# Error estimates for dynamic augmented Lagrangian boundary condition enforcement, with application to immersogeometric fluid-structure interaction 

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#### Abstract

In this work, we analyze the convergence of the recent numerical method for enforcing fluidstructure interaction (FSI) kinematic constraints in the immersogeometric framework for cardiovascular fluid-structure interaction. In the immersogeometric framework, the structure is modeled as a thin shell, and its influence on the fluid subproblem is imposed as a forcing term. This force has the interpretation of a Lagrange multiplier field supplemented by penalty forces, in an augmented Lagrangian formulation of the FSI kinematic constraints. Because of the non-matching fluid and structure discretizations used, no discrete inf-sup condition can be assumed. To avoid solving (potentially unstable) discrete saddle point problems, the penalty forces are treated implicitly and the multiplier field is updated explicitly. In the present contribution, we introduce the term dynamic augmented Lagrangian (DAL) to describe this time integration scheme. We formulate the DAL algorithm for a linearized parabolic model problem, analyze the regularity of solutions to this problem, and provide error estimates for the DAL method in both the $L^{\infty}\left(H^{1}\right)$ and $L^{\infty}\left(L^{2}\right)$ norms. We also prove error estimates for a recently-proposed extension of the DAL method, with improved conservation properties. Numerical experiments indicate that the derived estimates are sharp and that the results of the model problem analysis can be extrapolated to the setting of nonlinear FSI, for which the numerical method was originally proposed.


Keywords: immersogeometric method, fluid-structure interaction (FSI), augmented Lagrangian method, parabolic initial-boundary value problem, sub-optimal error estimates, order of

[^0]
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## 1. Introduction

Recent years have seen great interest in numerical analysis of fluid-structure interaction (FSI) [1-26] due to its relevance to structural [27, 28], biomedical [29], and other engineering applications [30]. In a recent series of articles [31-36], we developed a framework for simulating FSI dynamics of thin, flexible shell structures immersed in a viscous, incompressible fluid, where we assume that the thin structure can cut through the fluid meshes and the fluid/structure meshes do not have to match each other on the fluid-structure interface [37-39]. The target application was bioprosthetic heart valve [40] analysis. Heart valves are anatomical structures in the heart, regulating the direction of blood flow. Bioprosthetic heart valves are artificial replacements for diseased valves that mimic the structure of native valves: they consist of several thin elastic leaflets that are pushed open by flow in one direction and shut by flow in the other direction. FSI analysis could become an important tool for understanding and designing bioprosthetic heart valves; however, this problem class presents special difficulties [41-43]: (1) due to the thinness of the heart valve leaflets, shell models are typically employed for efficiency. However, the high-order derivatives in some shell models (e.g., Kirchhoff-Love shells [44]) require additional smoothness on the numerical solutions; (2) in heart valve problems the fluid and structure densities are close. Convergence of coupled solvers is problematic because of the "added-mass" effect [45]; (3) the fluid velocity gradients and blood pressure are discontinuous across the leaflets, which is difficult to approximate in unfitted discretizations; (4) the leaflets undergo large deformations in each cardiac cycle, which causes significant changes in the geometry of the region occupied by fluid.

To accommodate large deformations of and contact between bioprosthetic heart valve leaflets, we pursued an immersed boundary numerical method (see, e.g., [16-18, 46]), in which the fluid and structure are discretized separately and coupled in the numerical method. To simplify the translation of bioprosthetic heart valve designs into analysis models, we used an isogeometric discretization [47,48] of the bioprosthetic heart valve leaflets, representing discrete approximations to the valve's deformation with the same spline function space used to design its geometry. Moreover, isogeometric spline spaces can be as smooth as the geometry allows, which permits straightforward discretization of fourth-order thin shell models. We introduced the term immersogeometric analysis in [31], to identify the idea of directly using design geometries as immersed boundaries. References [28, 49-51] explore immersogeometric analysis in several application areas, using a variety of numerical methods.

This paper focuses on the particular numerical method introduced in [31], and refinements of it developed in subsequent work [36]. This numerical method is specialized for problems in which the structure is modeled geometrically as a surface of co-dimension one to the fluid sub-
problem domain. The interaction between the shell structure and fluid is mediated by a Lagrange multiplier field, which enforces the constraint that the fluid and shell structure velocities match along the shell structure's midsurface. Immersed boundary numerical methods based on this distributed Lagrange multiplier concept have their origins in the work of Glowinski and collaborators [52, 53], and continue to be studied today by Boffi, Gastaldi and Cavallini [22, 54]. In [53], a Lagrange-multiplier based fictitious domain method is presented and validated on a fluid-solid coupling problem, where both the fluid and solid subdomains are in 2D. In this method, the fluid flow equations are enforced in the whole domain, including the solid subdomain, and the solid boundary is constrained using a distributed Lagrange multiplier. In [22], the authors provided optimal error estimates for the Lagrange-multiplier based fictitious domain method on a simplified linear model for the solid. Besides the case when the fluid and solid have the same dimension, the authors have further discussed the case of a thin solid immersed in a fluid. However, in [22] the error estimates depend on the problem regularity, i.e., the smoothness of the solutions, which was not discussed in the paper. Moreover, the Lagrange-multiplier based fictitious domain method entails the construction of a saddle point problem, which is fraught with practical difficulties in the discrete setting. Drawing inspiration from discrete optimization [55, 56] and solution methods for contact problems [57] (where non-matching discretizations are the norm), we first attempted to use an augmented Lagrangian iteration. This iteration avoids directly solving a saddle point problem by introducing auxiliary penalty forces, then alternating between 1 : solving an unconstrained problem, with fixed Lagrange multipliers, and 2: using the penalty forces to update the Lagrange multipliers. Finding the convergence of this iteration unreliable, we truncated the original augmented Lagrangian iteration, updating the Lagrange multiplier only once each time step. This can be reinterpreted as an implicit discretization of the feedback forcing method of Goldstein et al. [58-62], in which the fluid-structure forcing (i.e. Lagrange multiplier) is governed by a stiff differential equation in time, essentially penalizing the time integral of fluid-structure velocity discrepancy [31, Section 4.3]. Feedback forcing is a rate form of the fluid-structure displacement penalties [63, Section 4.2.1] used quite widely to simulate airbag inflation [64], heart valve FSI [65-68], and other fluid-thin structure interaction phenomena [69-71].

Retaining the Lagrange multiplier viewpoint has allowed us to devise stabilization schemes [34, 36] that would not clearly emerge from the picture of accumulating a fluid-structure displacement difference in rate form and penalizing it. In this paper, we introduce the term dynamic augmented Lagrangian (DAL) to describe the resulting family of numerical methods for imposing the Dirichlet boundary conditions on the fluid-structure interface. The idea of DAL was initially introduced for immersogeometric analysis in [31], and came from a combination of heuristic analogies to feedback methods and results of numerical experiments. In [35, Section 3], we began to undertake a numerical analysis of DAL, by introducing a scalar parabolic model problem with a

Dirichlet boundary condition applied on a surface cutting through the interior of the domain. Our analysis of the model problem in [35] proceeded as follows: First, we related the model problem to a feedback forcing regularization. Second, we analyzed the error in discretizing this regularized problem in space and time. However, the dependence of the regularized problem's coefficients on refinement parameters introduced many technical difficulties into the analysis, and numerical experiments indicated that the resulting error bounds were not sharp.

The present paper aims to address the solution regularity for the immersed thin solid problem, and based on this regularity further investigate the convergence rates for DAL methods. We provide a novel analysis of the original DAL on the parabolic model problem introduced in [35] and, for the first time, analyze a recent extension of DAL [36]. In this extension, the multiplier is projected onto a coarser space, to circumvent the trade-off between stability and conservation that results from the original DAL [34, 35]. The new analysis does not rely on passing through a regularized problem, and arrives at sharp bounds. We begin by reviewing the fluid-thin structure interaction problem setting (Section 2) and the immersogeometric FSI framework (Section 3) in which the two DAL methods are developed and applied. Then, in Section 4, we recall the linear parabolic model problem introduced in [35, Section 3.1.1] and study the regularity of its solutions. Section 5 analyzes the convergence of the original DAL and the projection-based DAL methods in the context of this model problem. Numerical testing in Section 6 supports the analysis on linear problems, and in Section 7 the numerical results on a novel benchmark 2D FSI problem indicates that the conclusions of the model problem analysis extrapolate to much more complicated problems. Section 8 summarizes our findings and discusses future research.

## 2. Mathematical model for FSI

This section defines the mathematical model of fluid-thin structure interaction in which the DAL methods we consider here were originally developed. We also outline the discretizations used for the fluid and thin structure subproblems.

### 2.1. Augmented Lagrangian formulation of FSI

We begin with a versatile augmented Lagrangian framework for FSI [72], which we specialize to the case of thin immersed structures. The region occupied by fluid is denoted $\Omega \subset \mathbb{R}^{3}$, and the deformed structure geometry at time $t$ is modeled by the 2 D surface $\Gamma_{t} \subset \Omega$. Let $\mathbf{u}$ and $p$ denote the fluid velocity and pressure fields. Let $\mathbf{y}$ denote the structure displacement relative to some reference configuration, $\Gamma_{0}$. The structure velocity is denoted $\boldsymbol{\eta} \equiv \dot{\mathbf{y}}$. The fluid-structure kinematic constraint that $\mathbf{u}=\boldsymbol{\eta}$ on $\Gamma_{t}$ is enforced using the augmented Lagrangian

$$
\begin{equation*}
\int_{\Gamma_{t}} \lambda \cdot(\mathbf{u}-\boldsymbol{\eta}) d \Gamma+\frac{1}{2} \int_{\Gamma_{t}} \beta|\mathbf{u}-\boldsymbol{\eta}|^{2} d \Gamma, \tag{2.1}
\end{equation*}
$$

in which $\lambda$ is a Lagrange multiplier and $\beta \geq 0$ is a penalty parameter. This results in the following variational problem: Find $\mathbf{u} \in \mathcal{S}_{u}, p \in \mathcal{S}_{p}, \mathbf{y} \in \mathcal{S}_{d}$, and $\boldsymbol{\lambda} \in \mathcal{S}_{\ell}$ such that, for all $\mathbf{w}_{1} \in \mathcal{V}_{u}, q \in \mathcal{V}_{p}$, $\mathbf{w}_{2} \in \mathcal{V}_{d}$, and $\delta \boldsymbol{\lambda} \in \mathcal{V}_{\ell}$

$$
\begin{array}{r}
B_{1}\left(\{\mathbf{u}, p\},\left\{\mathbf{w}_{1}, q\right\}\right)-F_{1}\left(\left\{\mathbf{w}_{1}, q\right\}\right)+\int_{\Gamma_{t}} \mathbf{w}_{1} \cdot \boldsymbol{\lambda} d \Gamma+\int_{\Gamma_{t}} \mathbf{w}_{1} \cdot \beta(\mathbf{u}-\boldsymbol{\eta}) d \Gamma=0, \\
B_{2}\left(\mathbf{y}, \mathbf{w}_{2}\right)-F_{2}\left(\mathbf{w}_{2}\right)-\int_{\Gamma_{t}} \mathbf{w}_{2} \cdot \boldsymbol{\lambda} d \Gamma-\int_{\Gamma_{t}} \mathbf{w}_{2} \cdot \beta(\mathbf{u}-\boldsymbol{\eta}) d \Gamma=0, \\
\int_{\Gamma_{t}} \delta \boldsymbol{\lambda} \cdot(\mathbf{u}-\boldsymbol{\eta}) d \Gamma=0, \tag{2.4}
\end{array}
$$

where $\mathcal{S}_{u}, \mathcal{S}_{p}, \mathcal{S}_{d}$, and $\mathcal{S}_{\ell}$ are trial solution spaces for the different solution components and $\mathcal{V}_{u}$, $\mathcal{V}_{p}, \mathcal{V}_{d}$, and $\mathcal{V}_{\ell}$ are the corresponding test spaces. $B_{1}, B_{2}, F_{1}$, and $F_{2}$ are semi-linear forms and linear functionals defining the fluid and structure subproblems.

### 2.2. Fluid subproblem

We assume the fluid to be incompressible and Newtonian, with the following weak formulation

$$
\begin{align*}
B_{1}(\{\mathbf{u}, p\},\{\mathbf{w}, q\})= & \int_{\Omega} \mathbf{w} \cdot \rho_{f}\left(\left.\frac{\partial \mathbf{u}}{\partial t}\right|_{\mathbf{x}}+\mathbf{u} \cdot \nabla \mathbf{u}\right) d \Omega+\int_{\Omega} \varepsilon(\mathbf{w}): \sigma_{f} d \Omega \\
& +\int_{\Omega} q \nabla \cdot \mathbf{u} d \Omega-\gamma \int_{\Gamma_{N}} \mathbf{w} \cdot \rho_{f}\left\{\mathbf{u} \cdot \mathbf{n}_{f}\right\}_{-} \mathbf{u} d \Gamma  \tag{2.5}\\
F_{1}(\{\mathbf{w}, q\})= & \int_{\Omega} \mathbf{w} \cdot \rho_{f} \mathbf{f}_{1} d \Omega+\int_{\Gamma_{N}} \mathbf{w} \cdot \mathbf{T}_{1} d \Gamma \tag{2.6}
\end{align*}
$$

where $\rho_{1}$ is the mass density of the fluid, $\varepsilon(\cdot)$ is the symmetric gradient, $\sigma_{f}=-p \mathbf{I}+2 \mu \varepsilon(\mathbf{u})$ is the fluid Cauchy stress, $\mu$ is the dynamic viscosity, $\mathbf{f}_{1}$ is a body force, $\mathbf{T}_{1}$ is a traction on $\Gamma_{N} \subset \partial \Omega$, and $\mathbf{n}_{f}$ is the unit outward-facing normal to $\Omega . \partial(\cdot) /\left.\partial t\right|_{\mathbf{x}}$ indicates time differentiation holding $\mathbf{x} \in \Omega$ fixed. In the last term of (2.5), the function $\{\cdot\}_{-}$extracts the negative part of $(\cdot)$ :

$$
\{x\}_{-}=\left\{\begin{array}{rr}
0 & x>0  \tag{2.7}\\
x & \text { otherwise }
\end{array} .\right.
$$

This enhances the stability of the problem in situations where flow enters through $\Gamma_{N}$ [35, 73]. The dimensionless coefficient $\gamma$ controls the strength of stabilization.

Our past work on immersogeometric analysis has discretized this subproblem in a number of ways. In [31], we used the variational multiscale (VMS) formulation, with equal-order pressure and velocity interpolation. This suffered from difficulties with mass loss in the discrete fluid solution. In [35], we circumvented this issue by applying a modification of the divergence conforming (or div-conforming) discretization described in Evans and Hughes [74-76] and based on work by

Buffa et al. [77, 78]. We refer interested readers to [74] for further information on div-conforming B-splines and [35, Section 2.2] for details of the implementation we used for immersogeometric FSI analysis. Numerical FSI examples in this paper use the div-conforming fluid discretization.

### 2.3. Thin structure subproblem

We define the forms $B_{2}$ and $F_{2}$ for the structure subproblem by assuming Kirchhoff-Love thin shell kinematics [44, 79, 80]:

$$
\begin{equation*}
B_{2}(\mathbf{y}, \mathbf{w})=\left.\int_{\Gamma_{t}} \mathbf{w} \cdot \rho_{s} h_{\mathrm{th}} \frac{\partial^{2} \mathbf{y}}{\partial t}\right|_{\mathbf{X}} d \Gamma+\int_{\Gamma_{0}} \int_{-h_{\mathrm{th}} / 2}^{h_{\mathrm{th}} / 2} D_{\mathbf{w}} \mathbf{E}: \mathbf{S} d \xi d \Gamma \tag{2.8}
\end{equation*}
$$

and

$$
\begin{equation*}
F_{2}(\mathbf{w})=\int_{\Gamma_{t}} \mathbf{w} \cdot \rho_{s} h_{\mathrm{th}} \mathbf{f}_{2} d \Gamma+\int_{\Gamma_{t}} \mathbf{w} \cdot \mathbf{h}^{\mathrm{net}} d \Gamma, \tag{2.9}
\end{equation*}
$$

where $\rho_{s}$ is mass density, $\mathbf{f}_{2}$ is a body force, $h_{\mathrm{th}}$ is the thickness of the shell, and $\xi$ is a coordinate parameterizing the through-thickness direction. The elasticity term is referred to the reference configuration; $\mathbf{E}$ is the Green-Lagrange strain tensor,

$$
\begin{equation*}
D_{\mathbf{w}} \mathbf{E}(\mathbf{y})=\left.\frac{d}{d \epsilon} \mathbf{E}(\mathbf{y}+\epsilon \mathbf{w})\right|_{\epsilon=0}, \tag{2.10}
\end{equation*}
$$

and $\mathbf{S}$ is the second Piola-Kirchhoff stress tensor. The last term of $F_{2}$ combines tractions prescribed on both sides of $\Gamma_{t}: \mathbf{h}^{\text {net }}=\mathbf{h}\left(\xi=-h_{\mathrm{th}} / 2\right)+\mathbf{h}\left(\xi=+h_{\mathrm{th}} / 2\right)$. The time derivative $\partial(\cdot) /\left.\partial t\right|_{\mathbf{X}}$ is taken holding $\mathbf{X} \in \Gamma_{0}$ fixed. $\mathbf{S}$ can be computed from $\mathbf{E}$ using an arbitrary constitutive model. Computing $\mathbf{E}$ from the midsurface displacement $\mathbf{y}$ relies on kinematic assumptions detailed in [44, 79, 80].

In the resulting thin shell subproblem, the smoothness of the solution space is especially important because of the high-order derivatives of $\mathbf{y}$ and $\mathbf{w}$ resulting from the mapping of $\mathbf{y}$ to $\mathbf{E}$. For $B_{2}(\mathbf{w}, \mathbf{y})$ to remain bounded, $\mathbf{w}$ and $\mathbf{y}$ must be at least in $H^{2}(\Gamma)$. Therefore, in this paper we employ isogeometric analysis (IGA) spline spaces to discretize the thin shell subproblem in space, because IGA accommodates the additional smoothness required of numerical solutions.

## 3. Dynamic augmented Lagrangian methods for fluid-structure coupling

This section discusses how we discretize the constraint coupling the subproblems. We describe two variants of the DAL method: the original DAL approach proposed in [31] (including a stabilization technique added in [34]) and a modified version proposed in [36], which projects the Lagrange multiplier onto a coarse discrete space.

### 3.1. The original DAL approach

In [31], we formally separated the constraint $\mathbf{u}=\boldsymbol{\eta}$ on $\Gamma$ into no-penetration and no-slip components:

$$
\begin{equation*}
\mathbf{u} \cdot \mathbf{n}_{f}=-\boldsymbol{\eta} \cdot \mathbf{n}_{s} \tag{3.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbf{u}-\left(\mathbf{u} \cdot \mathbf{n}_{f}\right) \mathbf{n}_{f}=\boldsymbol{\eta}-\left(\boldsymbol{\eta} \cdot \mathbf{n}_{s}\right) \mathbf{n}_{s} \tag{3.2}
\end{equation*}
$$

where $\mathbf{n}_{s}=-\mathbf{n}_{f}$. 3.1) and (3.2) are enforced by normal and tangential components of $\boldsymbol{\lambda}$, respectively, as well as the penalty forcing $\beta(\mathbf{u}-\boldsymbol{\eta})$. In the applications with a closed structure separating regions of different pressures (e.g. an aortic heart valve during diastole), the thin structure must be able to prevent leakage to maintain the correct qualitative solution behavior. On the other hand, the no-slip constraint is less essential; its strong enforcement may even be detrimental to solution quality on coarse meshes [81-84]. We therefore discretized these constraint components differently. For the no-slip constraint, we simply rely on imposing consistency with a penalty integral term and neglect the corresponding component of the Lagrange multiplier field. For the no-penetration constraint, we retain a scalar Lagrange multiplier field on $\Gamma$, denoted $\lambda=\boldsymbol{\lambda} \cdot \mathbf{n}_{f}$, to strengthen enforcement of non-penetration. Because $\Gamma_{t}$ can cut through the fluid domain arbitrarily, it would be difficult to construct inf-sup stable combinations of discrete velocity and multiplier spaces. We discretize $\lambda$ as a set of samples at quadrature points, which may be viewed as coefficients in a linear combination of piecewise-constant basis functions, each supported on the patch of surface area associated with a quadrature point (cf. Pinsky's interpretation [85, (6.6)-(6.7)] of discretizing the plastic multiplier at quadrature points in a three-field formulation of elastoplasticity). However, we do not place any upper bound on the density of quadrature points relative to the fluid and structure discretizations, so no discrete inf-sup condition can be assumed. To stabilize the formulation, we introduced (in [34]) a relaxation parameter, $r$, to regularize the no-penetration constraint residual:

$$
\begin{equation*}
(\mathbf{u}-\boldsymbol{\eta}) \cdot \mathbf{n}_{f} \quad \rightarrow \quad(\mathbf{u}-\boldsymbol{\eta}) \cdot \mathbf{n}_{f}-\frac{r}{\beta} \lambda . \tag{3.3}
\end{equation*}
$$

This "perturbed Lagrangian" approach for multiplier stabilization has previously been applied to contact problems [86]. The use of a perturbed residual and a multiplier discretized at quadrature points is clearly equivalent to a penalty method, in which the penalty parameter scales like $\sim$ $\tau_{\mathrm{NOR}}^{B} / r$. The reason for retaining an unknown multiplier field is that it allows us to formulate a semi-implicit time discretization that does not rely on solving a severely ill-conditioned problem when $r \ll 1$. (In the initial formulation of [31], $r$ was in fact zero.)

These modifications of the constraint lead to the following problem: Find $\mathbf{u} \in \mathcal{S}_{u}, p \in \mathcal{S}_{p}$,
$\mathbf{y} \in \mathcal{S}_{d}$, and $\lambda \in \mathcal{S}_{\ell}$ such that, for all test functions $\mathbf{w}_{1} \in \mathcal{V}_{u}, q \in \mathcal{V}_{p}, \mathbf{w}_{2} \in \mathcal{V}_{d}$, and $\delta \lambda \in \mathcal{V}_{\ell}$

$$
\begin{align*}
& B_{1}\left(\left\{\mathbf{w}_{1}, q\right\},\{\mathbf{u}, p\}\right)-F_{1}\left(\left\{\mathbf{w}_{1}, q\right\}\right)+B_{2}\left(\mathbf{w}_{2}, \mathbf{y}\right)-F_{2}\left(\mathbf{w}_{2}\right) \\
& \quad+\int_{\Gamma_{t}}\left(\mathbf{w}_{1}-\mathbf{w}_{2}\right) \cdot \lambda \mathbf{n}_{f} d \Gamma \\
& \quad+\int_{\Gamma_{t}}\left(\mathbf{w}_{1}-\mathbf{w}_{2}\right) \cdot \tau_{\mathrm{NOR}}^{B}\left((\mathbf{u}-\boldsymbol{\eta}) \cdot \mathbf{n}_{f}\right) \mathbf{n}_{f} d \Gamma \\
& \quad+\int_{\Gamma_{t}}\left(\mathbf{w}_{1}-\mathbf{w}_{2}\right) \cdot \tau_{\mathrm{TAN}}^{B}\left((\mathbf{u}-\boldsymbol{\eta})-\left((\mathbf{u}-\boldsymbol{\eta}) \cdot \mathbf{n}_{f}\right) \mathbf{n}_{f}\right) d \Gamma \\
& \quad+\int_{\Gamma_{t}} \delta \lambda \cdot\left((\mathbf{u}-\boldsymbol{\eta}) \cdot \mathbf{n}_{f}-\frac{r \lambda}{\tau_{\mathrm{NOR}}^{B}}\right) d \Gamma=0 . \tag{3.4}
\end{align*}
$$

Applying spatial discretizations to the subproblems of (3.4), as discussed in previous sections, allows us to represent the semi-discrete solutions as vectors of basis function coefficients, which is convenient for discussing time integration algorithms. When describing time stepping procedures for the fully-discrete problem, we use $n$ to denote the time step index, and denote the spatially discretized fluid velocity solution as a vector of basis function coefficients $\mathbf{U}^{n}$. Likewise, the spatially discretized fluid velocity time derivative, fluid pressure, structure displacement, and displacement time derivatives are denoted $\dot{\mathbf{U}}^{n}, \mathbf{P}^{n}, \mathbf{Y}^{n}, \dot{\mathbf{Y}}^{n}$, and $\ddot{\mathbf{Y}}^{n}$. We denote the multiplier at time level $n$ as $\lambda^{n}$.

The time discretization proposed in [31, 34] proceeds in two phases at each time step. First, $\lambda$ is held fixed at $\lambda^{n}$, and the penalty-coupled problem is solved implicitly. Second, $\lambda^{n+1}$ is computed explicitly. More precisely, the algorithm of [31, 34] considered solution variables at time level $n$ known, and first solved the following problem for all $(n+1)$-level unknowns except $\lambda^{n+1}$ :

$$
\begin{align*}
& \operatorname{Res}\left(\mathbf{U}^{n+\alpha_{f}}, \dot{\mathbf{U}}^{n+\alpha_{m}}, \mathbf{Y}^{n+\alpha_{f}}, \dot{\mathbf{Y}}^{n+\alpha_{f}}, \ddot{\mathbf{Y}}^{n+\alpha_{m}}, \mathbf{P}^{n+1}, \lambda^{n}\right)=\mathbf{0},  \tag{3.5}\\
& \mathbf{U}^{n+1}=\mathbf{U}^{n}+\Delta t\left((1-\gamma) \dot{\mathbf{U}}^{n}+\gamma \dot{\mathbf{U}}^{n+1}\right),  \tag{3.6}\\
& \dot{\mathbf{U}}^{n+\alpha_{m}}=\dot{\mathbf{U}}^{n}+\alpha_{m}\left(\dot{\mathbf{U}}^{n+1}-\dot{\mathbf{U}}^{n}\right),  \tag{3.7}\\
& \mathbf{U}^{n+\alpha_{f}}=\mathbf{U}^{n}+\alpha_{f}\left(\mathbf{U}^{n+1}-\mathbf{U}^{n}\right),  \tag{3.8}\\
& \mathbf{Y}^{n+1}=\mathbf{Y}^{n}+\Delta t \dot{\mathbf{Y}}^{n}+\frac{\Delta t^{2}}{2}\left((1-2 \beta) \ddot{\mathbf{Y}}^{n}+2 \beta \ddot{\mathbf{Y}}^{n+1}\right),  \tag{3.9}\\
& \dot{\mathbf{Y}}^{n+1}=\dot{\mathbf{Y}}^{n}+\Delta t\left((1-\gamma) \ddot{\mathbf{Y}}^{n}+\gamma \ddot{\mathbf{Y}}^{n+1}\right),  \tag{3.10}\\
& \ddot{\mathbf{Y}}^{n+\alpha_{m}}=\ddot{\mathbf{Y}}^{n}+\alpha_{m}\left(\ddot{\mathbf{Y}}^{n+1}-\ddot{\mathbf{Y}}^{n}\right),  \tag{3.11}\\
& \dot{\mathbf{Y}}^{n+\alpha_{f}}=\dot{\mathbf{Y}}^{n}+\alpha_{f}\left(\dot{\mathbf{Y}}^{n+1}-\dot{\mathbf{Y}}^{n}\right),  \tag{3.12}\\
& \mathbf{Y}^{n+\alpha_{f}}=\mathbf{Y}^{n}+\alpha_{f}\left(\mathbf{Y}^{n+1}-\mathbf{Y}^{n}\right), \tag{3.13}
\end{align*}
$$

where $\alpha_{m}, \alpha_{f}, \beta$, and $\gamma$ are time integration parameters. $\operatorname{Res}(\ldots)$ is the nonlinear algebraic residual
corresponding to discretization of (3.4) with $\lambda$ fixed and $\mathcal{V}_{\ell}=\emptyset$. The subproblems are still coupled through the penalty term, but in a more gentle way, which can be resolved using block iteration [35, Section 4]. Equations (3.5)-(3.13) come from the generalized $-\alpha$ method of time integration [87]. In [31-34], we followed [88, Section 4.4] by using a subset of generalized- $\alpha$ methods parameterized by $\rho_{\infty} \in[0,1]$, which controls numerical damping and defines the four free parameters as

$$
\begin{align*}
\alpha_{m} & =\frac{1}{2}\left(\frac{3-\rho_{\infty}}{1+\rho_{\infty}}\right)  \tag{3.14}\\
\alpha_{f} & =\frac{1}{1+\rho_{\infty}},  \tag{3.15}\\
\gamma & =\frac{1}{2}+\alpha_{m}-\alpha_{f},  \tag{3.16}\\
\beta & =\frac{1}{4}\left(1+\alpha_{m}-\alpha_{f}\right)^{2} . \tag{3.17}
\end{align*}
$$

The mathematical interpretation of $\rho_{\infty}$ is it is the spectral radius of the amplification matrix as $\Delta t \rightarrow \infty$; see [87]. In FSI examples within this work, we maintain a direct analogy to the linear model problems analyzed in Section 5 by using the backward Euler method instead. This can be conveniently implemented within the generalized- $\alpha$ predictor-multi-corrector scheme of [88] by setting the generalized $-\alpha$ parameters to $\alpha_{m}=\alpha_{f}=\gamma=\beta=1$ and modifying the displacement predictor to be consistent with the backward Euler method.

The second, explicit phase of the DAL time stepping procedure is to update the Lagrange multiplier:

$$
\begin{equation*}
\lambda^{n+1} \leftarrow \frac{\lambda^{n}+\tau_{\mathrm{NOR}}^{B}\left(\mathbf{u}^{n+\alpha_{f}}-\boldsymbol{\eta}^{n+\alpha_{f}}\right) \cdot \mathbf{n}_{f}^{n+\alpha_{f}}}{1+r} \tag{3.18}
\end{equation*}
$$

The present semi-implicit algorithm is in fact stable in an energetic sense, as discussed physically in [34, Section 3.2] and mathematically in [35]. In summary, then, the original DAL approach employs the following two-phase algorithm in each time step:

1. Solve (3.5)-(3.13) with $\lambda^{n}$ fixed.
2. Update $\lambda^{n+1}$ explicitly, by (3.18).

Regarding the choices of the penalty parameters, we suggested, based on dimensional analysis and physical considerations [35, Section 5.2.1], to set

$$
\begin{equation*}
\tau_{\mathrm{TAN}}^{B}=C_{\mathrm{TAN}} \frac{\mu}{h}, \tag{3.19}
\end{equation*}
$$

and

$$
\begin{equation*}
\tau_{\mathrm{NOR}}^{B}=\max \left\{C_{\mathrm{NOR}}^{\mathrm{inert}} \frac{\rho_{1} h}{\Delta t}, C_{\mathrm{NOR}}^{\mathrm{visc}} \frac{\mu}{h}\right\} \tag{3.20}
\end{equation*}
$$

where $C_{\mathrm{TAN}}, C_{\mathrm{NOR}}^{\mathrm{inert}}$, and $C_{\mathrm{NOR}}^{\mathrm{visc}}$ are dimensionless $O(1)$ constants and $h$ is the fluid element diameter.

When analyzing this original DAL method for the linear parabolic problem in Section 5, we consider the normal penalty only and take the asymptotic value of the parameter as $\tau_{\mathrm{NOR}}^{B}=\beta=O(1 / h)$.

### 3.2. The projection-based DAL

As discussed in [35], the original DAL method requires $r>0$ for a stable spatial discretization. In [36], we suggested a stabilized method which projects the Lagrangian multiplier $\lambda$ onto a coarse element space satisfying an inf-sup condition. This spatial discretization remains stable while satisfying the kinematic conservation law

$$
\begin{equation*}
\int_{\Gamma_{t}}(\mathbf{u}-\boldsymbol{\eta}) \cdot \mathbf{n}_{f} d \Gamma=0 \tag{3.21}
\end{equation*}
$$

which can only be recovered in the unstable limit of $r \rightarrow 0$ with the original DAL approach.
In this section, we first describe the stabilized immersogeometric framework with fully projected Lagrangian multiplier, which is equivalent to the $r \rightarrow \infty$ limit of the formulation from [36]. Then we discuss the semi-implicit time integration method for this immersogemetric framework, in which the projection-based DAL method is introduced and employed. Further discussions on the inf-sup stability for the projection-based DAL method as well as its error estimates will be provided in Section 5 .

We start with a modification of the algorithm provided in (3.4), by introducing the projection operator $P$ as an $L^{2}$ projection from $\mathcal{V}_{\ell}$ to a finite-dimensional subspace $\mathcal{V}_{H}$ : For $\lambda \in \mathcal{V}_{\ell}$,

$$
\begin{equation*}
\langle P \lambda, \delta \lambda\rangle=\langle\lambda, \delta \lambda\rangle, \quad \forall \delta \lambda \in \mathcal{V}_{H}, \tag{3.22}
\end{equation*}
$$

where $\langle\cdot, \cdot\rangle$ is the inner product in $L^{2}(\Gamma)$. The space $\mathcal{V}_{H}$ is defined on a coarse mesh of $\Gamma$, with element size $H>h$. In this paper, we keep $C \leq H / h$ where $C>1$ is a sufficiently large constant required by the inf-sup condition, as will be discussed in Section 5. Practically, having $H \gg h$ as $h \rightarrow 0$ would be sufficient for the projection-based DAL method to converge. We also denote the complementary commutative projector of $P$ as $P^{\perp}=I-P$, then we have,

$$
\begin{equation*}
\left\langle P \lambda, P^{\perp} \delta \lambda\right\rangle=\left\langle P^{\perp} \lambda, P \delta \lambda\right\rangle=0, \quad \forall \lambda, \delta \lambda \in \mathcal{V}_{\ell} . \tag{3.23}
\end{equation*}
$$

The original immersogeometric framework can then be modified, and we employ the spatial discretization as: Find $\mathbf{u} \in \mathcal{S}_{u}, p \in \mathcal{S}_{p}, \mathbf{y} \in \mathcal{S}_{d}$, and $\lambda \in \mathcal{V}_{H}$ such that, for all test functions $\mathbf{w}_{1} \in \mathcal{V}_{u}$,
$q \in \mathcal{V}_{p}, \mathbf{w}_{2} \in \mathcal{V}_{d}$, and $\delta \lambda \in \mathcal{V}_{H}$

$$
\begin{align*}
& B_{1}\left(\left\{\mathbf{w}_{1}, q\right\},\{\mathbf{u}, p\} ; \hat{\mathbf{u}}\right)-F_{1}\left(\left\{\mathbf{w}_{1}, q\right\}\right)+B_{2}\left(\mathbf{w}_{2}, \mathbf{y}\right)-F_{2}\left(\mathbf{w}_{2}\right) \\
& \quad+\int_{\Gamma_{t}}\left(\mathbf{w}_{1}-\mathbf{w}_{2}\right) \cdot \mathbf{n}_{f} \lambda d \Gamma \\
& \quad+\int_{\Gamma_{t}}\left(\mathbf{w}_{1}-\mathbf{w}_{2}\right) \cdot \tau_{\mathrm{NOR}}^{B}\left((\mathbf{u}-\boldsymbol{\eta}) \cdot \mathbf{n}_{f}\right) \mathbf{n}_{f} d \Gamma \\
& \quad+\int_{\Gamma_{t}}\left(\mathbf{w}_{1}-\mathbf{w}_{2}\right) \cdot \tau_{\mathrm{TAN}}^{B}\left((\mathbf{u}-\boldsymbol{\eta})-\left((\mathbf{u}-\boldsymbol{\eta}) \cdot \mathbf{n}_{f}\right) \mathbf{n}_{f}\right) d \Gamma \\
& \quad+\int_{\Gamma_{t}} \delta \lambda(\mathbf{u}-\boldsymbol{\eta}) \cdot \mathbf{n}_{f} d \Gamma=0 . \tag{3.24}
\end{align*}
$$

In this method, we also separate the penalty terms as the tangential and the normal parts, and the tangential and the normal penalty parameters are given as in equations (3.19) and (3.20).

We now consider DAL time discretization of this formulation. The effect of the projection is to modify the explicit multiplier update step of the original DAL method. The update formula (3.18) is replaced by

$$
\begin{equation*}
\lambda^{n+1} \leftarrow P\left(\lambda^{n}+\tau_{\mathrm{NOR}}^{B}\left(\mathbf{u}^{n+\alpha_{f}}-\eta^{n+\alpha_{f}}\right) \cdot \mathbf{n}_{f}^{n+\alpha_{f}}\right) \tag{3.25}
\end{equation*}
$$

Summarizing, then, the projection-based DAL method uses the following two-phase time stepping procedure:

1. Solve (3.5)-(3.13) with $\lambda^{n}$ fixed.
2. Update $\lambda^{n+1}$ by (3.25).

For general $\mathcal{V}_{H}$, the second step would require inverting a matrix. However, this step can be made explicit by considering $\mathcal{V}_{H}$ to consist of piecewise constants on the elements of the coarse boundary mesh, rendering the projection matrix diagonal. As explained in [36], the approximation power of $\mathcal{V}_{H}$ is largely unimportant.

## 4. A model problem and its regularity

References [34, 35] discuss the stability of the original DAL integrator, and the latter provides some initial error estimates in the context of a linear parabolic model problem. However, the estimates of [35] are not sharp and depend on unproven assumptions about the regularity of solutions to the model problem. To further investigate the convergence rate of the original DAL and to develop error estimates for the projection-based DAL, we formulate these methods for a linear model problem. To ensure general applicability of the derived error estimates, we first provide analysis of the regularity of solutions to this model problem.


Figure 1: The domain $\Omega$ and the immersed boundary $\Gamma$.

