Towards a theory of BV functions in abstract Wiener spaces

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Abstract

Functions of bounded variation in an abstract Wiener space, i.e., an infinite dimensional Banach space endowed with a Gaussian measure and a related differentiable structure, have been introduced by M. Fukushima and M. Hino using Dirichlet forms, and their properties have been studied with tools from stochastics. In this paper we reformulate, with purely analytical tools, the definition and the main properties of BV functions, and start investigating further properties.

1. Introduction

Functions of bounded variation of one independent variable have been introduced by C. Jordan in 1881, and subsequently studied by various Authors. After several attempts (let us refer to the historical note in [4, Section 3.12]), generalisations to $\mathbb{R}^n$ began in the fifties of 20^{th} Century, starting from E. De Giorgi's fundamental paper [11]. Subsequently, the structure of BV functions has been deeply understood, and applications have been found in many fields. We shall quote some of them in Section 2.

More recently, generalisations have been developed in different contexts, such as manifolds, Carnot groups, fractals, and general metric measure spaces, with interesting applications also in these frameworks. We refer to [5] and the references there for a review of these topics. These theories rely on the doubling property of the underlying measure, hence cannot cover the case of an infinite dimensional Banach space endowed with a probability measure. In fact, in this case clearly the doubling condition is not satisfied, as it implies that balls are totally bounded.

A definition of BV functions in abstract Wiener spaces has been given by M. Fukushima in [19], M. Fukushima and M. Hino in [20], and is based upon Dirichlet form theory, see [21], [27]. The starting point of these papers has been a characterisation of sets with finite perimeter in finite dimensions in terms of the behaviour of suitable stochastic processes (see [18]), and in fact the tools used in [19], [20] come also from stochastics. In this paper our main aim is to compare the finite and infinite dimensional theory of BV functions from a purely analytical point of view, closer to the classical setting. We recover all the (analytical) results by M. Fukushima and M. Hino and start investigating further properties.

The importance of generalising the classical notion of perimeter and variation has been pointed

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out in several occasion by E. De Giorgi: we refer to [13], where the infinite dimensional context is explicitly quoted. There are several motivations for studying BV functions in Banach spaces, e.g., isoperimetric inequalities and mass concentration, see [23], [25], infinite dimensional analysis and semigroups (see e.g. [9], [10]), differential systems like $\dot{\mathbf{X}}(t,x) = \mathbf{b}(\mathbf{X}(t,x))$ with BV vector field $\mathbf{b}$, see [3] for the Sobolev case, and the aforementioned statistical mechanics and stochastic processes, but further meaningful variational problems could likely be formulated in this context.

The paper is organised as follows: in the first section we review the theory of BV functions on $\mathbb{R}^n$ and show the equivalence of four definitions of total variation. We focus in particular on the tools used in the proofs, in order to stress that almost none of them is available in infinite dimensions. In Section 3 we describe the Wiener space setting and the tools useful to rephrase the possible definitions of total variation. In Section 4 we define BV functions in Wiener spaces and discuss their basic properties. Finally, in Section 5 we report on further results and describe several open problems that should be addressed in order to give good description of BV useful to applications.

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2. A review of BV functions in $\mathbb{R}^n$

There are various ways of defining BV functions on $\mathbb{R}^n$, which we discuss in the following theorem. They are useful in different contexts, and we add a few comments after sketching the proof of the following statement. A detailed analysis of BV functions in $\mathbb{R}^n$ is available in [4], [15], [16], [22], [33]. We denote by $\mathrm{Lip}(\mathbb{R}^n)$ the space of Lipschitz continuous functions on $\mathbb{R}^n$.

Theorem 2.1 *Let $u \in L^1(\mathbb{R}^n)$. The following are equivalent:*

1 *there exist real finite measures μ_j, $j = 1, \ldots, n$, on $\mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} u D_j \phi dx = -\int_{\mathbb{R}^n} \phi d\mu_j, \quad \forall \phi \in C_c^1(\mathbb{R}^n), \quad (1)$$

i.e., the distributional gradient $Du = \mu$ is an $\mathbb{R}^n$-valued measure with finite total variation $|Du|(\mathbb{R}^n)$;

2 *the following holds:*

$$V(u) := \sup\Big\{ \int_{\mathbb{R}^n} u \operatorname{div} \phi dx : \phi \in [C_c^1(\mathbb{R}^n)]^n, \|\phi\|_\infty \le 1 \Big\} < \infty;$$

3 *the following holds:*

$$L(u) := \inf\Big\{ \liminf_{h\to\infty} \int_{\mathbb{R}^n} |\nabla u_h| dx : u_h \in \mathrm{Lip}(\mathbb{R}^n),\ u_h \stackrel{L^1}{\to} u \Big\} < \infty;$$

4 *if $(T(t))_{t\ge 0}$ denotes the heat semigroup in $\mathbb{R}^n$, then*

$$\mathcal{I}[u] := \lim_{t\to 0} \int_{\mathbb{R}^n} |\nabla T(t) u| dx < \infty. \quad (2)$$

Moreover, $|Du|(\mathbb{R}^n) = V(u) = L(u) = \mathcal{I}[u]$.

If one of (hence all) the conditions in Theorem 2.1 holds, we say that $u \in BV(\mathbb{R}^n)$. Moreover, if $E \subset \mathbb{R}^n$ and $|D\chi_E|(\mathbb{R}^n)$ is finite, we say that E is a set with finite perimeter, and use the notation $P(E)$ (perimeter of E) for the total variation of the measure $D\chi_E$.

Let us point out that the first definition of BV has been given by E. De Giorgi in [11] through condition **4**, which he showed to be equivalent to **1**. Condition **2** follows at once by computing the total variation of Du. De Giorgi did not use the terminology of semigroups, and in fact he wrote condition **4** in a convolution form, without mentioning that the convolution kernel was the heat kernel.

Proof.

4 ⇒ 3 Simply, notice that $u_h = T(t_h)u$, with $t_h \to 0$, can be used in **3**. In particular, $L(u) \le \mathcal{I}[u]$.

3 ⇒ 2 Let $(u_h) \subset L^1(\mathbb{R}^n)$ be such that $u_h \stackrel{L^1}{\to} u$ and $\int |\nabla u_h| dx \to L(u)$. By w^*-compactness of measures, the sequence $h \mapsto \nabla u_h dx$ weakly* converges to a measure μ as $h \to \infty$; then $|\mu|(\mathbb{R}^n) \le L(u)$, and

$$\begin{aligned}\int_{\mathbb{R}^n} u \operatorname{div} \phi dx &= \lim_{h\to\infty} \int_{\mathbb{R}^n} u_h \operatorname{div} \phi dx \\ &= -\lim_{h\to\infty} \int_{\mathbb{R}^n} \langle \nabla u_h, \phi \rangle dx \\ &= -\int_{\mathbb{R}^n} \phi d\mu\end{aligned}$$

for every $\phi \in [C_c^1(\mathbb{R}^n)]^n$, so that $V(u) \le L(u)$.

2 ⇒ 1 If $V(u) < \infty$ then, by the Riesz representation theorem, the functionals $\phi \mapsto \int uD_j\phi$ are measures μ_j and it is not difficult to show that (1) holds.

1 ⇒ 4 Since the heat semigroup $(T(t))_{t\geq 0}$ is contractive in $L^1(\mathbb{R}^n)$ and the commutation relation $T(t)\nabla = \nabla T(t)$ holds, the function $t \mapsto \|\nabla T(t)u\|_{L^1(\mathbb{R}^n)}$ is decreasing and the limit in (2) exists (finite or not). Moreover, denoting by $G_t(x) = (4\pi t)^{-n/2}\exp\{-|x|^2/4t\}$ the Gauss-Weierstrass kernel, we have

$$\begin{aligned}\|\nabla T(t)u\|_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n}\Big|\nabla_x\int_{\mathbb{R}^n} G_t(x-y)u(y)dy\Big|dx\\ &= \int_{\mathbb{R}^n}\Big|-\int_{\mathbb{R}^n}\nabla_y G_t(x-y)u(y)dy\Big|dx\\ &\leq \int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} G_t(x-y)dx\Big)d|Du|(y) = |Du|(\mathbb{R}^n),\end{aligned}$$

whence $\mathcal{I}[u] \leq |Du|(\mathbb{R}^n)$.

□

Notice that the above conditions can be used in different contexts, according to the underlying structure. For instance, **1** and **2** require a differentiable structure, and can be used in manifolds and Carnot groups (with a suitable notion of divergence), **4** requires a differential structure as well, together with some semigroup theory and works (with some limitations: see [29], [8]) on Riemannian manifolds, whereas it is not clear in Carnot groups. Finally, **3** seems to be the more flexible definition, requires only a notion of Lipschitz continuity, and in fact can be used in metric measure spaces (with a doubling condition), see [28], [5] and the references there. On the other hand, under condition **3** the question arises to understand if a gradient, as a vector, can be defined.

We cannot discuss the huge amount of variational problems that can be settled in BV, and confine ourselves to a generic quotation of integral functionals with linear growth in the gradient,

$$F(u) = \int f(x,u,Du)dx, \tag{3}$$

with e.g. $|f(x,u,\xi)| \leq a(x) + b(x)|\xi|$, $a \in L^1$, b continuous and f (convex in ξ) suitably extended for measure gradients Du, and geometric problems, e.g, isoperimetric and shape optimisation problems. Applications come, to quote only a few among the more recent ones, from variational models in elasto-plasticity (possibly with fractures), image smoothing and segmentation (e.g. Mumford-Shah functional). One of the first big success of the theory of sets of finite perimeter has been the complete solution of the isoperimetric problem (see [12]), i.e., the proof that equality holds in the isoperimetric inequality

$$\text{meas}\,(E) \leq c_n|D\chi_E|^{\frac{n}{n-1}} \tag{4}$$

if and only if E is a ball, *among all sets with finite perimeter*, where $c_n = n^{-\frac{n}{n-1}}\omega_n^{-\frac{1}{n-1}}$ and ω_n is the volume of the unit ball. We shall come back to this inequality, but point out that it entails the continuous embedding $W^{1,1}(\mathbb{R}^n) \hookrightarrow L^{\frac{n}{n-1}}(\mathbb{R}^n)$.

