CLASSIFICATION OF STABLE SOLUTIONS FOR BOUNDARY VALUE PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS ON RIEMANNIAN MANIFOLDS WITH NONNEGATIVE RICCI CURVATURE

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Abstract. We present a geometric formula of Poincaré type, which is inspired by a classical work of Sternberg and Zumbrun, and we provide a classification result of stable solutions of linear elliptic problems with nonlinear Robin conditions on Riemannian manifolds with nonnegative Ricci curvature.

The result obtained here is a refinement of a result recently established by Bandle, Mastrolia, Monticelli and Punzo.

1. Introduction

The study of partial differential equations on manifolds has a long tradition in analysis and geometry, see e.g. [27, 31, 32, 25, 1]. The interest for such topic may come from different perspectives: on the one hand, at a local level, classical equations with variable coefficients can be efficiently comprised into the manifold setting, allowing more general and elegant treatments; in addition, at a global level, the geometry of the manifold can produce new interesting phenomena and interplay with the structure of the solutions thus creating a novel scenario for the problems into consideration.

Of course, given the complexity of the topic, the different solutions of a given partial differential equation on a manifold can give rise to a rather wild "zoology" and it is important to try to group the solutions into suitable "classes" and possibly to classify all the solution belonging to a class.

In this spirit, very natural classes of solutions in a variational setting arise from energy considerations. The simplest class in this framework is probably that of "minimal solutions", namely the class of solutions which minimize (or, more generally, local minimize) the energy functional.

On the other hand, it is often useful to look at a more general class than minimal solutions, that is the class of solutions at which the second derivative of the energy functional is nonnegative. These solutions are called "stable" (see e.g. [13]). Of course, the class of stable solutions contains that of minimal solutions, but the notion of stability is often

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in concrete situations more treatable than that of minimality: for instance, it is typically very difficult to establish whether or not a given solution is minimal, since one in principle should compare its energy with that of all the possible competitors, while a stability check could be more manageable, relying only on a single, and sometimes sufficiently explicit, second derivative bound.

The goal of this paper is to study the case of a linear elliptic equation on a domain of a Riemannian manifold with nonnegative Ricci curvature, endowed with nonlinear boundary data. We will consider stable solutions in this setting and provide sufficient conditions to ensure that they are necessarily constant.

The framework in which we work is the following. Let M be a connected m-dimensional Riemannian manifold endowed with a smooth Riemannian metric $g = (g_{ij})$. We denote by Δ the Laplace-Beltrami operator induced by g. Let $\Omega \subset M$ be a compact orientable domain and ν be the outer normal vector of $\partial\Omega$ lying in the tangent space T_pM for any $p \in \partial\Omega$. We assume that $\partial\Omega$ is orientable for the outer normal to be well defined and continuous.

In this paper we study the solutions to the following boundary value problem:

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ \partial_\nu u + h(u) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $f, h \in C^1(\mathbb{R})$ and $\partial_\nu u := g(\nabla u, \nu)$. Similar problems have been investigated in [11, 2, 3, 24] mainly taking $h(t) = \alpha t$, $\alpha \in \mathbb{R}$. We stress that taking a general $h \in C^1(\mathbb{R})$ in (1.1) is not only interesting from a purely mathematical point of view (the boundary condition becomes nonlinear) but it also has an interest in some models coming from the real world. Indeed, in Appendix A, we provide a sketchy biological motivation for equation (1.1).

As usual, we consider the volume term induced by g, that is, in local coordinates,

$$dV = \sqrt{|g|}\, dx^1 \wedge \cdots \wedge dx^m,$$

where $\{dx^1, \dots, dx^m\}$ is the basis of 1-forms dual to the vector basis $\{\partial_1, \dots, \partial_m\}$, and $|g| = \det(g_{ij}) \geq 0$. We also denote by $d\sigma$ the volume measure on $\partial\Omega$ induced by the embedding $\partial\Omega \hookrightarrow M$.