On the domain as defined in Figure 1, a parabolic interface problem is considered: Given $f \in L^{2}(\Omega)$, find $u=u(\mathbf{x}, t)$ satisfying

$$
\begin{align*}
\frac{\partial u_{i}}{\partial t}-\nabla \cdot\left(c \nabla u_{i}\right) & =f(\mathbf{x}, t), \quad \text { in } \Omega_{i} \times(0, T], i \in\{1,2\},  \tag{4.1a}\\
u(\mathbf{x}, t) & =g(\mathbf{x}, t), \quad \text { on } \Gamma,  \tag{4.1b}\\
u(\mathbf{x}, 0) & =u_{0}(\mathbf{x}), \quad \text { in } \Omega,  \tag{4.1c}\\
u(\mathbf{x}, t) & =0, \quad \text { on } \partial \Omega . \tag{4.1d}
\end{align*}
$$

Here $\Omega, \Omega_{i}$, and $\Gamma$ are shown in Figure 1; $c$ is a linear transformation that we further consider to be multiplication by a constant, $c(\mathbf{v})=\kappa \mathbf{v} ; u_{i}=\left.u\right|_{\Omega_{i}}$, and $g(\mathbf{x}, t)$ can be seen as analogous to the shell structure velocity. In the thin structure problem, the fluid pressure is not continuous across the thin structure, and the second spatial derivative of fluid velocity is not well-defined on the interface; to state a problem for $u$ without reference to the subdomains (which, in practice, may not be well-defined if, e.g., $\Gamma$ is not closed), we re-write the problem in a weak formulation, in which we introduce the Lagrange multiplier $\lambda=\kappa\left(\frac{\partial u_{1}}{\partial \mathbf{n}_{1}}\right)+\kappa\left(\frac{\partial u_{2}}{\partial \mathbf{n}_{2}}\right)$ and consider the parabolic evolution problem in a weak formulation:

$$
\begin{align*}
\left(u_{t}, v\right)+a(u, v)-\langle\lambda, v\rangle & =(f, v), \quad \text { for all test functions } v \in X,  \tag{4.2a}\\
\langle u, \delta \lambda\rangle & =\langle g, \delta \lambda\rangle, \quad \text { for all test functions } \delta \lambda \in M,  \tag{4.2b}\\
u(\mathbf{x}, 0) & =u_{0}(\mathbf{x}), \quad \text { in } \Omega,  \tag{4.2c}\\
u(\mathbf{x}, t) & =0, \quad \text { on } \partial \Omega . \tag{4.2d}
\end{align*}
$$

Here $(\cdot, \cdot)$ and $\langle\cdot, \cdot\rangle$ denote the inner product in $L^{2}(\Omega)$ and in $L^{2}(\Gamma)$, respectively. $X=\left\{w \in L^{2}(\Omega)\right.$ :
$w \in H^{1}(\Omega)$ and $w=0$ on $\left.\partial \Omega\right\}, M=L^{2}(\Gamma)$, and the bilinear operator $a(u, v)$ is given by

$$
a(u, v):=\int_{\Omega} \kappa(\nabla u) \cdot \nabla v,
$$

which is elliptic, i.e., $a(u, u)=\kappa\|\nabla u\|_{L^{2}(\Omega)}^{2}$, and continuous on $H^{1}(\Omega \backslash \Gamma)$, i.e., $a(u, v) \leq$ $\kappa\|\nabla u\|_{L^{2}(\Omega)}\|\nabla v\|_{L^{2}(\Omega)}$.

For the above parabolic problem, we can obtain its regularity by the following results, for $\Omega \in \mathbb{R}^{2}$ and $\Omega \in \mathbb{R}^{3}$ :

Theorem 4.1. For $\Omega \in \mathbb{R}^{2}$ or $\mathbb{R}^{3}$ with finite diameter (diam $\Omega<\infty$ ), when the boundaries $\partial \Omega_{j}$, $j=1,2$ and the force loading $f$ and boundary condition $g$ are all smooth, the solutions $u_{j}(j=1,2)$ to (4.2) are smooth in both space and time [89. Section 7.1.3]. We have $H^{3 / 2-\epsilon}$ regularity (for all $\epsilon>0$ ) for the function

$$
u: \Omega \longrightarrow \mathbb{R},\left.\quad u\right|_{\Omega_{j}}=u_{j}, j=1,2,
$$

and its time derivatives: $\frac{\partial u}{\partial t}, \frac{\partial^{2} u}{\partial t^{2}}$.
Proof: We will discuss the 2D and 3D cases separately. When $\Omega \in \mathbb{R}^{2}$, we need to check $\nabla u \in H^{1 / 2-\epsilon}(\Omega)$ for all $\epsilon>0$, i.e. the Gagliardo seminorm

$$
\|\nabla u\|_{H^{1 / 2-\epsilon}(\Omega)}=\int_{\Omega} \int_{\Omega} \frac{|\nabla u(x)-\nabla u(y)|^{2}}{|x-y|^{3-2 \epsilon}} d x d y
$$

is finite for all $\epsilon>0$. Since both $u_{1}, u_{2}$ are smooth, it follows $|\nabla u(x)-\nabla u(y)|^{2} \in L^{\infty}(\Omega)$, thus it suffices to check

$$
(\forall \epsilon>0) \quad \int_{\Omega \times \Omega} \frac{1}{|x-y|^{3-2 \epsilon}} d x d y<+\infty .
$$

As

$$
\begin{equation*}
\Omega \times \Omega=\left(\Omega_{1} \times \Omega_{1}\right) \cup\left(\Omega_{2} \times \Omega_{1}\right) \cup\left(\Omega_{1} \times \Omega_{2}\right) \cup\left(\Omega_{2} \times \Omega_{2}\right), \tag{4.3}
\end{equation*}
$$

and $u_{j}$ is smooth on $\Omega_{j}, j=1,2$, it suffices to check

$$
(\forall \epsilon>0) \quad \int_{\Omega_{1} \times \Omega_{2}} \frac{1}{|x-y|^{3-2 \epsilon}} d x d y<+\infty .
$$

Let $D:=\operatorname{diam} \Omega$, which we assumed to be finite. Thus for all $x \in \Omega$ it holds $\Omega \subseteq B(x, D)$, and

$$
\int_{\Omega_{1} \times \Omega_{2}} \frac{1}{|x-y|^{3-2 \epsilon}} d x d y \leq \int_{\Omega_{1}} \int_{\Omega_{2} \cap B(x, D)} \frac{1}{|x-y|^{3-2 \epsilon}} d y d x .
$$

The convergence of this integral entirely depends on what happens when $x$ and $y$ are close. Thus
one can define $\Omega_{\delta}:=\left\{x \in \Omega_{1}: \operatorname{dist}(x, \Gamma) \geq \delta\right\}$, then

$$
\begin{align*}
\int_{\Omega_{1}} \int_{\Omega_{2} \cap B(x, D)} \frac{1}{|x-y|^{3-2 \epsilon}} d y d x \leq & \int_{\left\{x \in \Omega_{1}: d \operatorname{dist}(x, \Gamma) \geq 1\right\}} \int_{\Omega_{2}} \frac{1}{|x-y|^{3-2 \epsilon}} d y d x \\
& +\sum_{n=1}^{+\infty} \int_{\Omega_{\frac{1}{n+1}} \backslash \Omega_{\frac{1}{n}}} \int_{\Omega_{2} \cap B(x, D)} \frac{1}{|x-y|^{3-2 \epsilon}} d y d x . \tag{4.4}
\end{align*}
$$

Clearly,

$$
\int_{\left\{x \in \Omega_{1}: \operatorname{dist}(x, \Gamma) \geq 1\right\}} \int_{\Omega_{2}} \frac{1}{|x-y|^{3-2 \epsilon}} d y d x<+\infty
$$

To estimate $\sum_{n=1}^{+\infty} \int_{\Omega_{\frac{1}{n+1}} \backslash \Omega_{\frac{1}{n}}} \int_{\Omega_{2} \cap B(x, D)} \frac{1}{|x-y|^{3-2 \epsilon}} d y d x$, we first note that since $\Gamma$ is smooth, for $n$ sufficiently large the area of the "strip" $\Omega_{\frac{1}{n+1}} \backslash \Omega_{\frac{1}{n}}$ is estimated by $\mathcal{H}^{2}\left(\Omega_{\frac{1}{n+1}} \backslash \Omega_{\frac{1}{n}}\right) \leq \frac{\mathcal{H}^{1}(\Gamma)}{n(n+1)}$, where $\mathcal{H}^{1}$ (resp. $\mathcal{H}^{2}$ ) denotes the 1-Hausdorff measure (resp. 2-Hausdorff measure), which intuitively corresponds to the perimeter (resp. area). Direct computation then gives

$$
\begin{aligned}
\int_{\Omega_{\frac{1}{n+1}} \backslash \Omega_{\frac{1}{n}}} \int_{\Omega_{2} \cap B(x, D)} & \frac{1}{|x-y|^{3-2 \epsilon}} d y d x=\int_{\Omega_{\frac{1}{n+1}} \backslash \Omega_{\frac{1}{n}}} \int_{\Omega_{2} \cap B(0, D)} \frac{1}{|y|^{3-2 \epsilon}} d y d x \\
& \leq \frac{\mathcal{H}^{1}(\Gamma)}{n(n+1)} \int_{\Omega_{2} \cap B(0, D)} \frac{1}{\left.|y|\right|^{3-2 \epsilon}} d y=\frac{2 \pi \mathcal{H}^{1}(\Gamma)}{n(n+1)} \int_{\frac{1}{n+1}}^{D} \rho^{2 \epsilon-2} d \rho \\
& =\frac{2 \pi \mathcal{H}^{1}(\Gamma)}{(1-2 \epsilon) n(n+1)}\left((n+1)^{1-2 \epsilon}-D^{1-2 \epsilon}\right) .
\end{aligned}
$$

Thus

$$
\sum_{n=1}^{+\infty} \int_{\Omega_{\frac{1}{n+1}} \backslash \Omega_{\frac{1}{n}}} \int_{\Omega_{2} \cap B(x, D)} \frac{1}{|x-y|^{3-2 \epsilon}} d y d x \leq \sum_{n=1}^{+\infty} \frac{2 \pi \mathcal{H}^{1}(\Gamma)}{(1-2 \epsilon) n(n+1)}\left((n+1)^{1-2 \epsilon}-D^{1-2 \epsilon}\right)
$$

which converges for all $\epsilon>0$. Thus both right-hand side terms in (4.4) are finite, and $u \in H^{3 / 2-\epsilon}(\Omega)$ for all $\epsilon>0$. We have then proved the regularity for $u$ in the 2D case. For the regularity of its second time derivative, we have the Gagliardo semi-norm as

$$
\left[\nabla u_{t t}\right]_{H^{1 / 2-\epsilon}(\Omega)}^{2}=\int_{\Omega \times \Omega} \frac{\left|\nabla u_{t t}(x)-\nabla u_{t t}(y)\right|^{2}}{|x-y|^{3-2 \epsilon}} d x d y
$$

Since $u \in C^{\infty}\left(\Omega_{j} \times(0, T)\right)$ for all $T>0$ and $j=1,2$, we have

$$
\begin{equation*}
\sum_{j=1}^{2} \int_{\Omega_{j} \times \Omega_{j}} \frac{\left|\nabla u_{t t}(x)-\nabla u_{t t}(y)\right|^{2}}{|x-y|^{3-2 \epsilon}} d x d y<+\infty \tag{4.5}
\end{equation*}
$$

and $\nabla u_{t t} \in L^{\infty}(\Omega)$. Thus

$$
\int_{\Omega_{1} \times \Omega_{2}} \frac{\left|\nabla u_{t t}(x)-\nabla u_{t t}(y)\right|^{2}}{|x-y|^{3-2 \epsilon}} d x d y \leq \int_{\Omega_{1} \times \Omega_{2}} \frac{2\left\|\nabla u_{t t}\right\|_{L^{\infty}(\Omega)}^{2}}{|x-y|^{3-2 \epsilon}} d x d y,
$$

and it is thus enough to prove

$$
\begin{equation*}
\int_{\Omega_{1} \times \Omega_{2}} \frac{1}{|x-y|^{3-2 \epsilon}} d x d y<+\infty \tag{4.6}
\end{equation*}
$$

Set

$$
\Omega_{1}^{r}:=\left\{x \in \Omega_{1}: \operatorname{dist}(x, \Gamma)=r\right\}, \quad \Omega_{2}(x, \rho):=\Omega_{2} \cap \partial B(x, \rho),
$$

and the integral in 4.6 is rewritten as

$$
\int_{\Omega_{1} \times \Omega_{2}} \frac{1}{|x-y|^{3-2 \epsilon}} d x d y=\int_{0}^{D} \int_{\Omega_{1}^{r}} \int_{r}^{D} \int_{\Omega_{2}(x, \rho)} \rho^{-3+2 \epsilon} d \mathcal{H}_{\left\llcorner\Omega_{2}(x, \rho)\right.}^{1}(y) d \rho d \mathcal{H}_{\left\llcorner\Omega_{r}^{1}\right.}^{1}(x) d r
$$

Noting that $\sup _{D \geq r \geq 0} \mathcal{H}^{1}\left(\Omega_{1}^{r}\right)<+\infty$ due to $\mathcal{H}^{1}(\Gamma)<+\infty$, and $\mathcal{H}^{1}\left(\Omega_{2}(x, \rho)\right) \leq 2 \pi \rho$ for all $x \in \Omega_{1}$ and $\rho>0$, we get

$$
\begin{aligned}
& \int_{0}^{D} \int_{\Omega_{1}^{r}} \int_{r}^{D} \int_{\Omega_{2}(x, \rho)} \rho^{-3+2 \epsilon} d \mathcal{H}_{\left\llcorner\Omega_{2}(x, \rho)\right.}^{1}(y) d \rho d \mathcal{H}_{\left\llcorner\Omega_{r}^{1}\right.}^{1}(x) d r \leq 2 \pi \int_{0}^{D} \int_{\Omega_{1}^{r}} \int_{r}^{D} \rho^{-2+2 \epsilon} d \rho d \mathcal{H}_{\left\llcorner\Omega_{r}^{1}\right.}^{1}(x) d r \\
& \quad \leq 2 \pi(1-2 \epsilon)^{-1} \int_{0}^{D} \int_{\Omega_{1}^{r}} r^{-1+2 \epsilon} d \mathcal{H}_{\left\llcorner\Omega_{r}^{1}\right.}^{1}(x) d r \leq 2 \pi(1-2 \epsilon)^{-1}\left(\sup _{D \geq r \geq 0} \mathcal{H}^{1}\left(\Omega_{1}^{r}\right)\right) \int_{0}^{D} r^{-1+2 \epsilon} d r,
\end{aligned}
$$

which converges for all $\epsilon>0$. Thus (4.6) is proven. The proof of

$$
\int_{\Omega_{2} \times \Omega_{1}} \frac{1}{|x-y|^{2+2 s}} d x d y<+\infty
$$

is completely identical, and combining with (4.5) and (4.6) we infer $\left[\nabla u_{t t}\right]_{H^{1 / 2-\epsilon}(\Omega)}^{2}<+\infty$ for all $\epsilon>0$. Now the theorem for the 2D case has been completely proved.

When $\Omega \in \mathbb{R}^{3}$, we need to show that

$$
[\nabla u]_{H^{1 / 2-\epsilon}(\Omega)}:=\int_{\Omega \times \Omega} \frac{|\nabla u(x)-\nabla u(y)|^{2}}{|x-y|^{4-2 \epsilon}} d x d y<+\infty
$$

for all $\epsilon>0$. Similar as in (4.3), $\Omega$ is split into four parts, and since $u_{j} \in C^{\infty}\left(\Omega_{j}\right), j=1,2$, we have

$$
\sum_{j=1}^{2} \int_{\Omega_{j} \times \Omega_{j}} \frac{|\nabla u(x)-\nabla u(y)|^{2}}{|x-y|^{4-2 \epsilon}} d x d y<+\infty .
$$

Then it suffices to show

$$
\begin{equation*}
\int_{\Omega_{1} \times \Omega_{2}} \frac{|\nabla u(x)-\nabla u(y)|^{2}}{|x-y|^{4-2 \epsilon}} d x d y<+\infty . \tag{4.7}
\end{equation*}
$$

Again, in view of $u_{j} \in C^{\infty}\left(\Omega_{j}\right), j=1,2$, we have $\nabla u(x)-\nabla u(y) \in L^{\infty}(\Omega)$, i.e., there exists a constant $M$ such that $|\nabla u(x)-\nabla u(y)|<M$. Let

$$
\Omega_{1}^{r}:=\left\{x \in \Omega_{1}: \operatorname{dist}(x, \Gamma)=r\right\},
$$

and note immediately that since $\Omega$ is bounded, we have $\sup _{r} \mathcal{H}^{2}\left(\Omega_{1}^{r}\right)<+\infty$. Moreover, we can partition $\Omega_{1}, \Omega_{2}$ as follows:

$$
\Omega_{1}=\bigcup_{r>0} \Omega_{1}^{r}, \quad \Omega_{2}=\bigcup_{\rho>0}\left(\Omega_{2} \cap \partial B(x, \rho)\right) \quad \text { for all } x \in \Omega_{1} .
$$

Note that for any $r>0$ and $x \in \Omega_{1}^{r}$, the closest point in $\Omega_{2}$ is at distance at least $r$, hence $\Omega_{2} \cap$ $\partial B(x, \rho)=\emptyset$ for all $\rho<r$. Thus the integral (4.7) satisfies

$$
\begin{align*}
& \int_{\Omega_{1} \times \Omega_{2}} \frac{|\nabla u(x)-\nabla u(y)|^{2}}{|x-y|^{4-2 \epsilon}} d x d y \\
& \quad \leq 2 M \int_{0}^{D} \int_{\Omega_{1}^{r}} \int_{r}^{+\infty} \int_{\Omega_{2} \cap \partial B(x, \rho)}|x-y|^{-4+2 \epsilon} d \mathcal{H}_{\left\llcorner\Omega_{2} \cap \partial B(x, \rho)\right.}^{2}(y) d \rho d \mathcal{H}_{\left\llcorner\Omega_{1}^{r}\right.}^{2}(x) d r, \tag{4.8}
\end{align*}
$$

where $D$ denotes the diameter of $\Omega$. Direct computation then gives

$$
\begin{align*}
\int_{0}^{D} \int_{\Omega_{1}^{r}} & \int_{r}^{+\infty} \int_{\Omega_{2} \cap \partial B(x, \rho)}|x-y|^{-4+2 \epsilon} d \mathcal{H}_{\left\llcorner\Omega_{2} \cap \partial B(x, \rho)\right.}^{2}(y) d \rho d \mathcal{H}_{\left\llcorner\Omega_{1}^{r}\right.}^{2}(x) d r \\
& \leq \int_{0}^{D} \int_{\Omega_{1}^{r}} \int_{r}^{+\infty} \rho^{-4+2 \epsilon} \mathcal{H}^{2}\left(\Omega_{2} \cap \partial B(x, \rho)\right) d \rho d \mathcal{H}_{\left\llcorner\Omega_{1}^{r}\right.}^{2}(x) d r \\
& \leq 4 \pi \int_{0}^{D} \int_{\Omega_{1}^{r}} \int_{r}^{+\infty} \rho^{-2+2 \epsilon} d \rho d \mathcal{H}_{\left\llcorner\Omega_{1}^{r}\right.}^{2}(x) d r=4 \pi(1-2 \epsilon)^{-1} \int_{0}^{D} \int_{\Omega_{1}^{r}} r^{-1+2 \epsilon} d \mathcal{H}_{\left\llcorner\Omega_{1}^{r}\right.}^{2}(x) d r \\
& \leq\left(2 \pi(1-2 \epsilon)^{-1} \sup _{r} \mathcal{H}^{2}\left(\Omega_{1}^{r}\right)\right) \int_{0}^{D} r^{-1+2 \epsilon} d r \tag{4.9}
\end{align*}
$$

with the last integral converging if and only if $\epsilon>0$. We have then proved the regularity for $u$ in the 3D case; for the time derivative of $u$ one can similarly write out the Gagliardo semi-norm as

$$
\left[\nabla u_{t t}\right]_{H^{1 / 2-\epsilon}(\Omega)}^{2}=\int_{\Omega \times \Omega} \frac{\left|\nabla u_{t t}(x)-\nabla u_{t t}(y)\right|^{2}}{|x-y|^{4-2 \epsilon}} d x d y .
$$

Since $u \in C^{\infty}\left(\Omega_{j} \times(0, T)\right), T>0, j=1,2$ and $\nabla u_{t t} \in L^{\infty}(\Omega)$, we have

$$
\begin{equation*}
\sum_{j=1}^{2} \int_{\Omega_{j} \times \Omega_{j}} \frac{\left|\nabla u_{t t}(x)-\nabla u_{t t}(y)\right|^{2}}{|x-y|^{4-2 \epsilon}} d x d y<+\infty \tag{4.10}
\end{equation*}
$$

and

$$
\int_{\Omega_{1} \times \Omega_{2}} \frac{\left|\nabla u_{t t}(x)-\nabla u_{t t}(y)\right|^{2}}{|x-y|^{4-2 \epsilon}} d x d y \leq \int_{\Omega_{1} \times \Omega_{2}} \frac{2\left\|\nabla u_{t t}\right\|_{L^{\infty}(\Omega)}^{2}}{|x-y|^{4-2 \epsilon}} d x d y
$$

Tus it suffices to prove

$$
\begin{equation*}
\int_{\Omega_{1} \times \Omega_{2}} \frac{1}{|x-y|^{4-2 \epsilon}} d x d y<+\infty . \tag{4.11}
\end{equation*}
$$

Following the procedure in (4.8) and 4.9), one can show that 4.11) converges for all $\epsilon>0$. Therefore, $\left[\nabla u_{t t}\right]_{H^{1 / 2-\epsilon}(\Omega)} \leq+\infty$ for all $\epsilon>0$. We have finished the proof of regularity for the 3D cases.

## 5. Convergence of the dynamic augmented Lagrangian (DAL) methods

In this section, we aim to estimate, a priori, the error for the two DAL methods introduced in Section 3, when they are applied to the problem given in Section 4 . Both the original DAL method (in Section 5.1) and the projection-based DAL method (in Section 5.2) will be analyzed. For the original DAL method, the spatial error for an elliptic interface problem is estimated in Section 5.1.1, and the error of a fully discretized scheme for a parabolic interface problem is provided in Section 5.1.2. Note that, for the original DAL method, we will mainly focus on the case that $r>0$ so the convergence and stability are not be constrained by the inf-sup condition. More investigations on the inf-sup condition will be involved in Section 5.2, where we provide a proof for the error estimate of the projection-based DAL method on the elliptic interface problem, followed by the estimates for the errors on the parabolic interface problem. Throughout this section, we consider the symbol " $C$ " to indicate a generic constant that is independent of $h, H$, and $\Delta t$, but may have different numerical values in different situations.

### 5.1. Original dynamic augmented Lagrangian (DAL) method

For the original DAL method, integrating by parts and introducing the Lagrange multiplier $\lambda=\frac{\kappa}{1+r}\left(\frac{\partial u_{1}}{\partial \mathbf{n}_{1}}+\frac{\partial u_{2}}{\partial \mathbf{n}_{2}}\right)$, we derive an alternative weak formulation for this interface problem: Find $(u, \lambda) \in X \times M$ such that for $t \in(0, T]$,

$$
\begin{align*}
\left(u_{t}, w\right)+a(u, w)+(1+r)\langle\lambda, w\rangle & =(f, w), \quad \forall w \in X,  \tag{5.1a}\\
\langle u-g, \delta \lambda\rangle & =0, \quad \forall \delta \lambda \in M \tag{5.1b}
\end{align*}
$$

where $X=\left\{w \in H^{1}(\Omega)\right.$ and $w=0$ on $\left.\partial \Omega\right\}, M=L^{2}(\Gamma),(\cdot, \cdot)$ denotes the inner product in $L^{2}(\Omega)$ and $\langle\cdot, \cdot\rangle$ is the inner product in $L^{2}(\Gamma)$. According to the discussions in Section 4 , the bilinear operator $a(\cdot, \cdot)$ is coercive and continuous, i.e, $a(u, u) \geq \kappa\|\nabla u\|_{L^{2}(\Omega)}^{2}$, and $a(u, v) \leq \kappa\|\nabla u\|_{L^{2}(\Omega)}\|\nabla v\|_{L^{2}(\Omega)}$.

To state the numerical method, we associate with $\Omega$ a regular triangulation $\mathcal{T}_{h}(\Omega)$ consisting of elements $T$ of mesh sizes $h=\max _{\mathcal{T}_{h}(\Omega)} h_{T}$. The low regularity of the solution to the model problem implies that the minimum distance between $u$ and a discrete solution on an unfitted mesh will converge at low order, so, for simplicity, we assume that the space $X$ is approximated by linear finite elements, viz.

$$
\begin{equation*}
X_{h}(\Omega)=\left\{w_{h} \in P_{1}(T) \mid w_{h}=0 \text { on } \partial \Omega\right\} . \tag{5.2}
\end{equation*}
$$

Let $W_{h}(\Gamma)$ be the restriction of $X_{h}(\Omega)$ to $\Gamma$. The original DAL method can now be applied to the weak problem (5.1): Find $\left(u_{h}^{n+1}, \lambda_{h}^{n+1}\right) \in X_{h} \times W_{h}$ such that

$$
\begin{align*}
&\left(\frac{u_{h}^{n+1}-u_{h}^{n}}{\Delta t}, w_{h}\right)+a\left(u_{h}^{n+1}, w_{h}\right)+\left\langle\lambda_{h}^{n}, w_{h}\right\rangle+\beta\left\langle u_{h}^{n+1}-g\left(t^{n+1}\right), w_{h}\right\rangle \\
&=\left(f\left(t^{n+1}\right), w_{h}\right), \quad \forall w_{h} \in X_{h},  \tag{5.3a}\\
&\left\langle\lambda_{h}^{n+1}, \delta \lambda_{h}\right\rangle=\frac{1}{1+r}\left(\left\langle\lambda_{h}^{n}, \delta \lambda_{h}\right\rangle+\beta\left\langle u_{h}^{n+1}-g\left(t^{n+1}\right), \delta \lambda_{h}\right\rangle\right), \quad \forall \delta \lambda_{h} \in W_{h}, \tag{5.3b}
\end{align*}
$$

where $r$ and $\beta$ are suitably chosen penalty parameters. Since $\left.w_{h}\right|_{\Gamma} \in W_{h}$, we can take $\delta \lambda_{h}=\left.w_{h}\right|_{\Gamma}$ in (5.3b) and combining it with (5.3a) yields

$$
\begin{equation*}
\left(\frac{u_{h}^{n+1}-u_{h}^{n}}{\Delta t}, w_{h}\right)+a\left(u_{h}^{n+1}, w_{h}\right)+(1+r)\left\langle\lambda_{h}^{n+1}, w_{h}\right\rangle=\left(f\left(t^{n+1}\right), w_{h}\right), \tag{5.4}
\end{equation*}
$$

and (5.3b) can be written as

$$
\begin{equation*}
\Delta t\left\langle\frac{\lambda_{h}^{n+1}-\lambda_{h}^{n}}{\Delta t}, \delta \lambda_{h}\right\rangle+r\left\langle\lambda_{h}^{n+1}, \delta \lambda_{h}\right\rangle=\beta\left\langle u_{h}^{n+1}-g\left(t^{n+1}\right), \delta \lambda_{h}\right\rangle . \tag{5.5}
\end{equation*}
$$

In the ensuing derivations, we will use the following mesh-dependent half-norms as suggested in [90]:

$$
\begin{align*}
& \|\lambda\|_{1 / 2, h}^{2}=\frac{1}{h}\langle\lambda, \lambda\rangle  \tag{5.6}\\
& \|\lambda\|_{-1 / 2, h}^{2}=h\langle\lambda, \lambda\rangle \tag{5.7}
\end{align*}
$$

### 5.1.1. Static: Elliptic interface problem

We first study the error estimate for the static problem

$$
\begin{align*}
a(u, w)+(1+r)\langle\lambda, w\rangle & =(f, w), & & \forall w \in X,  \tag{5.8a}\\
\langle u-g, \delta \lambda\rangle & =0, & & \forall \delta \lambda \in M, \tag{5.8b}
\end{align*}
$$

and a discretization corresponding to the steady limit of the original DAL method

$$
\begin{align*}
a\left(u_{h}, w_{h}\right)+(1+r)\left\langle\lambda_{h}, w_{h}\right\rangle & =\left(f, w_{h}\right), & \forall w_{h} \in X_{h},  \tag{5.9a}\\
r\left\langle\lambda_{h}, \delta \lambda_{h}\right\rangle & =\beta\left\langle u_{h}-g, \delta \lambda_{h}\right\rangle & \forall \delta \lambda_{h} \in W_{h} . \tag{5.9b}
\end{align*}
$$

Denoting

$$
\begin{equation*}
\mathcal{A}\left(u_{h}, \lambda_{h} ; w_{h}, \delta \lambda_{h}\right)=a\left(u_{h}, w_{h}\right)+(1+r)\left\langle\lambda_{h}, w_{h}\right\rangle-(1+r)\left\langle u_{h}, \delta \lambda_{h}\right\rangle+\frac{r(1+r)}{\beta}\left\langle\lambda_{h}, \delta \lambda_{h}\right\rangle \tag{5.10}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{F}\left(w_{h}, \delta \lambda_{h}\right)=\left(f, w_{h}\right)-(1+r)\left\langle g, \delta \lambda_{h}\right\rangle, \tag{5.11}
\end{equation*}
$$

the formulation (5.9) can be written as

$$
\begin{equation*}
\mathcal{A}\left(u_{h}, \lambda_{h} ; w_{h}, \delta \lambda_{h}\right)=\mathcal{F}\left(w_{h}, \delta \lambda_{h}\right), \quad \forall w_{h} \in X_{h}, \delta \lambda_{h} \in W_{h} . \tag{5.12}
\end{equation*}
$$

Combining the weak formulation of the static problem (5.8) with (5.9), one can obtain an alternative definition of the static solution from the original DAL method

$$
\begin{equation*}
a\left(u-u_{h}, w_{h}\right)+(1+r)\left\langle\lambda-\lambda_{h}, w_{h}\right\rangle-(1+r)\left\langle u-u_{h}, \delta \lambda_{h}\right\rangle=\frac{r(1+r)}{\beta}\left\langle\lambda_{h}, \delta \lambda_{h}\right\rangle, \tag{5.13}
\end{equation*}
$$

which is equivalent to

$$
\begin{array}{r}
a\left(u-u_{h}, w_{h}\right)+(1+r)\left\langle\lambda-\lambda_{h}, w_{h}\right\rangle-(1+r)\left\langle u-u_{h}, \delta \lambda_{h}\right\rangle+\frac{r(1+r)}{\beta}\left\langle\lambda-\lambda_{h}, \delta \lambda_{h}\right\rangle \\
=\frac{r(1+r)}{\beta}\left\langle\lambda, \delta \lambda_{h}\right\rangle \tag{5.14}
\end{array}
$$

or

$$
\begin{equation*}
\mathcal{A}\left(u-u_{h}, \lambda-\lambda_{h} ; w_{h}, \delta \lambda_{h}\right)=\frac{r(1+r)}{\beta}\left\langle\lambda, \delta \lambda_{h}\right\rangle . \tag{5.15}
\end{equation*}
$$

For given $u(t)$ and $\lambda(t)$, we can define a perturbed mixed elliptic projection $\left(\hat{u}_{h}, \hat{\lambda}_{h}\right)$ as the solution of

$$
\begin{equation*}
\mathcal{A}\left(u-\hat{u}_{h}, \lambda-\hat{\lambda}_{h} ; w_{h}, \delta \lambda_{h}\right)=\frac{r(1+r)}{\beta}\left\langle\lambda, \delta \lambda_{h}\right\rangle, \quad \forall w_{h} \in X_{h}, \delta \lambda_{h} \in W_{h} . \tag{5.16}
\end{equation*}
$$

Then the estimates in this section also hold true for the mixed elliptic projection.
In the estimates, we employ the mesh-dependent norm

$$
\begin{equation*}
\left\|\left(u_{h}, \lambda_{h}\right)\right\|^{2}=\left(\nabla u_{h}, \nabla u_{h}\right)+h\left\langle\lambda_{h}, \lambda_{h}\right\rangle=\left\|\nabla u_{h}\right\|_{L^{2}(\Omega)}^{2}+\left\|\lambda_{h}\right\|_{-1 / 2, h}^{2} . \tag{5.17}
\end{equation*}
$$

With $r=O(1)$ and $\beta=\frac{l}{h}=O(1 / h)$, we can then prove the coercivity and the boundedness properties of $\mathcal{A}(\cdot, \cdot ; \cdot, \cdot)$, as follow:

Lemma 5.1. For all $w_{h} \in X_{h}, \delta \lambda_{h} \in W_{h}, \mathcal{A}$ satisfies the coercivity property:

$$
\begin{equation*}
\mathcal{A}\left(w_{h}, \delta \lambda_{h} ; w_{h}, \delta \lambda_{h}\right) \geq C\| \|\left(w_{h}, \delta \lambda_{h}\right)\| \|^{2} . \tag{5.18}
\end{equation*}
$$

Proof: From the definition of $\mathcal{A}$ we have

$$
\begin{align*}
& \mathcal{A}\left(w_{h}, \delta \lambda_{h} ; w_{h}, \delta \lambda_{h}\right) \\
& \quad=a\left(w_{h}, w_{h}\right)+(1+r)\left\langle\delta \lambda_{h}, w_{h}\right\rangle-(1+r)\left\langle w_{h}, \delta \lambda_{h}\right\rangle+\frac{r(1+r)}{\beta}\left\langle\delta \lambda_{h}, \delta \lambda_{h}\right\rangle \\
& \quad=a\left(w_{h}, w_{h}\right)+\frac{r(1+r) h}{l}\left\langle\delta \lambda_{h}, \delta \lambda_{h}\right\rangle \\
& \quad \geq \kappa\left\|\nabla w_{h}\right\|_{L^{2}(\Omega)}^{2}+\frac{r(1+r)}{l}\left\|\delta \lambda_{h}\right\|_{-1 / 2, h}^{2} \\
& \quad \geq C\left\|\left(w_{h}, \delta \lambda_{h}\right)\right\|^{2} . \tag{5.19}
\end{align*}
$$

We have then finished the proof.
Lemma 5.2. Let $(v, \mu) \in X \times M$ and $\left(w_{h}, \delta \lambda_{h}\right) \in X_{h} \times W_{h}$, the following inequality holds

$$
\begin{equation*}
\mathcal{A}\left(v, \mu ; w_{h}, \delta \lambda_{h}\right) \leq C\left(\| \|(v, \mu)\| \|+\|\mu\|_{L^{2}(\Gamma)}+\|v\|_{1 / 2, h}\right)\left\|\left(w_{h}, \delta \lambda_{h}\right)\right\| \| . \tag{5.20}
\end{equation*}
$$

Proof: The definition of $\mathcal{A}$ yields

$$
\begin{align*}
& \mathcal{A}\left(v, \mu ; w_{h}, \delta \lambda_{h}\right) \\
= & a\left(v, w_{h}\right)+(1+r)\left\langle\mu, w_{h}\right\rangle-(1+r)\left\langle v, \delta \lambda_{h}\right\rangle+\frac{r(1+r)}{\beta}\left\langle\mu, \delta \lambda_{h}\right\rangle \\
& \leq \kappa\|\nabla v\|_{L^{2}(\Omega)}\left\|\nabla w_{h}\right\|_{L^{2}(\Omega)}+(1+r)\|\mu\|_{L^{2}(\Gamma)}\left\|w_{h}\right\|_{L^{2}(\Gamma)}+(1+r)\|v\|_{1 / 2, h}\left\|\delta \lambda_{h}\right\|_{-1 / 2, h} \\
& +\frac{r(1+r)}{l}\|\mu\|_{-1 / 2, h}\left\|\delta \lambda_{h}\right\|_{-1 / 2, h} \\
\leq & C\left(\|\nabla v\|_{L^{2}(\Omega)}\|+\| \mu\left\|_{L^{2}(\Gamma)}+\right\| v\left\|_{1 / 2, h}+\right\| \mu \|_{-1 / 2, h}\right)\left(\left\|\nabla w_{h}\right\|_{L^{2}(\Omega)}+\left\|w_{h}\right\|_{L^{2}(\Gamma)}\right. \\
& \left.+\left\|\delta \lambda_{h}\right\|_{-1 / 2, h}\right) . \tag{5.21}
\end{align*}
$$

Since $w_{h}=0$ on $\partial \Omega$, the trace inequality

$$
\begin{equation*}
\left\|w_{h}\right\|_{L^{2}(\Gamma)} \leq C\left\|w_{h}\right\|_{H^{1}(\Omega)} \leq C\left(\left\|\nabla w_{h}\right\|_{L^{2}(\Omega)}+\left\|w_{h}\right\|_{L^{2}(\Omega)}\right) \tag{5.22}
\end{equation*}
$$

and Poincaré's inequality

$$
\begin{equation*}
\left\|w_{h}\right\|_{L^{2}(\Omega)}^{2} \leq C\left\|\nabla w_{h}\right\|_{L^{2}(\Omega)} \tag{5.23}
\end{equation*}
$$

imply

$$
\begin{equation*}
\left\|w_{h}\right\|_{L^{2}(\Gamma)} \leq C\left\|\nabla w_{h}\right\|_{L^{2}(\Omega)} . \tag{5.24}
\end{equation*}
$$

Given that $h \ll 1$, one can obtain

$$
\begin{align*}
& \mathcal{A}\left(v, \mu ; w_{h}, \delta \lambda_{h}\right) \\
\leq & C\left(\|\nabla \nu\|_{L^{2}(\Omega)}\|+\| \mu\left\|_{L^{2}(\Gamma)}+\right\| v\left\|_{1 / 2, h}+h\right\| \mu \|_{L^{2}(\Gamma)}\right)\left(\left\|\nabla w_{h}\right\|_{L^{2}(\Omega)}+\left\|\delta \lambda_{h}\right\|_{-1 / 2, h}\right) \\
\leq & C\left(\|\|(v, \mu)\|\|+\|\mu\|_{L^{2}(\Gamma)}+\|v\|_{1 / 2, h}\right)\left\|\left(w_{h}, \delta \lambda_{h}\right)\right\| \| . \tag{5.25}
\end{align*}
$$