In view of the investigation of properties listed in Theorem 2.1 in spaces of infinite dimensions, let us have a closer look at the tools used to prove it. Basically, they are the Riesz representation theorem $\mathscr{M}(\mathbb{R}^n) = (C_b(\mathbb{R}^n))^*$ and the related w^*-compactness of measures; the integration by parts formula (1), where the divergence plays the role of the adjoint operator of the gradient; the regularising properties of the heat semigroup, that ensure that $T(t)u \in W^{1,1}(\mathbb{R}^n)$ if $u \in L^1(\mathbb{R}^n)$, as well as the contractivity of $(T(t))_{t\geq 0}$ in L^1 and the commutation property $\nabla T(t) = T(t)\nabla$.

If X is an (infinite dimensional) Banach space, none of the above properties holds: linear functionals on $C_b(X)$ are *finitely* additive measures, and, accordingly, w^*-compactness holds for finitely additive measures, there is no generalisation of the Lebesgue measure, it is not obvious which is the right semigroup to be used in condition **4**. Furthermore, (1) can be written in a vectorial form taking a test function $\Phi = (\phi_1,\ldots,\phi_n)$ and defining the divergence operator $\text{div}\,\Phi = \sum_j D_j\phi_j$:

$$\int_{\mathbb{R}^n} u\,\text{div}\,\Phi dx = -\int_{\mathbb{R}^n}\sum_{j=1}^n \phi_j d\mu_j; \tag{5}$$

hence, a div operator is to be found in infinite dimensions in such a way that (a suitable form of) (5) holds.

3. Wiener space setting

In this section we describe our setting: given an (infinite dimensional) separable Banach space X, we denote by $\|\cdot\|_X$ its norm and by $B_X(x,r) = \{y \in X : \|y-x\|_X < r\}$ the open ball centred at $x \in X$ and with radius $r > 0$. X^* denotes the topological dual, with duality $\langle\cdot,\cdot\rangle$. By $C(X)$ we denote the space of all continuous functions on X and by $C_b(X)$ the space of all bounded continous functions.

Given the elements $x_1^*, \dots, x_m^*$ in X^*, we denote by $\Pi_{x_1^*,\dots,x_m^*} : X \to \mathbb{R}^m$ the finite dimensional projection of X onto $\mathbb{R}^m$ induced by the elements $x_1^*, \dots, x_m^*$, that is the map

$$\Pi_{x_1^*,\dots,x_m^*} x = (\langle x, x_1^*\rangle, \dots, \langle x, x_m^*\rangle),$$

also denoted by $\Pi_m : X \to \mathbb{R}^m$ if it is not necessary to specify the elements $x_1^*, \dots, x_m^*$. The symbol $\mathcal{F}C_b^k(X)$ denotes the space of k times continuously differentiable cylindrical functions with bounded derivatives up to the order k, that is $u \in \mathcal{F}C_b^k(X)$ if $u(x) = v(\Pi_m x)$ for some $v \in C_b^k(\mathbb{R}^m)$.

We divide this section in some subsections; first of all we recall some notion of measure theory, with particular emphasis on the peculiarities of the infinite dimensional (i.e., non locally compact) setting, then we pass to the definition and description of abstract Wiener spaces. In the third subsection we discuss the integration by parts formula and recall the definition of gradient and divergence. Finally, we introduce Sobolev classes and the Ornstein-Uhlenbeck semigroup together with some of their basic properties.

3.1. *Infinite dimensional measure theory*

We denote by $\mathscr{B}(X)$ the Borel σ-algebra and by $\mathcal{E}(X)$ the cylindrical σ-algebra generated by X^*, that is the σ-algebra generated by the sets of the form $E = \Pi_m^{-1} B$ with $B \in \mathscr{B}(\mathbb{R}^m)$. Since X is separable, these families coincide, see [32, Theorem I.2.2], even if we fix a sequence $(x_i^*) \subset X^*$ which separates the points in X and use only elements from that sequence to generate Π_m. We shall make later on some special choice of (x_i^*), induced by a Gaussian probability measure γ in X.

We also denote by $\mathscr{M}(X, Y)$ the set of countably additive measures on X with values on a Banach space Y with finite total variation, $\mathscr{M}(X)$ if $Y = \mathbb{R}$. We denote by $|\mu|$ the total variation measure of μ, defined by

$$|\mu|(B) := \sup\Big\{ \sum_{h=1}^n \|\mu(B_h)\|_Y ; B = \bigcup_{h=1}^n B_h \Big\}, \quad (6)$$

for every $B \in \mathscr{B}(X)$, where the union is a disjoint union. Notice that, using the polar decomposition, there is a unit $|\mu|$-measurable vector field $\sigma : X \to Y$ such that $\mu = \sigma|\mu|$, and then the equality

$$|\mu|(X) = \sup\Big\{ \int_X \langle \sigma, \phi\rangle d|\mu|, \phi \in C_b(X, Y^*), \\ \|\phi(x)\|_{Y^*} \le 1 \ \forall x \in X \Big\}, \quad (7)$$

holds, where $\langle , \rangle$ denotes the duality between Y and Y^*. Finally, let us define the sup of (the total variation of) an arbitrary family of measures $\{\mu_\alpha,\ \alpha \in I\}$ by setting

$$\bigvee_{\alpha \in I} |\mu_\alpha|(A) = \sup\Big\{ \sum_{n=1}^\infty |\mu_{\alpha_n}|(A_n) \Big\},$$

where the supremum runs along all the countable pairwise disjoint partitions $A = \bigcup_n A_n$ and all the choices of the sequence $(\alpha_n) \subset I$.

3.2. *The abstract Wiener space*

Assume that a centred Gaussian measure γ is defined on X. This means that γ is a probability measure and for all $x^* \in X^*$ the law $x^*_\# \gamma$ is a centred Gaussian measure on $\mathbb{R}$, that is, the Fourier transform of γ is given by

$$\hat\gamma(x^*) = \int_X \exp\{-i\langle x, x^*\rangle\} d\gamma(x) \\ = \exp\Big\{ -\frac{1}{2}\langle Qx^*, x^*\rangle \Big\} \quad \forall x^* \in X^*,$$

where $Q \in \mathcal{L}(X^*, X)$ is the covariance operator. The covariance operator is a symmetric and positive operator uniquely determined by the relation

$$\langle Qx^*, y^*\rangle = \int_X \langle x, x^*\rangle \langle x, y^*\rangle d\gamma(x), \ \forall x^*, y^* \in X^* \quad (8)$$

and we also write $\mathscr{N}(0, Q)$ for γ. The fact that the operator Q defined by (8) is bounded is a consequence of Fernique's Theorem (see e.g. [6, Theorem 2.8.5]), asserting the existence of a positive $\beta > 0$ such that

$$\int_X \exp\{\beta \|x\|_X^2\} d\gamma(x) < \infty;$$

as another consequence of this we get also that any $x^* \in X^*$ defines a function $x \mapsto \langle x, x^*\rangle$ that belongs to $L^p(X, \gamma)$ for all $p \ge 1$, and even more, see (vi) in Section 3. In particular, we can think of any $x^* \in X^*$ as an element of $L^2(X, \gamma)$. Let us denote by $R^* : X^* \to L^2(X, \gamma)$ the embedding, $R^* x^*(x) = \langle x, x^*\rangle$. The space $\mathscr{H} = L^2(X, \gamma)$ is called the *reproducing kernel* of the Gaussian measure γ and $R^* X^*$ turns out to be dense in it. The above definition is motivated by the fact that if we consider the operator $R : \mathscr{H} \to X$ whose adjoint is R^*, then

$$R\hat{h} = \int_X \hat{h}(x)x d\gamma(x), \tag{9}$$

where the last integral has to be understood as a Bochner integral. In fact, denoting by $[\cdot,\cdot]_{\mathscr{H}}$ the inner product in $\mathscr{H}$, the equality

$$\begin{aligned}\langle R\hat{h}, x^*\rangle &= [\hat{h}, R^*x^*]_{\mathscr{H}} \\ &= \int_X \hat{h}(x)\langle x, x^*\rangle d\gamma(x) \\ &= \Big\langle \int_X \hat{h}(x)x d\gamma(x), x^*\Big\rangle,\end{aligned}$$

that holds for all $x^* \in X^*$, implies (9). With the definition of R, R^* we obtain directly by (8) the decomposition $Q = RR^*$:

$$\begin{aligned}\langle RR^*x^*, y^*\rangle &= [R^*x^*, R^*y^*]_{\mathscr{H}} \\ &= \int_X \langle x, x^*\rangle\langle x, y^*\rangle d\gamma(x) \\ &= \langle Qx^*, y^*\rangle.\end{aligned}$$

It can be proved that the operator R is compact and even more, i.e., that R belongs to the ideal $\gamma(\mathscr{H}, X)$ of γ-Radonifying, or Gaussian-Radonifying, operators, a particular important subclass of compact operators, see e.g. [26] (γ-Radonifying operators reduce to Hilbert-Schmidt operators when X is a Hilbert space). We point out that the previous statement shows that not all symmetric positive operators in $\mathcal{L}(X^*, X)$ are covariance operators of a Gaussian measure, but only those admitting a decomposition $Q = RR^*$ with $R \in \gamma(\mathscr{H}, X)$ for some separable Hilbert space $\mathscr{H}$. In particular, in the case when X is Hilbert, Q has to be a trace operator. This remark shows that another way is possible: one can start with $R \in \gamma(\mathscr{H}, X)$ for some separable Hilbert space $\mathscr{H}$ and define the covariance operator $Q = RR^*$; from this, a unique centred Gaussian measure on X is defined with covariance Q. In any case, the measure γ is concentrated on the separable subspace of X defined as the closure of $R\mathscr{H}$ in X. The space $H = R\mathscr{H}$ is particularly important and is called the Cameron-Martin space; it is a Hilbert space with inner product defined by

$$[h_1, h_2]_H = [\hat{h}_1, \hat{h}_2]_{\mathscr{H}}$$

for all $h_1, h_2 \in H$, where $h_i = R\hat{h}_i$, $i = 1, 2$.

