condition a la Bakry-Émery, as well as their measured Gromov-Hausdorff limits, and Alexandrov spaces.

In this RCD setting, there is a very natural generalized heat flow, which is the L^2-gradient flow of the Cheeger energy. In particular, all those chatacterizations on manifold are known to be valid in appropriate weak sense on metric measure space (see [5–7]). Furthermore, more entropy-like (internal energy) functionals have been studied in [8] and [21], which could be used to study more problems such as $\mathrm{RCD}^*(k,N)$ condition.

Besides the Boltzmann entropy (and other internal energy functionals), another important example is the (potential energy) functional

$$U(\cdot) : \mathcal{P}_2(X) \ni \mu \mapsto \int_X u(x)\,\mathrm{d}\mu(x),$$

where u is a lower semicontinuous function on $\mathbb{R}^n$ (or Hilbert space) whose negative part has squared-distance growth (see [4]). It is known that each of the characterizations concerning entropy/heat flow has a parallel description for $U(\cdot)$ and its gradient flow, and they characterize the convexity of u. Then we would like to know if we can characterize the convexity of $U(\cdot)$ in the setting of (non-smooth) metric measure space, in a similar way as we know about $\mathrm{Ent}_{\mathfrak{m}}(\cdot)$. In this direction, several results have been obatined by Sturm, Ketterer etc., in [44], [33], [41] and [27]. However, these results just answer our question partially. So a complete study of this problem is still needed, which is the first motivation of the current work.

On the other hand, in the study of Ricci-limit spaces, i.e. measured Gromov–Hausdorff limits of Riemannian manifolds with Ricci curvature uniformly bounded from below, two (almost) rigidity theorems "(almost) splitting theorem" and " (almost) volume cone implies (almost) metric cone theorem" play important roles (see Cheeger-Colding's papers [14–17]). In the proofs of these rigidity theorems on Ricci-limtit spaces, as well as on $\mathrm{RCD}(k,\infty)$ and $\mathrm{RCD}^*(k,N)$ spaces (see e.g. [19], [22]), the analysis on some special K-convex functions play key roles. For example, in "volume cone implies metric cone theorem" (see [14], [19]), the target function is the distance function $u := \frac{1}{2}\mathrm{d}^2(\cdot,\mathrm{O})$ where O is a fixed point. We know $\mathrm{Hess}_u = \mathrm{Id}_N$, such that u is a "N-convex function". In "splitting theorem" (see [18], [22]), the target function is the Busemann function associated to a line which is harmonic, so that it should be regarded as a "0-convex function". In the case of the above mentioned non-smooth metric measure spaces, due to lack of regularity, the metric property could not be obtained directly from the existence of these K-convex functions.

However, the results in [35,44] and [33] concerning K-convex functions could not be used directly to study the rigidity theorems, since the pre-request of applying those results seems to be too restrictive in our situation. This encourages us to study K-convex functions deeply in RCD setting, so that we can learn the metric property of the space directly from the analytical properties of some special K-convex functions.

Before introducing the main result of this article, we should clarify the relationship between (Wasserstein) gradient flow of $U(\cdot)$ and the flow generated by the (non-smooth) vector field ∇u, as we identified the heat flow and the gradient flow of entropy before.

On one hand, Ambrosio-Trevisan extend the famous Di Perna-Lions theory to $\mathrm{RCD}(k,\infty)$ metric measure spaces in [11], they prove that the continuity equation

$$\partial_t \mu_t + \nabla \cdot (-\mu_t \nabla u) = 0, \tag{1.1}$$

is well posed under some assumptions on the Sobolev regularity of u. They prove the existence and uniqueness of the solution to (1.1) for any initial condition $\mu_0 \in \mathcal{P}(X)$ with $\mu_0 \leq C_0 \mathfrak{m}$. They also prove the existence of the regular Lagrangian flow $(F_t)_{t\in[0,T)}$ such that the flows $F_t(x), t > 0$ is non-branching and $\mu_t = (F_t)_\sharp \mu_0 \leq C_1 \mathfrak{m}$.

On the other hand, in [25] Gigli and the author study the absolutely continuous curves in Wasserstein space through its corresponding continuity equation on metric measure space. It is proved in [25] that (μ_t) solves (1.1) if and only if it is a gradient flow of $U : \mu \mapsto \int u \,\mathrm{d}\mu$ in Wasserstein space. In other words, (μ_t) is the gradient flow of U if and only if the velocity field of its continuity equation is $-\nabla u$.

The main results of this paper show that the following characterizations are equivalent (see Theorem 3.12, Theorem 3.14), where u is a scaler function with appropriate a priori regularities.

1) u is *infinitesimally K-convex*, i.e. $\mathrm{Hess}_u(\cdot,\cdot)$ which is the Hessian of u satisfies $\mathrm{Hess}_u(\nabla f, \nabla f) \geq K|\mathrm{D}f|^2$ $\mathfrak{m}$-a.e. for any $f \in W^{1,2}$.

2) u is *weakly K-convex*, i.e. $U(\cdot)$ is K-displacement convex.

3) ∇u is *K-monotone* in the sense that

$$\int \langle \nabla u, \nabla \varphi \rangle \,\mathrm{d}\mu^1 + \int \langle \nabla u, \nabla \varphi^c \rangle \,\mathrm{d}\mu^2 \geq K W_2^2(\mu^1, \mu^2).$$

for any $\mu^1, \mu^2 \in \mathcal{P}_2$ with bounded densities, where (φ, φ^c) is the Kantorovich potentials associated to (μ^1, μ^2). It can be seen that this concept is a natural generalization of the monotone vectors in Hilbert space.

4) The exponential contraction in Wasserstein distance:

$$W_2(\mu_t^1, \mu_t^2) \leq e^{-Kt} W_2(\mu_0^1, \mu_0^2), \quad \forall t > 0$$

holds for any two solutions $(\mu_t^1), (\mu_t^2)$ to the continuity equation (1.1), whose velocity fields are $-\nabla u$.

5) The regular Lagrangian flow (F_t) of $-\nabla u$ is well-defined on the entire space X, and exponential contraction:

$$\mathrm{d}(F_t(x), F_t(y)) \leq e^{-Kt}\mathrm{d}(x, y)$$

holds for any $x, y \in X$ and $t > 0$. We will see some applications of this property in Section 4.

6) For any $f \in W^{1,2}(X, \mathrm{d}, \mathfrak{m})$, we have $f \circ F_t \in W^{1,2}$ and

$$|\mathrm{D}(f \circ F_t)|(x) \leq e^{-Kt}|\mathrm{D}f| \circ F_t(x), \quad \mathfrak{m} - \text{a.e. } x \in X.$$

We divide the characterization theorem above into two theorems in Section 3, because the pre-requests on the regularity of u are slightly different. The first one is **Theorem** 3.12, which deals with the equivalence of 1) and 2). It has been proven (in e.g. [33], [27], [35]) when u is a test function (see section 2.2 for the definition). However, in many cases which are potential applications of the characterization theorem, e.g. in " splitting theorem" and "volume cone implies metric cone theorem", the functions only have lower differentiability and integrability. In Theorem 3.12, $u \in W^{2,2}_{\text{loc}}$ is only assumed to be locally bounded and $u(x) \geq -a - b\mathrm{d}^2(x, x_0)$ for some $a, b \in \mathbb{R}^+, x_0 \in X$. So it is possible to apply our characterization theorem to more functions on non-compact space.

The second one is **Theorem** 3.14, which deals with the equivalence of 2) $-$ 6). The well-posedness of this theorem requires the existence and uniqueness of regular Lagrangian flow on metric measure space, which is studied by Ambrosio-Trevisan (in [11]). For the potential application of the theorem, we also need to extend Ambrosio-Trevisan's result to a lager class of vector fields. This will be studied in Proposition 3.2. Consequently, we will see in **Theorem** 3.16 that the K-monotonicity of a (possibly) non-symmetric vector field $\mathbf{b}$ can be characterized in similar ways as $3), 4), 5), 6)$ above. We remark that these equivalent descriptions are new even on Riemannian manifold and Riemannian limit space. Due to lack of second order differentiation formula, and low regularity of the vector field, the usual argument in smooth setting fails to work under such non-smooth condition (see also Remark 3.18).

At last, we summarize the highlights and main innovations of this paper.

a) Equivalent characterizations to K-convexity of function.

b) Equivalent characterizations to K-monotonicity of non-symmetric vector field.

c) Improve the known results concerning K-convex function, and continuity equation on metric measure space.

d) Improve the understanding of K-convex function on Riemannian manifold.

e) A new approach to study rigidity theorems on spaces with lower Ricci curvature bound.

The organization of this paper is as following. In section 2 we review some basic results on optimal transport, Sobolev spaces and (co)tangent modules on metric measure space, and continuity equation on metric measure space studied in [11], [25]. In section 3, we prove our main theorems which characterize the K-convexity of functions and K-monotonicity of vector fields on metric measure spaces. In the last section, we apply our characterization theorem to prove two results, which are key steps in the proofs of "splitting theorems" and "from volume cone to metric cone theorem".

2 Preliminaries

2.1 Metric measure space and optimal transport

We recall some basic results concerning analysis on metric spaces and optimal transport theory. More detailed discussions could be found in [2], [4] and [45]. Basic assumptions on the metric measure space in this paper are:

Assumption 2.1. The metric measure space $M := (X, \mathrm{d}, \mathfrak{m})$ satisfies:

i) (X, d) is a complete and separable geodesic metric space,

ii) $\operatorname{supp} \mathfrak{m} = X$,

iii) $\mathfrak{m}$ is a d-Borel measure and gives finite value on bounded sets,

iv) $(X, \mathrm{d}, \mathfrak{m})$ has exponential volume growth: $\int e^{-\lambda \mathrm{d}^2(x,x_0)} \, \mathrm{d}\mathfrak{m}(x) < \infty$ for some $\lambda > 0, x_0 \in X$.

The local Lipschitz constant $\mathrm{lip}(f) : X \to [0, \infty]$ of a function f is defined by

$$\mathrm{lip}(f)(x) := \begin{cases} \overline{\lim}_{y \to x} \frac{|f(y)-f(x)|}{\mathrm{d}(x,y)}, & x \text{ is not isolated} \\ 0, & \text{otherwise.} \end{cases}$$

The space of continuous curves on $[0,1]$ with values in X is denoted by $\mathrm{C}([0,1], X)$ and equipped with the uniform distance. Its subspace consisting of geodesics is denoted by $\mathrm{Geo}(X)$. For $t \in [0,1]$ we denote by $e_t : \mathrm{C}([0,1], X) \mapsto X$ the "evaluation map" defined by

$$e_t(\gamma) := \gamma_t, \qquad \forall \gamma \in \mathrm{C}([0,1], X).$$

A curve $\gamma : [0,1] \to X$ is called absolutely continuous if there exists $f \in L^1([0,1])$ such that

$$\mathrm{d}(\gamma_s, \gamma_t) \le \int_t^s f(r) \, \mathrm{d}r, \qquad \forall t, s \in [0,1], \ t < s. \tag{2.1}$$

For an absolutely continuous curve γ, it can be proved that the limit $\lim_{h \to 0} \frac{\mathrm{d}(\gamma_{t+h}, \gamma_t)}{|h|}$ exists for a.e. t and thus defines a function, called metric speed and denoted by $|\dot\gamma_t|$, which is in $L^1([0,1])$. If $|\dot\gamma_t| \in L^2([0,1])$, we say that the curve is 2-absolutely continuous and denote the set of 2-absolutely continuous by $\mathrm{AC}^2([0,1], X)$.

The space of Borel probability measures on X is denoted by $\mathcal{P}(X)$ and $\mathcal{P}_2(X) \subset \mathcal{P}(X)$ is the space of probability measures with finite second order moment, i.e. $\mu \in \mathcal{P}_2(X)$ if $\mu \in \mathcal{P}(X)$ and $\int \mathrm{d}^2(x, x_0)\,\mathrm{d}\mu(x) < +\infty$ for some $x_0 \in X$. We equip $\mathcal{P}_2(X)$ with the L^2-transportation distance W_2, or 2-Wasserstein distance defined by:

$$W_2^2(\mu, \nu) := \inf \int \mathrm{d}^2(x, y)\,\mathrm{d}\pi(x, y), \tag{2.2}$$

where the inf is taken among all $\pi \in \mathcal{P}(X^2)$ whose marginals are μ, ν.

The measures which attain the infimum are called optimal transport plans and denoted by $\mathrm{Opt}(\mu, \nu)$. Given $\varphi : X \to \mathbb{R} \cup \{-\infty\}$, which is not identically $-\infty$, the c-transform $\varphi^c : X \to \mathbb{R} \cup \{-\infty\}$ is defined by

$$\varphi^c(y) := \inf_{x \in X} \frac{\mathrm{d}^2(x, y)}{2} - \varphi(x).$$

φ is said to be c-concave if it is not identically $-\infty$ and $\varphi = \psi^c$ for some $\psi : X \to \mathbb{R} \cup \{-\infty\}$. It is known that for $\mu, \nu \in \mathcal{P}_2(X)$, $W_2^2(\mu, \nu)$ can be obtained as maximization of the dual problem

$$\frac{1}{2} W_2^2(\mu, \nu) = \sup \int \varphi\,\mathrm{d}\mu + \int \varphi^c\,\mathrm{d}\nu, \tag{2.3}$$

where the sup is taken among all c-concave functions φ. Notice that the integrals on the right hand side are well posed because for any c-concave function φ and $\mu, \nu \in \mathcal{P}_2(X)$ we always have $\max\{\varphi, 0\} \in L^1(\mu)$ and $\max\{\varphi^c, 0\} \in L^1(\nu)$. The sup can be achieved and any maximizing φ is called Kantorovich potential from μ to ν. For any Kantorovich potential we have in particular $\varphi \in L^1(\mu)$ and $\varphi^c \in L^1(\nu)$. Equivalently, the sup in (2.3) can be taken among all $\varphi : X \to \mathbb{R}$ Lipschitz and bounded.

Absolutely continuous curves in $(\mathcal{P}_2, W_2)$ can be characterized by the following proposition:

Proposition 2.2 (Superposition principle, [36])**.** *Let* (X, d) *be a complete and separable metric space, and* $(\mu_t)_{t \in [0,1]} \in \mathrm{AC}^2([0, 1], \mathcal{P}_2)$*. Then there exists a measure* $\Pi \in \mathcal{P}(\mathrm{C}([0, 1], X))$ *concentrated on* $\mathrm{AC}^2([0, 1], X)$ *such that:*

$$\begin{aligned} (e_t)_\sharp \pi &= \mu_t, && \forall t \in [0, 1] \\ \int |\dot\gamma_t|^2\,\mathrm{d}\pi(\gamma) &= |\dot\mu_t|^2, && \textit{a.e. } t. \end{aligned}$$

Such a measure Π *associated to the curve* (μ_t) *is called a lifting of* (μ_t)*.*

2.2 Sobolev space and tangent module

The Sobolev space $W^{1,2}(M)$ is defined as in [5]. We say that $f \in L^2(X, \mathfrak{m})$ is a Sobolev function in $W^{1,2}(M)$ if there exists a sequence of Lipschitz functions

functions $\{f_n\} \subset L^2$, such that $f_n \to f$ and $\mathrm{lip}(f_n) \to G$ in L^2 for some $G \in L^2(X, \mathfrak{m})$. It is known that there exists a minimal function G in $\mathfrak{m}$-a.e. sense. We call this minimal G the minimal weak upper gradient (or weak gradient for simplicity) of f, and denote it by $|\mathrm{D}f|$. It is known that the locality holds for $|\mathrm{D}f|$, i.e. $|\mathrm{D}f| = |\mathrm{D}g|$ $\mathfrak{m}$-a.e. on the set $\{x \in X : f(x) = g(x)\}$. Similarly, we define local Sobolev space $W^{1,2}_{\mathrm{loc}}(M)$ which consists of functions $f \in L^2_{\mathrm{loc}}$ such that for any open set Ω with bounded closure, $f \in W^{1,2}(\Omega)$.

As a consequence of the definition above, we have the lower semi-continuity: if $(f_n)_n \subset W^{1,2}$ converge to some $f \in L^2$ in $\mathfrak{m}$-a.e. sense and such that $(|\mathrm{D}f_n|)_n$ is bounded in $L^2(X, \mathfrak{m})$, then $f \in W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ and

$$|\mathrm{D}f| \le G, \qquad \mathfrak{m}\text{-a.e.},$$

for every L^2-weak limit G of some subsequence of $(|\mathrm{D}f_n|)_n$.

We equip $W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ with the norm

$$\|f\|^2_{W^{1,2}(X,\mathrm{d},\mathfrak{m})} := \|f\|^2_{L^2(X,\mathfrak{m})} + \||\mathrm{D}f|\|^2_{L^2(X,\mathfrak{m})}.$$

It is known that $W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ is a Banach space, but not necessary a Hilbert space. We say that $(X, \mathrm{d}, \mathfrak{m})$ is an infinitesimally Hilbertian space if $W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ is a Hilbert space.

On an infinitesimally Hilbertian space M, we have a natural pointwise bilinear map defined by

$$[W^{1,2}(M)]^2 \ni (f, g) \mapsto \Gamma(f, g) := \frac{1}{4}\Big(|\mathrm{D}(f+g)|^2 - |\mathrm{D}(f-g)|^2\Big).$$

We have the following Leibniz rule (see Proposition 3.17 in [24] for a proof):

$$\Gamma(fg, h) = f\Gamma(g, h) + g\Gamma(f, h)$$

for any $f, g, h \in W^{1,2} \cap L^\infty$.

Then we can define the measure-valued Laplacian by duality (integration by part).

Definition 2.3 (Measure valued Laplacian, [23, 24])**.** The space $\mathrm{D}(\mathbf{\Delta}) \subset L^2_{\mathrm{loc}}(M)$ is the space of $f \in W^{1,2}_{\mathrm{loc}}(M)$ such that there is a measure μ satisfying

$$\int h \,\mathrm{d}\mu = -\int \Gamma(h, f) \,\mathrm{d}\mathfrak{m}, \forall h : M \mapsto \mathbb{R}, \ \text{ Lipschitz with bounded support.}$$

In this case the measure μ is unique and we shall denote it by $\mathbf{\Delta} f$. If $\mathbf{\Delta} f \ll m$, we denote its density by Δf. If $\Delta f \in L^2$, it can be seen that

$$\int \varphi \Delta f \,\mathrm{d}\mathfrak{m} = -\int \Gamma(\varphi, f) \,\mathrm{d}\mathfrak{m}$$

for any $\varphi \in W^{1,2}$.

Let $(f_n)_{n=0}^\infty \subset \mathrm{D}(\mathbf{\Delta})$. We say that (f_n) converge to f_∞ in $\mathrm{D}(\mathbf{\Delta})$ if $\Delta f_n \to \Delta f_\infty$ in L^2.

Remark 2.4. We do not assume that $\mathbf{\Delta} f$ has bounded total variation in this paper. Similarly, Δf is not necessarily L^1-integrable, but locally integrable.