We have then finished the proof.
We define $I_{h}$ as the linear interpolation operator defined on $X_{h}$, and the $L^{2}$ projection $\Pi_{h}$ from $L^{2}(\Gamma)$ to $W^{h}$ as

$$
\begin{equation*}
\int_{\Gamma}\left(\mu-\Pi_{h} \mu\right) \delta \lambda_{h}=0, \quad \forall \delta \lambda_{h} \in W^{h} . \tag{5.26}
\end{equation*}
$$

We then have the following properties [91]:
Lemma 5.3. For $u \in H^{3 / 2-\epsilon}(\Omega)$,

$$
\begin{gather*}
\left\|u-I_{h} u\right\|_{H^{1}(\Omega)} \leq C h^{1 / 2-\epsilon}\|u\|_{H^{3 / 2-\epsilon}(\Omega)},  \tag{5.27}\\
\left\|u-I_{h} u\right\|_{L^{2}(\Gamma)} \leq C h^{1-\epsilon}\|u\|_{H^{3 / 2-\epsilon}(\Omega)} . \tag{5.28}
\end{gather*}
$$

Lemma 5.4. For $\lambda \in H^{1 / 2-\epsilon}(\Gamma)$,

$$
\begin{gather*}
\left\|\lambda-\Pi_{h} \lambda\right\|_{L^{2}(\Gamma)} \leq C h^{1 / 2-\epsilon}\|\lambda\|_{H^{1 / 2-\epsilon}(\Gamma)}  \tag{5.29}\\
\left\|\lambda-\Pi_{h} \lambda\right\|_{H^{-1 / 2}(\Gamma)} \leq C h^{1 / 2}\|\lambda\|_{L^{2}(\Gamma)} .  \tag{5.30}\\
\left\|\lambda-\Pi_{h} \lambda\right\|_{L^{2}(\Gamma)} \leq C\|\lambda\|_{L^{2}(\Gamma)} . \tag{5.31}
\end{gather*}
$$

We also have the following Lemmas from [92]
Lemma 5.5. For $u \in H^{3 / 2-\epsilon}(\Omega)$, and $\lambda \in H^{1 / 2-\epsilon}(\Gamma)$,

$$
\begin{gather*}
\|u\|_{H^{1 / 2}(\Omega)} \leq C\|u\|_{H^{1}(\Omega)} \leq C\|u\|_{H^{3 / 2-\epsilon}(\Omega)}  \tag{5.32}\\
\|u\|_{H^{1 / 2}(\Gamma)} \leq C\|u\|_{H^{1}(\Omega)}  \tag{5.33}\\
\|\lambda\|_{L^{2}(\Gamma)} \leq C\|\lambda\|_{H^{1 / 2-\epsilon(\Gamma)}} \tag{5.34}
\end{gather*}
$$

Lemma 5.6. For any given function $\omega$, consider the elliptic problem

$$
\begin{align*}
-\nabla \cdot(c(\nabla(\phi))) & =\omega, & & \text { in } \Omega \backslash \Gamma,  \tag{5.35a}\\
\phi & =0, & & \text { on } \partial \Omega,  \tag{5.35b}\\
\phi & =0, & & \text { on } \Gamma . \tag{5.35c}
\end{align*}
$$

Since the boundaries of $\Omega_{1}$ and $\Omega_{2}$ are both smooth, letting $\phi$ be the solution, the restrictions $\phi_{j}:=\left.\phi\right|_{\Omega_{j}}, j=1,2$, are smooth. Set $\theta:=c\left(\nabla \phi_{1}\right) \cdot \mathbf{n}_{1}+c\left(\nabla \phi_{2}\right) \cdot \mathbf{n}_{2}$. Then the following inequality holds:

$$
\begin{equation*}
\|\phi\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\|\theta\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2} \leq C\|\omega\|_{L^{2}(\Omega)}^{2} . \tag{5.36}
\end{equation*}
$$

for some constant $C$.
Proof: We note first that $\phi$ satisfies $\left\|\phi_{j}\right\|_{H^{2}\left(\Omega_{j}\right)} \leq C\|\omega\|_{L^{2}\left(\Omega_{j}\right)}, j=1,2$. Thus

$$
\begin{equation*}
\|\phi\|_{L^{2}(\Omega)}^{2}=\|\phi\|_{L^{2}\left(\Omega_{1}\right)}^{2}+\|\phi\|_{L^{2}\left(\Omega_{2}\right)}^{2} \leq C\|\omega\|_{L^{2}(\Omega)}^{2} . \tag{5.37}
\end{equation*}
$$

By the trace theorem, since $\nabla \phi_{j} \in H^{1}\left(\Omega_{j}\right), j=1,2$, we have also

$$
\left\|\nabla \phi_{j} \cdot \mathbf{n}_{j}\right\|_{H^{1 / 2-\epsilon}(\Gamma)} \leq\left\|\nabla \phi_{j}\right\|_{H^{1 / 2-\epsilon}(\Gamma)} \leq C\left\|\nabla \phi_{j}\right\|_{H^{1-\epsilon}\left(\Omega_{j}\right)}, \quad j=1,2,
$$

where $\mathbf{n}_{j}$ denotes the outer (with respect to $\Omega_{j}$ ) normal derivative to $\Gamma$. Thus

$$
\begin{align*}
\|\theta\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})} & \leq\left(\left\|\nabla \phi_{1} \cdot \mathbf{n}_{1}\right\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})}+\left\|\nabla \phi_{2} \cdot \mathbf{n}_{2}\right\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})}\right) \\
& \leq C\left(\left\|\nabla \phi_{1}\right\|_{H^{1-\epsilon\left(\Omega_{1}\right)}}+\left\|\nabla \phi_{2}\right\|_{H^{1-\epsilon}\left(\Omega_{2}\right)}\right) \leq C\|\omega\|_{L^{2}(\Omega)} \tag{5.38}
\end{align*}
$$

It remains to estimate the Gagliardo semi-norm $\|\nabla \phi\|_{H^{1 / 2-\epsilon}(\Omega)}$. By definition,

$$
\begin{aligned}
\|\nabla \phi\|_{H^{1 / 2-\epsilon}(\Omega)}= & \int_{\Omega} \int_{\Omega} \frac{|\nabla \phi(x)-\nabla \phi(y)|^{2}}{|x-y| 3^{3-2 \epsilon}} d y d x \\
= & \int_{\Omega_{1}} \int_{\Omega_{1}} \frac{|\nabla \phi(x)-\nabla \phi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d y d x+\int_{\Omega_{2}} \int_{\Omega_{2}} \frac{|\nabla \phi(x)-\nabla \phi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d y d x \\
& +\int_{\Omega_{1}} \int_{\Omega_{2}} \frac{|\nabla \phi(x)-\nabla \phi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d y d x+\int_{\Omega_{1}} \int_{\Omega_{2}} \frac{|\nabla \phi(x)-\nabla \phi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d y d x .
\end{aligned}
$$

Since $\phi_{j} \in H^{2}\left(\Omega_{j}\right), j=1,2$, it follows

$$
\begin{align*}
\int_{\Omega_{j}} \int_{\Omega_{j}} \frac{|\nabla \phi(x)-\nabla \phi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d y d x & =\int_{\Omega_{j}} \int_{\Omega_{j}} \frac{\left|\nabla \phi_{j}(x)-\nabla \phi_{j}(y)\right|^{2}}{\left.|x-y|\right|^{3-2 \epsilon}} d y d x \\
& \leq C\left\|\nabla \phi_{j}\right\|_{H^{1}\left(\Omega_{j}\right)}^{2} \leq C\|\omega\|_{L^{2}(\Omega)}^{2}, \quad j=1,2 \tag{5.39}
\end{align*}
$$

To estimate $\int_{\Omega_{1}} \int_{\Omega_{2}} \frac{|\nabla \phi(x)-\nabla \phi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d y d x$, note that

$$
\begin{aligned}
\int_{\Omega_{1}} \int_{\Omega_{2}} \frac{|\nabla \phi(x)-\nabla \phi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d y d x= & \int_{\left\{x \in \Omega_{1}: \operatorname{dist}(x, \Gamma) \leq \delta\right\}} \int_{\Omega_{2}} \frac{|\nabla \phi(x)-\nabla \phi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d y d x \\
& +\int_{\left\{x \in \Omega_{1}: \operatorname{dist}(x, \Gamma)>\delta\right\}} \int_{\Omega_{2}} \frac{|\nabla \phi(x)-\nabla \phi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d y d x
\end{aligned}
$$

for all $\delta>0$. Since we already proved $\|\nabla \phi\|_{H^{1 / 2-\epsilon(\Omega)}}<+\infty$, there exists sufficiently small $\delta_{0}>0$ such that

$$
\begin{equation*}
\int_{\left\{x \in \Omega_{1}: \operatorname{dist}(x, \Gamma) \leq \delta_{0}\right\}} \int_{\Omega_{2}} \frac{|\nabla \phi(x)-\nabla \phi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d y d x \leq\|\omega\|_{L^{2}(\Omega)} \tag{5.40}
\end{equation*}
$$

On the other hand,

$$
\begin{align*}
& \int_{\left\{x \in \Omega_{1}: \operatorname{dist}(x, \Gamma)>\delta_{0}\right\}} \\
\leq & \frac{|\nabla \phi(x)-\nabla \phi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d y d x \\
\leq & \delta_{0}^{2 \epsilon-3} \int_{\left\{x \in \Omega_{1}: \operatorname{dist}(x, \Gamma)>\delta_{0}\right\}} \int_{\Omega_{2}}|\nabla \phi(x)-\nabla \phi(y)|^{2} d y d x \\
\leq & 2 \delta_{0}^{2 \epsilon-3} \int_{\Omega_{1}} \int_{\Omega_{2}}\left(|\nabla \phi(x)|^{2}+\left.\nabla \phi(y)\right|^{2}\right) d y d x  \tag{5.41}\\
\leq & 2 \delta_{0}^{2 \epsilon-3} \mathcal{H}^{2}(\Omega)\left(\left\|\nabla \phi_{1}\right\|_{L^{2}\left(\Omega_{1}\right)}^{2}+\left\|\nabla \phi_{2}\right\|_{L^{2}\left(\Omega_{2}\right)}^{2}\right) \leq C\|\omega\|_{L^{2}(\Omega)}^{2},
\end{align*}
$$

where $\mathcal{H}^{2}$ denotes the 2 -Hausdorff measure. Combining (5.41) with 5.40 we obtain $\|\nabla \phi\|_{H^{1 / 2-\epsilon}(\Omega)} \leq C\|\omega\|_{L^{2}(\Omega)}^{2}$. Combining with (5.37), (5.38) and (5.39) we have proved (5.36).

With the above lemmas, we can now prove the error estimates as in the following theorem
Theorem 5.7. Let $(u, \lambda) \in X \times M$ be the solution of (5.8) and $\left(\hat{u}_{h}, \hat{\lambda}_{h}\right) \in X_{h} \times W_{h}$ be the solutions of (5.9). Then, for $r=O(1)$ and $\beta=\frac{l}{h}$, the following error estimates for the original DAL method hold true:

$$
\begin{gather*}
\left\|\left(u-\hat{u}_{h}, \lambda-\hat{\lambda}_{h}\right)\right\| \leq C h^{1 / 2-\epsilon}\left(\|u\|_{H^{3 / 2-\epsilon}(\Omega)}+\|\lambda\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})}\right),  \tag{5.42}\\
\left\|u-\hat{u}_{h}\right\|_{L^{2}(\Omega)} \leq C h^{1-2 \epsilon}\left(\|u\|_{H^{3 / 2-\epsilon}(\Omega)}+\|\lambda\|_{H^{1 / 2-\epsilon}(\Gamma)}\right) . \tag{5.43}
\end{gather*}
$$

Proof: From the triangle inequality, we obtain

$$
\begin{equation*}
\left\|\left\|\left(u-\hat{u}_{h}, \lambda-\hat{\lambda}_{h}\right)\right\|\right\| \leq\| \|\left(u-I_{h} u, \lambda-\Pi_{h} \lambda\right)\|+\|\left\|\left(I_{h} u-\hat{u}_{h}, \Pi_{h} \lambda-\hat{\lambda}_{h}\right)\right\| . \tag{5.44}
\end{equation*}
$$

Lemma 5.3 and Lemma 5.4 yield the estimate for the first term:

$$
\begin{align*}
\left\|\left(u-I_{h} u, \lambda-\Pi_{h} \lambda\right)\right\|^{2} & =\left\|\nabla\left(u-I_{h} u\right)\right\|_{L^{2}(\Omega)}^{2}+h\left\|\lambda-\Pi_{h} \lambda\right\|_{L^{2}(\Gamma)}^{2} \\
& \leq\left\|u-I_{h} \mathbf{u}\right\|_{H^{1}(\Omega)}^{2}+h\left\|\lambda-\Pi_{h} \lambda\right\|_{L^{2}(\Gamma)}^{2} \\
& \leq C h^{1-2 \epsilon}\left(\|\mathbf{u}\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+h\|\lambda\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}\right) . \tag{5.45}
\end{align*}
$$

Therefore, given $h \ll 1$, for the first term

$$
\begin{equation*}
\left\|\left\|\left(u-I_{h} u, \lambda-\Pi_{h} \lambda\right)\right\| \leq C h^{1 / 2-\epsilon}\right\| u \|_{H^{3 / 2-\epsilon}(\Omega)} . \tag{5.46}
\end{equation*}
$$

For the second term, we can use Lemma 5.1

$$
\begin{align*}
\left\|\left(I_{h} u-\hat{u}_{h}, \Pi_{h} \lambda-\hat{\lambda}_{h}\right)\right\|^{2} \leq & C \mathcal{A}\left(I_{h} u-\hat{u}_{h}, \Pi_{h} \lambda-\hat{\lambda}_{h} ; I_{h} u-\hat{u}_{h}, \Pi_{h} \lambda-\hat{\lambda}_{h}\right) \\
\leq & C\left(\mathcal{A}\left(u-\hat{u}_{h}, \lambda-\hat{\lambda}_{h} ; I_{h} u-\hat{u}_{h}, \Pi_{h} \lambda-\hat{\lambda}_{h}\right)\right. \\
& \left.-\mathcal{A}\left(u-I_{h} u, \lambda-\Pi_{h} \lambda ; I_{h} u-\hat{u}_{h}, \Pi_{h} \lambda-\hat{\lambda}_{h}\right)\right) . \tag{5.47}
\end{align*}
$$

From the alternative definition given in (5.16), the first term of (5.47) can be rewritten as

$$
\begin{align*}
\mathcal{A}\left(u-\hat{u}_{h}, \lambda-\hat{\lambda}_{h} ; I_{h} u-\hat{u}_{h}, \Pi_{h} \lambda-\hat{\lambda}_{h}\right) & =\frac{(1+r) r}{\beta}\left\langle\lambda, \Pi_{h} \lambda-\hat{\lambda}_{h}\right\rangle \\
& \leq \frac{(1+r) r}{l}\|\lambda\|_{-1 / 2, h}\left\|\Pi_{h} \lambda-\hat{\lambda}_{h}\right\|_{-1 / 2, h} \\
& \leq C\|\lambda\|_{-1 / 2, h}\| \|\left(I_{h} u-\hat{u}_{h}, \Pi_{h} \lambda-\hat{\lambda}_{h}\right) \| . \tag{5.48}
\end{align*}
$$

The estimate for the second part of (5.47) can be derived from Lemma 5.2

$$
\begin{align*}
& \mathcal{A}\left(u-I_{h} u, \lambda-\Pi_{h} \lambda ; I_{h} u-\hat{u}_{h}, \Pi_{h} \lambda-\hat{\lambda}_{h}\right) \\
\leq & C\left(\left\|\left\|\left(u-I_{h} u, \lambda-\Pi_{h} \lambda\right)\right\|\right\|+\left\|\lambda-\Pi_{h} \lambda\right\|_{L^{2}(\Gamma)}+\left\|u-I_{h} u\right\|_{1 / 2, h}\right)\left\|\left(I_{h} u-\hat{u}_{h}, \Pi_{h} \lambda-\hat{\lambda}_{h}\right)\right\| \| . \tag{5.49}
\end{align*}
$$

Combining (5.48) and (5.49), one can get

$$
\begin{align*}
& \left\|\left(I_{h} u-\hat{u}_{h}, \Pi_{h} \lambda-\hat{\lambda}_{h}\right)\right\| \\
\leq & C\left(\|\lambda\|_{-1 / 2, h}+\left\|\mid\left(u-I_{h} u, \lambda-\Pi_{h} \lambda\right)\right\| l+\left\|\lambda-\Pi_{h} \lambda\right\|_{L^{2}(\Gamma)}+\left\|u-I_{h} u\right\|_{1 / 2, h}\right) \\
\leq & C\left(h^{1 / 2}\|\lambda\|_{L^{2}(\Gamma)}+h^{1 / 2-\epsilon}\|u\|_{H^{3 / 2-\epsilon}(\Omega)}+h^{1 / 2-\epsilon}\|\lambda\|_{H^{1 / 2-\epsilon(\Gamma)}}+h^{1 / 2-\epsilon}\|u\|_{H^{3 / 2-\epsilon}(\Omega)}\right) \\
\leq & C h^{1 / 2-\epsilon}\left(\|u\|_{H^{3 / 2-\epsilon}(\Omega)}+\|\lambda\|_{H^{1 / 2-\epsilon}(\mathrm{T})}\right) . \tag{5.50}
\end{align*}
$$

To estimate the $L^{2}$ error, we apply the Aubin-Nitsche duality argument. We introduce the dual problem of finding $(\phi, \theta)$ as the solution of

$$
\begin{align*}
-\nabla \cdot(c(\nabla \phi)) & =u-\hat{u}_{h}, \quad \text { in } \Omega \backslash \Gamma,  \tag{5.51a}\\
\phi & =0, \quad \text { on } \partial \Omega,  \tag{5.51b}\\
\phi & =0, \quad \text { on } \Gamma,  \tag{5.51c}\\
\theta & =c\left(\nabla \phi_{1}\right) \cdot \mathbf{n}_{1}+c\left(\nabla \phi_{2}\right) \cdot \mathbf{n}_{2}, \quad \text { on } \Gamma, \tag{5.51d}
\end{align*}
$$

from Lemma 5.6, the following bound holds for $\phi$

$$
\begin{equation*}
\|\phi\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\|\theta\|_{L^{2}(\Gamma)}^{2} \leq C\left\|u-\hat{u}_{h}\right\|_{L^{2}(\Omega)}^{2} . \tag{5.52}
\end{equation*}
$$

Then, taking an inner product of (5.51a) and $u-\hat{u}_{h}$ gives

$$
\begin{align*}
\left\|u-\hat{u}_{h}\right\|_{L^{2}(\Omega)}^{2} & =a\left(\phi, u-\hat{u}_{h}\right)-\left\langle c\left(\nabla \phi_{1}\right) \cdot \mathbf{n}_{1}, u-\hat{u}_{h}\right\rangle-\left\langle c\left(\nabla \phi_{2}\right) \cdot \mathbf{n}_{2}, u-\hat{u}_{h}\right\rangle \\
& =a\left(\phi-I_{h} \phi, u-\hat{u}_{h}\right)+a\left(I_{h} \phi, u-\hat{u}_{h}\right)-\left\langle\theta, u-\hat{u}_{h}\right\rangle . \tag{5.53}
\end{align*}
$$

We can estimate the above three terms separately. Firstly

$$
\begin{align*}
a\left(\phi-I_{h} \phi, u-\hat{u}_{h}\right) & \leq C\left\|\nabla\left(\phi-I_{h} \phi\right)\right\|_{L^{2}(\Omega)}\left\|\nabla\left(u-\hat{u}_{h}\right)\right\|_{L^{2}(\Omega)} \\
& \leq C\left\|\phi-I_{h} \phi\right\|_{H^{1}(\Omega)}\left\|\nabla\left(u-\hat{u}_{h}\right)\right\|_{L^{2}(\Omega)} \\
& \leq C h^{1 / 2-\epsilon}\|\phi\|_{H^{3 / 2-\epsilon}(\Omega)}\left\|\nabla\left(u-\hat{u}_{h}\right)\right\|_{L^{2}(\Omega)} \\
& \leq C h^{1-2 \epsilon}\|\phi\|_{H^{3 / 2-\epsilon}(\Omega)}\left(\|u\|_{H^{3 / 2-\epsilon}(\Omega)}+\|\lambda\|_{H^{1 / 2-\epsilon}(\Gamma)}\right) \\
& \leq C h^{1-2 \epsilon}\left\|u-\hat{u}_{h}\right\|_{L^{2}(\Omega)}\left(\|u\|_{H^{3 / 2-\epsilon}(\Omega)}+\|\lambda\|_{H^{1 / 2-\epsilon}(\mathbb{T})}\right) . \tag{5.54}
\end{align*}
$$

Secondly, from (5.9a) and (5.8a)

$$
\begin{align*}
a\left(I_{h} \phi, u-\hat{u}_{h}\right) & =-(1+r)\left\langle\lambda-\hat{\lambda}_{h}, I_{h} \phi\right\rangle \\
& =(1+r)\left\langle\lambda-\hat{\lambda}_{h}, \phi-I_{h} \phi\right\rangle  \tag{5.55}\\
& \leq(1+r)\left\|\lambda-\hat{\lambda}_{h}\right\|_{L^{2}(\Gamma)}\left\|\phi-I_{h} \phi\right\|_{L^{2}(\Gamma)} \\
& \leq C h^{1-2 \epsilon}\left(\|u\|_{H^{3 / 2-\epsilon}(\Omega)}+\|\lambda\|_{H^{1 / 2-\epsilon(\Gamma)}}\right)\|\phi\|_{H^{3 / 2-\epsilon}(\Omega)} \\
& \leq C h^{1-2 \epsilon}\left\|u-\hat{u}_{h}\right\|_{L^{2}(\Omega)}\left(\|u\|_{H^{3 / 2-\epsilon}(\Omega)}+\|\lambda\|_{H^{1 / 2-\epsilon(\Gamma)}}\right) \tag{5.56}
\end{align*}
$$

where (5.55) was obtained based on the fact that $\phi=0$ on $\Gamma$. For the last term in (5.53), one can further divide it into two parts:

$$
\begin{align*}
\left\langle\theta, u-\hat{u}_{h}\right\rangle & =\left\langle\theta-\Pi_{h} \theta, u-\hat{u}_{h}\right\rangle+\left\langle\Pi_{h} \theta, u-\hat{u}_{h}\right\rangle \\
& =\left\langle\theta-\Pi_{h} \theta, u-\hat{u}_{h}\right\rangle+\frac{r}{\beta}\left\langle\Pi_{h} \theta, \hat{\lambda}_{h}\right\rangle \\
& \leq C\left\|\theta-\Pi_{h} \theta\right\|_{H^{-1 / 2}(\Gamma)}\left\|u-\hat{u}_{h}\right\|_{H^{1 / 2}(\Gamma)}+\frac{r h}{l}\left(\|\theta\|_{L^{2}(\Gamma)}+\left\|\theta-\Pi_{h} \theta\right\|_{L^{2}(\Gamma)}\right)\left\|\hat{\lambda}_{h}\right\|_{L^{2}(\Gamma)} \\
& \leq C h^{1 / 2}\|\theta\|_{L^{2}(\Gamma)}\left\|u-\hat{u}_{h}\right\|_{H^{1 / 2}(\Gamma)}+C \frac{r h}{l}\|\theta\|_{L^{2}(\Gamma)}\left\|\hat{d}_{h}\right\|_{L^{2}(\Gamma)} . \tag{5.57}
\end{align*}
$$

Since $u-\hat{u}_{h}=0$ on $\partial \Omega$, we have

$$
\begin{equation*}
\left\|u-\hat{u}_{h}\right\|_{H^{1 / 2}(\Gamma)} \leq C\left\|u-\hat{u}_{h}\right\|_{H^{1}(\Omega)} \leq C\left\|\nabla\left(u-\hat{u}_{h}\right)\right\|_{L^{2}(\Gamma)} . \tag{5.58}
\end{equation*}
$$

Together with the estimate for $\hat{\lambda}_{h}$

$$
\begin{equation*}
\left\|\hat{\lambda}_{h}\right\|_{L^{2}(\Gamma)} \leq\|\lambda\|_{L^{2}(\Gamma)}+\left\|\lambda-\hat{\lambda}_{h}\right\|_{L^{2}(\Gamma)} \leq C h^{-\epsilon}\left(\|u\|_{H^{3} / 2-\epsilon(\Omega)}+\|\lambda\|_{H^{1 / 2-\epsilon}(\Gamma)}\right), \tag{5.59}
\end{equation*}
$$

we obtain

$$
\begin{align*}
& \left\langle\theta, u-\hat{u}_{h}\right\rangle \\
\leq & C h^{1 / 2}\|\theta\|_{L^{2}(\Gamma)}\left\|\nabla\left(u-\hat{u}_{h}\right)\right\|_{L^{2}(\Gamma)}+C h^{1-\epsilon}\|\theta\|_{L^{2}(\Gamma)}\left(\|u\|_{H^{3 / 2-\epsilon}(\Omega)}+\|\lambda\|_{H^{1 / 2-\epsilon}(\Gamma)}\right) \\
\leq & C h^{1-\epsilon}\|\theta\|_{L^{2}(\Gamma)}\left(\|u\|_{H^{3 / 2-\epsilon}(\Omega)}+\|\lambda\|_{H^{1 / 2-\epsilon}(\Gamma)}\right) \\
\leq & C h^{1-\epsilon}\left\|u-\hat{u}_{h}\right\|_{L^{2}(\Omega)}\left(\|u\|_{H^{3 / 2-\epsilon}(\Omega)}+\|\lambda\|_{H^{1 / 2-\epsilon}(\Gamma)}\right) . \tag{5.60}
\end{align*}
$$

Combining (5.53)-(5.60), we get

$$
\begin{equation*}
\left\|u-\hat{u}_{h}\right\|_{L^{2}(\Omega)} \leq C h^{1-2 \epsilon}\left(\|u\|_{H^{3 / 2-\epsilon}(\Omega)}+\|\lambda\|_{H^{1 / 2-\epsilon}(\Gamma)}\right) . \tag{5.61}
\end{equation*}
$$

We have then finished the error estimate of the original DAL method for the static problem.
Remark 1. The analysis for linear finite elements can be extended to higher-order elements, but the convergence rate would not improve. Nevertheless, in practice, higher-order approximations of fluid mechanics may be beneficial for representing complex flow features away from the boundary; cf. the reasoning in [93, Section 3].

### 5.1.2. Fully discrete: Parabolic interface problem

From the analysis for the static problem in Section 5.1.1, we obtain the following Lemma which is useful for the analysis of the fully-discrete original DAL scheme in this section.

Lemma 5.8. For given $u$ and $\lambda$, we have the following estimate for their perturbed mixed elliptic projections $\left(\hat{u}_{h}, \hat{\lambda}_{h}\right)$ defined in (5.16):

$$
\begin{gather*}
\left\|\left(u-\hat{u}_{h}, \lambda-\hat{\lambda}_{h}\right)\right\|^{2} \leq C h^{1-2 \epsilon}\left(\|u\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\|\lambda\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}\right),  \tag{5.62}\\
\left\|u-\hat{u}_{h}\right\|_{L^{2}(\Omega)}^{2} \leq C h^{2-4 \epsilon}\left(\|u\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\|\lambda\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}\right) .  \tag{5.63}\\
\left\|\left\|\left(\frac{\partial u}{\partial t}-\frac{\partial \hat{u}_{h}}{\partial t}, \frac{\partial \lambda}{\partial t}-\frac{\partial \hat{\lambda}_{h}}{\partial t}\right)\right\|^{2} \leq C h^{1-2 \epsilon}\left(\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\mathrm{T})}^{2}\right),\right.  \tag{5.64}\\
\left\|\frac{\partial u}{\partial t}-\frac{\partial \hat{u}_{h}}{\partial t}\right\|_{L^{2}(\Omega)}^{2} \leq C h^{2-4 \epsilon}\left(\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}\right) . \tag{5.65}
\end{gather*}
$$

Let $\Delta t$ be the time step size and $T=N \Delta t$. We employ backward Euler time integration, and obtain the fully discrete scheme as in (5.3). Denoting $\bar{\partial}_{t} v^{n+1}=\frac{v^{n+1}-v^{n}}{\Delta t}$, the original fully discrete DAL scheme at the $n+1$-the time step can be rewritten: Given $\left(u_{h}^{n}, \lambda_{h}^{n}\right)$, find $\left(u_{h}^{n+1}, \lambda_{h}^{n+1}\right) \in X_{h} \times W_{h}$ such that

$$
\begin{align*}
\left(\bar{\partial}_{t} u_{h}^{n+1}, w_{h}\right)+a\left(u_{h}^{n+1}, w_{h}\right)+(1+r)\left\langle\lambda_{h}^{n+1}, w_{h}\right\rangle & =\left(f\left(t^{n+1}\right), w_{h}\right), \quad \forall w_{h} \in X_{h},  \tag{5.66a}\\
\Delta t\left\langle\bar{\partial}_{t} \lambda_{h}^{n+1}, \delta \lambda_{h}\right\rangle+r\left\langle\lambda_{h}^{n+1}, \delta \lambda_{h}\right\rangle & =\beta\left\langle u_{h}^{n+1}-g\left(t^{n+1}\right), \delta \lambda_{h}\right\rangle, \quad \forall \delta \lambda_{h} \in M_{h},  \tag{5.66b}\\
u_{h}^{0} & =I_{h} u_{0},  \tag{5.66c}\\
\lambda_{h}^{0} & =0 . \tag{5.66d}
\end{align*}
$$

The above fully-discrete method can also be written in an equivalent form: Find $\left(u_{h}^{n+1}, \lambda_{h}^{n+1}\right) \in$ $X_{h} \times W_{h}$ such that for all $w_{h} \in X_{h}, \delta \lambda_{h} \in W_{h}$,

$$
\begin{equation*}
\left(\bar{\partial}_{t} u_{h}^{n+1}, w_{h}\right)+\frac{\Delta t(1+r)}{\beta}\left\langle\bar{\partial}_{t} \lambda_{h}^{n+1}, \delta \lambda_{h}\right\rangle+\mathcal{A}\left(u_{h}^{n+1}, \lambda_{h}^{n+1} ; w_{h}, \delta \lambda_{h}\right)=\mathcal{F}\left(w_{h}, \delta \lambda_{h}\right) . \tag{5.67}
\end{equation*}
$$

Theorem 5.9. Let $(u, \lambda) \in X \times M$ be the solution of (4.1) and $\left(u_{h}^{n+1}, \lambda_{h}^{n+1}\right) \in X_{h} \times W_{h}$ be the solutions of (5.66). Then, for $r=O(1)$ and $\beta=\frac{l}{h}$, the following error estimates for the original DAL method hold true:

$$
\begin{align*}
& \|\left(u^{\left.n\left(t^{n+1}\right)-u_{h}^{n+1}, \lambda\left(t^{n+1}\right)-\lambda_{h}^{n+1}\right)\| \|^{2}} \begin{array}{rl}
\leq & C\left(\frac{h^{2-4 \epsilon}}{\Delta t}+\Delta t\right)\left(\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})}^{2}\right. \\
& \left.+\int_{0}^{t^{n+1}}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\mathrm{T})}^{2}+\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2} d t\right)
\end{array}\right.
\end{align*}
$$

and

$$
\begin{align*}
& \left\|u\left(t^{n+1}\right)-u_{h}^{n+1}\right\|_{L^{2}(\Omega)}^{2} \\
\leq & C\left(h^{2-4 \epsilon}+\Delta t^{2}\right)\left(\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}\right. \\
& \left.+\int_{0}^{t^{n+1}}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}+\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2} d t\right) . \tag{5.69}
\end{align*}
$$

Proof: With the perturbed mixed elliptic projection $\left(\hat{u}_{h}\left(t^{n+1}\right), \hat{\lambda}_{h}\left(t^{n+1}\right)\right)$ of $u\left(t^{n+1}\right)$ and $\lambda\left(t^{n+1}\right)$, set

$$
\begin{equation*}
u\left(t^{n+1}\right)-u_{h}^{n+1}=\left(u\left(t^{n+1}\right)-\hat{u}_{h}\left(t^{n+1}\right)\right)+\left(\hat{u}_{h}\left(t^{n+1}\right)-u_{h}^{n+1}\right)=\varphi^{n+1}+\theta^{n+1} \tag{5.70}
\end{equation*}
$$

$$
\begin{equation*}
\lambda\left(t^{n+1}\right)-\lambda_{h}^{n+1}=\left(\lambda\left(t^{n+1}\right)-\hat{\lambda}_{h}\left(t^{n+1}\right)\right)+\left(\hat{\lambda}_{h}\left(t^{n+1}\right)-\lambda_{h}^{n+1}\right)=\omega^{n+1}+\xi^{n+1} . \tag{5.71}
\end{equation*}
$$

The estimates of the interpolation errors $\varphi^{n+1}$ and $\omega^{n+1}$ are known from Lemma 5.18:

$$
\begin{gather*}
\left\|\left(\varphi^{n+1}, \omega^{n+1}\right)\right\|^{2} \leq C h^{2-2 \epsilon}\left(\left\|u\left(t^{n+1}\right)\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\lambda\left(t^{n+1}\right)\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}\right),  \tag{5.72}\\
\left\|\varphi^{n+1}\right\|_{L^{2}(\Omega)}^{2} \leq C h^{2-4 \epsilon}\left(\left\|u\left(t^{n+1}\right)\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\lambda\left(t^{n+1}\right)\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}\right) . \tag{5.73}
\end{gather*}
$$

For $\theta^{n+1}$, take $w=w_{h}$ in (5.1) and recall the definition of the perturbed mixed elliptic projection (5.16); one can obtain

$$
\begin{equation*}
\left(\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}, w_{h}\right)+a\left(\hat{u}_{h}\left(t^{n+1}\right), w_{h}\right)+(1+r)\left\langle\hat{\lambda}_{h}\left(t^{n+1}\right), w_{h}\right\rangle=\left(f\left(t^{n+1}\right), w_{h}\right) . \tag{5.74}
\end{equation*}
$$

Subtracting (5.66a) from (5.18) yields

$$
\begin{equation*}
\left(\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}-\bar{\partial}_{t} u_{h}^{n+1}, w_{h}\right)+a\left(\hat{u}_{h}\left(t^{n+1}\right)-u_{h}^{n+1}, w_{h}\right)+(1+r)\left\langle\hat{\lambda}_{h}\left(t^{n+1}\right)-\lambda_{h}^{n+1}, w_{h}\right\rangle=0 \tag{5.75}
\end{equation*}
$$

or, equivalently,

$$
\begin{equation*}
\left(\bar{\partial}_{t} \theta^{n+1}, w_{h}\right)+a\left(\theta^{n+1}, w_{h}\right)+(1+r)\left\langle\xi^{n+1}, w_{h}\right\rangle=-\left(\bar{\partial}_{t} \varphi^{n+1}, w_{h}\right)+\left(\bar{\partial}_{t} u\left(t^{n+1}\right)-\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}, w_{h}\right) . \tag{5.76}
\end{equation*}
$$

On the other hand, for $\xi^{n+1}$, taking $w_{h}=0$ in the perturbed mixed elliptic projection definition (5.16) yields

$$
\begin{equation*}
\beta\left\langle\hat{u}_{h}\left(t^{n+1}\right)-g\left(t^{n+1}\right), \delta \lambda_{h}\right\rangle-r\left\langle\hat{\lambda}_{h}\left(t^{n+1}\right), \delta \lambda_{h}\right\rangle=0 . \tag{5.77}
\end{equation*}
$$

We can now subtract (5.66b) from the above equation:

$$
\begin{equation*}
\Delta t\left\langle\bar{\partial}_{t} \lambda_{h}^{n+1}, \delta \lambda_{h}\right\rangle-r\left\langle\hat{\lambda}_{h}\left(t^{n+1}\right)-\lambda_{h}^{n+1}, \delta \lambda_{h}\right\rangle+\beta\left\langle\hat{u}_{h}\left(t^{n+1}\right)-u_{h}^{n+1}, \delta \lambda_{h}\right\rangle=0 \tag{5.78}
\end{equation*}
$$

or, equivalently,

$$
\begin{equation*}
\left\langle\theta^{n+1}, \delta \lambda_{h}\right\rangle=\frac{r}{\beta}\left\langle\xi^{n+1}, \delta \lambda_{h}\right\rangle+\frac{\Delta t}{\beta}\left\langle\bar{\partial}_{t} \xi^{n+1}, \delta \lambda_{h}\right\rangle-\frac{\Delta t}{\beta}\left\langle\bar{\partial}_{t} \lambda\left(t^{n+1}\right), \delta \lambda_{h}\right\rangle+\frac{\Delta t}{\beta}\left\langle\bar{\partial}_{t} \omega^{n+1}, \delta \lambda_{h}\right\rangle . \tag{5.79}
\end{equation*}
$$

Taking $w_{h}=\theta^{n+1}$ and $\delta \lambda_{h}=\xi^{n+1}$ and subtracting $(5.79) \times(1+r)$ from (5.76) yields

$$
\begin{align*}
& \left(\bar{\partial}_{t} \theta^{n+1}, \theta^{n+1}\right)+a\left(\theta^{n+1}, \theta^{n+1}\right)+\frac{\Delta t(1+r)}{\beta}\left\langle\bar{\partial}_{t} \xi^{n+1}, \xi^{n+1}\right\rangle+\frac{r(1+r)}{\beta}\left\langle\xi^{n+1}, \xi^{n+1}\right\rangle \\
= & -\left(\bar{\partial}_{t} \varphi^{n+1}, \theta^{n+1}\right)+\left(\bar{\partial}_{t} u\left(t^{n+1}\right)-\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}, \theta^{n+1}\right)-\frac{\Delta t(1+r)}{\beta}\left\langle\bar{\partial}_{t} \omega^{n+1}, \xi^{n+1}\right\rangle \\
& +\frac{\Delta t(1+r)}{\beta}\left\langle\bar{\partial}_{t} \lambda\left(t^{n+1}\right), \xi^{n+1}\right\rangle . \tag{5.80}
\end{align*}
$$

