DIRECT EPIPERIMETRIC INEQUALITIES FOR THE THIN OBSTACLE PROBLEM AND APPLICATIONS

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ABSTRACT. We introduce a new logarithmic epiperimetric inequality for the $2m$-Weiss' energy in any dimension and we recover with a simple direct approach the usual epiperimetric inequality for the $3/2$-Weiss' energy. In particular, even in the latter case, at difference from the classical statements, we do not assume any a-priori closeness to a special class of homogeneous function. In dimension 2, we also prove for the first time the classical epiperimetric inequality for the $(2m - 1/2)$-Weiss' energy, thus covering all the admissible energies.

As a first application, we classify the global λ-homogeneous minimizers of the thin obstacle problem, with $\lambda \in [3/2, 2 + c] \cup \bigcup_{m\in\mathbb{N}}(2m - c_m^-, 2m + c_m^+)$, showing as a consequence that the frequencies $3/2$ and $2m$ are isolated and thus improving on the previously known results. Moreover, we give an example of a new family of $(2m - 1/2)$-homogeneous minimizers in dimension higher than 2.

Secondly, we give a short and self-contained proof of the regularity of the free boundary of the thin obstacle problem, previously obtained by Athanasopoulos-Caffarelli-Salsa [2] for regular points and Garofalo-Petrosyan [11] for singular points. In particular we improve the C^1 regularity of the singular set with frequency $2m$ by an explicit logarithmic modulus of continuity.

Keywords: epiperimetric inequality, logarithmic epiperimetric inequality, monotonicity formula, thin obstacle problem, free boundary, singular points, frequency function

1. Introduction

In this paper we study the regular and singular parts of the free-boundary for solutions of the *thin-obstacle problem*, that is the minimizers of the Dirichlet energy

$$\mathcal{E}(u) := \int_{B_1} |\nabla u|^2 \, dx$$

in the class of admissible functions

$$\mathcal{A} := \left\{ u \in H^1(B_1) \, : \, u \geq 0 \text{ on } B_1', \; u(x', x_d) = u(x', -x_d) \text{ for every } (x', x_d) \in B_1 \right\},$$

with Dirichlet boundary conditions $u = w$ on ∂B_1. Here and in the rest of the paper $d \geq 2$ is the dimension of the space, $B_1 \subset \mathbb{R}^d$ denotes the unit ball, and $B_1' := B_1 \cap \{x_d = 0\}$; for any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we denote by x' the vector of the first $(d-1)$ coordinates, $x' = (x_1, \ldots, x_{d-1})$, and $w \in \mathcal{A}$ is a given boundary datum.
Given a minimizer $u \in \mathcal{A}$ of $\mathcal{E}$ with Dirichlet boundary conditions, the *coincidence set* $\Delta(u) \subset B_1'$ is defined as $\Delta(u) := \{(x', 0) \in B_1' \, : \, u(x', 0) = 0\}$ and the *free boundary* $\Gamma(u)$ *of* u is the topological boundary of the coincidence set in the relative topology of B_1'.

1.1. **State of the art.** Athanasopoulos and Caffarelli [1] proved that the optimal regularity of any local minimizer u is $C^{1,1/2}(B_1^+)$. Athanasopoulos, Caffarelli and Salsa pioneered the study of the regularity of the free boundary $\Gamma(u)$ in [2]. They showed in [2, Lemma 1] that for every $x_0 \in \Gamma(u)$ the *Almgren's frequency function*

$$(0, 1 - |x_0|) \ni r \mapsto N^{x_0}(r, u) := \frac{r \int_{B_r(x_0)} |\nabla u|^2 \, dx}{\int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{d-1}}$$

is monotone nondecreasing in r. Thus, the limit

$$N^{x_0}(0, u) := \lim_{r \to 0} N^{x_0}(r, u)$$

exists for every point $x_0 \in \Gamma(u)$ and the free boundary can be decomposed according to the value of the frequency function in zero. We denote the set of points of frequency $\lambda \in \mathbb{R}$ by

$$\mathcal{S}^\lambda(u) := \{x \in \Gamma(u) \,:\, N^x(0,u) = \lambda\}.$$

Using the frequency function one can split the free-boundary into the following three disjoint sets:

- the *regular free boundary* which consists of the points with the lowest possible frequency
$$\mathrm{Reg}(u) := \mathcal{S}^{3/2}(u)\,;$$
- the points with even integer frequency $\mathcal{S}^{2m}(u)$, whose union by definition constitutes the set of *singular points* $\mathrm{Sing}(u)$
$$\mathrm{Sing}(u) := \bigcup_{m\in\mathbb{N}} \mathcal{S}^{2m}(u);$$
- the remaining part, $\Gamma(u) \setminus \big(\mathrm{Reg}(u) \cup \mathrm{Sing}(u)\big)$, denoted in the literature by $\mathrm{Other}(u)$.

The first result on the regularity of the free boundary for the thin-obstacle problem is due to Athanasopoulos, Caffarelli and Salsa. In [2] they give a complete description of the blow-up limits at the points of frequency $3/2$ and prove that the regular free boundary $\mathrm{Reg}(u)$ is locally a $(d-2)$-dimensional $C^{1,\alpha}$ hypersurface in $\mathbb{R}^{d-1}$. Later the regular part of the free boundary has been shown to be C^∞ in [13, 14] and analogous results were extended to more general fractional laplacian (see [4]), of which the thin-obstacle is a particular example.

Garofalo and Petrosyan (cp. [11, Theorem 2.6.2]) showed that $\mathrm{Sing}(u)$ is precisely the set of points where the coincidence set is asymptotically negligible, that is

$$\mathrm{Sing}(u) = \left\{ x_0 \in \Gamma(u) \,:\, \lim_{r\to 0} \frac{\mathcal{H}^{d-1}(\Delta(u) \cap B'_r(x_0))}{\mathcal{H}^{d-1}(B'_r(x_0))} = 0 \right\}. \tag{1.1}$$

With the help of new monotonicity formulas of Weiss and Monneau type, Garofalo and Petrosyan showed that each set $\mathcal{S}^{2m}$ is contained in a countable union of C^1 manifolds in $\mathbb{R}^{d-1}$.

In general the set $\mathrm{Other}(u)$ is not empty and is not even small compared to the free boundary $\Gamma(u)$. Indeed, in dimension two the function $h(r,\theta) = r^{2m-1/2}\sin\left(\frac{1-4m}{2}\theta\right)$ is a global solution with frequency $2m - 1/2$ in zero. Using this example one can easily construct global solutions in any dimension $d \geq 2$ whose entire free-boundary is a $(d-2)$-dimensional plane consisting only of points with frequency $2m - 1/2$. Recently, Focardi and Spadaro [9] proved the $\mathcal{H}^{d-2}$-rectifiability of the set $\mathrm{Other}(u)$ and that it consists of points of frequency $2m - 1/2$ up to a set of zero $\mathcal{H}^{d-2}$ measure, but nothing is known up to now regarding its regularity in dimension $d > 2$. We notice that in some special cases, the set $\mathrm{Other}(u)$ might be empty. Indeed, Barrios, Figalli and Ros-Oton proved in [3] that this is precisely the case when the constraint $u(x',0) \geq 0$ is replaced by $u(x',0) \geq \varphi(x')$, for any $x' \in \mathbb{R}^{d-1}$, where φ is a non-zero superharmonic obstacle.

A different approach for the regularity of the free boundary was proposed by Garofalo-Petrosyan-Vega-Garcia [12] and Focardi-Spadaro [10], following the result of Weiss [18] for the classical obstacle problem. For points of the regular free boundary $x_0 \in \mathrm{Reg}(u) = \mathcal{S}^{3/2}$, they prove an epiperimetric inequality for the *Weiss' boundary adjusted energy*

$$\mathcal{W}_\lambda^{x_0}(r,u) := \frac{1}{r^{d-2+2\lambda}} \int_{B_r(x_0)} |\nabla u|^2\,dx - \frac{\lambda}{r^{d-1+2\lambda}} \int_{\partial B_r(x_0)} u^2\,d\mathcal{H}^{d-1}\,,$$

which allows to quantify the convergence of $\mathcal{W}_\lambda^{x_0}(r,u)$ as $r \to 0$ to be of Hölder type and provides an alternative proof of the $C^{1,\alpha}$ regularity of the free boundary. The epiperimetric inequality approach was first introduced by Reifenberg [15], White [19] and Taylor [17] in the context of minimal surfaces, later brought to the classical obstacle problem by Weiss [18] and recently developed in [16] with new contributions in the framework of free boundaries.

1.2. **Main results.** In this paper we show that the epiperimetric inequality is not as a property of the Weiss' thin-obstacle energy and its homogeneous minimizers, but it is a property of the family of Weiss' energies $\mathcal{W}_\lambda$, $\lambda = 3/2, 2m$ only. Indeed our approach *does not require any a-priori knowledge of the admissible blow-ups* (which in the previous results [12, 10] for regular points was assumed by requiring a suitable closeness to the already-known blow up), and actually yields their classification. Moreover, as usual, it gives a short, self-contained proof of the known regularity of $\mathrm{Reg}(u)$ and, thanks to the direct arguments at the basis of the epiperimetric inequality, allows to obtain a new logarithmic modulus of continuity for the singular set, which improves the results of [10, 12].

Moreover, this time assuming closeness to the blow-ups, we show a new direct epiperimetric inequality for the $(2m - 1/2)$-Weiss' energy in dimension 2, thus proving it at every free-boundary point. Furthermore we give an example of a new family of $(2m-1/2)$-homogeneous minimizer, which show why the generalization of our 2-dimensional proof to higher dimensions is not possible.

1.2.1. *Epiperimetric inequalities for $\mathcal{W}_\lambda$, $\lambda = 3/2, 2m$, in any dimension.* In this section we present our epiperimetric inequalities. Notice that, at difference from the existing literature, they hold for *any trace* c without any closeness assumption to the admissible blow ups. For the energy $\mathcal{W}_{3/2}$ we give a short and self-contained proof of the following statement.

Theorem 1 (Epiperimetric inequality for $\mathcal{W}_{3/2}$)**.** *Let $d \geq 2$ and $B_1 \subset \mathbb{R}^d$. Then for every $c \in H^1(\partial B_1)$ such that its 3/2-homogeneous extension $z(r,\theta) := r^{3/2}c(\theta)$ belongs to $\mathcal{A}$, there exists $v \in \mathcal{A}$ such that $v = c$ on ∂B_1 and*

$$\mathcal{W}_{3/2}(v) \leq \left(1 - \frac{1}{2d+3}\right)\mathcal{W}_{3/2}(z). \tag{1.2}$$

A similar statement was obtained in [12, 10], even though in these papers a further assumption is required (the closeness of the boundary datum c to the set of admissible blow ups of frequency $3/2$) and it is based on a contradiction argument. The proof of Theorem 1 exhibits instead an explicit energy competitor v, by choosing suitable homogeneous extensions for the different modes on the sphere; thus greatly simplifying the existing proofs. This approach is similar to the ones of [5], for the obstacle problem, and [16], for the Alt-Caffarelli functional in dimension 2.

In analogy to the results on the classical obstacle problem [5], our direct approach allows to obtain a logarithmic epiperimetric inequality for the family of energies $\mathcal{W}_{2m}$, $m \in \mathbb{N}$, in any dimension. This, together with [5], is the first instance in the literature (even in the context of minimal surfaces) of an epiperimetric inequality of logarithmic type, and the first instance in the context of the lower dimensional obstacle problems where an epiperimetric inequality for singular points has a direct proof. This result allows us to prove a complete and self-contained regularity result for $\mathrm{Sing}(u)$ and improve the known results by giving an explicit modulus of continuity. Further applications to other singular points of the thin obstacle problem and to the fractional obstacle problem for any $s \in (0,1)$ will be presented in [6, 7].

Theorem 2 (Logarithmic epiperimetric inequality for $\mathcal{W}_{2m}$)**.** *Let $d \geq 2$, $m \in \mathbb{N}$. For every function $c \in H^1(\partial B_1)$ such that its $2m$-homogeneous extension $z(r,\theta) = r^{2m}c(\theta)$ is in $\mathcal{A}$ and*

$$\int_{\partial B_1} c^2\, d\mathcal{H}^{d-1} \leq 1 \qquad \text{and} \qquad |\mathcal{W}_{2m}(z)| \leq 1\,, \tag{1.3}$$

there are a constant $\varepsilon = \varepsilon(d,m) > 0$ and a function $h \in \mathcal{A}$, with $h = c$ on ∂B_1, satisfying

$$\mathcal{W}_{2m}(h) \leq \mathcal{W}_{2m}(z)\big(1 - \varepsilon\,|\mathcal{W}_{2m}(z)|^\gamma\big), \qquad \text{where} \quad \gamma := \frac{d-2}{d}\,. \tag{1.4}$$

We notice that with our method the power $0 < \gamma < 1$ in (1.4) cannot be avoided, see for instance [5, Example 1]. This is essentially due to the possible convergence of polynomial of fixed degree $2m$ with low symmetry to ones with higher symmetry.

1.2.2. *Complete analysis of the free boundary points in dimension two.* In dimension $d = 2$, it is known that the only admissible values of the frequency at points of the free boundary are $3/2, 2m$ and $2m - 1/2$, for $m \in \mathbb{N}$. Theorem 1 and Theorem 2 already provide the classical epiperimetric inequalty for the points $3/2$ and $2m$; indeed, in the case $d = 2$, we have $\gamma = 0$ in (1.4). We complete the analysis in dimension two by proving an epiperimetric inequality also at the points of density $2m - 1/2$. Before we state the theorem, we recall that in this case the admissible blow up is (up to a constant and a change of orientation) of the form

$$h_{2m-1/2}(r,\theta) = r^{\frac{4m-1}{2}} \sin\left(\frac{1-4m}{2}\theta\right).$$

Assuming this time a closeness condition to $h_{2m-1/2}$ we can prove the following.

Theorem 3 (Epiperimetric inequality for points of frequency $2m - 1/2$ in dimension two)**.** *Let $d = 2$ and $m \in \mathbb{N}$. There exist constants $\delta > 0$ and $\kappa > 0$ such that the following claim holds. For every function $c \in H^1(\partial B_1)$ such that its $2m - 1/2$ homogeneous extension $z \in \mathcal{A}$ and satisfying*

$$\|c - h_{2m-1/2}\|_{L^2(\partial B_1)} \leq \delta\,, \tag{1.5}$$

there exists $h \in \mathcal{A}$ such that $h|_{\partial B_1} = c$ and

$$\mathcal{W}_{2m-1/2}(h) \leq (1-\kappa)\mathcal{W}_{2m-1/2}(z). \tag{1.6}$$

In dimension $d = 2$, the regularity of the free boundary (namely, the fact that they are isolated in the line) can be obtained also with softer arguments than our epiperimetric inequality; however, the previous result allows for instance to show the $C^{1,\alpha}$ decay of u on the unique blow up at each free boundary point and also provides an alternative, self-contained approach.