We have the following proposition characterizing the curvature-dimensions conditions $\mathrm{RCD}(k,\infty)$ and $\mathrm{RCD}^*(k,N)$ through non-smooth Bakry-Émery theory. We recall that a space is $\mathrm{RCD}(k,\infty)/\mathrm{RCD}^*(k,N)$ if it is a $\mathrm{CD}(K,\infty)/\mathrm{CD}^*(K,N)$ space which are defined by Lott-Sturm-Villani in [37, 42, 43] and Bacher-Sturm in [12], equipped with an infinitesimally Hilbertian Sobolev space. For more details, see [6] and [3].

We define $\mathrm{TestF}(M) \subset W^{1,2}(M)$, the set of test functions by

$$\mathrm{TestF}(M) := \Big\{ f \in \mathrm{D}(\mathbf{\Delta}) \cap L^\infty : f \in W^{1,2} \cap W^{1,\infty} \ \text{ and } \ \Delta f \in W^{1,2}(M) \cap L^\infty(M) \Big\}.$$

It is known that $\mathrm{TestF}(M)$ is dense in $W^{1,2}(M)$ when M is $\mathrm{RCD}(k,\infty)$.

Let $f, g \in \mathrm{TestF}(M)$. We know (see [40]) that $\Gamma(f,g) \in \mathrm{D}(\mathbf{\Delta})$, and the measure $\mathbf{\Gamma}_2(f,g)$ is well-defined by

$$\mathbf{\Gamma}_2(f,g) = \frac{1}{2}\mathbf{\Delta}\Gamma(f,g) - \frac{1}{2}\big(\Gamma(f,\Delta g) + \Gamma(g,\Delta f)\big)\,\mathfrak{m},$$

and we put $\mathbf{\Gamma}_2(f) := \mathbf{\Gamma}_2(f,f)$. Then we have the following Bochner inequality on metric measure space, which can be regarded as a variant definition of $\mathrm{RCD}(k,\infty)$ and $\mathrm{RCD}^*(k,N)$ conditions.

We recall the Sobolev-to-Lipschitz property, which is a fundamental prerequisite for Bakry-Émery theory, see [7] and [26] for more discussion about this property.

Definition 2.5 (Sobolev-to-Lipschitz property)**.** We say that a metric measure space $(X,\mathrm{d},\mathfrak{m})$ has Sobolev to Lipschitz property if for any function $f \in W^{1,2}(X)$ with $|\mathrm{D}f| \in L^\infty(X)$, we can find a function $\tilde{f}$ such that $f = \tilde{f}$ $\mathfrak{m}$-a.e. and $\mathrm{Lip}(\tilde{f}) = \operatorname{ess\,sup} |\mathrm{D}f|$.

Proposition 2.6 (Bakry-Émery condition, [6, 7], [21])**.** *Let $M = (X,\mathrm{d},\mathfrak{m})$ be a* $\mathrm{RCD}^*(k,N)$ *space with $k \in \mathbb{R}$ and $N \in [1,\infty]$. Then*

$$\mathbf{\Gamma}_2(f) \geq \Big(k|\mathrm{D}f|^2 + \frac{1}{N}(\Delta f)^2\Big)\,\mathfrak{m}$$

for any $f \in \mathrm{TestF}(M)$.

Conversely, let $M = (X,\mathrm{d},\mathfrak{m})$ be an infinitesimally Hilbertian space satisfying Sobolev-to-Lipschitz property, fulfils the Assumption 2.1. Then it is a $\mathrm{RCD}^*(k,N)$ *space with $k \in \mathbb{R}$ and $N \in [1,\infty]$ if*

$$\frac{1}{2}\int |\mathrm{D}f|^2 \Delta\varphi \,\mathrm{d}\mathfrak{m} - \int \langle \nabla f, \nabla \Delta f\rangle \varphi \,\mathrm{d}\mathfrak{m} \geq k \int |\mathrm{D}f|^2 \varphi \,\mathrm{d}\mathfrak{m} + \frac{1}{N}\int (\Delta f)^2 \varphi \,\mathrm{d}\mathfrak{m}$$

for any $\varphi \in \mathrm{D}_{L^\infty}(\Delta)$ and $f \in \mathrm{D}_{W^{1,2}}(\Delta)$, where

$$\mathrm{D}_{L^\infty}(\Delta) := \Big\{ \varphi : \Delta\varphi \in L^2 \cap L^\infty, \varphi \in W^{1,2} \cap L^\infty \Big\},$$

and

$$\mathrm{D}_{W^{1,2}}(\Delta) := \Big\{ \varphi : \varphi \in W^{1,2}, \Delta\varphi \in W^{1,2} \Big\}.$$

Next, we will review the concepts of "tangent/cotangent vector field" in non-smooth setting. Firstly we recall the definition and basic properties of L^∞-module.

Definition 2.7 (L^2-normed L^∞-module)**.** Let $M = (X, \mathrm{d}, \mathfrak{m})$ be a metric measure space. A L^2-normed $L^\infty(M)$ module is a Banach space $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ equipped with a bilinear map

$$\begin{aligned} L^\infty(M) \times \mathbf{B} &\mapsto \mathbf{B}, \\ (f, v) &\mapsto f \cdot v \end{aligned}$$

such that

$$\begin{aligned} (fg) \cdot v &= f \cdot (g \cdot v), \\ \mathbf{1} \cdot v &= v \end{aligned}$$

for every $v \in \mathbf{B}$ and $f, g \in L^\infty(M)$, where $\mathbf{1} \in L^\infty(M)$ is the function identically equals to 1 on X, and a "pointwise norm" $|\cdot| : \mathbf{B} \mapsto L^2(M)$ which maps $v \in \mathbf{B}$ to a non-negative function in $L^2(M)$ such that

$$\begin{aligned} \|v\|_{\mathbf{B}} &= \||v|\|_{L^2} \\ |f \cdot v| &= |f||v|, \quad \mathfrak{m} - \text{a.e.} \end{aligned}$$

for every $f \in L^\infty(M)$ and $v \in \mathbf{B}$.

It can be seen that the L^2-normed L^∞-module has the following properties:

Locality: for any $v \in \mathbf{B}$ and Borel sets $\{A_i\}_{i \in \mathbb{N}} \subset X$ we have

$$\chi_{A_i} \cdot v = 0, \quad \forall i \in \mathbb{N} \Rightarrow \chi_{\cup_i A_i} \cdot v = 0.$$

Gluing: for every sequence $(v_i)_{i \in \mathbb{N}} \subset \mathbf{B}$ and sequence of Borel sets $\{A_i\}_i$ such that

$$\chi_{A_i \cap A_j} \cdot v_i = \chi_{A_i \cap A_j} \cdot v_j, \quad \forall i, j, \quad \text{and} \quad \overline{\lim_{n \to \infty}} \|\sum_{i=1}^n \chi_{A_i} \cdot v_i\|_{\mathbf{B}} < \infty,$$

there exists $v \in \mathbf{B}$ such that

$$\chi_{A_i} \cdot v = \chi_{A_i} \cdot v_i, \quad \forall i, \quad \text{and} \quad \|v\|_{\mathbf{B}} \le \underline{\lim_{n \to \infty}} \|\sum_{i=1}^n \chi_{A_i} \cdot v_i\|_{\mathbf{B}}.$$

Then we define the tangent (and cotangent) modules of M, which are particular examples of L^2-normed module. We define the "Pre-Cotangent Module" $\mathcal{PCM}$ as the set consisting of the elements $\{(A_i, f_i)\}_{i \in \mathbb{N}}$, where $\{A_i\}_{i \in \mathbb{N}}$ is a Borel partition of X, and $\{f_i\}_i$ are Sobolev functions such that $\sum_i \int_{A_i} |\mathrm{D} f_i|^2 < \infty$.

We define an equivalence relation on $\mathcal{PCM}$ by

$$\{(A_i, f_i)\}_{i \in \mathbb{N}} \sim \{(B_j, g_j)\}_{j \in \mathbb{N}} \quad \text{if} \quad |\mathrm{D}(g_j - f_i)| = 0, \quad \mathfrak{m} - \text{a.e. on } A_i \cap B_j.$$

We denote the equivalence class of $\{(A_i, f_i)\}_{i \in \mathbb{N}}$ by $[(A_i, f_i)]$. In particular, we call $[(X, f)]$ the differential of a function $f \in W^{1,2}$ and denote it by $\mathrm{d}f$.

Then we define the following operations:

a) $[(A_i, f_i)] + [(B_i, g_i)] := [(A_i \cap B_j, f_i + g_j)]$,

b) Multiplication by scalars: $\lambda[(A_i, f_i)] := [(A_i, \lambda f_i)]$,

c) Multiplication by simple functions: $(\sum_j \lambda_j \chi_{B_j})[(A_i, f_i)] := [(A_i \cap B_j, \lambda_j f_i)]$,

d) Pointwise norm: $|[(A_i, f_i)]| := \sum_i \chi_{A_i} |\mathrm{D} f_i|$,

where χ_A denote the characteristic function on the set A.

It can be seen that all the operations above are continuous on $\mathcal{PCM}/\sim$ with respect to the norm $\|[(A_i, f_i)]\| := \sqrt{\int |[(A_i, f_i)]|^2 \,\mathrm{d}\mathfrak{m}}$ and the $L^\infty(M)$-norm on the space of simple functions. Therefore we can extend them to the completion of $(\mathcal{PCM}/\sim, \|\cdot\|)$ and we denote this completion by $L^2(T^*M)$. As a consequence of our definition, we can see that $L^2(T^*M)$ is the $\|\cdot\|$ closure of $\{\sum_{i\in I} a_i \mathrm{d} f_i : |I| < \infty, a_i \in L^\infty(M), f_i \in W^{1,2}\}$ (see Proposition 2.2.5 in [23]). It can also be seen from the definition and the infinitesimal Hilbertianity assumption on M that $L^2(T^*M)$ is a Hilbert space equipped with the inner product induced by $\|\cdot\|$. Moreover, $\Big(L^2(T^*M), \|\cdot\|, |\cdot|\Big)$ is a L^2-normed module according to the Definition 2.7, which we shall call cotangent module of M.

We define the tangent module $L^0(TM)$ as $\mathrm{Hom}_{L^\infty(M)}(L^2(T^*M), L^0(M))$, i.e. $T \in L^0(TM)$ if it is a linear map from $L^2(T^*M)$ to $L^0(M)$ as Banach spaces satisfying the L^∞-homogeneity:

$$T(fv) = fT(v), \quad \forall v \in L^2(T^*M), \quad f \in L^\infty(M),$$

and continuity:

$$T(v) \le G|v| \quad \mathfrak{m} - \text{a.e.}, \qquad \forall\, v \in L^2(T^*M)$$

for some $G \in L^0$. The smallest function G satisfying this property will be denoted by $|T|$. For example, for any $f \in W^{1,2}_{\mathrm{loc}}(M)$, we know that there exists an element in $L^0(TM)$ which we denote by ∇f such that $\nabla f(\mathrm{d}g) = \Gamma(f, g) \le |\mathrm{D}f||\mathrm{D}g|$ for any $g \in W^{1,2}$. So $|\nabla f| = |\mathrm{D}f| \in L^2_{\mathrm{loc}}$.

We define $L^2(TM)$ as the space consisting of vectors $T \in L^0(TM)$ such that $|T| \in L^2(M)$. It can be seen that $L^2(TM)$ has a natural L^2-normed $L^\infty(M)$-module structure, and it is isometric to $L^2(T^*M)$ both as a module and a Hilbert space. We denote the corresponding element of $\mathrm{d}f$ in $L^2(TM)$ by ∇f and call it the gradient of f (see also the Riesz theorem for Hilbert modules in Chapter 1 of [23]). The natural pointwise norm on $L^2(TM)$ (we also denote it by $|\cdot|$) satisfies $|\nabla f| = |\mathrm{d}f| = |\mathrm{D}f|$. It is also known that $\{\sum_{i\in I} a_i \nabla f_i : |I| < \infty, a_i \in L^\infty(M), f_i \in W^{1,2}\}$ is dense in $L^2(TM)$. In other words, since we have a pointwise inner product $\langle \cdot, \cdot \rangle : [L^2(T^*M)]^2 \mapsto L^1(M)$ such that

$$\langle \mathrm{d}f, \mathrm{d}g \rangle := \Gamma(f, g) = \frac{1}{4}\Big(|\mathrm{D}(f+g)|^2 - |\mathrm{D}(f-g)|^2\Big),$$

we can then define the gradient ∇g as the unique element in $L^2(TM)$ such that $\nabla g(\mathrm{d}f) := \langle \mathrm{d}f, \mathrm{d}g\rangle$, $\mathfrak{m}$-a.e. for every $f \in W^{1,2}(M)$. Therefore, $L^2(TM)$ inherits a pointwise inner product from $L^2(T^*M)$ and we still use $\langle\cdot,\cdot\rangle$ to denote it. We define $L^2_{\mathrm{loc}}(TM)$ as those $\mathbf{b} \in L^0(TM)$ such that $|\mathbf{b}| \in L^2_{\mathrm{loc}}(M)$. It can be seen that $L^2_{\mathrm{loc}}(TM)$ inherits a pointwise inner product from $L^2(TM)$.

Next we review the definition and basic properties of the covariant derivatives and Sobolev spaces $W^{2,2}(M), H^{2,2}(M)$ and $W^{1,2}_C(TM)$, $H^{1,2}_C(TM)$. It is proved in Lemma 3.2 of [40] that $\langle\nabla f, \nabla g\rangle \in \mathrm{D}(\boldsymbol{\Delta}) \subset W^{1,2}(M)$ for any $f, g \in \mathrm{TestF}(M)$. Therefore we can define the Hessian of $f \in \mathrm{TestF}(M)$, which is a bilinear map: $\mathrm{Hess}_f : \{\nabla g : g \in \mathrm{TestF}(M)\}^2 \mapsto L^0(M)$ by

$$2\mathrm{Hess}_f(\nabla g, \nabla h) = \langle\nabla g, \nabla\langle\nabla f, \nabla h\rangle\rangle + \langle\nabla h, \nabla\langle\nabla f, \nabla g\rangle\rangle - \langle\nabla f, \nabla\langle\nabla g, \nabla h\rangle\rangle \quad (2.4)$$

for any $g, h \in \mathrm{TestF}(M)$. It is known that Hess_f can be extended to a continuous symmetric $L^\infty(M)$-bilinear map on $[L^2(TM)]^2$ with values in $L^0(M)$.

We denote the pointwise scalar product of two tensors $X, Y \in L^2(TM)\otimes L^2(TM)$ by $X : Y$. It can be seen that $|X|^2_{\mathrm{HS}} := \sqrt{X : X}$ is the Hilbert-Schmidt norm of X. We recall that the distributional divergence can be defined through integration by part.

Definition 2.8 (Distributional divergence, [11, 23])**.** The domain of divergence $\mathrm{D}(\mathrm{div}) \subset L^2_{\mathrm{loc}}(TM)$ is the space of all $X \in L^2_{\mathrm{loc}}(TM)$ for which there exists a function $f \in L^2_{\mathrm{loc}}(X, \mathfrak{m})$ such that

$$\int fg\,\mathrm{d}\mathfrak{m} = -\int \langle X, \nabla g\rangle\,\mathrm{d}\mathfrak{m}, \qquad \forall g \quad \text{Lipschitz with bounded support.}$$

In this case, we call (the unique) f the divergence of X and denote it by $\mathrm{div}X$.

It can be seen (see section 2.3.3, [23]) that $\mathrm{div}(\varphi X) := \langle\nabla\varphi, X\rangle + f\mathrm{div}X$ for $\varphi \in \mathrm{Lip}(M) \cap L^\infty$ and $X \in \mathrm{D}(\mathrm{div})$.

Definition 2.9 (Sobolev space $W^{1,2}_{C,\mathrm{loc}}(TM)$)**.** The Sobolev space $W^{1,2}_{C,\mathrm{loc}}(TM)$ is the space of all $X \in L^2_{\mathrm{loc}}(TM)$ for which there exists a $T \in L^2_{\mathrm{loc}}(TM) \otimes L^2_{\mathrm{loc}}(TM)$ such that

$$\int hT : (\nabla g_1 \otimes \nabla g_2)\,\mathrm{d}\mathfrak{m} = -\int \langle X, \nabla g_2\rangle \mathrm{div}(h\nabla g_1) - h\mathrm{Hess}_{g_2}(X, \nabla g_1)\,\mathrm{d}\mathfrak{m}$$

for any $g_1, g_2, h \in \mathrm{TestF}(M)$. In this case we call T the covariant derivative of X and denote it by ∇X. We endow $W^{1,2}_{C,\mathrm{loc}}(TM)$ with the (extended) norm $\|\cdot\|_{W^{1,2}_C(TM)}$ defined by

$$\|X\|^2_{W^{1,2}_C(TM)} := \|X\|^2_{L^2(TM)} + \||\nabla X|_{\mathrm{HS}}\|^2_{L^2(M)}.$$

We define $W^{1,2}_C(TM)$ as those $X \in W^{1,2}_{C,\mathrm{loc}}(TM)$ with finite norm.

We recall that the class of test vector fields $\mathrm{TestV}(M) \subset L^2(TM)$ is defined as

$$\mathrm{TestV}(M) := \Big\{ \sum_{i=1}^{n} g_i \nabla f_i : n \in \mathbb{N}, f_i, g_i \in \mathrm{TestF}(M), i = 1, ..., n \Big\}.$$

It can be proved that $\mathrm{TestV}(M)$ is dense in $L^2(TM)$ when M is $\mathrm{RCD}(k, \infty)$ (see [23]).

It can be seen that $\mathrm{TestV}(M) \subset W_C^{1,2}(TM)$. In particular, for any $f \in \mathrm{TestF}(M)$ we have $\nabla f \in W_C^{1,2}(TM)$ and $(\nabla \nabla f)^b = \mathrm{Hess}_f$ where b is the isomorphism from $L^2(TM) \otimes L^2(TM)$ to $L^2(T^*M) \otimes L^2(T^*M)$.

We define $W^{2,2}_{\mathrm{loc}}(M)$ as the space of functions $f \in W^{1,2}_{\mathrm{loc}}(M)$ with $\nabla f \in W^{1,2}_{C,\mathrm{loc}}(TM)$, equipped with the (extended) norm

$$\|f\|^2_{W^{2,2}(M)} := \||\mathrm{D} f|\|^2_{L^2(M)} + \||\nabla \nabla f|_{\mathrm{HS}}\|^2_{L^2(M)}.$$

We define $W^{2,2}(M)$ as the subspace of $W^{2,2}_{\mathrm{loc}}(M)$ consisting of vectors with finite norm. We call $(\nabla \nabla f)^b$ the Hessian of f and denote it by Hess_f. It can be seen that this notation is compatible with (2.4) when $f \in \mathrm{TestF}$. We define $H^{2,2}(M) \subset W^{2,2}(M)$ as the $W^{2,2}$- closure of $\mathrm{TestF}(M)$.

Definition 2.10 (Sobolev space $H_C^{1,2}(TM)$)**.** We define the Sobolev space $H_C^{1,2}(TM) \subset W_C^{1,2}(TM)$ as the $W_C^{1,2}(TM)$-closure of $\mathrm{TestV}(M)$.