With this notation, the Fourier transform of the Gaussian measure γ becomes

$$\hat{\gamma}(x^*) = \exp\Big\{-\frac{1}{2}\|\hat{x^*}\|^2_{\mathscr{H}}\Big\}, \qquad \forall x^* \in X^*,$$

where $\hat{x^*} = R^*x^*$. This definition can be easily extended to the whole of $\mathscr{H}$ by density. Notice that the assumption that X is separable is not restrictive, as otherwise we can replace it with the separable space $X_1 = \overline{H}$ where γ is concentrated. In this way, the Cameron-Martin space H is dense in X. Using the embedding $R^*X^* \subset \mathscr{H}$, we shall say that a family $\{x_j^*\}$ of elements of X^* is orthonormal if the corresponding family $\{R^*x_j^*\}$ is orthonormal in $\mathscr{H}$. It can be proved that $\gamma(H) = 0$, see [6, Theorem 2.4.7]; we also notice that the unit Hilbert ball $B_H(0,1) = R\hat{B}_{\mathscr{H}}(0,1)$ with $\hat{B}_{\mathscr{H}}(0,1) = \{\hat{h} \in \mathscr{H} : \|\hat{h}\|_{\mathscr{H}} < 1\}$ is a pre-compact subset of X since R is compact. Since X and X^* are separable, starting from a sequence in X^* dense in H, we may construct an orthonormal basis (h_j) in H with $h_j = Qx_j^*$. Set also $H_m = \operatorname{span}\{h_1, \ldots, h_m\}$, and define $X^\perp = \ker \Pi_{x_1^*,\ldots,x_m^*}$ and X_m the (m-dimensional) complementary space. Accordingly, we have the canonical decomposition $\gamma = \gamma_m \otimes \gamma^\perp$ of the measure γ, and notice that these Gaussian measures are *rotation invariant*, i.e., if $\varrho : X \times X \to X \times X$ is given by $\varrho(x, y) = (\cos\vartheta x + \sin\vartheta y, -\sin\vartheta x + \cos\vartheta y)$ for some $\vartheta \in \mathbb{R}$, then $\varrho_\#(\gamma \otimes \gamma) = \gamma \otimes \gamma$ and the same holds for γ_m, $\gamma^\perp$. We shall use, in particular, the following equality:

$$\begin{aligned}&\int_{X^\perp}\int_{X^\perp} u(\cos\vartheta x + \sin\vartheta y) d\gamma^\perp(x) d\gamma^\perp(y) \\ &= \int_{X^\perp} u(x) d\gamma^\perp(x), \ \forall u \in L^1(X^\perp, \gamma^\perp),\end{aligned} \tag{10}$$

which is obtained by the above relation by integrating the function $u \otimes 1$ on $X^\perp \times X^\perp$. For every function $u \in L^1(X, \gamma)$ its *canonical cylindrical approximations* are defined as the conditional expectations relative to the σ-algebras Σ_m generated by $\{\langle x, \hat{h}_1\rangle, \ldots, \langle x, \hat{h}_m\rangle\}$,

$$u^m = \mathbb{E}_m u \text{ s.t. } \int_A u d\gamma = \int_A u^m d\gamma \tag{11}$$

for all $A \in \Sigma_m$. Then, $u^m \to u$ in $L^1(X, \gamma)$ and γ-a.e. (see e.g. [6, Corollary 3.5.2]). More explicitly, we set

$$\begin{aligned}\mathbb{E}_m u(x) &= \int_X u(P_m x + (I - P_m)y) d\gamma(y) \\ &= \int_{X^\perp} u(P_m x + y') d\gamma^\perp(y'),\end{aligned}$$

where P_m is the projection onto X_m. Notice that the restriction of γ to Σ_m is invariant under translations along all the vectors in $X^\perp$, hence we may write $\mathbb{E}_m u(x) = v(P_m x)$ for some function v, and, with an abuse of notation, $\mathbb{E}_m u(x_m)$ instead of $\mathbb{E}_m u(x)$.

The importance of the Cameron-Martin space relies mainly on the following fact, that is crucial in the integration by parts formula discussed in the next subsection; if we consider for $h \in X$ the translated measure

$$\gamma_h(B) = \gamma(B - h), \qquad B \in \mathscr{B}(X),$$

then γ_h is absolutely continuous with respect to γ if and only if $h \in H$ and in this case, with the usual notation $h = R\hat{h}$, $\hat{h} \in \mathscr{H}$, we have, see e.g. [6, Corollary 2.4.3],

$$d\gamma_h(x) = \exp\Big\{\hat{h}(x) - \frac{1}{2}\|h\|_H^2\Big\}d\gamma(x). \tag{12}$$

It is also important to notice that if we define for any $\lambda \in \mathbb{R}$ the measure

$$\gamma_\lambda(B) = \gamma(\lambda B), \qquad \forall B \in \mathscr{B}(X)$$

then $\gamma_\lambda \ll \gamma_\sigma$ if and only if $|\lambda| = |\sigma|$ (see for instance [6, Example 2.7.4]).

Let us now present the prototype of abstract Wiener space. It was introduced by N. Wiener as the space of trajectories of Brownian motion, endowed with a Gaussian probability distribution. By $f \in AC([0,1])$ we mean that f is absolutely continuous, i.e., $f'(t)$ exists a.e. in $[0,1]$ and $f(t) = f(0) + \int_0^t f'(s)ds$.

Example 3.1 The *standard Wiener space* is the triple (X, H, γ), where

$$\begin{aligned} X &= \{f \in C([0,1];\mathbb{R}) \,:\, f(0) = 0\}, \\ &\|\cdot\|_X = \|\cdot\|_\infty \\ H &= \Big\{f \in AC([0,1]) \cap X : \int_0^1 |f'(t)|^2 dt < \infty\Big\}, \\ &[f,g]_H = \int_0^1 f'(t)g'(t)\,dt. \end{aligned}$$

Then one considers an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of H such that $e_n \in H_0$ for any n, where

$$H_0 := \{h \in H \,:\, h'' \text{ is a measure}\}.$$

For example one can take

$$\begin{aligned} &e_1(t) = t, \\ &e_n(t) = \frac{\sqrt{2}}{(n-1)\pi}\sin(n-1)\pi t \quad n \geq 2. \end{aligned}$$

The measure γ on X is then characterised by the equality

$$\int_X \exp\{-i\langle x, h\rangle\}d\gamma(x) = \exp\Big\{-\frac{1}{2}\|h\|_H^2\Big\}, h \in H_0.$$

This example can also be modified in order that both X and H are Hilbert spaces: it suffices to consider $X = L^2([0,1])$ and $H = W^{1,2}([0,1])$.

3.3. *Gradient and divergence*

Let us now discuss an integration by parts formula that is the equivalent of (1) in the present context. For $h \in X$, define

$$\partial_h f(x) = \lim_{t\to 0}\frac{f(x+th) - f(x)}{t}$$

(whenever the limit exists); we look for an operator ∂_h^* such that for every $f, g \in \mathcal{F}C_b^1(X)$ the equality

$$\int_X g(x)\partial_h f(x)d\gamma(x) = -\int_X f(x)\partial_h^* g(x)d\gamma(x)$$

holds. Starting from the incremental ratio, we get

$$\begin{aligned} \int_X &\frac{f(x+th) - f(x)}{t}g(x)d\gamma(x) = \\ &-\int_X f(y)\frac{g(y) - g(y-th)}{t}d\gamma(y) \\ &+\int_X f(x)g(x)d\mu_t(x) \end{aligned}$$

where μ_t is the measure

$$\mu_t = \frac{1}{t}\Big(\mathscr{N}(0,Q) - \mathscr{N}(-th,Q)\Big).$$

From Cameron-Martin formula (12) we know that $\mu_t \ll \gamma$ if and only if $h \in H$. In this case, we can pass to the limit as $t \to 0$, getting

$$\begin{aligned} \lim_{t\to 0} -\int_X f(y)\frac{g(y) - g(y-th)}{t}d\gamma(y) \\ = -\int_X f(y)\partial_h g(y)d\gamma(y) \\ \lim_{t\to 0}\int_X f(x)g(x)d\mu_t(x) \\ = \int_X f(x)g(x)\hat{h}(x)d\gamma(x). \end{aligned}$$

Therefore, ∂_h^* is well defined if (and only if) $h \in H$, and

$$\partial_h^* g(x) = \partial_h g(x) - g(x)\hat{h}(x),$$

where as usual $h = R\hat{h}$. Let us now define the gradient and the divergence operators. For $f \in \mathcal{F}C_b^1$, the H-gradient of f, denoted by $\nabla_H f$, is the map from X into H defined by

$$[\nabla_H f(x), h]_H = \partial_h f(x), \quad h \in H,$$

where $\partial_h f(x)$ is defined as before. Notice that if $f(x) = f_m(\Pi_m x)$ with $f_m \in C^1(\mathbb{R}^m)$, then

$$\partial_h f(x) = \nabla f_m(\Pi_m x)\cdot \Pi_m h.$$

If we fix an orthonormal basis $(h_j)_{j\in\mathbb{N}}$ of H, we can write

$$\nabla_H f(x) = \sum_{j\in\mathbb{N}} \partial_j f(x) h_j, \qquad \partial_j = \partial_{h_j},$$

where it is important to notice that the directional derivative ∂_h is computed by normalizing h with respect to the norm in H. Considering the space $\mathcal{F}C_b^1(X,H)$, we may define $-\nabla_H^*$, the adjoint operator of ∇_H, as the linear map from $\mathcal{F}C_b^1(X,H)$ to $\mathcal{F}C_b(X)$ such that

$$\begin{aligned}\nabla_H^*\phi(x) &= \sum_{j\in\mathbb{N}}\partial_j^*\phi_j(x)\\ &= \sum_{j\in\mathbb{N}}\partial_j\phi_j(x)-\phi_j(x)\hat{h}_j(x).\end{aligned}$$

Finally, we denote by $C_b^1(X)$ the $\mathbb{R}$-valued functions in $\phi\in C_b(X)$ such that all the derivatives $\partial_h\phi$, $h\in H$, are continuous in X. Analogously, $C_b^1(X,H)$ denotes the H-valued functions whose components are in $C_b^1(X)$.