Since

$$
\begin{equation*}
\left(v^{n+1}-v^{n}, v^{n+1}\right)=\frac{1}{2}\left\|v^{n+1}\right\|^{2}-\frac{1}{2}\left\|v^{n}\right\|^{2}+\frac{1}{2}\left\|v^{n+1}-v^{n}\right\|^{2} \geq \frac{1}{2}\left\|v^{n+1}\right\|^{2}-\frac{1}{2}\left\|v^{n}\right\|^{2}, \tag{5.81}
\end{equation*}
$$

for the left hand side of (5.80) we have

$$
\begin{align*}
& \left(\bar{\partial}_{t} \theta^{n+1}, \theta^{n+1}\right)+a\left(\theta^{n+1}, \theta^{n+1}\right)+\frac{\Delta t(1+r)}{\beta}\left\langle\bar{\partial}_{t} \xi^{n+1}, \xi^{n+1}\right\rangle+\frac{r(1+r)}{\beta}\left\langle\xi^{n+1}, \xi^{n+1}\right\rangle \\
\geq & \frac{1}{2 \Delta t}\left(\left\|\theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}-\left\|\theta^{n}\right\|_{L^{2}(\Omega)}^{2}\right)+a\left(\theta^{n+1}, \theta^{n+1}\right)+\frac{1+r}{2 \beta}\left(\left\|\xi^{n+1}\right\|_{L^{2}(\Gamma)}^{2}-\left\|\xi^{n}\right\|_{L^{2}(\Gamma)}^{2}\right) \\
& +\frac{r(1+r)}{\beta}\left\|\xi^{n+1}\right\|_{L^{2}(\Gamma)}^{2} \\
\geq & \frac{1}{2 \Delta t}\left(\left\|\theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}-\left\|\theta^{n}\right\|_{L^{2}(\Omega)}^{2}\right)+\kappa\left\|\nabla \theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\frac{1+r}{2 l}\left(\left\|\xi^{n+1}\right\|_{-1 / 2, h}^{2}-\left\|\xi^{n}\right\|_{-1 / 2, h}^{2}\right) \\
& +\frac{r(1+r)}{l}\left\|\xi^{n+1}\right\|_{-1 / 2, h}^{2} . \tag{5.82}
\end{align*}
$$

From the fact that $\theta^{n+1}=0$ on $\partial \Omega$, and applying the Poincaré's inequality

$$
\begin{equation*}
\left\|\theta^{n+1}\right\|_{L^{2}(\Omega)}^{2} \leq C_{\mathrm{Po}}\left\|\nabla \theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}, \tag{5.83}
\end{equation*}
$$

we can then have the inequality for the right hand side of (5.80):

$$
\begin{align*}
& -\left(\bar{\partial}_{t} \varphi^{n+1}, \theta^{n+1}\right)+\left(\bar{\partial}_{t} u\left(t^{n+1}\right)-\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}, \theta^{n+1}\right)-\frac{\Delta t(1+r)}{\beta}\left\langle\bar{\partial}_{t} \omega^{n+1}, \xi^{n+1}\right\rangle \\
& +\frac{\Delta t(1+r)}{\beta}\left\langle\bar{\partial}_{t} \lambda\left(t^{n+1}\right), \xi^{n+1}\right\rangle \\
\leq & \frac{\kappa}{2 C_{\mathrm{Po}}}\left\|\theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\frac{C_{\mathrm{Po}}}{\kappa}\left\|\bar{\partial}_{t} \varphi^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\frac{C_{\mathrm{Po}}}{\kappa}\left\|\bar{\partial}_{t} u\left(t^{n+1}\right)-\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}\right\|_{L^{2}(\Omega)}^{2} \\
& +\frac{r(1+r)}{2 l}\left\|\xi^{n+1}\right\|_{-1 / 2, h}^{2}+\frac{\Delta t^{2}(1+r)}{r l}\left\|\bar{\partial}_{t} \omega^{n+1}\right\|_{-1 / 2, h}^{2}+\frac{\Delta t^{2}(1+r)}{r l}\left\|\bar{\partial}_{t} \lambda\left(t^{n+1}\right)\right\|_{-1 / 2, h}^{2} \\
\leq & \frac{\kappa}{2}\left\|\nabla \theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\frac{C_{\mathrm{Po}}}{\kappa}\left\|\bar{\partial}_{t} \varphi^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\frac{C_{\mathrm{Po}}}{\kappa}\left\|\bar{\partial}_{t} u\left(t^{n+1}\right)-\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}\right\| \|_{L^{2}(\Omega)}^{2} \\
& +\frac{r(1+r)}{2 l}\left\|\xi^{n+1}\right\|_{-1 / 2, h}^{2}+\frac{\Delta t^{2}(1+r)}{r l}\left\|\bar{\partial}_{t} \omega^{n+1}\right\|_{-1 / 2, h}^{2}+\frac{\Delta t^{2}(1+r)}{r l}\left\|\bar{\partial}_{t} \lambda\left(t^{n+1}\right)\right\|_{-1 / 2, h}^{2} . \tag{5.84}
\end{align*}
$$

Putting together the inequalities for the left and right hand sides, (5.80) yields

$$
\begin{align*}
& \quad \frac{1}{2 \Delta t}\left(\left\|\theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}-\left\|\theta^{n}\right\|_{L^{2}(\Omega)}^{2}\right)+\frac{\kappa}{2}\left\|\nabla \theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\frac{1+r}{2 l}\left(\left\|\xi^{n+1}\right\|_{-1 / 2, h}^{2}-\left\|\xi^{n}\right\|_{-1 / 2, h}^{2}\right) \\
& \\
& +\frac{r(1+r)}{2 l}\left\|\xi^{n+1}\right\|_{-1 / 2, h}^{2} \\
& \leq \frac{C_{\mathrm{Po}}}{\kappa}\left\|\bar{\partial}_{t} \varphi^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\frac{C_{\mathrm{Po}}}{\kappa}\left\|\bar{\partial}_{t} u\left(t^{n+1}\right)-\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}\right\|_{L^{2}(\Omega)}^{2}+\frac{\Delta t^{2}(1+r)}{r l}\left\|\bar{\partial}_{t} \omega^{n+1}\right\|_{-1 / 2, h}^{2}  \tag{5.85}\\
& \\
& \quad+\frac{\Delta t^{2}(1+r)}{r l}\left\|\bar{\partial}_{t} \lambda\left(t^{n+1}\right)\right\|_{-1 / 2, h}^{2},
\end{align*}
$$

or, equivalently,

$$
\begin{align*}
& \left(\left\|\theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}-\left\|\theta^{n}\right\|_{L^{2}(\Omega)}^{2}\right)+\Delta t \kappa\left\|\nabla \theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\frac{\Delta t(1+r)}{l}\left(\left\|\xi^{n+1}\right\|_{-1 / 2, h}^{2}-\left\|\xi^{n}\right\|_{-1 / 2, h}^{2}\right) \\
& +\frac{\Delta t r(1+r)}{l}\left\|\xi^{n+1}\right\|_{-1 / 2, h}^{2} \\
& \leq C\left(\Delta t\left\|\bar{\partial}_{t} \varphi^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\Delta t\left|\bar{\partial}_{t} u\left(t^{n+1}\right)-\frac{\partial u}{\partial t}\right|_{t^{n+1}}\left\|_{L^{2}(\Omega)}^{2}+\Delta t^{3}\right\| \bar{\partial}_{t} \omega^{n+1} \|_{-1 / 2, h}^{2}\right. \\
& \left.+\Delta t^{3}\left\|\bar{\partial}_{t} \lambda\left(t^{n+1}\right)\right\|_{-1 / 2, h}^{2}\right) . \tag{5.86}
\end{align*}
$$

We will now try to bound each term in the right hand side of (5.86):

$$
\begin{align*}
\left\|\bar{\partial}_{t} \varphi^{n+1}\right\|_{L^{2}(\Omega)}^{2} & =\left\|\frac{1}{\Delta t} \int_{t^{n}}^{t^{n+1}} \frac{\partial \varphi}{\partial t} d t\right\|_{L^{2}(\Omega)}^{2} \leq \frac{1}{\Delta t} \int_{t^{n}}^{t^{n+1}}\left\|\frac{\partial \varphi}{\partial t}\right\|_{L^{2}(\Omega)}^{2} d t \\
& \leq \frac{C h^{2-4 \epsilon}}{\Delta t} \int_{t^{n}}^{t^{n+1}}\left(\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})}^{2}\right) d t  \tag{5.87}\\
\left\|\bar{\partial}_{t} u\left(t^{n+1}\right)-\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}\right\|_{L^{2}(\Omega)}^{2} & =\left\|\frac{1}{\Delta t} \int_{t^{n}}^{t^{n+1}}\left(t-t^{n}\right) \frac{\partial^{2} u}{\partial t^{2}} d t\right\|_{L^{2}(\Omega)}^{2} \leq\left\|\frac{1}{\Delta t} \int_{t^{n}}^{t^{n+1}} \Delta t\left|\frac{\partial^{2} u}{\partial t^{2}}\right| d t\right\|_{L^{2}(\Omega)}^{2} \\
& \leq \Delta t \int_{t^{n}}^{t^{n+1}}\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2} d t .  \tag{5.88}\\
& =\frac{1}{\Delta t} \int_{t^{n}}^{t^{n}}\left\|_{t^{n+1}}^{n+1}\right\|_{-1 / 2, h}^{2} \\
& \leq \frac{\partial \omega}{\partial t}\left\|_{-1 / 2, h}^{2} \int_{t^{n}}^{t^{n+1}} \frac{\partial \omega}{\partial t} d t\right\|_{L^{2}(\Gamma)}^{2} \leq \frac{h}{\Delta t} \int_{t^{n}}^{t^{n+1}}\left\|\frac{\partial \omega}{\partial t}\right\|_{L^{2}(\Gamma)}^{2} d t \\
& \int_{t^{n}}^{t^{n^{n+1}}}\left(\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}\right) d t  \tag{5.89}\\
& \leq \frac{C h}{\Delta t} \int_{t^{n}}^{t^{n+1}}\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2} d t
\end{align*}
$$

Substituting the above inequalities into (5.86) yields

$$
\begin{align*}
& \left(\left\|\theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}-\left\|\theta^{n}\right\|_{L^{2}(\Omega)}^{2}\right)+\Delta t \kappa\left\|\nabla \theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\frac{\Delta t(1+r)}{l}\left(\left\|\xi^{n+1}\right\|_{-1 / 2, h}^{2}-\left\|\xi^{n}\right\|_{-1 / 2, h}^{2}\right) \\
& +\frac{\Delta t r(1+r)}{l}\left\|\xi^{n+1}\right\|_{-1 / 2, h}^{2} \\
\leq & C \int_{t^{n}}^{t^{n+1}}\left(\left(h^{2-4 \epsilon}+h^{1-\epsilon} \Delta t^{2}+h \Delta t^{2}\right)\left(\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\mathrm{T})}^{2}\right)+\Delta t^{2}\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2}\right) d t . \tag{5.91}
\end{align*}
$$

Summing (5.91) up from the first step to the $(n+1)^{\text {th }}$ step, we find that

$$
\begin{align*}
& \left\|\theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\Delta t \kappa \sum_{i=0}^{n}\left\|\nabla \theta^{i+1}\right\|_{L^{2}(\Omega)}^{2}+\frac{\Delta t(1+r)}{l}\left\|\xi^{n+1}\right\|_{-1 / 2, h}^{2} \\
& \quad+\frac{\Delta t r(1+r)}{l} \sum_{i=0}^{n}\left\|\xi^{i+1}\right\|_{-1 / 2, h}^{2} \\
& \leq C \int_{0}^{t^{n+1}}\left(\left(h^{2-4 \epsilon}+h^{1-2 \epsilon} \Delta t^{2}+h \Delta t^{2}\right)\left(\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})}^{2}\right)\right. \\
& \left.\quad+\Delta t^{2}\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2}\right) d t+\left\|\theta^{0}\right\|_{L^{2}(\Omega)}^{2}+\frac{\Delta t(1+r)}{l}\left\|\xi^{0}\right\|_{-1 / 2, h}^{2} . \tag{5.92}
\end{align*}
$$

While taking the initial values of $u_{h}^{0}$ and $\lambda_{h}^{0}$ as in (5.66), and since $h \ll 1$, one has the estimates for the initial errors

$$
\begin{gather*}
\left\|\theta^{0}\right\|_{L^{2}(\Omega)}=\left\|\hat{u}_{h}\left(t^{0}\right)-I_{h} u_{0}\right\|_{L^{2}(\Omega)} \leq\left\|u_{0}-I_{h} u_{0}\right\|_{L^{2}(\Omega)}+\left\|\hat{u}_{h}\left(t^{0}\right)-u_{0}\right\|_{L^{2}(\Omega)} \\
\leq C h^{1-2 \epsilon}\left(\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}+\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon(\Gamma)}}\right)  \tag{5.93}\\
\left\|\xi^{0}\right\|_{-1 / 2, h}=\left\|\hat{\lambda}_{h}\left(t^{0}\right)-0\right\|_{-1 / 2, h} \leq\left\|\lambda_{0}\right\|_{-1 / 2, h}+\left\|\hat{\lambda}_{h}\left(t^{0}\right)-\lambda_{0}\right\|_{-1 / 2, h} \\
\leq C h^{1 / 2-\epsilon}\left(\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}+\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}\right) . \tag{5.94}
\end{gather*}
$$

Substituting the initial error estimates into (5.92) and using the fact that $h \ll 1$ yields

$$
\begin{align*}
& \left\|\theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\Delta t \kappa \sum_{i=0}^{n}\left\|\nabla \theta^{i+1}\right\|_{L^{2}(\Omega)}^{2}+\frac{\Delta t(1+r)}{l}\left\|\xi^{n+1}\right\|_{-1 / 2, h}^{2} \\
& \quad+\frac{\Delta t r(1+r)}{l} \sum_{i=0}^{n}\left\|\xi^{i+1}\right\|_{-1 / 2, h}^{2} \\
& \leq C \int_{0}^{t^{n+1}}\left(\left(h^{2-4 \epsilon}+h^{1-2 \epsilon} \Delta t^{2}+h \Delta t^{2}\right)\left(\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2}\right)+\Delta t^{2}\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2}\right) d t \\
& \quad+C\left(h^{2-4 \epsilon}+h^{1-2 \epsilon} \Delta t\right)\left(\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2}\right)  \tag{5.95}\\
& \leq C \int_{0}^{t^{n+1}}\left(h^{2-4 \epsilon}\left(\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon \epsilon(\Omega)}}^{2}+\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2}\right)+\Delta t^{2}\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2}\right) d t \\
& \quad+C\left(h^{2-4 \epsilon}+h^{1-2 \epsilon} \Delta t\right)\left(\left\|u_{0}\right\|_{H^{3 / 2-\epsilon(\Omega)}}^{2}+\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2}\right) . \tag{5.96}
\end{align*}
$$

Therefore

$$
\begin{align*}
& \sum_{i=0}^{n}\left\|\nabla \theta^{i+1}\right\|_{L^{2}(\Omega)}^{2}+\sum_{i=0}^{n}\left\|\xi^{i+1}\right\|_{-1 / 2, h}^{2} \\
\leq & \frac{C}{\Delta t} \int_{0}^{t^{n+1}}\left(h^{2-4 \epsilon}\left(\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon(\Omega)}}^{2}+\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon(\mathrm{\Gamma})}}^{2}\right)+\Delta t^{2}\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2}\right) d t \\
& +\frac{C}{\Delta t}\left(h^{2-4 \epsilon}+h^{1-2 \epsilon} \Delta t\right)\left(\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})}^{2}\right) \tag{5.97}
\end{align*}
$$

and

$$
\begin{align*}
\left\|\theta^{n+1}\right\|_{L^{2}(\Omega)}^{2} & \leq C \int_{0}^{t^{n+1}}\left(h^{2-4 \epsilon}\left(\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})}^{2}\right)+\Delta t^{2}\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2}\right) d t \\
& +C\left(h^{2-4 \epsilon}+h^{1-2 \epsilon} \Delta t\right)\left(\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})}^{2}\right) . \tag{5.98}
\end{align*}
$$

The inequality (5.97) yields

$$
\begin{align*}
& \left\|\nabla \theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\left\|\xi^{n+1}\right\|_{-1 / 2, h}^{2} \\
\leq & \frac{C}{\Delta t} \int_{0}^{t^{n+1}}\left(h^{2-4 \epsilon}\left(\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})}^{2}\right)+\Delta t^{2}\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2}\right) d t \\
& +\frac{C}{\Delta t}\left(h^{2-4 \epsilon}+h^{1-2 \epsilon} \Delta t\right)\left(\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon}(\mathrm{T})}^{2}\right) . \tag{5.99}
\end{align*}
$$

Together with the estimates for the interpolation errors $\varphi^{n+1}$ and $\omega^{n+1}$, we have obtained an error
estimate in the $|||(\cdot, \cdot)|||$-norm:

$$
\begin{align*}
& \left\|\left\|\left(u\left(t^{n+1}\right)-u_{h}^{n+1}, \lambda\left(t^{n+1}\right)-\lambda_{h}^{n+1}\right) \mid\right\|^{2}\right. \\
& \leq C\left(\frac{h^{2-4 \epsilon}}{\Delta t}+h^{1-2 \epsilon}\right)\left(\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})}^{2}\right) \\
& +C \int_{0}^{t^{n+1}}\left(\frac{h^{2-4 \epsilon}}{\Delta t}\left(\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon(\mathrm{C})}}^{2}\right)+\Delta t\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2}\right) d t \\
& +C h^{1-2 \epsilon}\left(\left\|u\left(t^{n+1}\right)\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\lambda\left(t^{n+1}\right)\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2}\right) \\
& \leq C\left(\frac{h^{2-4 \epsilon}}{\Delta t}+h^{1-2 \epsilon}+\Delta t\right)\left(\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}+\int_{0}^{t^{n+1}}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}\right. \\
& \left.+\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})}^{2}+\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2} d t\right) \\
& \leq C\left(\frac{h^{2-4 \epsilon}}{\Delta t}+\Delta t\right)\left(\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2}+\int_{0}^{t^{n+1}}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}\right. \\
& \left.+\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}+\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2} d t\right) . \tag{5.100}
\end{align*}
$$

On the other hand, combining (5.98) with the estimate for $\varphi^{n+1}$, we obtain an error estimate in the $L^{2}$ norm:

$$
\begin{align*}
& \left\|u\left(t^{n+1}\right)-u_{h}^{n+1}\right\|_{L^{2}(\Omega)}^{2} \\
\leq & 2\left(\left\|\varphi^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\left\|\theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}\right) \\
\leq & C\left(h^{2-4 \epsilon}\left(\left\|u\left(t^{n+1}\right)\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\lambda\left(t^{n+1}\right)\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}\right)+\left(h^{2-4 \epsilon}+h^{1-2 \epsilon} \Delta t\right)\left(\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}\right.\right. \\
& \left.+\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}\right)+\int_{0}^{t^{n+1}}\left(h^{2-4 \epsilon}\left(\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2}\right)\right. \\
& \left.\left.+\Delta t^{2}\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2}\right) d t\right) \\
\leq & C\left(h^{2-4 \epsilon}+\Delta t^{2}\right)\left(\left\|u\left(t^{n+1}\right)\right\|_{H^{3 / 2-\epsilon(\Omega)}}^{2}+\left\|\lambda\left(t^{n+1}\right)\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2}+\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}\right. \\
& \left.+\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2}+\int_{0}^{t^{n^{n+1}}}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2}+\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2} d t\right) \\
\leq & C\left(h^{2-4 \epsilon}+\Delta t^{2}\right)\left(\left\|u_{0}\right\|_{H^{3 / 2-\epsilon(\Omega)}}^{2}+\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2}+\int_{0}^{t^{n+1}}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}\right. \\
& \left.+\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2}+\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2} d t\right) . \tag{5.101}
\end{align*}
$$

We have then finished the error estimate of the fully-discrete original DAL method.
Theorem 5.1.2 has indicated the following corollary:
Corollary 1. In the original fully-discrete DAL method, the optimal time step size is $\Delta t=O(h)$, which yields the error estimates:

$$
\begin{gather*}
\left\|\nabla\left(u\left(t^{n}\right)-u_{h}^{n}\right)\right\|_{L^{2}(\Omega)}^{2}+h\left\|\lambda\left(t^{n}\right)-\lambda_{h}^{n}\right\|_{L^{2}(\Gamma)}^{2} \leq C h^{1-4 \epsilon},  \tag{5.102}\\
\left\|u\left(t^{n}\right)-u_{h}^{n}\right\|_{L^{2}(\Omega)}^{2} \leq C h^{2-4 \epsilon} . \tag{5.103}
\end{gather*}
$$

Or, equivalently, $h^{1 / 2-2 \epsilon}$ order of accuracy for the $\left\||(\cdot, \cdot) \||-\right.$ norm and $h^{1-2 \epsilon}$ order for $u$ in the $L^{2}(\Omega)-$ norm.

Remark 5.10. Due to the lack of strong consistency of the method and the problem regularity in the static problem, we lose $h^{1 / 2+\epsilon}$ order for the $\|\mid(\cdot, \cdot)\|$-norm and $h^{1+2 \epsilon}$ order for the $L^{2}$-norm, as shown in (5.42) and (5.43).

Remark 5.11. The $\frac{h^{2-4 \epsilon}}{\Delta t}$ in 5.68) suggests that, when taking $\Delta t$ smaller than $O\left(h^{2}\right)$, one might expect to see large errors in the $H^{1}$-norm. A similar concern was raised by our earlier analysis [35]. However, numerical experiments in Section 6.4 suggest that the method remains robust in the limit of temporal over-refinement.

Remark 5.12. In practical numerical implementations, we are using nodal quadrature to obtain a diagonal mass matrix for what is essentially an $L^{2}(\Gamma)$ projection. The Lagrange multiplier is represented as a collection of samples at quadrature points on $\Gamma$ (i.e., amplitudes of the Dirac measure) and the residual of the perturbed constraint equation is tested independently at each point. In such implementations, $W_{h}$ is enriched and the interpolation error should be much smaller than considered in the analysis here. Since coercivity condition (5.18) still holds, such an enrichment of $W_{h}$ should not affect the error estimates of $u$.

### 5.2. Projection-based dynamic augmented Lagrangian (DAL) method

For the projection-based DAL method, introducing the Lagrange multiplier yields an alternative weak formulation for this interface problem: Find $(u, \lambda) \in X \times M$ such that for $t \in(0, T]$,

$$
\begin{align*}
\left(u_{t}, w\right)+a(u, w)+\langle\lambda, w\rangle & =(f, w), \quad \forall w \in X,  \tag{5.104a}\\
\langle u-g, \delta \lambda\rangle & =0, \quad \forall \delta \lambda \in M, \tag{5.104b}
\end{align*}
$$

where $a(u, w), X, M$ are defined as in the previous section. For the numerical method, we consider the regular triangulation $\mathcal{T}_{h}(\Omega)$ and the space

$$
\begin{equation*}
X_{h}(\Omega)=\left\{w_{h} \in P_{1}(T) \mid w_{h}=0 \text { on } \partial \Omega\right\} . \tag{5.105}
\end{equation*}
$$

For the projection, we also consider a coarse, piecewise constant function space $W_{H}(\Gamma)$ on the interface, which may have jump discontinuities at element boundaries. We also assume that $X_{h}(\Omega)$ and $W_{H}(\Gamma)$ are all strongly regular [92]. The Lagrange multiplier method with projection-based stabilization technique in Section 3.2 can now be applied on the weak problem (5.104): Find $\left(u_{h}^{n+1}, \lambda_{H}^{n+1}\right) \in X_{h} \times W_{H}$ such that for all $w_{h} \in X_{h}$ and $\delta \lambda_{H} \in W_{H}$,

$$
\begin{array}{r}
\left(\frac{u_{h}^{n+1}-u_{h}^{n}}{\Delta t}, w_{h}\right)+a\left(u_{h}^{n+1}, w_{h}\right)+\left\langle\lambda_{H}^{n}, w_{h}\right\rangle+\beta\left\langle u_{h}^{n+1}-g\left(t^{n+1}\right), w_{h}\right\rangle=\left(f\left(t^{n+1}\right), w_{h}\right) \\
\left\langle\lambda_{H}^{n+1}, \delta \lambda_{H}\right\rangle=\left\langle\lambda_{H}^{n}, \delta \lambda_{H}\right\rangle+\beta\left\langle P\left(u_{h}^{n+1}-g\left(t^{n+1}\right)\right), \delta \lambda_{H}\right\rangle \tag{5.106b}
\end{array}
$$

From 5.106 b ) and the orthogonality property of the projection operator $P$, we have

$$
\begin{align*}
\left\langle\lambda_{H}^{n+1}, w_{h}\right\rangle & =\left\langle\lambda_{H}^{n+1}, P\left(w_{h}\right)\right\rangle \\
& =\left\langle\lambda_{H}^{n}, P\left(w_{h}\right)\right\rangle+\beta\left\langle P\left(u_{h}^{n+1}-g\left(t^{n+1}\right)\right), P\left(w_{h}\right)\right\rangle \\
& =\left\langle\lambda_{H}^{n}, w_{h}\right\rangle+\beta\left\langle u_{h}^{n+1}-g\left(t^{n+1}\right), w_{h}\right\rangle-\beta\left\langle P^{\perp}\left(u_{h}^{n+1}-g\left(t^{n+1}\right)\right), P^{\perp}\left(w_{h}\right)\right\rangle . \tag{5.107}
\end{align*}
$$

Substituting into (5.106a) yields

$$
\begin{align*}
& \left(\frac{u_{h}^{n+1}-u_{h}^{n}}{\Delta t}, w_{h}\right)+a\left(u_{h}^{n+1}, w_{h}\right)+\left\langle\lambda_{H}^{n+1}, P\left(w_{h}\right)\right\rangle+\beta\left\langle P^{\perp}\left(u_{h}^{n+1}\right), P^{\perp}\left(w_{h}\right)\right\rangle \\
= & \left(f\left(t^{n+1}\right), w_{h}\right)+\beta\left\langle P^{\perp}\left(g\left(t^{n+1}\right)\right), P^{\perp}\left(w_{h}\right)\right\rangle \tag{5.108}
\end{align*}
$$

and (5.106b) can be written as

$$
\begin{equation*}
\Delta t\left\langle\frac{\lambda_{H}^{n+1}-\lambda_{H}^{n}}{\Delta t}, \delta \lambda_{H}\right\rangle=\beta\left\langle P\left(u_{h}^{n+1}-g\left(t^{n+1}\right)\right), \delta \lambda_{H}\right\rangle \tag{5.109}
\end{equation*}
$$

Combining (5.108) with (5.109) with $\delta \lambda_{H}=P\left(w_{h}\right)$ we can obtain

$$
\begin{align*}
& \left(\frac{u_{h}^{n+1}-u_{h}^{n}}{\Delta t}, w_{h}\right)-\Delta t\left\langle\frac{\lambda_{H}^{n+1}-\lambda_{H}^{n}}{\Delta t}, w_{h}\right\rangle+a\left(u_{h}^{n+1}, w_{h}\right)+\left\langle\lambda_{H}^{n+1}, P\left(w_{h}\right)\right\rangle+\beta\left\langle u_{h}^{n+1}, w_{h}\right\rangle \\
= & \left(f\left(t^{n+1}\right), w_{h}\right)+\beta\left\langle g\left(t^{n+1}\right), w_{h}\right\rangle, \tag{5.110}
\end{align*}
$$

In the following derivations, we will use the mesh-dependent half norm based on the mesh $X_{h}$

$$
\begin{equation*}
\|\lambda\|_{1 / 2, h}^{2}=\frac{1}{h}\langle\lambda, \lambda\rangle, \quad\|\lambda\|_{-1 / 2, h}^{2}=h\langle\lambda, \lambda\rangle \tag{5.111}
\end{equation*}
$$

and the norms defiend on the coarse mesh $W_{H}$ :

$$
\begin{equation*}
\|\lambda\|_{1 / 2, H}^{2}=\frac{1}{H}\langle\lambda, \lambda\rangle, \quad\|\lambda\|_{-1 / 2, H}^{2}=H\langle\lambda, \lambda\rangle . \tag{5.112}
\end{equation*}
$$

### 5.2.1. Static: Elliptic interface problem

We now study the error estimates for the projection-based DAL method defined in Section 3.2. As in the above sections, we start with the static problem (5.104). The steady limit of the projection-based DAL method can be written as

$$
\begin{align*}
a\left(u_{h}, w_{h}\right)+\left\langle\lambda_{H}, w_{h}\right\rangle+\beta\left\langle u_{h}, w_{h}\right\rangle & =\left(f, w_{h}\right)+\beta\left\langle g, w_{h}\right\rangle, \quad \forall w_{h} \in X_{h}  \tag{5.113a}\\
\left\langle u_{h}-g, \delta \lambda_{H}\right\rangle & =0, \quad \forall \delta \lambda_{H} \in W_{H} . \tag{5.113b}
\end{align*}
$$

Defining

$$
\begin{equation*}
\mathcal{A}\left(u_{h}, \lambda_{H} ; w_{h}, \delta \lambda_{H}\right)=a\left(u_{h}, w_{h}\right)+\left\langle\lambda_{H}, w_{h}\right\rangle-\left\langle u_{h}, \delta \lambda_{H}\right\rangle+\beta\left\langle u_{h}, w_{h}\right\rangle \tag{5.114}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{F}\left(w_{h}, \delta \lambda_{H}\right)=\left(f, w_{h}\right)-\left\langle g, \delta \lambda_{H}\right\rangle+\beta\left\langle g, w_{h}\right\rangle, \tag{5.115}
\end{equation*}
$$

the formulation (5.113) can be written as

$$
\begin{equation*}
\mathcal{A}\left(u_{h}, \lambda_{H} ; w_{h}, \delta \lambda_{H}\right)=\mathcal{F}\left(w_{h}, \delta \lambda_{H}\right), \quad \forall w_{h} \in X_{h}, \delta \lambda_{H} \in W_{H} . \tag{5.116}
\end{equation*}
$$

Combining (5.113) with the weak formulation from the static problem

$$
\begin{align*}
a(u, w)+\langle\lambda, w\rangle & =(f, w), \quad \forall w \in X,  \tag{5.117a}\\
\langle u-g, \delta \lambda\rangle & =0, \quad \forall \delta \lambda \in M, \tag{5.117b}
\end{align*}
$$

one can obtain an alternative definition of the static solution

$$
\begin{equation*}
a\left(u-u_{h}, w_{h}\right)+\left\langle\lambda-\lambda_{H}, w_{h}\right\rangle-\left\langle u-u_{h}, \delta \lambda_{H}\right\rangle+\beta\left\langle u-u_{h}, w_{h}\right\rangle=0, \tag{5.118}
\end{equation*}
$$

or

$$
\begin{equation*}
\mathcal{A}\left(u-u_{h}, \lambda-\lambda_{H} ; w_{h}, \delta \lambda_{H}\right)=0 . \tag{5.119}
\end{equation*}
$$

Therefore, one can similarly define a mixed elliptic projection $\left(\tilde{u}_{h}, \tilde{\lambda}_{H}\right)$ for given $u(t)$ and $\lambda(t)$ as the solution of

$$
\begin{equation*}
\mathcal{A}\left(u-\tilde{u}_{h}, \lambda-\tilde{\lambda}_{H} ; w_{h}, \delta \lambda_{H}\right)=0, \quad \forall w_{h} \in X_{h}, \delta \lambda_{H} \in W_{H} . \tag{5.120}
\end{equation*}
$$

Then the estimates in this section also hold true for the mixed elliptic projection.
In the estimates, we employ the mesh-dependent norm

$$
\begin{equation*}
\left\|\left\|\left(u_{h}, \lambda_{H}\right)\right\|_{p}^{2}=\right\| \nabla u_{h}\left\|_{L^{2}(\Omega)}^{2}+\right\| u_{h}\left\|_{1 / 2, h}^{2}+\right\| \lambda_{H} \|_{-1 / 2, h}^{2} \tag{5.121}
\end{equation*}
$$

Before introducing the main lemmas for the continuity property and the inf-sup condition, we first introduce the following lemma:

Lemma 5.13. For any given function $\mu_{H} \in W_{H}$, denote by $\chi$ the solution of the Robin problem for the differential equation

$$
\begin{align*}
-\nabla \cdot(c(\nabla(\chi))) & =0, & & \text { in } \Omega \backslash \Gamma,  \tag{5.122a}\\
\chi & =0, & & \text { on } \partial \Omega,  \tag{5.122b}\\
c\left(\nabla \chi_{1}\right) \cdot \mathbf{n}_{1}+c\left(\nabla \chi_{2}\right) \cdot \mathbf{n}_{2}+\beta \chi & =\mu_{H}, & & \text { on } \Gamma . \tag{5.122c}
\end{align*}
$$

Then

$$
\begin{equation*}
a(\chi, \chi)+\beta\langle\chi, \chi\rangle=\left\langle\chi, \mu_{H}\right\rangle \tag{5.123}
\end{equation*}
$$

and there exist constants $0<C_{1}<C_{2}<\infty$ such that

$$
\begin{equation*}
C_{1}\left\langle\chi, \mu_{H}\right\rangle \leq\left\|\mu_{H}\right\|_{-1 / 2, h}^{2} \leq C_{2}\left\langle\chi, \mu_{H}\right\rangle . \tag{5.124}
\end{equation*}
$$

Proof: One can obtain (5.123) by applying the test function $\chi$ to (5.122a) and integrating. For (5.124), we first want to prove the following inequality

$$
\begin{equation*}
\left\|\mu_{H}\right\|_{-1 / 2, h}^{2} \leq C_{2}\left\langle\chi, \mu_{H}\right\rangle \tag{5.125}
\end{equation*}
$$

Take $v_{h}=\Pi_{h}\left(\mu_{H}\right)$, then we have

$$
\begin{equation*}
\left\|v_{h}\right\|_{L^{2}(\Gamma)} \leq\left\|\mu_{H}\right\|_{L^{2}(\Gamma)} . \tag{5.126}
\end{equation*}
$$

We now show that for $h / H$ sufficiently small, the following statement holds:

$$
\begin{equation*}
\left\|\mu_{H}\right\|_{L^{2}(\Gamma)}^{2} \leq C\left\langle v_{h}, \mu_{H}\right\rangle \tag{5.127}
\end{equation*}
$$

Since $\mu_{H}$ is piecewise constant, with possible jump discontinuities at element boundaries, we have
$\mu_{H} \in H^{1 / 2-\epsilon}(\Gamma)$ for any $\epsilon \geq 0$. Therefore, with the assumption that $W_{H}$ is strongly regular and when $h / H$ is sufficiently small,

$$
\begin{equation*}
\left\|\mu_{H}-v_{h}\right\|_{L^{2}(\mathrm{~T})} \leq C h^{1 / 2-\epsilon}\left\|\mu_{H}\right\|_{H^{1 / 2-\epsilon}(\mathrm{T})} \leq C\left(\frac{h}{H}\right)^{1 / 2-\epsilon}\left\|\mu_{H}\right\|_{L^{2}(\mathrm{~T})} \leq \frac{1}{2}\left\|\mu_{H}\right\|_{L^{2}(\mathrm{~T})} . \tag{5.128}
\end{equation*}
$$

Then

$$
\begin{align*}
\left\|\mu_{H}\right\|_{L^{2}(\Gamma)}^{2} & =\left\langle v_{h}, \mu_{H}\right\rangle+\left\langle\mu_{H}-v_{h}, \mu_{H}\right\rangle=\left\langle v_{h}, \mu_{H}\right\rangle+\left\|\mu_{H}-v_{h}\right\|_{L^{2}(\Gamma)}^{2} \\
& \leq\left\langle v_{h}, \mu_{H}\right\rangle+\frac{1}{4}\left\|\mu_{H}\right\|_{L^{2}(\Gamma)}^{2}, \tag{5.129}
\end{align*}
$$

which implies

$$
\begin{equation*}
\left\|\mu_{H}\right\|_{L^{2}(\Gamma)}^{2} \leq \frac{4}{3}\left\langle v_{h}, \mu_{H}\right\rangle, \tag{5.130}
\end{equation*}
$$

giving (5.127).
By an inverse imbedding theorem, there exists a linear mapping $\mathcal{M}$ of $H^{1 / 2}(\Gamma)$ into $H^{1}(\Omega)$ such that for any function $\theta \in H^{1 / 2}(\Gamma)$ we have $\mathcal{M}(\theta)=\theta$ on $\Gamma$ and $\|\mathcal{M}(\theta)\|_{H^{1}(\Omega)} \leq C\|\theta\|_{H^{1 / 2}(\Gamma)}$. Since $v_{h}=\Pi_{h}\left(\mu_{H}\right)$ is continuous, we have $v_{h} \in H^{1 / 2}(\Gamma)$ and therefore

$$
\begin{equation*}
\left\|\mathcal{M}\left(v_{h}\right)\right\|_{H^{1}(\Omega)} \leq C\left\|v_{h}\right\|_{H^{1 / 2}(\Gamma)} \leq C\left\|v_{h}\right\|_{1 / 2, h} . \tag{5.131}
\end{equation*}
$$

The above bounds yield

$$
\begin{align*}
\left\|\mu_{H}\right\|_{L^{2}(\Gamma)}^{2} & \leq C\left\langle\mu_{H}, v_{h}\right\rangle=C\left(a\left(\chi, \mathcal{M}\left(v_{h}\right)\right)+\beta\left\langle\chi, v_{h}\right\rangle\right) \\
& \leq C\left(\left\|\mathcal{M}\left(v_{h}\right)\right\|_{H^{1}(\Omega)}\|\chi\|_{H^{1}(\Omega)}+\beta\left\|v_{h}\right\|_{L^{2}(\Gamma)}\|\chi\|_{L^{2}(\Gamma)}\right) \\
& \leq C\left(\left\|v_{h}\right\|_{1 / 2, h}\|\chi\|_{H^{1}(\Omega)}+\frac{1}{h}\left\|\mu_{H}\right\|_{L^{2}(\Gamma)}\|\chi\|_{L^{2}(\Gamma)}\right) \\
& \leq C\left(\frac{1}{\sqrt{h}}\left\|\mu_{H}\right\|_{L^{2}(\Gamma)}\|\chi\|_{H^{1}(\Omega)}+\frac{1}{\sqrt{h}}\left\|\mu_{H}\right\|_{L^{2}(\Gamma)}\|\chi\|_{1 / 2, h}\right), \tag{5.132}
\end{align*}
$$

which, after dividing through by $\left\|\mu_{H}\right\|_{L^{2}(\Gamma)} / \sqrt{h}$, implies (5.125):

$$
\begin{align*}
\left\|\mu_{H}\right\|_{-1 / 2, h} & \leq C\left(\|\chi\|_{H^{1}(\Omega)}+\|\chi\|_{1 / 2, h}\right) \\
& \leq C \sqrt{a(\chi, \chi)+\beta\langle\chi, \chi\rangle} \\
& =C \sqrt{\left\langle\mu_{H}, \chi\right\rangle} . \tag{5.133}
\end{align*}
$$