As customary, we say that u is a weak solution to (1.1) if $u \in C^1(\overline{\Omega})$ and

$$\int_\Omega \langle \nabla u, \nabla \varphi \rangle \, dV + \int_{\partial\Omega} h(u)\varphi \, d\sigma = \int_\Omega f(u)\varphi \, dV, \qquad \text{for any } \varphi \in C^1(\Omega). \tag{1.2}$$

Moreover, we say that a weak solution u is stable if

$$\int_\Omega |\nabla \varphi|^2 \, dV + \int_{\partial\Omega} h'(u)\varphi^2 \, d\sigma - \int_\Omega f'(u)\varphi^2 \, dV \geq 0, \qquad \text{for any } \varphi \in C^1(\Omega). \tag{1.3}$$

In order to state our result we recall below some classical notions in Riemannian geometry. Given a vector field X, we denote

$$|X| = \sqrt{\langle X, X \rangle}.$$

Also (see, for instance Definition 3.3.5 in [25]), it is customary to define the Hessian of a smooth function φ as the symmetric 2-tensor given in a local patch by

$$(H_\varphi)_{ij} = \partial^2_{ij}\varphi - \Gamma^k_{ij}\partial_k\varphi,$$

where Γ^k_{ij} are the Christoffel symbols, namely

$$\Gamma^k_{ij} = \frac{1}{2} g^{hk}\left(\partial_i g_{hj} + \partial_j g_{ih} - \partial_h g_{ij}\right).$$

Given a tensor A, we define its norm by $|A| = \sqrt{AA^*}$, where A^* is the adjoint.

The above quantities are related to the Ricci tensor Ric via the Bochner-Weitzenböck formula (see, for instance, [4] and references therein):

$$\frac{1}{2}\Delta|\nabla\varphi|^2 = |H_\varphi|^2 + \langle\nabla\Delta\varphi, \nabla\varphi\rangle + \mathrm{Ric}(\nabla\varphi, \nabla\varphi). \tag{1.4}$$

Finally, we let II and H denote the second fundamental tensor and the mean curvature of the embedding $\partial\Omega \hookrightarrow \Omega$ in the direction of the outward unit normal vector field ν, respectively.

We are now in position to state our main result:

Theorem 1.1. *Let $u \in C^3(\overline{\Omega})$ be a stable solution to (1.1). Assume that the Ricci curvature is nonnegative in Ω, and that, for any $p \in \partial\Omega$,*

(1.5) *the quadratic form $II - h'(u)\,\tilde{g}$ on the tangent space $T_p(\partial\Omega)$ is nonpositive definite,*

where $\tilde{g}$ is the induced metrics on $\partial\Omega$. If

$$\int_{\partial\Omega}\Big(h(u)\,f(u) + (m-1)\,\big(h(u)\big)^2\,H + h'(u)\big(h(u)\big)^2\Big)\,d\sigma \le 0, \tag{1.6}$$

then u is constant in Ω.

Remark 1.2. Theorem 1.1 has been proved in [2, Theorem 4.5] in the particular case in which $h(t) := \alpha t$, for some $\alpha \in \mathbb{R}$. We point out that, with this particular choice of h, Theorem 1.1 here weakens the assumptions on the sign of α of [2, Theorem 4.5]. Indeed, in [2, Theorem 4.5], our Theorem 1.1 is proved under the following additional assumptions (see conditions (i) and (ii) on page 137 of [2]):

- $\alpha > 0$, or
- $\alpha < 0$ and u does not change sign in $\overline{\Omega}$.

Interestingly, these additional assumptions are not required in our Theorem 1.1. In this sense, not only Theorem 1.1 here is new since it comprises the case of a nonlinear boundary reaction, but it also improves the results previously known in the literature for the linear case.

One of the advantages of the approach that we take is indeed its flexibility, being able to deal at the same time with the linear and nonlinear cases, and to obtain new results in both situations.

The proof of Theorem 1.1 is based on a geometric Poincaré-type inequality, which we state in this setting as follows:

Theorem 1.3. *Let u be stable weak solution to* (1.1)*. Then,*

$$\begin{aligned}&\int_\Omega \Big(\mathrm{Ric}(\nabla u,\nabla u)+|H_u|^2-\big|\nabla|\nabla u|\big|^2\Big)\varphi^2\,dV\\ &\qquad-\int_{\partial\Omega}\left(\frac{1}{2}\left\langle\nabla|\nabla u|^2,\nu\right\rangle+h'(u)|\nabla u|^2\right)\varphi^2\,d\sigma\\ &\le\int_\Omega|\nabla u|^2|\nabla\varphi|^2\,dV,\end{aligned}\tag{1.7}$$

for any $\varphi\in C^\infty(\Omega)$.