As an extension of the result in [40], we have the following proposition concerning $H_C^{1,2}(TM)$ vectors.

Proposition 2.11 (Proposition 3.4.6, [23])**.** *Let $X \in H_C^{1,2}(TM)$. Then $\langle X, Y \rangle \in W^{1,2}(M)/W^{1,2}_{\mathrm{loc}}(M)$ for any $Y \in W_C^{1,2}(TM)/W^{1,2}_{C,\mathrm{loc}}(TM)$. In particular,*

$$\nabla Y : (\nabla g \otimes \nabla h) = \langle \nabla g, \nabla \langle Y, \nabla h \rangle \rangle - \mathrm{Hess}_h(Y, \nabla g)$$

for any $h \in \mathrm{TestF}(M)$.

We define the symmetric part of ∇X by

$$\nabla^s X : (\nabla f \otimes \nabla g) := \frac{1}{2}\Big(\nabla X : (\nabla f \otimes \nabla g) + \nabla X : (\nabla g \otimes \nabla f) \Big)$$

for any $f, g \in \mathrm{TestF}(M)$. In particular, for $X \in W_C^{1,2}(TM)$ we know

$$\nabla^s X : (\nabla f \otimes \nabla f) = \langle \nabla f, \nabla \langle X, \nabla f \rangle \rangle - \frac{1}{2} \langle X, \nabla |\mathrm{D} f|^2 \rangle$$

for any $f, g \in \mathrm{TestF}(M)$.

We have the following improved Bochner inequality, a more refined version for $\mathrm{RCD}^*(k, N)$ space could be found in [31].

Proposition 2.12 (Improved Bochner inequality, [23]). *Let $M = (X, \mathrm{d}, \mathfrak{m})$ be a* $\mathrm{RCD}(k, \infty)$ *space. Then for any $f \in \mathrm{TestF}(M)$ we have*

$$\mathbf{\Gamma}_2(f) \geq \left(K|\mathrm{D}f|^2 + |\mathrm{Hess}_f|^2_{\mathrm{HS}}\right)\mathfrak{m}$$

where $|\mathrm{Hess}_f|_{\mathrm{HS}}$ is the Hilbert-Schmidt norm of the Hessian (as a bi-linear map). In the case of $\mathrm{RCD}^*(k, N)$ *space, $|\mathrm{Hess}_f|_{\mathrm{HS}}$ can be computed by local coordinate (see Proposition 2.19 below).*

We also have the following important results.

Proposition 2.13 (Corollary 3.3.9, Proposition 3.3.18, [23]). *Let $M = (X, \mathrm{d}, \mathfrak{m})$ be a* $\mathrm{RCD}(k, \infty)$ *space. Then for any $f \in W^{1,2}(M)$ with $\Delta f \in L^2$, we have*

$$\||\mathrm{Hess}_f|_{\mathrm{HS}}\|^2_{L^2} \leq \|\Delta f\|^2_{L^2} - k\||\mathrm{D}f|\|^2_{L^2}.$$

Furthermore, we know $\overline{\left\{f : f \in W^{1,2}, \Delta f \in L^2\right\}}^{W^{2,2}} = H^{2,2} \subset W^{2,2}$.

At the end of this part, we review some useful knowledge about the dimension of M, which is understood as the dimension of $L^2(TM)$ as a L^∞-module. The readers who are familiar with the so-called "Lipschitz differentiable space" studied firstly by Cheeger, could find that the following results have their counterparts in [13].

Definition 2.14 (Local independence). Let B be a Borel set with positive measure. We denote the subset of $L^2(TM)$ consisting of those v such that $\chi_{B^c} v = 0$ by $L^2(TM)_{|_B}$. We say that $\{v_i\}_1^n \subset L^2(TM)$ is independent on B if

$$\sum_{i=1}^n f_i v_i = 0, \quad \mathfrak{m} - \text{a.e. on } B$$

holds if and only if $f_i = 0$ $\mathfrak{m}$-a.e. on B for each i.

Definition 2.15 (Local span and generators). Let B be a Borel set in X and $V := \{v_i\}_{i\in I} \subset L^2(TM)$. The span of V on B, denoted by $\mathrm{Span}_B(V)$, is the subset of $L^2(TM)_{|_B}$ with the following property: there exist a Borel decomposition $\{B_n\}_{n\in\mathbb{N}}$ of B, families of vectors $\{v_{i,n}\}_{i=1}^{m_n} \subset L^2(TM)$ and functions $\{f_{i,n}\}_{i=1}^{m_n} \subset L^\infty(M)$, $n = 1, 2, ...$, such that

$$\chi_{B_n} v = \sum_{i=1}^{m_n} f_{i,n} v_{i,n}$$

for each n. We call the closure of $\mathrm{Span}_B(V)$ *the space generated by V on B*.

We say that $L^2(TM)$ is finitely generated if there exists a finite set $\{v_1, ..., v_n\}$ spanning $L^2(TM)$ on X, and locally finitely generated if there is a partition $\{E_i\}$ of X such that $L^2(TM)_{|_{E_i}}$ is finitely generated for every $i \in \mathbb{N}$. It can be seen (in [23], Proposition 1.4.4) that we have well-defined basis and dimension on metric measure space.

Definition 2.16 (Local basis and dimension)**.** We say that a finite set $v_1, ..., v_n$ is a basis on Borel set B if it is independent on B and $\mathrm{Span}_B\{v_1, ..., v_n\} = L^2(TM)_{|B}$. If $L^2(TM)$ has a basis of cardinality n on B, we say that it has dimension n on B, or that its local dimension on B is n. If $L^2(TM)$ does not admit any local basis of finite cardinality on any subset of B with positive measure, we say that $L^2(TM)$ has infinite dimension on B.

Proposition 2.17 (Theorem 1.4.11, [23])**.** *Let $(X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}(k, \infty)$ metric measure space. Then there exists a unique decomposition $\{E_n\}_{n\in\mathbb{N}\cup\{\infty\}}$ of X such that*

- *For any $n \in \mathbb{N}$ and any $B \subset E_n$ with finite positive measure, $L^2(TM)$ has a unit orthogonal basis $\{e_{i,n}\}_{i=1}^n$ on B,*
- *For every subset B of E_∞ with finite positive measure, there exists a set of unit orthogonal vectors $\{e_{i,B}\}_{i\in\mathbb{N}\cup\{\infty\}} \subset L^2(TM)_{|B}$ which generates $L^2(TM)_{|B}$,*

where unit orthogonal of a countable set $\{v_i\}_i \subset L^2(TM)$ on B means $\langle v_i, v_j\rangle = \delta_{ij}$ $\mathfrak{m}$-a.e. on B.

Definition 2.18 (Analytic Dimension)**.** We say that the dimension of $L^2(TM)$ is k if $k = \sup\{n : \mathfrak{m}(E_n) > 0\}$ where $\{E_n\}_{n\in\mathbb{N}\cup\{\infty\}}$ is the decomposition given in Proposition 2.17. We define the analytic dimension of M as the dimension of $L^2(TM)$ and denote it by $\dim_{\max} M$.

Combining Proposition 3.2 in [31] and Proposition 2.13, we have the following result concerning the analytic dimension of $\mathrm{RCD}^*(k, N)$ space.

Proposition 2.19. *Let $M = (X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}^*(k, N)$ metric measure space. Then $\dim_{\max} M \leq N$. Furthermore, if the local dimension on a Borel set E is N, we have $\mathrm{trHess}_f(x) = \Delta f(x)$ $\mathfrak{m}$-a.e. $x \in E$ for every $f \in W^{1,2}(M)$ with $\Delta f \in L^2$.*

2.3 Continuity equation on metric measure space

In this part we introduce some recent results about the continuity equation on metric measure space, more detailed discussions could be found in [25]. We assume that the metric measure space $(X, \mathrm{d}, \mathfrak{m})$ is $\mathrm{RCD}(k, \infty)$. Under this assumption, we know $W^{1,2}(X, \mathrm{d}, \mathfrak{m})$ is separable (see [1]) so that the continuity equation could be defined pointwisely, and we can prove that Wasserstein geodesics are C^1-continuous.

We start by recalling the definition of weak solution to the continuity equation in non-smooth setting.

Definition 2.20 (Solutions to $\partial_t \mu_t = L_t$)**.** Let $(X, \mathrm{d}, \mathfrak{m})$ be a metric measure space. Assume that (μ_t) is a W_2-continuous curve with bounded compression (i.e. $\mu_t \leq C\mathfrak{m}$ for some constant C), and $\{L_t\}_{t\in[0,1]}$ is a family of maps from $S^2(X)$ to $\mathbb{R}$.

We say that (μ_t) solves the continuity equation

$$\partial_t \mu_t = L_t, \tag{2.5}$$

provided:

i) for a.e. $t \in [0,1]$, $S^2 \ni f \mapsto L_t(f)$ is a bounded linear functional, and $\|L_t\| \in L^2([0,1])$,

ii) for every $f \in L^1 \cap S^2(X)$ the map $t \mapsto \int f \, \mathrm{d}\mu_t$ is in absolutely continuous and the identity

$$\frac{\mathrm{d}}{\mathrm{d}t} \int f \, \mathrm{d}\mu_t = L_t(f),$$

holds for a.e. t.

In the following proposition we will see that the continuity equation characterizes 2-absolute continuity.

Proposition 2.21 (Continuity equation on metric measure space, [25]). *Let $(X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}(k, \infty)$ space, (μ_t) be a continuous curve with bounded compression in Wasserstein space. Then the following are equivalent.*

i) (μ_t) is 2-absolutely continuous w.r.t. W_2.

ii) There is a family of maps $\{L_t\}_{t\in[0,1]}$ from $S^2(X)$ to $\mathbb{R}$ such that (μ_t) solves the continuity equation (2.5).

Furthermore, if the above characterizations hold, we have

$$\|L_t\| = |\dot{\mu}_t|, \qquad a.e.\ t \in [0,1].$$

As an application of the Proposition 2.21, we can prove the following result concerning the derivative of $W_2^2(\cdot, \nu)$ along an absolutely continuous curve.

Proposition 2.22 (Derivative of $W_2^2(\cdot, \nu)$, Proposition 3.10, [25]). *Let $(X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}(k, \infty)$ space. Let $(\mu_t) \subset \mathcal{W}_2(X)$ be an absolutely continuous curve with bounded compression, ν with bounded support and notice that $t \mapsto \frac{1}{2}W_2^2(\mu_t, \nu)$ is absolutely continuous. Then the for a.e. $t \in [0,1]$ the formula*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t, \nu) = L_t(\varphi_t), \tag{2.6}$$

holds, where φ_t is any Kantorovich potential from μ_t to ν.

Next, we discuss more about the geodesics in Wasserstein space. Firstly, we review the Hopf-Lax formula.

Definition 2.23.

$$\mathrm{Q}_t(\phi)(x) := \begin{cases} \inf_{y\in X} c(x,y) + \phi(y) & t > 0 \\ \phi(x) & t = 0 \end{cases} \tag{2.7}$$

where $c(x,y) = \frac{\mathrm{d}^2(x,y)}{2t}$, $t > 0$.

It is known that $t \mapsto \mathrm{Q}_t(f)$ is a continuous semigroup for any lower semi-continuous and bounded function f. In particular, $\lim_{t\to 0} \mathrm{Q}_t(f) = f$. Furthermore, we have the following metric Hamilton-Jacobi equation.

Lemma 2.24 (Subsolution of Hamilton-Jacobi equation). *For every $x \in X$ it holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Q}_t(f)(x) + \frac{1}{2}|\mathrm{D}\mathrm{Q}_t(f)|^2(x) = 0$$

with at most countably many exceptions in $(0, +\infty)$.

We have the following proposition concerning the evolution of Kantorovich potentials by Hopf-Lax formula (see Theorem 7.36 in [45] or Theorem 2.18 in [2] for a proof).

Proposition 2.25 (Evolution of Kantorovich potentials). *Let (X, d) be a metric space, $(\mu_t)_t$ a W_2-geodesic in Wasserstein space and φ a Kantorovich potential from μ_0 to μ_1. Then for every $t \in [0, 1]$:*

1) the function $t\mathrm{Q}_t(-\varphi)$ is a Kantorovich potential from μ_t to μ_0,

2) the function $(1-t)\mathrm{Q}_{1-t}(-\varphi^c)$ is a Kantorovich potential from μ_t to μ_1.

Moreover, we know the evolution of Kantorovich potential is related to the continuity equation of the corresponding geodesic.

Proposition 2.26 (Geodesics, [25]). *Let (μ_t) be a geodesic with bounded compression such that μ_0, μ_1 have bounded supports, and φ a Kantorovich potential from μ_0 to μ_1 which are bounded supported. Then*

$$\partial_t \mu_t + \nabla \cdot (\nabla \phi_t \mu_t) = 0,$$

where $\phi_t := -\mathrm{Q}_{1-t}(-\varphi^c)$ for every $t \in [0, 1]$.

Similarly,

$$\partial_t \mu_t + \nabla \cdot (\nabla \varphi_t \mu_t) = 0,$$

where $\varphi_t := \mathrm{Q}_t(-\varphi)$ for every $t \in [0, 1]$.

At last, we recall a result about C^1-regularity of geodesics.

Proposition 2.27 (Weak C^1-regularity for geodesics, Proposition 5.7 [25] and Corollary 5.7 [22]). *Let $(\mu_t) \subset \mathcal{P}_2(X)$ be a geodesic with bounded compression. Assume further that μ_0, μ_1 have bounded supports. We denote the density of μ_t by ρ_t, then for any $t \in [0, 1]$ and any sequence $(t_n) \subset [0, 1]$ converging to t, there exists a subsequence (t_{n_k}) such that*

$$\rho_{t_{n_k}} \to \rho_t, \quad \mathfrak{m} - a.e.$$

as $k \to \infty$. Furthermore, (μ_t) is a weakly C^1 curve in the sense that $t \mapsto \int f \,\mathrm{d}\mu_t$ is C^1 for any $f \in W^{1,2}$.

3 Main results

3.1 Regular Lagrangian flow

In this part we firstly review the existence and uniqueness theory of continuity equation, and *regular Lagrangian flows* (RLF for short) on metric measure space studied by Ambrosio-Trevisan in [11]. Then we prove some basic properties which will be used in the proof of our main theorems.

Proposition 3.1 (Regular Lagrangian flow and continuity equation, Ambrosio-Trevisan, [11])**.** *Let* $\mathbf{b} \in L^2_{\mathrm{loc}}(TM)$ *be with* $|\mathbf{b}| \in L^2 + L^\infty$, $\mathbf{b} \in W^{1,2}_{C,\mathrm{loc}}(TM)$ *with* $|\nabla \mathbf{b}|_{\mathrm{HS}} \in L^2$, $\mathrm{div}\mathbf{b} \in L^2 + L^\infty$ *and* $\big(\mathrm{div}\mathbf{b}\big)^- \in L^\infty$*. There exists a measurable map (which we call regular Lagrangian flow)* $F : X \times [0,T] \mapsto X$ *such that*

1) F_t *is a semigroup in the sense that* $F_{t+s}(x) = F_t \circ F_s(x)$ *and* $F_0(x) = x$ $\mathfrak{m}$*-a.e. for any* $s,t \in [0,T]$*.*

2) There exists a constant $C_0(T)$ *such that* $(F_t)_\sharp(\mathfrak{m}) \le C_0\mathfrak{m}$ *for all* $t \in [0,T]$*.*

3) For any initial condition $\mu_0 = f\mathfrak{m}$ *with* $f \in L^1 \cap L^\infty$, $\mu_t := (F_t)_\sharp\mu_0$ *is a solution to the continuity equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int g\,\mathrm{d}\mu_t = \int \langle \nabla g, \mathbf{b}\rangle\,\mathrm{d}\mu_t, \quad L^1-a.e.\ t \in (0,T), \quad \lim_{t\to 0}\int g\,\mathrm{d}\mu_t = \int g\,\mathrm{d}\mu_0$$

for any $g \in \mathrm{Lip}(X,\mathrm{d}) \cap L^\infty$*. We also know that* $\frac{\mathrm{d}\mu_t}{\mathrm{d}\mathfrak{m}} \in L^1 \cap L^\infty$*.*

4) For $\mathfrak{m}$*-a.e.* x, $|\dot{F}_t|(x) = |\mathbf{b}|(x)$ *a.e.* $t \in (0,T)$*.*

5) Let $f \in W^{1,2}(X,\mathrm{d},\mathfrak{m})$*. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} f \circ F_t(x) = \langle \mathbf{b}, \nabla f\rangle \circ F_t(x)$$

for $L^1 \times \mathfrak{m}$*-a.e.* (t,x)*.*

6) Let $\mu_0 = f\mathfrak{m}$, $\mu_t = (F_t)_\sharp\mu_0$ *with* $f \in L^2$*. Then*

$$\left\|\frac{\mathrm{d}\mu_t}{\mathrm{d}\mathfrak{m}}\right\|_{L^2} \le e^{C_1 t}\|f\|_{L^2}$$

for some constant C_1 *which depends only on* $\|(\mathrm{div}\mathbf{b})^-\|_{L^\infty}$*.*

7) F_t *is unique/non-branching in the sense that if* $\bar{F}_t$ *is another map satisfying the properties above, then* $(\bar{F}_t)_\sharp\mu = (F_t)_\sharp\mu$ *for any* $\mu \in \mathcal{P}(X)$ *with bounded density.*

In some potential applications, we do not have the global $L^2 + L^\infty$-bound for $|\mathbf{b}|$, $\mathrm{div}\mathbf{b}$ or global L^∞-bound for $\big(\mathrm{div}\mathbf{b}\big)^-$. The following proposition tells us that the theory concerning the existence and uniqueness of regular Lagrangian flow still works in some special situations, see also Theorem 4.2, [27] for an example in this direction.

Proposition 3.2. *Let* $\mathbf{b} \in W^{1,2}_{C,\mathrm{loc}}(TM)$*. Assume that* $|\mathbf{b}| \leq C_0 \mathrm{d}(x, x_0) + C_1$ *for some* $C_0, C_1 > 0, x_0 \in X$*, and* $|\nabla \mathbf{b}|_{\mathrm{HS}} \in L^2(\Omega)$*,* $\mathrm{div}\mathbf{b} \in L^2(\Omega) + L^\infty(\Omega)$*,* $(\mathrm{div}\mathbf{b})^- \in L^\infty(\Omega)$ *for any bounded set* Ω*. Then there exists a unique regular Lagrangian flow associate to the vector field* $\mathbf{b}$*.*

Proof. Let $\mu \in \mathcal{P}_2(X)$ be an arbitrary measure with bounded density. We assume that $\mathrm{supp}\,\mu \in B_R(x_0)$ for some $R \geq 1$. Let χ be a cut-off function in Lemma 6.7, [9] such that χ is Lipschitz and

a) $0 \leq \chi \leq 1$, χ supports on $B_{3R}(x_0)$ and $\chi = 1$ on $B_{2R}(x_0)$,

b) $\Delta\chi \in L^\infty$ and $|\mathrm{D}\chi|^2 \in W^{1,2}$.