For the other side of (5.124), since

$$
\|\chi\|_{1 / 2, h}=\frac{1}{\sqrt{h}}\|\chi\|_{L^{2}(\mathrm{~T})} \leq C \sqrt{a(\chi, \chi)+\beta\langle\chi, \chi\rangle},
$$

we have

$$
\begin{align*}
\left\langle\chi, \mu_{H}\right\rangle & \leq\|\chi\|_{1 / 2, h}\left\|\mu_{H}\right\|_{-1 / 2, h} \\
& \leq C \sqrt{a(\chi, \chi)+\beta\langle\chi, \chi\rangle}\left\|\mu_{H}\right\|_{-1 / 2, h} \\
& =C \sqrt{\left\langle\chi, \mu_{H}\right\rangle}\left\|\mu_{H}\right\|_{-1 / 2, h}, \tag{5.134}
\end{align*}
$$

and thus

$$
\left\langle\chi, \mu_{H}\right\rangle \leq C\left\|\mu_{H}\right\|_{-1 / 2, h}^{2} .
$$

We have finished the proof.
Lemma 5.14. Assuming that $\Omega$ has finite diameter, then for any given function $\mu_{H} \in W_{H}$, denote by $\chi$ the solution of the Robin problem in (5.122), we have $\chi \in H^{3 / 2-\epsilon}(\Omega)$. Moreover, for $1 \leq s \leq$ $3 / 2-\epsilon$, the following inequality holds

$$
\begin{equation*}
\|\chi\|_{H^{s}(\Omega)} \leq C \sqrt{h}\left\|\mu_{H}\right\|_{H^{s-1}(\Gamma)} . \tag{5.135}
\end{equation*}
$$

Proof: For $s=1$, Lemma 5.2.1 gives

$$
\begin{equation*}
\|\chi\|_{H^{1}(\Omega)}^{2} \leq C a(\chi, \chi) \leq C\left\langle\chi, \mu_{H}\right\rangle \leq C\left\|\mu_{H}\right\|_{-1 / 2, h}^{2} . \tag{5.136}
\end{equation*}
$$

For the case $1<s=3 / 2-\epsilon$, we separate the proof for 2D and 3D cases. Since $\mu_{H}$ is piecewise constant, $\mu_{H} \in H^{1 / 2-\epsilon}(\Gamma)$ for any $\epsilon \geq 0$. Note that for any piece-wise constant $\mu_{H}$, we can approximate in $H^{1 / 2-\epsilon}(\Gamma)$ with a sequence of smooth functions which are obtained by mollifying $\mu_{H}$ on arbitrarily small sets near the jump points. Therefore, we first prove the estimate for $\mu_{H} \in H^{1-\epsilon}(\Gamma)$. Later we will extend the proof to $\mu_{H} \in H^{1 / 2-\epsilon}(\Gamma)$ for all $\epsilon>0$. For the 2D case, the Gagliardo semi-norm can be written as

$$
[\nabla \chi]_{H^{1 / 2-\epsilon(\Omega)}}^{2}:=\int_{\Omega \times \Omega} \frac{|\nabla \chi(x)-\nabla \chi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d x d y .
$$

Note that

$$
\begin{aligned}
\int_{\Omega \times \Omega} \frac{|\nabla \chi(x)-\nabla \chi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d x d y=\sum_{j=1}^{2} & \int_{\Omega_{j} \times \Omega_{j}} \frac{|\nabla \chi(x)-\nabla \chi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d x d y \\
& +\int_{\Omega_{1} \times \Omega_{2}} \frac{|\nabla \chi(x)-\nabla \chi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d x d y \\
& +\int_{\Omega_{2} \times \Omega_{1}} \frac{|\nabla \chi(x)-\nabla \chi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d x d y .
\end{aligned}
$$

It suffices to show

$$
\begin{align*}
& \int_{\Omega_{1} \times \Omega_{1}} \frac{|\nabla \chi(x)-\nabla \chi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d x d y=\left\|\chi_{k}\right\|_{H^{3 / 2-\epsilon}\left(\Omega_{1}\right)}^{2}<C h\left\|\mu_{H}\right\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})}^{2}  \tag{5.137}\\
& \int_{\Omega_{1} \times \Omega_{2}} \frac{|\nabla \chi(x)-\nabla \chi(y)|^{2}}{|x-y| 3^{3-2 \epsilon}} d x d y<C h\left\|\mu_{H}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2} \tag{5.138}
\end{align*}
$$

as the proof for the other two integral terms is completely analogous. Here the symbol " $C$ " may denote different constants in different equations, but they are all independent of $h$. To prove (5.137), we rewrite the problem as

$$
\begin{align*}
-\nabla \cdot(c(\nabla(\chi))) & =0, \quad \text { in } \Omega \backslash \Gamma,  \tag{5.139}\\
\chi & =0, \quad \text { on } \partial \Omega,  \tag{5.140}\\
\chi_{k} & =\frac{1}{\beta}\left(\mu_{H}-j\right) \quad \text { on } \Gamma, \tag{5.141}
\end{align*}
$$

where $j:=\kappa \nabla\left(\chi_{1}-\chi_{2}\right) \cdot \mathbf{n}_{1}$ is the normal jump of the gradient across $\Gamma$. To estimate $\|j\|_{L^{2}(\Gamma)}$, we claim

$$
\begin{equation*}
\|j\|_{L^{2}(\Gamma)} \leq\left\|\mu_{H}\right\|_{L^{2}(\Gamma)} \tag{5.142}
\end{equation*}
$$

By integration by parts we have

$$
\begin{aligned}
\int_{\Gamma} \chi_{1} \nabla \chi_{1} \cdot \mathbf{n}_{1} d \mathcal{H}_{\llcorner\Gamma}^{1} & =\int_{\Omega_{1}}\left[\chi_{1} \Delta \chi_{1}+\left|\nabla \chi_{1}\right|^{2}\right] d x=\left\|\nabla \chi_{1}\right\|_{L^{2}\left(\Omega_{1}\right)}^{2}, \\
-\int_{\Gamma} \chi_{2} \nabla \chi_{2} \cdot \mathbf{n}_{1} d \mathcal{H}_{\llcorner\Gamma}^{1} & =\int_{\Omega_{1}}\left[\chi_{2} \Delta \chi_{2}+\left|\nabla \chi_{2}\right|^{2}\right] d x=\left\|\nabla \chi_{2}\right\|_{L^{2}\left(\Omega_{2}\right)}^{2},
\end{aligned}
$$

hence

$$
\begin{aligned}
\|\nabla \chi\|_{L^{2}(\Omega)}^{2} & =\int_{\Gamma} \chi \nabla\left(\chi_{1}-\chi_{2}\right) \cdot \mathbf{n}_{1} d \mathcal{H}_{\llcorner\Gamma}^{1}=\int_{\Gamma} \frac{1}{\beta}\left(\mu_{H}-\nabla\left(\chi_{1}-\chi_{2}\right) \cdot \mathbf{n}_{1}\right)\left(\nabla\left(\chi_{1}-\chi_{2}\right) \cdot \mathbf{n}_{1}\right) d \mathcal{H}_{\llcorner\Gamma}^{1} \\
& =\frac{h}{l} \int_{\Gamma}\left[\mu_{H} \nabla\left(\chi_{1}-\chi_{2}\right) \cdot \mathbf{n}_{1}-\left|\nabla\left(\chi_{1}-\chi_{2}\right) \cdot \mathbf{n}_{1}\right|^{2}\right] d \mathcal{H}_{\llcorner\Gamma}^{1},
\end{aligned}
$$

which then gives

$$
\begin{aligned}
\left\|\nabla\left(\chi_{1}-\chi_{2}\right) \cdot \mathbf{n}_{1}\right\|_{L^{2}(\Gamma)}^{2} & =\int_{\Gamma}\left|\nabla\left(\chi_{1}-\chi_{2}\right) \cdot \mathbf{n}_{1}\right|^{2} d \mathcal{H}_{\llcorner\Gamma}^{1} \leq \int_{\Gamma} \mu_{H} \nabla\left(\chi_{1}-\chi_{2}\right) \cdot \mathbf{n}_{1} d \mathcal{H}_{\llcorner\Gamma}^{1} \\
& \leq\left\|\mu_{H}\right\|_{L^{2}(\Gamma)}\left\|\nabla\left(\chi_{1}-\chi_{2}\right) \cdot \mathbf{n}_{1}\right\|_{L^{2}(\Gamma)},
\end{aligned}
$$

which proves (5.142). This implies $j \in H^{-1 / 2}(\Gamma)$, and by the trace theorem we have

$$
\|j\|_{H^{-1 / 2}(\mathrm{\Gamma})} \leq C\|\chi\|_{H^{1}(\Omega)} \leq C h^{1 / 2}\left\|\mu_{H}\right\|_{L^{2}(\Gamma)}
$$

Since $\mu_{H} \in H^{1 / 2}(\Gamma)$, we have also

$$
\begin{aligned}
\|\nabla \chi\|_{L^{2}(\Omega)}^{2} & \leq \frac{h}{l} \int_{\Gamma}\left[\mu_{H} j-|j|^{2}\right] d \mathcal{H}_{\llcorner\Gamma}^{1} \leq \frac{h}{l}\left\|\mu_{H}\right\|_{H^{1 / 2}(\Gamma)}\|j\|_{H^{-1 / 2}(\Gamma)}-\frac{h}{l}\|j\|_{L^{2}(\Gamma)}^{2} \\
& \leq C h\left\|\mu_{H}\right\|_{H^{1 / 2}(\Gamma)}\|\chi\|_{H^{1}(\Omega)}-\frac{h}{l}\|j\|_{L^{2}(\Gamma)}^{2} \\
& \leq C h^{3 / 2}\left\|\mu_{H}\right\|_{H^{1 / 2}(\Gamma)}\left\|\mu_{H}\right\|_{L^{2}(\Gamma)}-\frac{h}{l}\|j\|_{L^{2}(\Gamma)}^{2},
\end{aligned}
$$

hence

$$
\begin{aligned}
\|\chi\|_{H^{1}(\Omega)} & \leq C h\left\|\mu_{H}\right\|_{H^{1 / 2}(\Gamma)}, \\
\|j\|_{L^{2}(\Gamma)}^{2} & \leq C h^{1 / 2}\left\|\mu_{H}\right\|_{H^{1 / 2}(\Gamma)}\left\|\mu_{H}\right\|_{L^{2}(\Gamma)}, \\
\left\|h\left(\mu_{H}-j\right)\right\|_{H^{1 / 2}(\Gamma)} & =l\|\chi\|_{H^{1 / 2}(\Gamma)} \leq C\|\chi\|_{H^{1}(\Omega)} \leq C h\left\|\mu_{H}\right\|_{H^{1 / 2}(\Gamma)} \\
& \Longrightarrow\left\|\mu_{H}-j\right\|_{H^{1 / 2}(\Gamma)} \leq C\left\|\mu_{H}\right\|_{H^{1 / 2}(\Gamma)} \\
& \Longrightarrow\|j\|_{H^{1 / 2}(\Gamma)} \leq\left\|\mu_{H}-j\right\|_{H^{1 / 2}(\Gamma)}+\left\|\mu_{H}\right\|_{H^{1 / 2}(\Gamma)} \leq C\left\|\mu_{H}\right\|_{H^{1 / 2}(\Gamma)} .
\end{aligned}
$$

This last inequality is the most crucial point in the proof of this lemma: the solution $u$ is thus the solution of the transmission-reflection problem

$$
\begin{align*}
-\nabla \cdot(c(\nabla(\chi))) & =0, \quad \text { in } \Omega \backslash \Gamma,  \tag{5.143}\\
\chi_{1}=\chi_{2} & =0, \quad \text { on } \partial \Omega,  \tag{5.144}\\
\chi_{1}=\chi_{2} & =\chi, \quad \text { on } \Gamma,  \tag{5.145}\\
\nabla\left(\chi_{1}-\chi_{2}\right) \cdot \mathbf{n}_{1} & =j \in H^{1 / 2}(\Gamma), \quad \text { on } \Gamma, \tag{5.146}
\end{align*}
$$

and since the Laplacian is clearly a coercive form, by [94, Theorem 5.3.7] $\chi_{k} \in H^{2}\left(\Omega_{k}\right), k=1,2$ and

$$
\left\|\chi_{k}\right\|_{H^{2}\left(\Omega_{k}\right)} \leq C\left(\left\|\chi_{k}\right\|_{H^{1}\left(\Omega_{j}\right)}+\|j\|_{H^{1 / 2}(\Gamma)}\right), \quad k=1,2
$$

and by the interpolation inequality, for $k=1,2$,

$$
\begin{align*}
\left\|\chi_{k}\right\|_{H^{3 / 2-\epsilon}\left(\Omega_{k}\right)} & \leq C\left\|\chi_{k}\right\|_{H^{2}\left(\Omega_{k}\right)}^{1 / 2-\epsilon}\left\|\chi_{k}\right\|_{H^{1}\left(\Omega_{k}\right)}^{1 / 2+\epsilon} \\
& \leq C\left(\left\|\chi_{k}\right\|_{H^{1}\left(\Omega_{k}\right)}+\|j\|_{H^{1 / 2}(\Gamma)}\right)^{1 / 2-\epsilon}\left\|\chi_{k}\right\|_{H^{1}\left(\Omega_{k}\right)}^{1 / 2+\epsilon} \\
& \leq C\left(\left\|\chi_{k}\right\|_{H^{1}\left(\Omega_{k}\right)}+\|j\|_{H^{1 / 2}(\Gamma)}^{1 / 2-}\left\|\chi_{k}\right\|_{H^{1}\left(\Omega_{k}\right)}^{1 / 2+\epsilon}\right) \\
& \leq C h^{1 / 2}\left\|\mu_{H}\right\|_{L^{2}(\Gamma)}+C\left(\left\|\mu_{H}\right\|_{H^{1 / 2}(\Gamma)}^{1 / 2-\epsilon} h^{1 / 2+\epsilon}\left\|\mu_{H}\right\|_{H^{1 / 2}(\Gamma)}^{1 / 2+\epsilon}\right) \\
& \leq C h^{1 / 2}\left\|\mu_{H}\right\|_{L^{2}(\Gamma)}+C h^{1 / 2+\epsilon}\left\|\mu_{H}\right\|_{H^{1 / 2}(\Gamma)}, \\
& \leq C h^{1 / 2}\left\|\mu_{H}\right\|_{L^{2}(\Gamma)}+C h^{1 / 2}\left\|\mu_{H}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}, \tag{5.147}
\end{align*}
$$

which concludes the proof of (5.137).
To prove (5.138), we note first that

$$
\begin{aligned}
\int_{\Omega_{1} \times \Omega_{2}} \frac{|\nabla \chi(x)-\nabla \chi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d x d y & =\int_{\left(\Omega_{1} \times \Omega_{2}\right) \cap\{|x-y|>h\}} \frac{|\nabla \chi(x)-\nabla \chi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d x d y \\
& +\int_{\left(\Omega_{1} \times \Omega_{2}\right) \cap\{|x-y| \leq h\}} \frac{|\nabla \chi(x)-\nabla \chi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d x d y
\end{aligned}
$$

and we aim to prove

$$
\begin{align*}
& \int_{\left(\Omega_{1} \times \Omega_{2}\right) \cap\{|x-y|>h\}} \frac{|\nabla \chi(x)-\nabla \chi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d x d y \leq C h\left\|\mu_{H}\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2}  \tag{5.148}\\
& \int_{\left(\Omega_{1} \times \Omega_{2}\right) \cap\{|x-y| \leq h\}} \frac{|\nabla \chi(x)-\nabla \chi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d x d y \leq C h\left\|\mu_{H}\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2} \tag{5.149}
\end{align*}
$$

To prove (5.148), we note that

$$
\int_{\left(\Omega_{1} \times \Omega_{2}\right) \cap\{|x-y|>h\}} \frac{|\nabla \chi(x)-\nabla \chi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d x d y \leq 2 \int_{\left(\Omega_{1} \times \Omega_{2}\right) \cap\{|x-y|>h\}} \frac{|\nabla \chi(x)|^{2}+|\nabla \chi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d x d y
$$

Set

$$
Q: \Omega_{1} \longrightarrow[0,+\infty), \quad Q(x):=\int_{\Omega_{2} \cap\{|x-y|>h\}} \frac{|\nabla \chi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d y
$$

and note

$$
\begin{aligned}
Q(x) & =\int_{\mathbb{R}^{2}} \frac{\left|\mathbf{1}_{\Omega_{2}}(y) \nabla \chi(y)\right|^{2} \mathbf{1}_{\{|x-y|>h\}}(y)}{|x-y|^{3-2 \epsilon}} d y=\left|\mathbf{1}_{\Omega_{2}} \nabla \chi\right|^{2} * \mathbf{1}_{\{| |>h\}}|\cdot|^{-(3-2 \epsilon)}, \\
\|Q\|_{L^{1}\left(\Omega_{1}\right)} & =\int_{\Omega_{1} \times \Omega_{2}} \frac{|\nabla \chi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d x d y,
\end{aligned}
$$

where $|\cdot|$ denotes the Euclidean norm function of $\mathbb{R}^{2}$, hence

$$
\|Q\|_{L^{1}\left(\Omega_{1}\right)} \leq\left.\| \| \mathbf{1}_{\Omega_{2}} \nabla \chi\right|^{2}\left\|_{L^{1}\left(\mathbb{R}^{2}\right)}\right\|\left\|\left.\cdot\right|^{-(3-2 \epsilon)}\right\|_{L^{1}\left(\mathbb{R}^{2} \backslash B(0, h)\right)} .
$$

Clearly

$$
\left\|\left|\mathbf{1}_{\Omega_{2}} \nabla \chi\right|^{2}\right\|_{L^{1}\left(\mathbb{R}^{2}\right)}=\int_{\Omega_{2}}|\nabla \chi(y)|^{2} d y=\|\nabla \chi\|_{L^{2}\left(\Omega_{2}\right)}^{2},
$$

and

$$
\left|\left||\cdot|^{-(3-2 \epsilon)} \|_{L^{1}\left(\mathbb{R}^{2} \backslash B(0, h)\right)}=2 \pi \int_{h}^{+\infty} \rho^{-2+2 \epsilon} d \rho=\frac{2 \pi h^{-1+2 \epsilon}}{1-2 \epsilon} .\right.\right.
$$

Thus since $W_{H}$ is strongly regular and $h \ll H$,

$$
\begin{aligned}
\|Q(x)\|_{L^{1}\left(\Omega_{1}\right)} & \leq\|\nabla \chi\|_{L^{2}\left(\Omega_{2}\right)}^{2} \frac{2 \pi h^{-1+2 \epsilon}}{1-2 \epsilon} \leq 2 \pi(1-2 \epsilon)^{-1} h^{1+2 \epsilon}\left\|\mu_{H}\right\|_{H^{1 / 2}(\Gamma)}^{2} \\
& \leq C h^{1+2 \epsilon} H^{-2 \epsilon}\left\|\mu_{H}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2} \leq C h\left\|\mu_{H}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}
\end{aligned}
$$

which proves (5.148). We now proceed to prove (5.149): Denote by

$$
\Omega_{k}(r):=\left\{x \in \Omega_{k}: \operatorname{dist}(x, \Gamma) \leq r\right\}, \quad k=1,2
$$

and note that

$$
\left\{(x, y) \in \Omega_{1} \times \Omega_{2}:|x-y| \leq h\right\} \subseteq \Omega_{1}(h) \times \Omega_{2}(h)
$$

With the assumption that $\mu_{H} \in H^{1-\epsilon}(\Gamma)$, we have,

$$
\begin{aligned}
\|\nabla \chi\|_{L^{2}(\Omega)}^{2} & \leq h \int_{\Gamma}\left[\mu_{H} j-|j|^{2}\right] d \mathcal{H}_{\llcorner\Gamma}^{1} \leq h\left\|\mu_{H}\right\|_{H^{1-\epsilon}(\Gamma)}\|j\|_{H^{\epsilon-1}(\Gamma)}-h\|j\|_{L^{2}(\Gamma)}^{2} \\
& =h\left\|\mu_{H}\right\|_{H^{1-\epsilon}(\Gamma)}\|j\|_{L^{2}(\Gamma)}^{2}\|j\|_{H^{1-\epsilon}(\Gamma)} \\
& \Longrightarrow\|j\| j \|_{L^{2}(\Gamma)}^{2} \\
& \Longrightarrow\left\|\left\|_{H^{1-\epsilon}(\Gamma)} \leq\right\| \mu_{H}\right\|_{H^{1-\epsilon}(\Gamma)}
\end{aligned}
$$

As

$$
|\nabla \chi(x)-\nabla \chi(y)|^{2} \leq C\left(\left|\nabla \chi(x)-\nabla \chi\left(x_{\Gamma}\right)\right|^{2}+\left|\nabla \chi\left(x_{\Gamma}\right)-\nabla \chi\left(y_{\Gamma}\right)\right|^{2}+\left|\nabla \chi(y)-\nabla \chi\left(y_{\Gamma}\right)\right|^{2}\right),
$$

where $x_{\Gamma}$ is the projection of $x$ on $\Gamma$. Thus

$$
\begin{aligned}
& \int_{\left(\Omega_{1} \times \Omega_{2}\right) \cap\{|x-y| \leq h\}} \frac{|\nabla \chi(x)-\nabla \chi(y)|^{2}}{|x-y|^{3-2 \epsilon}} d x d y \\
= & \int_{\left(\Omega_{1} \times \Omega_{2}\right) \cap\{|x-y| \leq h\}} \frac{\left|\nabla \chi(x)-\nabla \chi\left(x_{\Gamma}\right)\right|^{2}}{|x-y|^{3-2 \epsilon}} d x d y \\
& +\int_{\left(\Omega_{1} \times \Omega_{2}\right) \cap\{|x-y| \leq h\}} \frac{\left|\nabla \chi\left(x_{\Gamma}\right)-\nabla \chi\left(y_{\Gamma}\right)\right|^{2}}{|x-y|^{3-2 \epsilon}} d x d y \\
& +\int_{\left(\Omega_{1} \times \Omega_{2}\right) \cap\{|x-y| \leq h\}} \frac{\left|\nabla \chi\left(y_{\Gamma}\right)-\nabla \chi(y)\right|^{2}}{|x-y|^{3-2 \epsilon}} d x d y
\end{aligned}
$$

Clearly $|x-y| \geq\left|x-x_{\Gamma}\right|$, and by [94, Theorem 4.12], for fractional regularity results for solutions of elliptic equations), we have Hölder regularity of $\nabla^{2} \chi$ :

$$
\begin{aligned}
\left\|\nabla^{2} \chi\right\|_{C^{0,1 / 2-\epsilon}\left(\Omega_{k}\right)} & \leq\left\|h\left(\mu_{H}-j\right)\right\|_{H^{1-\epsilon}(\Gamma)}+\|j\|_{H^{1-\epsilon}(\Gamma)} \quad k=1,2 \\
& \leq C\|\chi\|_{H^{3 / 2-\epsilon}\left(\Omega_{k}\right)}+\left\|\mu_{H}\right\|_{H^{1-\epsilon}(\Gamma)} \\
& \leq C\left\|\mu_{H}\right\|_{H^{1 / 2-\epsilon}(\Gamma)} \sqrt{h}+\left\|\mu_{H}\right\|_{H^{1-\epsilon}(\mathrm{\Gamma})} \leq C\left\|\mu_{H}\right\|_{H^{1-\epsilon}(\mathrm{T})} .
\end{aligned}
$$

Since $\Omega$ has finite diameter, $\sup \left|\nabla^{2} \chi\right| \leq C\left\|\nabla^{2} \chi\right\|_{C^{0,1 / 2-\epsilon}\left(\Omega_{k}\right)} \leq C\left\|\mu_{H}\right\|_{H^{1-\epsilon}(\Gamma)}$

$$
\begin{aligned}
& \sup _{\Omega_{k}}\left|\nabla^{2} \chi\right| \leq C\left\|\nabla^{2} \chi\right\|_{C^{0,1 / 2-\epsilon}\left(\Omega_{k}\right)} \leq C\left\|\mu_{H}\right\|_{H^{1-\epsilon}(\mathrm{T})} \\
& \Longrightarrow\left|\nabla \chi(x)-\nabla \chi\left(x_{\Gamma}\right)\right|^{2} \leq C\left\|\nabla^{2} \chi\right\|_{C^{0,1 / 2-\epsilon}\left(\Omega_{k}\right)}^{2} h^{2} \leq C\left\|\mu_{H}\right\|_{H^{1-\epsilon}(\Gamma)}^{2} h^{2} .
\end{aligned}
$$

Thus with the fact the $W_{H}$ is strongly regular and $r_{y}:=\operatorname{dist}(y, \Gamma)$,

$$
\begin{gathered}
\int_{\Omega_{2}(h)} \int_{B(y, h)} \frac{\left|\nabla \chi(x)-\nabla \chi\left(x_{\Gamma}\right)\right|^{2}}{|x-y|^{3-2 \epsilon}} d x d y \leq C h^{2}\left\|\mu_{H}\right\|_{H^{1-\epsilon(\Gamma)}}^{2} \int_{\Omega_{2}(h)} \int_{r_{y}}^{h} \rho^{-2+2 \epsilon} d \rho d y \\
\quad \leq C h^{2}\left\|\mu_{H}\right\|_{H^{1-\epsilon}(\Gamma)}^{2} \int_{0}^{h} r_{y}^{-1+2 \epsilon} d y_{T} \leq C h^{2+2 \epsilon}\left\|\mu_{H}\right\|_{H^{1-\epsilon}(\Gamma)}^{2} \\
\quad \leq C \frac{h^{2+2 \epsilon}}{H}\left\|\mu_{H}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2} \leq C h^{1+2 \epsilon}\left\|\mu_{H}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}
\end{gathered}
$$

Thus the integrals

$$
\int_{\left(\Omega_{1} \times \Omega_{2}\right) \cap\{|x-y| \leq h\}} \frac{\left|\nabla \chi(x)-\nabla \chi\left(x_{\Gamma}\right)\right|^{2}}{|x-y|^{3-2 \epsilon}} d x d y, \quad \int_{\left(\Omega_{1} \times \Omega_{2}\right) \cap\{|x-y| \leq h\}} \frac{\left|\nabla \chi\left(y_{\Gamma}\right)-\nabla \chi(y)\right|^{2}}{|x-y|^{3-2 \epsilon}} d x d y
$$

are bounded by $C h\left\|\mu_{H}\right\|_{H^{1 / 2-\epsilon}(\mathrm{I})}^{2}$. The last remaining term to estimate is

$$
\int_{\left(\Omega_{1} \times \Omega_{2}\right) \cap\{|x-y| \leq h\}} \frac{\left|\nabla \chi\left(x_{\Gamma}\right)-\nabla \chi\left(y_{\Gamma}\right)\right|^{2}}{|x-y|^{3-2 \epsilon}} d x d y .
$$

Since $h \ll 1$, the projection of any point in $\Omega_{k}(h)(k=1,2)$ on $\Gamma$ is uniquely determined. Thus we can introduce a set of coordinate as follows: given $x \in \Omega_{1}(h), x$ is uniquely determined by its projection $x_{\Gamma}$, and by $x_{\|}:=\left|x-x_{\Gamma}\right|$. Similarly for $y \in \Omega_{2}(h)$. Thus

$$
\begin{aligned}
\int_{\left.\left(\Omega_{1} \times \Omega_{2}\right) \cap| | x-y \mid \leq h\right\}} & \frac{\left|\nabla \chi\left(x_{\Gamma}\right)-\nabla \chi\left(y_{\Gamma}\right)\right|^{2}}{|x-y|^{3-2 \epsilon}} d x d y \leq \int_{\Omega_{1}(h) \times \Omega_{2}(h)} \frac{\left|\nabla \chi\left(x_{\Gamma}\right)-\nabla \chi\left(y_{\Gamma}\right)\right|^{2}}{|x-y|^{3-2 \epsilon}} d x d y \\
& =\int_{0}^{h} \int_{0}^{h} \int_{\Gamma \times \Gamma} \frac{\left|\nabla \chi\left(x_{\Gamma}\right)-\nabla \chi\left(y_{\Gamma}\right)\right|^{2}}{|x-y|^{3-2 \epsilon}} d x_{\Gamma} d y_{\Gamma} d x_{\|} d y_{\|} \\
& =h^{2} \int_{\Gamma \times \Gamma} \frac{\left|j\left(x_{\Gamma}\right)-j\left(y_{\Gamma}\right)\right|^{2}}{|x-y|^{3-2 \epsilon}} d x_{\Gamma} d y_{\Gamma} \\
& =h^{2}\|j\|_{H^{1-\epsilon(\Gamma)}}^{2} \leq C\left\|\mu_{H}\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2} h .
\end{aligned}
$$

Thus (5.149) is proven. Summing together (5.137), (5.138), (5.148) and (5.149) we obtain

$$
\begin{equation*}
\|\chi\|_{H^{3 / 2-\epsilon}(\Omega)} \leq C\left\|\mu_{H}\right\|_{H^{1 / 2-\epsilon}(\Gamma)} \sqrt{h} \tag{5.150}
\end{equation*}
$$

which concludes the proof in the case $\mu_{H} \in H^{1 / 2}(\Gamma)$.
When $\mu_{H} \notin H^{1-\epsilon}(\Gamma)$, since $\mu_{H}$ is a piecewise constant function we have $\mu_{H} \notin H^{1 / 2-\epsilon}(\Gamma)$ for all $\epsilon>0$. Then we can approximate $\mu_{H}$ with a sequence of smoother functions $\mu_{H}^{(n)} \subseteq H^{1-\varepsilon}(\Gamma)$ such that $\mu_{H}^{(n)} \rightarrow \mu_{H}$ in $H^{1 / 2-\epsilon}(\Gamma)$. Repeating the above arguments for $\mu_{H}^{(n)}$, denoting by $\chi^{(n)}$ the solution of

$$
\begin{align*}
-\nabla \cdot\left(c\left(\nabla\left(\chi^{(n)}\right)\right)\right) & =0, & & \text { in } \Omega \backslash \Gamma,  \tag{5.151}\\
\chi_{1}^{(n)}=\chi_{2}^{(n)} & =0, & & \text { on } \partial \Omega,  \tag{5.152}\\
\chi_{1}^{(n)}=\chi_{2}^{(n)} & =\chi^{(n)}, & & \text { on } \Gamma,  \tag{5.153}\\
\nabla\left(\chi_{1}^{(n)}-\chi_{2}^{(n)}\right) \cdot \mathbf{n}_{1}+\beta \chi^{(n)} & =\mu_{H}^{(n)}, & & \text { on } \Gamma, \tag{5.154}
\end{align*}
$$

inequality (5.150) gives

$$
\left\|\chi^{(n)}\right\|_{H^{3 / 2-\epsilon}(\Omega)} \leq C^{\prime} h^{1 / 2}\left\|\mu_{H}^{(n)}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}
$$

where the right-hand side quantities is uniformly bounded from above as $n \rightarrow+\infty$. Thus the sequence $\chi^{(n)}$ is uniformly bounded in $H^{3 / 2-\epsilon}(\Omega)$, hence upon extracting a subsequence (which we
do not relabel) $\chi^{(n)} \rightarrow \chi$ in $H^{3 / 2-\epsilon}(\Omega)$, with $\chi$ being the solution of (5.122). Thus by the lowersemicontinuity of the norm $\|\cdot\|_{H^{3 / 2-\epsilon(\Omega)}}$ we infer

$$
\|\chi\|_{H^{3 / 2-\epsilon}(\Omega)} \leq \liminf _{n \rightarrow+\infty} C^{\prime} h^{1 / 2}\left\|\mu_{H}^{(n)}\right\|_{H^{1 / 2-\epsilon}(\mathrm{T})}=C^{\prime} h^{1 / 2}\left\|\mu_{H}\right\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})},
$$

concluding the proof for 2D case. The 3D case can be similarly proved with the above procedure, except that the Gagliardo semi-norm should be modified as

$$
[\nabla \chi]_{H^{1 / 2-\epsilon(\Omega)}}^{2}:=\int_{\Omega \times \Omega} \frac{|\nabla \chi(x)-\nabla \chi(y)|^{2}}{|x-y|^{4-2 \epsilon}} d x d y .
$$

We have then finished the proof.
With $\beta=\frac{l}{h}=\mathrm{O}(1 / h)$, we now prove the continuity property and the inf-sup condition of $\mathcal{A}(\cdot, \cdot ; \cdot, \cdot)$ as follows:

Lemma 5.15. For all $(v, \mu) \in X \times M$, $\left(w_{h}, \delta \lambda_{H}\right) \in X_{h} \times W_{H}, \mathcal{A}\left(v, \mu ; w_{h}, \delta \lambda_{H}\right)$ satisfies the continuous property:

$$
\begin{equation*}
\mathcal{A}\left(v, \mu ; w_{h}, \delta \lambda_{H}\right) \leq C\left(\|\nabla v\|_{L^{2}(\Omega)}+\|v\|_{1 / 2, h}+\|\mu\|_{L^{2}(\Gamma)}\right)\left\|\left(w_{h}, \delta \lambda_{H}\right)\right\|_{p} . \tag{5.155}
\end{equation*}
$$

Proof: From the definition of $\mathcal{A}$ we have

$$
\begin{align*}
& \mathcal{A}\left(v, \mu ; w_{h}, \delta \lambda_{H}\right) \\
= & a\left(v, w_{h}\right)+\left\langle\mu, w_{h}\right\rangle-\left\langle v, \delta \lambda_{H}\right\rangle+\beta\left\langle v, w_{h}\right\rangle \\
\leq & C\left(\|\nabla v\|_{L^{2}(\Omega)}\left\|\nabla w_{h}\right\|_{L^{2}(\Omega)}+\left\|w_{h}\right\|_{L^{2}(\Gamma)}\|\mu\|_{L^{2}(\Gamma)}+\|v\|_{1 / 2, h}\left\|\delta \lambda_{H}\right\|_{-1 / 2, h}\right. \\
& \left.+\|v\|_{1 / 2, h}\left\|w_{h}\right\|_{1 / 2, h}\right) \\
\leq & C\left(\|\nabla v\|_{L^{2}(\Omega)}\left\|\nabla w_{h}\right\|_{L^{2}(\Omega)}+\left\|\nabla w_{h}\right\|_{L^{2}(\Omega)}\|\mu\|_{L^{2}(\Gamma)}+\|v\|_{1 / 2, h}\left\|\delta \lambda_{H}\right\|_{-1 / 2, h}\right. \\
& \left.+\|v\|_{1 / 2, h}\left\|w_{h}\right\|_{1 / 2, h}\right) \\
\leq & C\left(\|\nabla v\|_{L^{2}(\Omega)}+\|v\|_{1 / 2, h}+\|\mu\|_{L^{2}(\Gamma)}\right)\left\|\left(w_{h}, \delta \lambda_{H}\right)\right\| \|_{p} . \tag{5.156}
\end{align*}
$$

We have then finished the proof.
Lemma 5.16. Let $\left(v_{h}, \mu_{H}\right) \in X_{h} \times W_{H}$. When $h / H$ is small enough, the following inf-sup property holds:

$$
\begin{equation*}
\sup _{\left\|\left(w_{h}, \delta \lambda_{H}\right)\right\|_{p}=1} \mathcal{A}\left(v_{h}, \mu_{H} ; w_{h}, \delta \lambda_{H}\right) \geq C\| \|\left(v_{h}, \mu_{H}\right) \|_{p} \tag{5.157}
\end{equation*}
$$

Proof: For a given $\mu_{H}$, denote by $y$ the solution of the Robin problem for the differential
equation

$$
\begin{align*}
-\nabla \cdot(c(\nabla(y))) & =0, & & \text { in } \Omega \backslash \Gamma,  \tag{5.158a}\\
y & =0, & & \text { on } \partial \Omega,  \tag{5.158b}\\
c\left(\nabla y_{1}\right) \cdot \mathbf{n}_{1}+c\left(\nabla y_{2}\right) \cdot \mathbf{n}_{2}+\beta y & =\mu_{H}, & & \text { on } \Gamma . \tag{5.158c}
\end{align*}
$$

Then

$$
\begin{equation*}
a\left(v_{h}, y\right)=\left\langle v_{h}, \mu_{H}\right\rangle-\beta\left\langle v_{h}, y\right\rangle \tag{5.159}
\end{equation*}
$$

and from Lemma 5.2.1, the following bound holds for $y$ :

$$
\begin{equation*}
\left\|\mu_{H}\right\|_{-1 / 2, h}^{2} \leq C\left\langle\mu_{H}, P(y)\right\rangle \tag{5.160}
\end{equation*}
$$