3.4. *Sobolev spaces and the Ornstein-Uhlenbeck semigroup*

There are several definitions of Sobolev spaces on Wiener spaces. Before giving a definition convenient for our purposes, let us define the L^p spaces. Given a Banach space F, for $p\geq 1$ we denote by $L^p(X,\gamma,F)$ the space of γ-measurable F-valued functions f from X to F such that $\|f\|_F^p$ is γ-summable; in the case $F=\mathbb{R}$ we simply write $L^p(X,\gamma)$. The operator ∇_H is a closable operator in $L^p(X,\gamma)$, hence we may define the Sobolev space $\mathbb{D}^{1,p}(X,\gamma)$ as the domain of the closure of ∇_H in $L^p(X,\gamma)$. Notice that the space denoted by $\mathbb{D}^{1,p}(X,\gamma)$ by Fukushima is denoted $W^{p,1}(X,\gamma)$ in [6]. Anyway, these spaces coincide, see [6, Section 5.2] for all the above statements.

On L^p spaces we may define the Ornstein-Uhlenbeck semigroup $(T_t)_{t\geq 0}$, which is the natural semigroup to be used instead of the heat semigroup. It is defined by Mehler's formula

$$T_tu(x)=\int_X u\left(e^{-t}x+\sqrt{1-e^{-2t}}y\right)d\gamma(y) \qquad (13)$$

for all $u\in L^1(X,\gamma)$, $t>0$. Unlike the heat semigroup in Euclidean spaces, T_tu does not belong to $\mathbb{D}^{1,1}(X,\gamma)$ for all $u\in L^1(X,\gamma)$. But, as in $\mathbb{R}^n$, according to the isoperimetric inequality (4), the embedding $W^{1,1}(\mathbb{R}^n)\hookrightarrow L^p(\mathbb{R}^n)$ holds for $p\leq n(n-1)^{-1}$, also in our case $u\in\mathbb{D}^{1,1}(X,\gamma)$ ensures some more summability on u. The dependence of the classical Sobolev embedding upon the space dimension explains why one is forced to look at something outside the L^p scale. Let us start from the *Gaussian isoperimetric inequality*, see [23]. Let $E\subset X$, and set $B_r=\{x\in H:\|x\|_H<r\}$, $E_r=E+B_r$; then

$$\Phi^{-1}(\gamma(E_r))\geq\Phi^{-1}(\gamma(E))+r, \qquad (14)$$
$$\Phi(t)=\int_{-\infty}^t\frac{e^{-s^2/2}}{\sqrt{2\pi}}ds.$$

We sketch here why this inequality implies the isoperimetric inequality; the complete proof of it will follow from Remark 5.5. We introduce the function

$$\begin{aligned}\mathscr{U}(t)&=(\Phi'\circ\Phi^{-1})(t)\\ &\approx t\sqrt{2\log(1/t)},\ t\to 0;\end{aligned}$$

from (14) we obtain that

$$\begin{aligned}\gamma(E_r)&\geq\Phi(\Phi^{-1}(\gamma(E))+r)\\ &=\gamma(E)+r\Phi'(\Phi(\gamma(E)))+o(r)\\ &=\gamma(E)+r\mathscr{U}(\gamma(E))+o(r),\end{aligned}$$

and then

$$\liminf_{r\to 0}\frac{\gamma(E_r)-\gamma(E)}{r}\geq\mathscr{U}(\gamma(E)).$$

The quantity on the left hand side is related to the Minkowski content of the set E and it is known, in the Euclidean case, to agree with the perimeter only under additional regularity on E. For instance, if $X=\mathbb{R}^m$, $\gamma=G_m$ the standard centred Gaussian measure on $\mathbb{R}^m$ and E a set with smooth boundary, then

$$\begin{aligned}P_{G_m}(E)&=\lim_{r\to 0}\frac{G_m(E_r)-G_m(E)}{r}\\ &\geq\mathscr{U}(G_m(E)).\end{aligned} \qquad (15)$$

It is also possible to prove in this case that equality holds if and only if E is an hyperplane; the first proof of (14) has been given by Sudakov and Tsirel'son [31] using an approximation of Gaussian measure with orthogonal projections of uniform measures on spheres, a second proof was given by Borell [7] using a Brunn-Minkowski inequality and finally Ehrhard [14] proved it using a symmetrization technique.

Remark 3.2 We point out that the right Minkowski content uses enlargements E_r of the set E with respect to balls of H and not of X. The reason of this can be explained as follows; the Gaussian measure γ introduces some anisotropy on X due to the covariance operator Q. This anysotropy is compensated in the definition of total variation and perimeter by the gradient ∇_H, since it is defined using vectors that have unit H-norm. The corresponding compensation in the computation of the Minkowski content is achieved by using the balls of H.

The isoperimetric inequality implies also the following inequality

$$\|\nabla f\|_{L^1} \geq \int_0^\infty \mathscr{U}\left(\gamma(\{|f|>s\})\right) ds,$$

and the embedding of $\mathbb{D}^{1,1}(X,\gamma)$ into the Orlicz space

$$\begin{aligned} L\log^{1/2}L(X,\gamma) &:= \{u : X \to R \text{ measurable} : \\ &\qquad A_{1/2}(|u|) \in L^1(X,\gamma)\}, \\ A_{1/2}(t) &= \int_0^t \log^{1/2}(1+s)ds \end{aligned}$$

follows, see [20, Proposition 3.2]. Let us also introduce the complementary function Ψ of $A_{1/2}$ by the formula

$$\begin{aligned} \Psi(y) &= \int_0^y \left(A'_{1/2}\right)^{-1}(t)dt \\ &= \int_0^y \left(e^{t^2}-1\right)dt \end{aligned}$$

and the space

$$\begin{aligned} L^\Psi(X,\gamma) = \Big\{ & g \text{ measurable} : \Psi(\alpha|g|) \in L^1(X,\gamma) \\ & \text{for some } \alpha > 0 \Big\}. \end{aligned}$$

From the fact that for $x \geq 0$ and $y \geq 0$, $xy \leq A_{1/2}(x) + \Psi(y)$, it is possible to obtain the following properties (for the general theory of Orlicz spaces see for instance [30]):

(i) $L\log^{1/2}L(X,\gamma)$ and $L^\Psi(X,\gamma)$ are Banach spaces under the norms

$$\begin{aligned} \|f\|_{L\log^{1/2}L(X,\gamma)} = \inf\Big\{ & \alpha > 0 : \\ & \int_X A_{1/2}\left(|f|/\alpha\right) d\gamma \leq 1 \Big\}, \end{aligned}$$

$$\|g\|_{L^\Psi} := \inf\left\{\alpha > 0 : \int_X \Psi\left(|g|/\alpha\right) d\gamma \leq 1\right\};$$

(ii) for $f \in L\log^{1/2}L(X,\gamma)$ and $g \in L^\Psi(X,\gamma)$ we have

$$\|fg\|_{L^1} \leq 2\,\|f\|_{L\log^{1/2}L(X,\gamma)}\,\|g\|_{L^\Psi}\,, \tag{16}$$

$$\|fg\|_{L^1} \leq \left(\left\|A_{1/2}(|f|)\right\|_{L^1} + 1\right)\|g\|_{L^\Psi}\,. \tag{17}$$

To see (16), simply use

$$|ab| \leq A_{1/2}(a) + \Psi(b)$$

with

$$a = \frac{|f(x)|}{\|f\|_{L\log^{1/2}L(X,\gamma)}}, \quad b = \frac{|g(x)|}{\|g\|_{L^\Psi}}$$

and integrate. To prove (17), set

$$a = |f(x)|, \qquad b = \frac{|g(x)|}{\|g\|_{L^\Psi}};$$

integrating and taking into account that $\|\Psi(|g|/\|g\|_{L^\Psi})\|_{L^1} = 1$, (17) follows.

(iii) If $(f_h)_{h\in\mathbb{N}}$ converges to f in $L\log^{1/2}L(X,\gamma)$, then

$$\lim_{h\to\infty}\int_X g f_h d\gamma = \int_X g f d\gamma$$

for any $g \in L^\Psi(X,\gamma)$;

(iv) since $g \equiv 1 \in L^\Psi(X,\gamma)$, by (16), we see that $L\log^{1/2}L(X,\gamma)$ is continuously embedded in $L^1(X,\gamma)$;

(v) if $f \in \mathbb{D}^{1,1}(X,\gamma)$, we have that

$$\begin{aligned} \int_X \|\nabla_H f\|_H d\gamma = \sup\Big\{ & \int_X f\nabla_H^*\phi d\gamma : \\ & \phi \in \mathcal{F}C_b^1(X,H), \|\phi(x)\|_H \leq 1\Big\}; \end{aligned} \tag{18}$$

this property essentially follows from the embedding of $\mathbb{D}^{1,1}(X,\gamma)$ in $L\log^{1/2}L(X,\gamma)$; in fact, the embedding and point (iii) ensure that for any $\phi \in \mathcal{F}C_b^1(X,H)$ the following integration by parts formula holds for $f \in \mathbb{D}^{1,1}(X,\gamma)$

$$\begin{aligned} \int_X [\nabla_H f(x), \phi(x)]_H d\gamma(x) = & \\ - \int_X f(x)\nabla_H^*\phi(x)d\gamma(x). & \end{aligned} \tag{19}$$

(vi) It follows from Fernique's theorem that the function $x \mapsto \langle x, x^*\rangle$ belongs to $L^\Psi(X,\gamma)$. As a consequence, by (ii), if f belongs to $L\log^{1/2}L(X,\gamma)$ then the function $f(\cdot)\langle\cdot,x^*\rangle$ is summable.