Then we know that $\chi\mathbf{b} \in \mathrm{D}(\mathrm{div})$, and it satisfies $\mathrm{div}(\chi\mathbf{b}) = \langle \mathbf{b}, \nabla\chi\rangle + \chi\mathrm{div}\mathbf{b} \in L^2 + L^\infty$, so that

$$\|\big(\mathrm{div}(\chi\mathbf{b})\big)^-\|_{L^\infty} \leq \||\mathbf{b}||\nabla\chi|\|_{L^\infty} + \|\chi\big(\mathrm{div}\mathbf{b}\big)^-\|_{L^\infty} < \infty,$$

and $|\chi\mathbf{b}| \in L^2 + L^\infty$, $\nabla(\chi\mathbf{b}) = \chi\nabla\mathbf{b} + \nabla\chi\otimes\mathbf{b} \in L^2(TM)\otimes L^2(TM)$. By Proposition 3.1 we know the regular Lagrangian flow associated to $\chi\mathbf{b}$ exists. We denote this flow by $\bar{F}_t$. We know that the curve $\mu_t := (\bar{F}_t)_\sharp\mu$ is the unique solution to the continuity equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_t + \mathrm{div}(\chi\mathbf{b}\mu_t) = 0, \qquad \mu_0 = \mu.$$

In particular, when $\mathrm{supp}\,\mu_t \subset B_{2R}$, we know $|\mathbf{b}|(x) \leq 2C_0R + C_1$ for $x \in \mathrm{supp}\,\mu_t$. From 4) of Proposition 3.1, we know $\mathrm{supp}\,\mu_t \subset B_{2R}$ when $t \in [0, \frac{2R-R}{2C_0R+C_1}]$. So

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_t + \mathrm{div}(\mathbf{b}\mu_t) = 0, \qquad t \in [0, \frac{1}{2C_0 + C_1}], \quad \mu_0 = \mu. \tag{3.1}$$

Then for any $T > 0$, we can find a solution to the continuity equation (3.1) for $t \in [0, T]$ by repeating the construction above for finite times. It can be seen from the construction that this solution is unique.

Finally, we can prove the existence and uniqueness of regular Lagrangian flow using Theorem 8.3 in [11] and the proof therein. □

For convenience, we will not distinguish the regular Lagrangian flow (F_t) and the curve of measures push-forward by F_t. We will see in Proposition 3.4 that the curve push-forward by F_t is C^1. To prove this result, we firstly recall a useful lemma.

Lemma 3.3 ("Weak-strong" convergence, Lemma 5.11, [22])**.** *Let* $(X, \mathrm{d}, \mathfrak{m})$ *be an infinitesimally Hilbertian space. Assume that:*

i) Let $(\mu_n) \subset \mathcal{P}(X)$ *be a sequence with uniformly bounded densities, such that* $\rho_n \to \rho$ $\mathfrak{m}$*-a.e. for some probability density* ρ*, where* ρ_n *is the density of* μ_n*.*

ii) Let $(f_n) \subset W^{1,2}$ *be a sequence such that:*

$$\sup_{n\in\mathbb{N}} \int |\mathrm{D}f_n|^2 \,\mathrm{d}\mathfrak{m} < \infty,$$

and assume that $f_n \to f$ $\mathfrak{m}$*-a.e. for some Borel function* f*.*

Then for any $\mathbf{b} \in L^2(TM)$*, we have*

$$\lim_{n\to\infty} \int \langle \nabla f_n, \mathbf{b} \rangle \, \mathrm{d}\mu_n = \int \langle \nabla f, \mathbf{b} \rangle \, \mathrm{d}\mu,$$

where $\mu := \rho\mathfrak{m}$*.*

Proof. If $\mathbf{b} = \nabla g$ for some $g \in W^{1,2}$, the assertion is proved in Lemma 5.11, [22]. For any $\epsilon > 0$, we can find $v_\epsilon \in \mathrm{TestV}$ with $v_\epsilon = \sum_i^N a_i g_i$ such that $\|\mathbf{b} - v_\epsilon\| < \epsilon$. Then we have

$$\begin{aligned}
\lim_{n\to\infty} \int \langle \nabla f_n, \mathbf{b} \rangle \, \mathrm{d}\mu_n &= \lim_{n\to\infty} \int \langle \nabla f_n, \mathbf{b} - v_\epsilon \rangle \, \mathrm{d}\mu_n + \lim_{n\to\infty} \int \langle \nabla f_n, v_\epsilon \rangle \, \mathrm{d}\mu_n \\
&\leq C\epsilon + \lim_{n\to\infty} \sum_i^N \int \langle \nabla f_n, \nabla g_i \rangle \, a_i \mathrm{d}\mu_n \\
&= C\epsilon + \sum_i^N \int \langle \nabla f, \nabla g_i \rangle \, a_i \mathrm{d}\mu \\
&\leq C\epsilon + C_1\epsilon + \int \langle \nabla f, \mathbf{b} \rangle \, \mathrm{d}\mu.
\end{aligned}$$

Letting $\epsilon \to 0$ and considering the opposite inequality we prove the assertion. □

Proposition 3.4. *Let* (F_t) *be a regular Lagrangian flow associated to* $\mathbf{b} \in W^{1,2}_{C,\mathrm{loc}}$ *defined as in Proposition 3.1. Assume that* μ_0 *has bounded density and bounded support. Then* $\mu_t := (F_t)_\sharp \mu_0$ *fulfils the hypothesis in Lemma 3.3 to be a* C^1 *curve.*

Proof. Let $\mu_t := (F_t)_\sharp \mu_0$ be a RLF with $\rho_t := \frac{\mathrm{d}\mu_t}{\mathrm{d}\mathfrak{m}} \in L^\infty$. By 6) of Proposition 3.1 we know $\overline{\lim}_{t\to 0} \|\rho_t\|_{L^2} \leq \|\rho_0\|_{L^2}$. It is known that the functional $\mathcal{P}_2 \ni \mu \mapsto \int \left(\frac{\mathrm{d}\mu_t}{\mathrm{d}\mathfrak{m}}\right)^2 \mathrm{d}\mathfrak{m}$ is lower semi-continuous w.r.t Wasserstein distance. So the function $t \mapsto \|\rho_t\|_2$ is lower semi-continuous. Then we have $\lim_{t\to 0} \|\rho_t\|_{L^2} = \|\rho_0\|_{L^2}$.

Since $\rho_t \to \rho_0$ weakly in duality with $C_b(X)$ and (ρ_t) are uniformly bounded in L^2. We know that $\rho_t \to \rho_0$ weakly in $L^2(X, \mathfrak{m})$. Combining with $\lim_{t\to 0} \|\rho_t\|_{L^2} = \|\rho_0\|_{L^2}$ we know $\rho_t \to \rho_0$ in L^2 strongly, and in L^p strongly for any $p \in [1, \infty)$.

From semi-group property, we know $t \mapsto \rho_t$ is continuous in L^1. For any t, $(t_n)_n \geq 0$ such that $t_n \to t$, we know there exists a subsequence $(t_{n_k})_k$ such that $\rho_{t_{n_k}} \to \rho_t$ $\mathfrak{m}$-a.e. as $k \to \infty$. Therefore, by Proposition 3.1 and Lemma 3.3 we get

$$\lim_{t_{n_k}\to t} \frac{\mathrm{d}}{\mathrm{d}r} \int f \, \mathrm{d}\mu_r|_{r=t_{n_k}} = \lim_{k\to\infty} \int \langle \nabla f, \mathbf{b} \rangle \, \mathrm{d}\mu_{t_{n_k}} = \int \langle \nabla f, \mathbf{b} \rangle \, \mathrm{d}\mu_t$$

for any $f \in W^{1,2}$. So (μ_t) is a C^1 curve . □

The following simple lemma is a complement to the Proposition 3.1.

Lemma 3.5. *Let* $f \in W^{1,2}$*,* $\mathbf{b} \in L^2_{\mathrm{loc}}(TM)$*. We assume that* $(F_t)_t$ *is the regular Lagrangian flow associated to* $\mathbf{b}$*. If* $f \circ F_t \in W^{1,2}$ *for any* $t > 0$*. Then for all* $t \in [0, T]$*,*

$$\langle \mathbf{b}, \nabla f \rangle \circ F_t(x) = \langle \mathbf{b}, \nabla (f \circ F_t)(x), \quad \mathfrak{m} - a.e.\ x \in X.$$

Proof. Let $\mu_0 \in \mathcal{P}(X)$ be an arbitrary measure with bounded density and bounded support. We define $\mu_t = (F_t)_\sharp \mu_0, t > 0$. From the definition of continuity equation and Proposition 3.1, we know

$$\frac{\mathrm{d}}{\mathrm{d}t} \int f \, \mathrm{d}\mu_t = \int \langle \mathbf{b}, \nabla f \rangle \, \mathrm{d}\mu_t = \int \langle \mathbf{b}, \nabla f \rangle \circ F_t \, \mathrm{d}\mu_0$$

for a.e. $t \in [0, T]$. By Proposition 3.4 above we know this formula holds for all t. Meanwhile, since $f \circ F_{t+h} \in W^{1,2}$ for any $h > 0$, we know

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}h} \int f \, \mathrm{d}\mu_{t+h}|_{h=0} &= \frac{\mathrm{d}}{\mathrm{d}h} \int f \circ F_t \, \mathrm{d}\mu_h|_{h=0} \\ &= \int \langle \mathbf{b}, \nabla (f \circ F_t) \rangle \, \mathrm{d}\mu_0. \end{aligned}$$

Then we have

$$\int \langle \mathbf{b}, \nabla (f \circ F_t) \rangle \, \mathrm{d}\mu_0 = \int \langle \mathbf{b}, \nabla f \rangle \circ F_t \, \mathrm{d}\mu_0.$$

As μ_0 is arbitrary, we know $\langle \mathbf{b}, \nabla f \rangle \circ F_t = \langle \mathbf{b}, \nabla (f \circ F_t)$, $\mathfrak{m}$-a.e.. $\square$

3.2 K-convex functions and K-monotone vectors

Firstly we introduce some notions/concepts to characterize the convexity of functions, and the monotonicity of vector fields in non-smooth setting.

The first one is a zero order characterization.

Definition 3.6 (Weak K-convexity)**.** Let $u \in L^1_{\mathrm{loc}}(X, \mathfrak{m})$. We say that u is *weakly K-convex* if the functional $U(\cdot) : \mathcal{P}_2 \ni \mu \mapsto \int_X u \, \mathrm{d}\mu$ is K-convex on Wasserstein space in the sense that

$$U(\mu_t) \le (1-t)U(\mu_0) + tU(\mu_1) - \frac{K}{2}(1-t)tW_2^2(\mu_0, \mu_1) \tag{3.2}$$

for any $t \in [0, 1]$ along any geodesic $(\mu_t) \subset (\mathcal{P}_2, W_2)$, where μ_0, μ_1 have bounded densities and bounded supports.

The second one is a first order characterization.

Definition 3.7 (K-monotonicity)**.** We say that a vector field $\mathbf{b} \in L^2_{\mathrm{loc}}(TM)$ is *K-monotone* if

$$\int \langle \mathbf{b}, \nabla \varphi \rangle \, \mathrm{d}\mu^1 + \int \langle \mathbf{b}, \nabla (\varphi)^c \rangle \, \mathrm{d}\mu^2 \ge K W_2^2(\mu^1, \mu^2).$$

for any $\mu^1, \mu^2 \in \mathcal{P}_2$ with bounded densities and bounded supports, where (φ, φ^c) is the Kantorovich potentials relative to (μ^1, μ^2).

Remark 3.8. If $\mathbf{b} \in L^2(TM)$, by the following metric Brenier's theorem, we can replace the condition "$\mu^1, \mu^2 \in \mathcal{P}_2$ with bounded densities and bounded densities" in Definition 3.7 by "bounded densities" only.

Similarly, by metric Brenier's theorem we can rephrase Definition 3.6 as: for any $\mu_0, \mu_1 \in \mathcal{P}(X)$ with bounded densities and bounded supports, there exists a geodesic $(\mu_t) \subset (\mathcal{P}_2, W_2)$ connecting μ_0, μ_1 such that the inequality (3.2) holds.

Proposition 3.9 (Metric Brenier's theorem, [6,39])**.** *Let $(X, \mathrm{d}, \mathfrak{m})$ be a* $\mathrm{RCD}(k, \infty)$ *metric measure space, and $\mu, \nu \in \mathcal{P}_2$ be absolutely continuous w.r.t. $\mathfrak{m}$. Let φ be a Kantorovich potential relative to (μ, ν). Then the geodesic connecting μ and ν is unique. The lifting Π of this geodesic (μ_t) is induced by a map and Π concentrates on a set of non-branching geodesics. Moreover, for Π-a.e. $\gamma \in \mathrm{Geo(X)}$ we have*

$$\mathrm{d}(\gamma_0, \gamma_1) = \mathrm{lip}(\varphi)(\gamma_0) = |\mathrm{D}\varphi|(\gamma_0).$$

In particular, we have

$$W_2^2(\mu, \nu) = \int |\mathrm{D}\varphi|^2 \, \mathrm{d}\mu.$$

Moreover, we know the locality of Kantorovich potentials, i.e.

$$|\mathrm{D}(\varphi - \bar{\varphi})| = 0 \qquad \mathfrak{m} - a.e. \ on \ \operatorname{supp} \mu$$

for any $\varphi, \bar{\varphi}$ which are both Kantorovich potentials from μ to ν.

Next, we introduce the concept of infinitesimal K-monotonicity of a vector field $\mathbf{b} \in W^{1,2}_{C,\mathrm{loc}}(TM)$, which is a second order characterization. We recall that the Hessian of a test function f could be defined by

$$2\mathrm{Hess}_f(\nabla g_1, \nabla g_2) = \langle \nabla \langle \nabla f, \nabla g_1 \rangle, \nabla g_2 \rangle + \langle \nabla \langle \nabla f, \nabla g_2 \rangle, \nabla g_1 \rangle - \langle \nabla \langle \nabla g_1, \nabla g_2 \rangle, \nabla f \rangle,$$

and the covariant derivative of a vector field $\mathbf{b} \in W^{1,2}_{C,\mathrm{loc}}(TM)$ can be represented as:

$$\nabla \mathbf{b} : (\nabla g_1 \otimes \nabla g_2) = \langle \nabla \langle \mathbf{b}, \nabla g_2 \rangle, \nabla g_1 \rangle - \mathrm{Hess}_{g_2}(\nabla g_1, \mathbf{b}),$$

where $g_1, g_2 \in \mathrm{TestF}$.

Definition 3.10 (Infinitesimal K-monotonicity)**.** Let $\mathbf{b} \in W^{1,2}_{C,\mathrm{loc}}(TM)$ be a vector field. We say that $\mathbf{b}$ is *infinitesimally K-monotone* if

$$\nabla^s \mathbf{b} : (X \otimes X) = \nabla \mathbf{b} : (X \otimes X) \geq K|X|^2 \qquad \mathfrak{m} - \text{a.e.}$$

for any $X \in L^2(TM)$.

Definition 3.11 (Infinitesimal K-convexity)**.** We say that f is *infinitesimally K-convex* if $\nabla f \in W^{1,2}_{C,\mathrm{loc}}(TM)$ and ∇f is infinitesimally K-monotone. In other words, f is infinitesimally K-convex if $f \in W^{2,2}_{\mathrm{loc}}$ and $\mathrm{Hess}_f(\nabla g, \nabla g) \geq K|\mathrm{D}g|^2$ for any $g \in \mathrm{TestF}$.

Next we prove the first theorem in this article. When $u \in \mathrm{TestF}$, this result has been proven in Theorem 7.1 in [33] (see also Lemma 2.1 in [35], Theorem 3.3 in [27]). In Theorem 3.12, we will remove some bounds on $u, \nabla u$, and the condition $\Delta u \in W^{1,2}$ in the former proofs. Similar to the former ones in [33], [35] etc., the proof of the current theorem is also based on Bochner's inequality on metric measure space and the original definition of $\mathrm{CD}(k,\infty)$ condition.

Theorem 3.12. *Let $M := (X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}(k,\infty)$ metric measure space, $u \in W^{2,2}_{\mathrm{loc}}(X, \mathrm{d}, \mathfrak{m})$. Assume further that $u \in L^\infty_{\mathrm{loc}}(M)$ and $u(x) \geq -a - b\mathrm{d}^2(x, x_0)$ for some $a, b \in \mathbb{R}$, $x_0 \in X$. Then the following are equivalent*

i) u is infinitesimally K-convex,

ii) u is weakly K-convex.

Proof. First of all, we rewrite the Bochner's formula in Proposition 2.6 in the following weak form. We recall that $\mathrm{D}_{L^\infty}(\Delta) := \Big\{\varphi : \Delta\varphi \in L^2 \cap L^\infty, \varphi \in W^{1,2} \cap L^\infty\Big\}$.

For any $f \in \mathrm{TestF}(M)$, $\varphi \in \mathrm{D}_{L^\infty}(\Delta)$, we define

$$\begin{aligned}
\Gamma_2(f;\varphi) &:= \int \varphi \,\mathrm{d}\mathbf{\Gamma}_2(f) && (3.3)\\
&= \frac{1}{2}\int \varphi \,\mathrm{d}\mathbf{\Delta}|\mathrm{D}f|^2 - \int \langle \nabla f, \nabla \Delta f\rangle \varphi \,\mathrm{d}\mathfrak{m} && (3.4)\\
&= -\frac{1}{2}\int \langle \nabla |\mathrm{D}f|^2, \nabla\varphi\rangle \,\mathrm{d}\mathfrak{m} - \int \langle \nabla f, \nabla \Delta f\rangle \varphi \,\mathrm{d}\mathfrak{m} && (3.5)\\
&= \frac{1}{2}\int |\mathrm{D}f|^2 \Delta\varphi \,\mathrm{d}\mathfrak{m} - \int \langle \nabla f, \nabla \Delta f\rangle \varphi \,\mathrm{d}\mathfrak{m}. && (3.6)
\end{aligned}$$

If $\varphi \in \mathrm{Lip}$, we know $\varphi\nabla f \in \mathrm{D}(\mathrm{div})$, hence

$$\begin{aligned}
\Gamma_2(f;\varphi) &= \frac{1}{2}\int |\mathrm{D}f|^2 \Delta\varphi \,\mathrm{d}\mathfrak{m} + \int \mathrm{div}(\varphi\nabla f)\Delta f \,\mathrm{d}\mathfrak{m}\\
&= \frac{1}{2}\int |\mathrm{D}f|^2 \Delta\varphi \,\mathrm{d}\mathfrak{m} + \int (\Delta f)^2\varphi \,\mathrm{d}\mathfrak{m} + \int \langle \nabla\varphi, \nabla f\rangle \Delta f \,\mathrm{d}\mathfrak{m}\\
&=: \Gamma_2'(f;\varphi).
\end{aligned}$$

By Proposition 2.6 we know

$$\Gamma_2'(f;\varphi) = \Gamma_2(f;\varphi) \geq k\int |\mathrm{D}f|^2\varphi \,\mathrm{d}\mathfrak{m} \tag{3.7}$$

for any $f \in \mathrm{TestF}(M)$, $\varphi \in \mathrm{D}_{L^\infty}(\Delta) \cap \mathrm{Lip}$, $\varphi \geq 0$.