One can then obtain the bound for $\mathcal{A}\left(v_{h}, \mu_{H} ; u_{h}+y, 2 \mu_{H}\right)$ :

$$
\begin{align*}
\mathcal{A}\left(v_{h}, \mu_{H} ; v_{h}+y, 2 \mu_{H}\right)= & a\left(v_{h}, v_{h}\right)+a\left(v_{h}, y\right)+\left\langle\mu_{H}, P\left(v_{h}\right)\right\rangle+\left\langle\mu_{H}, P(y)\right\rangle-\left\langle P\left(v_{h}\right), 2 \mu_{H}\right\rangle \\
& +\beta\left\langle v_{h}, v_{h}\right\rangle+\beta\left\langle v_{h}, y\right\rangle \\
= & a\left(v_{h}, v_{h}\right)+\left\langle v_{h}, \mu_{H}\right\rangle+\left\langle\mu_{H}, P\left(v_{h}\right)\right\rangle+\left\langle\mu_{H}, P(y)\right\rangle-\left\langle P\left(v_{h}\right), 2 \mu_{H}\right\rangle \\
& +\beta\left\langle v_{h}, v_{h}\right\rangle \\
= & a\left(v_{h}, v_{h}\right)+\left\langle\mu_{H}, P(y)\right\rangle+\beta\left\langle v_{h}, v_{h}\right\rangle \\
\geq & a\left(v_{h}, v_{h}\right)+C^{*}| | \mu_{H} \|_{-1 / 2, h}^{2}+\beta\left\langle v_{h}, v_{h}\right\rangle . \tag{5.161}
\end{align*}
$$

where the constant $C^{*}$ is associated with the inequality (5.160). Taking $y_{h}=I_{h}(y)$, the following estimate holds

$$
\begin{gathered}
\left\|y_{h}-y\right\|_{H^{1}(\Omega)} \leq C h^{1 / 2-\epsilon}\|y\|_{H^{3 / 2-\epsilon}(\Omega)} \leq h^{1-\epsilon}\left\|\mu_{H}\right\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})} \leq C \frac{h^{1-\epsilon}}{H^{1 / 2-\epsilon}}\left\|\mu_{H}\right\|_{L^{2}(\Gamma)} \leq C\left(\frac{h}{H}\right)^{1 / 2-\epsilon}\left\|\mu_{H}\right\|_{-1 / 2, h}, \\
\left\|y_{h}-y\right\|_{L^{2}(\Gamma)} \leq C h^{1-\epsilon}\|y\|_{H^{3 / 2-\epsilon}(\Omega)} \leq C \sqrt{h}\left(\frac{h}{H}\right)^{1 / 2-\epsilon}\left\|\mu_{H}\right\|_{-1 / 2, h} .
\end{gathered}
$$

When $w_{h}=v_{h}+y_{h}, \delta \lambda_{H}=2 \mu_{H}$, we have

$$
\begin{align*}
& \mathcal{A}\left(v_{h}, \mu_{H} ; u_{h}+y_{h}, 2 \mu_{H}\right)= \mathcal{A}\left(v_{h}, \mu_{H} ; u_{h}+y, 2 \mu_{H}\right)+a\left(v_{h}, y_{h}-y\right)+\left\langle\mu_{H}, P\left(y_{h}-y\right)\right\rangle \\
&+\beta\left\langle v_{h}, y_{h}-y\right\rangle \\
& \geq \kappa\left\|\nabla v_{h}\right\|_{L^{2}(\Omega)}^{2}+C^{*}\left\|\mu_{H}\right\|_{-1 / 2, h}^{2}+\beta\left\langle v_{h}, v_{h}\right\rangle+a\left(v_{h}, y_{h}-y\right) \\
&+\left\langle\mu_{H}, P\left(y_{h}-y\right)\right\rangle+\beta\left\langle v_{h}, y_{h}-y\right\rangle . \tag{5.162}
\end{align*}
$$

We then bound the last three terms on the right hand side. Firstly

$$
\begin{align*}
a\left(v_{h}, y_{h}-y\right) & \geq-\frac{\kappa}{2}\left\|\nabla v_{h}\right\|_{L^{2}(\Omega)}-C\left\|\nabla\left(y_{h}-y\right)\right\|_{L^{2}(\Omega)}^{2} \\
& \geq-\frac{\kappa}{2}\left\|\nabla v_{h}\right\|_{L^{2}(\Omega)}-C\left(\frac{h}{H}\right)^{1-2 \epsilon}\left\|\mu_{H}\right\|_{-1 / 2, h}^{2} \tag{5.163}
\end{align*}
$$

For the second term one can obtain

$$
\begin{align*}
\left\langle\mu_{H}, P\left(y_{h}-y\right)\right\rangle & \geq-\frac{1}{2} C^{*}\left\|\mu_{H}\right\|_{-1 / 2, h}^{2}-C\left\|y_{h}-y\right\|_{1 / 2, h}^{2} \\
& \geq-\frac{1}{2} C^{*}\left\|\mu_{H}\right\|_{-1 / 2, h}^{2}-C\left(\frac{h}{H}\right)^{1-2 \epsilon}\left\|\mu_{H}\right\|_{-1 / 2, h}^{2} . \tag{5.164}
\end{align*}
$$

For the last term

$$
\begin{align*}
\beta\left\langle v_{h}, y_{h}-y\right\rangle & \geq-\frac{l}{2 h}\left\|v_{h}\right\|_{L^{2}(\Gamma)}^{2}-\frac{C}{h}\left\|y_{h}-y\right\|_{L^{2}(\Gamma)}^{2} \\
& \geq-\frac{l}{2}\left\|v_{h}\right\|_{1 / 2, h}^{2}-C\left(\frac{h}{H}\right)^{1-2 \epsilon}\left\|\mu_{H}\right\|_{-1 / 2, h}^{2} . \tag{5.165}
\end{align*}
$$

Substituting the above three estimates into (5.162) yields

$$
\begin{align*}
& \mathcal{A}\left(v_{h}, \mu_{H} ; u_{h}+y_{h}, 2 \mu_{H}\right) \\
& \geq \kappa\left\|\nabla v_{h}\right\|_{L^{2}(\Omega)}^{2}+C^{*}\left\|\mu_{H}\right\|_{-1 / 2, h}^{2}+\beta\left\langle v_{h}, v_{h}\right\rangle+a\left(v_{h}, y_{h}-y\right) \\
& \quad+\left\langle\mu_{H}, P\left(y_{h}-y\right)\right\rangle+\beta\left\langle v_{h}, y_{h}-y\right\rangle \\
& \geq \kappa\left\|\nabla v_{h}\right\|_{L^{2}(\Omega)}^{2}+C^{*}\left\|\mu_{H}\right\|_{-1 / 2, h}^{2}+\beta\left\langle v_{h}, v_{h}\right\rangle-\frac{\kappa}{2}\left\|\nabla v_{h}\right\|_{L^{2}(\Omega)}^{2}-C\left(\frac{h}{H}\right)^{1-2 \epsilon}\left\|\mu_{H}\right\|_{-1 / 2, h}^{2} \\
& \quad-\frac{1}{2} C^{*}\left\|\mu_{H}\right\|_{-1 / 2, h}^{2}-C\left(\frac{h}{H}\right)^{1-2 \epsilon}\left\|\mu_{H}\right\|_{-1 / 2, h}^{2}-\frac{l}{2}\left\|v_{h}\right\|_{1 / 2, h}^{2}-C\left(\frac{h}{H}\right)^{1-2 \epsilon}\left\|\mu_{H}\right\|_{-1 / 2, h}^{2} \\
& \geq \kappa\left\|\nabla v_{h}\right\|_{L^{2}(\Omega)}^{2}+\frac{C^{*}}{2}\left\|\mu_{H}\right\|_{-1 / 2, h}^{2}+l\left\|v_{h}\right\|_{1 / 2, h}^{2}-\frac{\kappa}{2}\left\|\nabla v_{h}\right\|_{L^{2}(\Omega)}^{2}-C\left(\frac{h}{H}\right)^{1-2 \epsilon}\left\|\mu_{H}\right\|_{-1 / 2, h}^{2} \\
& \quad-\frac{l}{2}\left\|v_{h}\right\|_{1 / 2, h}^{2} \\
& =  \tag{5.166}\\
& =C\left(\left\|\nabla v_{h}\right\|_{L^{2}(\Omega)}^{2}+\left(C^{*}-C_{1}\left(\frac{h}{H}\right)^{1-2 \epsilon}\right)\left\|\mu_{H}\right\|_{-1 / 2, h}^{2}+\left\|v_{h}\right\|_{1 / 2, h}^{2}\right) .
\end{align*}
$$

On the other hand, since

$$
\begin{gathered}
\|\nabla y\|_{L^{2}(\Omega)}^{2} \leq C\left\|\mu_{H}\right\|_{-1 / 2, h}^{2} \\
\|y\|_{1 / 2, h}^{2}=\frac{1}{l}\left(-a(y, y)+\left\langle y, \mu_{H}\right\rangle\right) \leq C\left\langle y, \mu_{H}\right\rangle \leq C\left\|\mu_{H}\right\|_{-1 / 2, h}^{2}
\end{gathered}
$$

$$
\left\|\nabla\left(y-y_{h}\right)\right\|_{L^{2}(\Omega)}^{2} \leq C\|y\|_{H^{1}(\Omega)}^{2} \leq C\|\nabla y\|_{L^{2}(\Omega)}^{2} \leq C\left\|\mu_{H}\right\|_{-1 / 2, h}^{2},
$$

and

$$
\left\|y-y_{h}\right\|_{1 / 2, h}^{2}=\frac{1}{h}\left\|y-y_{h}\right\|_{L^{2}(\Gamma)}^{2} \leq C\|y\|_{H^{1}(\Omega)}^{2} \leq C\|\nabla y\|_{L^{2}(\Omega)}^{2} \leq C\left\|\mu_{H}\right\|_{-1 / 2, h}^{2},
$$

we have the following estimate

$$
\begin{align*}
& \left\|\nabla w_{h}\right\|_{L^{2}(\Omega)}^{2}+\left\|\delta \lambda_{H}\right\|_{-1 / 2, h}^{2}+\left\|w_{h}\right\|_{1 / 2, h}^{2} \\
= & \left\|\nabla v_{h}+\nabla y_{h}\right\|_{L^{2}(\Omega)}^{2}+\left\|2 \mu_{H}\right\|_{-1 / 2, h}^{2}+\left\|v_{h}+y_{h}\right\|_{1 / 2, h}^{2} \\
\leq & C\left(\left\|\nabla v_{h}\right\|_{L^{2}(\Omega)}^{2}+\|\nabla y\|_{L^{2}(\Omega)}^{2}+\left\|\nabla\left(y-y_{h}\right)\right\|_{L^{2}(\Omega)}^{2}+\left\|\mu_{H}\right\|_{-1 / 2, h}^{2}+\left\|v_{h}\right\|_{1 / 2, h}^{2}\right. \\
& \left.+\left\|y-y_{h}\right\|_{1 / 2, h}^{2}+\|y\|_{1 / 2, h}^{2}\right) \\
\leq & C\left(\left\|\nabla v_{h}\right\|_{L^{2}(\Omega)}^{2}+\left\|\mu_{H}\right\|_{-1 / 2, h}^{2}+\left\|v_{h}\right\|_{1 / 2, h}^{2}\right) . \tag{5.167}
\end{align*}
$$

Therefore, we have

$$
\begin{equation*}
\mathcal{A}\left(v_{h}, \mu_{H} ; w_{h}, \delta \lambda_{H}\right) \geq C\| \|\left(v_{h}, \mu_{H}\right)\left\|_{p}\right\|\left\|\left(w_{h}, \delta \lambda_{H}\right)\right\|_{p} . \tag{5.168}
\end{equation*}
$$

and we have finished the proof.
With the above lemmas, we can prove the error estimates in the following theorem
Theorem 5.17. Let $(u, \lambda) \in X \times M$ be the solution of (5.117) and $\left(\tilde{u}_{h}, \tilde{\lambda}_{H}\right) \in X_{h} \times W_{H}$ be the solutions of (5.113). Then for $r=O(1)$ and $\beta=\frac{l}{h}$, when $h / H$ is small, the following estimates hold true:

$$
\begin{array}{r}
\left\|\left(u-\tilde{u}_{h}, \lambda-\tilde{\lambda}_{H}\right)\right\|_{p} \leq C\left(h^{1 / 2-\epsilon}\|u\|_{H^{3 / 2-\epsilon}(\Omega)}+H^{1 / 2-\epsilon}\|\lambda\|_{H^{1 / 2-\epsilon}(\mathrm{T})}\right), \\
\left\|u-\tilde{u}_{h}\right\|_{L^{2}(\Omega)} \leq C\left(h^{1 / 2-\epsilon}+H^{1-\epsilon}\right)\left(h^{1 / 2-\epsilon}\|u\|_{H^{3 / 2-\epsilon}(\Omega)}+H^{1 / 2-\epsilon}\|\lambda\|_{H^{1 / 2-\epsilon}(\mathrm{T})}\right) . \tag{5.170}
\end{array}
$$

Proof: Similarly, as in the proof for the original DAL method, from the triangle inequality we first obtain

$$
\begin{equation*}
\left\|\left\|\left(u-\tilde{u}_{h}, \lambda-\tilde{\lambda}_{H}\right)\right\|_{p} \leq\right\|\left\|\left(u-I_{h} u, \lambda-\Pi_{H} \lambda\right)\right\|_{p}+\| \|\left(I_{h} u-\tilde{u}_{h}, \Pi_{H} \lambda-\tilde{\lambda}_{H}\right) \|_{p} . \tag{5.171}
\end{equation*}
$$

For the first term, given $h \ll 1$, we have

$$
\begin{align*}
\left\|\left(u-I_{h} u, \lambda-\Pi_{H} \lambda\right)\right\|_{p}^{2} & =\left\|\nabla\left(u-I_{h} u\right)\right\|_{L^{2}(\Omega)}^{2}+\left\|\lambda-\Pi_{H} \lambda\right\|_{-1 / 2, h}^{2}+\left\|u-I_{h} u\right\|_{1 / 2, h}^{2} \\
& \leq\left\|u-I_{h} u\right\|_{H^{1}(\Omega)}^{2}+h\left\|\lambda-\Pi_{H} \lambda\right\|_{L^{2}(\Gamma)}^{2}+\frac{1}{h}\left\|u-I_{h} u\right\|_{L^{2}(\Gamma)}^{2} \\
& \leq C h^{1-2 \epsilon}\|u\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+C h H^{1-2 \epsilon}\|\lambda\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2} \\
& \leq C h^{1-2 \epsilon}\|u\|_{H^{3 / 2-\epsilon(\Omega)}}^{2} . \tag{5.172}
\end{align*}
$$

For the second term, the proof of Lemma 5.16 shows that there exists $\left(w_{h}, \delta \lambda_{H}\right) \in X_{h} \times W_{H}$ such that $\left\|\mid\left(w_{h}, \delta \lambda_{H}\right)\right\|_{p}=1$ and

$$
\begin{align*}
& \left\|\mid\left(I_{h} u-\tilde{u}_{h}, \Pi_{H} \lambda-\tilde{\lambda}_{H}\right)\right\|_{p} \\
\leq & C \mathcal{A}\left(I_{h} u-\tilde{u}_{h}, \Pi_{H} \lambda-\tilde{\lambda}_{H} ; w_{h}, \delta \lambda_{H}\right) \\
= & C\left(\mathcal{A}\left(u-\tilde{u}_{h}, \lambda-\tilde{\lambda}_{H} ; w_{h}, \delta \lambda_{H}\right)-\mathcal{A}\left(u-I_{h} u, \lambda-\Pi_{H} \lambda ; w_{h}, \delta \lambda_{H}\right)\right) . \tag{5.173}
\end{align*}
$$

From the alternative definition given in (5.120), the first term of (5.173) should be equal to 0 . The estimate for the second part of (5.173) can be derived from Lemma 5.15 .

$$
\begin{align*}
& \mathcal{A}\left(u-I_{h} u, \lambda-\Pi_{H} \lambda ; w_{h}, \delta \lambda_{H}\right) \\
\leq & C\left(\left\|\nabla\left(u-I_{h} u\right)\right\|_{L^{2}(\Omega)}+\left\|\lambda-\Pi_{H} \lambda\right\|_{L^{2}(\Gamma)}+\left\|u-I_{h} u\right\|_{1 / 2, h}\right)\left\|\left(w_{h}, \delta \lambda_{H}\right)\right\|_{p} \\
= & C\left(\left\|\nabla\left(u-I_{h} u\right)\right\|_{L^{2}(\Omega)}+\left\|\lambda-\Pi_{H} \lambda\right\|_{L^{2}(\Gamma)}+\left\|u-I_{h} u\right\|_{1 / 2, h}\right) . \tag{5.174}
\end{align*}
$$

(5.173) and (5.174) yield

$$
\begin{align*}
& \|\left(\left(I_{h} u-\tilde{u}_{h}, \Pi_{H} \lambda-\tilde{\lambda}_{H}\right) \|_{p}\right. \\
\leq & C\left(\left\|\nabla\left(u-I_{h} u\right)\right\|_{L^{2}(\Omega)}+\left\|\lambda-\Pi_{H} \lambda\right\|_{L^{2}(\Gamma)}+\left\|u-I_{h} u\right\|_{1 / 2, h}\right) \\
\leq & C\left(h^{1 / 2-\epsilon}\|u\|_{H^{3 / 2-\epsilon}(\Omega)}+H^{1 / 2-\epsilon}\|\lambda\|_{H^{1 / 2-\epsilon}(\Gamma)}\right), \tag{5.175}
\end{align*}
$$

which finishes the estimate for the $H^{1}$ error. To estimate the $L^{2}$ error, we apply an Aubin-Nitsche duality argument similar to that used for the original DAL method, by introducing the dual problem of finding $(\phi, \theta)$ such that

$$
\begin{align*}
-\nabla \cdot(c(\nabla(\phi))) & =u-\tilde{u}_{h}, \quad \text { in } \Omega \backslash \Gamma,  \tag{5.176a}\\
\phi & =0, \quad \text { on } \partial \Omega,  \tag{5.176b}\\
\phi & =0, \quad \text { on } \Gamma,  \tag{5.176c}\\
\theta & =c\left(\nabla \phi_{1}\right) \cdot \mathbf{n}_{1}+c\left(\nabla \phi_{2}\right) \cdot \mathbf{n}_{2}, \quad \text { on } \Gamma . \tag{5.176d}
\end{align*}
$$

From Lemma 5.6, the following bound holds for $\phi$ :

$$
\begin{equation*}
\|\phi\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\|\theta\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})}^{2} \leq C\left\|u-\tilde{u}_{h}\right\|_{L^{2}(\Omega)}^{2} . \tag{5.177}
\end{equation*}
$$

Taking an inner product of (5.176a) and $u-\tilde{u}_{h}$ gives

$$
\begin{equation*}
\left\|u-\tilde{u}_{h}\right\|_{L^{2}(\Omega)}^{2}=a\left(\phi-I_{h} \phi, u-\tilde{u}_{h}\right)+a\left(I_{h} \phi, u-\tilde{u}_{h}\right)-\left\langle\theta, u-\tilde{u}_{h}\right\rangle . \tag{5.178}
\end{equation*}
$$

We can estimate the above three terms separately. Firstly,

$$
\begin{align*}
a\left(\phi-I_{h} \phi, u-\tilde{u}_{h}\right) & \leq C\left\|\nabla\left(\phi-I_{h} \phi\right)\right\|_{L^{2}(\Omega)}\left\|\nabla\left(u-\tilde{u}_{h}\right)\right\|_{L^{2}(\Omega)} \\
& \leq C\left\|\phi-I_{h} \phi\right\|_{H^{1}(\Omega)}\left\|\nabla\left(u-\tilde{u}_{h}\right)\right\|_{L^{2}(\Omega)} \\
& \leq C h^{1 / 2-\epsilon}\|\phi\|_{H^{3 / 2-\epsilon}(\Omega)}\left(h^{1 / 2-\epsilon}\|u\|_{H^{3 / 2-\epsilon}(\Omega)}+H^{1 / 2-\epsilon}\|\lambda\|_{H^{1 / 2-\epsilon}(\Gamma)}\right) \\
& \leq C h^{1 / 2-\epsilon}\left\|u-\hat{u}_{h}\right\|_{L^{2}(\Omega)}\left(h^{1 / 2-\epsilon}\|u\|_{H^{3 / 2-\epsilon}(\Omega)}+H^{1 / 2-\epsilon}\|\lambda\|_{H^{1 / 2-\epsilon}(\Gamma)}\right) . \tag{5.179}
\end{align*}
$$

Secondly, from (5.120) and the fact that $\phi=0$ on $\Gamma$, one can obtain

$$
\begin{align*}
a\left(I_{h} \phi, u-\tilde{u}_{h}\right) & =-\left\langle\lambda-\tilde{\lambda}_{H}, I_{h} \phi\right\rangle-\beta\left\langle I_{h} \phi, u-\tilde{u}_{h}\right\rangle \\
& =\left\langle\lambda-\tilde{\lambda}_{H}, \phi-I_{h} \phi\right\rangle+\beta\left\langle\phi-I_{h} \phi, u-\tilde{u}_{h}\right\rangle \\
& \leq \frac{C}{h^{1 / 2}}\left\|\lambda-\tilde{\lambda}_{H}\right\|_{-1 / 2, h}\left\|\phi-I_{h} \phi\right\|_{L^{2}(\Gamma)}+\frac{C}{h^{1 / 2}}\left\|u-\tilde{u}_{h}\right\|_{1 / 2, h}\left\|\phi-I_{h} \phi\right\|_{L^{2}(\Gamma)} \\
& \leq C h^{-1 / 2}\left(h^{1 / 2-\epsilon}\|u\|_{H^{3 / 2-\epsilon}(\Omega)}+H^{1 / 2-\epsilon}\|\lambda\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})}\right)\left\|\phi-I_{h} \phi\right\|_{L^{2}(\Gamma)} \\
& \leq C h^{1 / 2-\epsilon}\left(h^{1 / 2-\epsilon}\|u\|_{H^{3 / 2-\epsilon}(\Omega)}+H^{1 / 2-\epsilon}\|\lambda\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})}\right)\|\phi\|_{H^{3 / 2-\epsilon}(\Omega)} \\
& \leq C h^{1 / 2-\epsilon}\left\|u-\tilde{u}_{h}\right\|_{L^{2}(\Omega)}\left(h^{1 / 2-\epsilon}\|u\|_{H^{3 / 2-\epsilon}(\Omega)}+H^{1 / 2-\epsilon}\|\lambda\|_{H^{1 / 2-\epsilon}(\Gamma)}\right) \tag{5.180}
\end{align*}
$$

For the last term in (5.178), from the fact that $\left\langle\Pi_{H} \theta, u-\tilde{u}_{h}\right\rangle=0$, we divide it into two parts:

$$
\begin{align*}
\left\langle\theta, u-\tilde{u}_{h}\right\rangle & =\left\langle\theta-\Pi_{H} \theta, u-\tilde{u}_{h}\right\rangle+\left\langle\Pi_{H} \theta, u-\tilde{u}_{h}\right\rangle \\
& =\left\langle\theta-\Pi_{H} \theta, u-\tilde{u}_{h}\right\rangle \\
& \leq C\left\|\theta-\Pi_{H} \theta\right\|_{H^{-1 / 2}(\mathrm{\Gamma})}\left\|u-\tilde{u}_{h}\right\|_{H^{1 / 2}(\mathrm{~T})} \\
& \leq C\left\|\theta-\Pi_{H} \theta\right\|_{H^{-1 / 2}(\mathrm{\Gamma})}\left\|u-\tilde{u}_{h}\right\|_{H^{1}(\Omega)} \\
& \leq C H^{1-\epsilon}\|\theta\|_{H^{1 / 2-\epsilon(\Gamma)}}\left(h^{1 / 2-\epsilon}\|u\|_{H^{3 / 2-\epsilon}(\Omega)}+H^{1 / 2-\epsilon}\|\lambda\|_{H^{1 / 2-\epsilon(\Gamma)}}\right) \\
& \leq C H^{1-\epsilon}\left\|u-\tilde{u}_{h}\right\|_{L^{2}(\Omega)}\left(h^{1 / 2-\epsilon}\|u\|_{H^{3 / 2-\epsilon}(\Omega)}+H^{1 / 2-\epsilon}\|\lambda\|_{H^{1 / 2-\epsilon}(\mathrm{T})}\right) . \tag{5.181}
\end{align*}
$$

Combining (5.178)-5.181), one can get

$$
\left\|u-\tilde{u}_{h}\right\|_{L^{2}(\Omega)} \leq C\left(h^{1 / 2-\epsilon}+H^{1-\epsilon}\right)\left(h^{1 / 2-\epsilon}\|u\|_{H^{3 / 2-\epsilon}(\Omega)}+H^{1 / 2-\epsilon}\|\lambda\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})}\right) .
$$

We have then finished the error estimate of the projection-based DAL method on the static problem.

### 5.2.2. Fully discrete: Parabolic interface problem

Based on the analysis for the static problem in Section5.2.1, we will have the following lemma for the projection-based DAL method:

Lemma 5.18. For given $u$ and $\lambda$, we have the following estimates for their mixed elliptic projection ( $\tilde{u}_{h}, \tilde{\lambda}_{H}$ ) defined in (5.120):

$$
\begin{gather*}
\left\|\left\|\left(u-\tilde{u}_{h}, \lambda-\tilde{\lambda}_{H}\right)\right\|_{p}^{2} \leq C\left(h^{1-2 \epsilon}\|u\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\|\lambda\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2}\right),\right.  \tag{5.182}\\
\left\|u-\tilde{u}_{h}\right\|_{L^{2}(\Omega)}^{2} \leq C\left(h^{1-2 \epsilon}+H^{2-2 \epsilon}\right)\left(h^{1-2 \epsilon}\|u\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\|\lambda\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2}\right),  \tag{5.183}\\
\left\|\left\|\left(\frac{\partial u}{\partial t}-\frac{\partial \tilde{u}_{h}}{\partial t}, \frac{\partial \lambda}{\partial t}-\frac{\partial \tilde{\lambda}_{H}}{\partial t}\right)\right\|\right\|_{p}^{2} \leq C\left(h^{1-2 \epsilon}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon(\Omega)}}^{2}+H^{1-2 \epsilon}\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2}\right),  \tag{5.184}\\
\left\|\frac{\partial u}{\partial t}-\frac{\partial \tilde{u}_{h}}{\partial t}\right\|_{L^{2}(\Omega)}^{2} \leq C\left(h^{1-2 \epsilon}+H^{2-2 \epsilon}\right)\left(h^{1-2 \epsilon}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon(\Omega)}}^{2}+H^{1-2 \epsilon}\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2}\right) . \tag{5.185}
\end{gather*}
$$

Let $\Delta t$ be the time step size and $T=N \Delta t$. We employ the backward Euler method for time integration, and obtain the following fully discrete scheme for the projection-based DAL method at the $(n+1)$-th time step: Given $\left(u_{h}^{n}, \lambda_{H}^{n}\right)$, find $\left(u_{h}^{n+1}, \lambda_{H}^{n+1}\right) \in X_{h} \times W_{H}$ such that

$$
\begin{align*}
& \left(\bar{\partial}_{t} u_{h}^{n+1}, w_{h}\right)-\Delta t\left\langle\bar{\partial}_{t} \lambda_{H}^{n+1}, w_{h}\right\rangle+a\left(u_{h}^{n+1}, w_{h}\right)+\left\langle\lambda_{H}^{n+1}, w_{h}\right\rangle+\beta\left\langle u_{h}^{n+1}, w_{h}\right\rangle \\
= & \left(f\left(t^{n+1}\right), w_{h}\right)+\beta\left\langle g\left(t^{n+1}\right), w_{h}\right\rangle, \quad \forall w_{h} \in X_{h},  \tag{5.186a}\\
& \Delta t\left\langle\bar{\partial}_{t} \lambda_{H}^{n+1}, \delta \lambda_{H}\right\rangle=\beta\left\langle u_{h}^{n+1}-g\left(t^{n+1}\right), \delta \lambda_{H}\right\rangle, \quad \forall \delta \lambda_{H} \in W_{H},  \tag{5.186b}\\
& u_{h}^{0}=I_{h} u_{0},  \tag{5.186c}\\
& \lambda_{H}^{0}=0 . \tag{5.186d}
\end{align*}
$$

The above fully-discrete method can be written in an equivalent form: Find $\left(u_{h}^{n+1}, \lambda_{H}^{n+1}\right) \in X_{h} \times W_{H}$ such that $\forall w_{h} \in X_{h}, \delta \lambda_{H} \in W_{H}$,

$$
\begin{equation*}
\left(\bar{\partial}_{t} u_{h}^{n+1}, w_{h}\right)-\Delta t\left\langle\bar{\partial}_{t} \lambda_{H}^{n+1}, w_{h}\right\rangle+\frac{\Delta t}{\beta}\left\langle\bar{\partial}_{t} \lambda_{H}^{n+1}, \delta \lambda_{H}\right\rangle+\mathcal{A}\left(u_{h}^{n+1}, \lambda_{H}^{n+1} ; w_{h}, \delta \lambda_{H}\right)=\mathcal{F}\left(w_{h}, \delta \lambda_{H}\right) \tag{5.187}
\end{equation*}
$$

To prove the error estimate for the above fully-discrete method, we first introduce a lemma for the inf-sup condition.

Lemma 5.19. When $h / H$ is small enough, there exists a constant $C$ such that for any given function $\mu_{H} \in W_{H}$, the following inf-sup condition holds true:

$$
\begin{equation*}
\sup _{w_{h} \in X_{h}} \frac{\left\langle w_{h}, \mu_{H}\right\rangle}{\left\|w_{h}\right\|_{H^{1}(\Omega)}+\left\|w_{h}\right\|_{-1 / 2, h}} \geq C\left\|\mu_{H}\right\|_{-1 / 2, h} . \tag{5.188}
\end{equation*}
$$

Proof: Set $v_{h}=0$ in (5.157). The supremum is clearly attained for $\delta \lambda_{H}=0$, producing (5.188).

Theorem 5.20. Let $(u, \lambda) \in X \times M$ be the solution of (4.1) and $\left(u_{h}^{n+1}, \lambda_{H}^{n+1}\right) \in X_{h} \times W_{H}$ be the solutions of (5.186), then when the fixed element ratio $h / H$ is small enough and $\beta=\frac{l}{h}$, the following error estimates for the projection-based DAL method hold true:

$$
\begin{align*}
& \left\|\left(u\left(t^{n+1}\right)-u_{h}^{n+1}, \lambda\left(t^{n+1}\right)-\lambda_{H}^{n+1}\right)\right\|_{p}^{2} \\
& \leq C\left(\frac{h^{1-2 \epsilon}+H^{2-2 \epsilon}}{\Delta t}+1\right)\left(h^{1-2 \epsilon}\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon}(\mathrm{T})}^{2}\right. \\
& \left.\quad+\int_{0}^{t^{n+1}} h^{1-2 \epsilon}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\mathrm{T})}^{2}+\Delta t\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2} d t\right) \tag{5.189}
\end{align*}
$$

and

$$
\begin{align*}
& \left\|u\left(t^{n+1}\right)-u_{h}^{n+1}\right\|_{L^{2}(\Omega)}^{2} \\
& \leq \\
& C\left(h^{1-2 \epsilon}+H^{2-2 \epsilon}+\Delta t\right)\left(h^{1-2 \epsilon}\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}\right.  \tag{5.190}\\
& \left.\quad+\int_{0}^{t^{n+1}} h^{1-2 \epsilon}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})}^{2}+\Delta t\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2} d t\right) .
\end{align*}
$$

Proof: With the mixed elliptic projection $\left(\tilde{u}_{h}\left(t^{n+1}\right), \tilde{\lambda}_{H}\left(t^{n+1}\right)\right)$ for $u\left(t^{n+1}\right)$ and $\lambda\left(t^{n+1}\right)$, set

$$
\begin{gather*}
u\left(t^{n+1}\right)-u_{h}^{n+1}=\left(u\left(t^{n+1}\right)-\tilde{u}_{h}\left(t^{n+1}\right)\right)+\left(\tilde{u}_{h}\left(t^{n+1}\right)-u_{h}^{n+1}\right)=\varphi^{n+1}+\theta^{n+1},  \tag{5.191}\\
\lambda\left(t^{n+1}\right)-\lambda_{H}^{n+1}=\left(\lambda\left(t^{n+1}\right)-\tilde{\lambda}_{H}\left(t^{n+1}\right)\right)+\left(\tilde{\lambda}_{H}\left(t^{n+1}\right)-\lambda_{H}^{n+1}\right)=\omega^{n+1}+\xi^{n+1} \tag{5.192}
\end{gather*}
$$

The estimates of $\varphi^{n+1}$ and $\omega^{n+1}$ are know from Lemma 5.18 .

$$
\begin{array}{r}
\left\|\left(\varphi^{n+1}, \omega^{n+1}\right)\right\|_{p}^{2} \leq C\left(h^{1-2 \epsilon}\left\|u\left(t^{n+1}\right)\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\lambda\left(t^{n+1}\right)\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}\right) \\
\left\|\varphi^{n+1}\right\|_{L^{2}(\Omega)}^{2} \leq C\left(h^{1-2 \epsilon}+H^{2-2 \epsilon}\right)\left(h^{1-2 \epsilon}\left\|u\left(t^{n+1}\right)\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\lambda\left(t^{n+1}\right)\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}\right) . \tag{5.194}
\end{array}
$$

For $\theta^{n+1}$, with the definition of elliptic projection (5.120) we have

$$
\begin{equation*}
a\left(u\left(t^{n+1}\right)-\tilde{u}_{h}\left(t^{n+1}\right), w_{h}\right)+\left\langle\lambda\left(t^{n+1}\right)-\tilde{\lambda}_{H}\left(t^{n+1}\right), w_{h}\right\rangle+\beta\left\langle u\left(t^{n+1}\right)-\tilde{u}_{h}\left(t^{n+1}\right), w_{h}\right\rangle=0 . \tag{5.195}
\end{equation*}
$$

Taking $w=w_{h}$ in (5.104) and combining it with the above equation, one can obtain

$$
\begin{align*}
& \left(\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}, w_{h}\right)+a\left(\tilde{u}_{h}\left(t^{n+1}\right), w_{h}\right)+\left\langle\tilde{\lambda}_{H}\left(t^{n+1}\right), w_{h}\right\rangle+\beta\left\langle\tilde{u}_{h}\left(t^{n+1}\right), w_{h}\right\rangle \\
& \quad=\left(f\left(t^{n+1}\right), w_{h}\right)+\beta\left\langle g\left(t^{n+1}\right), w_{h}\right\rangle . \tag{5.196}
\end{align*}
$$

Subtracting (5.186a) from the above equation yields

$$
\begin{align*}
& \left(\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}-\bar{\partial}_{t} u_{h}^{n+1}, w_{h}\right)+\Delta t\left\langle\bar{\partial}_{t} \lambda_{H}^{n+1}, w_{h}\right\rangle+a\left(\tilde{u}_{h}\left(t^{n+1}\right)-u_{h}^{n+1}, w_{h}\right) \\
& \quad+\left\langle\tilde{\lambda}_{H}\left(t^{n+1}\right)-\lambda_{H}^{n+1}, w_{h}\right\rangle+\beta\left\langle\tilde{u}_{h}\left(t^{n+1}\right)-u_{h}^{n+1}, w_{h}\right\rangle=0 \tag{5.197}
\end{align*}
$$

or, equivalently,

$$
\begin{align*}
& \left(\bar{\partial}_{t} \theta^{n+1}, w_{h}\right)+a\left(\theta^{n+1}, w_{h}\right)+\left\langle\xi^{n+1}, w_{h}\right\rangle+\beta\left\langle\theta^{n+1}, w_{h}\right\rangle \\
= & -\left(\bar{\partial}_{t} \varphi^{n+1}, w_{h}\right)+\left(\bar{\partial}_{t} u\left(t^{n+1}\right)-\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}, w_{h}\right)+\Delta t\left\langle\bar{\partial}_{t} \xi^{n+1}, w_{h}\right\rangle \\
& +\Delta t\left\langle\bar{\partial}_{t} \omega^{n+1}, w_{h}\right\rangle-\Delta t\left\langle\bar{\partial}_{t} \lambda\left(t^{n+1}\right), w_{h}\right\rangle . \tag{5.198}
\end{align*}
$$

From the fact that $\theta^{n+1}=0$ on $\partial \Omega$ and Poincaré's inequality, one can obtain

$$
\begin{equation*}
\left\|\theta^{n+1}\right\|_{L^{2}(\Omega)}^{2} \leq C_{\mathrm{Po}}\left\|\nabla \theta^{n+1}\right\|_{L^{2}(\Omega)}^{2} \tag{5.199}
\end{equation*}
$$