1.2.3. *Application of the epiperimetric inequalities I: homogeneous minimizers and admissible frequencies.* A very important and not yet well-understood question in the contest of the thin-obstacle problem is the study of the admissible frequencies at free-boundary points. Indeed nothing is known, except for the gap between $3/2$ and 2 (see [2]) and the recent result of Focardi and Spadaro [9], where they establish that the collection of free-boundary points with frequency different than $3/2$, $2m$ and $2m - 1/2$, is a set of $\mathcal{H}^{d-2}$ measure zero. It is conjectured that these are the only admissible frequencies, but not even the gap between 2 and the subsequent admissible frequency was known. Indeed, thanks to Theorem 4 below, we are able to recover the gap $3/2 - 2$ and to prove the new result that *the frequencies $2m$ are isolated for every $m \in \mathbb{N}$*, where the gap is given by explicit constants.

We say that $\lambda \in \mathbb{R}$ is an *admissible frequency* if there is a solution $u \in H^1(B_1)$ of the thin-obstacle problem and a point $x_0 \in \Gamma(u)$ such that $N^{x_0}(0) = \lambda$. For a minimizer u and an admissible frequency $\lambda = N^{x_0}(0)$, the monotonicity of the frequency function implies that, up to a subsequence, $\|u_{r,x_0}\|^{-1}_{L^2(\partial B_1)} u_{r,x_0}$ converges, as $r \to 0$, weakly in $H^1(B_1)$ and strongly in $L^2(B_1) \cap L^2(\partial B_1)$ to a λ-homogeneous global solution $p : \mathbb{R}^d \to \mathbb{R}$ such that $\|p\|_{L^2(\partial B_1)} = 1$. In particular, if we denote by

$$\mathcal{K}_\lambda := \{u \in H^1(B_1) : u \text{ is a nonzero } \lambda\text{-homogeneous minimizer of } \mathcal{E} \text{ in } \mathcal{A}\}$$

we have that

$$\text{if } \lambda \text{ is an admissible frequency, then } \mathcal{K}_\lambda \neq \emptyset. \tag{1.7}$$

A complete description of the spaces $\mathcal{K}_\lambda$ and the admissible frequencies is known only in dimension two, where the only possible values of λ are $3/2$, $2m$, and $2m - 1/2$ for $m \in \mathbb{N}_+$. However, as a consequence of our logarithmic epiperimetric inequality we can describe the set $\mathcal{K}_\lambda$ for values of λ close to $2m$.

Theorem 4 (λ-homogeneous minimizers)**.** *Let $d \geq 2$. Then for every $m \in \mathbb{N}$ there exist constants $c_m^\pm > 0$, depending only on d and m, such that*

$$\mathcal{K}_\lambda = \emptyset \qquad \textit{for every } \lambda \in (3/2, 2) \cup \bigcup_{m\in\mathbb{N}} \big((2m - c_m^-, 2m) \cup (2m, 2m + c_m^+)\big). \tag{1.8}$$

Moreover, setting

$$h_e(x) := \left(2(x'\cdot e) - \sqrt{(x'\cdot e)^2 + x_d^2}\right)\sqrt{\sqrt{(x'\cdot e)^2 + x_d^2} + x\cdot e} = \mathrm{Re}(x'\cdot e + i|x_d|)^{3/2}\ , \qquad (1.9)$$

we have

$$\mathcal{K}_{3/2} = \{C\, h_e \,:\, e \in \mathbb{S}^{d-1} \quad and \quad C > 0\}\,, \qquad (1.10)$$

$$\begin{aligned}\mathcal{K}_{2m} = \{C\, p_{2m} \,:\, p_{2m} \quad &\text{is a } 2m\text{-homogeneous harmonic polynomial,}\\ &p_{2m} \geq 0 \text{ on } B_1', \quad \|p_{2m}\|_{L^2(\partial B_1)} = 1 \quad \text{and} \quad C > 0\}\,.\end{aligned} \qquad (1.11)$$

Remark 1.1. Theorem 4 and (1.7) imply that the frequencies $3/2$ and $2m$, for every $m \in \mathbb{N}$, are isolated, and in particular $N^{x_0} \notin (3/2, 2) \cup \bigcup_{m\in\mathbb{N}} \big((2m - c_m^-, 2m) \cup (2m + c_m^+)\big)$ for every $x_0 \in \Gamma(u)$, where u is a minimizer of the obstacle problem for general obstacle ϕ.

At difference with respect to other results where gaps of this kind are established, the arguments leading to the constants c_m are never by contradiction, hence the constants c_m can be tracked in the proofs (see Remark 6.2 for an explicit example).

We wish to stress that the classes $\mathcal{K}_{2m}$ and $\mathcal{K}_{3/2}$ were already characterized (see [2, 11]) and that typically this characterization is needed to prove an epiperimetric inequality. However our epiperimetric inequalities are a property of the energies $\mathcal{W}_\lambda$, and not of a class of blow-ups, and as such allow us to characterize the $\mathcal{K}_\lambda$ as a corollary.

Remark 1.2. Finally we notice that (1.1) follows immediately from Theorem 4, the classification, thus giving an alternative proof to the one of [11].

The characterization of the class $\mathcal{K}_{2m-1/2}$ in dimension higher than 2 remains a major open problem. The main difficulty is in the fact that it combines the characteristics of $\mathcal{K}_{3/2}$ and $\mathcal{K}_{2m}$. Indeed, on the one hand its elements with maximal symmetry must be 0 on half the hyperplane $\{x_d = 0\}$, as the elements of $\mathcal{K}_{3/2}$, thus suggesting that they should appear as blow-ups at flat points. On the other hand we can show that, as in the case of $\mathcal{K}_{2m}$, there is a continuous family of different elements of $\mathcal{K}_{2m-1/2}$ which are 0 on half of $\{x_d = 0\}$, thus showing that they are not isolated. We give an example of such a family in dimension three in the following example. Examples in any dimensions can be constructed by extending the three-dimensional solutions invariantly with respect to the remaining $d - 3$ coordinates.

Example 1. *Let $d = 3$ and $m > 1$. Then for every $t \in [0, 1]$ the function*

$$h_t(r,\theta,\varphi) = -r^{2m-1/2}\sin^{2m-5/2}\theta\left[\sin^2\theta\sin\left((2m - 1/2)\,\varphi\right) + t\left((4m-2)\cos^2\theta - 1\right)\sin\left((2m - 5/2)\,\varphi\right)\right],$$

is in $\mathcal{K}_{2m-1/2}$, where the coordinates $r > 0$, $\varphi \in (0, 2\pi)$ and $\theta \in (0, \pi)$ are such that

$$x_1 = r\sin\theta\cos\varphi, \quad x_2 = r\cos\theta, \quad x_3 = r\sin\theta\sin\varphi,$$

is a smooth parametrization of $\mathbb{R}^3 \setminus (\{x_3 = 0\} \cap \{x_1 \geq 0\})$.

The proof is a straightforward computation. The function h_t is harmonic on the set $\mathbb{R}^3 \setminus (\{x_3 = 0\} \cap \{x_1 \geq 0\})$, $h_t = 0$ on $\{\varphi = 0\} = \{x_3 = 0\} \cap \{x_1 > 0\}$ and $h_t \geq 0$ on $\{\varphi = \pi\} = \{x_3 = 0\} \cap \{x_1 < 0\}$. Moreover, $\frac{\partial h_t}{\partial \varphi} \leq 0$ on the set $\{\varphi = 0\}$. Thus, $\frac{\partial h_t}{\partial x_3} \leq 0$ on the set $\{x_1 > 0\} \cap \{x_3 = 0\}$, h_t is superharmonic on B_1 and so, it is a minimizer of $\mathcal{E}$ in $\mathcal{A}$.

1.2.4. *Application of the epiperimetric inequalities II: regularity of the free boundary in any dimension.* Using the epiperimetric inequalities Theorem 1 and Theorem 2 we prove the following regularity result, valid in any dimension.

Theorem 5 (Regularity of the Regular and Singular set)**.** *Let $u \in \mathcal{A}_w$ be a minimizer of the thin-obstacle energy $\mathcal{E}$.*

(i) *There exists a dimensional constant $\alpha > 0$ such that* $\mathrm{Reg}(u)$ *is in B_1' a $C^{1,\alpha}$ regular open submanifold of dimension $(d-2)$.*

(ii) *For every* $m \in \mathbb{N}$ *and* $k = 0, \dots, d-2$, $S_k^{2m}(u)$ *is contained in the union of countably many submanifolds of dimension* k *and class* $C^{1,log}$. *In particular* $\mathrm{Sing}(u)$ *is contained in the union of countably many submanifolds of dimension* $(d-2)$ *and class* $C^{1,log}$.

Remark 1.3. If we consider minimizers $u \in H^1(B_1^+)$ with Dirichlet boundary conditions of the more general thin-obstacle problem, where we minimize the energy $\mathcal{E}$ in the class of admissible functions

$$\mathcal{A}^\phi := \{u \in H^1(B_1^+) \,:\, u \geq \phi \text{ on } B_1',\ u(x', x_d) = u(x', -x_d) \text{ for every } (x', x_d) \in B_1\},$$

with $\phi \in C^{l,\beta}(B_1', \mathbb{R}+)$, then an analogous statement holds, that is

(i) there exists a dimensional constant $0 < \alpha \leq \beta$ such that $\mathrm{Reg}(u)$ is in B_1' a $C^{1,\alpha}$ regular submanifold of dimension $(d-2)$,

(ii) for every $2m < l$ and $k = 0, \dots, d-2$, $S_k^{2m}(u)$ is contained in the union of countably many submanifolds of dimension k and class $C^{1,log}$.

This result can be proved as a standard application of our various epiperimetric inequalities and the almost minimality of the blow-ups at a point of the free-boundary, which follows from the regularity of the obstacle (see for instance [5]). In particular it provides an improvement in the regularity of $\mathcal{S}^{2m}$, $2m < l$, from C^1 to $C^{1,\log}$ of the results of [11, 3].

1.3. **Organization of the paper.** The paper is organized as follows. We introduce notation and classical results in Section 2, while Sections 3, 4, and 5 are devoted to the proofs of the epiperimetric inequalities from Theorems 1, 2 and 3, respectively. Section 6 contains the proof of Theorem 4, which is new and follows from our direct approach to the epiperimetric inequality. Section 7 is dedicated to the proof of Theorem 5 which is based on arguments of classical flavor and which is adapted to the logarithmic case.

2. Preliminaries

In this section we recall some properties of the solutions of the thin-obstacle problem, the frequency function, the Weiss' boundary adjusted functional and we deal with some preliminary computations.

2.1. **Regularity of minimizers.** The optimal regularity of the solutions of the thin obstacle problem was proved in [1]. We recall the precise estimate in the following theorem.

Theorem 2.1 (Optimal regularity of minimizers [1])**.** *Let* $u \in \mathcal{A}$ *be a minimizer of* $\mathcal{E}$ *with Dirichlet boundary conditions. Then* $u \in C^{1,1/2}(B_{1/2}^+)$ *and there exists a dimensional constant* $C_d > 0$ *such that*

$$\|u\|_{C^{1,1/2}(B_{1/2}^+)} \leq C_d \|u\|_{L^2(B_1)}.$$

2.2. **Properties of the frequency function.** Let $u \in H^1(B_1)$ be a minimizer of the thin-obstacle energy and $x_0 \in \Gamma(u)$. Then we introduce the quantities

$$D^{x_0}(r) := \int_{B_r(x_0)} |\nabla u|^2\,dx, \quad H^{x_0}(r) := \int_{\partial B_r(x_0)} u^2\,d\mathcal{H}^{d-1} \quad \text{and} \quad N^{x_0}(r) := \frac{r\,D^{x_0}(r)}{H^{x_0}(r)},$$

where $0 < r < 1 - |x_0|$. Furthermore in this notation we have

$$\mathcal{W}_\lambda^{x_0}(r,u) = \frac{1}{r^{d-2+2\lambda}} D^{x_0}(r) - \frac{\lambda}{r^{d-1+2\lambda}} H^{x_0}(r) := \mathcal{W}_\lambda^{x_0}(r).$$

In the following we will need the monotonicity of N, which can be found in [2], and of $\mathcal{W}_\lambda$, which can be found in [10] in the case of frequency $3/2$. For the sake of completeness we give here a proof in the general case.

Lemma 2.2 (Properties of the frequency function)**.** *Let* $u \in H^1(B_1)$ *be a minimizer of* $\mathcal{E}$ *and* $x_0 \in \Gamma(u)$, *then the following properties hold.*

- *The functions $N^{x_0}(r)$ and $\mathcal{W}^{x_0}_\lambda(r)$, for any $\lambda > 0$, are monotone nondecreasing and in particular*

$$\frac{d}{dr}\mathcal{W}^{x_0}_\lambda(r) = \frac{(d-2+2\lambda)}{r}\big(\mathcal{W}_\lambda(z_r) - \mathcal{W}_\lambda(u_r)\big) + \frac{1}{r}\int_{\partial B_1} (\nabla u_r \cdot \nu - \lambda\, u_r)^2 \, d\mathcal{H}^{d-1}\,, \tag{2.1}$$

where $u_r(x) := \dfrac{u(x_0+rx)}{r^\lambda}$ *and* $z_r(x) := |x|^\lambda\, u_r\left(x/|x|\right)$.