We denote the space of test functions with bounded support by $\mathrm{TestF}_{\mathrm{bs}}(M) \subset \mathrm{TestF}(M)$, then we will see that $\mathrm{TestF}_{\mathrm{bs}}(M)$ is dense in $\mathrm{TestF}(M)$.

Let $\chi_n \in \mathrm{TestF}$, $n \in \mathbb{N}$ be cut-off functions (see Lemma 6.7, [9]) such that

a) $0 \leq \chi_n \leq 1$, χ_n supports on $B_{3n}(x_0)$ and $\chi_n = 1$ on $B_n(x_0)$,

b) $\mathrm{Lip}(\chi_n) \leq \frac{1}{n}$,

c) $\Delta\chi_n \in L^\infty$ uniformly in n and $|\mathrm{D}\chi_n|^2 \in W^{1,2}$.

For any $f \in \mathrm{TestF}$ we define $f_n := \chi_n f$. Then we know $\nabla f_n = f\nabla\chi_n + \chi_n \nabla f$, $\Delta f_n = f\Delta\chi_n + \chi_n\Delta f + 2\langle\nabla f, \nabla\chi_n\rangle$. Hence we know $f_n \in \mathrm{TestF}_{\mathrm{bs}}$, $f_n \to f$ in $W^{1,2}$ and $\Delta f_n \to \Delta f$ in L^2. So we know $\mathrm{TestF}_{\mathrm{bs}}$ is dense in TestF with respect to both $W^{1,2}$ and $\mathrm{D}(\mathbf{\Delta})$ topology.

We define $\mathrm{TestF}(M^u)$ as the space of test functions on $M^u := (X, \mathrm{d}, e^{-u}\mathfrak{m})$. Since u is locally bounded, we know $W^{1,2}_{\mathrm{loc}}(M) = W^{1,2}_{\mathrm{loc}}(M^u)$. It can be seen (by Leibniz rule) that

$$\Big\{f \in \mathrm{D}(\mathbf{\Delta}) : \mathbf{\Delta} f \in L^2_{\mathrm{loc}}(M)\Big\} \cap L^\infty_{\mathrm{loc}}(M) = \Big\{f \in \mathrm{D}(\mathbf{\Delta}^{M^u}) : \mathbf{\Delta}^{M^u} f \in L^2_{\mathrm{loc}}(M^u)\Big\} \cap L^\infty_{\mathrm{loc}}(M^u),$$

and $\mathbf{\Delta}^{M^u} f = \mathbf{\Delta} f - \langle \nabla u, \nabla f\rangle\, \mathfrak{m}$ for any $f \in \big\{f \in \mathrm{D}(\mathbf{\Delta}) : \mathbf{\Delta} f \in L^2_{\mathrm{loc}}(M)\big\} \cap L^\infty_{\mathrm{loc}}(M)$. In fact, for any $f \in \big\{f \in \mathrm{D}(\mathbf{\Delta}) : \mathbf{\Delta} f \in L^2_{\mathrm{loc}}(M)\big\} \cap L^\infty_{\mathrm{loc}}(M)$ and $\varphi \in \mathrm{Lip}$ with bounded support, we know

$$\begin{aligned}
\int \langle \nabla f, \nabla\varphi\rangle e^{-u}\,\mathrm{d}\mathfrak{m} &= \int \langle \nabla f, e^{-u}\nabla\varphi\rangle\,\mathrm{d}\mathfrak{m}\\
\text{(by Leibniz rule)} \quad &= \int \langle \nabla f, \nabla(e^{-u}\varphi)\rangle\,\mathrm{d}\mathfrak{m} - \int \langle \nabla f, \nabla e^{-u}\rangle\varphi\,\mathrm{d}\mathfrak{m}\\
&= -\int e^{-u}\varphi\Delta f\,\mathrm{d}\mathfrak{m} + \int \langle\nabla f, \nabla u\rangle\varphi e^{-u}\,\mathrm{d}\mathfrak{m}.
\end{aligned}$$

So $f \in \mathrm{D}(\mathbf{\Delta}^{M^u})$ and $\Delta^{M^u} f = \Delta f - \langle\nabla u, \nabla f\rangle \in L^2_{\mathrm{loc}}$. Conversely, we can prove the assertion concerning $\mathbf{\Delta}^{M^u}$ in the same way.

For any $f \in \mathrm{TestF}(M^u)$, $\varphi \in \mathrm{D}_{L^\infty}(\Delta^{M^u}) \cap W^{1,2} \cap \mathrm{Lip}(M^u)$, we define $\Gamma_2^u(f;\varphi)$ as in (3.3) by replacing Δ by $\Delta^u := \Delta - \langle\nabla u, \nabla\cdot\rangle$, and $\mathfrak{m}$ by $e^{-u}\mathfrak{m}$. We claim that the following assertions are equivalent:

1) $\Gamma_2(f;\varphi) \geq k\int|\mathrm{D}f|^2\varphi\,\mathrm{d}\mathfrak{m}$, and $\int \mathrm{Hess}_u(\nabla f, \nabla f)\varphi\,\mathrm{d}\mathfrak{m} \geq K\int|\mathrm{D}f|^2\varphi\,\mathrm{d}\mathfrak{m}$ for any $f \in \mathrm{TestF}(M)$, $\varphi \in \mathrm{D}_{L^\infty}(\Delta) \cap \mathrm{Lip}(M)$, $\varphi \geq 0$,

2) $\Gamma_2(f;\varphi) \geq k\int|\mathrm{D}f|^2\varphi\,\mathrm{d}\mathfrak{m}$, and $\int \mathrm{Hess}_u(\nabla f, \nabla f)\varphi\,\mathrm{d}\mathfrak{m} \geq K\int|\mathrm{D}f|^2\varphi\,\mathrm{d}\mathfrak{m}$ for any $f \in \mathrm{TestF}_{\mathrm{bs}}(M)$, $\varphi \in \mathrm{D}_{L^\infty}(\Delta) \cap \mathrm{Lip}(M)$, $\varphi \geq 0$ with bounded support,

3) $\Gamma_2^{mu}(f;\varphi) \geq (mK + k)\int|\mathrm{D}f|^2\varphi\, e^{-mu}\mathrm{d}\mathfrak{m}$ for any $m \in \mathbb{N}$, $f \in \mathrm{TestF}_{\mathrm{bs}}(M)$, $\varphi \in \mathrm{D}_{L^\infty}(\Delta) \cap \mathrm{Lip}(M)$, $\varphi \geq 0$ with bounded support,

4) $\Gamma_2^{mu}(f;\varphi) \geq (mK + k)\int|\mathrm{D}f|^2\varphi\, e^{-mu}\mathrm{d}\mathfrak{m}$ for any $m \in \mathbb{N}$, $f \in \mathrm{D}_{W^{1,2}(M^u)}(\Delta^u)$, $\varphi \in \mathrm{D}_{L^\infty}(\Delta^u)$, $\varphi \geq 0$.

1) $\Longleftrightarrow$ 2) is a direct consequence of the density of $\mathrm{TestF}_{\mathrm{bs}}$ in TestF.

To prove 2) $\Longrightarrow$ 3), it is sufficient to prove

$$\Gamma_2^{mu}(f;\varphi) = \Gamma_2(f; e^{-mu}\varphi) + m\int \mathrm{Hess}_u(\nabla f, \nabla f)\varphi\, e^{-mu}\mathrm{d}\mathfrak{m}$$

for any $f \in \mathrm{TestF_{bs}}(M)$, $\varphi \in \mathrm{D}_{L^\infty}(\Delta) \cap \mathrm{Lip}(M)$, $\varphi \geq 0$. By Proposition 2.11 we know $\langle \nabla u, \nabla f\rangle \in W^{1,2}$, and the Hessian of $u \in W^{2,2}$ could be written in the form of (2.4). So by a direct computation we have

$$\begin{aligned}
\Gamma_2^{mu}(f;\varphi) &= \frac{1}{2}\int |\mathrm{D}f|^2 \Delta^{M^u}\varphi\, e^{-mu}\mathrm{d}\mathfrak{m} - \int \langle \nabla f, \nabla \Delta^{M^u} f\rangle \varphi\, e^{-mu}\mathrm{d}\mathfrak{m}\\
&= \frac{1}{2}\int |\mathrm{D}f|^2 (\Delta - m\nabla u)\varphi\, e^{-mu}\mathrm{d}\mathfrak{m} - \int \langle \nabla f, \nabla(\Delta f - m\langle \nabla u, \nabla f\rangle)\rangle \varphi\, e^{-mu}\mathrm{d}\mathfrak{m}\\
&= \Gamma_2(f; e^{-mu}\varphi) + m\int \mathrm{Hess}_u(\nabla f, \nabla f)\varphi\, e^{-mu}\mathrm{d}\mathfrak{m}.
\end{aligned}$$

Conversely, we claim that for any $\varphi \in \mathrm{D}_{L^\infty}(\Delta) \cap \mathrm{Lip}(M)$, $\varphi \geq 0$ with bounded support, we can find $\varphi_n \in \mathrm{D}_{L^\infty}(\Delta)\cap\mathrm{Lip}(M)$, $\varphi \geq 0$ with bounded support such that $\varphi_n e^{-mu} \to \varphi$ in $W^{1,2}$. For this aim, we use a well known approximation procedure. For any $f \in L^2$, we define

$$\mathrm{h}_\epsilon f := \frac{1}{\epsilon}\int_0^\infty \kappa(r/\epsilon)\mathcal{H}_r f\,\mathrm{d}r = \int_0^\infty \kappa(s)\mathcal{H}_{\epsilon s} f\,\mathrm{d}s. \quad \epsilon > 0,$$

where $(\mathcal{H}_t)$ is the heat flow, and $\kappa \in C_c^\infty((0,\infty))$ with $\kappa \geq 0$ and $\int_0^\infty \kappa(r)\,\mathrm{d}r = 1$. It can be checked that $\Delta \mathrm{h}_\epsilon f \in L^2 \cap L^\infty$, $\mathrm{h}_\epsilon f \in \mathrm{TestF}$ if $f \in L^2 \cap L^\infty$. We also know that $\mathrm{h}_\epsilon f \to f$ both in $W^{1,2}$ and in $\mathrm{D}(\boldsymbol{\Delta})$ as $\epsilon \downarrow 0$.

Now we turn back to our problem. Since u, e^{-u} are locally finite, we can approximate ηe^{mu} by test functions ϕ_n, where $\eta \in \mathrm{TestF}$ has bounded support and $\eta = 1$ on $\mathrm{supp}\,\varphi$. Then $\varphi_n := \varphi\phi_n$ achieve our aim. Assume that 3) holds, we know

$$\Gamma_2(f; e^{-mu}\varphi_n) + m\int \mathrm{Hess}_u(\nabla f, \nabla f)\varphi_n\, e^{-mu}\mathrm{d}\mathfrak{m} \geq (mK + k)\int |\mathrm{D}f|^2 \varphi_n\, e^{-mu}\mathrm{d}\mathfrak{m}$$

Letting $n \to \infty$, combining with (3.5) and L^2_{loc}-integrability of $|\mathrm{Hess}_u|_{\mathrm{HS}}$ we have

$$\Gamma_2(f;\varphi) + m\int \mathrm{Hess}_u(\nabla f, \nabla f)\varphi\,\mathrm{d}\mathfrak{m} \geq (mK + k)\int |\mathrm{D}f|^2\varphi\,\mathrm{d}\mathfrak{m}. \tag{3.8}$$

Letting $m = 0$, we know $\Gamma_2(f;\varphi) \geq k\int |\mathrm{D}f|^2\varphi\,\mathrm{d}\mathfrak{m}$. Dividing m on both sides of (3.8) and letting $m \to \infty$, we prove $\mathrm{Hess}_u \geq K$.

To prove 3) $\Longrightarrow$ 4) it is sufficient to approximate f, φ in 4). For this aim, we firstly assume that $f \in L^\infty$ and $\varphi \in \mathrm{Lip}$, then we can use the approximation technique above again. Let $f \in \mathrm{D}_{W^{1,2}(M^u)}(\Delta^u) \cap L^\infty$. For any $n \in \mathbb{N}$, we can find $a_n > n$ such that

$$\|\chi_{a_n} f - f\|_{W^{1,2}(M^u)} + \|\Delta^u(\chi_{a_n} f - f)\|_{L^2(M^u)} < \frac{1}{n}.$$

Since $\chi_{a_n} f \in L^2 \cap L^\infty(M)$, we know from the above mentioned approximation procedure that $\mathrm{h}_\epsilon(\chi_{a_n} f) \to \chi_{a_n} f$ both in $W^{1,2}(M)$ and in $\mathrm{D}(\boldsymbol{\Delta})$ as $\epsilon \downarrow 0$. In particular, we know $\chi_{a_n}\mathrm{h}_\epsilon(\chi_{a_n} f) \to \chi_{a_n}\chi_{a_n} f = \chi_{a_n} f$ in $W^{1,2}(M)$ and in $\mathrm{D}(\boldsymbol{\Delta})$. As both $\chi_{a_n}\mathrm{h}_\epsilon(\chi_{a_n} f)$ and $\chi_{a_n} f$ are bounded supported, we know the convergences also hold in $W^{1,2}(M^u)$ and in $\mathrm{D}(\boldsymbol{\Delta^u})$.

Therefore there exits $0 < b_n < \frac{1}{n}$ such that

$$\|\chi_{a_n} f \mathrm{h}_{b_n}(\chi_{a_n} f) - \chi_{a_n} f\|_{W^{1,2}(M^u)} + \|\Delta\big(\chi_{a_n} \mathrm{h}_{b_n}(\chi_{a_n} f)\big) - \Delta(\chi_{a_n} f)\|_{L^2(M^u)} < \frac{1}{n}.$$

We define $f_n := \chi_{a_n} \mathrm{h}_{b_n}(\chi_{a_n} f)$. It can be seen that $f_n \in \mathrm{TestF_{bs}}(M)$ and $f_n \to f$ in both in $W^{1,2}(M^u)$ and $\mathrm{D}(\mathbf{\Delta^u})$ as $n \to \infty$. Similarly, for any $\varphi \in \mathrm{D}_{L^\infty}(\Delta^u) \cap \mathrm{Lip}(M^u)$, $\varphi \geq 0$, we can define $\varphi_n := \chi_{a'_n} \mathrm{h}_{b'_n}(\chi_{a'_n}\varphi)$ in the same way for some a'_n, b'_n.

It can be checked that $\Gamma_2^u(f_n, \varphi_n) \to \Gamma_2^u(f, \varphi)$ and $\int |\mathrm{D} f_n|^2 \varphi_n \, e^{-u} \mathrm{d}\mathfrak{m} \to \int |\mathrm{D} f|^2 \varphi \, e^{-u} \mathrm{d}\mathfrak{m}$ as $n \to \infty$. Then we have 4) for such functions f, φ. By an approximation using heat flow, we can remove the assumption $\varphi \in \mathrm{Lip}$ (see e.g. Proposition 3.6, [27]). We can also remove the assumption $f \in L^\infty$ by a simple truncation argument (see e.g. Theorem 4.8, [21]). Then we prove 4) for all the required functions f and φ.

Finally, it can be checked that the test functions f, φ in 3) are included in the test functions in 4), so we also have 4) $\Longrightarrow$ 3).

Now we can complete the proof of the theorem:

<u>*i)* $\Longrightarrow$ *ii)*.</u>

If u is infinitesimally K-convex. Combining with the fact that M is $\mathrm{RCD}(k, \infty)$, we know 4) holds. As M has Sobolev-to-Lipschitz property, so $M^{mu} := (X, \mathrm{d}, e^{-mu}\mathfrak{m})$ also has such property. Since $u(x) \geq -a - b\mathrm{d}^2(x, x_0)$, we know $e^{-mu}\mathfrak{m}$ has exponential volume growth. By Proposition 2.6, we know M^{mu} is $\mathrm{RCD}(k + mK, \infty)$ space. Therefore (by the original definition of $\mathrm{CD}(k, \infty)$ condition, see [42]) we have

$$\mathrm{Ent}_{e^{-mu}\mathfrak{m}}(\mu_t) \leq (1-t)\mathrm{Ent}_{e^{-mu}\mathfrak{m}}(\mu_0) + t\mathrm{Ent}_{e^{-mu}\mathfrak{m}}(\mu_1) - \frac{mK + k}{2} t(1-t) W_2^2(\mu_0, \mu_1) \tag{3.9}$$

for any geodesic (μ_t) in Wasserstein space with bounded densities and bounded supports. Dividing m on both sides of (3.9) and letting $m \to \infty$, combining with the fact $\mathrm{Ent}_{e^{-mu}\mathfrak{m}}(\mu_t) = \mathrm{Ent}_\mathfrak{m}(\mu_t) + m\int u \, \mathrm{d}\mu_t$, we know u is weakly K-convex.

<u>*ii)* $\Longrightarrow$ *i)*.</u>

If u is weakly K-convex, we know (from the definition) that the metric measure space $M^{mu} := (X, \mathrm{d}, e^{-mu}\mathfrak{m})$ is $\mathrm{RCD}(k + mK, \infty)$ for any $m \in N$. By Proposition 2.6 we have 4), thus we get 1). By the density of test functions in 1) we know $\mathrm{Hess}_u \geq K$.

□

3.3 Equivalent characterizations

In this part we will prove the main results in this paper. The first theorem characterizes the K-convex functions on $\mathrm{RCD}(k, \infty)$ space. Due to lack of knowledge about the regularity of weak K-convex functions, we assume a priori that u has the following regularities.

Assumption 3.13. Basic assumptions on u are the following:

i) $u \in L^1_{\mathrm{loc}}(X, \mathfrak{m})$ and lower semi-continuous,

ii) $u(x) \geq -a - b\mathrm{d}^2(x, x_0)$ for some $a, b \in \mathbb{R}$, $x_0 \in X$.

Assumptions i) and ii) ensures that the functional $\mathcal{P}_2 \ni \mu \mapsto \int u \, \mathrm{d}\mu$ is lower semi-continuous, and not identically $-\infty$.

iii) $\nabla u \in L^2_{\mathrm{loc}}(TM)$,

iv) there exists a unique regular Lagrangian flow associated to $-\nabla u$.

Theorem 3.14. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}(k, \infty)$ metric measure space, u be a function fulfils Assumption 3.13. We denote the regular Lagrangian flow associated to $-\nabla u$ by (F_t). Then the following characterizations are equivalent.*

1) u is weakly K-convex.

2) ∇u is K-monotone.

3) the exponential contraction in Wasserstein distance:

$$W_2(\mu_t^1, \mu_t^2) \leq e^{-Kt} W_2(\mu_0^1, \mu_0^2), \quad \forall t > 0$$

holds for any two absolutely continuous curves $(\mu_t^1), (\mu_t^2) \subset (\mathcal{P}_2, W_2)$ with bounded compression, whose velocity fields are $-\nabla u$.