Combining it with lemma 5.19, there exists a $v_{h} \in X_{h}$ satisfying

$$
\begin{array}{rl} 
& \left\|\xi^{n+1}\right\|_{-1 / 2, h}\left(\left\|v_{h}\right\|_{H^{1}(\Omega)}+\left\|v_{h}\right\|_{1 / 2, h}\right) \leq C\left\langle\xi^{n+1}, v_{h}\right\rangle \\
= & C\left(-\left(\bar{\partial}_{t} \theta^{n+1}, v_{h}\right)-a\left(\theta^{n+1}, v_{h}\right)-\beta\left\langle\theta^{n+1}, v_{h}\right\rangle+\Delta t\left\langle\bar{\partial}_{t} \xi^{n+1}, v_{h}\right\rangle\right. \\
& +\Delta t\left\langle\bar{\partial}_{t} \omega^{n+1}, v_{h}\right\rangle-\Delta t\left\langle\bar{\partial}_{t} \lambda\left(t^{n+1}\right), v_{h}\right\rangle-\left(\bar{\partial}_{t} \varphi^{n+1}, v_{h}\right) \\
& \left.+\left(\bar{\partial}_{t} u\left(t^{n+1}\right)-\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}, v_{h}\right)\right) \\
\leq C & C\left\|\bar{\partial}_{t} \theta^{n+1}\right\|_{L^{2}(\Omega)}\left\|v_{h}\right\|_{L^{2}(\Omega)}+\left\|\nabla \theta^{n+1}\right\|_{L^{2}(\Omega)}\left\|\nabla v_{h}\right\|_{L^{2}(\Omega)}+\beta\left\|\theta^{n+1}\right\|_{L^{2}(\Gamma)}\left\|v_{h}\right\|_{L^{2}(\Gamma)} \\
& +\Delta t\left\|\bar{\partial}_{t} \xi^{n+1}\right\|_{L^{2}(\Gamma)}\left\|v_{h}\right\|_{L^{2}(\Gamma)}+\Delta t\left\|\bar{\partial}_{t} \omega^{n+1}\right\|_{L^{2}(\Gamma)}\left\|v_{h}\right\|_{L^{2}(\Gamma)}+\Delta t\left\|\bar{\partial}_{t} \lambda\left(t^{n+1}\right)\right\|_{L^{2}(\Gamma)}\left\|v_{h}\right\|_{L^{2}(\Gamma)} \\
& \left.+\left\|\bar{\partial}_{t} \varphi^{n+1}\right\|_{L^{2}(\Omega)}\left\|v_{h}\right\|_{L^{2}(\Omega)}+\left\|\bar{\partial}_{t} u\left(t^{n+1}\right)-\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}\right\|_{L^{2}(\Omega)}\left\|v_{h}\right\|_{L^{2}(\Omega)}\right) \\
\leq C\left(\left\|\bar{\partial}_{t} \theta^{n+1}\right\|_{L^{2}(\Omega)}\left\|v_{h}\right\|_{H^{1}(\Omega)}+\left\|\nabla \theta^{n+1}\right\|_{L^{2}(\Omega)}\left\|v_{h}\right\|_{H^{1}(\Omega)}+\sqrt{\beta}\left\|\theta^{n+1}\right\|_{L^{2}(\Gamma)}\left\|v_{h}\right\|_{1 / 2, h}\right. \\
& \left.+\Delta t \sqrt{h}\left\|\bar{\partial}_{t} \xi^{n+1}\right\|_{L^{2}(\Gamma)}\left\|v_{h}\right\|_{1 / 2, h}+\Delta t \sqrt{h}\left\|\bar{\partial}_{t} \omega^{n+1}\right\|_{L^{2}(\Gamma)}\right)\left\|v_{h}\right\|_{1 / 2, h} \\
& +\Delta t \sqrt{h}\left\|\bar{\partial}_{t} \lambda\left(t^{n+1}\right)\right\|_{L^{2}(\Gamma)}\left\|v_{h}\right\|_{1 / 2, h}+\left\|\bar{\partial}_{t} \varphi^{n+1}\right\|_{L^{2}(\Omega)}\left\|v_{h}\right\|_{H^{1}(\Omega)} \\
& \left.+\left\|\bar{\partial}_{t} u\left(t^{n+1}\right)-\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}\right\|\left\|_{L^{2}(\Omega)}\right\| v_{h} \|_{H^{1}(\Omega)}\right) \\
\quad \leq C\left(\left\|v_{h}\right\|_{H^{1}(\Omega)}+\left\|v_{h}\right\|_{1 / 2, h}\right)\left(\left\|\bar{\partial}_{t} \theta^{n+1}\right\|_{L^{2}(\Omega)}+\left\|\nabla \theta^{n+1}\right\|_{L^{2}(\Omega)}+\sqrt{\frac{1}{h}}\left\|\theta^{n+1}\right\|_{L^{2}(\Gamma)}\right. \\
& +\Delta t \sqrt{h}\left\|\bar{\partial}_{t} \xi^{n+1}\right\|_{L^{2}(\Gamma)}+\Delta t \sqrt{h}\left\|\bar{\partial}_{t} \omega^{n+1}\right\|_{L^{2}(\Gamma)}+\Delta t \sqrt{h}\left\|\bar{\partial}_{t} \lambda\left(t^{n+1}\right)\right\|_{L^{2}(\Gamma)} \\
& \left.+\left\|\bar{\partial}_{t} \varphi^{n+1}\right\|_{L^{2}(\Omega)}+\left\|\bar{\partial}_{t} u\left(t^{n+1}\right)-\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}\right\| \|_{L^{2}(\Omega)}\right) \tag{5.200}
\end{array}
$$

or, equivalently,

$$
\begin{align*}
\left\|\xi^{n+1}\right\|_{-1 / 2, h}^{2} \leq & C_{m}\left(\left\|\bar{\partial}_{t} \theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\left\|\nabla \theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\beta\left\|\theta^{n+1}\right\|_{L^{2}(\Gamma)}^{2}+\Delta t^{2} h\left\|\bar{\partial}_{t} \xi^{n+1}\right\|_{L^{2}(\Gamma)}^{2}\right. \\
& +\Delta t^{2} h\left\|\bar{\partial}_{t} \omega^{n+1}\right\|_{L^{2}(\Gamma)}^{2}+\Delta t^{2} h\left\|\bar{\partial}_{t} \lambda\left(t^{n+1}\right)\right\|_{L^{2}(\Gamma)}^{2}+\left\|\bar{\partial}_{t} \varphi^{n+1}\right\|_{L^{2}(\Omega)}^{2} \\
& \left.+\left\|\bar{\partial}_{t} u\left(t^{n+1}\right)-\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}\right\|_{L^{2}(\Omega)}^{2}\right) . \tag{5.201}
\end{align*}
$$

On the other hand, for $\xi^{n+1}$, take $\delta \lambda=\delta \lambda_{H}$ in (5.104). Then, with the elliptic projection definition (5.120), we have

$$
\begin{equation*}
\left\langle\tilde{u}_{h}\left(t^{n+1}\right)-u\left(t^{n+1}\right), \delta \lambda_{H}\right\rangle=0 \tag{5.202}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
\left\langle\tilde{u}_{h}\left(t^{n+1}\right)-g\left(t^{n+1}\right), \delta \lambda_{H}\right\rangle=0 . \tag{5.203}
\end{equation*}
$$

Subtracting (5.186b) from the above equation yields

$$
\begin{equation*}
\Delta t\left\langle\bar{\partial}_{t} \lambda_{H}^{n+1}, \delta \lambda_{H}\right\rangle+\beta\left\langle\tilde{u}_{h}\left(t^{n+1}\right)-u_{h}^{n+1}, \delta \lambda_{H}\right\rangle=0 \tag{5.204}
\end{equation*}
$$

or, equivalently,

$$
\begin{equation*}
\left\langle\theta^{n+1}, \delta \lambda_{H}\right\rangle=\frac{\Delta t}{\beta}\left\langle\bar{\partial}_{t} \xi^{n+1}, \delta \lambda_{H}\right\rangle-\frac{\Delta t}{\beta}\left\langle\bar{\partial}_{t} \lambda\left(t^{n+1}\right), \delta \lambda_{H}\right\rangle+\frac{\Delta t}{\beta}\left\langle\bar{\partial}_{t} \omega^{n+1}, \delta \lambda_{H}\right\rangle . \tag{5.205}
\end{equation*}
$$

Take $w_{h}=\theta^{n+1}$ and $\delta \lambda_{H}=\xi^{n+1}$, subtracting (5.205) from (5.198) yields

$$
\begin{align*}
& \left(\bar{\partial}_{t} \theta^{n+1}, \theta^{n+1}\right)+a\left(\theta^{n+1}, \theta^{n+1}\right)+\frac{\Delta t}{\beta}\left\langle\bar{\partial}_{t} \xi^{n+1}, \xi^{n+1}\right\rangle+\beta\left\langle\theta^{n+1}, \theta^{n+1}\right\rangle \\
= & -\left(\bar{\partial}_{t} \varphi^{n+1}, \theta^{n+1}\right)+\left(\bar{\partial}_{t} u\left(t^{n+1}\right)-\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}, \theta^{n+1}\right)+\Delta t\left\langle\bar{\partial}_{t} \xi^{n+1}, \theta^{n+1}\right\rangle+\Delta t\left\langle\bar{\partial}_{t} \omega^{n+1}, \theta^{n+1}\right\rangle \\
& -\Delta t\left\langle\bar{\partial}_{t} \lambda\left(t^{n+1}\right), \theta^{n+1}\right\rangle-\frac{\Delta t}{\beta}\left\langle\bar{\partial}_{t} \omega^{n+1}, \xi^{n+1}\right\rangle+\frac{\Delta t}{\beta}\left\langle\bar{\partial}_{t} \lambda\left(t^{n+1}\right), \xi^{n+1}\right\rangle . \tag{5.206}
\end{align*}
$$

For the left hand side, we have

$$
\begin{align*}
& \left(\bar{\partial}_{t} \theta^{n+1}, \theta^{n+1}\right)+a\left(\theta^{n+1}, \theta^{n+1}\right)+\frac{\Delta t}{\beta}\left\langle\bar{\partial}_{t} \xi^{n+1}, \xi^{n+1}\right\rangle+\beta\left\langle\theta^{n+1}, \theta^{n+1}\right\rangle \\
\geq & \frac{1}{2 \Delta t}\left(\left\|\theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}-\left\|\theta^{n}\right\|_{L^{2}(\Omega)}^{2}\right)+\frac{\Delta t}{2}\left\|\bar{\partial}_{t} \theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\kappa\left\|\nabla \theta^{n+1}\right\|_{L^{2}(\Omega)}^{2} \\
& +\frac{1}{2 l}\left(\left\|\xi^{n+1}\right\|_{-1 / 2, h}^{2}-\left\|\xi^{n}\right\|_{-1 / 2, h}^{2}\right)+\frac{\Delta t^{2}}{2 l}\left\|\bar{\partial}_{t} \xi^{n+1}\right\|_{-1 / 2, h}^{2}+\beta\left\|\theta^{n+1}\right\|_{L^{2}(\Gamma)}^{2} \tag{5.207}
\end{align*}
$$

For the right hand side, when the mesh is strongly regular we can combine it with the inequality

$$
\begin{aligned}
& -\left(\bar{\partial}_{t} \varphi^{n+1}, \theta^{n+1}\right)+\left(\bar{\partial}_{t} u\left(t^{n+1}\right)-\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}, \theta^{n+1}\right)+\Delta t\left\langle\bar{\partial}_{t} \xi^{n+1}, \theta^{n+1}\right\rangle+\Delta t\left\langle\bar{\partial}_{t} \omega^{n+1}, \theta^{n+1}\right\rangle \\
& -\Delta t\left\langle\bar{\partial}_{t} \lambda\left(t^{n+1}\right), \theta^{n+1}\right\rangle-\frac{\Delta t}{\beta}\left\langle\bar{\partial}_{t} \omega^{n+1}, \xi^{n+1}\right\rangle+\frac{\Delta t}{\beta}\left\langle\bar{\partial}_{t} \lambda\left(t^{n+1}\right), \xi^{n+1}\right\rangle \\
\leq & \frac{\kappa}{2 C_{\mathrm{Po}}}\left\|\theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\frac{C_{\mathrm{Po}}}{\kappa}\left\|\bar{\partial}_{t} \varphi^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\frac{C_{\mathrm{Po}}}{\kappa}\left\|\bar{\partial}_{t} u\left(t^{n+1}\right)-\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}\right\|_{L^{2}(\Omega)}^{2} \\
& +\frac{\Delta t^{2}}{3 \beta}\left\|\bar{\partial}_{t} \xi^{n+1}\right\|_{L^{2}(\Gamma)}^{2}+\frac{3 \beta}{4}\left\|\theta^{n+1}\right\|_{L^{2}(\Gamma)}^{2}+\frac{\Delta t}{2 h}\left\|\theta^{n+1}\right\|_{L^{2}(\Gamma)}^{2}+h \Delta t\left\|\bar{\partial}_{t} \omega^{n+1}\right\|_{L^{2}(\Gamma)}^{2} \\
& +h \Delta t\left\|\bar{\partial}_{t} \lambda\left(t^{n+1}\right)\right\|_{L^{2}(\Gamma)}^{2}+\frac{\Delta t h}{4 C_{m}}\left\|\xi^{n+1}\right\|_{L^{2}(\Gamma)}^{2}+\frac{2 h \Delta t C_{m}}{l^{2}}\left\|\bar{\partial}_{t} \omega^{n+1}\right\|_{L^{2}(\Gamma)}^{2} \\
& +\frac{2 h \Delta t C_{m}}{l^{2}}\left\|\bar{\partial}_{t} \lambda\left(t^{n+1}\right)\right\|_{L^{2}(\Gamma)}^{2} \\
\leq & \frac{\kappa}{2 C_{\mathrm{P}_{0}}}\left\|\theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\frac{C_{\mathrm{P}_{\mathrm{P}}}^{K}}{\kappa}\left\|\bar{\partial}_{t} \varphi^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\frac{C_{\mathrm{Po}}}{\kappa}\left\|\bar{\partial}_{t} u\left(t^{n+1}\right)-\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}\right\|_{L^{2}(\Omega)}^{2} \\
& +\frac{\Delta t^{2}}{3 l}\left\|\bar{\partial}_{t} \xi^{n+1}\right\|_{-1 / 2, h}^{2}+\frac{3 l+2 \Delta t}{4 h}\left\|\theta^{n+1}\right\|_{L^{2}(\Gamma)}^{2}+\frac{\Delta t}{4 C_{m}}\left\|\xi^{n+1}\right\|_{-1 / 2, h}^{2} \\
& +\frac{\Delta t\left(l^{2}+2 C_{m}\right)}{l^{2}}\left\|\bar{\partial}_{t} \omega^{n+1}\right\|_{-1 / 2, h}^{2}+\frac{\Delta t\left(l^{2}+2 C_{m}\right)}{l^{2}}\left\|\bar{\partial}_{t} \lambda\left(t^{n+1}\right)\right\|_{-1 / 2, h}^{2} \\
\leq & \frac{\kappa}{2}\left\|\nabla \theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\frac{C_{\mathrm{Po}}}{\kappa}\left\|\bar{\partial}_{t} \varphi^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\frac{C_{\mathrm{Po}}}{\kappa}\left\|\bar{\partial}_{t} u\left(t^{n+1}\right)-\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}\right\|_{L^{2}(\Omega)}^{2} \\
& +\frac{\Delta t^{2}}{3 l}\left\|\bar{\partial}_{t} \xi^{n+1}\right\|_{-1 / 2, h}^{2}+\frac{3 l+2 \Delta t}{4 h}\left\|\theta^{n+1}\right\|_{L^{2}(\Gamma)}^{2}+\frac{\Delta t}{4}\left(\left\|\bar{\partial}_{t} \theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}\right. \\
& +\left\|\nabla \theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\beta\left\|\theta^{n+1}\right\|_{L^{2}(\Gamma)}^{2}+\Delta t^{2} h\left\|\bar{\partial}_{t} \xi^{n+1}\right\|_{L^{2}(\Gamma)}^{2}+\Delta t^{2} h\left\|\bar{\partial}_{t} \omega^{n+1}\right\|_{L^{2}(\Gamma)}^{2} \\
& \left.+\Delta t^{2} h\left\|\bar{\partial}_{t} \lambda\left(t^{n+1}\right)\right\|_{L^{2}(\Gamma)}^{2}+\left\|\bar{\partial}_{t} \varphi^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\left\|\bar{\partial}_{t} u\left(t^{n+1}\right)-\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}\right\|_{L^{2}(\Omega)}^{2}\right) \\
& +\frac{\Delta t\left(l^{2}+2 C_{m}\right)}{l^{2}}\left\|\bar{\partial}_{t} \omega^{n+1}\right\|_{-1 / 2, h}^{2}+\frac{\Delta t\left(l^{2}+2 C_{m}\right)}{l^{2}}\left\|\bar{\partial}_{t} \lambda\left(t^{n+1}\right)\right\|_{-1 / 2, h}^{2}
\end{aligned}
$$

When $\Delta t \ll 1$, we have

$$
\begin{aligned}
& -\left(\bar{\partial}_{t} \varphi^{n+1}, \theta^{n+1}\right)+\left(\bar{\partial}_{t} u\left(t^{n+1}\right)-\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}, \theta^{n+1}\right)+\Delta t\left\langle\bar{\partial}_{t} \xi^{n+1}, \theta^{n+1}\right\rangle \\
& +\Delta t\left\langle\bar{\partial}_{t} \omega^{n+1}, \theta^{n+1}\right\rangle-\Delta t\left\langle\bar{\partial}_{t} \lambda\left(t^{n+1}\right), \theta^{n+1}\right\rangle-\frac{\Delta t}{\beta}\left\langle\bar{\partial}_{t} \omega^{n+1}, \xi^{n+1}\right\rangle \\
& +\frac{\Delta t}{\beta}\left\langle\bar{\partial}_{t} \lambda\left(t^{n+1}\right), \xi^{n+1}\right\rangle \\
\leq & \frac{\kappa}{2}\left\|\nabla \theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\frac{C_{\mathrm{Po}}}{\kappa}\left\|\bar{\partial}_{t} \varphi^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\frac{C_{\mathrm{Po}}}{\kappa}\left\|\bar{\partial}_{t} u\left(t^{n+1}\right)-\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}\right\| \|_{L^{2}(\Omega)}^{2} \\
& +\frac{\Delta t^{2}}{3 l}\left\|\bar{\partial}_{t} \xi^{n+1}\right\|_{-1 / 2, h}^{2}+\frac{3 l}{4 h}\left\|\theta^{n+1}\right\|_{L^{2}(\Gamma)}^{2}+\frac{\Delta t}{4}\left\|\bar{\partial}_{t} \theta^{n+1}\right\|_{L^{2}(\Omega)}^{2} \\
& +\frac{\Delta t\left(l^{2}+2 C_{m}\right)}{l^{2}}\left\|\bar{\partial}_{t} \omega^{n+1}\right\|_{-1 / 2, h}^{2}+\frac{\Delta t\left(l^{2}+2 C_{m}\right)}{l^{2}}\left\|\bar{\partial}_{t} \lambda\left(t^{n+1}\right)\right\|_{-1 / 2, h}^{2} .
\end{aligned}
$$

Putting the inequalities for the left hand side and the right hand side together, 5.206) yields

$$
\begin{align*}
& \quad \frac{1}{2 \Delta t}\left(\left\|\theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}-\left\|\theta^{n}\right\|_{L^{2}(\Omega)}^{2}\right)+\frac{\Delta t}{4}\left\|\bar{\partial}_{t} \theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\frac{\kappa}{2}\left\|\nabla \theta^{n+1}\right\|_{L^{2}(\Omega)}^{2} \\
& \\
& +\frac{1}{2 l}\left(\left\|\xi^{n+1}\right\|_{-1 / 2, h}^{2}-\left\|\xi^{n}\right\|_{-1 / 2, h}^{2}\right)+\frac{\Delta t^{2}}{6 l}\left\|\bar{\partial}_{t} \xi^{n+1}\right\|_{-1 / 2, h}^{2}+\frac{l}{4 h}\left\|\theta^{n+1}\right\|_{L^{2}(\Gamma)}^{2} \\
& \leq \frac{C_{\mathrm{P}}}{\kappa}\left\|\bar{\partial}_{t} \varphi^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\frac{C_{\mathrm{P} 0}}{\kappa}\left\|\bar{\partial}_{t} u\left(t^{n+1}\right)-\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}\right\|_{L^{2}(\Omega)}^{2}+\frac{\Delta t\left(l^{2}+2 C_{m}\right)}{l^{2}}\left\|\bar{\partial}_{t} \omega^{n+1}\right\|_{-1 / 2, h}^{2}  \tag{5.208}\\
& \\
& \quad+\frac{\Delta t\left(l^{2}+2 C_{m}\right)}{l^{2}}\left\|\bar{\partial}_{t} \lambda\left(t^{n+1}\right)\right\|_{-1 / 2, h}^{2}
\end{align*}
$$

or, equivalently,

$$
\begin{align*}
& \left(\left\|\theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}-\left\|\theta^{n}\right\|_{L^{2}(\Omega)}^{2}\right)+\frac{\Delta t^{2}}{2}\left\|\bar{\partial}_{t} \theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\kappa \Delta t\left\|\nabla \theta^{n+1}\right\|_{L^{2}(\Omega)}^{2} \\
& \quad+\frac{\Delta t}{l}\left(\left\|\xi^{n+1}\right\|_{-1 / 2, h}^{2}-\left\|\xi^{n}\right\|_{-1 / 2, h}^{2}\right)+\frac{\Delta t^{3}}{3 l}\left\|\bar{\partial}_{t} \xi^{n+1}\right\|_{-1 / 2, h}^{2}+\frac{\Delta t l}{2 h}\left\|\theta^{n+1}\right\|_{L^{2}(\Gamma)}^{2} \\
& \leq C\left(\Delta t\left\|\bar{\partial}_{t} \varphi^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\Delta t\left\|\bar{\partial}_{t} u\left(t^{n+1}\right)-\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}\right\|_{L^{2}(\Omega)}^{2}+\Delta t^{2}\left\|\bar{\partial}_{t} \omega^{n+1}\right\|_{-1 / 2, h}^{2}\right. \\
& \left.\quad+\Delta t^{2}\left\|\bar{\partial}_{t} \lambda\left(t^{n+1}\right)\right\|_{-1 / 2, h}^{2}\right) . \tag{5.209}
\end{align*}
$$

The terms in the right hand side can be bounded as

$$
\begin{align*}
&\left\|\bar{\partial}_{t} \varphi^{n+1}\right\|_{L^{2}(\Omega)}^{2}=\left\|\frac{1}{\Delta t} \int_{t^{n}}^{t^{n+1}} \frac{\partial \varphi}{\partial t} d t\right\|_{L^{2}(\Omega)}^{2} \leq \frac{1}{\Delta t} \int_{t^{n}}^{t^{n+1}}\left\|\frac{\partial \varphi}{\partial t}\right\|_{L^{2}(\Omega)}^{2} d t \\
& \leq \frac{C\left(h^{1-2 \epsilon}+H^{2-2 \epsilon}\right)}{\Delta t} \int_{t^{n}}^{t^{n+1}}\left(h^{1-2 \epsilon}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})}^{2}\right) d t  \tag{5.210}\\
&\left\|\bar{\partial}_{t} u\left(t^{n+1}\right)-\left.\frac{\partial u}{\partial t}\right|_{t^{n+1}}\right\|_{L^{2}(\Omega)}^{2} \leq \Delta t \int_{t^{n}}^{t^{n+1}}\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2} d t  \tag{5.211}\\
&\left\|\bar{\partial}_{t} \omega^{n+1}\right\|_{-1 / 2, h}^{2}=h\left\|\frac{1}{\Delta t} \int_{t^{n}}^{t^{n+1}} \frac{\partial \omega}{\partial t} d t\right\|_{L^{2}(\Gamma)}^{2} \leq \frac{h}{\Delta t} \int_{t^{n}}^{t^{n+1}}\left\|\frac{\partial \omega}{\partial t}\right\|_{L^{2}(\Gamma)}^{2} d t \\
& \quad=\frac{1}{\Delta t} \int_{t^{n}}^{t^{n+1}}\left\|\frac{\partial \omega}{\partial t}\right\|_{-1 / 2, h}^{2} d t \\
& \leq \frac{C}{\Delta t} \int_{t^{n}}^{t^{n+1}}\left(h^{1-2 \epsilon}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}\right) d t  \tag{5.212}\\
&\left\|\bar{\partial}_{t} \lambda\left(t^{n+1}\right)\right\|_{-1 / 2, h}^{2} \leq \frac{C h}{\Delta t} \int_{t^{n}}^{t^{n+1}}\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2} d t \tag{5.213}
\end{align*}
$$

Combining with $h, H \ll 1$, the above bounds lead to

$$
\begin{align*}
& \left(\left\|\theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}-\left\|\theta^{n}\right\|_{L^{2}(\Omega)}^{2}\right)+\frac{\Delta t^{2}}{2}\left\|\bar{\partial}_{t} \theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\kappa \Delta t\left\|\nabla \theta^{n+1}\right\|_{L^{2}(\Omega)}^{2} \\
& +\frac{\Delta t}{l}\left(\left\|\xi^{n+1}\right\|_{-1 / 2, h}^{2}-\left\|\xi^{n}\right\|_{-1 / 2, h}^{2}\right)+\frac{\Delta t^{3}}{3 l}\left\|\bar{\partial}_{t} \xi^{n+1}\right\|_{-1 / 2, h}^{2}+\frac{\Delta t}{2 h}\left\|\theta^{n+1}\right\|_{L^{2}(\Gamma)}^{2} \\
\leq & C \int_{t^{n}}^{t^{n+1}}\left(\left(h^{1-2 \epsilon}+H^{2-2 \epsilon}+\Delta t\right)\left(h^{1-2 \epsilon}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon(\Omega)}}^{2}+\left(h+H^{1-2 \epsilon}\right)\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2}\right)\right. \\
& \left.+\Delta t^{2}\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2}\right) d t \\
\leq & C \int_{t^{n}}^{t^{n+1}}\left(\left(h^{1-2 \epsilon}+H^{2-2 \epsilon}+\Delta t\right)\left(h^{1-2 \epsilon}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon(\Omega)}}^{2}+H^{1-2 \epsilon}\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2}\right)\right. \\
& \left.+\Delta t^{2}\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2}\right) d t . \tag{5.214}
\end{align*}
$$

Summing (5.214) up from the first step to the $(n+1)^{\text {st }}$ step gives

$$
\begin{align*}
& \left\|\theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\frac{\Delta t^{2}}{2} \sum_{i=0}^{n}\left\|\bar{\partial}_{t} \theta^{i+1}\right\|_{L^{2}(\Omega)}^{2}+\kappa \Delta t \sum_{i=0}^{n}\left\|\nabla \theta^{i+1}\right\|_{L^{2}(\Omega)}^{2}+\frac{\Delta t}{l}\left\|\xi^{n+1}\right\|_{-1 / 2, h}^{2} \\
& \quad+\frac{\Delta t^{3}}{3 l} \sum_{i=0}^{n}\left\|\bar{\partial}_{t} \xi^{i+1}\right\|_{-1 / 2, h}^{2}+\frac{\Delta t l}{2} \sum_{i=0}^{n}\left\|\theta^{i+1}\right\|_{1 / 2, h}^{2} \\
& \leq C \int_{0}^{t^{n+1}}\left(\left(h^{1-2 \epsilon}+H^{2-2 \epsilon}+\Delta t\right)\left(h^{1-2 \epsilon}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})}^{2}\right)\right. \\
&  \tag{5.215}\\
& \left.+\Delta t^{2}\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2}\right) d t+\left\|\theta^{0}\right\|_{L^{2}(\Omega)}^{2}+\frac{\Delta t}{l}\left\|\xi^{0}\right\|_{-1 / 2, h}^{2} .
\end{align*}
$$

While taking the initial values of $u_{h}^{0}$ and $\lambda_{H}^{0}$ as in (5.186), we have estimates for the initial errors similar to (5.93)-(5.94) from the analysis of the original DAL method:

$$
\begin{align*}
&\left\|\theta^{0}\right\|_{L^{2}(\Omega)}^{2}=\left\|\tilde{u}_{h}(0)-I_{h} u_{0}\right\|_{L^{2}(\Omega)}^{2} \leq 2\left\|\tilde{u}_{h}(0)-u_{0}\right\|_{L^{2}(\Omega)}^{2}+2\left\|u_{0}-I_{h} u_{0}\right\|_{L^{2}(\Omega)}^{2} \\
& \leq C\left(h^{1-2 \epsilon}+H^{2-2 \epsilon}\right)\left(h^{1-2 \epsilon}\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}\right)  \tag{5.216}\\
&\left\|\xi^{0}\right\|_{-1 / 2, h}^{2}=\left\|\tilde{\lambda}_{H}(0)\right\|_{-1 / 2, h}^{2} \leq 2\left\|\tilde{\lambda}_{H}(0)-\lambda(0)\right\|_{-1 / 2, h}^{2}+2\|\lambda(0)\|_{-1 / 2, h}^{2} \\
& \quad \leq C\left(h^{1-2 \epsilon}\left\|u_{0}\right\|_{H^{3 / 2-\epsilon(\Omega)}}^{2}+H^{1-2 \epsilon}\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2}\right) . \tag{5.217}
\end{align*}
$$

Substituting the initial error estimates into (5.215) gives

$$
\begin{align*}
\| & \theta^{n+1}\left\|_{L^{2}(\Omega)}^{2}+\frac{\Delta t^{2}}{2} \sum_{i=0}^{n}\right\| \overline{\partial_{t}} \theta^{i+1}\left\|_{L^{2}(\Omega)}^{2}+\kappa \Delta t \sum_{i=0}^{n}\right\| \nabla \theta^{i+1}\left\|_{L^{2}(\Omega)}^{2}+\frac{\Delta t}{l}\right\| \xi^{n+1} \|_{-1 / 2, h}^{2} \\
& +\frac{\Delta t^{3}}{3 l} \sum_{i=0}^{n}\left\|\bar{\partial}_{t} \xi^{i+1}\right\|_{-1 / 2, h}^{2}+\frac{\Delta t l}{2} \sum_{i=0}^{n}\left\|\theta^{i+1}\right\|_{1 / 2, h}^{2} \\
\leq & C \int_{0}^{t^{n+1}}\left(\left(h^{1-2 \epsilon}+H^{2-2 \epsilon}+\Delta t\right)\left(h^{1-2 \epsilon}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2}\right)+\Delta t^{2}\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2}\right) d t \\
& +C\left(h^{1-2 \epsilon}+H^{2-2 \epsilon}+\Delta t\right)\left(h^{1-2 \epsilon}\left\|u_{0}\right\|_{H^{3 / 2-\epsilon(\Omega)}}^{2}+H^{1-2 \epsilon}\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2}\right), \tag{5.218}
\end{align*}
$$

which yields the estimate for $\theta^{n+1}$ in the $L^{2}$ norm

$$
\begin{align*}
\left\|\theta^{n+1}\right\|_{L^{2}(\Omega)}^{2} & \leq C \int_{0}^{t^{n+1}}\left(\left(h^{1-2 \epsilon}+H^{2-2 \epsilon}+\Delta t\right)\left(h^{1-2 \epsilon}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})}^{2}\right)\right. \\
& \left.+\Delta t^{2}\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2}\right) d t+C\left(h^{1-2 \epsilon}+H^{2-2 \epsilon}+\Delta t\right)\left(h^{1-2 \epsilon}\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}\right. \\
& \left.+H^{1-2 \epsilon}\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon(\mathrm{\Gamma})}}^{2}\right) . \tag{5.219}
\end{align*}
$$

Combining (5.219) with the estimate for $\varphi^{n+1}$, we obtain the error estimate in the $L^{2}$ norm

$$
\begin{align*}
& \left\|u\left(t^{n+1}\right)-u_{h}^{n+1}\right\|_{L^{2}(\Omega)}^{2} \\
\leq & 2\left(\left\|\varphi^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\left\|\theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}\right) \\
\leq & C\left(\left(h^{1-2 \epsilon}+H^{2-2 \epsilon}\right)\left(h^{1-2 \epsilon}\left\|u\left(t^{n+1}\right)\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\lambda\left(t^{n+1}\right)\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}\right)+\left(h^{1-2 \epsilon}+H^{2-2 \epsilon}\right.\right. \\
& +\Delta t)\left(h^{1-2 \epsilon}\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}\right)+\int_{0}^{t^{n+1}}\left(( h ^ { 1 - 2 \epsilon } + H ^ { 2 - 2 \epsilon } + \Delta t ) \left(h^{1-2 \epsilon}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}\right.\right. \\
& \left.\left.\left.+H^{1-2 \epsilon}\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}\right)+\Delta t^{2}\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2}\right) d t\right) \\
\leq & C\left(h^{1-2 \epsilon}+H^{2-2 \epsilon}+\Delta t\right)\left(h^{1-2 \epsilon}\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}+\int_{0}^{t^{n+1}}\left(h^{1-2 \epsilon}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}\right.\right. \\
& \left.\left.+H^{1-2 \epsilon}\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}+\Delta t\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2}\right) d t\right) . \tag{5.220}
\end{align*}
$$

On the other hand, from (5.218), we have also obtained

$$
\begin{align*}
& \sum_{i=0}^{n}\left\|\nabla \theta^{i+1}\right\|_{L^{2}(\Omega)}^{2}+\sum_{i=0}^{n}\left\|\theta^{i+1}\right\|_{1 / 2, h}^{2}+\left\|\xi^{n+1}\right\|_{-1 / 2, h}^{2} \\
\leq & \frac{C}{\Delta t} \int_{0}^{t^{n+1}}\left(\left(h^{1-2 \epsilon}+H^{2-2 \epsilon}+\Delta t\right)\left(h^{1-2 \epsilon}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2}\right)+\Delta t^{2}\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2}\right) d t \\
& +\frac{C\left(h^{1-2 \epsilon}+H^{2-2 \epsilon}+\Delta t\right)}{\Delta t}\left(h^{1-2 \epsilon}\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2}\right) \tag{5.221}
\end{align*}
$$

which yields

$$
\begin{align*}
& \left\|\nabla \theta^{n+1}\right\|_{L^{2}(\Omega)}^{2}+\left\|\theta^{n+1}\right\|_{1 / 2, h}^{2}+\left\|\xi^{n+1}\right\|_{-1 / 2, h}^{2} \\
\leq & C\left(\frac{h^{1-2 \epsilon}+H^{2-2 \epsilon}}{\Delta t}+1\right)\left(h^{1-2 \epsilon}\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})}^{2}\right. \\
& \left.+\int_{0}^{t^{n+1}} h^{1-2 \epsilon}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\mathrm{\Gamma})}^{2}+\Delta t\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2} d t\right) \tag{5.222}
\end{align*}
$$

Together with the estimates for $\varphi^{n+1}$ and $\omega^{n+1}$, we have obtained the error estimate in the $\left|||(\cdot, \cdot)|| \|_{p}\right.$-norm:

$$
\begin{align*}
& \left.\|\left(u^{n+1}\right)-u_{h}^{n+1}, \lambda\left(t^{n+1}\right)-\lambda_{H}^{n+1}\right) \|_{p}^{2} \\
\leq & C\left(\frac{h^{1-2 \epsilon}+H^{2-2 \epsilon}}{\Delta t}+1\right)\left(h^{1-2 \epsilon}\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}+\int_{0}^{t^{n+1}} h^{1-2 \epsilon}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}\right. \\
& \left.+H^{1-2 \epsilon}\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}+\Delta t\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2} d t\right)+C\left(h^{1-2 \epsilon}\left\|u\left(t^{n+1}\right)\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\lambda\left(t^{n+1}\right)\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}\right) \\
\leq & C\left(\frac{h^{1-2 \epsilon}+H^{2-2 \epsilon}}{\Delta t}+1\right)\left(h^{1-2 \epsilon}\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon(\Gamma)}}^{2}+\int_{0}^{t^{n+1}} h^{1-2 \epsilon}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}\right. \\
& \left.+H^{1-2 \epsilon}\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}+\Delta t\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2} d t\right) . \tag{5.223}
\end{align*}
$$

We have finished the error estimate of the projection-based DAL method.
Theorem 5.2.2 has indicated the following corollary:
Corollary 2. In the projection-based fully-discrete DAL method, the optimal time step size is $\Delta t=$ $O(h)=O(H)$, which yields the error estimates:

$$
\begin{gather*}
\left\|\nabla\left(u\left(t^{n}\right)-u_{h}^{n}\right)\right\|_{L^{2}(\Omega)}^{2}+\frac{1}{h}\left\|u\left(t^{n}\right)-u_{h}^{n}\right\|_{L^{2}(\Gamma)}^{2}+h\left\|\lambda\left(t^{n}\right)-\lambda_{h}^{n}\right\|_{L^{2}(\Gamma)}^{2} \leq C h^{1-4 \epsilon},  \tag{5.224}\\
\left\|u\left(t^{n}\right)-u_{h}^{n}\right\|_{L^{2}(\Omega)}^{2} \leq C h^{2-4 \epsilon} . \tag{5.225}
\end{gather*}
$$

That is, in similar fashion to the original DAL method, with these settings we have $h^{1 / 2-2 \epsilon}$ order of accuracy for the $\left\|\|(\cdot, \cdot)\|_{p}\right.$-norm and $h^{1-2 \epsilon}$ order for $u$ in the $L^{2}(\Omega)$-norm.

Remark 5.21. Given the problem regularity $u \in H^{3 / 2-\epsilon}(\Omega)$, the error estimates in both DAL methods are consistent with the error estimates in [22]. Therefore, although the original augmented Lagrangian iteration was truncated in the DAL methods, the error estimates have shown that comparing with the algorithms in [22] where the Lagrange multiplier was updated exactly, such a
truncation in the iterations did not affect the convergence rates for $u$, due to the fact that the error introduced by this truncation is actually of the same order with the error from the degraded solution regularity.

## 6. Numerical results for model problems

The basic results of the error analysis for the original DAL method are already supported by the numerical experiments of [35, Section 3.1.8]. When taking $\Delta t=O(h)$, half-order convergence for the $H^{1}(\Omega)$ error of $u$ and first-order convergence for the $L^{2}(\Omega)$ error of $u$ were observed. For the error of $\lambda$, we have observed no convergence of the $L^{2}(\Gamma)$ error. For further details, we refer the interested reader to [35]. In this section, we test the results of the analysis for the projection-based DAL method, then explore how robust the results are by applying them in situations outside the scope of the linearized model problem analysis presented earlier. In section 6.1 and section 6.2 , two types of element spaces for the Lagrange multiplier are investigated. Moreover, in section 6.3 we demonstrate the kinematic conservation property of the projection-based DAL method, which was the original motivation for developing this method. Lastly, in section 6.4 we investigate the stability of the temporal over-refinement cases as discussed in Remark 5.11. To demonstrate the robustness, in section 6.4 the original DAL method with $r=0$ is employed, which represents the case with the worst stability among the DAL methods.