We notice that formula (1.7) relates the stability condition of the solution with the principal curvatures and the tangential gradient of the corresponding level set. Since this formula bounds a weighted L^2-norm of any $\varphi\in C^1(\Omega)$ plus a boundary term by a weighted L^2-norm of its gradient, we may consider this formula as a weighted Poincaré type inequality.

The idea of using weighted Poincaré inequalities to deduce quantitative and qualitative information on the solutions of a partial differential equation has been originally introduced by Sternberg and Zumbrun in [29, 30] in the context of the Allen-Cahn equation, and it has been extensively exploited to prove symmetry and rigidity results, see e.g. [14, 19, 20]. See also [15, 17, 18, 22, 23, 28] for applications to Riemannian and sub-Riemannian manifolds, [6] for problems involving the Ornstein-Uhlenbeck operator, [7, 16] for semilinear equations with unbounded drift and [21, 8, 9, 10] for systems of equations.

Recently, in [12, 11], the cases of Neumann conditions for boundary reaction-diffusion equations and of Robin conditions for linear and quasilinear equations have been studied, using a Poincaré inequality that involves also suitable boundary terms.

We point out that Theorem 1.1 comprises the classical case of the Laplacian in the Euclidean space with homogeneous Neumann data, which was studied in the celebrated papers [5, 26]. In this spirit, our Theorem 1.1 can be seen as a nonlinear version of the results of [5, 26] on Riemannian manifolds (and, with respect to [5, 26], we perform a technically different proof, based on Theorem 1.3).

The next two sections are devoted to the proofs of Theorems 1.3 and 1.1 respectively.

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2. Proof of Theorem 1.3

Applying (1.3) with φ replaced by $|\nabla_g u|\varphi$, we get

$$\begin{aligned}
&\int_\Omega f'(u)|\nabla u|^2\varphi^2\,dV \\
\leq\ & \int_\Omega \Big(\big|\nabla|\nabla u|\big|^2\varphi^2 + |\nabla u|^2|\nabla\varphi|^2 + 2\varphi|\nabla u|\langle\nabla\varphi,\nabla|\nabla u|\rangle\Big)\,dV \\
&\quad + \int_{\partial\Omega} h'(u)|\nabla u|^2\varphi^2\,d\sigma \\
=\ & \int_\Omega \Big(\big|\nabla|\nabla u|\big|^2\varphi^2 + |\nabla u|^2|\nabla\varphi|^2 + \frac{1}{2}\langle\nabla\varphi^2,\nabla|\nabla u|^2\rangle\Big)\,dV + \int_{\partial\Omega} h'(u)|\nabla u|^2\varphi^2 d\sigma.
\end{aligned}$$

Therefore, integrating by parts the third term in the last line, we get

$$\begin{aligned}
&\int_\Omega f'(u)|\nabla u|^2\varphi^2\,dV \\
\leq\ & \int_\Omega \Big(\big|\nabla|\nabla u|\big|^2\varphi^2 + |\nabla u|^2|\nabla\varphi|^2 - \frac{1}{2}\varphi^2\Delta|\nabla u|^2\Big)\,dV \\
&\quad + \frac{1}{2}\int_{\partial\Omega}\varphi^2\left\langle\nabla|\nabla u|^2,\nu\right\rangle\,d\sigma + \int_{\partial\Omega} h'(u)|\nabla u|^2\varphi^2\,d\sigma.
\end{aligned}$$