For our purposes, the following properties of the Ornstein-Uhlenbeck semigroup are relevant: T_t is strongly continuous in $L\log^{1/2}L(X,\gamma)$ and $T_t u \in \mathbb{D}^{1,1}(X,\gamma)$ for any $u \in L\log^{1/2}L(X,\gamma)$ (see [20, Proposition 3.6]). Moreover, it is important that the Ornstein-Uhlenbeck semigroup is a contraction semigroup and the following commutation relation holds for any $u \in \mathbb{D}^{1,1}(X,\gamma)$

$$\nabla_H T_t u = e^{-t} T_t \nabla_H u, \qquad t > 0. \tag{20}$$

Therefore, we get

$$\begin{aligned} \nabla_H T_{t+s} u &= \nabla_H T_t(T_s u) \\ &= e^{-t} T_t \nabla_H T_s u, \end{aligned}$$

for any $u \in L\log^{1/2}L(X,\gamma)$, see [6, Proposition 5.4.8]. It also follows from (20) that

$$\int_X T_t f \nabla_H^* \phi d\gamma = e^{-t}\int_X f \nabla_H^*(T_t\phi) d\gamma, \tag{21}$$

for all $f \in L^1(X,\gamma)$, $\phi \in \mathcal{F}C_b^1(X,H)$. In fact, writing as usual

$$\phi(x) = \sum_{j=1}^n \phi_j(x) h_j,$$

and using the symmetry of T_t and (20), we get

$$\begin{aligned}\int_X T_t f \nabla_H^* \phi \, d\gamma &= \int_X f T_t(\nabla_H^* \phi) d\gamma \\ &= \int_X f \sum_{j=1}^n T_t(\partial_j^* \phi_j) d\gamma \\ &= e^{-t} \int_X f \nabla_H^*(T_t \phi) d\gamma.\end{aligned}$$

Another important consequence of (20) is that if $u \in \mathbb{D}^{1,1}(X,\gamma)$ then

$$\lim_{t\to 0} \|\nabla_H T_t u - \nabla_H u\|_{L^1(X,\gamma)} = 0. \tag{22}$$

Finally, notice that if u^m are the canonical cylindrical approximations of a function $u \in L\log^{1/2}L(X,\gamma)$ defined in (11) then

$$\int_X \|\nabla_H T_t u^m\|_H d\gamma \le \int_X \|\nabla_H T_t u\|_H d\gamma \ \forall\, t > 0. \tag{23}$$

To prove (23), let us first notice that, by the rotational invariance of $\gamma^\perp$,

$$T_t \mathbb{E}_m u = \mathbb{E}_m T_t u.$$

Indeed,

$$\begin{aligned} T_t\mathbb{E}_m u(x) &= \int_X \mathbb{E}_m u(e^{-t}x + \sqrt{1-e^{-2t}}z) d\gamma(z) \\ &= \int_X\int_{X^\perp} u(e^{-t}P_m x + \sqrt{1-e^{-2t}}P_m z + y') \\ &\qquad d\gamma^\perp(y') d\gamma(z)\end{aligned}$$

and

$$\begin{aligned} \mathbb{E}_m T_t u(x) &= \int_{X^\perp} T_t u(P_m x + w') d\gamma^\perp(w') \\ & \int_{X^\perp}\int_X u(e^{-t}(P_m x + w') + \sqrt{1-e^{-2t}}z) \\ &\qquad d\gamma^\perp(w') d\gamma(z) \\ & \int_{X_m}\int_{X^\perp}\int_{X^\perp} u(P_m(e^{-t}x + \sqrt{1-e^{-2t}}z) \\ &\qquad + (e^{-t}w' + \sqrt{1-e^{-2t}}z')) \\ &\qquad d\gamma^\perp(w') d\gamma^\perp(z') d\gamma_m(z_m) \\ &= \int_X\int_{X^\perp} u(e^{-t}P_m x + \sqrt{1-e^{-2t}}P_m z + y') \\ &\qquad d\gamma^\perp(y') d\gamma(z).\end{aligned}$$

From the above commutation relation it follows that the vector $\nabla_H T_t \mathbb{E}_m u = \nabla_H \mathbb{E}_m T_t u$ coincides with its projection ∇_m on H_m, since for any function v the equality

$$\begin{aligned}\nabla_m &\int_{X^\perp} v(x_m, x') d\gamma^\perp(x') \\ &= \int_{X^\perp} \nabla_m v(x_m, x') d\gamma^\perp(x')\end{aligned}$$

holds. Moreover, by Jensen's inequality we have

$$\begin{aligned}\|\nabla_H T_t \mathbb{E}_m u(x)\|_H &= \|\mathbb{E}_m(\nabla_m T_t u)(x)\|_H \\ &= \Big\| \int_{X^\perp} \nabla_m T_t u(P_m x + x') d\gamma^\perp(x')\Big\|_H \\ &\le \int_{X^\perp} \|\nabla_m T_t u(P_m x + x')\|_H d\gamma^\perp(x') \\ &= \mathbb{E}_m \|\nabla_m T_t u(x)\|_H\end{aligned}$$

Summarising, we get

$$\begin{aligned}\int_X &\|\nabla_H T_t u^m\|_H d\gamma \\ &= \int_X \|\mathbb{E}_m(\nabla_m T_t u)(x_m)\|_H d\gamma(x_m, x') \\ &= \int_{X_m} \|\mathbb{E}_m(\nabla_m T_t u)(x_m)\|_H d\gamma_m(x_m) \\ &\le \int_{X_m} \mathbb{E}_m(\|\nabla_m T_t u\|_H)(x_m) d\gamma_m(x_m) \\ &= \int_{X^\perp}\int_{X_m} \|\nabla_m T_t u(x_m, x^\perp)\|_H d\gamma(x_m) d\gamma^\perp(x') \\ &\le \int_X \|\nabla_H T_t u(x)\|_H d\gamma(x).\end{aligned}$$

4. *BV* functions in infinite dimensions

We have collected in the preceding section the tools we need in order to discuss BV functions in the Wiener space setting. The $BV(X,\gamma)$ class can be defined as follows.

Definition 4.1 *Let $u \in L\log^{1/2}L(X,\gamma)$. We say that $u \in BV(X,\gamma)$ if there exists a unique measure $\mu \in \mathscr{M}(X,H)$ such that for any $\phi \in C_b^1(X)$ we have*

$$\int_X u(x)\partial_j^*\phi(x) d\gamma(x) = -\int_X \phi(x) d\mu_j(x) \ \forall j \in \mathbb{N}, \tag{24}$$

where $\mu_j = [\mu, h_j]_H$.

Remark 4.2 If $u \in BV(X,\gamma)$, we denote by $D_H u$ the measure μ and, by (7), its total variation is given by

$$|D_H u|(X) := \sup\Big\{ \int_X u \nabla_H^* \Phi d\gamma;\ \Phi \in \mathcal{F}C_b^1(X,H), \\ \|\Phi(x)\|_H \le 1\Big\} < \infty.$$

Notice that from (vi) in Section 3 it follows that the functions $u\partial_j^* g$, $u\nabla_H^* g$ are γ-summable. The $L\log^{1/2}L(X,\gamma)$ membership hypothesis will be discussed later. Let us see an equivalent way of defining the BV class.

Proposition 4.3 *Let $u \in L\log^{1/2}L(X,\gamma)$; then, $u \in BV(X,\gamma)$ if and only if for every $h \in H$ there is a real measure μ_h such that*

$$\int_X u(x)\partial_h^*\phi(x)d\gamma(x) = -\int_X \phi(x)d\mu_h(x) \qquad (25)$$

for all $\phi \in C_b^1(X)$, with $\bigvee_{\|h\|_H=1} |\mu_h|$ finite.

Proof. If $u \in BV(X,\gamma)$ then the existence of μ_h for all $h \in H$ follows from the linearity of the ∂_h operator with respect to h, and the boundedness of $|\mu_h|$ from the finiteness of $|D_H u|$.

Conversely, define $\mu_j = \mu_{h_j}$ and $\mu = (\mu_j)_{j=1}^\infty$ for an orthonormal basis (h_j) of H. Then, $\mu_h = [\mu, h]$ and (24) holds for all $\phi \in \mathscr{F}C_b^1(X)$. Finally, it can be extended to general $\phi \in C_b^1(X,H)$ by considering its canonical cylindrical approximations ϕ^m and passing to the limit by dominated convergence. $\square$

Before stating the main result, we relate BV functions in $\mathbb{R}^m$ with cylindrical functions in X. We denote by G_m the standard Gaussian distribution on $\mathbb{R}^m$ and define the total variation of a function $v \in L^1(\mathbb{R}^m, G_m)$ by

$$|D_{G_m} v|(\mathbb{R}^m) = \sup\Big\{ \int_{\mathbb{R}^m} v\nabla^*\phi dG_m : \phi \in [C_b^1(\mathbb{R}^m)]^m,\ \|\phi\|_\infty \le 1\Big\}. \qquad (26)$$

Proposition 4.4 *Let $u \in L\log^{1/2}L(X,\gamma)$ be a cylindrical function,*

$$u(x) = v(\Pi_{x_1^*,\dots,x_m^*}x)$$

with $x_j^* \in X^*$ and $v \in BV(\mathbb{R}^m, G_m)$; then $u \in BV(X,\gamma)$ and, if $(R^*x_j^*)_{j=1,\dots,m}$ are orthonormal in $\mathscr{H}$, then

$$|D_H u|(X) = |D_{G_m} v|(\mathbb{R}^m).$$

Proof. Let us assume that $x_j^* \in X^*$ are selected in such a way that $e_j := \Pi x_j^*$ are orthonormal in $\mathbb{R}^m$; then, if we denote by Π the map $\Pi_{x_1^*,\dots,x_m^*} : X \to \mathbb{R}^m$ and by Π^* its adjoint, we have that for any $\xi \in \mathbb{R}^m$

$$\begin{aligned}\langle \Pi Q\Pi^*\xi, e_i\rangle &= \langle \Pi RR^*(\xi_1 x_1^* + \dots + \xi_m x_m^*), e_i\rangle \\ &= \langle RR^*(\xi_1 x_1^* + \dots + \xi_m x_m^*), x_i^*\rangle \\ &= [R^*(\xi_1 x_1^* + \dots + \xi_m x_m^*), R^*x_i]_{\mathscr{H}} \\ &= \xi_i,\end{aligned}$$

that is $\Pi Q\Pi^* = I_m$. Then the push-forward measure $\Pi_\#\gamma$ coincides with G_m. This implies that, setting $H_m = \mathrm{span}\{Qx_1^*,\dots,Qx_m^*\}$ and $y = \Pi x$,

$$\begin{aligned}|D_H u|(X) &= \sup\Big\{ \int_X u(x)\nabla^*\phi(x)d\gamma(x) : \phi \in \mathcal{F}C_b^1(X,H),\ \|\phi\|_H \le 1\Big\} \\ &= \sup\Big\{ \int_X u(x)\nabla^*\phi(x)d\gamma(x) : \phi \in \mathcal{F}C_b^1(X,H_m),\ \|\phi\|_H \le 1\Big\} \\ &= \sup\Big\{ \int_{\mathbb{R}^m} v(y)\nabla^*\psi(y)dG_m(y) : \psi \in [C_b^1(\mathbb{R}^m)]^m,\ \|\psi\|_\infty \le 1\Big\},\end{aligned}$$

and the thesis follows. $\square$

We are now in a position to prove the analogue of Theorem 2.1 in the present context.