### 6.1. Confirming estimates for the projection-based DAL method: foreground definition for $W_{H}$

To test the conclusions of our error analysis for the projection-based DAL method, we construct a particular linearized model problem. We set the space dimension to two and choose $a(u, v)=$ $(\nabla u, \nabla v)_{L^{2}(\Omega)} . \Omega$ is the square $(-W / 2, W / 2)^{2} \subset \mathbb{R}^{2}$, with $W=2.5$, and $\Gamma$ is the unit circle $\{\mathbf{x} \in$ $\left.\mathbb{R}^{2}:\|\mathbf{x}\|_{\ell^{2}}=1\right\}$. The time interval terminates at $T=0.1$. The initial temperature distribution is

$$
\begin{equation*}
u_{0}(\mathbf{x})=u_{\text {Bess }}(\mathbf{x})+u_{\text {sin }}(\mathbf{x}) \tag{6.1}
\end{equation*}
$$

where

$$
u_{\mathrm{Bess}}(\mathbf{x})= \begin{cases}J_{0}\left(R\|\mathbf{x}\|_{\ell^{2}}\right) & \|\mathbf{x}\|_{\ell^{2}}<1  \tag{6.2}\\ 0 & \text { otherwise }\end{cases}
$$

and

$$
\begin{equation*}
u_{\mathrm{sin}}(\mathbf{x})=\sin \left(\frac{2 \pi x_{1}}{W}\right) \sin \left(\frac{2 \pi x_{2}}{W}\right) \tag{6.3}
\end{equation*}
$$

where $R$ is the first root of the Bessel function $J_{0}$. Given $g=\left.u\right|_{\Gamma}$ and $f=0$, this problem has a time-dependent analytic solution as

$$
\begin{equation*}
u(\mathbf{x}, t)=u_{\mathrm{Bess}}(\mathbf{x}) e^{-R^{2} t}+u_{\sin }(\mathbf{x}) e^{-\left(\frac{8 \pi^{2} t}{W^{2}}\right)} \tag{6.4}
\end{equation*}
$$

This solution exhibits discontinuous first derivatives of $u$ at $\Gamma$ and is smooth everywhere else, i.e., $u \in H^{3 / 2-\epsilon}(\Omega)$; this matches the spatial regularity that is shown to hold over a large class of problem data in Section 4. The exact multiplier $\lambda$ is constant in space and decaying in time, corresponding to the jump in normal derivative of the Bessel function component of $u .{ }^{\top}$

To define the space $W_{H}(\Gamma)$, in this section we divide $\Gamma$ into elements of size $\sim H$, as illustrated in the left plot of Figure 2. In the following developments, we denote this type of definition for $W_{H}$ as the "foreground" construction [36]. In contrast with the "foreground" construction in this section, in the right plot of Figure 2 we illustrate another choice of Lagrange multiplier space construction, i.e., the "background" construction. In the background construction, the Lagrange multiplier is discretized in the trace on $\Gamma$ of a space of degree-zero $B$-splines with every other knot's multiplicity set to zero along each direction (i.e. the space of constants on clusters of $2^{d}$ Bézier elements, see [36] for further details), and its numerical performance will be discussed in the next section.


Figure 2: Sketches illustrating foreground (left) and background (right) definitions of the boundary Lagrange multiplier. In each case, $\Gamma_{E}$ indicates a section of $\Gamma$ that serves as an element domain in the multiplier mesh.

We now discretize the temperature field with a linear uniform B-spline space of $2^{M} \times 2^{M}$ elements, for $M \in\{3, \ldots, 11\}$. The Lagrange multiplier is approximated with a piecewise constant function on $2^{M-3}$ elements that evenly divide the arc length of $\Gamma$, i.e., $\lambda_{H}$ is represented with a single constant element in the computations with the coarsest $8 \times 8$ background mesh $(M=3)$, and a constant ratio of $H / h$ is maintained during refinement. Defining $H=2 \pi /\left(2^{M-3}\right)$, i.e., the arc length of one element, $H / h \approx 20$. If we take $H$ to be the arc length of one of these elements and let $h=W / 2^{M}$, then we have $H \sim h$. We set $\beta=1 / h$. The discrete initial condition is set by nodal interpolaton of $u_{0}$. Integrals over $\Gamma$ are evaluated using $32 \times 2^{M}$ evenly-spaced quadrature points. The time step is proportional to $h$, viz. $\Delta t=T / 2^{M}$. An illustrative snapshot of a numerical solution is shown in Figure 3 .

[^1]

Figure 3: Annotated snapshot of a numerical solution to the scalar parabolic test problem.

The convergence results of projection-based DAL method are illustrated in Figure 4. The left plot of Figure 4 shows the convergence of the $L^{2}(\Omega)$ and $H^{1}(\Omega)$ norms of the error $u(T)-u_{h}^{N}$, suggesting convergence rates of $1 / 2$ in $H^{1}(\Omega)$ and 1 in $L^{2}(\Omega)$ for both cases, as expected from the analysis. The right plot of Figure 4 shows the convergence of the $L^{2}(\Gamma)$ norm of the multiplier error $\lambda_{H}-\lambda$ at time $T$. Although this error does not diverge, it is only bounded under refinement. This observation is consistent with convergence at a rate of $1 / 2$ in the weaker $\|\cdot\|_{-1 / 2, h}$ norm.


Figure 4: Convergence of $u$ and $\lambda$ using the projection-based DAL, with respect to element size $h$ and $\Delta t=O(h)$. Left: $L^{2}(\Omega)$ and $H^{1}(\Omega)$ errors of $u(T)$. Right: $L^{2}(\Gamma)$ error of $\lambda(T)$.

The lack of convergence of the Lagrange multiplier in a strong norm is also clear from comparing plots of the Lagrange multiplier at various levels of refinement. The multiplier field $\lambda(T)$ is shown at several refinement levels in Figure 5, as a function of the arc length along $\Gamma$. While there is an obvious improvement from the coarsest discretizations to more refined ones, the magnitudes of oscillations in the asymptotic regime do not decrease, as is consistent with the $O(1) L^{2}(\Gamma)$ error
shown in the right plot of Figure 4


Figure 5: The Lagrange multiplier as a function of arc length along $\Gamma$, at various levels $M$ of refinement.

### 6.2. Confirming estimates for the projection-based DAL method: background definition for $W_{H}$

It would at first appear very intuitive to define the space $W_{H}(\Gamma)$ by dividing $\Gamma$ into elements of size $\sim H$, as in the numerical experiments of Section 6.1. However, this use of a "foreground" construction may lead to difficulties in practice. In the FSI case, one might immerse a structure into a background mesh of the fluid domain that is refined in a highly non-uniform and/or anisotropic way, to, e.g., capture boundary layers near a fitted boundary. A foreground construction of $W_{H}$ that is appropriate for a structure immersed in one part of $\Omega$ may become inappropriate as the structure moves to a less-refined region or changes its orientation relative to anisotropic refinement of the background mesh. Puso et al. [95] previously compared foreground and background definitions of Lagrange multiplier spaces for immersed discretizations and found that defining the Lagrange multiplier using the background mesh was typically a more stable choice. In [36], we chose a background-based definition of the Lagrange multiplier for isogeometric background discretizations, which will be investigated in this section.

We now test the convergence of the projection-based DAL method with this background multiplier space. Aside from changing the definition of $W_{H}$, we repeat the experiment of Section 6.1. The convergence of the error in temperature is shown in the left plot of Figure 6; we obtain the same convergence rates as we did when using the foreground multiplier space. The $O(1) L^{2}(\Gamma)$ error for the multiplier $\lambda$, shown in the right plot of Figure 6, is also consistent with the results of Section 6.1 .

The multiplier field in this case appears to have oscillations of larger amplitude, as shown in Figure 7. The thin spikes in Figure 7 correspond to poorly cut elements of the background mesh. If we define $H=2 h$, irrespective of how $\Gamma$ cuts through elements, we are reducing the ratio $H / h$ by about a factor of 10 relative to the experiment of Section 6.1, so it is not surprising that we would


Figure 6: Convergence results of $u$ and $\lambda$ while using a background-mesh-derived multiplier space in the projectionbased DAL method, with respect to element size $h$ and $\Delta t=O(h)$. Left: convergence of of $L^{2}(\Omega)$ and $H^{1}(\Omega)$ errors of $u(T)$. Right: convergence of the $L^{2}(\Gamma)$ error of $\lambda(T)$.
see larger oscillations, even without considering the issue of cut cells. While these oscillations and spikes may look alarming, they do not appear to impede the convergence of the temperature solution. Further, unlike multiplier oscillations in the $r=0$ limit of the original DAL method, these spurious features do not grow in an unbounded way under temporal refinement, or in the limit of $T \rightarrow \infty$.


Figure 7: The Lagrange multiplier as a function of arc length along $\Gamma$, at various levels $M$ of refinement, when using the background mesh to define $W_{H}$. Note that some of the functions plotted extend beyond the bounds on the vertical axis.

### 6.3. Advantages of the projection-based DAL method

As elaborated in [36], the purpose of the projection-based DAL method is to ensure kinematic conservation in the steady limit without needing to reduce the parameter $r$ to an unstable value near zero. We demonstrate this conservation property on the problem introduced in Section 6.1 by
looking at the conservation error (cf. (3.21))

$$
\begin{equation*}
E_{\Gamma}=\int_{\Gamma}\left(u_{h}^{N}-g(T)\right) d \Gamma \tag{6.5}
\end{equation*}
$$

in the limit of $\Delta t \rightarrow 0$ at fixed $h$. (The steady limit of $T \rightarrow \infty$ for this problem is trivial.) We use the $M=4$ meshes of $\Omega$ and $\Gamma$ defined in Section 6.1 and consider $\Delta t=T / 2^{M_{t}}$, for $M_{t} \in\{3, \ldots, 10\}$. For computations using the original DAL method, we set $r=12^{23}$ The resulting values of $\log \left|E_{\Gamma}\right|$ are plotted in Figure 8 . It is clear that the projection-based DAL recovers kinematic conservation, with the conservation error converging as $O(\Delta t)$, while the conservation error from the original DAL does not converge.


Figure 8: Convergence of $\log \left|E_{\Gamma}\right|$ as $\Delta t \rightarrow 0$ at fixed $h$.

### 6.4. Over-refinement in time

As mentioned in Remark 5.11, for the original DAL method, error estimates suggest a lower bound for stability on the time step in terms of $h$ as $\Delta t>O\left(h^{2-4 \epsilon}\right)$. The possibility of error blowing up under temporal refinement is a serious practical concern in nonlinear FSI simulations, as reducing $\Delta t$ on a fixed spatial discretization may be necessary to ensure stability or rapid convergence. (See, e.g., the stability analysis in [35, Section 4] of the block iterative procedure used to resolve the implicit step of the time integrator.) In this section, we explore the consequences of refining too quickly in time, relative to the spatial discretization.

[^2]To exacerbate the potential small-time-step instability, we consider the original DAL method with limiting case of $r=0$ and run two experiments using the scalar parabolic model problem. First, we refine in both space and time, with $\Delta t \sim h^{2}$. Then, we refine in time while holding $h$ fixed. Based on the bound (5.68), we might expect to see $H^{1}(\Omega)$ errors in temperature fail to converge in the first case, and diverge in the second case. The particular problem we consider is the same as that used in Section 6.1. We use (part of) the same sequence of spatial meshes for the temperature field, and attempt to cast a spotlight on any possible unstable behavior of the multiplier by setting $\beta=10^{5} / \mathrm{h}$.

The convergence of error in temperature at time $T$ is shown for $\Delta t=T /\left(2^{M} / 4\right)^{2}$ in the left plot of Figure 9. The temperature error when taking $\Delta t \rightarrow 0$ on the $M=4$ spatial mesh is shown in the right plot of Figure 9 . We see from these results that the apparent small-time-step divergence of the error bound (5.68) appears to be a false alarm, suggesting that either the predicted divergence only occurs in select pathological cases, or the theoretical estimate is overly pessimistic.


Figure 9: Convergence results of $u$ with over-refinement in time, with the original DAL method and $r=0$. Left: Convergence of $L^{2}(\Omega)$ and $H^{1}(\Omega)$ errors in $u(T)$ when taking $\Delta t \sim h^{2}$. Right: Convergence of $L^{2}(\Omega)$ error in $u(T)$ when taking $\Delta t \rightarrow 0$ at fixed $h$. (For $h$ fixed, all norms of $u_{h}^{N}$ are equivalent.)

## 7. A benchmark problem and numerical results for nonlinear FSI

In this section we test how well the model problem analysis extrapolates to the setting of nonlinear FSI, on a benchmark 2D FSI problem with manufactured solution. Briefly, we first derived a benchmark problem with analytic solution that exhibits the regularity expected in practice and satisfies all kinematic constraints, then substituting it into the strong form of the governing equations, to obtain a source term. This is complicated by several factors in the case of unsteady fluid-thin structure interaction. First, the overall method presumes that the solution is stable, but this cannot be assured in nonlinear problems, especially with thin structures (that may buckle) and fluid flow (that may become turbulent). Second, the expected regularity of the fluid solution in immersed FSI applications is less than that needed by the strong form of the governing equations.

We address the first of these challenges by considering flow at low Reynolds numbers, to avoid turbulence, and formulating the thin structure as a prestressed membrane. In particular, we select the constitutive law of the structure in $(2.8)$ to be

$$
\begin{equation*}
\int_{-h_{\mathrm{th}} / 2}^{h_{\mathrm{th}} / 2} D_{\mathbf{w}} \mathbf{E}: \mathbf{S} d \xi=\int_{-h_{\mathrm{th}} / 2}^{h_{\mathrm{th}} / 2} D_{\mathbf{w}} \varepsilon:\left(\mathbf{n}+\mathbf{n}^{\mathrm{pre}}\right) d \xi \tag{7.1}
\end{equation*}
$$

where $\varepsilon$ is membrane strain [79, (3.34)], $\mathbf{n}$ is the membrane resultant force [79, (3.38)], and $\mathbf{n}^{\text {pre }}$ is a prescribed membrane prestress (cf. [96, (5)], in the 3D solid setting). To resolve the second difficulty, we construct the benchmark problem in the way that the distributional parts of the fluid solution derivatives are always induced by fluid-structure coupling, rather than imposed in the manufacturing process, through an artificial concentrated source term. To be specific, the benchmark 2D FSI problem is constructed through the following steps:

- In Section 7.1, define the shell structure displacement, and define a solenoidal fluid velocity field with a discontinuous gradient at the deformed shell structure position. Then calculate the body force $\mathbf{f}_{1}$ on the fluid from the strong form of the fluid equation.
- In Section 7.2, compute the jump in fluid traction, $-\boldsymbol{\lambda}$, at the shell structure, due to the jump in velocity gradient and an arbitrary pressure difference.
- In Section 7.3, prescribe $\boldsymbol{-} \boldsymbol{\lambda}$ as a fixed traction on the structure, then solve for the remaining body force, $\mathbf{f}_{2}$, based on the strong form of the shell equation.

Lastly, in Section 7.4 we test the convergence of the computational results from DAL methods in the derived benchmark 2D FSI problem, with respect to the manufactured analytic solution.

### 7.1. Choosing structure displacement and velocity solutions

The fluid subproblem domain is $\Omega=(0, L)^{2}$ and the initial shell structure midsurface divides $\Omega$ in half: $\Gamma_{0}=\{L / 2\} \times(0, L)$. We parameterize this midsurface by $\xi_{1}=X_{2}$, so that the conversion between curvilinear and local Cartesian coordinates in the reference configuration [79, (3.41)] is simple. We want to manufacture a shell structure displacement solution

$$
\begin{equation*}
\mathbf{y}(\mathbf{X}, t)=V t Y\left(X_{2}\right) \mathbf{e}_{1}, \tag{7.2}
\end{equation*}
$$

where $V>0$ is a constant and

$$
\begin{equation*}
Y(x)=x(L-x)=L x-x^{2} . \tag{7.3}
\end{equation*}
$$

Note that the expression for $\mathbf{y}$ remains well-defined for $\mathbf{X} \notin \Gamma_{0}$, which will be useful in the construction of the conforming fluid velocity field. A sketch of the problem is shown in the left plot of Figure 10 .


Figure 10: A sketch of the benchmark FSI problem settings. Left: The fluid domain with the deformed structure, where arrows indicate the surrounding fluid velocity field. Right: Shell structure parameterization.

Remark 2. If bending is included in the shell structure, one can avoid issues with the timedependent boundary condition on the shell structure bending moment by replacing (7.3) with

$$
\begin{equation*}
Y(x)=\frac{1}{2}\left(1-\cos \left(\frac{2 \pi x}{L}\right)\right), \tag{7.4}
\end{equation*}
$$

or some other function for which $Y^{\prime}(0)=Y^{\prime}(L)=0$.
The fluid velocity field must conform to the selected structure displacement. Further, to have regularity representative of expected applications, we want the pressure and fluid velocity gradients to be discontinuous at $\Gamma_{t}$. For $\mathbf{x} \in \Omega$ with $x_{1}<L / 2+V t Y\left(x_{2}\right)$ (i.e., $\mathbf{x}$ is to the left of $\Gamma_{t}$ ), we set

$$
\begin{equation*}
\mathbf{u}^{\text {left }}(\mathbf{x}, t)=V Y\left(x_{2}\right) \mathbf{e}_{1} \tag{7.5}
\end{equation*}
$$

On the other hand, for $\mathbf{x}$ to the right of $\Gamma_{t}$,

$$
\begin{equation*}
\mathbf{u}^{\text {right }}(\mathbf{x}, t)=V Y\left(x_{2}\right) \mathbf{e}_{1}+\mathbf{u}_{\text {shear }}(\mathbf{x}, t), \tag{7.6}
\end{equation*}
$$

with

$$
\begin{equation*}
\mathbf{u}_{\text {shear }}(\mathbf{x}, t)=\mathbf{F}\left(\phi^{-1}(\mathbf{x}, t)\right) \mathbf{U}_{\text {shear }}\left(\phi^{-1}(\mathbf{x}), t\right) . \tag{7.7}
\end{equation*}
$$

Here $\mathbf{U}_{\text {shear }}(\mathbf{X}, t)=\frac{V}{L}\left(X_{1}-L / 2\right) \mathbf{e}_{2}$ is a velocity field with uniform shear stress in the structure's reference configuration, $\phi(\mathbf{X}, t)=\mathbf{X}+\mathbf{y}(\mathbf{X}, t)$ is the volume preserving motion which extends the shell structure's midsurface motion to $\Omega$, and $\mathbf{F}=\frac{\partial \phi}{\partial \mathbf{X}}$ is the deformation gradient of $\phi$. Because $\mathbf{U}_{\text {shear }}$ is solenoidal and $\mathbf{u}_{\text {shear }}$ is its pushforward by $\phi$ using the Piola transform (with $\operatorname{det} \mathbf{F}=1$ ),
$\mathbf{u}_{\text {shear }}$ is also solenoidal. Specifically

$$
\begin{align*}
\mathbf{u}_{\text {shear }}(\mathbf{x}, t) & =F_{12}\left(\mathbf{U}_{\text {shear }}\right)_{2} \mathbf{e}_{1}+F_{22}\left(\mathbf{U}_{\text {shear }}\right)_{2} \mathbf{e}_{2} \\
& =\frac{V\left(x_{1}-\left(\operatorname{Vt} Y\left(x_{2}\right)+L / 2\right)\right)}{L}\left(\operatorname{Vt} Y^{\prime}\left(x_{2}\right) \mathbf{e}_{1}+\mathbf{e}_{2}\right) . \tag{7.8}
\end{align*}
$$

To manufacture the fluid and structure solutions, we will need $\nabla \mathbf{u}, \Delta \mathbf{u}$, and $\partial_{t} \mathbf{u}$. Taking derivatives of the solutions to the left and right sides of the (deformed) structure, we have, for the points $\mathbf{x}$ to the left of $\Gamma_{t}$,

$$
\nabla \mathbf{u}^{\mathrm{left}}=\left(\begin{array}{cc}
0 & V Y^{\prime}  \tag{7.9}\\
0 & 0
\end{array}\right), \quad \Delta \mathbf{u}^{\mathrm{left}}=\binom{V Y^{\prime \prime}}{0}, \quad \partial_{t} \mathbf{u}^{\mathrm{left}}=\mathbf{0}
$$

and for the points $\mathbf{x}$ to the right of $\Gamma_{t}$

$$
\begin{gather*}
\nabla \mathbf{u}^{\text {right }}=\left(\begin{array}{c|c}
\frac{V^{2} t}{L} Y^{\prime} & V Y^{\prime}+\frac{V^{2} t}{L}\left(x_{1} Y^{\prime \prime}-V t\left(Y^{\prime} Y^{\prime}+Y Y^{\prime \prime}\right)-\frac{L}{2} Y^{\prime \prime}\right) \\
\hline \frac{V}{L} & -\frac{V^{2} t}{L} Y^{\prime}
\end{array}\right),  \tag{7.10}\\
\Delta \mathbf{u}^{\text {right }}=\binom{V Y^{\prime \prime}+\frac{V^{2} t}{L}\left(x_{1} Y^{\prime \prime \prime}-V t\left(3 Y^{\prime} Y^{\prime \prime}+Y Y^{\prime \prime \prime}\right)-\frac{L}{2} Y^{\prime \prime \prime}\right)}{-\frac{V^{2} t}{L} Y^{\prime \prime}},  \tag{7.11}\\
\partial_{t} \mathbf{u}^{\text {right }}=\frac{1}{L}\binom{V^{2} x_{1} Y^{\prime}-2 V^{3} t Y Y^{\prime}-\frac{V^{2} L}{2} Y^{\prime}}{-V^{2} Y} . \tag{7.12}
\end{gather*}
$$

We can then calculate the body force term for the whole fluid domain

$$
\begin{equation*}
\rho_{f} \mathbf{f}_{1}=\rho_{f}\left(\partial_{t} \mathbf{u}+\mathbf{u} \cdot \nabla \mathbf{u}\right)-\mu \Delta \mathbf{u}-\nabla p \tag{7.13}
\end{equation*}
$$

where $\nabla p$ is an arbitrary irrotational pressure gradient. For simplicity, we assume that $\nabla p=0$ to the left and right of the structure, with a pressure jump $P$ across the deformed structure $\Gamma_{t}$, i.e., $p^{\text {left }}-p^{\text {right }}=P$.

### 7.2. Obtaining the traction jump on the structure

In this section we prescribe the fluid traction jump across the structure as a source term on the structure. With the gradient $\nabla \mathbf{u}$ computed above, we can write the viscous stress in the fluid as a function of the shell structure midsurface parameter $\xi_{1}$ :

$$
\begin{equation*}
\boldsymbol{\tau}\left(\xi_{1}, t\right)=\mu\left(\nabla \mathbf{u}\left(\mathbf{X}\left(\xi_{1}\right)+\mathbf{y}\left(\xi_{1}, t\right)\right)+\left(\nabla \mathbf{u}\left(\mathbf{X}\left(\xi_{1}\right)+\mathbf{y}\left(\xi_{1}, t\right)\right)\right)^{T}\right) \tag{7.14}
\end{equation*}
$$

We now set the fluid-induced traction load on the structure subproblem as:

$$
\begin{equation*}
-\boldsymbol{\lambda}=\left(\tau^{\text {right }} \mathbf{n}_{s}-\boldsymbol{\tau}^{\text {left }} \mathbf{n}_{s}\right)+P \mathbf{n}_{s} \tag{7.15}
\end{equation*}
$$

where $P$ is the pressure jump, and $\mathbf{n}_{s}$ is the unit vector normal to the deformed structure:

$$
\begin{equation*}
\mathbf{n}_{s}=\frac{1}{\sqrt{1+\left(V t Y^{\prime}\right)^{2}}}\binom{1}{-V t Y^{\prime}} . \tag{7.16}
\end{equation*}
$$

### 7.3. Manufacturing the shell structure solution

We now manufacture a shell structure solution by determining the remaining prescribed forcing needed to obtain the solution $\mathbf{y}=\operatorname{VtY}\left(X_{2}\right) \mathbf{e}_{1}$ in the presence of the exact fluid subproblem solution. To apply shell theory to a 1D structure, consider it to be extruded in the $x_{3}$ direction, along which all problem variables are constant, as illustrated in the right plot of Figure 10 .

We need to first derive the Euler-Lagrange form of the shell structure's virtual work principle. To define the membrane strain, we define the covariant basis vectors in the reference and current configurations:

$$
\begin{equation*}
\text { In the reference configuration: } \quad \mathbf{A}_{1}=\mathbf{e}_{2}, \quad \mathbf{A}_{2}=\mathbf{e}_{3}, \tag{7.17}
\end{equation*}
$$

$$
\begin{equation*}
\text { In the current configuration: } \quad \mathbf{a}_{1}=\frac{\partial \mathbf{x}}{\xi_{1}}=\binom{V t Y^{\prime}}{1}, \quad \mathbf{a}_{2}=\mathbf{A}_{2} . \tag{7.18}
\end{equation*}
$$

The midsurface metric tensor in the reference configuration is identity due to the choice of a Cartesian parameterization. The (pulled-back) midsurface metric tensor in the deformed configuration is

$$
g_{\alpha \beta} \mathbf{A}^{\alpha} \otimes \mathbf{A}^{\beta}=\left(\begin{array}{cc}
\left(1+\left(V t Y^{\prime}\right)^{2}\right) & 0  \tag{7.19}\\
0 & 1
\end{array}\right),
$$

where $\mathbf{A}^{\alpha}$ are the contravariant basis vectors such that $\mathbf{A}_{\alpha} \cdot \mathbf{A}^{\beta}=\delta_{\alpha}{ }^{\beta}$. Since the displacements in the $x_{3}$-direction are constrained to be zero, the membrane strain is then

$$
\varepsilon=\frac{1}{2}(\mathbf{g}-\mathbf{I})=\left(\begin{array}{cc}
\frac{1}{2}\left(V t Y^{\prime}\right)^{2} & 0 \\
0 & 0
\end{array}\right) .
$$

We can compute the extension resultants $n_{\alpha \beta}=C_{\alpha \beta \gamma \delta} \varepsilon_{\gamma \delta} h_{\mathrm{th}}$ [79, (3.38)] with the given material tensor $C_{\alpha \beta \gamma \delta}$, and obtain

$$
\left(\begin{array}{l}
n_{11}  \tag{7.20}\\
n_{22} \\
n_{12}
\end{array}\right)=\frac{E h_{\mathrm{th}}}{1-v^{2}}\left(\begin{array}{ccc}
1 & v & 0 \\
v & 1 & 0 \\
0 & 0 & \frac{1-v}{2}
\end{array}\right)\left(\begin{array}{c}
\varepsilon_{11} \\
0 \\
0
\end{array}\right)=\frac{E h_{\mathrm{th}}}{2\left(1-v^{2}\right)}\left(V t Y^{\prime}\right)^{2}\left(\begin{array}{c}
1 \\
v \\
0
\end{array}\right)
$$

where $E$ and $v$ are the Young's modulus and Poisson's ratio of the membrane material. We now compute the membrane strain variation with respect to the displacement test function $\mathbf{w}$ :

$$
\begin{equation*}
D_{\mathbf{w}} \varepsilon_{11}=\left.\frac{d}{d \epsilon} \frac{1}{2}\left(\left|\frac{\partial(\mathbf{X}+(\mathbf{y}+\epsilon \mathbf{w}))}{\partial \xi_{1}}\right|^{2}-1\right)\right|_{\epsilon=0}=\frac{\partial(\mathbf{X}+\mathbf{y})}{\partial \xi_{1}} \cdot \frac{\partial \mathbf{w}}{\partial \xi_{1}} . \tag{7.21}
\end{equation*}
$$

The internal work term of the variational problem is then

$$
\begin{equation*}
\int_{\Gamma_{0}}\left(\mathbf{n}+\mathbf{n}^{\mathrm{pre}}\right): D_{\mathbf{w}} \varepsilon d \Gamma=\int_{\Gamma_{0}} n_{11} \frac{\partial(\mathbf{X}+\mathbf{y})}{\partial \xi_{1}} \cdot \frac{\partial \mathbf{w}}{\partial \xi_{1}} d \Gamma \tag{7.22}
\end{equation*}
$$

With the assumption that the test function $\mathbf{w}$ vanishes at the end points of $\Gamma_{0}$, we can integrate by parts to get

$$
\begin{equation*}
-\int_{\Gamma_{0}} \frac{\partial}{\partial \xi_{1}}\left(\left(n_{11}+n_{11}^{\mathrm{pre}}\right) \frac{\partial}{\partial \xi_{1}}(\mathbf{X}+\mathbf{y})\right) \cdot \mathbf{w} d \Gamma \tag{7.23}
\end{equation*}
$$

which yields the strong form of the shell structure subproblem

$$
\begin{equation*}
h_{\mathrm{th}}\left(\rho_{s}\right)_{0} \ddot{\mathbf{y}}-\frac{\partial}{\partial \xi_{1}}\left(\left(n_{11}+n_{11}^{\mathrm{pre}}\right) \frac{\partial}{\partial \xi_{1}}(\mathbf{X}+\mathbf{y})\right)=-\sqrt{g_{11}} \boldsymbol{\lambda}+h_{\mathrm{th}}\left(\rho_{s}\right)_{0} \mathbf{f}_{2} \tag{7.24}
\end{equation*}
$$

where $\sqrt{g_{11}}$ transfers the fluid traction $\boldsymbol{\lambda}$ to the reference configuration. Splitting into components, substituting in the manufactured solution in (7.2), and assuming that the presstress does not vary in space, we get

$$
\begin{gather*}
h_{\mathrm{th}}\left(\rho_{s}\right)_{0}\left(\mathbf{f}_{2}\right)_{1}=-\frac{\partial^{2}}{\partial \xi_{1}^{2}}\left(\frac{3 E h_{\mathrm{th}}}{2\left(1-v^{2}\right)}(V t)^{3}\left(\frac{\partial Y}{\partial \xi_{1}}\right)^{2}+n_{11}^{\mathrm{pre}} V t\right) Y+\sqrt{g_{11}} \lambda_{1}  \tag{7.25}\\
h_{\mathrm{th}}\left(\rho_{s}\right)_{0}\left(\mathbf{f}_{2}\right)_{2}=-\frac{E h_{\mathrm{th}}}{\left(1-v^{2}\right)}(V t)^{2} \frac{\partial Y}{\partial \xi_{1}} \frac{\partial^{2} Y}{\partial \xi_{1}^{2}}+\sqrt{g_{11}} \lambda_{2} . \tag{7.26}
\end{gather*}
$$

### 7.4. Numerical results

To test the convergence of the projection-based DAL approach to the manufactured solution in the benchmark problem, we take a test problem with the following parameters: $V=1, L=1$, $\Delta P=1, \mu=0.1, \rho_{f}=1,\left(\rho_{s}\right)_{0}=1, n_{11}^{\mathrm{pre}}=10, E=1, h_{\mathrm{th}}=0.1, v=0.3$, and $T=1$. We then discretize the fluid domain into a uniform grid of $2^{N} \times 2^{N}$ lowest-order div-conforming Bspline elements [77, 78], where $N=4, \ldots, 8$. Weakly-consistent advective stabilization of the form [35, (41)-(43)] is included, but its effect is minimal at such a low Reynolds number. The corresponding time steps are $\Delta t=T / 2^{N}$. The structure is discretized into $19 \times 2^{N-4}$ linear elements along the $\xi_{1}$ direction, to ensure some degree of mismatch with the fluid mesh. The FSI penalty parameter $\beta$ on $\Gamma_{t}$ is set to $10 \times 2^{N-4}$. The coarse-scale Lagrange multiplier space is discrtetized in the background fashion proposed in [36]. Dirichlet boundary conditions on $\partial \Omega$ are enforced
using a penalty method, with penalty parameter $1000 \times 2^{N-4}$. Quadrature on $\Gamma_{t}$ is performed using a three-point Gaussian quadrature rule in each element of the membrane structure, without regard to how the structure elements intersect the fluid elements. The nonlinear problem in the implicit step of the semi-implicit algorithm is solved using a fixed number of block iterations [35] Section 4].

A representative numerical solution $(N=6)$ is shown in Figure 11, demonstrating the correct qualitative behavior of the solution. The convergence of the $L^{2}(\Omega)$ and $H^{1}(\Omega)$ norms of the velocity errors are shown in the left plot Figure 12, these errors appear to converge at the rates predicted by the analysis of the model problem. The Lagrange multiplier fields as functions of $x_{2}$ are shown in the right plot Figure 12, it is evident that there is some $O(1)$ error in $L^{2}(\Gamma)$, which is again consistent with the analysis of the model problem, in which the multiplier converges in a weak, mesh-dependent norm. The $L^{\infty}$ deviation from a constant is much less pronounced here than in numerical tests with the model problem.


Figure 11: An annotated snapshot at $t=T$ of a numerical approximation to the manufactured solution. Streamlines indicate fluid velocity. Contours show pressure on a scale from $\leq-0.7$ (blue) to $\geq+0.7$ (red), reproducing the expected $\Delta P$ of 1 , with moderate over- and under-shoot near the structure. (The range of the scale exceeds $\Delta P$ to illustrate the over- and under-shoot phenomena.)

## 8. Conclusion

In this paper we presented two dynamic augmented Lagrangian (DAL) methods: the original DAL method (of [31, 34]) and the projection-based DAL method (of [36]), for enforcing Dirichlet boundary conditions. Both methods were previously developed within immersogeometric frameworks for fluid-thin structure interaction simulations. In the present work, we have for the first time addressed regularity of immersed thin structure problems for a simplified linearized parabolic model problem. In this model, the computational domain $\Omega$ is separated by a co dimension-one interface $\Gamma$, where a Dirichlet boundary condition is enforced. We have shown that for both 2D and 3D cases, when all the boundaries are smooth, the solution $u(\mathbf{x}, t)$ for this problem has regularity


Figure 12: Computational results in the FSI benchmark problem, from projection-based DAL method. Left: Convergence of the fluid velocity error at time $T$ in the $H^{1}(\Omega)$ and $L^{2}(\Omega)$ norms. Right: The Lagrange multiplier at time $T$, for various meshes, interpolated linearly between samples at quadrature points on $\Gamma_{T}$.
$\frac{\partial^{k} u}{\partial t^{k}} \in H^{3 / 2-\epsilon}(\Omega)$, for $k=0,2$ and any $\epsilon>0$. With this regularity, we have provided sharp error estimates for the original DAL method: When the element size is $h$, taking the penalty parameters $r=O(1)$, and $\beta=O(1 / h)$, the following error estimates hold for the original DAL method:

$$
\begin{aligned}
& \left\|\left\|\left(u\left(t^{n+1}\right)-u_{h}^{n+1}, \lambda\left(t^{n+1}\right)-\lambda_{h}^{n+1}\right)\right\|\right\|^{2} \\
& \leq \\
& \quad C\left(\frac{h^{2-4 \epsilon}}{\Delta t}+\Delta t\right)\left(\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}\right. \\
& \left.\quad+\int_{0}^{t^{n+1}}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}+\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2} d t\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& \left\|u\left(t^{n+1}\right)-u_{h}^{n+1}\right\|_{L^{2}(\Omega)}^{2} \\
\leq & C\left(h^{2-4 \epsilon}+\Delta t^{2}\right)\left(\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}\right. \\
& \left.+\int_{0}^{t^{n+1}}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}+\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2} d t\right) .
\end{aligned}
$$

The results suggest the optimal choice of time step size as $\Delta t=O(h)$, which yields approximately half-order accuracy for $u$ in the $H^{1}(\Omega)$ norm, and first-order accuracy of $u$ in the $L^{2}(\Omega)$ norm. Moreover, in this paper we have for the first time provided error estimates for the projection-based DAL method: When the element size on $\Omega$ is $h$, and the element size for the Lagrange multiplier is $H$, large enough, with penalty parameter $\beta=O(1 / h)$, the following error estimates for the
projection-based DAL method hold:

$$
\begin{aligned}
& \left\|\left(u\left(t^{n+1}\right)-u_{h}^{n+1}, \lambda\left(t^{n+1}\right)-\lambda_{H}^{n+1}\right)\right\|_{p}^{2} \\
& \leq C\left(\frac{h^{1-2 \epsilon}+H^{2-2 \epsilon}}{\Delta t}+1\right)\left(h^{1-2 \epsilon}\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon}(\mathrm{T})}^{2}\right. \\
& \left.\quad+\int_{0}^{t^{n+1}} h^{1-2 \epsilon}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\mathrm{T})}^{2}+\Delta t\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2} d t\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& \left\|u\left(t^{n+1}\right)-u_{h}^{n+1}\right\|_{L^{2}(\Omega)}^{2} \\
& \leq C\left(h^{1-2 \epsilon}+H^{2-2 \epsilon}+\Delta t\right)\left(h^{1-2 \epsilon}\left\|u_{0}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\lambda_{0}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}\right. \\
& \left.\quad+\int_{0}^{t^{n+1}} h^{1-2 \epsilon}\left\|\frac{\partial u}{\partial t}\right\|_{H^{3 / 2-\epsilon}(\Omega)}^{2}+H^{1-2 \epsilon}\left\|\frac{\partial \lambda}{\partial t}\right\|_{H^{1 / 2-\epsilon}(\Gamma)}^{2}+\Delta t\left\|\frac{\partial^{2} u}{\partial t^{2}}\right\|_{L^{2}(\Omega)}^{2} d t\right) .
\end{aligned}
$$

Similarly, as in the original DAL method, the above error estimates suggest the optimal choice of time-step size as $\Delta t=O(h)=O(H)$, which yields approximately half-order accuracy for $u$ in the $H^{1}(\Omega)$ norm, and first-order accuracy of $u$ in the $L^{2}(\Omega)$ norm. Given the problem regularity, the convergence rate in $H^{1}(\Omega)$ is the same as the rates from the Lagrange-multiplier based fictitiousdomain method [22, 53], which is actually the best possible rate when using a quasi-uniform mesh of $\Omega$ that is not designed to conform to the boundary $\Gamma$.

In the numerical investigations, we have firstly verified the error estimates for both DAL methods, on numerical tests using the model problem. To test the applicability of the above error estimates in practice for more complicated applications, i.e., in the immersogemetric methods for nonlinear and large-displacement FSI, we have derived a manufactured solution for a fluid-thin structure problem with pressure jump. Numerical evidence has suggested that on this FSI problem, the approximate fluid velocity converges at the predicted rates to the manufactured solution.

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[^1]:    ${ }^{1}$ Although the exact multiplier is trivial to represent with any reasonable discrete space, we expect the discrete solution to be polluted by errors in $u$, as indicated by the bounds derived previously.

[^2]:    ${ }^{2}$ Note that practical heart valve FSI computations in our previous work required $r \ll 1$, or $r=0$. Small $r$ allows good kinematic convergence, but sacrifices stability. We do not recommend using $r=1$ in practice, but we use it in this section, to illustrate the drawback of considering " $r=O(1)$ ", as needed for the convergence analysis.
    ${ }^{3}$ Concerns over taking $\Delta t \rightarrow 0$ at fixed $h$ with the original DAL method should be allayed by the experiments of Section 6.4