Hence, recalling (1.4),

$$\begin{aligned}
&\int_\Omega f'(u)|\nabla u|^2\varphi^2\,dV \\
\leq\ & \int_\Omega \Big[\big|\nabla|\nabla u|\big|^2\varphi^2 + |\nabla u|^2|\nabla\varphi|^2 - \Big(|H_u|^2 + \langle\nabla\Delta u,\nabla u\rangle + \mathrm{Ric}(\nabla u,\nabla u)\Big)\varphi^2\Big]\,dV \\
&\quad + \frac{1}{2}\int_{\partial\Omega}\varphi^2\left\langle\nabla|\nabla u|^2,\nu\right\rangle\,d\sigma + \int_{\partial\Omega} h'(u)|\nabla u|^2\varphi^2\,d\sigma.
\end{aligned} \tag{2.1}$$

Now, by differentiating the equation in (1.1), we see that

$$-\nabla\Delta u = f'(u)\nabla u.$$

Plugging this information into (2.1), we conclude that

$$\begin{aligned}
0\ \leq\ & \int_\Omega \Big[\big|\nabla|\nabla u|\big|^2\varphi^2 + |\nabla u|^2|\nabla\varphi|^2 - \Big(|H_u|^2 + \mathrm{Ric}(\nabla u,\nabla u)\Big)\varphi^2\Big]\,dV + \\
&+\frac{1}{2}\int_{\partial\Omega}\varphi^2\left\langle\nabla|\nabla u|^2,\nu\right\rangle d\sigma + \int_{\partial\Omega} h'(u)|\nabla u|^2\varphi^2 d\sigma,
\end{aligned}$$

which completes the proof of Theorem 1.3. □

3. Proof of Theorem 1.1

In this section we provide the proof of Theorem 1.1. We first state the following result, that proves Theorem 3.4 of [2] in the more general case in which h is any C^1 function.

Theorem 3.1. *Let $w \in C^3(\overline{\Omega})$ satisfy*

$$\partial_\nu w + h(w) = 0 \quad \text{on } \partial\Omega, \tag{3.1}$$

for some $h \in C^1(\mathbb{R})$. Then

$$\frac{1}{2}\frac{\partial}{\partial\nu}|\nabla w|^2 = II(\tilde\nabla w, \tilde\nabla w) - h'(w)|\tilde\nabla w|^2 - h(w)\, H_w(\nu,\nu) \quad \text{on } \partial\Omega,$$

where $\tilde\nabla w := \nabla w - g(\nabla w, \nu)\nu$ is the tangential gradient with respect to $\partial\Omega$, and H_w is the Hessian matrix of the function w.

Proof. We let $\{e_i\}$, with $i \in \{1, \dots, m\}$, be a Darboux frame along $\partial\Omega$, that is such that $e_m := \nu$. In this setting, conditions (3.1) reads

$$w_m = -h(w) \quad \text{on } \partial\Omega. \tag{3.2}$$

Also, for any i, $j \in \{1, \dots, m-1\}$, we define

$$H_{ij} := g\big(II(e_i, e_j), \nu\big).$$

Then, reasoning as in the proof of formula (3.32) in [2], we obtain that, for any $j \in \{1, \dots, m-1\}$,

$$w_{jm} = \sum_{i=1}^{m-1} H_{ij} w_i - h'(w) w_j \quad \text{on } \partial\Omega.$$

Therefore, multiplying both terms by w_j, we get

$$w_{jm}\, w_j = \sum_{i=1}^{m-1} H_{ij} w_i\, w_j - h'(w) w_j^2 \quad \text{on } \partial\Omega. \tag{3.3}$$

On the other hand, for any $i \in \{1, \dots, m\}$,

$$\frac{1}{2}\big(|\nabla w|^2\big)_i = \sum_{j=1}^{m} w_j\, w_{ji} = \sum_{i=1}^{m-1} w_j\, w_{ji} + w_m\, w_{mi} = \sum_{i=1}^{m-1} w_j\, w_{ji} - h(w) w_{mi},$$

where we used (3.2) in the last passage.