- *For every $N^{x_0}(0) > \lambda$, the function $\dfrac{H^{x_0}(r)}{r^{d-1+2\lambda}}$ is monotone nondecreasing and in particular*

$$\frac{d}{dr}\left(\frac{H^{x_0}(r)}{r^{d-1+2\lambda}}\right) = 2\frac{\mathcal{W}^{x_0}_\lambda(r)}{r}\,. \tag{2.2}$$

Proof. For the monotonicity of $\mathcal{W}_\lambda$, dropping the index x_0, we recall the identities

$$D'(r) = (d-2)\,\frac{D(r)}{r} + 2\int_{\partial B_r} (\partial_\nu u)^2 \, d\mathcal{H}^{d-1} \tag{2.3}$$

$$H'(r) = (d-1)\,\frac{H(r)}{r} + 2\int_{\partial B_r} u\,\partial_\nu u \, d\mathcal{H}^{d-1} \tag{2.4}$$

$$D(r) = \int_{\partial B_r} u\,\partial_\nu u \, d\mathcal{H}^{d-1}\,.$$

Then, similarly to [10], we compute

$$\begin{aligned}\mathcal{W}'_\lambda(r) &= \frac{D'(r)}{r^{d-2+2\lambda}} - (d-2+2\lambda)\,\frac{D(r)}{r^{d-1+2\lambda}} - \lambda\,\frac{H'(r)}{r^{d-1+2\lambda}} + \lambda\,(d-1+2\lambda)\,\frac{H(r)}{r^{d-1+2\lambda}} \qquad (2.5)\\ &\overset{(2.4)}{=} -\frac{(d-2+2\lambda)}{r}\,\mathcal{W}_\lambda(r) - \lambda\,(d-2+2\lambda)\,\frac{H(r)}{r^{d+2\lambda}} + \underbrace{\frac{D'(r)}{r^{d-2+2\lambda}} + 2\lambda^2\,\frac{H(r)}{r^{d+2\lambda}} - 2\lambda\,\frac{D(r)}{r^{d-1+2\lambda}}}_{=:I(r)}\,.\end{aligned}$$

Next a simple computation shows that

$$\begin{aligned}I(r) &= \frac{1}{r}\int_{\partial B_1} \left(|\nabla u_r|^2 - 2\lambda\, u_r\,\partial_\nu u_r + 2\lambda^2\, u_r^2\right) d\mathcal{H}^{d-1}\\ &= \frac{1}{r}\int_{\partial B_1} \left[(|\partial_\nu u_r| - \lambda\, u_r)^2 + |\nabla_\theta u_r|^2 + \lambda^2\, u_r^2\right] d\mathcal{H}^{d-1}\\ &= \frac{1}{r}\int_{\partial B_1} (|\partial_\nu u_r| - \lambda\, u_r)^2 \, d\mathcal{H}^{d-1} + (d-2+2\lambda)\int_{B_1} |\nabla z_r|^2\end{aligned}$$

which, together with (2.5), implies

$$\mathcal{W}'_\lambda(r) = \frac{(d-2+2\lambda)}{r}\,(\mathcal{W}_\lambda(z_r) - \mathcal{W}_\lambda(u_r)) + \frac{1}{r}\int_{\partial B_1} (\nabla u_r\cdot\nu - \lambda\, u_r)^2 \, d\mathcal{H}^{d-1}\,.$$

In particular, if u minimizes $\mathcal{E}$, then the monotonicity of $\mathcal{W}_\lambda$ follows.

For the second bullet, we can compute

$$\begin{aligned}\frac{d}{dr}\left(\frac{H(r)}{r^{d-1+2\lambda}}\right) &= \frac{H'(r)}{r^{d-1+2\lambda}} - (d-1+2\lambda)\,\frac{H(r)}{r^{d+2\lambda}}\\ &\overset{(2.4)}{=} (d-1)\frac{H(r)}{r^{d-2+2\lambda}} + \frac{2}{r^{d-1+2\lambda}}\int_{\partial B_r} u\,\partial_\nu u\, d\mathcal{H}^{d-1} - (d-1+2\lambda)\,\frac{H(r)}{r^{d+2\lambda}}\\ &\overset{(2.2)}{=} 2\,\frac{D(r)}{r^{d-1+2\lambda}} - (2\lambda)\,\frac{H(r)}{r^{d+2\lambda}} = \frac{2}{r}\mathcal{W}_\lambda(r)\,.\end{aligned}$$

Notice that $\mathcal{W}_\lambda(r) = \dfrac{H(r)}{r^{d-1+2\lambda}}(N(r)-\lambda)$, so that if $N(0) > \lambda$, then $\mathcal{W}_\lambda(r)$ is positive, by monotonicity of $N(r)$, and the claim follows. □

2.3. **Blow-up sequences, blow-up limits and admissible frequencies.** Given a function $u \in \mathcal{A}$ minimizing the energy $\mathcal{E}$ and a point $x_0 \in \mathcal{S}^\lambda$, we define the *blow-up sequence of u at x_0* by $u_{x_0,r}(x) := \frac{u(x_0+rx)}{r^\lambda}$. Using the monotonicity of N^{x_0} and H^{x_0} it is easy to see that

$$\int_{B_1} |\nabla u_{x_0,r}|^2\,dx = \frac{1}{r^{d-2+2\lambda}} \int_{B_r(x_0)} |\nabla u|^2\,dx = N^{x_0}(r)\,\frac{H^{x_0}(r)}{r^{d-1+2\lambda}} \le N^{x_0}(1)\,H^{x_0}(1)\,.$$

It follows that there exists a subsequence $(u_{x_0,r_k})_k$ and a function u_{x_0}, which depends on the subsequence, such that u_{x_0,r_k} converges weakly in $H^1(B_1)$ and strongly in $L^2(B_1) \cap L^2(\partial B_1)$ to some function $p_{x_0} \in \mathcal{A}$. Furthermore by Theorem 2.1 we have that the convergence is $C^{1,\alpha}_{loc}(B_1)$, for every $\alpha < 1/2$, and by the minimality of u, it is also strong in $H^1(B_1)$. A standard argument using the monotonicity of $\mathcal{W}^{x_0}_\lambda$ then shows that p_{x_0} is a λ-homogeneous global minimizer of $\mathcal{E}$. We say that p_{x_0} is a blow-up limit of u at x_0 and we denote by $\mathcal{K}^{x_0}(u)$ the set of all possible blow-up limits of u at x_0.

2.4. **Fourier expansion of the Weiss' energy.** On the $(d-1)$-dimensional sphere $\partial B_1 \subset \mathbb{R}^d$ we consider the Laplace-Beltrami operator $\Delta_{\partial B_1}$. Recall that the spectrum of $\Delta_{\partial B_1}$ is discrete and is given by the decreasing sequence of eigenvalues (counted with the multiplicity)

$$0 = \lambda_1 \le \lambda_2 \le \dots \le \lambda_k \le \dots$$

The corresponding normalized eigenfunctions $\phi_k : \partial B_1 \to \mathbb{R}$ are the solutions of the PDEs

$$-\Delta_{\partial B_1}\phi_k = \lambda_k \phi_k \quad \text{on} \quad \partial B_1, \qquad \int_{\partial B_1} \phi_k^2\,d\mathcal{H}^{d-1} = 1.$$

For every $\mu \in \mathbb{R}$ we will use the notation

$$\lambda(\mu) = \mu(\mu + d - 2), \tag{2.6}$$

and we will denote by α_k the unique positive real number such that $\lambda(\alpha_k) = \lambda_k$. It is easy to check that the homogeneous function $u_k(r,\theta) = r^{\alpha_k}\phi_k(\theta)$ is harmonic in $\mathbb{R}^d$ if and only if its trace ϕ_k is an eigenfunction on the sphere corresponding to the eigenvalue λ_k. Moreover, it is well known that in any dimension the homogeneities α_k are natural numbers and the functions u_k are harmonic polynomials of homogeneity α_k. Furthermore for every $\lambda \ge 0$ eigenvalue of the Laplace-Beltrami operator on the sphere, we define

$$E(\lambda) := \left\{ \phi \in H^1(\partial B_1) \,:\, -\Delta_{\partial B_1}\phi = \lambda\,\phi \quad \text{and} \quad \|\phi\|_{L^2(\partial B_1)} \ne 0 \right\},$$

that is $E(\lambda)$ is the eigenspace of $\Delta_{\partial B_1}$ associated to the eigenvalue λ intersected with the unit sphere. We write the energy of a homogeneous function in terms of its Fourier coefficients; a similar lemma can be found in [5, Lemma 2.1], but we report the short proof for completeness.

Lemma 2.3. *Let $d \ge 2$, $\alpha \ge \mu > 0$ and*

$$\kappa_{\alpha,\mu} := \frac{\alpha - \mu}{\alpha + \mu + d - 2}\,. \tag{2.7}$$

With the notations above, let $\psi = \sum_{j=1}^\infty c_j\phi_j \in H^1(\partial B_1)$, and let $\varphi_\alpha(r,\theta) := r^\alpha\psi(\theta)$ be the α-homogeneous extension of ψ in B_1. Then we have

$$\mathcal{W}_\mu(\varphi_\mu) = \frac{1}{2\mu + d - 2}\sum_{j=1}^\infty (\lambda_j - \lambda(\mu))\,c_j^2\,, \tag{2.8}$$

$$\mathcal{W}_\mu(\varphi_\alpha) - (1 - \kappa_{\alpha,\mu})\mathcal{W}_\mu(\varphi_\mu) = \frac{\kappa_{\alpha,\mu}}{d + 2\alpha - 2}\sum_{j=1}^\infty(-\lambda_j + \lambda(\alpha))c_j^2\,. \tag{2.9}$$

Proof. Since $\|\varphi_j\|_{L^2(\partial B_1)} = 1$ and $\|\nabla_\theta \varphi_j\|^2_{L^2(\partial B_1)} = \lambda_j$ for every $j \in \mathbb{N}$, we have

$$\mathcal{W}_\mu(\varphi_\alpha) = \sum_{j=1}^\infty c_j^2 \left(\int_0^1 r^{d-1}\, dr \int_{\partial B_1} d\mathcal{H}^{d-1} \left[\alpha^2 r^{2\alpha-2}\phi_j^2(\theta) + r^{2\alpha-2}|\nabla_\theta \phi_j|^2(\theta)\right] - \mu \int_{\partial B_1} \phi_j^2(\theta)\, d\mathcal{H}^{d-1} \right)$$
$$= \sum_{j=1}^\infty c_j^2 \left(\frac{\alpha^2 + \lambda_j}{d + 2\alpha - 2} - \mu \right),$$

where in the above identity $d\theta$ stands for the Hausdorff measure $\mathcal{H}^{d-1}$ on the sphere ∂B_1. Setting $\alpha = \mu$, we get (2.8). We now notice that for every λ we have

$$\left(\frac{\alpha^2 + \lambda}{d + 2\alpha - 2} - \mu \right) - (1 - \kappa_{\alpha,\mu}) \left(\frac{\mu^2 + \lambda}{d + 2\mu - 2} - \mu \right) = \frac{\kappa_{\alpha,\mu}}{d + 2\alpha - 2}(\lambda_\alpha - \lambda),$$

which shows (2.9). □

2.5. **Energy of homogeneous minimizers.** In this subsection we prove a lemma about the energy of homogeneous minimizers which will be useful in their classification.

Lemma 2.4. *Let $\mu \geq 0$ and $t \in \mathbb{R}$. If the trace $c \in H^1(\partial B_1)$ is such that the $(\mu+t)$-homogeneous extension $r^{\mu+t}c(\theta) \in \mathcal{A}$ and is a solution of the thin-obstacle problem, then*

$$\mathcal{W}_\mu(r^{\mu+t}c) = t\|c\|^2_{L^2(\partial B_1)} \qquad \textit{and} \qquad \mathcal{W}_\mu(r^\mu c) = \left(1 + \frac{t}{2\mu + d - 2}\right) \mathcal{W}_\mu(r^{\mu+t}c). \tag{2.10}$$

Proof. Since the Weiss energy vanishes for minimizers with the corresponding homogeneity, $\mathcal{W}_{\mu+t}(r^{\mu+t}c(\theta)) = 0$, by (2.8) we get that

$$\|\nabla_\theta c\|^2_{L^2(\partial B_1)} = \lambda(\mu + t)\|c\|^2_{L^2(\partial B_1)}.$$

Hence, we have $\mathcal{W}_\mu(r^{\mu+t}c) = \mathcal{W}_{\mu+t}(r^{\mu+t}c) + t\|c\|^2_{L^2(\partial B_1)} = t\|c\|^2_{L^2(\partial B_1)}$ and by Lemma 2.3 (2.8)

$$\mathcal{W}_\mu(r^\mu c) = \frac{1}{2\mu + d - 2}(\|\nabla_\theta c\|^2_{L^2(\partial B_1)} - \lambda(\mu)\|c\|^2_{L^2(\partial B_1)}) = \frac{\lambda(\mu+t) - \lambda(\mu)}{2\mu + d - 2}\|c\|^2_{L^2(\partial B_1)}$$
$$= \left(1 + \frac{t}{2\mu + d - 2}\right) t\|c\|^2_{L^2(\partial B_1)} = \left(1 + \frac{t}{2\mu + d - 2}\right) \mathcal{W}_\mu(r^{\mu+t}c).$$

□

3. Epiperimetric inequality for $\mathcal{W}_{3/2}$: Proof of Theorem 1.

In this section, after some preliminary considerations about $3/2$-homogeneous minimizers of $\mathcal{E}$, we prove the epiperimetric inequality at regular points Theorem 1.

3.1. **The function $h_{3/2}$.** In [2] Athanasopoulos, Caffarelli and Salsa show that there are no point of frequency smaller than $3/2$. On the other hand, one can easily construct global $3/2$-homogeneous solution for which the point 0 is on the free boundary. In dimension two, one such a solution expressed in polar coordinates is $h_{3/2}(r, \theta) = r^{3/2}\cos(3\theta/2)$, for $r > 0$ and $\theta \in (-\pi, \pi)$. In $\mathbb{R}^d$, it is sufficient to consider the two-dimensional solution $h_{3/2}$ extended invariantly in the remaining $d-2$ coordinates. More generally, for a given direction $e \in \partial B_1 \cap \{x_d = 0\}$ we consider the function h_e in (1.9), which is a $3/2$-homogeneous global solution of the thin obstacle problem. With a slight abuse of the notation, in polar coordinates, we will sometimes write $h_e(r, \theta) = r^{3/2}h_e(\theta)$. We notice that h_e has the following properties:

(i) The $L^2(\partial B_1)$-projection of $h_e(\theta)$ on the space of linear functions is non-zero. We may suppose that it is given by the trace of the function $x \mapsto C\, x \cdot e$, for some constant $C > 0$. Notice that the space of linear functions coincides with the eigenspace of the spherical laplacian corresponding to the multiple eigenvalue $\lambda_2 = \cdots = \lambda_d = d - 1$. Thus, h_e has a non-zero $(d-1)$-mode on the sphere.

(ii) h_e is harmonic on $B_1 \setminus (\{x_d = 0\} \cap \{x \cdot e > 0\})$. Thus, an integration by parts gives that, for every $\psi \in H^1(B_1)$ such that $\psi = 0$ on $\{x_d = 0\} \cap \{x \cdot e < 0\}$ we have

$$\int_{B_1} \nabla h_e \cdot \nabla \psi \, dx - \frac{3}{2} \int_{\partial B_1} h_e \psi \, d\mathcal{H}^{d-1} = 0.$$

In particular, $\mathcal{W}_{3/2}(h_e) = 0$.

(iii) The derivative $\frac{\partial h_e}{\partial x_d}$ has a jump across the set $\{x_d = 0\} \cap \{x \cdot e < 0\}$. The distributional laplacian of h_e on B_1, applied to the test function $\psi \in H^1(B_1)$, is given by

$$\int_{B_1} \psi \Delta h_e \, dx = 2 \int_{B_1' \cap \{x \cdot e < 0\}} \left| \frac{\partial h_e}{\partial x_d} \right| \psi \, d\mathcal{H}^{d-1} = 2 \int_{B_1'} \psi(x', 0) \frac{3}{\sqrt{2}} (x' \cdot e)_-^{1/2} \, dx' .$$

3.2. **Proof of Theorem 1.** Since the trace c is even with respect to the plane $\{x_d = 0\}$, its projection on the eigenspace of linear functions $E(\lambda_2) \subset H^1(B_1)$ is of the form $c_1 \, x \cdot e$ for some constant $c_1 > 0$ and some $e \in \partial B_1 \cap \{x_d = 0\}$. Let $C > 0$ be such that the $L^2(\partial B_1)$-projections of $C \, h_e$ and c on the eigenspace of linear functions $E(\lambda_2)$ are the same.