4) the flow (F_t) associate to $-\nabla u$ is well-defined for all $x \in X$ such that the exponential contraction holds in the sense that:

$$\mathrm{d}(F_t(x), F_t(y)) \leq e^{-Kt} \mathrm{d}(x, y)$$

for any $x, y \in X$ and $t > 0$.

5) for any $f \in W^{1,2}(X, \mathrm{d}, \mathfrak{m})$, we have $f \circ F_t \in W^{1,2}$ for any $t > 0$, and

$$|\mathrm{D}(f \circ F_t)|(x) \leq e^{-Kt} |\mathrm{D}f| \circ F_t(x), \quad \mathfrak{m} - a.e. x \in X.$$

Furthermore, if $u \in L^\infty_{\mathrm{loc}} \cap W^{2,2}_{\mathrm{loc}}$, then one of the above characterizations holds if and only if :

6) u is infinitesimally K-convex.

Proof. 1) $\Longrightarrow$ 2): Let $\mu_0, \mu_1 \in \mathcal{P}_2$ be any two measures with bounded densities and bounded supports. We consider the (unique) geodesic $(\mu_t)_{t \in [0,1]}$ from μ_0 to μ_1. From weak K-convexity, we know

$$U(\mu_s) \leq \frac{(1-s)}{1-t} U(\mu_t) + \frac{s-t}{1-t} U(\mu_1) - \frac{K}{2} \frac{(1-s)(s-t)}{1-t} W_2^2(\mu_0, \mu_1), \quad \forall s \in [t, 1] \tag{3.10}$$

where $U(\mu) = \int u \, \mathrm{d}\mu$. Therefore,

$$\frac{U(\mu_s) - U(\mu_t)}{s-t} \leq \frac{1}{1-t} \Big[U(\mu_1) - U(\mu_t) \Big] - \frac{K}{2} \frac{(1-s)}{1-t} W_2^2(\mu_0, \mu_1) \tag{3.11}$$

Letting $s \downarrow t$ and $t \downarrow 0$ in (3.11), by Proposition 2.21,Proposition 2.26, the C^1 continuity of geodesic in Proposition 2.27, and lower semicontinuity of U we obtain

$$-\int \langle \nabla u, \nabla \varphi \rangle \,\mathrm{d}\mu_0 \leq U(\mu_1) - U(\mu_0) - \frac{K}{2} W_2^2(\mu_0, \mu_1) \tag{3.12}$$

where φ is the Kantorovich potential from μ_0 to μ_1. Similarly, by changing the role of μ_1 and μ_0 we obtain

$$-\int \langle \nabla u, \nabla \varphi^c \rangle \,\mathrm{d}\mu_1 \leq U(\mu_0) - U(\mu_1) - \frac{K}{2} W_2^2(\mu_0, \mu_1). \tag{3.13}$$

Combining (3.12) and (3.13) we obtain

$$\int \langle \nabla u, \nabla \varphi \rangle \,\mathrm{d}\mu_0 + \int \langle \nabla u, \nabla \varphi^c \rangle \,\mathrm{d}\mu_1 \geq K W_2^2(\mu_0, \mu_1).$$

Since μ_0, μ_1 are arbitrary, we know ∇u is K-monotone.

$\underline{2) \Longrightarrow 1)}$: From Proposition 3.9 we know the uniqueness of geodesics, so by a classical approximation argument, it is sufficient to prove

$$U(\mu_{\frac{1}{2}}) \leq \frac{1}{2} U(\mu_0) + \frac{1}{2} U(\mu_1) - \frac{K}{8} W_2^2(\mu_0, \mu_1)$$

for any geodesic $(\mu_t) \subset (\mathcal{P}_2, W_2)$, where μ_0, μ_1 have bounded densities.

By Proposition 2.25 and Proposition 2.26 we know

$$\begin{aligned} U(\mu_{\frac{1}{2}}) - U(\mu_0) &= \int_0^{\frac{1}{2}} \Big(\frac{\mathrm{d}}{\mathrm{d}r} \int u \,\mathrm{d}\mu_r \Big) \,\mathrm{d}r \\ &= \int_0^{\frac{1}{2}} \frac{1}{1-2r} \Big(\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0} \int u \,\mathrm{d}\mu_{r+s(1-2r)} \Big) \,\mathrm{d}r \\ &= -\int_0^{\frac{1}{2}} \frac{1}{1-2r} \Big(\int \langle \nabla u, \nabla \varphi_{r,1-r} \rangle \,\mathrm{d}\mu_r \Big) \,\mathrm{d}r \end{aligned}$$

where $\varphi_{r,1-r}$ is the Kanrotovich potential relative to (μ_r, μ_{1-r}).

Similarly, we have

$$U(\mu_1) - U(\mu_{\frac{1}{2}}) = \int_{\frac{1}{2}}^1 \frac{1}{2r-1} \Big(\int \langle \nabla u, \nabla (\varphi_{1-r,r})^c \rangle \,\mathrm{d}\mu_r \Big) \,\mathrm{d}r.$$

By a change of variable, we know

$$U(\mu_1) - U(\mu_{\frac{1}{2}}) = \int_0^{\frac{1}{2}} \frac{1}{1-2r} \Big(\int \langle \nabla u, \nabla (\varphi_{r,1-r})^c \rangle \,\mathrm{d}\mu_{1-r} \Big) \,\mathrm{d}r.$$

Combining the results above, we obtain

$$
\begin{aligned}
&\frac{1}{2}U(\mu_0)+\frac{1}{2}U(\mu_1)-U(\mu_{\frac{1}{2}})\\
=\;&\frac{1}{2}\Big(U(\mu_0)-U(\mu_{\frac{1}{2}})\Big)+\frac{1}{2}\Big(U(\mu_1)-U(\mu_{\frac{1}{2}})\Big)\\
=\;&\frac{1}{2}\int_0^{\frac{1}{2}}\frac{1}{1-2r}\Big(\int\langle\nabla u,\nabla(\varphi_{1-r,r})^c\rangle\,\mathrm{d}\mu_r+\int\langle\nabla u,\nabla(\varphi_{r,1-r})^c\rangle\,\mathrm{d}\mu_{1-r}\Big)\,\mathrm{d}r\\
\geq\;&\frac{1}{2}\int_0^{\frac{1}{2}}\frac{1}{1-2r}K(1-2r)^2W^2(\mu_0,\mu_1)\,\mathrm{d}r\\
=\;&\frac{K}{8}W^2(\mu_0,\mu_1),
\end{aligned}
$$

which is the thesis.

1) $\Longrightarrow$ 3): Let $\mu_0\in\mathcal{P}_2(X)$ be a measure with bounded density and bounded support, (μ_t) be the RLF associated to $-\nabla u$ starting from μ_0. Assume $\mu_t, t\in[0,T]$ have uniformly bounded supports. We claim that (μ_t) is an EVI_K-gradient flow of U in the following sense:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t,\nu)+\frac{K}{2}W_2^2(\mu_t,\nu)\leq U(\nu)-U(\mu_t),\quad \text{for all } t>0 \tag{3.14}$$

for any $\nu\in\mathcal{P}_2(X)$. It is sufficient to prove (3.14) for any ν with bounded density and compact support (see Proposition 2.21, [6]).

By Proposition 2.22 we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t,\nu)=-\int\langle\nabla u,\nabla\varphi_t\rangle\,\mathrm{d}\mu_t \tag{3.15}$$

for a.e. $t>0$, where φ_t is the Kantorovich potential from μ_t to ν. From (3.12), we know

$$-\int\langle\nabla u,\nabla\varphi_t\rangle\,\mathrm{d}\mu_t\leq U(\nu)-U(\mu_t)-\frac{K}{2}W_2^2(\mu_t,\nu),\qquad\forall\, t\geq 0. \tag{3.16}$$

Combining (3.16) and (3.15) we know (3.14) holds for a.e. $t>0$. To prove the claim, it is sufficient to prove the C^1-continuity of the function $t\mapsto W_2^2(\mu_t,\nu)$. So we need to prove

$$\lim_{h\to 0}\int\langle\nabla u,\nabla\varphi_{t+h}\rangle\,\mathrm{d}\mu_{t+h}=\int\langle\nabla u,\nabla\varphi_t\rangle\,\mathrm{d}\mu_t$$

for any given t. In fact, from Proposition 3.4 and the compactness of $\operatorname{supp}\nu$ we can apply Lemma 2.3 in [3] to obtain the compactness/stability of Kantorovich potentials. Combining with Proposition 3.4, Proposition 3.9 and uniform boundedness of $\operatorname{supp}\mu_t$, we can prove the convergence using Lemma 3.3.

Let (ν_t) be another RLF associated to $-\nabla u$ starting from ν_0, where ν_0 has bounded density and bounded support such that $\nu_t, t\in[0,T]$ have uniformly bounded supports. Then by Theorem 4.0.4 in [4] we have the exponential contraction:

$$W_2(\mu_t,\nu_t)\leq e^{-Kt}W_2(\mu_0,\nu_0) \tag{3.17}$$

for any t.

For arbitrary $\mu_0, \nu_0 \in \mathcal{P}_2$ with bounded density, we can restrict μ_0, ν_0 on those points $x \in X$ such that $F_t(x) \subset B_R(x_0)$ for any $t \in [0, T]$, where $x_0 \in X$, $R > 0$. Then we can renormalise μ_0, ν_0 and denote them by μ_0^R, ν_0^R. We push-forward μ_0^R, ν_0^R by F_t and denote them by $(\mu_t^R), (\nu_t^R)$. From the results above we know (3.17) holds for $(\mu_t^R), (\nu_t^R)$.

Letting $\mathbb{R} \to \infty$ we know μ_0^R, ν_0^R converge to μ_0, ν_0 in $(\mathcal{P}_2, W_2)$. From the completeness of $(\mathcal{P}_2, W_2)$, we know $(\mu_t^R), (\nu_t^R)$ converge to some $(\mu_t), (\nu_t)$. It can be seen from the uniqueness of RLF that $\mu_t = (F_t)_\sharp \mu_0$ and $\nu_t = (F_t)_\sharp \nu_0$. So (3.17) holds for $(\mu_t), (\nu_t)$.

<u>3) $\Longrightarrow$ 4)</u>: Let $x \in X$ be an arbitrary point. From exponential contraction, by a typical approximation argument we know the flow of $-\nabla u$ from $\delta_x \in \mathcal{P}_2$ is uniquely defined. In fact, for any $x \in X$, we can find a sequence $(\mu^n) \subset \mathcal{P}_2$ such that $\lim_{n\to\infty} W_2(\mu_n, \delta_x) = 0$. From (3.17) we know the flow of $-\nabla u$ from μ^n, which is denoted by (μ_t^n), converges uniformly to a curve as $n \to \infty$. It can be seen that this limit curve is independent of the choice of (μ^n). We denote this curve by $\big(U_t(x)\big)_t \subset \mathcal{P}_2(X)$. Now we claim that $U_t(x)$ supports on a single point in X. Actually, assume that $\operatorname{supp} U_{t_0}(x)$ has at least two points $a, b \in X$ for some $t_0 > 0$. Let $\Pi^n \in \mathcal{P}(C([0,\infty), X)$ be the lifting of $(F_t)_\sharp\big(\frac{1}{\mathfrak{m}(B_{\frac{1}{n}}(x))}\mathfrak{m}|_{B_{\frac{1}{n}}(x)}\big)$. Since the RLF is non-branching, we know there exists $\Gamma^{1,n}, \Gamma^{2,n} \in \operatorname{supp}\Pi^n$ with positive measures such that $\inf\{\mathrm{d}(\gamma_{t_0}^1, \gamma_{t_0}^2) : \gamma^1 \in \Gamma^{1,n}, \gamma^2 \in \Gamma^{2,n}\} > \frac{1}{2}\mathrm{d}(a,b) > 0$ when n big enough. Then, by renormalization, we find two sequences of curves $\mu_t^{i,n} := (e_t)_\sharp\big(\frac{1}{\Pi^n(\Gamma^{i,n})}\Pi^n|_{\Gamma^{i,n}}\big), i = 1, 2$, such that $\mu_0^{i,n} \to \delta_x$ but $\mu_{t_0}^{1,n} \neq \mu_{t_0}^{2,n}$ which contradicts to the uniqueness of $U_t(x)$. We still use $U_t(x)$ to denote this single point.

Let $x \in X$ be a point where the curve $(F_t(x))_t$ is well-defined (i.e. $(F_t(x))_t$ is an absolutely continuous curve in X), where (F_t) is the RLF associated to $-\nabla u$. From the construction procedure of U_t and the uniqueness of $U_t(x)$ we know $U_t(x) = F_t(x)$.

Therefore, we can extend F_t to the whole space in the following way. For any $x \in X$, we define $(F_t)_\sharp \delta_x = U_t(x) = \delta_{F_t(x)}$. Finally, apply (3.17) again with $\mu_0 = \delta_x, \mu_1 = \delta_y$ we prove 4).

<u>4) $\Longrightarrow$ 5)</u>: Since $f \in W^{1,2}$, we know there exists a sequence $(f_n) \subset \mathrm{Lip}(X)$ such that $f_n \to f$ and $|\mathrm{lip}(f_n)| \to |\mathrm{D}f|$ in L^2. Then we have

$$\begin{aligned} \lim_{n\to\infty} \int |f \circ F_t - f_n \circ F_t|(x)^2 \,\mathrm{d}\mathfrak{m} &= \lim_{n\to\infty} \int |f - f_n|^2(x) \,\mathrm{d}(F_t)_\sharp \mathfrak{m} \\ &\leq C \lim_{n\to\infty} \int |f - f_n|^2 \,\mathrm{d}\mathfrak{m} \\ &= 0 \end{aligned}$$

where we use the fact that $(F_t)_\sharp \mathfrak{m} \leq C\mathfrak{m}$ in the second step. Similarly, we can prove that $|\mathrm{lip}(f_n)| \circ F_t$ converge to $|\mathrm{D}f| \circ F_t$ in L^2

From the hypothesis, we know

$$\begin{aligned}
|\mathrm{lip}(f_n\circ F_t)|(x) &= \varlimsup_{y\to x}\frac{|f_n\circ F_t(y)-f_n\circ F_t(x)|}{\mathrm{d}(y,x)}\\
&= \varlimsup_{y\to x}\frac{|f_n\circ F_t(y)-f_n\circ F_t(x)|}{\mathrm{d}(F_t(x),F_t(y))}\frac{\mathrm{d}(F_t(x),F_t(y))}{\mathrm{d}(y,x)}\\
&\leq \varlimsup_{y\to x}\frac{|f_n\circ F_t(y)-f_n\circ F_t(x)|}{\mathrm{d}(F_t(x),F_t(y))}\varlimsup_{y\to x}\frac{\mathrm{d}(F_t(x),F_t(y))}{\mathrm{d}(y,x)}\\
&\leq |\mathrm{lip}(f_n)|\circ F_t(x)e^{-Kt}.
\end{aligned}$$

Then we know

$$\begin{aligned}
\varlimsup_{n\to\infty}\int|\mathrm{lip}(f_n\circ F_t)|^2\,\mathrm{d}\mathfrak{m} &\leq \lim_{n\to\infty}e^{-2Kt}\int|\mathrm{lip}(f_n)|^2\circ F_t\,\mathrm{d}\mathfrak{m}\\
&= e^{-2Kt}\int|\mathrm{D}f|^2\circ F_t\,\mathrm{d}\mathfrak{m}.
\end{aligned}$$

Hence by definition we know $f\circ F_t\in W^{1,2}$.

Moreover, let G be a weak limit of a subsequence of $(\mathrm{lip}(f_n\circ F_t))_n$ in L^2. By pointwise minimality of weak gradient, we know $|\mathrm{D}(f\circ F_t)|\leq G\leq e^{-Kt}|\mathrm{D}f|\circ F_t$ $\mathfrak{m}$-a.e..

5) $\Longrightarrow$ 3): The strategy used in this proof is similar to the ones in [25] and [34], so we sketch the proof. We just need to prove 3) for μ_0^1,μ_0^2 with the form $\mu_0^1=f\mathfrak{m}$ and $\mu_0^2=g\mathfrak{m}$, where f,g are Lipschitz functions with bounded support. Now let $\varphi\in L^\infty\cap\mathrm{Lip}$ be with bounded support, $\Pi^0\in\mathcal{P}(C([0,1],X)$ be the lifting of the geodesic $(\nu_r^0)_r$ connecting μ_0^1 and μ_0^2. We denote $(F_t)_\sharp\nu_r^0$ by ν_r^t and denote the lifting of (ν_r^t) by Π^t. We also denote the velocity field of $(\nu_r^t)_r$ by $\nabla\phi_r^t$.

For any $r\in[0,1],h>0$, we have

$$\begin{aligned}
&\Big|\int Q_{r+h}(\varphi)\,\mathrm{d}\nu_{r+h}^t-\int Q_r(\varphi)\,\mathrm{d}\nu_r^t\Big|\\
\leq\;&\Big|\int Q_{r+h}(\varphi)\,\mathrm{d}\nu_{r+h}^t-\int Q_r(\varphi)\,\mathrm{d}\nu_{r+h}^t\Big|+\Big|\int Q_r(\varphi)\,\mathrm{d}\nu_{r+h}^t-\int Q_r(\varphi)\,\mathrm{d}\nu_r^t\Big|\\
\leq\;& C\int\big|Q_{r+h}(\varphi)-Q_r(\varphi)\big|\,\mathrm{d}\mathfrak{m}+\Big|\int Q_r(\varphi)\,\mathrm{d}\nu_{r+h}^t-\int Q_r(\varphi)\,\mathrm{d}\nu_r^t\Big|.
\end{aligned}$$

Then we know that $r\mapsto\int Q_r(\varphi)\,\mathrm{d}\nu_r^t$ is absolutely continuous, so it is differentiable almost everywhere. Using weak Leibniz rule (Lemma 4.3.4, [4]) we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}r}\int Q_r(\varphi)\,\mathrm{d}\nu_r^t\\
=\;&\lim_{h\to 0}\frac{\int Q_{r+h}(\varphi)\,\mathrm{d}\nu_{r+h}^t-\int Q_r(\varphi)\,\mathrm{d}\nu_r^t}{h}\\
\leq\;&\varlimsup_{h\to 0}\frac{\int Q_{r+h}(\varphi)\,\mathrm{d}\nu_r^t-\int Q_r(\varphi)\,\mathrm{d}\nu_r^t}{h}+\varlimsup_{h\to 0}\frac{\int Q_r(\varphi)\,\mathrm{d}\nu_r^t-\int Q_r(\varphi)\,\mathrm{d}\nu_{r-h}^t}{h}
\end{aligned}$$

for a.e. r.