As a consequence,

$$\frac{1}{2}\frac{\partial}{\partial\nu}|\nabla w|^2 = \sum_{j=1}^{m-1} w_j\, w_{jm} - h(w) w_{mm} \quad \text{on } \partial\Omega.$$

From this and (3.3) we thus obtain

$$\frac{1}{2}\frac{\partial}{\partial\nu}|\nabla w|^2 = \sum_{i,j=1}^{m-1} H_{ij} w_i\, w_j - h'(w) \sum_{j=1}^{m-1} w_j^2 - h(w) w_{mm} \quad \text{on } \partial\Omega,$$

which implies the desired result. ∎

Now we recall that Δ is the Laplace-Beltrami operator of the manifold (M, g), and we let $\tilde{\Delta}$ be the Laplace-Beltrami operator of the manifold $\partial\Omega$ endowed with the induced metric by the embedding $\partial\Omega \hookrightarrow M$. It holds that

$$\Delta w = \tilde{\Delta} w - (m-1)\, H \,\frac{\partial w}{\partial \nu} + H_w(\nu,\nu). \tag{3.4}$$

With this, we can prove the following result:

Lemma 3.2. *Let $u \in C^3(\overline{\Omega})$ be a stable solution of* (1.1). *Then*

$$\begin{aligned}
&\int_\Omega \Big(\mathrm{Ric}(\nabla u, \nabla u) + |H_u|^2 - \big|\nabla|\nabla u|\big|^2\Big)\varphi^2\, dV \\
&\qquad - \int_{\partial\Omega} \Big(I\!I(\tilde{\nabla} u, \tilde{\nabla} u) - h'(u)|\tilde{\nabla} u|^2 + h(u)\, f(u) + (m-1)\big(h(u)\big)^2 H + h'(u)\big(h(u)\big)^2\Big)\, \varphi^2\, d\sigma \\
&\le \int_\Omega |\nabla u|^2 |\nabla\varphi|^2\, dV - \int_{\partial\Omega} h(u)\, \Big\langle \tilde{\nabla} u, \tilde{\nabla}\varphi^2 \Big\rangle\, d\sigma,
\end{aligned} \tag{3.5}$$

for any $\varphi \in C^\infty(\Omega)$.

Proof. From Theorems 1.3 and 3.1, for every stable weak solution u to (1.1) and for any $\varphi \in C^\infty(\Omega)$, we have that

$$\begin{aligned}
&\int_\Omega \Big(\mathrm{Ric}(\nabla u, \nabla u) + |H_u|^2 - \big|\nabla|\nabla u|\big|^2\Big)\varphi^2\, dV \\
&\qquad - \int_{\partial\Omega} \Big(I\!I(\tilde{\nabla} u, \tilde{\nabla} u) - h'(u)|\tilde{\nabla} u|^2 - h(u)\, H_u(\nu,\nu) + h'(u)|\nabla u|^2\Big)\varphi^2\, d\sigma \\
&\le \int_\Omega |\nabla u|^2 |\nabla\varphi|^2\, dV.
\end{aligned} \tag{3.6}$$

Now we use (3.4) to manipulate the integral on the boundary of Ω: in this way, we obtain from (3.6) that

$$\begin{aligned}
&\int_\Omega \Big(\mathrm{Ric}(\nabla u, \nabla u) + |H_u|^2 - \big|\nabla|\nabla u|\big|^2\Big)\varphi^2\, dV \\
&\qquad - \int_{\partial\Omega} \left[I\!I(\tilde{\nabla} u, \tilde{\nabla} u) - h'(u)|\tilde{\nabla} u|^2 - h(u)\left(\Delta u - \tilde{\Delta} u + (m-1)\, H\, \frac{\partial u}{\partial \nu}\right) + h'(u)|\nabla u|^2\right] \varphi^2\, d\sigma \\
&\le \int_\Omega |\nabla u|^2 |\nabla\varphi|^2\, dV.
\end{aligned}$$

Thus, recalling[1] (1.1), we conclude that

$$
\begin{aligned}
&\int_\Omega \Big(\mathrm{Ric}(\nabla u,\nabla u)+|H_u|^2-\big|\nabla|\nabla u|\big|^2\Big)\varphi^2\,dV\\
&\qquad -\int_{\partial\Omega}\Big(I\!I(\tilde\nabla u,\tilde\nabla u)-h'(u)|\tilde\nabla u|^2+h(u)\,f(u)+h(u)\,\tilde\Delta u+(m-1)\,\big(h(u)\big)^2\,H+h'(u)|\nabla u|^2\Big)\varphi^2\,d\sigma\\
&\le \int_\Omega |\nabla u|^2|\nabla\varphi|^2\,dV.
\end{aligned}
\tag{3.7}
$$