Consider the function $u_0 : B_1 \to \mathbb{R}$ given by $u_0(x) := |x_d|^{3/2}$. Since $u_0(\theta)$ is even, it is orthogonal to the eigenspace $E(\lambda_2)$. Let the constant $c_0 \in \mathbb{R}$ be such that the projections of $c - Ch_e$ and $c_0 u_0$ on the eigenspace $E(\lambda_1)$ are the same.

We can now deduce that $c : \partial B_1 \to \mathbb{R}$ can be decomposed in a unique way as $Ch_e + c_0 u_0$, which has the same low modes of c, and of ϕ, which contains only higher modes on ∂B_1

$$c = Ch_e + c_0 u_0 + \phi, \qquad \phi(\theta) = \sum_{\{j \,:\, \lambda_j > 2d\}} c_j \phi_j(\theta).$$

The competitor $v : B_1 \to \mathbb{R}$ is then given by

$$v(r, \theta) = C r^{3/2} h_e(\theta) + c_0 r^{3/2} u_0(\theta) + r^2 \phi(\theta). \tag{3.1}$$

We notice that $v \in \mathcal{A}$. Indeed, since $c > 0$ on the equator $\{x_d = 0\} \cap \partial B_1$ and since $C > 0$, assures that $v(r, \theta) \geq r^2 c(\theta)$ is non-negative on the $(d-1)$-dimensional ball $B_1' := \{x_d = 0\} \cap B_1$. We now compute the energies of $r^{3/2} c$ and v. For any $\phi \in H^1(\partial B_1)$ we claim that

$$\mathcal{W}_{3/2}(Ch_e + c_0 u_0 + \varphi_\alpha) = -\frac{3c_0^2}{4} \int_{B_1} |x_d| \, dx + \mathcal{W}_{3/2}(\psi) + \frac{1}{d + \alpha - \frac{1}{2}} \beta(\phi), \tag{3.2}$$

where $\varphi_\alpha(r, \theta) := r^\alpha \phi(\theta)$ denotes the α-homogeneous extension of ϕ and

$$\beta(\phi) := -\frac{3}{2} c_0 \int_{\partial B_1} \frac{\phi(\theta)}{\sqrt{|\theta_d|}} \, d\mathcal{H}^{d-1}(\theta) + \frac{12}{\sqrt{2}} C \int_{\partial B_1'} \phi(\theta')(\theta' \cdot e)_-^{1/2} \, d\mathcal{H}^{d-2}(\theta').$$

Indeed, expanding $\mathcal{W}_{3/2}$ and integrating by parts we get

$$\begin{aligned}
\mathcal{W}_{3/2}(Ch_e + c_0 u_0 + \varphi_\alpha) &= C^2 \mathcal{W}_{3/2}(h_e) + \mathcal{W}_{3/2}(c_0 u_0 + \varphi_\alpha) \\
&\quad + 2C \left(\int_{B_1} \nabla h_e \cdot \nabla (c u_0 + \varphi_\alpha) - \frac{3}{2} \int_{\partial B_1} h_e (c_0 u_0 + \varphi_\alpha) \right) \\
&= \mathcal{W}_{3/2}(c_0 u_0 + \varphi_\alpha) - 2C \int_{B_1} \varphi_\alpha \Delta h_e \, dx,
\end{aligned}$$

$$\begin{aligned}
\mathcal{W}_{3/2}(c_0 u_0 + \varphi_\alpha) &= c_0^2 \mathcal{W}_{3/2}(u_0) + \mathcal{W}_{3/2}(\varphi_\alpha) + 2c_0 \left(\int_{B_1} \nabla u_0 \cdot \nabla \varphi_\alpha - \frac{3}{2} \int_{\partial B_1} u_0 \varphi_\alpha \right) \\
&= c_0^2 \mathcal{W}_{3/2}(u_0) + \mathcal{W}_{3/2}(\varphi_\alpha) - 2c_0 \int_{B_1} \varphi_\alpha \Delta u_0 \, dx.
\end{aligned} \tag{3.3}$$

An integration by parts and the fact that $\Delta u_0(x) = \frac{3}{4} |x_d|^{-1/2}$ give that

$$\mathcal{W}_{3/2}(u_0) = -\int_{B_1} u_0 \Delta u_0 \, dx = -\frac{3}{4} \int_{B_1} |x_d| \, dx < 0 . \tag{3.4}$$

The homogeneity of φ_α and the precise expressions of Δu_0 and Δh_e give that

$$\int_{B_1} \varphi_\alpha \Delta u_0 \, dx = \int_{B_1} \varphi_\alpha \frac{3}{4}|x_d|^{-1/2}\, dx = \frac{1}{d+\alpha-\frac{1}{2}} \int_{\partial B_1} \phi(\theta)\frac{3}{4}|\theta_d|^{-1/2}\, d\mathcal{H}^{d-1}(\theta), \tag{3.5}$$

$$\int_{B_1} \varphi_\alpha \Delta h_e \, dx = -2\int_{B_1'} \varphi_\alpha(x',0)\frac{3}{\sqrt{2}}(x'\cdot e)_-^{1/2}\, dx' = -\frac{1}{d+\alpha-\frac{1}{2}}\frac{6}{\sqrt{2}}\int_{\partial B_1'} \phi(\theta')(\theta'\cdot e)_-^{1/2}\, d\mathcal{H}^{d-2}(\theta')\,. \tag{3.6}$$

Finally, by (3.3), (3.4), (3.5) and (3.6) we get (3.2). Applying (3.2) to $\alpha = 3/2$ and $\alpha = 2$ we get

$$\begin{aligned}\mathcal{W}_{3/2}(v) - \Big(1-\frac{1}{2d+3}\Big)\mathcal{W}_{3/2}(z) &\le -\frac{3\,c_0^2}{4(2d+3)}\int_{B_1}|x_d|\,dx + \mathcal{W}_{3/2}(\varphi_2) - \Big(1-\frac{1}{2d+3}\Big)\mathcal{W}_{3/2}(\varphi_{3/2})\\ &\le -\frac{3\,c_0^2}{4(2d+3)}\int_{B_1}|x_d|\,dx, \end{aligned}\tag{3.7}$$

where the last inequality is due to Lemma 2.3 with $\mu = 3/2$ and $\alpha = 2$. □

Remark 3.1. In this remark we are interested in the equality case of the epiperimetric inequality (1.2). Indeed, if there was an equality in (1.2), then by (3.7) we should have that $c_0 = 0$ and also

$$\mathcal{W}_{3/2}(r^2\phi(\theta)) - \left(1-\frac{1}{2d+3}\right)\mathcal{W}_{3/2}(r^{3/2}\phi(\theta)) = 0.$$

By Lemma 2.3, we get that ϕ is an eigenfunction on the sphere ∂B_1 corresponding to the eigenvalue $\lambda(2) = 2d$, that is the restriction of a 2-homogeneous harmonic polynomial. Moreover, since the trace is c is non-negative on $\partial B_1'$ and $h_e = 0$ on $B_1'\cap\{x\cdot e<0\}$ we get that $\phi\ge 0$ on $B_1'\cap\{x\cdot e<0\}$ and by the fact that ϕ is even, we get $\phi \ge 0$ on B_1'.

4. Logarithmic epiperimetric inequality for $\mathcal{W}_{2m}$: Proof of Theorem 2

If $\mathcal{W}_{2m}(z)\le 0$, the conclusion is trivial, taking $h\equiv z$. Thus in the proof we assume $\mathcal{W}_{2m}(z) > 0$.

Using the notations from Subsection 2.4 we may decompose the trace c in Fourier series on the sphere as $c(\theta) = \sum_{j=1}^\infty c_j\phi_j(\theta) = P(\theta) + \phi(\theta)$, where

$$P(\theta) \quad := \sum_{\{j\,:\,\alpha_j\le 2m\}} c_j\,\phi_j(\theta) \qquad \text{and} \qquad \phi(\theta) \quad := \sum_{\{j\,:\,\alpha_j > 2m\}} c_j\,\phi_j(\theta)\,. \tag{4.1}$$

Let

$$M := -\min\big\{\min\{P(\theta),0\} \;:\; \theta\in\partial B_1,\ \theta_d = 0\big\},$$

and let h_{2m} be an eigenfunction, corresponding to the homogeneity $2m$, such that $h_{2m}\equiv 1$ on the hyperplane $\{x_d = 0\}\cap\partial B_1$.

Remark 4.1 (Construction of h_{2m}). In order to construct such an eigenfuction we first notice that the eigenspace corresponding to the homogeneity $2m$ consists of the restrictions to the sphere of $2m$-homogeneous harmonic polynomials in $\mathbb{R}^d$. Thus it is sufficient to construct a $2m$-homogeneous harmonic polynomial whose restriction to the space $\{x_d = 0\}$ is precisely $\big(x_1^2 + \cdots + x_{d-1}^2\big)^m$. We define

$$h_{2m}(x_1,\dots,x_d) := \sum_{n=0}^m C_n x_d^{2n}(x_1^2+\cdots+x_{d-1}^2)^{m-n},$$

where $C_0 = 1$ and, for every $n\ge 1$, C_n is given by the formula

$$C_n := -\frac{2(m-n+1)(d-1+2m-2n)}{2n(2n-1)}C_{n-1},$$

which assures that h_{2m} is harmonic. It is immediate to check that C_n is explicitely given by

$$C_n = \frac{(-2)^n\,m!}{(2n)!\,(m-n)!}\prod_{j=1}^n(d-1+2m-2j),$$

which concludes the construction of h_{2m}.

The $2m$-homogeneous extension z of c can be written as

$$z(r,\theta) = r^{2m}P(\theta) + M\,r^{2m}h_{2m}(\theta) - M\,r^{2m}h_{2m}(\theta) + r^{2m}\phi(\theta).$$

Our competitor h is given by

$$h(r,\theta) = r^{2m}P(\theta) + M\,r^{2m}h_{2m}(\theta) - M\,r^{\alpha}h_{2m}(\theta) + r^{\alpha}\phi(\theta). \tag{4.2}$$

for some $\alpha > 2m$ to be chosen later. Notice that h is non-negative on the set $\{x_d = 0\} \cap \partial B_1$.

The homogeneity $\alpha \geq 2m$ depends on the trace and is determined through the inequality

$$\kappa_{\alpha,2m} := \frac{\alpha - 2m}{\alpha + 2m + d - 2} = \varepsilon \|\nabla_\theta \phi\|^{2\gamma}_{L^2(\partial B_1)}, \tag{4.3}$$

where we will choose $\varepsilon > 0$ to be small enough, but yet depending only on the dimension. We now prove the epiperimetric inequality (1.4). We proceed in three steps.

Step 1. There are explicit (given in (4.8)) constants C_1 and C_2, depending only on d and m, such that for every $\alpha \in (2m, 2m + 1/2)$ the following inequality does hold:

$$\mathcal{W}_{2m}(h) - (1 - \kappa_{\alpha,2m})\mathcal{W}_{2m}(z) \leq C_1 \kappa^2_{\alpha,2m}\, M^2 - C_2 \kappa_{\alpha,2m} \|\nabla_\theta \phi\|^2_{L^2(\partial B_1)}. \tag{4.4}$$

We set for simplicity

$$\begin{aligned}
\psi(r,\theta) &:= \sum_{\{j,\ \alpha_j < 2m\}} c_j r^{2m} \phi_j(\theta),\\
H_{2m}(r,\theta) &:= M\,r^{2m}h_{2m}(\theta) + \sum_{\{j,\ \alpha_j = 2m\}} c_j r^{2m} \phi_j(\theta),\\
\varphi(r,\theta) &:= -M\,r^{2m}h_{2m}(\theta) + \sum_{\{j,\ \alpha_j > 2m\}} c_j r^{2m} \phi_j(\theta),\\
\tilde\varphi(r,\theta) &:= -M\,r^{\alpha}h_{2m}(\theta) + \sum_{\{j,\ \alpha_j > 2m\}} c_j r^{\alpha} \phi_j(\theta).
\end{aligned} \tag{4.5}$$

Thus, h and z are given by

$$z = \psi + H_{2m} + \varphi \qquad \text{and} \qquad h = \psi + H_{2m} + \tilde\varphi.$$

We first notice that the harmonicity and $2m$-homogeneity of H_{2m} imply

$$\mathcal{W}_{2m}(z) = \mathcal{W}_{2m}(\psi + \varphi) \qquad \text{and} \qquad \mathcal{W}_{2m}(h) = \mathcal{W}_{2m}(\psi + \tilde\varphi).$$

Moreover, by definition ψ is orthogonal in $L^2(B_1)$ and $H^1(B_1)$ to both φ and $\tilde\varphi$. Thus, we get

$$\mathcal{W}_{2m}(z) = \mathcal{W}_{2m}(\psi) + \mathcal{W}_{2m}(\varphi) \qquad \text{and} \qquad \mathcal{W}_{2m}(h) = \mathcal{W}_{2m}(\psi) + \mathcal{W}_{2m}(\tilde\varphi).$$

We now notice that, since ψ contains only lower frequencies, we have $\mathcal{W}_{2m}(\psi) < 0$. Thus,

$$\begin{aligned}
\mathcal{W}_{2m}(h) - (1-\kappa_{\alpha,2m})\mathcal{W}_{2m}(z) &= \kappa_{\alpha,2m}\mathcal{W}(\psi) + \mathcal{W}_{2m}(\tilde\varphi) - (1-\kappa_{\alpha,2m})\mathcal{W}_{2m}(\varphi)\\
&\leq \mathcal{W}_{2m}(\tilde\varphi) - (1-\kappa_{\alpha,2m})\mathcal{W}_{2m}(\varphi).
\end{aligned}$$

By Lemma 2.3 we have that

$$\begin{aligned}\mathcal{W}_{2m}(\tilde\varphi)-(1-\kappa_{\alpha,2m})\mathcal{W}_{2m}(\varphi)&=M^2\|h_{2m}\|^2_{L^2(\partial B_1)}\frac{\kappa_{\alpha,2m}}{d+2\alpha-2}(-\lambda(2m)+\lambda(\alpha))\\&\qquad+\frac{\kappa_{\alpha,2m}}{d+2\alpha-2}\sum_{\{j\,,\,\alpha_j>2m\}}^{\infty}(-\lambda_j+\lambda(\alpha))c_j^2\\&=M^2\|h_{2m}\|^2_{L^2(\partial B_1)}\kappa^2_{\alpha,2m}\frac{(2m+\alpha+d-2)^2}{d+2\alpha-2}\\&\qquad+\frac{\kappa_{\alpha,2m}}{d+2\alpha-2}\sum_{\{j\,,\,\alpha_j>2m\}}^{\infty}(-\lambda_j+\lambda(\alpha))c_j^2.\end{aligned}\tag{4.6}$$