Theorem 4.5 *Given $u \in L\log^{1/2}L(X,\gamma)$, the following are equivalent:*

1 *u belongs to $BV(X,\gamma)$;*

2 *the following holds*

$$V_H(u) := \sup\Big\{ \int_X u\nabla_H^*\phi d\gamma;\ \phi \in \mathcal{F}C_b^1(X,H),\ \|\phi(x)\|_H \le 1\Big\} < \infty;$$

3 *the following holds*

$$L_H(u) := \inf\Big\{ \liminf_{n\to\infty} \int_X \|\nabla_H u_n\|_H d\gamma : u_n \in \mathbb{D}^{1,1},\ u_n \xrightarrow{L^1} u\Big\} < \infty;$$

4 *if $(T_t)_{t\ge0}$ denotes the Ornstein-Uhlenbeck semigroup in X, then*

$$\mathcal{I}[u] := \lim_{t\to0} \int_X \|\nabla T_t u\|_H d\gamma < \infty.$$

Moreover, $|\mu|(X) = V_H(u) = L_H(u) = \mathcal{I}[u]$, and if one (and then all) of the previous holds true, then we denote by $|D_H u|(X)$ their common value.

Proof. As we have pointed out, the tools we use in this proof are different from those used in the finite dimensional case of Theorem 2.1. For this reason, it is convenient to prove the implications in a different order.

4 ⇒ 1 First of all, let us show that the limit in condition **4** always exists. Let $u \in L\log^{1/2}L(X,\gamma)$ and fix a time $t > 0$. Consider the map

$$s \longmapsto \int_X \|\nabla_H T_{t+s}u\|_H\, d\gamma = e^{-t}\int_X \|\nabla_H T_s u\|_H\, d\gamma.$$

Observe that

$$\int_X \|\nabla_H T_t u\|_H \, d\gamma \le \liminf_{s\to 0} \int_X \|\nabla_H T_{t+s} u\|_H \, d\gamma$$
$$= e^{-t} \liminf_{s\to 0} \int_X \|\nabla_H T_s u\|_H \, d\gamma$$
$$\le \liminf_{s\to 0} \int_X \|\nabla_H T_s u\|_H \, d\gamma,$$

and then

$$\limsup_{t\to 0} \int_X \|\nabla_H T_t u\|_H \, d\gamma \le \liminf_{t\to 0} \int_X \|\nabla_H T_t u\|_H \, d\gamma$$

which means that $\lim_{t\to 0}\int_X \|\nabla_H T_t u\|_H \, d\gamma$ exists. Furthermore the above chain of inequalities gives

$$e^t \int_X \|\nabla_H T_t u\|_H \, d\gamma \le \lim_{s\to 0} \int_X \|\nabla_H T_s u\|_H \, d\gamma = \mathcal{I}[u].$$

We now divide the rest of the proof in three steps.
Step 1. (Finite dimensional case) Let $\gamma = \mathscr{N}(0, Q)$ be a Gaussian measure in $\mathbb{R}^m$, and T_t the Ornstein-Uhlenbeck semigroup defined as in (13). Let h be a unit eigenvector of the matrix Q, let K be the hyperplane orthogonal to h, and consider the factorisation $\gamma = \gamma_1 \otimes \gamma^\perp$. Denote points in $\mathbb{R}^m$ as $x + th$, with $x \in K$ and $t \in \mathbb{R}$. If $u \in L\log^{1/2}L(\mathbb{R}^m, \gamma)$ and

$$\mathcal{I}[u] := \lim_{t\to 0} \int_{\mathbb{R}^m} |\nabla T_t u| d\gamma < \infty,$$

then also

$$\mathcal{I}_h[u] := \lim_{t\to 0} \int_{\mathbb{R}^m} |\partial_h T_t u| d\gamma < \infty,$$

hence it can be proved essentially as in the Euclidean case, when the reference measure is the Lebesgue measure (see also [4, Section 3.11]) that the directional distributional derivative $\partial_h u$ and

$$|\partial_h u|(\mathbb{R}^m) = \int_K |D_{G_1} u_x|(\mathbb{R}) d\gamma^\perp(x) = \mathcal{I}_h[u]$$
$$\partial_h u(A) = \int_K D_{G_1} u_x(A_x) d\gamma^\perp(x),$$

where $u_x(t) = u(x + th)$, $A_x = \{t \in \mathbb{R} : \ x + th \in A\}$ and $D_{G_1} u_x$ is the distributional derivative of u_x defined as in Proposition 4.4. Moreover,

$$\int_X u(x) \partial_h^* g(x) d\gamma(x) = -\int_X g(x) d\mu_h(x)$$

and

$$\Big| \int_{\mathbb{R}^m} u \partial_h^* \phi d\gamma \Big| \le \mathcal{I}[u] \|\phi\|_\infty, \ \phi \in C_b^1(\mathbb{R}^m). \quad (27)$$

By the linearity of the differential, all the directional derivatives of u are measures, and u belongs to $BV(\mathbb{R}^m, \gamma)$.
Step 2. (Cylindrical functions) Let u be a cylindrical function in $L\log^{1/2}L(X, \gamma)$ such that **4** holds. By the previous step and Proposition 4.4, for every $h \in H$ there is a measure $\tilde{\mu}_h$ defined on the algebra of cylindrical Borel sets such that

$$\int_X u(x) \partial_h^* g(x) d\gamma(x) = -\int_X g(x) d\tilde{\mu}_h(x)$$

for all $g \in \mathcal{F}C_b^1(X)$.
Step 3. (General case) Let $u \in L\log^{1/2}L(X, \gamma)$ be such that **4** holds, and let u^m be the canonical cylindrical approximations given by (11). Fixing $\hat{h} \in X^*$, $h = R\hat{h} \in H$, and using basically the same notation as in Step 1, define

$$\Theta_u(X) = \int_K |D_{G_1} u_x|(\mathbb{R}) d\gamma^\perp(x),$$

where $|D_{G_1} u_x|(\mathbb{R})$ is defined in (26) with $m = 1$. Notice that for h fixed and u_x as above the functional $u \mapsto |D_{G_1} u_x|(\mathbb{R})$ is L^1-lower semicontinuous, because it is the supremum of L^1-continuous functionals. Let us show that $\Theta_u(X)$ is L^1-lower semicontinuous as well. Indeed, let $u_k \to u$ in $L^1(X, \gamma)$, and assume thet the limit of $\Theta_{u_k}(X)$ exists and is finite. Possibly passing to a subsequence, assume also that

$$\sum_{k=1}^\infty \|u_k - u\|_{L^1(X,\gamma)} < \infty. \quad (28)$$

From (28) and Fubini's theorem it follows that

$$\sum_{k=1}^\infty \|(u_k)_x - u_x\|_{L^1(\mathbb{R}, G_1)} < \infty$$

for $\gamma^\perp$-a.e. $x \in K$ too, whence $(u_k)_x \to u_x$ for $\gamma^\perp$-a.e. $x \in K$. Therefore, $|D_{G_1} u_x|(\mathbb{R}) \le \liminf_k |D_{G_1} (u_k)_x|(\mathbb{R})$ $\gamma^\perp$-a.e. and the lower semicontinuity of $\Theta_u(X)$ follows from Fatou's lemma. Summarising, we have

$$\Theta_u(X) \le \liminf_{m\to\infty} \Theta_{u^m}(X)$$
$$\le \liminf_{m\to\infty} \lim_{t\to 0} \int_X |\partial_h T_t u^m| d\gamma$$
$$\le \lim_{t\to 0} \int_X |\partial_h T_t u| d\gamma \le \mathcal{I}[u] < \infty,$$

by the monotonicity of the norm of the derivatives with respect to cylindrical projections. It follows that for $\gamma^\perp$-a.e. $x \in K$ the function u_x has bounded variation. By a Fubini argument, based on the identity $\gamma = \gamma_1 \otimes \gamma^\perp$, the 1-dimensional integration by parts formula yields that the measure

$$A \mapsto \int_K Du_x(A_x) d\gamma^\perp(x)$$

provides the distributional derivative $\partial_h u$ and extends $\tilde{\mu}_h$ from the cylindrical sets to the whole Borel σ-algebra. It follows from (27) that $|\partial_h u|(X) \le \mathcal{I}[u]$ for all $h \in H$, $\|h\|_H = 1$, whence

$$\bigvee_{\|h\|_H=1} |\mu_h|(X) \le \mathcal{I}[u],$$

and **1** follows from Proposition 4.3.

1 $\Rightarrow$ **2** Simply comparing the classes of competitors, we notice that $V_H(u) \le |D_H u|(X)$.

2 $\Rightarrow$ **3** Let $t_n \to 0$ and $u_n = T_{t_n} u$. Then, for all $\phi \in \mathcal{F}C_b^1(X,H)$ with $\|\phi\|_H \le 1$, from (21) we deduce

$$\begin{aligned}\int_X [\nabla_H u_n, \phi]_H d\gamma &= -\int_X T_{t_n} u \nabla^* \phi d\gamma \\ &= -e^{-t_n} \int_X u \nabla^*(T_{t_n}\phi) d\gamma \\ &\le V_H(u).\end{aligned}$$

Therefore, $\|\nabla_H u_n\|_{L^1(X,\gamma)} \le V_H(u)$. In particular, we have proved that $L_H(u) \le V_H(u)$.

3 $\Rightarrow$ **4** Let $(u_n)_{n\in\mathbb{N}}$ be such that $\|\nabla_H u_n\|_{L^1(X)} \to L_H(u)$. Then,

$$\begin{aligned}\int_X \|\nabla_H T_t u\|_H d\gamma &\le \liminf_{n\to\infty} \int_X \|\nabla_H T_t u_n\|_H d\gamma \\ &= e^{-t} \liminf_{n\to\infty} \int_X \|\nabla_H u_n\|_H d\gamma \\ &\le L_H(u).\end{aligned}$$

Observe that in particular we have proved that $\mathcal{I}[u,X] \le L_H(u)$.