Now we observe that

$$
|\nabla u|^2=|\tilde\nabla u|^2+\left|\frac{\partial u}{\partial\nu}\right|^2=|\tilde\nabla u|^2+\big(h(u)\big)^2\qquad\text{on }\partial\Omega.
$$

Plugging this information into (3.7), we obtain that

$$
\begin{aligned}
&\int_\Omega \Big(\mathrm{Ric}(\nabla u,\nabla u)+|H_u|^2-\big|\nabla|\nabla u|\big|^2\Big)\varphi^2\,dV\\
&\qquad -\int_{\partial\Omega}\Big(I\!I(\tilde\nabla u,\tilde\nabla u)+h(u)\,f(u)+h(u)\,\tilde\Delta u+(m-1)\,\big(h(u)\big)^2\,H+h'(u)\big(h(u)\big)^2\Big)\varphi^2\,d\sigma\\
&\le \int_\Omega |\nabla u|^2|\nabla\varphi|^2\,dV.
\end{aligned}
$$

Now we notice that

$$
\int_{\partial\Omega} h(u)\,\varphi^2\,\tilde\Delta u\,d\sigma=-\int_{\partial\Omega} h'(u)|\tilde\nabla u|^2\varphi^2\,d\sigma-\int_{\partial\Omega} h(u)\left\langle\tilde\nabla u,\tilde\nabla\varphi^2\right\rangle\,d\sigma,
$$

and therefore

$$
\begin{aligned}
&\int_\Omega \Big(\mathrm{Ric}(\nabla u,\nabla u)+|H_u|^2-\big|\nabla|\nabla u|\big|^2\Big)\varphi^2\,dV\\
&\qquad -\int_{\partial\Omega}\Big(I\!I(\tilde\nabla u,\tilde\nabla u)-h'(u)|\tilde\nabla u|^2+h(u)\,f(u)+(m-1)\,\big(h(u)\big)^2H+h'(u)\big(h(u)\big)^2\Big)\,\varphi^2\,d\sigma\\
&\le \int_\Omega |\nabla u|^2|\nabla\varphi|^2\,dV-\int_{\partial\Omega} h(u)\,\left\langle\tilde\nabla u,\tilde\nabla\varphi^2\right\rangle\,d\sigma,
\end{aligned}
$$

which proves the desired inequality. □

Before completing the proof of Theorem 1.1 we recall the following lemmata proved in [17, Lemma 5] and [17, Lemma 9], respectively.

Lemma 3.3. *For any smooth function $\varphi : M\to\mathbb{R}$, we have that*

$$
|H_\varphi|^2\ge\big|\nabla|\nabla\varphi|\big|^2\qquad\textit{almost everywhere.}\tag{3.8}
$$

Lemma 3.4. *Suppose that the Ricci curvature of M is nonnegative and that* Ric *does not vanish identically.*

Let u be a solution of (1.1)*, with*

$$
\mathrm{Ric}(\nabla u,\nabla u)(p)=0\qquad\textit{for any } p\in M.
$$

[1] Notice that, since u is regular enough, the equation holds true up to the boundary of Ω.

Then, u is constant.

With this, we are able to finish the proof of Theorem 1.1:

Proof of Theorem 1.1: Taking $\varphi \equiv 1$ in (3.5) we see that

$$\begin{aligned}\int_\Omega \Big(\mathrm{Ric}(\nabla u, \nabla u) + |H_u|^2 - \big|\nabla|\nabla u|\big|^2\Big)\, dV \\ \leq \int_{\partial\Omega} \Big(I\!I(\tilde\nabla u, \tilde\nabla u) - h'(u)|\tilde\nabla u|^2 + h(u)\, f(u) + (m-1)\big(h(u)\big)^2 H + h'(u)\big(h(u)\big)^2\Big)\, d\sigma.\end{aligned}$$