We now estimate the last term in the right-hand side of (4.7).

$$\begin{aligned}\sum(\lambda_j-\lambda(\alpha))c_j^2&=\sum\lambda_jc_j^2-\lambda(\alpha)\sum c_j^2\\&\ge\sum\lambda_jc_j^2-\frac{\lambda(\alpha)}{\lambda(2m+1)}\sum\lambda_jc_j^2=\frac{\lambda(2m+1)-\lambda(\alpha)}{\lambda(2m+1)}\|\nabla_\theta\phi\|^2_{L^2(\partial B_1)},\end{aligned}\tag{4.7}$$

where all the sums are over $\{j\,,\,\alpha_j>2m\}$. If we assume that $\alpha\in(2m,2m+{}^1\!/\!_2)$, then (4.7), together with (4.6), gives (4.4) with the constants

$$C_1=(4m+d)\|h_{2m}\|^2_{L^2(\partial B_1)}\qquad\text{and}\qquad C_2=\frac{\lambda(2m+1)-\lambda(2m+{}^1\!/\!_2)}{\lambda(2m+1)}.\tag{4.8}$$

In order to conclude the proof of Step 1 we now show that we can choose ε small enough (depending only on the dimension and m) such that the bounds (1.3) on the trace c imply that $\alpha<2m+{}^1\!/\!_2$. Indeed, by (4.7) with $\alpha=2m$ and Lemma 2.3 (2.8), we get

$$\|\nabla_\theta\phi\|^2_{L^2(\partial B_1)}\le\frac{\lambda(2m+1)}{4m+d-2}\sum_{\{j\,,\,\alpha_j>2m\}}(\lambda_j-\lambda(2m))c_j^2=\lambda(2m+1)\mathcal{W}_{2m}(r^{2m}\phi(\theta)).\tag{4.9}$$

Using that $c=P+\phi$, the orthogonality of P and ϕ on the sphere and Lemma 2.3, we get that

$$\begin{aligned}\mathcal{W}_{2m}(r^{2m}\phi(\theta))&=\mathcal{W}_{2m}(z)-\mathcal{W}_{2m}(r^{2m}P(\theta))\le\mathcal{W}_{2m}(z)+\frac{\lambda(2m)}{2m+d-2}\|P\|^2_{L^2(\partial B_1)}\\&\le\mathcal{W}_{2m}(z)+2m\|c\|^2_{L^2(\partial B_1)}\le1+2m,\end{aligned}$$

where the last inequality is due to (1.3). Together with (4.9) this gives the estimate

$$\|\nabla_\theta\phi\|^2_{L^2(\partial B_1)}\le(2m+1)^2(2m+d-1).$$

Thus, choosing $\varepsilon\le\big(2\lambda(2m+1)\big)^{-2}$, we finally obtain

$$\alpha-2m=(\alpha+2m+d-2)\varepsilon\|\nabla_\theta\phi\|^{2\gamma}_{L^2(\partial B_1)}\le2(2m+1)^2(2m+d-1)^2\varepsilon\le\frac12.$$

Step 2. There is a constant $C_3>0$, depending on d and m, such that

$$M^2\le C_3\|\nabla_\theta\phi\|^{2(1-\gamma)}_{L^2(\partial B_1)}.\tag{4.10}$$

We start by noticing that there is a constant L_m, depending only on d and m, such that the eigenfunctions corresponding to the low frequencies are globally L_m-Lipschitz continuous, that is

$$\|\nabla_\theta\phi_j\|_{L^\infty(\partial B_1)}\le L_m\,,\quad\text{for every}\quad j\in\mathbb{N}\quad\text{such that}\quad\alpha_j\le2m.$$

Now, since by hypothesis (1.3) the trace $c(\theta)$ is such that $\|P\|^2_{L^2(\partial B^1)}\le\|c\|^2_{L^2(\partial B^1)}\le1$, we have that all the constants c_j in the Fourier expansion of P are bounded by 1. Thus, the function

$P : \partial B_1 \to \mathbb{R}$ is L-Lipschitz continuous for some $L > 0$, depending only on d and m. Denoting by P_- the negative part of P, $P_-(\theta) = \min\{P(\theta), 0\}$, we get that

$$\int_{\mathbb{S}^{d-2}} P_-^2 \, d\mathcal{H}^{d-2} \geq C_d M^2 \left(\frac{M}{L}\right)^{d-2} = \frac{C_d}{L^{d-2}} M^d, \tag{4.11}$$

for some dimensional constant C_d. On the other hand, since $P + \phi$ is non-negative on $\mathbb{S}^{d-2} = \{x_d = 0\} \cap \partial B_1$ we get that

$$\int_{\mathbb{S}^{d-2}} \phi^2 \, d\mathcal{H}^{d-2} \geq \int_{\mathbb{S}^{d-2}} P_-^2 \, d\mathcal{H}^{d-2}. \tag{4.12}$$

Now, by the trace inequality on the sphere ∂B_1, there is a dimensional constant C_d such that

$$\begin{aligned} \int_{\mathbb{S}^{d-2}} \phi^2 \, d\mathcal{H}^{d-2} &\leq C_d \left(\int_{\mathbb{S}^{d-1}} |\nabla_\theta \phi|^2 \, d\mathcal{H}^{d-1} + \int_{\mathbb{S}^{d-1}} \phi^2 \, d\mathcal{H}^{d-1} \right) \\ &\leq C_d \left(1 + \frac{1}{\lambda(2m)}\right) \int_{\mathbb{S}^{d-1}} |\nabla_\theta \phi|^2 \, d\mathcal{H}^{d-1}, \end{aligned} \tag{4.13}$$

where the last inequality is due to the fact that in the Fourier expansion of ϕ there are only frequencies $\lambda_j > \lambda(2m)$. Combining (4.11), (4.12) and (4.13), we get (4.10).

Notice that in this step we used the non-negativity of the trace c (in the inequality (4.12)) and also the condition that c is bounded in $L^2(\partial B_1)$ (when we give the Lipschitz bound on P). More precisely, the constant C_3 depends on the norm $\|P\|_{L^2(\partial B_1)}$, which in turn is bounded by one.

Step 3. Conclusion of the proof of Theorem 2. By Step 1 (4.4) and Step 2 (4.10) we get

$$\begin{aligned} \mathcal{W}_{2m}(h) - (1 - \kappa_{\alpha,2m}) \mathcal{W}_{2m}(z) &\leq C_1 \kappa_{\alpha,2m}^2 M^2 - C_2 \kappa_{\alpha,2m} \|\nabla_\theta \phi\|_{L^2(\partial B_1)}^2 \\ &\leq \kappa_{\alpha,2m}^2 C_1 C_3 \|\nabla_\theta \phi\|_{L^2(\partial B_1)}^{2(1-\gamma)} - C_2 \kappa_{\alpha,2m} \|\nabla_\theta \phi\|_{L^2(\partial B_1)}^2 \\ &= \varepsilon \left(\varepsilon C_1 C_3 - C_2\right) \|\nabla_\theta \phi\|_{L^2(\partial B_1)}^{2+2\gamma}, \end{aligned} \tag{4.14}$$

where the last equality is due to the definition (4.3) of $\kappa_{\alpha,2m}$. Choosing $\varepsilon \leq \dfrac{C_2}{C_1 C_3}$, we get that

$$\mathcal{W}_{2m}(h) - \left(1 - \varepsilon \|\nabla_\theta \phi\|_{L^2(\partial B_1)}^{2\gamma}\right) \mathcal{W}_{2m}(z) \leq 0. \tag{4.15}$$

Using again Lemma 2.3, we have

$$\begin{aligned} \mathcal{W}_{2m}(z) &= \frac{1}{4m + d - 2} \sum_{j=1}^{\infty} (\lambda_j - \lambda(2m)) c_j^2 \\ &\leq \frac{1}{4m + d - 2} \sum_{\{j\,,\,\alpha_j > 2m\}}^{\infty} (\lambda_j - \lambda(2m)) c_j^2 \leq \sum_{\{j\,,\,\alpha_j > 2m\}}^{\infty} \lambda_j c_j^2 = \|\nabla_\theta \phi\|_{L^2(\partial B_1)}^2. \end{aligned}$$

which together with (4.15) gives

$$\mathcal{W}_{2m}(h) \leq \left(1 - \varepsilon \, \|\nabla_\theta \phi\|_{L^2(\partial B_1)}^{2\gamma}\right) \mathcal{W}_{2m}(z) \leq \big(1 - \varepsilon \, \mathcal{W}_{2m}^{\gamma}(z)\big) \mathcal{W}_{2m}(z),$$

which is precisely (1.4). □

We conclude this section with the following Remark, which will be useful for the characterization of the possible blow-up limits.

Remark 4.2. In the hypotheses of Theorem 2, we have the following, slightly stronger version of the logarithmic epiperimetric inequality:

$$\mathcal{W}_{2m}(h) \leq \mathcal{W}_{2m}(z) \big(1 - \varepsilon \, |\mathcal{W}_{2m}(z)|^{\gamma}\big) - \frac{C_2 \varepsilon}{2} \|\nabla_\theta \phi\|_{L^2(\partial B_1)}^{2+2\gamma}, \tag{4.16}$$

for which it is sufficient to choose $0 < \varepsilon < \frac{C_2}{2 C_1 C_3}$ in (4.14), ϕ being the function defined in (4.1) containing the higher modes of the trace c on the sphere ∂B_1.

5. Epiperimetric inequality for $\mathcal{W}_{2m-1/2}$ in dimension two: Proof of Theorem 3

We prove the theorem in several steps.

Step 1. Sectorial decomposition of $h_{2m-1/2}$. We notice that the function $h_{2m-1/2}$ has $4m-1$ half-lines from the origin along which it vanishes. These lines correspond to the angles

$$s_i := \frac{2i}{4m-1}\pi, \qquad \text{for } i = 1, ..., 4m-1,$$

and they individuate $4m-1$ circular sectors in B_1 corresponding to the nodal domains of $h_{2m-1/2}$. We consider the following $2m$ sets, which are invariant under the transformation $\theta \to -\theta$

$$S_j = \big\{(r,\theta) : r \in [0,1],\ \theta \in]s_{j-1}, s_j[\cup]2\pi - s_j, 2\pi - s_{j-1}[\big\},$$

where $j = 1, \dots, 2m$. Notice that $S_1, \dots, S_{2m-1}$ are unions of two sectors of angle $\frac{2\pi}{4m-1}$, while S_{2m} is the sector $\big\{(r,\theta) : r \in [0,1],\ \theta \in]s_{2m-1}, s_{2m}[\big\}$. We define the restrictions of $h_{2m-1/2}$ to these sectors for $j = 1, ..., 2m$

$$f_j(r,\theta) := \mathbb{1}_{S_j}(r,\theta) h_{2m-1/2}(r,\theta) = \Big(\mathbb{1}_{]s_{j-1},s_j[}(\theta) + \mathbb{1}_{]2\pi - s_j, 2\pi - s_{j-1}[}(\theta)\Big) h_{2m-1/2}(r,\theta).$$

We notice that, since $h_{2m-1/2}$ vanishes on $B_1 \cap \partial S_j$, the fuctions f_j are in $H^1(B_1)$. Moreover, they are $(2m - 1/2)$-homogeneous even functions, namely $f_j(r,\theta) = r^{2m-1/2} f_j(\theta)$ and $f_j(r,\theta) = f_j(r, -\theta)$. We claim that for any $b_1, ..., b_{2m} \in \mathbb{R}$

$$\mathcal{W}_{2m-1/2}\Big(\sum_{i=1}^{2m} b_i f_i\Big) = 0. \tag{5.1}$$

Indeed, since the energy $\mathcal{W}_{2m-1/2}$ is quadratic in its argument and for every $i \neq j$ the supports of f_i and f_j have negligible intersection, the energy of the linear combination is given by

$$\mathcal{W}_{2m-1/2}\Big(\sum_{i=1}^{2m} b_i f_i\Big) = \sum_{i=1}^{2m} b_i^2 \mathcal{W}_{2m-1/2}(f_i).$$

Moreover the functions f_i are harmonic in each S_j and vanish on the rays delimiting their support, that is on $\partial S_j \cap B_1$. Thus $\mathcal{W}_{2m-1/2}(f_i) = 0$ for every $i = 1, ..., 2m$, so that the previous inequality implies (5.1).

Step 2. Decomposition of the datum c. We claim that we can write c in a unique way as

$$c(\theta) = \sum_{i=1}^{2m} a_i f_i(\theta) + \tilde{c}(\theta) \qquad \text{on} \quad \partial B_1,$$

where

- $a_1, ..., a_{2m} \in \mathbb{R}$ and $a_{2m} > 0$
- $\tilde{c} \in H^1(\partial B_1)$ is even and it is orthogonal in $L^2(\partial B_1)$ to $1, \cos(\theta), ..., \cos((2m-1)\theta)$.

To prove this claim, we call L the span of 1, $cos(\theta)$,..., $\cos((2m-1)\theta)$, which is a linear subspace of $L^2(\partial B_1)$ of dimension $2m$. We set $P_L(c)$ to be the projection of c onto L. To show the existence of $a_1, ..., a_{2m} \in \mathbb{R}$, it is enough to prove that the $2m$ functions $P_L(f_1), ..., P_L(f_{2m})$ are linearly independent, so that their span gives the whole L. Hence, we take any linear combination $b_1 f_1 + ... + b_{2m} f_{2m}$, such that its projection on L is 0, aiming to prove that $b_1 = ... = b_{2m} = 0$. By (5.1), the energy of $b_1 f_1 + ... + b_{2m} f_{2m}$ is 0. On the other hand, since the function $b_1 f_1 + ... + b_{2m} f_{2m}$ is assumed to have only modes higher than $2m - 1/2$ on ∂B_1, its $(2m - 1/2)$-homogenous extension has nonnegative energy thanks to (2.8), and its energy is 0 if and only if $b_1 f_1 + ... + b_{2m} f_{2m} \equiv 0$. Hence, this must be the case. Hence we can write in a unique way $P_L(c)$ as a linear combination of $P_L(f_1), ..., P_L(f_{2m})$

$$P_L(c) = \sum_{i=1}^{2m} a_i P_L(f_i).$$

Since c is assumed to be close to $h_{2m-1/2}$ by (1.5), and since $h_{2m-1/2}$ is strictly positive on the support of f_{2m}, we can assume without loss of generality that $a_{2m} > 0$. Finally, we set $\tilde{c} = c - \sum_{i=1}^{2m} a_i f_i$.