□

As we have noticed, the hypothesis that u belongs to $L\log^{1/2}L(X,\gamma)$ gives a meaning to **1**, **4**. On the other hand, if $|D_H u|$ is finite, membership of u in $L\log^{1/2}L(X,\gamma)$ follows from the isoperimetric inequality as in the Sobolev case.

As a particular case, if $E \subset X$, $u = \chi_E$ and $|D_H \chi_E|(X)$ is finite, we say that E is a set of finite perimeter and set $P_\gamma(E) = |D_H \chi_E|(X)$. If $u \in BV(X,\gamma)$, according to general measure theory, a $|D_H u|$-measurable unit vector field $\sigma : X \to H$ exists such that the polar decomposition $D_H u = \sigma |D_H u|$ holds. Accordingly, we use the following notation:

$$\int_X [\phi, D_H u]_H = \int_X [\phi, \sigma]_H d|D_H u| \qquad (29)$$

for every $\phi \in C_b(X,H)$.

5. Further properties and open problems

In this section we describe a few properties of BV functions. Further results will be presented in a forthcoming paper. Thinking of $P_\gamma(E)$ as a measure on X, we denote by $P_\gamma(E,B)$ the $P_\gamma(E)$-measure of the Borel set B.

Corollary 5.1 *Let us fix $x^* \in X^*$ and $c \in \mathbb{R}$; then the sets $E = \{x \in X : \langle x, x^*\rangle \le c\}$ have finite perimeter with*

$$P_\gamma(E) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{c^2}{2\|Qx^*\|_H^2}\right\}.$$

Proof. Without loss of generality, we may assume that $\|Qx^*\|_H = 1$. The assertion simply follows by noticing that $\chi_E(x) = v(\langle x, x^*\rangle)$. Then the set E is a cylindrical set of the form

$$E = \Big\{x \in X : \langle x, x^*\rangle \in B\Big\},$$

with $B = \{s \in \mathbb{R} : s \le c\}$. This implies that

$$P_\gamma(E) = P_{G_1}(B, \mathbb{R}) = \frac{e^{-c^2/2}}{\sqrt{2\pi}}. \qquad \square$$

More generally, one can consider level sets of Lipschitz functions. There are two natural classes of Lipschitz continuous functions $F : X \to \mathbb{R}$, obtained by requiring a control of the incremental ratio with respect to the X or the H norms. Since, as remarked before, directional derivatives are computed normalizing with respect to the H-norm, in this context H-Lipschitz functions are more natural.

Definition 5.2 *A function $f : X \to \mathbb{R}$ is said to be H-Lipschitz continuous if it is $(\mathscr{B}(X), \mathscr{B}(\mathbb{R}))$-measurable and there exists a constant C such that for γ-a.e. x one has*

$$|f(x+h) - f(x)| \le C\|h\|_H, \quad \forall h \in H. \qquad (30)$$

It can be proved that for a H-Lipschitz function f there exists a full-measure set X_0 such that $X_0 + H = X_0$ and, for every $x \in X_0$, one has

$$|f(x+h) - f(x+k)| \le C\|h-k\|_H, \quad \forall h,k \in H.$$

In particular, f has a version such that the previous inequality is satisfied for every $x \in X$. Of course, by Fernique theorem any H-Lipschitz function f is γ

summable and, denoting by C its Lipschitz constant, for all $r > 0$ one has

$$\gamma\left(x : \left|f(x) - \int f d\gamma\right| > r\right) \le 2\exp\left(-\frac{r^2}{2C^2}\right)$$

(see [6, Section 5.11]). Moreover, a H-Lipschitz function f is in $\mathbb{D}^{1,p}(X,\gamma)$ for every $p \ge 1$, and in particular $f \in BV(X,\gamma)$.

An important result is the following coarea formula, which can be proved by following *verbatim* the proof of [15, Section 5.5].

Theorem 5.3 *If $u \in BV(X,\gamma)$, then for every Borel set $B \subset X$ the following equality holds:*

$$|D_H u|(B) = \int_{\mathbb{R}} P_\gamma(\{u > t\}, B)dt. \qquad (31)$$

The following approximation result for sets of finite perimeter is a consequence of the approximation in BV through smooth functions and the coarea formula.

Proposition 5.4 *Let $E \subset X$ be a set with finite perimeter; then there exists a sequence E_j of cylindrical sets $E_j = \Pi_{m_j}^{-1} B_j$, with $B_j \in \mathbb{R}^{m_j}$ smooth sets, such that*

$$\lim_{j\to\infty} \|\chi_{E_j} - \chi_E\|_{L^1(X,\gamma)} = 0$$

and

$$\lim_{j\to\infty} P_\gamma(E_j) = P_\gamma(E).$$

Proof. The density of $\mathbb{D}^{1,1}(X,\gamma)$ in variation in $BV(X,\gamma)$ and the density of smooth cylindrical functions in $\mathbb{D}^{1,1}(X,\gamma)$ imply the existence of a sequence (u_j) of smooth cylindrical functions with

$$u_j \to \chi_E \text{ in } L^1(X,\gamma),$$
$$\int_X \|\nabla_H u_j\|_H d\gamma \to P_\gamma(E).$$

The conclusion then follows from the coarea formula by taking smooth levels B_j of u_j. □

Due to the previous proposition, we say that E is a *smooth set* if $E = \Pi_m^{-1} B$ for some set $B \in \mathbb{R}^m$ with smooth boundary. As a consequence, we have that

$$P_\gamma(E) = \inf\left\{\liminf_{j\to+\infty} \int_{\partial B_j} G_{m_j}\, d\mathcal{H}^{m_j-1}\right\}$$

where the infimum is taken over all the sequences of smooth sets $B_j \subset \mathbb{R}^{m_j}$ such that $\Pi_{m_j}^{-1} B_j$ converges to E in $L^1(X,\gamma)$ and $\mathcal{H}^{m_j-1}$ is the (m_j-1)-dimensional Hausdorff measure, see [7].

Remark 5.5 As a corollary of Proposition 5.4 we can extend the isoperimetric inequality to all sets with finite perimeter; in fact, we can consider a sequence of smooth set $(E_j)_j$ converging to E in $L^1(X,\gamma)$ such that $P_\gamma(E_j) \to P_\gamma(E)$. For any $j \in \mathbb{N}$ we have $E_j = \Pi_{m_j}^{-1} B_j$ and then $\gamma(E_j) = G_{m_j}(B_j)$ and $P_\gamma(E_j) = P_{G_{m_j}}(B_j)$. Equation (15) then implies that $P_\gamma(E_j) \ge \mathscr{U}(\gamma(E_j))$ with equality if E_j is a hyperplane. We can choose a direction x^* and define

$$H_j = \{x \in X : \langle x, x^*\rangle \le \|Qx^*\|_H \Phi^{-1}(\gamma(E_j))\}$$

where Φ is the function defined in (14); for such hyperplane we have that $\gamma(H_j) = \gamma(E_j)$ and $P_\gamma(E_j) \ge P_\gamma(H_j)$. Taking the limit as $j \to +\infty$, we have that H_j converges in $L^1(X,\gamma)$ to the hyperplane

$$H = \{x \in X : \langle x, x^*\rangle \le \|Qx^*\|_H \Phi^{-1}(\gamma(E))\}$$

with the property that $\gamma(H) = \gamma(E)$ and

$$P_\gamma(E) \ge P_\gamma(H).$$

We point out that the choice of a direction is completely arbitrary; we also point out that the previous argument proves uniqueness of isoperimetric set in the class of cylindrical sets with finite perimeter. It is not clear to us if uniqueness still holds in the class of all sets with finite perimeter.

We recall that the Sobolev classes $\mathbb{D}^{1,p}(X,\gamma)$ are stable under composition with Lipschitz continuous functions, and $\nabla_H(f\circ u) = f'(u)\nabla_H u$ for every Lipschitz continuous function $f : \mathbb{R} \to \mathbb{R}$. The same holds for BV functions. The following is a quite rough result, when compared with the finite dimensional case. We shall add further comments later.

Proposition 5.6 *For any function $u \in BV(X,\gamma)$ and for any Lipschitz continuous function $f : \mathbb{R} \to \mathbb{R}$ we have $f\circ u \in BV(X,\gamma)$ and $|D_H(f\circ u)|(X) \le L|D_H u|(X)$, where L is the Lipschitz constant of f.*

Proof. Let $u \in BV(X,\gamma)$, and let $(u_n) \subset \mathbb{D}^{1,1}$ be a sequence such that $u_n \to u$ in L^1 and

$$\int_X \|\nabla_H u_n\|_H d\gamma \to |D_H u|(X).$$

Then, as f is a Lipschitz function, $f(u_n) \to f(u)$ in $L^1(X,\gamma)$ and

$$\begin{aligned}\limsup_{n\to\infty} \int_X \|\nabla_H(f(u_n))\|_H d\gamma &= \limsup_{n\to\infty} \int_X |f'(u_n)|\|\nabla_H u_n\|_H d\gamma \\ &\le L \lim_{n\to\infty} \int_X \|\nabla_H u_n\|_H d\gamma \\ &= L|D_H u|(X) < \infty. \qquad \square\end{aligned}$$

Let us now investigate the properties of sections of functions with bounded variation, extending the construction seen in the proof of **4** $\Rightarrow$ **1** in Theorem 4.5. Given the reproducing kernel $\mathscr{H}$, we can consider a decomposition of it given by $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$. This can be obtained, for instance, by fixing an orthonormal system $\{\hat{h}_j\}_{j\in\mathbb{N}}$ of $\mathscr{H}$, and by choosing a finite subset $I \subset \mathbb{N}$ with $\mathscr{H}_1 = \operatorname{span}(\hat{h}_j : j \in I)$ and $\mathscr{H}_2$ its orthogonal complement. Then $X_1 = R(\mathscr{H}_1)$ and $X_2 = \overline{R(\mathscr{H}_2)}$ (closure in X) are such that $X = X_1 \oplus X_2 \sim X_1 \times X_2$ and $\gamma = \gamma_1 \otimes \gamma_2$, where $\gamma_i(B) = \gamma(B \times X_j)$ for $B \subset X_i,\ i \neq j$.