Hence, using (1.5) and (1.6), we obtain that

$$\int_\Omega \Big(\mathrm{Ric}(\nabla u, \nabla u) + |H_u|^2 - \big|\nabla|\nabla u|\big|^2\Big)\, dV \leq 0. \tag{3.9}$$

On the other hand, by Lemma 3.3,

$$|H_u|^2 - \big|\nabla|\nabla u|\big|^2 \geq 0 \quad \text{ on } \Omega,$$

and so (3.9) gives

$$\int_\Omega \mathrm{Ric}(\nabla u, \nabla u)\, dV \leq 0,$$

which implies that

$$\mathrm{Ric}(\nabla u, \nabla u) = 0 \quad \text{ in } \Omega.$$

Thus, the conclusion follows from Lemma 3.4. □

Appendix A. A biological motivation for equation (1.1)

In this appendix, we show a simple model from mathematical biology which naturally produces equation (1.1). We describe it on a subset of the plane, which would provide a Euclidean model, but when set on a considerable portion of the Earth, it is also naturally set on a manifold.

Suppose that a population lives in a region $\Omega \subset \mathbb{R}^2$. Assume that in such region the population is subject to the usual logistic equation, with birth rate $\beta > 0$ and carrying capacity $K > 0$, and moves according to a Brownian motion.

In case the boundary of Ω acts as a perfect fence, people inside cannot go outside and viceversa, and the problem is naturally endowed with homogeneous Neumann conditions. Assume, on the other hand, that the boundary of Ω does not prevent, in principle, the individuals to go through. Assume also that other individuals live outside Ω (and, for simplicity, suppose that the number of these individuals is not really relevant to the problem, since they found their own living resources outside Ω) and that their incoming in Ω can be influenced by the number of individuals on $\partial\Omega$: for instance, suppose that the individuals living outside Ω (in the proximity of $\partial\Omega$) see the ones on the boundary and are attracted by them, possibly in a quantitative way which is described by the nonlinear function h (e.g., the more people along the boundary, the higher the attraction, and, for instance, the abundance of people may act as an additional spur to get in). In this way, the number of people going in through the boundary is provided by the nonlinearity h via the relation $-\partial_\nu u = h(u)$ (since ν is the exterior normal, we take into account this sign convention here). The model would then provide the equation

$$\begin{cases} \partial_t u = \Delta u + \beta\, u\left(1 - \frac{u}{K}\right) & \text{in } \Omega \times (0, +\infty), \\ \partial_\nu u + h(u) = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(\cdot, 0) = u_0(\cdot) & \text{in } \Omega. \end{cases} \tag{A.1}$$

In this framework, $u(x,t)$ represents the population density at the point $x \in \Omega$ at time $t > 0$, and u_0 is the initial density in Ω of this population. Notice that stationary solutions of (A.1) satisfy (1.1) with $f(u) := \beta\, u\left(1 - \frac{u}{K}\right)$. In this setting, condition (1.5) is satisfied, for instance, if the curvature of Ω is uniformly bounded and $h(u) := M(e^u - 1)$, with $M > 0$ sufficiently large (strong attraction regime of a "very social" species).

It is of course rather unexpected to also satisfy (1.6) in such a model: this is indeed the content of Theorem 1.1 which would give that if both conditions (1.5) and (1.6) are fulfilled, then any stable population u is necessarily constant (and, conversely, if u vanishes identically, then both conditions (1.5) and (1.6) can be satisfied for large M, and null solutions are stable for small β).

Another case of interest is that in which (A.1) holds in a large portion of the sphere: for instance, if we take $\epsilon > 0$ sufficiently small and

$$\begin{aligned} &\Omega := \{x = (x_1, x_2, x_3) \in S^2 \text{ s.t. } x_3 < 1 - \epsilon\}, \\ \text{with} \quad &S^2 := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \text{ s.t. } x_1^2 + x_2^2 + x_3^2 < 1\}, \end{aligned}$$

we have that both $I\!I$ and H are negative and their absolute value is comparable to $1/\epsilon$. This says that both conditions (1.5) and (1.6) are satisfied in this case, which implies that stable solutions are necessarily constant in view of Theorem 1.1.

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