Step 3. Choice of an energy competitor and computation of the energy. We let $\alpha > 2m - 1/2$ to be chosen later and we define an energy competitor for c as

$$h(r,\theta) := \sum_{j=1}^{2m} a_j f_j(r,\theta) + r^\alpha \tilde{c}(\theta) = r^{\frac{4m-1}{2}}(c - \tilde{c}) + r^\alpha \tilde{c}.$$

The energy of h can be written as

$$\begin{aligned}\mathcal{W}_{2m-1/2}(h) =& \mathcal{W}_{2m-1/2}\big(r^{\frac{4m-1}{2}}(c-\tilde{c})\big) + \mathcal{W}_{2m-1/2}\big(r^\alpha \tilde{c}\big)\\ &+ 2\int_{B_1} \nabla\big(r^{\frac{4m-1}{2}}(c-\tilde{c})\big)\cdot\nabla\big(r^\alpha\tilde{c}\big)\,d\mathcal{H}^2 - (4m-1)\int_{\partial B_1}(c-\tilde{c})\tilde{c}\,d\mathcal{H}^1.\end{aligned}$$

By the deinition of $\tilde{c}$ and Step 1 we have that the first term in the right-hand side vanishes:

$$\mathcal{W}_{2m-1/2}(h) = \mathcal{W}_{2m-1/2}\big(r^\alpha\tilde{c}\big) + 2\sum_{j=1}^{2m} a_j\left(\int_{B_1}\nabla f_j\cdot\nabla\big(r^\alpha\tilde{c}\big)\,d\mathcal{H}^2 - (4m-1)\int_{\partial B_1} f_j\tilde{c}\,d\mathcal{H}^1\right). \quad (5.2)$$

We rewrite the middle term integrating by parts, and using that $\Delta f_j = 0$ on $\{f_j \neq 0\}$

$$\begin{aligned}\int_{B_1}\nabla f_j\cdot\nabla\big(r^\alpha\tilde{c}\big)\,d\mathcal{H}^2 &= \int_{S_j} r^\alpha\tilde{c}(\theta)\,\Delta f_j + \int_{\partial S_j}\frac{\partial f_j}{\partial n}\,r^\alpha\tilde{c}\\ &= 2\int_{\partial B_1^+\cap S_j}\frac{\partial f_j}{\partial r}r^\alpha\tilde{c}(\theta)\,d\mathcal{H}^1 + 2\int_{\{\theta=s_j\}}\frac{1}{r}\frac{\partial f_j}{\partial\theta}r^\alpha\tilde{c}(\theta)\,d\mathcal{H}^1 - 2\int_{\{\theta=s_{j-1}\}}\frac{1}{r}\frac{\partial f_j}{\partial\theta}r^\alpha\tilde{c}(\theta)\,d\mathcal{H}^1.\end{aligned}$$

Now since f_j is $(2m - 1/2)$-homogeneous we can write $f_j(r,\theta) = r^{2m-1/2} f_j(\theta)$ and we get that

$$2\int_{\partial B_1^+\cap S_j}\frac{\partial f_j}{\partial r}r^\alpha\tilde{c}(\theta)\,d\mathcal{H}^1 = 2\int_{\partial B_1^+\cap S_j}\partial_r\big(r^{2m-1/2}f_j(\theta)\big)r^\alpha\tilde{c}(\theta)\,d\mathcal{H}^1 = (4m-1)\int_{\partial B_1^+} f_j\tilde{c}\,d\mathcal{H}^1,$$

$$\begin{aligned}2\int_{\{\theta=s_j\}}\frac{1}{r}\frac{\partial f_j}{\partial\theta}r^\alpha\tilde{c}(\theta)\,d\mathcal{H}^1 - 2\int_{\{\theta=s_{j-1}\}}\frac{1}{r}\frac{\partial f_j}{\partial\theta}r^\alpha\tilde{c}(\theta)\,d\mathcal{H}^1&\\ = 2\int_0^1 r^{\alpha+\frac{4m-3}{2}}\big(\partial_\theta f_j(s_j)\tilde{c}(s_j) - \partial_\theta f_j(s_{j-1})\tilde{c}(s_{j-1})\big)\,dr&\\ = \frac{2}{\alpha+2m-1/2}\big(\partial_\theta f_j(s_j)\tilde{c}(s_j) - \partial_\theta f_j(s_{j-1})\tilde{c}(s_{j-1})\big).&\end{aligned}$$

Hence we can rewrite (5.2) as

$$\mathcal{W}_{2m-1/2}(h) = \mathcal{W}_{2m-1/2}\big(r^\alpha\tilde{c}\big) + \frac{2}{\alpha+2m-1/2}\sum_{j=1}^{2m} a_j\big(\partial_\theta f_j(s_j)\tilde{c}(s_j) - \partial_\theta f_j(s_{j-1})\tilde{c}(s_{j-1})\big). \quad (5.3)$$

Since the previous two equalities hold also when $\alpha = 2m - 1/2$, we see that

$$\mathcal{W}_{2m-1/2}(z) = \mathcal{W}_{2m-1/2}\big(r^{\frac{4m-1}{2}}\tilde{c}\big) + \frac{2}{4m-1}\sum_{j=1}^{2m} a_j\big(\partial_\theta f_j(s_j)\tilde{c}(s_j) - \partial_\theta f_j(s_{j-1})\tilde{c}(s_{j-1})\big). \quad (5.4)$$

Step 4. Conclusion. Setting $\kappa_{\alpha,2m-1/2}$ according to (2.7), a suitable linear combination between the last terms in (5.3) and (5.4) is 0, because by the defintion of $\kappa_{\alpha,2m-1/2}$ we have

$$\frac{2}{\alpha+2m-1/2} - (1-\kappa_{\alpha,2m-1/2})\frac{2}{4m-1} = 0. \quad (5.5)$$

Putting together (5.3), (5.4) and (5.5), we find

$$\mathcal{W}_{2m-1/2}(h) - (1-\kappa_{\alpha,2m-1/2})\mathcal{W}_{2m-1/2}(z) = \mathcal{W}_{2m-1/2}\big(r^\alpha\tilde{c}\big) - (1-\kappa_{\alpha,2m-1/2})\mathcal{W}_{2m-1/2}\big(r^{\frac{4m-1}{2}}\tilde{c}\big).$$

Thanks to Lemma 2.3, in particular to (2.9), we obtain that

$$\mathcal{W}_{2m-1/2}\big(r^{\alpha}\tilde{c}\big) - (1-\kappa_{\alpha,2m-1/2})\mathcal{W}_{2m-1/2}\big(r^{\frac{4m-1}{2}}\tilde{c}\big) \le 0,$$

because by definition $\tilde{c}$ is orthogonal to $1, \cos(\theta), ..., \cos((2m-1)\theta)$ (which, in dimension 2, are the only eigenfunctions with corresponding homogeneity less than or equal to $2m - 1/2$). □

6. Admissible frequencies for the thin-obstacle problem

We first prove an easy version of the epiperimetric inequality useful for negative energies. Then we use this result, together with Lemma 2.4 and Theorems 1 and 2 to conclude the proof of Theorem 4.

6.1. **Epiperimetric inequality for negative energies.** The following proposition gives an epiperimetric inequality for negative energies.

Proposition 6.1 (Epiperimetric inequality for negative energies)**.** *Let $d \ge 2$, $c \in H^1(\partial B_1)$ be a function such that its $2m$-homogeneous extension $z(r,\theta) := r^{2m}c(\theta) \in \mathcal{A}$ and $\|c\|_{L^2(\partial B_1)} = 1$.*

Then there exist a constant $\varepsilon = \varepsilon(d,m) > 0$ and a function $h \in \mathcal{A}$ with $h = c$ on ∂B_1 and

$$\mathcal{W}_{2m}(h) \le (1+\varepsilon)\mathcal{W}_{2m}(z). \tag{6.1}$$

Proof. For $j \in \mathbb{N}$, let ϕ_j be the eigenfunctions of the Laplacian on ∂B_1, λ_j and α_j the corresponding eigenvalues and homogeneities (see Subsection 2.4). We decompose c in Fourier as

$$c = \sum_{j=1}^{\infty} c_j\phi_j = \sum_{\{j\,:\,\alpha_j<2m\}} c_j\,\phi_j + \sum_{\{j\,:\,\alpha_j=2m\}} c_j\,\phi_j + \sum_{\{j\,:\,\alpha_j>2m\}} c_j\,\phi_j =: c_< + c_= + c_>$$

We consider the maximum of the negative part of $c_<$

$$M := -\min\big\{ \min\{c_<(\theta), 0\} \;:\; \theta \in \partial B_1,\ \theta_d = 0\big\}.$$

Since Q contains only low Fourier frequencies, M is controlled by $\|Q\|_{L^2(\partial B_1)}$, namely, there is a constant $C_1 := C_1(d,m) > 0$ such that

$$M^2 \le \Big(\sum_{\{j\,:\,\alpha_j<2m\}} |c_j|\Big)^2 \le C_1 \sum_{\{j\,:\,\alpha_j<2m\}} c_j^2 = C_1\|Q\|^2_{L^2(\partial B_1)}. \tag{6.2}$$

Let $\alpha := \alpha(d,m) \in (2m-1, 2m)$ to be chosen later and let

$$\varepsilon := \frac{2m-\alpha}{\alpha+2m+d-2} > 0. \tag{6.3}$$

Let h_{2m} be the eigenfunction built in Remark 4.1, corresponding to the homogeneity $2m$, such that $h_{2m} \equiv 1$ on the hyperplane $\{x_d = 0\} \cap \partial B_1$. We set for simplicity

$$h_{<,\mu}(r,\theta) := (c_<(\theta) + Mh_{2m}(\theta))r^{\mu} \qquad \text{for } \mu = 2m, \alpha,$$

$$h_=(r,\theta) := (c_=(\theta) - Mh_{2m}(\theta))r^{2m}, \qquad h_>(r,\theta) := c_> r^{2m}\phi_j(\theta).$$

We notice that z can be written as a sum of these objects and we introduce the energy competitor h, obtaining by extending the lower modes of c with homogeneity α and leaving the rest unchanged

$$z = h_{<,2m} + h_= + h_> \qquad \text{and} \qquad h = h_{<,\alpha} + h_= + h_>. \tag{6.4}$$

Since $0 \le h_{<,\alpha} \ge h_{<,2m}$ on B_1', we have that $h \ge z \ge 0$ in B_1'; moreover, $h = z$ on ∂B_1.

Next, we compute the energy of z and h. Since $h_=$ is harmonic and $2m$-homogenous, and since $h_>$ is orthogonal in $L^2(B_1)$ and $H^1(B_1)$ to $h_{<,\mu}$, for $\mu = 2m, \alpha$ we have

$$\mathcal{W}_{2m}(h_{<,\mu} + h_= + h_>) = \mathcal{W}_{2m}(h_{<,\mu} + h_= + h_>) = \mathcal{W}_{2m}(h_{<,\mu}) + \mathcal{W}_{2m}(h_>).$$

Thus, we rewrite the quantity in (6.1) and we observe that $\mathcal{W}_{2m}(h_>) \ge 0$ by Lemma 2.3

$$\begin{aligned}\mathcal{W}_{2m}(h) - (1+\varepsilon)\mathcal{W}_{2m}(z) &= \mathcal{W}_{2m}(h_{<,\alpha}) - (1+\varepsilon)\mathcal{W}_{2m}(h_{<,2m}) - \varepsilon\mathcal{W}_{2m}(h_>)\\ &\le \mathcal{W}_{2m}(h_{<,\alpha}) - (1+\varepsilon)\mathcal{W}_{2m}(h_{<,2m}).\end{aligned} \tag{6.5}$$

Denoting by λ the function in (2.6) by Lemma 2.3 we rewrite the right-hand side as

$$\mathcal{W}_{2m}(h_{<,\alpha}) - (1+\varepsilon)\mathcal{W}_{2m}(h_{<,2m}) = M^2\|h_{2m}\|^2_{L^2(\partial B_1)}\frac{\varepsilon(-\lambda(2m)+\lambda(\alpha))}{d+2\alpha-2} - \frac{\varepsilon}{d+2\alpha-2}\sum_{\{j\,,\,\alpha_j<2m\}}^{\infty}(-\lambda_j+\lambda(\alpha))c_j^2. \quad (6.6)$$

Since $2m - 1/2 < \alpha < 2m$, then setting $C_2 := \lambda(2m-1/2) - \lambda(2m-1) > 0$, such that

$$\sum_{\{j\,,\,\alpha_j<2m\}}(\lambda(\alpha)-\lambda_j)c_j^2 \geq C_2\sum_{\{j\,,\,\alpha_j<2m\}}c_j^2 = C_2\|Q\|^2_{L^2(\partial B_1)} \geq \frac{C_2}{C_1}M^2, \quad (6.7)$$

where in the last inequality we used (6.2). Since $-\lambda(2m)+\lambda(\alpha) = \varepsilon(2m+\alpha+d-2)^2$, combining (6.5), (6.7) and (6.6) we get

$$\begin{aligned}\mathcal{W}_{2m}(h) - (1+\varepsilon)\mathcal{W}_{2m}(z) &\leq M^2\|h_{2m}\|^2_{L^2(\partial B_1)}\frac{\varepsilon^2(2m+\alpha+d-2)^2}{d+2\alpha-2} - \frac{\varepsilon C_2 M^2}{C_1(d+2\alpha-2)}\\ &\leq \frac{M^2\varepsilon}{d+2\alpha-2}\Big(\|h_{2m}\|^2_{L^2(\partial B_1)}(4m+d)^2\varepsilon - \frac{C_2}{C_1}\Big). \quad (6.8)\end{aligned}$$

Choosing $\varepsilon := \varepsilon(d,m)$ small enough, namely α sufficiently close to $2m$ by the choice of ε in (6.3), we find that the right-hand side in (6.8) is less than or equal to 0, that is (6.1). □

6.2. **Proof of Theorem 4.** We divide the proof in two steps.

6.2.1. *Frequencies* $\frac{3}{2}$ *and* $2m$. We first prove (1.10). Let $c : \partial B_1 \to \mathbb{R}$ be the trace of a $3/2$-homogeneous non-trivial global solution $z \in \mathcal{K}_{3/2}$ of the thin-obstacle problem. Let v be the competitor defined in (3.1). By the optimality of z and the epiperimetric inequality (1.2) we get that

$$0 = \mathcal{W}_{3/2}(z) \leq \mathcal{W}_{3/2}(v) \leq \left(1 - \frac{1}{2d+3}\right)\mathcal{W}_{3/2}(z) = 0,$$

and in particular both the inequalities are in fact equalities. By Remark 3.1 we get that $z = Ch_e + r^{3/2}\phi$, where $C \geq 0$, $e \in \partial B_1'$ and $\phi : \partial B_1 \to \mathbb{R}$ is an eigenfunction of the sperical Laplacian, corresponding to the eigenvalue $\lambda(2) = 2d$, and such that $\phi \geq 0$ on $\partial B_1'$. Thus, we have

$$\begin{aligned}0 = \mathcal{W}_{3/2}(z) &= \mathcal{W}_{3/2}(h) + \mathcal{W}_{3/2}(r^{3/2}\phi) + 2C\left(\int_{B_1}\nabla h_e\cdot\nabla(r^{3/2}\phi(\theta)) - \frac{3}{2}\int_{\partial B_1}h_e\phi\,d\mathcal{H}^{d-1}\right)\\ &\geq \mathcal{W}_{3/2}(h) + \mathcal{W}_{3/2}(r^{3/2}\phi) \geq \mathcal{W}_{3/2}(r^{3/2}\phi) \geq 0,\end{aligned}$$

where, by Lemma 2.3 the last inequality is an equality if and only if $\phi \equiv 0$. Thus, $z = Ch_e$ for some $e \in \partial B_1'$ and $C \geq 0$. Since $0 \neq \mathcal{K}_{3/2}$ we get that $C > 0$, which concludes the proof of (1.10).