Given a function $u : X \to \mathbb{R}$ we define, for every fixed $x_2 \in X_2$, the function $u_{x_2} : X_1 \to \mathbb{R}$ as $u_{x_2}(x_1) = u(x_1, x_2)$. We also define the gradient ∇_{H_1} in the obvious way and the divergence $\nabla^*_{H_1}$ as the adjoint of ∇_{H_1}. Given a regular function $u : X \to \mathbb{R}$, the operator ∇_{H_1} is nothing but the finite dimensional gradient of the regular function u_{x_2}. After defining:

$$|D_{H_1}u|_\gamma(X) = \sup\Big\{ \int_X u(x)\nabla^*_{H_1}\phi(x)d\gamma(x) : \phi \in \mathcal{F}C^1_b(X, H_1), \|\phi(x)\|_{H_1} \leq 1\Big\},$$

we can then state and prove the following result.

Proposition 5.7 *Let $u \in L\log^{1/2}L(X,\gamma)$; then*

$$|D_{H_1}u|(X) = \int_{X_2} |D_{H_1}u_{x_2}|(X_1)d\gamma_2(x_2).$$

Proof. If we fix

$$\phi \in \mathcal{F}C^1_b(X, H_1)$$

with

$$\|\phi(x)\|_{H_1} \leq 1,$$

we may write

$$\begin{aligned}&\int_X u(x)\nabla^*_{H_1}\phi(x)d\gamma(x)\\ &= \int_{X_2} d\gamma(x_2)\int_{X_1} u_{x_2}(x_1)\nabla^*_{H_1}\phi_{x_2}(x_1)d\gamma_1(x_1)\\ &\leq \int_{X_2} |D_{H_1}u_{x_2}|(X_1)d\gamma_2(x_2),\end{aligned}$$

whence

$$|D_{H_1}u|(X) \leq \int_{X_2} |D_{H_1}u_{x_2}|(X_1)d\gamma_2(x_2).$$

For the reverse inequality, we may assume that $|D_{H_1}u|(X) < \infty$; Theorem 4.5 can be restated by saying that there exists a measure $\mu_{H_1} = D_{H_1}u \in \mathscr{M}(X, H_1)$ such that for all $\phi \in \mathcal{F}C^1_b(X, H_1)$

$$\int_X u\nabla^*_{H_1}\phi d\gamma = -\int_X [\phi, dD_{H_1}u]_{H_1}$$

and

$$\lim_{t\to 0}\int_X \|\nabla_{H_1}T_t u(x)\|_{H_1}d\gamma(x) = |D_{H_1}u|(X)$$

hold. We also notice that

$$|D_{H_1}T_t u|(X) = \int_X \|\nabla_{H_1}T_t u(x)\|_{H_1}d\gamma(x).$$

Since

$$\begin{aligned}&\lim_{t\to 0}\int_{X_2} \|T_t u_{x_2} - u_{x_2}\|_{L^1(X_1,\gamma_1)}d\gamma_2(x_2)\\ &= \lim_{t\to 0}\int_X |T_t u(x) - u(x)|d\gamma(x) = 0,\end{aligned}$$

there exists $t_n \to 0$ such that

$$\lim_{n\to\infty}\int_{X_1} |(T_{t_n}u)_{x_2} - u_{x_2}|d\gamma_1(x_1) = 0$$

for γ_2-a.e. $x_2 \in X_2$. By lower semicontinuity we get

$$\begin{aligned}&\int_{X_2} |D_{H_1}u_{x_2}|(X_1)d\gamma_2(x_2)\\ &\leq \int_{X_2}\liminf_{n\to\infty}\int_{X_1} \|\nabla_{H_1}T_{t_n}u_{x_2}(x_1)\|_{H_1}d\gamma_1(x_1)d\gamma_2(x_2)\\ &\leq \liminf_{n\to\infty}\int_{X_2}\int_{X_1} \|\nabla_{H_1}T_{t_n}u_{x_2}(x_1)\|_{H_1}d\gamma_1(x_1)d\gamma_2(x_2)\\ &= \lim_{n\to\infty}\int_X \|\nabla_{H_1}T_{t_n}u(x)\|_{H_1}d\gamma(x) = |D_{H_1}u|(X).\end{aligned}$$

□

The finite dimensional slicing can be used to give an equivalent characterization of functions with bounded variation.

Proposition 5.8 *For every $u \in L^1(X,\gamma)$, the following equality holds:*

$$|D_H u|(X) = \sup_K\Big\{|D_K u|(X)\Big\}, \qquad (32)$$

where the supremum is taken over all the finite dimensional subspaces K of H. In particular, $u \in BV(X,\gamma)$ if and only if $\sup_K |D_K u|(X) < +\infty$.

Proof. The inequality $\geq$ is obvious, hence set $M = \sup|D_K u|(X)$, K as above, assume $M < \infty$ and prove the converse. For every $\phi \in \mathcal{F}C^1_b(X, H)$ there is a finite dimensional subspace K of H such that $\phi(x) \in K$ for every $x \in X$, and then

$$\int_X u\nabla^*_H\phi d\gamma = \int_X u\nabla^*_K\phi d\gamma \leq M,$$

and by the arbitrariness of ϕ, $|D_H u|(X) \leq M$ follows, and the proof is complete. □

Remark 5.9 It is easily seen that if we fix an orthonormal basis (h_j) of H, and denote by H_n the span of $\{h_1, \ldots, h_n\}$, then a simplified form of (32) holds, namely

$$|D_H u|(X) = \sup_n \Big\{ |D_{H_n} u|(X) \Big\}.$$

Indeed, the inequality $\geq$ is obvious from Proposition 5.8. Concerning the opposite inequality, we may assume $|D_H u|(X) < \infty$ (hence $u \in BV(X, \gamma)$), and for $\varepsilon > 0$ fixed we may find $\phi \in \mathcal{F}C_b(X, H)$, $\|\phi\|_H \leq 1$, such that

$$\int_X [\phi, D_H u]_H \geq |D_H u|(X) - \varepsilon.$$

Therefore, to conclude it suffices to approximate uniformly the function ϕ by functions $\phi_n \in \mathcal{F}C_b(X, H_n)$.

Open problems We have presented in this note only a translation of the possible definition of total variation of a function into the context of a Wiener space, and a few simple analytical properties. Having in mind how rich is the theory of sets of finite perimeter and BV functions in the finite dimensional case of $\mathbb{R}^n$, it is easy to raise many natural questions. As mentioned in Section 2, BV functions have proved to be very useful when dealing with variational problems where either the energy functional to be minimised has a linear growth with respect to the norm of the gradient, or the competitors are expected to exhibit discontinuity surfaces. In fact, in these cases Sobolev classes are not suitable: they lack good compactness properties of the sublevels of functionals with linear growth, and do not allow for discontinuities along hypersurfaces (i.e., manifolds of codimension 1).

Concerning the first point, consider a functional as in (3): this expression makes sense (under some mild regularity assumptions on f) if, e.g., $u \in \mathbb{D}^{1,1}(X, \gamma)$, but for general $u \in BV(X, \gamma)$ a suitable as explicit as possible formula should be found for the relaxed functional

$$\overline{F}(u) := \inf \Big\{ \liminf_{n \to \infty} \int_X f(x, u_n, \nabla_H u_n) d\gamma : \quad u_n \in \mathbb{D}^{1,1},\ u_n \xrightarrow{L^1} u \Big\}. \tag{33}$$

A related issue concerns compactness properties of sequences of BV functions. On one hand, the finiteness of the measure γ suggests that bounded sets could be relatively compact, but on the other hand this is known to be false without further assumptions in the Sobolev space $\mathbb{D}^{1,2}(X, \gamma)$, at least when using the definition recalled here (a different definition of Sobolev spaces is given in [9]).

Before adding further comments on the relaxation problem, let us discuss some related analytical and geometrical features. First, it is clear from general results in measure theory that for $u \in BV(X, \gamma)$ the measure $D_H u$ has a Lebesgue decomposition as $D_H u = U d\gamma + D_H^s u$, where we denote by $U \in L^1(X, \gamma, H)$ the density of the absolutely continuous part and by $D_H^s u$ the singular part. A better understanding of the pointwise values of U and the structure of $D_H^s u$ is still missing.

For instance, in the important particular case of characteristic functions $u = \chi_E$, the whole of $D_H u$ is singular, and in finite dimensions it is known that it is concentrated on the *reduced boundary* of E (the relevant part of the boundary from the measure theoretic point of view, which turns out to have some mild regularity properties) and is absolutely continuous with respect to the codimension 1 Hausdorff measure $\mathcal{H}^{n-1}$, the density being the *approximate normal*. It is natural to ask if these properties can be rephrased in the Wiener setting, explicitly, to look for a good definition of *rectifiability*, approximate normal and reduced boundary of a set, and to compare $D\chi_E$ with the existing notions of surface measures, such as those in [1], which relies on regular parametrisations of the embedded manifolds, or in [17], where a Carathéodory type construction based on coverings, closer to the (spherical) Hausdorff measure $\mathcal{H}^{n-1}$, is studied.

Let us come to general BV functions. In this case, in $\mathbb{R}^n$ it is known that the density U can be recovered as the *approximate differential* of u. It is also known that $D^s u$ further decomposes into a *jump part*, concentrated on a rectifiable set J_u sharing the same geometric properties as the reduced boundary, and a *Cantor part* lying on a set of intermediate dimension, and that the jump part is absolutely continuous with respect to the (restriction to J_u of the) Hausdorff measure $\mathcal{H}^{n-1}$, the density being the *jump* of u, i.e., the difference between two one-sided *approximate limits* $u^+,\ u^-$. Apart from its intrinsic interest, this decomposition is important in order to go further in the analysis of the functional $\overline{F}$ in (33): if f is independent of u then an integral representation of $\overline{F}(u)$ could likely be achieved by using only the Lebesgue decomposition of $D_H u$, but if f depends explicitly upon u then a more detailed analysis of the gradient is probably necessary, as it has been the case in $\mathbb{R}^n$.

Further issues could come from concrete problems: are there functionals for which variational problems are naturally well-posed on $BV(X, \gamma)$? Is it possible to extend the results obtained in [3] for Sobolev fields to the BV case? In the finite dimensional case, see [2], deep properties of BV have ben used.

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