By Hamilton-Jacobi equation in Lemma 2.24, Proposition 2.27 and dominated convergence theorem we have

$$\lim_{h\to 0}\frac{\int Q_{r+h}(\varphi)\,\mathrm{d}\nu_r^t - \int Q_r(\varphi)\,\mathrm{d}\nu_r^t}{h} = \int -\frac{1}{2}|\mathrm{D}Q_r(\varphi)|^2\mathrm{d}\nu_r^t$$

for a.e. $r\in(0,1)$. From Proposition 2.27 we know

$$\lim_{h\to 0}\frac{\int Q_r(\varphi)\,\mathrm{d}\nu_{r+h}^t - \int Q_r(\varphi)\,\mathrm{d}\nu_r^t}{h} = \int \langle\nabla(Q_r(\varphi)\circ F_t),\nabla\phi_r^0\rangle\mathrm{d}\nu_r^0$$

for all r.

Combining with the computations above we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}r}\int Q_r(\varphi)\,\mathrm{d}\nu_r^t \le \int -\frac{1}{2}|\mathrm{D}Q_r(\varphi)|^2\mathrm{d}\nu_r^t + \int \langle\nabla(Q_r(\varphi)\circ F_t),\nabla\phi_r^0\rangle\mathrm{d}\nu_r^0$$

for a.e. $r\in(0,1)$.

Then we have the following estimate:

$$\begin{aligned}
& \int \varphi^c(y)\,\mathrm{d}\mu_t^2(y) + \int \varphi(x)\,\mathrm{d}\mu_t^1(x)\\
=\ & \int \Big(Q_1(-\varphi)(y) - Q_0(-\varphi)(x)\Big)\,\mathrm{d}\Pi^t\\
=\ & \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r}\int \Big(Q_r(-\varphi)\mathrm{d}\nu_r^t\Big)\,\mathrm{d}r\\
\le\ & \int_0^1\int -\frac{1}{2}|\mathrm{D}Q_r(-\varphi)|^2\mathrm{d}\nu_r^t\,\mathrm{d}r\\
& \qquad + \int_0^1\int \langle\nabla(Q_r(-\varphi)\circ F_t),\nabla\phi_r^0\rangle\mathrm{d}\nu_r^0\,\mathrm{d}r\\
\text{Young's inequality}\quad \le\ & \int_0^1\int -\frac{1}{2}|\mathrm{D}Q_r(-\varphi)|^2\mathrm{d}\nu_r^t\,\mathrm{d}r\\
& \qquad + \frac{1}{2}\int_0^1\int e^{2Kt}|\mathrm{D}(Q_r(-\varphi)\circ F_t)|^2\,\mathrm{d}\nu_r^0\mathrm{d}r\\
+\ & \frac{1}{2}e^{-2Kt}\int_0^1\int |\mathrm{D}\phi_r^0|^2\,\mathrm{d}\nu_r^0\mathrm{d}r\\
\text{hypothesis 5)}\quad \le\ & \frac{1}{2}e^{-2Kt}\int_0^1\int |\mathrm{D}\phi_r^0|^2\,\mathrm{d}\nu_r^0\mathrm{d}r\\
\text{Proposition 2.21} + \text{Proposition 2.26}\quad =\ & \frac{1}{2}e^{-2Kt}W_2^2(\mu_0^1,\mu_0^2).
\end{aligned}$$

Since φ is arbitrary, we know $W_2^2(\mu_t^1,\mu_t^2)\le e^{-2Kt}W_2^2(\mu_0^1,\mu_0^2)$.

$4)+5)\Longrightarrow 2)$: Let $\mu_0,\nu_0\in\mathcal{P}_2$ be measures with compact support and bounded density. We consider the RLFs $(\mu_t)_{t\in[0,T]}$ and $(\nu_t)_{t\in[0,T]}$ starting from μ_0,ν_0 respectively, where $T>0$. From Proposition 3.1 we know the measures $\mu_t,\nu_t,t\in[0,T]$ have uniformly bounded densities. From 4) we know μ_t,ν_t have compact supports for all $t \in [0,T]$. Since $\operatorname{supp}\mu_0$ and $\operatorname{supp}\nu_0$ are bounded, we know from 4) that the supports of μ_t, ν_t $t \in [0,T]$ are uniformly bounded.

We denote by $(\theta_r)_r$ the geodesic from μ_0 to ν_0, and denote the velocity field of $(\theta_r)_r$ by $\nabla\phi_r$. Let $\delta_r : [0,1] \mapsto [0,1]$ be a C^1 function (to be determined) with $\delta(i) = i, i = 0, 1$. We define an interpolation $(F_{tr})_\sharp\theta_{\delta_r}$ and denoted it by η^t_r.

Then we estimate $W_2^2(\mu_0, \nu_t)$ using a similar method as we used in 5) $\Longrightarrow$ 3). For any $\varphi \in L^\infty \cap \mathrm{Lip}$ with bounded support, we have

$$
\begin{aligned}
& \int \varphi^c(y)\,\mathrm{d}\nu_t(y) + \int \varphi(x)\,\mathrm{d}\mu_0(x) \\
= & \int \varphi^c(y)\,\mathrm{d}\eta^t_1(y) + \int \varphi(x)\,\mathrm{d}\eta^t_0(x) \\
= & \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} \int \Big(Q_r(-\varphi) \circ F_{tr}\Big)\mathrm{d}\theta_{\delta_r}\,\mathrm{d}r \\
= & \int_0^1 \int -\frac{1}{2}|\mathrm{D}Q_r(-\varphi)|^2\,\mathrm{d}\eta^t_r\mathrm{d}r + \int_0^1 \delta'_r \int \langle \nabla(Q_r(-\varphi) \circ F_{tr}), \nabla\phi_{\delta_r}\rangle\,\mathrm{d}\theta_{\delta_r}\mathrm{d}r \\
- & t\int_0^1 \int \langle \nabla(Q_r(-\varphi), \nabla u\rangle\,\mathrm{d}\eta^t_r\mathrm{d}r \\
= & \int_0^1 \int -\frac{1}{2}|\mathrm{D}\big(Q_r(-\varphi) + tu\big)|^2\,\mathrm{d}\eta^t_r\mathrm{d}r + \int_0^1 \delta'_r \int \langle \nabla\big((Q_r(-\varphi) + tu) \circ F_{tr}\big), \nabla\phi_{\delta_r}\rangle\,\mathrm{d}\theta_{\delta_r}\mathrm{d}r \\
+ & \int_0^1 \int \frac{1}{2}t^2|\mathrm{D}u|^2\,\mathrm{d}\eta^t_r\mathrm{d}r - t\int_0^1 \delta'_r \int \langle \nabla(u \circ F_{tr}), \nabla\phi_{\delta_r}\rangle\,\mathrm{d}\theta_{\delta_r}\mathrm{d}r \\
\leq & \int_0^1 \frac{1}{2}\big(\delta'_r\big)^2 e^{-2Krt} \int |\mathrm{D}\phi_{\delta_r}|^2\,\mathrm{d}\theta_{\delta_r}\mathrm{d}r + \int_0^1 \int \Big[\frac{1}{2}t^2\langle \nabla(u \circ F_{tr}), \nabla u\rangle - t\delta'_r\langle \nabla(u \circ F_{tr}), \nabla\phi_{\delta_r}\rangle\Big]\,\mathrm{d}\theta_{\delta_r}\mathrm{d}r \\
:= & A(t) + tB(t).
\end{aligned}
$$

We then choose

$$
\delta(r) := \frac{e^{2Krt} - 1}{e^{2Kt} - 1},
$$

so that $\delta'(r) = R_K(t)e^{2Krt}$ where

$$
R_K(t) := \frac{2Kt}{e^{2Kt} - 1} \quad \text{if } K \neq 0, \qquad R_0(t) = 1.
$$

Then we have

$$
\begin{aligned}
A(t) &= \int_0^1 \frac{1}{2}\big(\delta'_r\big)^2 e^{-2Krt} \int |\mathrm{D}\phi_{\delta_r}|^2\,\mathrm{d}\theta_{\delta_r}\mathrm{d}r \\
&= \frac{R_K(t)}{2} \int_0^1 \delta'_r \int |\mathrm{D}\phi_{\delta_r}|^2\,\mathrm{d}\theta_{\delta_r}\mathrm{d}r \\
&= \frac{R_K(t)}{2} \int_0^1 \int |\mathrm{D}\phi_r|^2\,\mathrm{d}\theta_r\mathrm{d}r \\
&= \frac{1}{2}R_K^2(t)W_2^2(\mu_0, \nu_0)
\end{aligned}
$$

It can be seen from Proposition 2.27 and Proposition 3.4 that $B(t)$ is continuous in t. In fact, by direct computation we can even prove:

$$\begin{aligned} U(\mu_0)-U(\nu_t) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r}\int \Big((-u)\circ F_{rt}\Big)\,\mathrm{d}\theta_{\delta_r}\mathrm{d}r \\ &= \int_0^1 \int \Big[t\langle \nabla (u\circ F_{tr}), \nabla u\rangle - \delta_r'\langle \nabla (u\circ F_{tr}), \nabla \phi_{\delta_r}\rangle\Big]\,\mathrm{d}\theta_{\delta_r}\mathrm{d}r \\ &\geq B(t). \end{aligned}$$

Combining the results above, we obtain

$$W_2^2(\mu_0,\nu_t)\leq R_K^2(t)W_2^2(\mu_0,\nu_0)+2tB(t).$$

Dividing $t>0$ on both sides and letting $t\to 0$, together with the formula $R_K(t)=1-\frac{Kt}{2}+o(t)$ we obtain

$$\frac{\mathrm{d}^+}{\mathrm{d}t}W_2^2(\mu_0,\nu_t)_{|_{t=0}}\leq B(0)-\frac{K}{2}W_2^2(\mu_0,\nu_0).$$

Since $t\mapsto W_2^2(\mu_0,\nu_t)$ is C^1 (see the proof of 1) $\Longrightarrow$ 3)), we know

$$-\int \langle \nabla u, \nabla \varphi^c_{0,0}\ \rangle\,\mathrm{d}\nu_0 \leq B(0)-\frac{K}{2}W_2^2(\mu_0,\nu_0), \tag{3.18}$$

where $B(0)=-\int_0^1\int\langle\nabla u,\nabla\phi^0_r\rangle\,\mathrm{d}\nu^0_r\mathrm{d}r$.

Using the same argument we can also prove

$$-\int \langle \nabla u, \nabla \varphi_{0,0}\ \rangle\,\mathrm{d}\mu_0 \leq C(0)-\frac{K}{2}W_2^2(\mu_0,\nu_0), \tag{3.19}$$

where $C(0)=\int_0^1\int\langle\nabla u,\nabla\phi^0_{1-r}\rangle\,\mathrm{d}\nu^0_{1-r}\mathrm{d}r=\int_0^1\int\langle\nabla u,\nabla\phi^0_r\rangle\,\mathrm{d}\nu^0_r\mathrm{d}r=-B(0)$.

Combining (3.18) and (3.19) we obtain

$$\int \langle \nabla u, \nabla \varphi^c_{0,0}\ \rangle\,\mathrm{d}\nu_0 + \int \langle \nabla u, \nabla \varphi_{0,0}\ \rangle\,\mathrm{d}\mu_0 \geq KW_2^2(\mu_0,\nu_0). \tag{3.20}$$

Finally, by an approximation by compactly supported measures and metirc Brenier's theorem, we know (3.20) holds for all μ_0,ν_0 with bounded support and bounded density, so ∇u is K-monotone.

6) $\Longleftrightarrow$ 1): This is a direct consequence of Theorem 3.12. □

Remark 3.15. Let f be a smooth function f on a Riemannian manifold (M,g), and (γ_t) be a smooth curve. We know the map $t\to f(\gamma_t)$ is smooth and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}f(\gamma_t)=\mathrm{Hess}_f(\gamma_t',\gamma_t')+\langle\nabla_{\gamma_t'}\gamma_t',\nabla f\rangle.$$

In particular, if (γ_t) is a geodesic, we know $\nabla_{\gamma_t'}\gamma_t' = 0$, then we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(\gamma_t) = \mathrm{Hess}_f(\gamma_t', \gamma_t').$$

We then know that the second order derivative along geodesic characterizes the convexity of a function f.

On $\mathrm{RCD}^*(k,N)$ spaces, we can use the second order differentiation formula developed by Gigli-Tamanini (see [30]) to study the convexity of $H^{2,2}$ functions. However, it is still unknown to us whether we can do the same in $\mathrm{RCD}(k,\infty)$ case or not.

Theorem 3.16. *Let $M := (X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}(k,\infty)$ space, $\mathbf{b} \in L^2_{\mathrm{loc}}(TM)$. We assume there exits a unique regular Lagrangian flow associated to $-\mathbf{b}$, which is denoted by (F_t). Then the following descriptions are equivalent.*

1) $\mathbf{b}$ is K-monotone.

2) the exponential contraction in Wasserstein distance:

$$W_2(\mu_t^1, \mu_t^2) \le e^{-Kt} W_2(\mu_0^1, \mu_0^2), \quad \forall t > 0$$

holds for any two curves $(\mu_t^1), (\mu_t^2)$ whose velocity fields are $-\mathbf{b}$.

3) the everywhere-defined RLF (F_t) of $-\mathbf{b}$, and the exponential contraction:

$$\mathrm{d}(F_t(x), F_t(y)) \le e^{-Kt}\mathrm{d}(x,y)$$

for any $x, y \in X$ and $t > 0$.

4) for any $f \in W^{1,2}(X, \mathrm{d}, \mathfrak{m})$, we have $f \circ F_t \in W^{1,2}$ and:

$$|\mathrm{D}(f \circ F_t)|(x) \le e^{-Kt}|\mathrm{D}f| \circ F_t(x), \quad \mathfrak{m} - a.e.\ x \in X$$

where (F_t) is the RLF of $-\mathbf{b}$

Proof. We can prove 2) $\Longrightarrow$ 3) $\Longrightarrow$ 4) $\Longrightarrow$ 2) and 4) $\Longrightarrow$ 1) in the same ways as in the proof of Theorem 3.14.

1) $\Longrightarrow$ 2): Let $\mu_0, \nu_0 \in \mathcal{P}_2$ be measures with bounded support and bounded density, $(\mu_t), (\nu_t)$ be the solutions to the continuity equation with velocity field $-\mathbf{b}$, with initial datum μ_0 and ν_0 respectively. It can be seen from Proposition 3.1 that μ_t, ν_t have bounded densities for any $t > 0$. Fix $T > 0$, we denote the lifting of $(\mu_t)_{t\in[0,T]}$ by $\Pi \in \mathcal{P}(\mathrm{AC}([0,T], X))$. Let $\Gamma \subset \mathrm{AC}([0,T], X)$ be the support of Π. For any $\epsilon > 0$, we can find $\Gamma_\epsilon \subset \Gamma$ which is compact in $\mathrm{C}([0,T], X)$ such that $\Pi(\Gamma \setminus \Gamma_\epsilon) < \epsilon$, and $\Gamma_\epsilon \subset B_R(x_0)$ for some $x_0 \in X$ and $R > \frac{1}{\epsilon}$. Then we define

$$\mu_t^\epsilon := (e_t)_\sharp\Big(\frac{1}{\Pi(\Gamma_\epsilon)}\Pi|_{\Gamma_\epsilon}\Big), \qquad \epsilon > 0, \quad t \in [0,T].$$

It can be seen that $\operatorname{supp}\mu_t^\epsilon = e_t(\Gamma_\epsilon)$ is compact for any $t \in [0,T]$ and

$$\lim_{\epsilon \to 0} W^2(\mu_0, \mu_0^\epsilon) = 0.$$

So without loss of generality we could assume that μ_t, ν_t support on compact sets for any $t \in [0,T]$. Furthermore, we may also assume that μ_t, ν_t have uniformly bounded supports for $t \in [0,T]$.

Then for any $s \geq 0$, by Proposition 2.22 we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t, \nu_s) = -\int \langle \mathbf{b}, \nabla\varphi_{t,s}\rangle \,\mathrm{d}\mu_t \tag{3.21}$$

for a.e. $t > 0$, where $\varphi_{t,s}$ is the Kantorovich potential from μ_t to ν_s. Similarly, fix a t we know

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{2}W_2^2(\mu_t, \nu_s) = -\int \langle \mathbf{b}, \nabla\phi_{s,t}\rangle \,\mathrm{d}\nu_s \tag{3.22}$$

for a.e. $s > 0$, where $\phi_{s,t}$ is the Kantorovich potential from ν_s to μ_t.

Now we claim that $t \mapsto -\int \langle \mathbf{b}, \nabla\varphi_{t,s}\rangle \,\mathrm{d}\mu_t$ is continuous for any s. We just need to prove

$$\lim_{h\to 0}\int \langle \mathbf{b}, \nabla\varphi_{t+h,s}\rangle \,\mathrm{d}\mu_{t+h} = \int \langle \mathbf{b}, \nabla\varphi_{t,s}\rangle \,\mathrm{d}\mu_t$$

for a given t.

By Proposition 3.4 and the compactness assumption on $\operatorname{supp}\nu_s$, we can apply Lemma 2.3 in [3] to obtain the compactness of Kantorovich potentials. Combining with Proposition 3.4 we know the convergence from Lemma 3.3.

Similarly, we can prove that $s \mapsto \int \langle \mathbf{b}, \nabla\phi_{s,t}\rangle \,\mathrm{d}\nu_s$ is continuous. Therefore we know (3.21) and (3.22) hold for all t and s respectively. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{2}W_2^2(\mu_t, \nu_s)|_{s=t} = -\int \langle \mathbf{b}, \nabla\phi_{t,t}\rangle \,\mathrm{d}\nu_t \tag{3.23}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{2}W_2^2(\mu_r, \nu_t)|_{r=t} = -\int \langle \mathbf{b}, \nabla\varphi_{t,t}\rangle \,\mathrm{d}\mu_t. \tag{3.24}$$

Furthermore, we know $t \mapsto W_2^2(\mu_t, \nu_t)$ is differentiable for a.e. $t \in [0,T]$. Using the formula in Lemma 4.3.4, [4] we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t, \nu_t) &= \frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{2}W_2^2(\mu_r, \nu_t)|_{r=t} + \frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{2}W_2^2(\mu_t, \nu_s)|_{s=t} \\ &= -\int \langle \mathbf{b}, \nabla\varphi_{t,t}\rangle \,\mathrm{d}\mu_t - \int \langle \mathbf{b}, \nabla\phi_{t,t}\rangle \,\mathrm{d}\nu_t\end{aligned}$$

for a.e. $t \in [0,T]$.

From the definition of K-monotonicity we know

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}W_2^2(\mu_t, \nu_t) \leq -KW_2^2(\mu_t, \nu_t)$$

for a.e. $t \in [0,T]$. Finally, by Grönwall's lemma we obtain the exponential contraction

$$W_2(\mu_t, \nu_t) \leq e^{-Kt}W_2(\mu_0, \nu_0) \tag{3.25}$$

for any $t \in [0,T]$.

$\square$

Remark 3.17. The velocity field $\mathbf{b}$ in this theorem could be replaced by a time-dependent field $\mathbf{b}_t$, with minor modifications to the current proof.