We now prove (1.11). Suppose that $c \in H^1(\partial B_1)$ is the trace of $2m$-homogeneous non-trivial global solution of the thin-obstacle problem. Let h be the competitor from (4.2). By the optimality of $r^{2m}c(\theta)$ and the improved version of the logarithmic epiperimetric inequality (4.16) we have

$$0 = \mathcal{W}_{2m}(r^{2m}c) \leq \mathcal{W}_{2m}(h) \leq \mathcal{W}_{2m}(r^{2m}c)\big(1-\varepsilon\,|\mathcal{W}_{2m}(r^{2m}c)|^\gamma\big) - \varepsilon_2\|\nabla_\theta\phi\|^{2+2\gamma}_{L^2(\partial B_1)} = -\varepsilon_2\|\nabla_\theta\phi\|^{2+2\gamma}_{L^2(\partial B_1)}.$$

Thus, necessarily $\|\nabla_\theta\phi\|_{L^2(\partial B_1)} = 0$, that is the Fourier expansion of c on the sphere ∂B_1 contains only low frequencies: $c(\theta) = \sum_{\{j\,,\,\alpha_j\leq 2m\}} c_j\phi_j(\theta)$. Now by Lemma 2.3 we get

$$0 = \mathcal{W}_{2m}(r^{2m}c) = \frac{1}{4m+d-2}\sum_{\{j\,,\,\alpha_j\leq 2m\}}(\lambda_j - \lambda(2m))c_j^2 \leq 0,$$

and so all coefficients, corresponding to frequencies with $\alpha_j < 2m$, must vanish. Thus c is a non-zero eigenfunction on the sphere corresponding to the eigenvalue $\lambda(2m) = 2m(2m+d-2)$.

6.2.2. *Frequency gap.* Let us first prove that

$$\mathcal{K}_\lambda = \emptyset \text{ for every } \lambda \in (3/2, 2).$$

Let $\lambda = 3/2 + t \in (3/2, 2)$ be an admissible frequency and $c \in H^1(\partial B_1)$ a non-trivial function whose $(3/2+t)$-homogeneous extension $r^{3/2+t}c(\theta) \in \mathcal{K}_{3/2+t}$ is a solution of the thin-obstacle problem. Let v be the competitor from (3.1). By the minimality of $r^{3/2+t}c(\theta)$, Theorem 1 and Lemma 2.4, applied with $\mu = 3/2$, we have that

$$\mathcal{W}_{3/2}(r^{3/2+t}c) \le \mathcal{W}_{3/2}(v) \le \left(1 - \frac{1}{2d+3}\right)\mathcal{W}_{3/2}(r^{3/2}c) = \left(1 - \frac{1}{2d+3}\right)\left(1 + \frac{t}{d+1}\right)\mathcal{W}_{3/2}(r^{3/2+t}c).$$

Since $\mathcal{W}_{3/2}(r^{3/2+t}c) > 0$, we get

$$\left(1 - \frac{1}{2d+3}\right)\left(1 + \frac{t}{d+1}\right) \ge 1,$$

which implies that $t \ge 1/2$ and concludes the proof of the claim.

We now fix $m \in \mathbb{N}_+$. We will show that there are constants $c_m^+ > 0$ and $c_m^- > 0$, depending only on d and m, such that

$$\mathcal{K}_\lambda = \emptyset \text{ for every } \lambda \in (2m - C_-, 2m + C_+) \setminus \{2m\}.$$

Let $\lambda = 2m + t$ be an admissible frequency and $c \in H^1(\partial B_1)$, $\|c\|_{L^2(B_1)} = 1$, a trace whose $(2m+t)$-homogeneous extension $r^{2m+t}c(\theta)$ is a minimizer of the thin-obstacle problem.

Suppose first that $t > 0$. Let h be the competitor from (4.2). By the minimality of $r^{2m+t}c$, Theorem 2 and Lemma 2.4, applied with $\mu = 2m$, we have that

$$\mathcal{W}_{2m}(r^{2m+t}c) \le \mathcal{W}_{2m}(h) \le (1 - \varepsilon t^\gamma)\,\mathcal{W}_{2m}(r^{2m}c) = (1 - \varepsilon t^\gamma)\left(1 + \frac{t}{4m+d-2}\right)\mathcal{W}_{3/2}(r^{3/2+t}c),$$

where for the first inequality we used that $\mathcal{W}_{2m}(r^{2m}c) \ge \mathcal{W}_{2m}(r^{2m+t}c) = t > 0$. By the positivity of $\mathcal{W}_{2m}(r^{2m+t}c)$, we get

$$(1 - \varepsilon t^\gamma)\left(1 + \frac{t}{4m+d-2}\right) \ge 1,$$

which provides us with the constant c_m^+.

Let now $t < 0$. Let h be the competitor from (6.4). By the minimality of $r^{2m+t}c$, Proposition 6.1 and Lemma 2.4, applied with $\mu = 2m$, we have

$$\mathcal{W}_{2m}(r^{2m+t}c) \le \mathcal{W}_{2m}(h) \le (1 + \varepsilon)\,\mathcal{W}_{2m}(r^{2m}c) = (1 + \varepsilon)\left(1 + \frac{t}{4m+d-2}\right)\mathcal{W}_{3/2}(r^{3/2+t}c).$$

Now since $\mathcal{W}_{2m}(r^{2m+t}c) = t < 0$ we get that

$$(1 + \varepsilon)\left(1 + \frac{t}{4m+d-2}\right) \le 1, \tag{6.9}$$

which gives us $c_m^- = \varepsilon(4m+d-2)/(1+\varepsilon)$, where ε is the constant from Proposition 6.1. □

Remark 6.2. Taking for instance $d = 3$, $m = 2$, we show how the constants in Theorem 4 can be made explicit. The polynomial h_4 of Remark 4.1 is given by $\frac{32}{3}x_3^4 - 10x_3^2(x_1^2 + x_2^2) + (x_1^2 + x_2^2)^2$ (and $\|h_4\|_{L^2(\partial B_1)} \sim 9.6$), the constant $C_1 = 16$ in (6.2) is the number of eigenfunctions with homogeneity less than 4, the constant C_2 in (6.7) is $15/4$. Hence, the optimal ε in (6.8) and the corresponding c_2^- deduced from (6.9) are given by

$$\varepsilon = \frac{C_2}{C_1\|h_4\|^2_{L^2(\partial B_1)}11^2} \qquad \text{and} \qquad c_2^- = \frac{9\varepsilon}{1+\varepsilon} \ge 0.0015.$$

7. Regularity of the regular and singular parts of the free-boundary

The first part of Theorem 5 was first proved in [2]. Once we have the epiperimetric inequality (1.2), it follows by a standard argument that can be found for example in [10, 12]. So we proceed with the proof of (ii). We start with the following proposition.

7.1. **Rate of convergence of the blow-up sequences.** Before starting the proof we remark that, by a simple scaling argument, if in Theorem 2 we replace the condition (1.3) with

$$\int_{\partial B_1} c^2 \, d\mathcal{H}^{d-1} \leq \Theta \qquad \text{and} \qquad |\mathcal{W}_{2m}(z)| \leq \Theta,$$

for some $\Theta > 0$, then the epiperimetric inequality (1.4) still holds, with ε replaced by $\varepsilon\,\Theta^{-\gamma}$. We will use this in the first step of the proof of the following

Proposition 7.1 (Decay of the Weiss' energy)**.** *Let $u \in H^1(B_1)$ be a minimizer of $\mathcal{E}$. Then for every $m \in \mathbb{N}$ and every compact set $K \Subset B_1' \cap \mathcal{S}^{2m}$, there is a constant $C := C(m, d, K, \|u\|_{H^1(B_1)}) > 0$ such that for every free boundary point $x_0 \in \mathcal{S}^{2m} \cap K$, the following decay holds*

$$\|u_{x_0,t} - u_{x_0,s}\|_{L^1(\partial B_1)} \leq C\,(-\log(t))^{-\frac{1-\gamma}{2\gamma}} \qquad \textit{for all} \quad 0 < s < t < \operatorname{dist}(K, \partial B_1)\,. \tag{7.1}$$

In particular the blow-up limit of u at x_0 is unique.

Proof. We divide the proof in three steps.

Step 1. Applicability of epiperimetric inequality at every scale. Let $\int_{B_1} u^2\,dx = \Theta_0$. Then, by the monotonicity of $\frac{H^{x_0}(r)}{r^{d-1+2\lambda}}$, for every λ (see Lemma 2.2), we deduce that

$$\Theta_0 \geq \int_{B_R(x_0)} u^2\,dx \geq \int_{R/2}^{R} H^{x_0}(r)\,dr \geq (R/2)^{d-1+2\lambda}\ \frac{H^{x_0}(R/2)}{R/2},$$

where $R := \operatorname{dist}(x_0, \partial B_1)$. In particular we have, using again the monotonicity of $\frac{H^{x_0}(r)}{r^{d-1+2\lambda}}$,

$$0 \leq \frac{H^{x_0}(r)}{r^{d-1+2\lambda}} \leq \left(\frac{2}{R}\right)^{d-1+2\lambda} \Theta_0\,, \qquad \text{for every } 0 < r < R/2 \text{ and } x_0 \in B_1\,.$$

For what concerns $\mathcal{W}_\lambda^{x_0}$ notice that

$$\mathcal{W}_\lambda^{x_0}(R) \leq R^{d-2+2\lambda} \int_{B_1} |\nabla u|^2\,dx\,, \qquad \text{for every } x_0 \in B_1\,.$$

Taking $\lambda = 2m$, it follows that for every $0 < r_0 \leq 1$ and for every $x_0 \in B_{1-r_0}$, we can apply (1.4) for every $0 < r < r_0$ and every rescaling $u_{x_0,r}(x) = \dfrac{u(x_0 + rx)}{r^{2m}}$ with Θ depending on r_0, d, m and $\|u\|_{H^1(B_1)}$.

Step 2. Closeness of the blow ups for a given point x_0. Let $r_0 > 0$ and $x_0 \in B_{1-r_0}$ and let $r \in (0, r_0]$. Then by Step 1 we can apply (1.4) to $u_{x_0,r}$ for every $0 < r < r_0$. We claim that

$$\|u_{x_0,t} - u_{x_0,s}\|_{L^1(\partial B_1)} \leq C\,(-\log(t/r_0))^{-\frac{1-\gamma}{2\gamma}} \qquad \text{for all} \quad 0 < s < t < r_0\,.$$

We assume $x_0 = 0$ without loss of generality, we fix $m \in \mathbb{N}$ and write $\mathcal{W}(r) = \mathcal{W}_{2m}^{x_0}(r, u)$. By (2.1)

$$\frac{d}{dr}\mathcal{W}(r) = \frac{(d-2+4m)}{r}\,(\mathcal{W}(z_r) - \mathcal{W}(r)) + \underbrace{\frac{1}{r}\int_{\partial B_1} (\nabla u_r \cdot \nu - 2m\,u_r)^2\ d\mathcal{H}^{d-1}}_{=:f(r)} \tag{7.2}$$

and the epiperimetric inequality of Theorem 2, there exists a radius $r_0 > 0$ such that for every $r \leq r_0$

$$\frac{d}{dr}\mathcal{W}(r) \geq \frac{d-2+4m}{r}\big(\mathcal{W}(z_r) - \mathcal{W}(r)\big) + f(r) \geq \frac{c}{r}\mathcal{W}(r)^{1+\gamma} + 2f(r) \tag{7.3}$$

where $c = \varepsilon\,\Theta^{-\gamma}(d-2+4m)$ and $\gamma \in (0,1)$ is a dimensional constant. In particular we obtain that

$$\frac{d}{dr}\Big(\frac{-1}{\gamma\mathcal{W}(r)^\gamma} - c\log r\Big) = \frac{1}{\mathcal{W}(r)^{1+\gamma}}\frac{d}{dr}\mathcal{W}(r) - \frac{c}{r} \geq \frac{1}{\mathcal{W}(r)^{1+\gamma}} f(r) \geq 0 \tag{7.4}$$

and this in turn implies that $-\mathcal{W}(r)^{-\gamma} - c\gamma\log r$ is an increasing function of r, namely that $\mathcal{W}(r)$ decays as

$$\mathcal{W}(r) \leq (\mathcal{W}(r_0)^{-\gamma} + c\gamma\log r_0 - c\gamma\log r)^{\frac{-1}{\gamma}} \leq (-c\gamma\log(r/r_0))^{\frac{-1}{\gamma}}\,. \tag{7.5}$$

For any $0 < s < t < r_0$ we estimate the L^1 distance between the blow-up at scales s and t through the Cauchy-Schwarz inequality and the monotonicity formula (7.2)

$$\begin{aligned}\int_{\partial B_1} |u_t - u_s|\, d\mathcal{H}^{d-1} &\le \int_{\partial B_1}\int_s^t \frac{1}{r}\,|x\cdot\nabla u_r - 2u_r|\, dr\, d\mathcal{H}^{n-1}\\ &\le \big(d\omega_d\big)^{1/2}\int_s^t r^{-1/2}\left(\frac{1}{r}\int_{\partial B_1}|x\cdot\nabla u_r - 2u_r|^2\, d\mathcal{H}^{d-1}\right)^{1/2} dr\\ &\le \left(\frac{d\omega_d}{2}\right)^{1/2}\int_s^t r^{-1/2}(\mathcal{W}'(r))^{1/2}\, dr \qquad (7.6)\\ &\le \left(\frac{d\omega_d}{2}\right)^{1/2}(\log(t)-\log(s))^{1/2}(\mathcal{W}(t)-\mathcal{W}(s))^{1/2}\,.\end{aligned}$$

Let $0 < s < t < r_0/2$ and $0 \le j \le i$ be such that $s/r_0 \in [2^{-2^{i+1}}, 2^{-2^i})$ and $t/r_0 \in [2^{-2^{j+1}}, 2^{-2^j})$. Applying the previous estimate (7.5) to the exponentially dyadic decomposition, we obtain

$$\begin{aligned}\int_{\partial B_1}|u_t-u_s|\, d\mathcal{H}^{d-1} &\le \int_{\partial B_1}\Big|u_t - u_{2^{-2^{j+1}}r_0}\Big|\, d\mathcal{H}^{d-1}\\ &\quad+\int_{\partial B_1}\Big|u_{2^{-2^i}r_0}-u_s\Big|\, d\mathcal{H}^{d-1} + \sum_{k=j+1}^{i-1}\int_{\partial B_1}\Big|u_{2^{-2^{k+1}}r_0}-u_{2^{-2^k}r_0}\Big|\, d\mathcal{H}^{d-1}\\ &\le C\sum_{k=j}^{i}\Big(\log\big(2^{-2^k}\big)-\log\big(2^{-2^{k+1}}\big)\Big)^{1/2}\Big(\mathcal{W}\big(2^{-2^k}r_0\big)-\mathcal{W}\big(2^{-2^{k+1}}r_0\big)\Big)^{1/2}\\ &\le C\sum_{k=j}^{i}2^{k/2}\mathcal{W}\big(2^{-2^k}r_0\big)^{1/2} \le C\sum_{k=j}^{i}2^{(1-1/\gamma)k/2}\\ &\le C2^{(1-1/\gamma)i/2}\le C(-\log(t/r_0))^{\frac{\gamma-1}{2\gamma}}\,, \qquad (7.7)\end{aligned}$$

where C is a constant, depending on d, m, r_0 and $\|u\|_{H^1(B_1)}$, that may vary from line to line.

Step 3. Conclusion. We notice that for $t \le r_0^2$, we have $\log(t/r_0) \le \frac{1}{2}\log t$, so that

$$\|u_{x_0,t}-u_{x_0,s}\|_{L^1(\partial B_1)} \le C\,(-\log(t))^{-\frac{1-\gamma}{2\gamma}} \qquad \text{for every} \quad 0<s<t<r_0^2.$$

Since $u_{x_0,t}$ is bounded in $L^2(\partial B_1)$ for every $t \le r_0$, by possibly enlarging the constant C, the above inequality holds for $0 < s < t < r_0$. □

7.2. **Non-degeneracy of the blow-up.** We now use the previous Proposition to prove that the blow-up limits are non-trivial. This is the only part of the proof of Theorem 5 where the frequency of the point plays a role.

Lemma 7.2 (Non-degeneracy). *Let $u \in H^1(B_1)$ be a minimizer of $\mathcal{E}$ and let $x_0 \in \mathcal{S}^\lambda$, where $\lambda \in \{3/2\} \cup \{2m : m \in \mathbb{N}\}$. Then the following strict lower bound holds*

$$H_0^{x_0} := \lim_{r\to 0}\frac{H^{x_0}(r)}{r^{d-1+2\lambda}} > 0\,.$$

In particular, since by the strong $L^2(\partial B_1)$ convergence of $u_{x_0,r}$ to the unique blow up p_{x_0} we have $H_0 := \|p_{x_0}\|^2_{L^2(\partial B_1)}$, it follows that p_{x_0} is non-trivial.

Proof. Without loss of generality we can suppose that $x_0 = 0$. We give the proof for $\lambda := 2m = N(0)$ for some $m \in \mathbb{N}$, the case $\lambda = 3/2$ being analogous. Assume by contradiction that

$$h(r) := \left(\frac{H(r)}{r^{d-1}}\right)^{1/2} = o(r^\lambda)$$

and consider the sequence $u_r(x) := \dfrac{u(rx)}{h(r)}$. It follows that $\|u_r\|_{L^2(\partial B_1)} = 1$ for every r, and so, by the monotonicity of the frequency function

$$\int_{B_1} |\nabla u_r|^2 \, dx = \frac{1}{r^{d-2}} D(r) \le N(1) \, \frac{1}{r^{d-1}} H(r) \le N(1) \,,$$

so that, up to a not relabeled subsequence, u_r converges weakly in $H^1(B_1)$ and strongly in $L^2(\partial B_1)$ to some function $p_\lambda \in H^1(B_1)$ such that $\|p_\lambda\|_{L^2(\partial B_1)} = 1$. Moreover, since $N(0) = \lambda$, p_λ is a λ-homogeneous function. Notice also that due to Theorem 2.1 the convergence is locally uniform in B_1. Next, for every u_r consider its blow-up sequence $[u_r]_\rho(x) := \rho^{-\lambda} u_r(\rho x)$. By Proposition 7.1, we know that, for every $r > 0$, there exists a unique blow-up limit $p_{\lambda,r} = \lim_{\rho\to 0}[u_r]_\rho$. Moreover, since all the functions u_r are uniformly bounded in $H^1(B_1)$, $\|u_r\|^2_{H^1(B_1)} \le N(1) + 1$, there is a constant C depending on the dimension, λ and $N(1)$ such that

$$\|[u_r]_t - p_{\lambda,r}\|^2_{L^2(\partial B_1)} \le C \,(-\log(t))^{-\frac{1-\gamma}{\gamma}} \qquad \text{for all} \quad 0 < t < 1 \,, \tag{7.8}$$

where we used the regularity of u to replace the L^1-norm from Proposition 7.1 with the L^2-norm. Using our contradiction assumption and the strong convergence of $[u_r]_\rho$ to $p_{\lambda,r}$ in $L^2(\partial B_1)$, we have

$$\|p_{\lambda,r}\|_{L^2(\partial B_1)} = \lim_{\rho\to 0} \frac{1}{\rho^{d-1+2\lambda}} \int_{\partial B_\rho} u_r^2 \, d\mathcal{H}^{d-1} = \frac{r^{d-1+2\lambda}}{\int_{\partial B_r} u^2 \, d\mathcal{H}^{d-1}} \, \lim_{\rho\to 0} \frac{1}{(r\,\rho)^{d-1+2\lambda}} \int_{\partial B_{r\rho}} u^2 \, d\mathcal{H}^{d-1} = 0$$

for every $r > 0$. It follows that, for fixed $\rho > 0$ (that we will choose small enough), we have

$$\begin{aligned}
1 = \frac{1}{\rho^{d-1+2\lambda}} \int_{\partial B_\rho} p_\lambda^2 \, d\mathcal{H}^{d-1} &\le \frac{2}{\rho^{d-1+2\lambda}} \int_{\partial B_\rho} |p_\lambda - u_r|^2 \, d\mathcal{H}^{d-1} + \frac{2}{\rho^{d-1+2\lambda}} \int_{\partial B_\rho} u_r^2 \, d\mathcal{H}^{d-1} \\
&= \frac{2}{\rho^{d-1+2\lambda}} \int_{\partial B_\rho} |p_\lambda - u_r|^2 \, d\mathcal{H}^{d-1} + 2 \int_{\partial B_1} [u_r]_\rho^2 \, d\mathcal{H}^{d-1} \\
&\le \frac{2}{\rho^{d-1+2\lambda}} \int_{\partial B_\rho} |p_\lambda - u_r|^2 \, d\mathcal{H}^{d-1} + C \,(-\log(\rho))^{-\frac{1-\gamma}{\gamma}},
\end{aligned}$$

where the first equality follows from the λ-homogeneity of p_λ and the last inequality from the rate of decay of $[u_r]_\rho$ to $p_{\lambda,r} \equiv 0$ in (7.8). Choosing first $\rho > 0$ and then $r = r(\rho) > 0$ we reach a contradiction. □

7.3. **Proof of Theorem 5.** Let $m \in \mathbb{N}_+$ be fixed and let be $x_1, x_2 \in \mathcal{S}^{2m}$. Let p_{x_1} and p_{x_2} be the unique blow-ups of u at x_1 and x_2 respectively. Then we can write $p_{x_1} = \lambda_1 \, p_1$ and $p_{x_2} = \lambda_2 \, p_2$, where p_1 and p_2 are normalized such that $p_1, p_2 \in \mathcal{H}_{2m} \setminus \{0\}$. Notice that

$$\|p_1 - p_2\|_{L^\infty(B_1)} \le c(d) \int_{\partial B_1} |p_1(x) - p_2(x)| \, d\mathcal{H}^{d-1}(x) \,, \tag{7.9}$$

since $\|p_1\|_{L^2(\partial B_1)} = 1 = \|p_2\|_{L^2(\partial B_1)}$ and they are $2m$-homogeneous.

Next notice by the triangular inequality

$$\|p_{x_1} - p_{x_2}\|_{L^1(\partial B_1)} \le \|u_{x_1,r} - p_{x_1}\|_{L^1(\partial B_1)} + \|u_{x_1,r} - u_{x_2,r}\|_{L^1(\partial B_1)} + \|u_{x_2,r} - p_{x_2}\|_{L^1(\partial B_1)}$$

Recalling that $u \in C^{1,1/2}$ and that $\nabla u(x_1) = 0$, we estimate the term in the middle with

$$\begin{aligned}
\|u_{x_1,r} - u_{x_2,r}\|_{L^1(\partial B_1)} &\le \int_{\partial B_1} \int_0^1 \frac{|\nabla u(x_1 + rx + t(x_2 - x_1))||x_2 - x_1|}{r^{2m}} \, dt \, d\mathcal{H}^{d-1}(x) \\
&\le C\|u\|_{C^{1,1/2}(B_r(x_1))} \frac{(r + |x_2 - x_1|)^{1/2} \, |x_2 - x_1|}{r^{2m}} \le C|x_1 - x_2|^{1/8m},
\end{aligned} \tag{7.10}$$

where we have set $r := |x_1 - x_2|^{1/4m}$. Moreover, if we assume that r_0 satisfies the inequality $|r_0|(-\log|r_0|)^{-\frac{1-\gamma}{2\gamma}} \le \mathrm{dist}(\{x_1, x_2\}, \partial B_1)$, then by Proposition 7.1 we see that

$$\|u_{x_1,r} - p_{x_1}\|_{L^1(\partial B_1)} + \|u_{x_2,r} - p_{x_2}\|_{L^1(\partial B_1)} \le C\,(-\log(r))^{-\frac{1-\gamma}{2\gamma}} = C\big(-\log|x_1 - x_2|\big)^{-\frac{1-\gamma}{2\gamma}} \tag{7.11}$$

Putting together this inequality with (7.10) and (7.11), we find

$$\|p_{x_1} - p_{x_2}\|_{L^1(\partial B_1)} \le C(-\log|x_1 - x_2|)^{-\frac{1-\gamma}{2\gamma}}\,. \tag{7.12}$$

Next, using (2.2) and (7.5) we can estimate

$$\frac{d}{dr}\left(\frac{H^{x_i}(r)}{r^{d-1+4m}}\right) = 2\frac{\mathcal{W}^{x_i}_{2m}(r)}{r} \le \frac{C}{r(-\gamma\log(r/r_0))^{\frac{1}{\gamma}}}\,,$$

which integrated gives

$$\frac{H^{x_i}(t)}{t^{d-1+4m}} - \lambda^2_{x_i} \le C\,(-\log t)^{-\frac{1-\gamma}{\gamma}} \qquad \text{for all } 0 < t < \mathrm{dist}(x_i, \partial B_1)\,. \tag{7.13}$$

Notice that in the previous integration we have used the fact that, by definition of p_i and by the strong convergence in $L^2(\partial B_1)$ of the blow ups we have

$$\lim_{r\to 0}\frac{H^{x_i}(r)}{r^{d-1+4m}} = \|p_{x_i}\|^2_{L^2(\partial B_1)} = \lambda^2_{x_i}\,.$$

Using (7.10) together with (7.13), we get

$$\begin{aligned}
|\lambda_{x_1} - \lambda_{x_2}|^2 &\le C\left|\lambda^2_{x_1} - \frac{H^{x_1}(r)}{r^{d-1+4m}}\right| + C\left|\frac{H^{x_1}(r)}{r^{d-1+4m}} - \frac{H^{x_2}(r)}{r^{d-1+4m}}\right| + C\left|\lambda^2_{x_2} - \frac{H^{x_2}(r)}{r^{d-1+4m}}\right| \\
&\le C\,(-\log(r))^{-\frac{1-\gamma}{\gamma}} + C\int_{\partial B_1}|u^2_{x_1,r} - u^2_{x_2,r}|\,d\mathcal{H}^{d-1} \\
&\le C\,(-\log(r))^{-\frac{1-\gamma}{\gamma}} + C\,\|u_{x_1,r} - u_{x_2,r}\|^2_{L^1(\partial B_1)} \overset{(7.10)}{\le} C(-\log|x_1 - x_2|)^{-\frac{1-\gamma}{\gamma}}\,,
\end{aligned} \tag{7.14}$$

where the choice of r is the same as above.

Finally, using (7.9), (7.12) and (7.14) we easily conclude that

$$\|p_1 - p_2\|_{L^\infty(B_1)} \le C\,(-\log|x_1 - x_2|)^{-\frac{1-\gamma}{\gamma}} \qquad \text{for every } x_1, x_2 \in K \cap \mathcal{S}^{2m} \Subset B_1 \tag{7.15}$$

where the constant C depends on $m, d, \mathrm{dist}(K, \partial B_1)$.

Now consider the collection of points $\mathcal{S}^{2m}_k$, for some $m \in \mathbb{N}$ and $0 < k < d-2$ and notice that, for every $K \Subset B_1 \cap \mathcal{S}^{2m}_k$, we can apply the Whitney extension theorem [8, Whitney extension theorem] to extend the function $(\tilde{p}_x)_{x\in K} \subset \mathcal{H}_{2m}$, where $\lambda_x\tilde{p}_x = p_x$ is the unique blow up at x to get a function $F \in C^{2m,\log}(\mathbb{R}^d)$, such that $\partial^\alpha F(x) = \partial^\alpha\tilde{p}_x(0)$. Since $x \in \mathcal{S}^{2m}_k$ and the blow-ups are non-degenerate (see Lemma 7.2), there are $d-1-k$ linearly independent vectors $e_i \in \mathbb{R}^{d-1}$, $i = 1, \dots, d-1-k$, such that

$$e_i \cdot \nabla_{x'}\tilde{p}_x \neq 0 \qquad \text{on } \mathbb{R}^d\,.$$

It follows that there are multi-indices β_i of order $|\beta_i| = 2m-1$, such that $\partial_{e_i}\partial^{\beta_i}F(x) = \partial_{e_i}\partial^{\beta_i}\tilde{p}_x(0) \neq 0$. On the other hand

$$\mathcal{S}^{2m} \cap K = K \subset \bigcap_{i=1}^{d-1-k}\{\partial^{\beta_i}F = 0\}$$

so that an application of the implicit function theorem in a neighborhood of each point $x \in K$ combined with the arbitrary choice of K yields that for every $x \in \mathcal{S}^{2m}$ there exists $r = r(x) > 0$ such that

$$\mathcal{S}^{2m}_k \cap B_r(x) \quad \text{is contained in a } k\text{-dimensional } C^{1,\log} \text{ submanifold}\,.$$

From here the conclusion follows. □

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