Remark 3.18. One would ask if the infinitesimal K-monotonicity of $\mathbf{b}$ is equivalent to the characterizations in the Theorem 3.16. Due to lack of Sobolev regularity of $f \circ F_t$ when $f \in W^{1,2}$, we can not prove the theorem from the infinitesimal K-monotonicity of $\mathbf{b}$ using the classical semigroup argument in Bakry-Émery theory. But in some special situations, we can achieve this goal.

Case 1. When $\mathbf{b}$ is a harmonic vector field on $\mathrm{RCD}(k,\infty)$ space, it is proved by Gigli-Rigoni (in [29]) that $f \circ F_t \in \mathrm{TestF}$ if $f \in \mathrm{TestF}$, and F_t induces an isometry. Formally speaking, in this case the Hille-Yoshida theorem works for the generator $L^n := \frac{1}{n}\Delta - \mathbf{b}$ with $n \in \mathbb{N}$. Then the corresponding semigroup $\mathrm{P}^n_t f$ converge to $f \circ F_t$ (by Lemma 3.5). Combining the gradient estimate of $\mathrm{P}^n_t f$ which can be proven by considering the modified $\mathbf{\Gamma}_2$ w.r.t $\frac{1}{n}\Delta - \mathbf{b}$, we can prove 4) in Theorem 3.16.

Case 2. On $\mathrm{RCD}^*(k,N)$ spaces, using the second order differentiation formula developed by Gigli-Tamanini (see [30]) we can easily prove that infinitesimal K-monotonicity is equivalent to K-monotonicity.

At the end of this section, we show that the K-monotonicity is stable with respect to measured Gromov-Hausdorff convergence. For simplicity, we adopt the notions from [10] without further explanation. Without loss of generality, we call that $\mathrm{RCD}(k,\infty)$ spaces $M_n := (X, \mathrm{d}, \mathfrak{m}_n)$ converge to $M := (X, \mathrm{d}, \mathfrak{m})$ in measured Gromov-Hausdorff topology if $\mathfrak{m}_n \to \mathfrak{m}$ weakly.

We define the countable class

$$\mathcal{H}_{\mathbb{Q}^+}\mathcal{A}_{\mathrm{bs}} := \Big\{\mathcal{H}_t f : f \in \mathcal{A}_{\mathrm{bs}}, t \in \mathbb{Q}^+\Big\} \subset \mathrm{Lip} \cap L^\infty,$$

where $\mathcal{A}_{\mathrm{bs}}$ is a sub-algebra of $\mathcal{A}$ consisting of functions with bounded support, where $\mathcal{A}$ is a $\mathbb{Q}$-vector space generated by

$$\min\Big\{\mathrm{d}(\cdot, x), k\Big\} \quad k \in \mathbb{Q} \cap [0,\infty],\ x \in D, D \text{ is dense in } X.$$

It can be seen (see e.g. [10]) that $\mathcal{H}_{\mathbb{Q}^+}\mathcal{A}_{\mathrm{bs}}$ is dense in $W^{1,2}$.

Corollary 3.19 (Stability of K-monotonicity)**.** *Let $\mathbf{b}_n \in W^{1,2}_C(TM_n), n \in \mathbb{N}$ be such that $\sup_n \|\mathbf{b}_n\|_{L^2(X,\mathfrak{m}_n)} < \infty$ and $\sup_n \|\mathrm{div}\mathbf{b}_n\|_{L^\infty(X,\mathfrak{m}_n)} < \infty$. If $(\mathbf{b}_n)_{n\in\mathbb{N}}$ are K-monotone and $\mathbf{b}_n(f)\mathfrak{m}_n \to \mathbf{b}(f)\mathfrak{m}$ as measures for all $f \in \mathcal{H}_{\mathbb{Q}^+}\mathcal{A}_{\mathrm{bs}}$, and*

$$\varlimsup_{n\to\infty} \int |\mathbf{b}_n|^2 \,\mathrm{d}\mathfrak{m}_n \le \int |\mathbf{b}|^2 \,\mathrm{d}\mathfrak{m}.$$

Then $\mathbf{b}$ is K-monotone.

Proof. From Theorem 8.2 in [10] we know the regular Lagrangian flow associated to $\mathbf{b}_n$ converge to the RLF of $\mathbf{b}$ in measure. We apply 2) of Theorem 3.16 with $\mathbf{b}_n$, from lower-semicontinuity of Wasserstein distance w.r.t weak topology, we know K-monotonicity of $\mathbf{b}_n$ implies K-monotonicity of $\mathbf{b}$. $\square$

4 Applications

In this section, we apply Theorem 3.14 to two special functions. Our aim is not to give complete proofs to the rigidity theorems which are already perfectly done, but to present how to use our result to connect the differential structure and metric structure on metric measure spaces in a different way. For simplicity, we will always start our discussion from the non-smooth differential equation concerning the K-convex functions.

Example 1: Splitting

Theorem. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}(k, \infty)$ metric measure space. If $\Delta u = 0$, and $|\mathrm{D}u| = 1$, then there exists a metric space Y such that X is isometric to $Y \times \mathbb{R}$.*

Proof. By a cut-off argument we can apply Corollary 2.13 to u, then we can prove that $\mathrm{Hess}_u = 0$. From Proposition 3.2 we know that the regular Lagrangian flows associated to ∇u and $-\nabla u$ exist, which are denoted by $(F_t^+)_{t\geq 0}$ and $(F_t^-)_{t\geq 0}$ respectively. By uniqueness of the RLF we know $F_t^+(F_s^-(x)) = F_s^-(F_t^+(x)) = F_{|t-s|}^{\mathrm{sign}(t-s)}$, where $\mathrm{sign}(t-s)$ is " $+$ " if $t - s \geq 0$ and is " $-$ " if $t - s < 0$. We define

$$F_t(x) := \begin{cases} F_t^+(x) & t \geq 0 \\ F_t^-(x) & t < 0 \end{cases} \tag{4.1}$$

Since $|\mathrm{D}f| = 1$ we know $\mathrm{Lip}(f) = 1$ from Sobolev-to-Lipschitz property. Then we can apply Theorem 3.12 and Theorem 3.14 to infinitesimally 0-convex functions u and $-u$. From 4) of Theorem 3.14 we know

$$\mathrm{d}(F_t(x), F_t(y)) \leq \mathrm{d}(x, y)$$

for any $x, y \in X$, $t \in \mathbb{R}$. So we have

$$\mathrm{d}(F_t(x), F_t(y)) \leq \mathrm{d}(x, y) = \mathrm{d}(F_{-t}(F_t(x)), F_{-t}(F_t(y))) \leq \mathrm{d}(F_t(x), F_t(y))$$

for any $x, y \in X$, $t \in \mathbb{R}$. Hence $\mathrm{d}(F_t(x), F_t(y)) = \mathrm{d}(x, y)$ for any $x, y \in X$, $t \in \mathbb{R}$. Therefore F_t induces an isometry between $u^{-1}(0)$ and $u^{-1}(t)$. Combining with the fact that $|\dot{F}_t|(x) = 1$, we know F_t induces a translation on the fibre $(F_t(x_0))_t$ for any $x_0 \in u^{-1}(0)$. It can also be checked that $u^{-1}(0)$ is totally geodesic.

Finally, by identifying the Sobolev spaces $W^{1,2}(\Phi^{-1}(X))$ and $W^{1,2}(\mathbb{R} \times u^{-1}(0))$, we know from the Sobolev-to-Lipschitz property that the map $\Phi : \mathbb{R} \times u^{-1}(0) \ni (t, x) \mapsto F_t(x) \in X$ is an isometry (see Section 6, [22]).

□

Remark 4.1. In "splitting theorem" (see [18], [22]), the function u is the Buseman function associated with a line. In [27] the function u is a solution to the equation $\Delta u = -u$, such that $\mathrm{Hess}_u = 0$.

Example 2: Volume cone implies metric cone

Theorem. *Let $(X, \mathrm{d}, \mathfrak{m})$ be a $\mathrm{RCD}(0, N)$ space with $\mathfrak{m} \ll \mathcal{H}^N$. If $\Delta u = N$, $|\mathrm{D}u|^2 = 2u$ and $u \leq C\mathrm{d}^2(\cdot, O)$ for some $O \in X$, $C > 0$, then (X, d) admits a warped product-like structure.*

Proof. Since $\mathfrak{m} \ll \mathcal{H}^N$, from the rectifiability theorem (see [38], [28] and [32]) we know $\dim_{\rm loc} = N$ is a constant. Then by Proposition 2.19, we know $u \in W^{1,2}_{C,\rm loc}$ and $\mathrm{trHess}_u(x) = \Delta u(x)$ $\mathfrak{m}$-a.e. $x \in X$. Hence Δ is a local operator so that we can represent it using local coordinate.

Since $\Delta u = N$, by Proposition 2.12 we know

$$N = \Delta u = \frac{1}{2}\Delta|\mathrm{D}u|^2 - \langle \nabla u, \nabla \Delta u\rangle \geq |\mathrm{Hess}_u|^2_{\mathrm{HS}}, \quad \mathfrak{m} - \textbf{a.e.}. \tag{4.2}$$

By Cauchy inequality and the fact that $\dim_{\rm loc} = N$ we know

$$|\mathrm{Hess}_u|^2_{\mathrm{HS}} \geq \frac{1}{N}(\mathrm{trHess}_f)^2 = \frac{1}{N}(\Delta f)^2 = N.$$

Combining with (4.2) we know $\mathrm{Hess}_u = \mathrm{Id}_N$.

Then we consider the regular Lagrangian flow associated to ∇u and $-\nabla u$, which are denoted by $(F_t^+)_{t\geq 0}$ and $(F_t^-)_{t\geq 0}$ respectively. We can also construct F_t as we did in the first example. We know both

$$\mathrm{d}(F_t(x), F_t(y)) \leq e^{-Nt}\mathrm{d}(x, y)$$

for any $x, y \in X$, $t > 0$, and

$$\mathrm{d}(F_t(x), F_t(y)) \leq e^{Nt}\mathrm{d}(x, y)$$

for any $x, y \in X$, $t < 0$.

Therefore, for any $t > 0$ we have

$$\mathrm{d}(F_t(x), F_t(y)) \leq e^{-Nt}\mathrm{d}(x, y) = e^{-Nt}\mathrm{d}\big(F_{-t}(F_t(x)), F_{-t}(F_t(y))\big) \leq e^{-Nt}e^{Nt}\mathrm{d}(F_t(x), F_t(y)).$$

Therefore we know $\mathrm{d}(F_t(x), F_t(y)) = e^{-Nt}\mathrm{d}(x, y)$. So (X, d) admits a warped product-like structure.

□

Remark 4.2. In “volume cone implies metric cone theorem ” (see [14], [19]), the target function u is the squared distance function $\frac{1}{2}\mathrm{d}^2(\cdot, \mathrm{O})$ where O is a fixed point. Then we know $|\mathrm{D}u|^2 = 2u = \mathrm{d}^2(\cdot, \mathrm{O})$. From the theorem above we know $|\dot{F}_t|(x) = |\mathrm{D}u| \circ F_t(x) = \mathrm{d}(F_t(x), \mathrm{O})$. We define $\Phi : \mathbb{R} \times u^{-1}(1) \ni (t, x) \mapsto F_t(x) \in X$. By identifying the Sobolev spaces $W^{1,2}(\Phi^{-1}(X))$ and $W^{1,2}(\mathbb{R} \times u^{-1}(1))$ (see [19] and [26]), we know from Sobolev-to-Lipschitz property that Φ is an isometry. So(X, d) admits a cone structure, and the point O is exactly the apex.

References

[1] L. Ambrosio, M. Colombo, and S. D. Marino, *Sobolev spaces in metric measure spaces: reflexivity and lower semicontinuity of slope*, Advanced Studies in Pure Mathematics: ”Variational methods for evolving objects”, 67 (2012), pp. 1–58.

[2] L. Ambrosio and N. Gigli, *A user's guide to optimal transport.* Modelling and Optimisation of Flows on Networks, Lecture Notes in Mathematics, Vol. 2062, Springer, 2011.

[3] L. Ambrosio, N. Gigli, A. Mondino, and T. Rajala, *Riemannian Ricci curvature lower bounds in metric measure spaces with σ-finite measure*, Trans. Amer. Math. Soc., 367 (2015), pp. 4661–4701.

[4] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, second ed., 2008.

[5] ——, *Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below*, Inventiones mathematicae, (2013), pp. 1–103.

[6] ——, *Metric measure spaces with riemannian Ricci curvature bounded from below*, Duke Math. J., 163 (2014), pp. 1405–1490.

[7] ——, *Bakry-Émery curvature-dimension condition and Riemannian Ricci curvature bounds*, Ann. Probab., 43 (2015), pp. 339–404.

[8] L. Ambrosio, A. Mondino, and G. Savaré, *Nonlinear diffusion equations and curvature conditions in metric measure spaces.* Preprint, arXiv:1509.07273. To appear on Mem. Amer. Math. Soc., 2015.

[9] ——, *On the Bakry-Émery condition, the gradient estimates and the local-to-global property of* $\mathrm{RCD}^*(k, N)$ *metric measure spaces*, J. Geom. Anal., 26 (2016), pp. 24–56.

[10] L. Ambrosio, F. Stra, and D. Trevisan, *Weak and strong convergence of derivations and stability of flows with respect to MGH convergence*, J. Funct. Anal., 272 (2017), pp. 1182–1229.

[11] L. Ambrosio and D. Trevisan, *Well-posedness of Lagrangian flows and continuity equations in metric measure spaces*, Anal. PDE, 7 (2014), pp. 1179–1234.

[12] K. Bacher and K.-T. Sturm, *Localization and tensorization properties of the curvature-dimension condition for metric measure spaces*, J. Funct. Anal., 259 (2010), pp. 28–56.

[13] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal., 9 (1999), pp. 428–517.

[14] J. Cheeger and T. H. Colding, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Ann. of Math. (2), 144 (1996), pp. 189–237.

[15] ——, *On the structure of spaces with Ricci curvature bounded below. I*, J. Differential Geom., 46 (1997), pp. 406–480.

[16] ——, *On the structure of spaces with Ricci curvature bounded below. II*, J. Differential Geom., 54 (2000), pp. 13–35.

[17] ——, *On the structure of spaces with Ricci curvature bounded below. III*, J. Differential Geom., 54 (2000), pp. 37–74.

[18] J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of non-negative Ricci curvature*, J. Differential Geometry, 6 (1971/72), pp. 119–128.

[19] G. De Philippis and N. Gigli, *From volume cone to metric cone in the nonsmooth setting*, Geom. Funct. Anal., 26 (2016), pp. 1526–1587.

[20] M. Erbar, *The heat equation on manifolds as a gradient flow in the Wasserstein space*, Ann. Inst. Henri Poincaré Probab. Stat., 46 (2010), pp. 1–23.

[21] M. Erbar, K. Kuwada, and K.-T. Sturm, *On the equivalence of the entropic curvature-dimension condition and Bochner's inequality on metric measure spaces*, Invent. Math., 201 (2015), pp. 993–1071.

[22] N. Gigli, *The splitting theorem in non-smooth context.* Preprint, arXiv:1302.5555., 2013.

[23] ——, *Non-smooth differential geometry.* Preprint, arXiv:1407.0809. To appear on Mem. Amer. Math. Soc., 2014.

[24] ——, *On the differential structure of metric measure spaces and applications*, Mem. Amer. Math. Soc., 236 (2015), pp. vi+91.

[25] N. Gigli and B.-X. Han, *The continuity equation on metric measure spaces*, Calc. Var. Partial Differential Equations, 53 (2015), pp. 149–177.

[26] ——, *Sobolev spaces on warped products.* Preprint, arXiv:1512.03177, 2015.

[27] N. Gigli, C. Ketterer, K. Kuwada, and S.-i. Ohta, *Rigidity for the spectral gap on $RCD(K,\infty)$-spaces.* Preprint, arXiv:1709.04017., 2017.

[28] N. Gigli and E. Pasqualetto, *Behaviour of the reference measure on RCD spaces under charts.* Preprint, arXiv:1607.05188, 2016.

[29] N. Gigli and C. Rigoni, *Recognizing the flat torus among $RCD^*(0,N)$ spaces via the study of the first cohomology group.* Preprint, arXiv:1705.04466, 2017.

[30] N. Gigli and L. Tamanini, *Second order differentiation formula on compact $RCD^*(K,N)$ spaces.* Preprint, arXiv:1701.03932, 2017.

[31] B.-X. Han, *Ricci tensor on $RCD^*(K,N)$ spaces.* Preprint, arXiv:1412.0441. To appear on J. Geom. Anal., 2017.

[32] M. Kell and A. Mondino, *On the volume measure of non-smooth spaces with ricci curvature bounded below*, Annali della Scuola Normale Superiore di Pisa, (2016).

[33] C. Ketterer, *Obata's rigidity theorem for metric measure spaces*, Anal. Geom. Metr. Spaces, 3 (2015), pp. 278–295.

[34] K. Kuwada, *Duality on gradient estimates and Wasserstein controls*, J. Funct. Anal., 258 (2010), pp. 3758–3774.

[35] J. Lierl and K.-T. Sturm, *Neumann heat flow and gradient flow for the entropy on non-convex domains.* Preprint, arXiv:1704.04164, 2017.

[36] S. Lisini, *Characterization of absolutely continuous curves in Wasserstein spaces*, Calc. Var. Partial Differential Equations, 28 (2007), pp. 85–120.

[37] J. Lott and C. Villani, *Ricci curvature for metric-measure spaces via optimal transport*, Ann. of Math. (2), 169 (2009), pp. 903–991.

[38] A. Mondino and A. Naber, *Structure theory of metric-measure spaces with lower ricci curvature bounds I.* Preprint, arXiv:1405.222, 2014.

[39] T. Rajala and K.-T. Sturm, *Non-branching geodesics and optimal maps in strong $CD(K,\infty)$-spaces*, Calc. Var. Partial Differential Equations, 50 (2014), pp. 831–846.

[40] G. Savaré, *Self-improvement of the Bakry-émery condition and Wasserstein contraction of the heat flow in $RCD(K,\infty)$ metric measure spaces*, Disc. Cont. Dyn. Sist. A, 34 (2014), pp. 1641–1661.

[41] N. Shanmugalingam, *Newtonian spaces: an extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoamericana, 16 (2000), pp. 243–279.

[42] K.-T. Sturm, *On the geometry of metric measure spaces. I*, Acta Math., 196 (2006), pp. 65–131.

[43] ——, *On the geometry of metric measure spaces. II*, Acta Math., 196 (2006), pp. 133–177.

[44] ——, *Gradient flows for semiconvex functions on metric measure spaces - existence, uniqueness and lipschitz continuity.* Preprint, arXiv:1410.3966, 2014.

[45] C. Villani, *Optimal transport. Old and new*, vol. 338 of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 2009.

[46] M.-K. von Renesse and K.-T. Sturm, *Transport inequalities, gradient estimates, entropy, and Ricci curvature*, Comm. Pure Appl. Math., 58 (2005), pp. 923–940.

Bang-Xian Han, Institute for applied mathematics, University of Bonn
Endenicher Allee 60 , D-53115 Bonn, Germany
Email: han@iam.uni-bonn.de